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Arbeitsgemeinschaft: Twistor D-Modules and the Decomposition Theorem

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Abstract. The purpose of this Arbeitsgemeinschaft is to introduce the notion of twistor $\mathcal{D}$-modules and their main properties. The guiding principle leading this discussion is Simpson's "meta-theorem", which gives a heuristic for generalizing (mixed) Hodge-theoretic results into (mixed) twistor-theoretic results. The strength of the twistor approach is that it enables to enlarge the scope of Hodge theory not only to arbitrary semi-simple perverse sheaves, equivalently semi-simple regular holonomic $\mathcal{D}$-modules via the Riemann-Hilbert correspondence, but also to possibly semi-simple *irregular* holonomic $\mathcal{D}$-modules. An overarching goal for this session is Mochizuki's proof of the decomposition theorem for semi-simple holonomic $\mathcal{D}$-modules on a smooth complex projective variety, first conjectured by Kashiwara in 1996.

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Introduction by the Organizers

The notion of twistor structure has been introduced by Simpson in [Sim97], following a letter to him by Deligne, in order to include the objects related by the Kobayashi-Hitchin correspondence on a compact Kähler manifold, namely stable Higgs bundles with vanishing characteristic classes and simple flat holomorphic bundles, into a larger family, so that to equip the enlarged moduli space of a hyperkähler structure, extending thereby the original construction of Hitchin [Hit87]. A more in-depth elaboration of this fundamental construction has recently been developed by Simpson [Sim08, Sim21]. The talks in this session will however not pursue in this direction, but the participants may consult these recent papers to better understand the backgrounds of the twistor approach.

Already in 1997, Simpson envisioned the following "meta-theorem":

Meta-theorem (Simpson, [Sim97])**.** *If the words "mixed Hodge structure" (resp. "variation of mixed Hodge structure") are replaced by the words "mixed twistor structure" (resp. "variation of mixed twistor structure") in the hypotheses and conclusions of any theorem in Hodge theory, then one obtains a true statement. The proof of the new statement will be analogous to the proof of the old statement.*

The aim of this session is to illustrate this meta-theorem with the proof of a conjecture made by Kashiwara in various talks around 1996 [Kas98], that is, the *decomposition theorem for semi-simple holonomic $\mathcal{D}$-modules*. One can summarize the twistor approach by saying that twistor theory provides with a theory of weights a category of objects that do not naturally exhibit a weight structure.

The decomposition theorem was first proved in [BBDG82] for pure perverse ℓ-adic sheaves on varieties over a finite field of characteristic $p \neq \ell$. It asserts that the push-forward by a proper morphism of such an object decomposes, in the derived category, into its perverse cohomology sheaves and each such is semi-simple. Furthermore, by a technique of reduction to characteristic p, Beilinson, Bernstein, Deligne and Gabber were able to extend it to semi-simple perverse sheaves on smooth complex projective varieties which are of geometric origin. M. Saito [Sai88, Sai90b] developed at the end of the eighties a completely new strategy to extend this result on complex varieties to any (semi-)simple perverse sheaf whose associated local system underlies a polarizable variation of $\mathbb{Q}$-Hodge structure, not necessarily of geometric origin. The theory of *mixed Hodge modules* is now widely used in Algebraic geometry.

The decomposition theorem for semi-simple perverse sheaves on smooth complex projective varieties has now two proofs. One is by Drinfeld [Dri01], extending the proof of [BBDG82] by reduction to characteristic p, relaxing the assumption of geometric origin by relying on a conjecture of de Jong, later proved by Böckle-Khare [BK06] and Gaitsgory [Gai07]. The other one, which will be the main topic of this session, applies the meta-theorem of Simpson to the strategy of M. Saito by introducing the category of polarizable twistor $\mathcal{D}$-modules. The starting point were the papers [Moc02, Sab05], and the proof was achieved in [Moc07]. One key point in M. Saito's theory is the use of Schmid's norm estimates and orbit theorems [Sch73] through the Hodge-Zucker theorem [Zuc79] yielding the Hodge theorem for the intersection complex of a polarizable variation of Hodge structure on a punctured compact Riemann surface. The "twistor analogues" of these results were provided by Simpson [Sim90].

Furthermore, Simpson also raised in [Sim90] the following:

"*A question is whether one could set up a correspondence in which some non-tame harmonic bundles correspond to systems of equations with irregular singularities.*"

The strength of the twistor approach is that it enables to enlarge the scope of Hodge theory not only to arbitrary semi-simple perverse sheaves, equivalently semi-simple regular holonomic $\mathcal{D}$-modules via the Riemann-Hilbert correspondence, on smooth complex projective varieties, but also to possibly irregular semi-simple holonomic $\mathcal{D}$-modules. In such a way, the analogy with the arithmetic theory of pure ℓ-adic perverse sheaves on varieties over finite fields is made stronger, as the latter does not restrict to tame objects, contrary to M. Saito's Hodge modules, whose associated $\mathcal{D}$-modules are known to be regular holonomic. For example, the analogue of the Katz-Laumon ℓ-adic Fourier transformation exists in the theory of mixed twistor $\mathcal{D}$-modules.

Simpson's insight has first been confirmed in dimension 1 [Sab99, BB04] and, after a first step in [Sab09], the full development of the theory of wild twistor $\mathcal{D}$-modules, both in the pure and the mixed case, has been achieved by T. Mochizuki in the sequence of works [Moc11, Moc15], extending [Moc07]. In particular, the monographs [Moc07, Moc11] provide the complete proof in the complex analytic setting of the conjecture of Kashiwara for semi-simple holonomic $\mathcal{D}$-modules (note that a wild analogue of Drinfeld's proof for the regular case still does not exist). An overview of this work is provided in [Moc14] (see also [Moc15, Chap. 1], and [Sab13] for a focus on the decomposition theorem). Let us also mention that the decomposition theorem in the Kähler setting, for regular holonomic $\mathcal{D}$-modules underlying a polarizable pure twistor $\mathcal{D}$-module, has recently been proved by T. Mochizuki [Moc22] (see also [Sai90a, Sai22] for the case of $\mathcal{O}_X$).

The introductory chapters of [Sab05], [Moc07] and [Moc11] are helpful for understanding how the various arguments fit together, leading to the proof of Kashiwara's conjecture.

Let us finally mention that Hodge module theory or twistor $\mathcal{D}$-module theory is not the only way to the decomposition theorem in complex algebraic geometry. For the case of regular holonomic $\mathcal{D}$-modules (or perverse sheaves) of geometric origin, so that Hodge theory is involved, we mention the work of de Cataldo and Migliorini [dCM02, dCM05, dCM09] (see also [Wil17]). A similar idea has been developed in [WY21] for proving the decomposition theorem in the case of a semi-simple local system on a smooth projective variety, relative to a morphism to another variety.

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Arbeitsgemeinschaft: Twistor D-Modules and the Decomposition Theorem

Table of Contents

Abstracts

Pure and mixed twistor structures, comparison with Hodge structures

Joakim Faergeman

In this talk, we introduce the notation of a mixed twistor structure. In particular, we investigate

- The relationship between mixed twistor structures and (complex) mixed Hodge structures.
- Basic properties of mixed twistor structures.

Let us clarify on point one above. If one has a vector bundle E on $\mathbb{P}^1$, one may take the fiber at $1 \in \mathbb{P}^1$ to obtain a complex vector space. This provides a functor between the category of mixed twistor structures and complex vector space. If furthermore E was $\mathbb{G}_m$-equivariant (for the natural $\mathbb{G}_m$-action on P1), one shows that $E_{|_1}$ is naturally equipped with two decreasing filtrations. Moreover, E is a pure twistor structure of weight w if and only if the two decreasing filtrations on $E_{|_1}$ are w-opposed. In this way, we obtain a functor

$$\{\mathbb{G}_m - \text{equivariant} MTS\} \to \text{MHS}$$

where the left hand side denotes the category of $\mathbb{G}_m$-equivariant mixed twistor structures and the right hand side denotes the category of mixed Hodge structures. One shows that the above functor is in fact an equivalence. The inverse functor is given by the Rees construction that takes a vector space V equipped with two decreasing filtrations and produces a vector bundle. The Rees construction naturally upgrades to a functor from the category of MHS to the category of $\mathbb{G}_m$-equivariant MTS. Next, let us clarify on bullet point two above. The main result of this talk is the fact that the category of mixed twistor structures is Abelian (in particular, the functor described above is an exact functor of Abelian categories). The proof is very similar to the usual proof that the category of (complex) mixed Hodge structures is Abelian. Namely, the crux of the proof consists of showing that any map of mixed twistor structures is strict. As a corollary, one obtains that the functors Gr_W and $-|_t$ are exact for $t \in \mathbb{P}^1$. Here Gr_W is the functor that takes a mixed twistor structure and produces the associated graded vector bundle. The functor $-|_t$ takes a mixed twistor structures and spits out the fiber at t of the underlying vector bundle.

Twistor structures over a complex manifold and polarizations

Marielle Ong

Variations of mixed twistor structures over X are locally free sheaves of modules over $X \times \mathbb{P}^1$, endowed with a strict filtration and a differential operator that respects the filtration and squares to zero. We say that a variation is pure if the associated graded pieces of the strict filtration are pure twistor structures over $\{x\} \times \mathbb{P}^1$ for $x \in X$ and are concentrated in one degree. Every variation gives rise to an underlying λ-connection by taking the fibers over points on $\mathbb{P}^1$. We then return to variations over a point and introduce the notion of polarizations of pure twistor structures. In fact, polarized pure twistor structures of weight 0 are equivalent to complex Hermitian vector spaces. Since Hodge structures give rise to twistor structures, we verify that polarizations of Hodge structure do indeed induce polarizations of twistor structures. We then generalize these notions to polarizations of variations of pure twistor structures over X. The twistor substructure of a polarized variation is polarized and is a direct summand. Consequently, the category of polarized variations of pure twistor structures over X is semisimple.

Harmonic bundles and equivalence with smooth polarized twistor structures of weight 0

Mads Bach Villadsen

Let (E, θ) be a holomorphic Higgs bundle on a complex manifold, with underlying $\mathcal{C}^\infty$-vector bundle H, and let h be a hermitian metric on H. Let ∂_E be the differential operator on H with symbol ∂ such that $\partial h(u, v) = h(\partial_E u, v) + h(u, \bar{\partial}_E)$ and $\bar{\partial} h(u, v) = h(\bar{\partial}_E u, v) + h(u, \partial_E)$, where $\bar{\partial}_E$ is the holomorphic structure for E. Let $\theta^\dagger$ be the adjoint of θ with respect to h, and let $\bar{\partial}_V = \bar{\partial}_E + \theta^\dagger$ and $\nabla = \partial_E + \theta$. If $\bar{\partial}_V^2 = 0$ and ∇ defines a flat holomorphic connection on the corresponding holomorphic bundle V, then (E, θ, h) is said to be a harmonic Higgs bundle, or equivalently, (V, ∇, h) is a harmonic flat bundle.

We discuss two concrete examples. For a variation of Hodge structures $(H = \bigoplus_{p+q=w} H^{p,q}, \nabla)$ with polarization k, the polarization together with Griffiths transversality shows that $\nabla(H^{p,q}) \subset (H^{p,q} \oplus H^{p-1,q+1}) \otimes A^{1,0}(X)$; let $\nabla = \partial_E + \theta$ be the corresponding decomposition. The corresponding decomposition of the holomorphic structure on H as a flat bundle yields $\overline{\partial}_E$ and $\theta^\dagger$. These operators, together with the Hodge metric h associated to k, give a harmonic Higgs bundle.

On a punctured disc, we write down equations for rank one harmonic Higgs bundles directly. Let L be the trivial holomorphic line bundle, with frame e. For $(a, \alpha) \in \mathbb{R} \times \mathbb{C}$, take the metric $h_a(e, e) = |z|^{2a}$ and the Higgs field $\theta = \alpha \frac{dz}{z}$. Then we explicitly compute the remaining operators discussed above, and show that this defines a harmonic bundle.

Finally, we discuss the equivalence, due to Simpson, between harmonic bundles on one side, and polarized variations of twistor structures of weight 0 on the other.

The non-Abelian Hodge correspondence and the Hodge-Simpson theorem

Ko Aoki

In this talk we see two global properties of harmonic bundles over a compact Kähler manifold (X, ω); one is the semisimplicity property and the other is the Hodge property. One key observation used in both proofs is the fact that any harmonic bundle satisfies the Kähler identities; more precisely, for a harmonic bundle $(H, \partial', \partial'', \theta', \theta'', h)$, we have

$$[\Lambda, D'] = i(D'')^*, \qquad [\Lambda, D''] = -i(D')^*,$$

where $D' = \partial' + \theta''$ and $D'' = \partial'' + \theta'$.

As explained in the previous lecture, any harmonic bundle gives us both a flat bundle and a Higgs bundle: The sum $\nabla = \partial' + \partial'' + \theta' + \theta''$ is a flat connection on H, while θ' induces the Higgs field ϑ on the holomorphic vector bundle $\mathcal{E} = (H, \partial'')$. The Corlette–Simpson correspondence determines when a given flat or Higgs bundle over a compact Kähler manifold (X, ω) underlies a harmonic bundle: Corlette's theorem [1] states that a flat bundle on a compact Kähler manifold admits a pluriharmonic metric if and only if it is semisimple, while Simpson's theorem [5] states that a Higgs bundle on a compact Kähler manifold admits a pluriharmonic metric if and only if it is polystable and has trivial Chern classes. It is worth noting that Mochizuki [3] proved the same type of characterization for λ-flat bundles.

We followed the proof of the "only if" part of Corlette's theorem. Fix a flat bundle (H, ∇). For a C^∞-automorphism g on H, we get another connection ${}^g\nabla$. We decompose it along the metric h as ${}^g\nabla = ({}^g\nabla)_h + {}^g\theta$ and consider the value

$$\|{}^g\theta\|^2 = \int_X \langle {}^g\theta, {}^g\theta \rangle \mathrm{Vol}_X,$$

which should be thought of as the energy of g. Then if h is pluriharmonic, the Kähler identities show that the energy function has a critical value at $g = \mathrm{id}$, which implies its semisimplicity.

There is an important corollary of the Corlette correspondence, which can be stated in algebraic geometry: For a map between smooth projective varieties $f \colon Y \to X$ and a local system $\mathbb{L}$, if $\mathbb{L}$ is semisimple, so is $f^*\mathbb{L}$. Indeed, we can easily see this by picking a pluriharmonic metric on the flat bundle associated to $\mathbb{L}$ since the pullback of a harmonic bundle is again a harmonic bundle. No algebraic proof of this fact is known. A natural question is what happens if we instead consider the pushforward of $\mathbb{L}$. Kashiwara's conjecture [2] considers a situation where we have a projective morphism between smooth varieties $f \colon Y \to X$ with a relatively ample line bundle $\mathcal{L}$. It states that for a holonomic D-module N on Y, the pushforward f_+N decomposes into the direct sum of its (shifted) cohomologies $f_+^i N[-i]$ and it satisfies the hard Lefschetz theorem with respect to $c_1(\mathcal{L})$, i.e., the morphism $c_1(\mathcal{L})^i \colon f_+^{-i}N \to f_+^i N$ is an equivalence. This conjecture has a long history, but ultimately it was solved by Mochizuki [4] using the theory of mixed twistor D-modules, which is the topic of this Arbeitsgemeinschaft. In this talk, we focus on a simple instance of this conjecture (or theorem) due to Simpson, which is the case where $X = *$ and N is regular.

We get back to our compact Kähler manifold (X, ω). For a flat bundle (H, ∇), its natural cohomology is the sheaf cohomology of the associated local system $\mathbb{L} = H^\nabla$. For a Higgs bundle $(\mathcal{E}, \vartheta)$, its natural cohomology is the Dolbeault cohomology, i.e., the hypercohomology of the complex $\mathcal{E} \to \Omega^1 \otimes \mathcal{E} \to \cdots$, where the differentials are given by $\vartheta \wedge -$. A standard resolution technique shows that these two are computed as the cohomologies of the complexes of the form

$$C^\infty(X; H) \to C^\infty(X; A^1 \otimes H) \to \cdots$$

with different differentials. Suppose that both arise from one harmonic bundle $(H, \partial', \partial'', \theta', \theta'', h)$. Then these are special cases of the family of complexes of the above form with the differentials given by $aD' + bD''$ for $(a, b) \in \mathbb{C}^2 \setminus 0$. Indeed, the case $(a, b) = (1, 1)$ corresponds to the sheaf cohomology of $\mathbb{L}$ while the case $(a, b) = (0, 1)$ corresponds to the Dolbeault cohomology of $(\mathcal{E}, \vartheta)$.

We followed the construction given in [6]. Again we consider $X \times \mathbb{P}^1$ as a space that is C^∞ in X and holomorphic in $\mathbf{P}^1$. Then we consider

$$H \boxtimes \mathcal{O} \to (A^1 \otimes H) \boxtimes \mathcal{O}(1) \to \cdots$$

where the differentials are given by $\lambda D' + \mu D''$, where λ and μ are the coordinate of $\mathbb{P}^1$ around 0 and ∞, respectively. We pushforward this complex along $X \times \mathbb{P}^1 \to \mathbb{P}^1$ and show that it splits into polarizable pure twistor structures of various weights. The point is again the Kähler identities, which enable us to use the harmonic representative technique as in the constant case. The Lefschetz theorem is clear again by the Kähler identities.

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The notion of $\mathcal{R}$-triple and various functors

Andreas Hohl

In this talk, we introduce the category of $\mathcal{R}$-triples, which will serve as a framework for the theory of twistor D-modules in later talks. The main references are [2, Chap. 1,2] and [1, Chap. 14].

We first define the sheaf of rings $\mathcal{R}_\mathcal{X}$, a certain ring of relative differential operators on $\mathcal{X} = X \times \mathbb{C}_\lambda$ (for X a complex manifold): Let $\mathcal{O}_\mathcal{X}$ be the sheaf of holomorphic functions on $\mathcal{X}$. Then $\mathcal{R}_\mathcal{X}$ is defined to be the $\mathbb{C}$-subalgebra of $\mathrm{End}_\mathbb{C}(\mathcal{O}_\mathcal{X})$ generated by $\mathcal{O}_\mathcal{X}$ and $\lambda p_X^* \Theta_X$, where $p_X \colon \mathcal{X} \to X$ is the projection and Θ_X is the tangent sheaf of X. Equivalently, one can think of $\mathcal{R}_\mathcal{X}$ as follows: The ring of differential operators $\mathcal{D}_X$ on X has a natural order filtration. If we denote by $R_F\mathcal{D}_X = \bigoplus_k (F_k\mathcal{D}_X)\lambda^k$ the associated Rees ring, we can set $\mathcal{R}_\mathcal{X} := \mathcal{O}_\mathcal{X} \otimes_{\mathcal{O}_X[\lambda]} R_F\mathcal{D}_X$.

We discuss some properties and operations for $\mathcal{R}_\mathcal{X}$-modules, such as the equivalence between left and right modules as well as direct and inverse image functors, whose constructions are mostly analogous to the theory of $\mathcal{D}$-modules. In particular, direct and inverse images are defined via a transfer bimodule $\mathcal{R}_{\mathcal{X}\to\mathcal{Y}}$. We also discuss the notion of holonomicity: Via the concept of good filtration, one can define a notion of *good* $\mathcal{R}_{\mathcal{X}}$-module and associate to each such module a characteristic variety, which is a subset of $T^*X \times \mathbb{C}_\lambda$. One calls the $\mathcal{R}_{\mathcal{X}}$-module holonomic if its characteristic variety is contained in $\Lambda \times \mathbb{C}_\lambda$ for some conic Lagrangian $\Lambda \subset T^*X$.

As an example, we firstly mention the case of a filtered holonomic $\mathcal{D}_X$-module, which (via the Rees construction) gives a holonomic $\mathcal{R}_{\mathcal{X}}$-module $\mathcal{M} := \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_X[\lambda]} R_F M$. Secondly, we show how a holomorphic vector bundle with a holomorphic family of λ-connections is naturally an $\mathcal{R}_{\mathcal{X}}$-module: Let E be a smooth vector bundle on $X \times \mathbb{C}_\lambda$ with holomorphic structure in the λ-direction given by

$$d''_\lambda \colon E \to E \otimes p_\lambda^{-1} A^{0,1}_{\mathbb{C}_\lambda}$$

and an operator

$$\mathbb{D} \colon E \to E \otimes p_X^{-1} A^1_X$$

satisfying $\mathbb{D}(f \cdot e) = e \otimes (\lambda d_X + \overline{d}_X) f + f\mathbb{D}(e)$ for some local (smooth) sections f of $\mathcal{C}^\infty_{\mathcal{X}}$ and e of E, respectively. Then a natural action of $\mathcal{R}_{\mathcal{X}}$ on $\mathcal{E} := \ker(\mathbb{D}^{0,1} + d''_\lambda)$ is defined by setting

$$\xi \cdot s := \mathbb{D}^{1,0}_{\lambda^{-1}\xi}(s)$$

for ξ a local section of $\lambda p_X^* \Theta_X$ and s a local section of $\mathcal{E}$. One can check that, via this definition, the λ-twisted Leibniz rule of $\mathbb{D}^{1,0}$ actually gives an (untwisted) Leibniz rule for the action of $\mathcal{R}_{\mathcal{X}}$, as desired.

Next, we introduce the notion of a sesquilinear pairing. For this, we define a certain "conjugation functor" for $\mathcal{R}_{\mathcal{X}}$-modules: First of all, let us remark that for any complex manifold Y, we have the complex conjugate manifold $\overline{Y}$ (the same underlying topological space equipped with the sheaf of functions that are antiholomorphic with respect to the complex structure on Y). We get a "naïve" conjugation functor turning an $\mathcal{O}_Y$- (or $\mathcal{D}_Y$-)module into an $\mathcal{O}_{\overline{Y}}$- (or $\mathcal{D}_{\overline{Y}}$)-module. (This is basically the pullback via the natural morphism of ringed spaces $\overline{Y} \to Y$ induced by complex conjugation.) Now, let us write $\Omega_0 = \mathbb{C}_\lambda = \mathbb{P}^1 \setminus \{\infty\}$ and $\Omega_\infty = \mathbb{P}^1 \setminus \{0\}$, then the antiholomorphic involution of $\mathbb{P}^1$ given by $\lambda \mapsto -1/\overline{\lambda}$ induces a morphism of ringed spaces $\Omega_\infty \to \overline{\Omega_0}$. Taking the product with $\mathrm{id}_{\overline{X}}$ and composing with the natural morphism of ringed spaces $\overline{X \times \Omega_0} \to X \times \Omega_0$ as above, we get a morphism of ringed spaces $\overline{X} \times \Omega_\infty \to X \times \Omega_0$. Pullback along this morphism turns a module $\mathcal{M}$ over $\mathcal{R}_{\mathcal{X}} = \mathcal{R}_{X\times\Omega_0}$ into a module denoted by $\overline{\mathcal{M}}$ over $\mathcal{R}_{\overline{X}\times\Omega_\infty}$.

Given two $\mathcal{R}_{\mathcal{X}}$-modules $\mathcal{M}_1, \mathcal{M}_2$, a sesquilinear pairing is then a pairing

$$C \colon \mathcal{M}_1|_{X\times S^1} \otimes_{\mathcal{O}_{\mathbb{C}_\lambda}|_{S^1}} \overline{\mathcal{M}_2|_{X\times S^1}} \to \mathcal{D}b_{X\times S^1/S^1}$$

with values in (relative) distributions.

With this notion in hand, we introduce the category of $\mathcal{R}$-triples, whose objects are of the form $(\mathcal{M}_1, \mathcal{M}_2, C)$, and mention some operations on them, such as Tate twists and proper direct images. We also mention the special case of smooth triples, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are locally free, which makes the pairing C take values in smooth functions and hence enables us to define an inverse image functor for these objects.

Finally, we formulate the notion of (pure) twistor structure in the language of $\mathcal{R}$-triples, first in the case where $\dim X = 0$, then in the case of an arbitrary complex manifold X (variations of pure twistor structures):

An $\mathcal{R}$-triple $(\mathcal{H}_1, \mathcal{H}_2, C)$ in the case $X = \{\mathrm{pt}\}$ is called a *pure twistor structure* of weight w if $\mathcal{H}_1$ and $\mathcal{H}_2$ are free, the pairing C is non-degenerate with values in $\mathcal{O}_{\mathbb{C}_\lambda}|_{S^1}$ and the $\mathcal{O}_{\mathbb{P}^1}$-module obtained by gluing $(\mathcal{H}_1|_\Delta)^\vee$ and $\overline{\mathcal{H}_2|_\Delta}$ via the resulting isomorphism $\overline{\mathcal{H}_2|_{S^1}} \simeq (\mathcal{H}_1|_{S^1})^\vee$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(w)^d$ for some $d \in \mathbb{Z}_{\geq 0}$.

For general X, one recalls that variations of twistor structures (as defined by Simpson) are related to families of λ-connections, which in turn give smooth $\mathcal{R}_\mathcal{X}$-modules by the above example. This motivates the definition in the language of $\mathcal{R}$-triples: A *variation of twistor structures* of weight w is a smooth $\mathcal{R}$-triple whose restriction over any $x \in X$ is a pure twistor structure as above.

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Specializability, S-decomposability, and canonical extensions

Yichen Qin

We will explain how to define specializability, S(support)-decomposability, and canonical extensions for $\mathcal{R}$-modules and $\mathcal{R}$-triples, following a similar approach for $\mathcal{D}$-modules. Throughout the lecture, we will emphasize the similarities and distinctions between the constructions in both settings.

The case of $\mathcal{D}$-modules

Let X be a smooth complex manifold and $f\colon X \to \mathbb{C}$ a holomorphic function. We denote $\mathcal{D}_X$ as the sheaf of differential operators and $\mathcal{M}$ as a (left) coherent $\mathcal{D}_X$-module.

Specializability. We start with a special case. Suppose there exists a smooth complex manifold X_0 such that $X = X_0 \times \mathbb{C}_t$ and $f = t$. The *V-filtration on $\mathcal{D}_X$ along t* is given locally on X, by

$$V_k\mathcal{D}_X := \{P \in \mathcal{D}_X \mid P(t)^i \subset (t)^{k+i} \quad \forall i\}. \tag{1}$$

It can be demonstrated that $V_k\mathcal{D}_X \cdot V_\ell\mathcal{D}_X \subset V_{k+\ell}\mathcal{D}_X$ holds for any $k, \ell \in \mathbb{Z}$. A *V-filtration on M along t* is an increasing, exhaustive, $\mathbb{Z}$-indexed filtration $U_\bullet\mathcal{M}$ of coherent $V_0\mathcal{D}_X$-modules, satisfying $V_\ell\mathcal{D}_X \cdot U_k\mathcal{M} \subset U_{k+\ell}\mathcal{M}$ for any $k, \ell \in \mathbb{Z}$.

We call $\mathcal{M}$ *specializable along* t if, locally on X, there exists a good V-filtration $U_\bullet\mathcal{M}$, and a Bernstein polynomial $b_U(s) \in \mathbb{C}[s]\backslash\{0\}$, with roots' real parts lying in the interval $[0,1)$, such that the following condition holds for any $k \in \mathbb{Z}$:

$$b_U(-(\partial_t \cdot t + k)) \cdot \mathrm{gr}_k^U \mathcal{M} = 0. \tag{2}$$

We can demonstrate that the V-filtration is locally unique by employing the goodness of $U_\bullet\mathcal{M}$. Consequently, specializability imples the global existence of a V-filtration on $\mathcal{M}$.

For simplicity, we assume $\mathcal{M}$ to be $\mathbb{R}$-specializable henceforth, meaning it is specializable along t and the roots of the Bernstein polynomial are all real numbers. We can refine the V-filtration to a filtration $V_\bullet\mathcal{M}$ indexed by $\mathbb{R}$, referred to as the *canonical* V*-filtration (Kashiwara–Malgrange filtration)*. The jumps of the V-filtration occur precisely at $A + \mathbb{Z}$ for a finite set $A \subset \mathbb{R}$, and the operator $(-\partial_t \cdot t - a)$ acts nilpotently on $\mathrm{gr}_a^V \mathcal{M}$ for each $a \in \mathbb{R}$.

Returning to the general case, for a smooth complex manifold X and a holomorphic function f on X, we denote the graph embedding $\iota_f \colon X \to X \times \mathbb{C}_t$ that maps x to $(x, f(x))$, with t being the coordinate of the $\mathbb{C}$ factor. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is considered ($\mathbb{R}$-)*specializable along* f if $\iota_{f,+}\mathcal{M}$ is ($\mathbb{R}$-)specializable along t.

The nearby and vanishing cycles. If $\mathcal{M}$ is specializable along f, then the *nearby cycles of* $\mathcal{M}$ *along* f are defined by $\psi_{f,\alpha}\mathcal{M} := \mathrm{gr}_\alpha^V(\iota_{f,+}\mathcal{M})$ for $\alpha \in [-1, 0)$, and the *vanishing cycle of* $\mathcal{M}$ *along* f is defined by $\phi_{f,0}\mathcal{M} := \mathrm{gr}_0^V(\iota_{f,+}\mathcal{M})$. Let can and var be the morphisms $-\partial_t \colon \psi_{f,-1}\mathcal{M} \to \phi_{f,0}\mathcal{M}$ and $t \colon \phi_{f,0}\mathcal{M} \to \psi_{f,-1}\mathcal{M}$ respectively. We have the following quiver:

$$\begin{tikzcd} \psi_{f,-1}\mathcal{M} \arrow[rr, bend left, "\mathrm{can}"] && \phi_{f,0}\mathcal{M} \arrow[ll, bend left, "\mathrm{var}"] \end{tikzcd} \tag{3}$$

In this context, the composition var $\circ$ can (resp. can $\circ$ var) acts on $\psi_{f,0}\mathcal{M}$ (resp. $\phi_{f,0}\mathcal{M}$) nilpotently, denoted by N. The *monodromy operator* on $\psi_{f,-1}$ (resp. $\phi_{f,0}$) is defined as $T = \exp(2\pi i(-t \cdot \partial_t))$ (resp. $T = \exp(2\pi i(-\partial_t \cdot t))$).

The minimal extension and S-decomposability. We call $\mathcal{M}$ a *minimal extension along* f if the map can: $\Psi_{f,-1}\mathcal{M} \to \Phi_{f,0}\mathcal{M}$ is surjective and the map var: $\Phi_{f,0}\mathcal{M} \to \Psi_{f,-1}\mathcal{M}$ is injective. This is equivalent to stating that $\mathcal{M}$ has no coherent subquotient $\mathcal{D}_X$-modules supported on $Z(f)$. Furthermore, $\mathcal{M}$ is called S*-decomposable along* f if $\Phi_{f,0} = \mathrm{im\,can} \oplus \ker \mathrm{var}$ holds true, which is equivalent to $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$, where $\mathcal{M}'$ is a minimal extension along f and $\mathcal{M}''$ is supported on $Z(f)$.

Nearby cycles and proper pushforward. For simplicity, we assume that $X = X_0 \times \mathbb{C}_t$, $Y = Y_0 \times \mathbb{C}_t$, and $f_0 \colon X_0 \to Y_0$ be a proper holomorphic map between smooth complex varieties. We put $f = f_0 \times \mathrm{id} \colon X \to Y$. Assume that $\mathcal{M}$ is specializable along t, then taking nearby cycles $\psi_{t,\alpha}$ and the vanishing cycle $\phi_{t,0}$ commutes with the proper pushforward $\mathcal{H}^i f_+$.

The case of $\mathcal{R}$-modules and $\mathcal{R}$-triples

Let $\mathcal{X} = X \times \mathbb{C}_\lambda$, and $\mathcal{M}$ a (left) coherent $\mathcal{R}_\mathcal{X}$-module, and $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$ a (left) coherent $\mathcal{R}$-triple. The main word that makes everything in this setting work is *strictness*. More precisely, $\mathcal{M}$ is said to be *strict at* λ_0 if it has no $(\lambda-\lambda_0)$-torsion, and is *strict* if it is strict at all points $\lambda_0 \in \mathbb{C}$. We call $\mathcal{T}$ strict if $\mathcal{M}_1$ and $\mathcal{M}_2$ are strict.

Specializability. For simplicity, assume that $X = X_0 \times \mathbb{C}_t$, with the general case being analogous to the situation of $\mathcal{D}_X$-modules. The V-filtration on $\mathcal{R}_\mathcal{X}$ along t is defined similarly to that on $\mathcal{D}_X$. For a given $\lambda_0 \in \mathbb{C}$, we define *V-filtrations at* λ_0 of $\mathcal{M}$ in a manner akin to the case of $\mathcal{D}_X$-modules, by replacing $\mathcal{D}_X$ with $\mathcal{R}_\mathcal{X}$ and requiring the filtration to be defined on a neighborhood of $X \times \lambda_0$.

Let $\eth_t = \lambda\partial_t$. A coherent $\mathcal{R}_\mathcal{X}$-module $\mathcal{M}$ is *specializable along t at* λ_0 if there exists a good V-filtration $U_\bullet^{(\lambda_0)}\mathcal{M}$ at λ_0, and a Bernstein polynomial $b_U(s)$, having roots as functions on λ, such that $b_U(-\eth_t \cdot t - k\lambda) \cdot \mathrm{gr}_k^{U^{(\lambda_0)}}\mathcal{M} = 0$ for any k. More precisely, there exists a finite set $A(\lambda_0) \subset \mathbb{R} \times \mathbb{C}$, such that the roots of $b_U(s)$ are

$$\{-\mathfrak{e}(\lambda, u) \mid u = (a, \alpha) \in A(\lambda_0) + \mathbb{Z} \times \{0\}, \quad 0 \leq \mathfrak{p}(\lambda_0, u) < 1\},$$

where $\mathfrak{p}(\lambda_0, (a, \alpha)) = a - 2\mathrm{Re}(\lambda_0\bar{\alpha})$ and $\mathfrak{e}(\lambda, (a, \alpha)) = \alpha - a\lambda - \bar{\alpha}\lambda^2$. The choice of Bernstein relations can be justified by the Basic example [4, Exe. 1.2]. We also have local uniqueness results [2, Lem. 22.3.4] for the filtration $U^{(\lambda_0)}$ and $A(\lambda_0)$, and we can refine $U^{(\lambda_0)}$ to the *canonical V-filtration along t at* λ_0 $V_\bullet^{(\lambda_0)}$ indexed by $\mathbb{R}$, such that $\prod_{\mathfrak{p}(\lambda_0,u)=c}(-\eth_t \cdot t + \mathfrak{e}(\lambda, u))$ acts unipotently on $\mathrm{gr}_c^{V^{(\lambda_0)}}\mathcal{M}$. As the roots of the Bernstein relations are in fact functions on λ, the V-filtration is not defined globally with respect to $\mathbb{C}_\lambda$.

We say that $\mathcal{M}$ is *specializable along* t if it is specializable along t at any $\lambda_0 \in \mathbb{C}_\lambda$ and the graded quotient $\mathrm{gr}_c^{U^{(\lambda_0)}}\mathcal{M}$ is strict for each $c \in \mathbb{R}$. In this case, the set $A(\lambda_0)$ does not depend on λ_0 and we denote by $\mathcal{KMS}(\mathcal{M}, t)$ the set $A + \mathbb{Z} \times \{0\}$. Moreover, $\mathcal{M}$ is called *strict specializable along* t if it is specializable along t and

(1) the morphisms $t\colon V_a^{(\lambda_0)}\mathcal{M} \to V_{a-1}^{(\lambda_0)}\mathcal{M}$ are isomorphism for $a < 0$,

(2) the morphisms $\eth_t\colon \mathrm{gr}_a^{V^{(\lambda_0)}}\mathcal{M} \to \mathrm{gr}_{a+1}^{V^{(\lambda_0)}}\mathcal{M}$ are surjective for $a > -1$.

For an $\mathcal{R}_\mathcal{X}$-triple $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$, it is specializable along t is $\mathcal{M}_1$ and $\mathcal{M}_2$ are so.

The nearby cycle and Vanishing cycles. Although the V-filtrations $V^{(\lambda_0)}\mathcal{M}$ are defined locally in terms of λ_0, the nearby cycles are defined globally. We define $\psi_{t,u}^{(\lambda_0)}\mathcal{M} \subset \mathrm{gr}_{\mathfrak{p}(\lambda_0,u)}^{V^{(\lambda_0)}}\mathcal{M}$ locally as the generalized eigenspace of $(-\eth_t \cdot t + \mathfrak{e}(\lambda_0, u))$, and we glue them by a compatibility lemma [2, Lem. 22.3.5] to get $\psi_{t,u}\mathcal{M}$.

In the case of $\mathcal{R}$-triples, we need to define the nearby cycles and the vanishing cycle for a sesquilinear form $C\colon \mathcal{M}_1 \mid_{X\times S^1} \times \mathcal{M}_2 \mid_{X\times S^1} \to \mathfrak{D}b_{X\times S^1/S^1}$. Sabbah defines it as the residue of a Mellin transform [3, (3.6.10)]. As for the vanishing cycle of the sesquilinear pairing $\phi_{t,0}C$, we cannot simply define it as $\psi_{t,(0,0)}C$ because $\psi_{t,(0,0)}C$ is 0 when $\mathcal{M}_i$ are supported on X_0, see [3, § 3.6b] for more details.

After defining the nearby and vanishing cycles of C, we introduce $\psi_{t,u}\mathcal{T}$ and $\phi_{t,0}\mathcal{T}$ as the triplets $(\psi_{t,u}\mathcal{M}_1, \psi_{t,u}\mathcal{M}_2, \psi_{t,u}C)$ and $(\psi_{t,(0,0)}\mathcal{M}_1, \psi_{t,(0,0)}\mathcal{M}_2, \phi_{t,0}C)$ respectively. Similar to the case of $\mathcal{D}_X$-modules, we have the can and var quiver for $\mathcal{T}$ like (3).

The minimal extension and the strict S-decomposability. These two notions are defined in a manner akin to those for $\mathcal{D}$-modules.

Nearby cycles and proper pushforward. Adopting the notation from the case of $\mathcal{D}_X$-modules, let $F = f \times \mathrm{id}\colon \mathcal{X} = X \times \mathbb{C}_\lambda \to \mathcal{Y} = Y \times \mathbb{C}_\lambda$. Compared to the case of $\mathcal{D}_X$-modules, we not only assume that $\mathcal{T}$ is strict specializable along t, but also require the strictness for $\mathcal{H}^i F_+ \psi_{t,u}\mathcal{T}$ for any $i \in \mathbb{Z}$ and $u \in \mathcal{KMS}(\mathcal{M}, t)$. Then, taking the nearby cycles $\psi_{t,u}$ commutes with the proper pushforward $\mathcal{H}^i F_+$ for any u [2, Lem. 22.10.5]. If, in addition, $\mathcal{T}$ and $\mathcal{H}^i F_+\mathcal{T}$ are strictly S-decomposable along t for any i, taking the nearby cycles $\phi_{t,0}$ commutes with the proper pushforward $\mathcal{H}^i F_+$[2, Lem. 22.10.5].

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Polarizable pure twistor $\mathcal{D}$-modules

Chuanhao Wei

I will first give the inductive definition of the category of (regular) pure twistor $\mathcal{D}$-modules using $\mathcal{R}$-triples. Some basic properties about such category will be given with sketched proof, including the fact that it is an abelian category. Then, I will introduce the notion of sesquilinear duality of some weight of an $\mathcal{R}$-triple. In the case that X is a point, I will give the definition of a polarized pure twistor structure of some weight w in terms of R-triples equipped with a sesquilinear duality of weight w. Then, the inductive definition of the category of polarizable (regular) twistor $\mathcal{D}$-modules will be given. The purely imaginary case will also be explained. Lastly, two main theorems: Decomposition Theorem (Kashiwara's Conjecture), and the generalization of Corlette-Simpson correspondence will be stated without proof.

Sketch of proofs of Statements 1, and the "easy" direction of Statement 2

Marco Hien

Part 1: Proof of the Decomposition Theorem. For a complex analytic manifold X we have the following categories of twistor D-modules:

- $\mathrm{MT}^{(\mathrm{r})}(X, w)^{(\mathrm{p})}$: polarizable pure regular twistor D-modules of weight w,
- $\mathrm{MLT}^{(\mathrm{r})}(X, w; 1)^{(\mathrm{p})}$: polarizable pure regular graded Lefschetz twistor D-modules of weight w (with Lefschetz degree 1).

By standard arguments and a result of Corlette and Simpson, the Decomposition Theorem for the derived direct image of a semisimple local system $\mathcal{F}$ with respect to a proper holomorphic map $f : U \to Y$ from an open subset $U \subset X$ of a smooth projective variety X into a complex manifold Y can be deduced from the following two results on twistor D-modules:

Theorem 1. *(Theorem 6.1.1. in* [Sab05]*) Let $f : X \to Y$ be a projective morphism of complex analytic manifolds and $(\mathcal{T}, \mathcal{S})$ a polarized object in $\mathrm{MT}^{(\mathrm{r})}(X, w)^{(\mathrm{p})}$. If $\mathcal{L}_c$ denotes the Lefschetz operator associated to a relative ample line bundle, the tuple*

$$(1) \qquad \Big(\bigoplus_i f_\dagger^i \mathcal{T}, \mathcal{L}_c, \bigoplus_i f_\dagger^i \mathcal{S}\Big)$$

is a polarized object in $\mathrm{MLT}^{(\mathrm{r})}(X, w; 1)^{(\mathrm{p})}$.

Theorem 2. *(Theorem 6.1.3 in* [Sab05]*) Any smooth polarized twistor structure of weight w on X is a polarized object of $\mathrm{MT}^{(\mathrm{r})}(X, w)^{(\mathrm{p})}$.*

In the situation of the Decomposition Theorem, a result of Corlette and Simpson yields that the $\mathcal{D}_X$-module associated to an irreducible local system $\mathcal{F}$ on X underlies a smooth polarized twistor structure of weight 0 to which Theorem 2 can be applied in order to prepare for a subsequent application of Theorem 1 to obtain the relative Hard Lefschetz result leading to the Decomposition Theorem.

Sketch of the proof of Theorem 1. The proof is realized by induction on the pair $(\dim(\mathrm{supp}(\mathcal{T}), \dim f(\mathrm{supp}(\mathcal{T})))$.

- $(n, m) = (1, 0)$ is treated seperately in a subsequent lecture.
- $(n, m) \Rightarrow (n+1, m+1)$: the essential case is to consider an object $(\mathcal{T}, \mathcal{S}) = (\mathcal{M}, \mathcal{M}, C, \mathrm{Id})$ on X of weight 0 with strict support Z of dimension $n+1$ such that $\dim f(Z) = m+1$. One has to prove that the inductive properties characterising the objects in $\mathrm{MLT}^{(\mathrm{r})}(X, 0; 1)^{(\mathrm{p})}$ hold for (1). For example, for any holomorphic function $t : V \to \mathbb{C}$ on some open $V \subset Y$, one of the tasks is to prove strict specializabilty and S-decomposability of $f_\dagger^i \mathcal{T}$ along $t = 0$. This is achieved by carefully analysing the behaviour of the direct images $f_\dagger^i \Psi_g(\mathcal{T})$ of the nearby cycles $\Psi_g(\mathcal{T})$ along $g = t \circ f$ taking into account the natural monodromy filtration on the latter and using

the inductive definition of $\mathrm{MT}^{(\mathrm{r})}(X,w)^{(\mathrm{p})}$ in terms of these nearby cycle functors.

- $(\leq n-1,0) \Rightarrow (n,0)$: without loss of generalization one can assume $X = \mathbb{P}^N$. The essential idea is to use a Lefschetz pencil with a suitable choice of axis A. The blow-up $\pi : \tilde{X} \to X$ of A induces a map $f : \tilde{X} \hookrightarrow X \times \mathbb{P}^1 \to \mathbb{P}^1$. For a generic choice of A the pull-back $\pi^+(\mathcal{T},\mathcal{S})$ of the object $(\mathcal{T},\mathcal{S})$ with strict support Z has strict support the blow-up $\tilde{Z}$ of Z and the fibres of $f|\tilde{Z}$ are of dimension $n-1$. This allows to apply the induction hypothesis by using a Gysin map for the inclusion of the fibre (as in [Moc11, 17.3]). Finally, the original pair $(\mathcal{T},\mathcal{S})$ is a direct summand of $\pi_+\pi^+(\mathcal{T},\mathcal{S})$.

Sketch of the proof of Theorem 2 by induction on $\dim(X)$. The cases $\dim(X) = 0$ or 1 are easily verified. For $\dim(X) \geq 2$ again one has to ensure the inductive properties with respect to taking nearby/vanishing cycles along any holomorphic function t. The case $t = x_1 \cdots x_r$ of a product of local coordinates can be verified by direct computations. The general case is reduced to the latter by blow-ups and ramifications using arguments similar to the proof of the final induction step describe above.

Part 2: Sketch of the proof of the "easy" direction of Statement 2. The statement to be proved is the following

Theorem 3. *(Theorem 4.2.12 in* [Sab05]*) Let X be a smooth projective complex variety and $(\mathcal{T} = (\mathcal{M}',\mathcal{M}'',C),\mathcal{S})$ be a polarized object of* $\mathrm{MT}^{(\mathrm{r})}(X,w)^{(\mathrm{p})}$. *Then the associated regular holonomic D-module $M := \Xi_{\mathrm{DR}}(\mathcal{M}'')$ is semisimple.*

Reduction of the proof of Theorem 3 to a theorem of Simpson. On a Riemann surface X with fixed punctures $S \subset X$ a filtered vector bundle consists of a vector bundle E on $X^\times = X \setminus S$ together with a family of extensions E_α to X for $\alpha \in \mathbb{R}$ such that locally at each $s \in S$ the filtration is left continuous, descending $E_{\alpha,s} \subset E_{\beta,s}$ for $a \geq \beta$ and $E_{\alpha+1,s} = z \cdot E_{\alpha,s}$ for a local coordinate z at s.

Denoting $\bar{E}$ the extension for the choice of $\alpha = 0$ and $\bar{E}(s)$ its fibre at s with the given filtration, one defines the degree of the filtered vector bundle as

$$\deg(E) = \deg(\bar{E}) + \sum_{s\in S}\sum_{0\leq\alpha<1} \dim \mathrm{gr}_\alpha \bar{E}(s)$$

and the slope as $\mathrm{slope}(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$.

A filtered regular Higgs bundle consists of a filtered vector bundle with a Higgs fields $\theta : E \to E \otimes \Omega^1_{X^\times}$ which is logarithmic with respect to the given extensions. It is called *stable* if the slope decreases strictly on any proper subbundle preserved by θ.

Similarly, a filtered regular meromorphic connection is a filtered vector bundle with a connection $\nabla : V \to V \otimes \Omega^1_{X^\times}$ logarithmic with respect to the given extensions. It is *stable* if the slope decreases strictly for any proper subbundle preserved by ∇. By the regular singular Riemann-Hilbert correspondence the latter correspond to stable filtered local systems.

In order to relate these objects one studies harmonic metrics on such bundles. Given a harmonic bundle $(E, \bar{\partial}, \theta, h)$ (with the complex structure induced by $\bar{\partial}$, the associated Higgs field θ and the harmonic metric h) on $X^{\times}$, locally at each puncture one can write θ in the form $\theta = f\frac{dz}{z}$. The harmonic bundle is called *tame* if the coefficients of the characteristic polynomial $\det(T - f)$ are holomorphic at $z = 0$. It is called *purely imaginary* if the eigenvalues of the residue of θ at all punctures are purely imaginary.

For a tame harmonic bundle on $X^{\times}$ as above, the parabolic filtration inside j_*V (with $j : X^{\times} \hookrightarrow X$ and V the holomorphic vector bundle) is defined by bounding the growth of sections with respect to the metric h in comparison with the growth of $|z|^{\alpha}$ for a local coordinate. Due to the tameness, the resulting parabolic filtration is a filtration by coherent modules and the associated connection ∇ is logarithmic with respect to these extensions.

If the tame harmonic bundle is purely imaginary, the parabolic filtration coincides with the canonical Kashiwara-Malgrange filtration on (V, ∇).

Simpson's Theorem (cp. [Sim90]) is a correspondence between

$$\begin{pmatrix} \text{stable filtered} \\ \text{regular Higgs} \\ \text{bundles of} \\ \text{degree } 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{irreducible} \\ \text{tame harmonic} \\ \text{bundles} \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{stable filtered} \\ \text{local systems} \\ \text{of degree } 0 \end{pmatrix}$$

From this one deduces a correspondence between purely imaginary tame harmonic bundles and semisimple local systems.

Now, the proof of Theorem 3 can be reduced to the latter correspondence by first noting that the given polarized object $(\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C), \mathcal{S})$ of $\mathrm{MT}^{(\mathrm{r})}(X, w)^{(\mathrm{p})}$, say with strict support Z, generically on $Z^0 \subset Z$ is smooth and the associated D-module is an intersection complex associated to a local system $\mathcal{L}$. It remains to prove that $\mathcal{L}$ is semisimple.

This is executed by induction on the dimension $\dim(Z)$. The case of a smooth Riemann surface Z follows by Simpson's correspondence. To apply the latter, one observes that the smooth twistor D-module on Z^0 corresponds to a harmonic bundle and that the harmonic metric constructed in this step is tame and purely imaginary. The non-smooth case can be handled analogously including the normalization of Z into the arguments.

The induction step follows rather easily by intersecting with a hyperplane section since it induces a surjection of the fundamental groups $\pi_1(Z^0 \cap H) \twoheadrightarrow \pi_1(Z^0)$ and hence semisimplicity of $\mathcal{L}$ on Z^0 follows from the semisimplicity of $\mathcal{L}|_{Z^0 \cap H}$.

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Tame harmonic bundles on curves: local properties

Carsten Neumann

The goal of this talk is to construct from a tame harmonic bundle attached to a variation of pure polarized twistor structure on the punctured disc $\Delta^* = \{|t| < 1\} \setminus \{0\}$ a pure polarized twistor $\mathcal{D}$-module on the whole disc Δ. One possible definition of tameness is the following: A harmonic bundle $(E, \overline{\partial_E}, \theta, h)$ is called *tame* if the eigenvalues λ_i of θ have poles of order at most one.

One important notion in this talk is that of filtered vector bundles on Δ^*: A filtered vector bundle on Δ^* is a vector bundle E with a decreasing filtration of j_*E by coherent sheaves E_a such that $E_a = \bigcap_{b<a} E_b$ and $E_{a+1} = tE_a$, where $j : \Delta^* \to \Delta$ is the inclusion. A filtered regular Higgs bundle is a filtered Higgs bundle such that

$$\theta : E_a \to E_a \otimes \Omega^1_\Delta(\log(\{0\})).$$

Similarly, a filtered regular flat bundle (V, ∇) is a flat bundle such that

$$\nabla : V_a \to V_a \otimes \Omega^1_\Delta(\log(\{0\})).$$

For a metrized holomorphic bundle (E, h), we define

$$E_a := \{\sigma \in j_*E \mid |\sigma|_h \leq Cr^{a-\epsilon} \text{ for all } \epsilon > 0\},$$

which gives us a decreasing filtration on

$$\widetilde{E} := \{\sigma \in j_*E \mid |\sigma|_h \text{ has moderate growth at } 0\}.$$

A metrized holomorphic bundle (E, h) is called *acceptable* if the curvature R_h of the metric connection satisfies

$$|R_h| \leq f + \frac{C}{|t|^2(\log|t|)^2}$$

near $t = 0$, where C is a constant and $f \in L^p$. In that case, the E_a are coherent sheaves and E is therefore a filtered vector bundle. Due to a theorem by Simpson [1], the tame harmonic bundle $(E, \overline{\partial_E}, \theta, h)$ is acceptable with either the holomorphic structures ∂_E or d'' and, furthermore, $((\widetilde{E}, \overline{\partial_E}), \theta)$ is a filtered regular Higgs bundle and $((\widetilde{E}, d''), \nabla)$ is a filtered regular flat bundle. (We use the correspondence between harmonic bundles $((E, \overline{\partial_E}, \theta, h)$ and flat bundles (V, ∇, h) with $\nabla = d' + d''$ its decomposition into operators of types (1,0) and (0,1).) A similar result holds for the connections $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ for each fixed $\lambda \in \mathbb{C}$, that is, the $\mathcal{E}^\lambda_a$ are locally free $\mathcal{O}_\Delta$-modules.

The main theorem of this talk is the following:

Theorem 1 ([2]). *The variation of polarized pure twistor structure of weight* 0 *attached to a harmonic bundle on* Δ^* *which is tame at the origin extends in a unique way as a polarized pure twistor* $\mathcal{D}$*-module on* Δ *whose underlying* $\mathcal{D}$*-module is the intermediate extension of the flat bundle underlying the variation.*

The construction goes as follows: Starting with a harmonic bundle $(E, \overline{\partial_E}, \theta, h)$, one has the $\mathcal{R}_{\Delta^* \times \mathbb{C}_\lambda}$-triple $\mathcal{T}(E) = (\mathcal{E}, \mathcal{E}, \mathcal{C}_0)$ with hermitian sesquilinear duality $\mathcal{S}(E) = (\mathrm{id}, \mathrm{id})$, where the $\mathcal{R}$-module $\mathcal{E}$ is induced by the deformed holomorphic bundle with its family of λ-connections and $\mathcal{C}_0$ is induced by the metric h. The desired twistor $\mathcal{D}$-module is the prolongment of this, which is a $\mathcal{R}_{\Delta\times\mathbb{C}_\lambda}$-triple $\mathfrak{T}(E) = (\mathfrak{E}, \mathfrak{E}, \mathfrak{C})$ with the hermitian sesquilinear duality $\mathfrak{S}(E)$. Then $\Xi_{DR}(\mathfrak{E})$ is the intermediate extension of the flat bundle underlying the variation.

The converse assertion, namely that a polarized pure twistor structure of weight 0 on Δ^* which is the restriction of a polarized pure twistor $\mathcal{D}$-module on Δ has an associated harmonic bundle which is tame at the origin, is simpler: The restriction of a polarized pure twistor $\mathcal{D}$-module of weight 0 to the punctured disc corresponds to a harmonic bundle and, by the regularity assumption, each negative step of the V-filtration is $\mathcal{O}$-locally free in a neighbourhood of $\lambda = 0$; since it is stable by the action of $t\partial_t$, it follows that the Higgs field, obtained by setting $\lambda = 0$ in the action of $t\eth_t$, satisfies the tameness assumption of [1].

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Decomposition theorem for curves

Yajnaseni Dutta

The main goal of this talk is to complete Step 1 from the proof of Statement 1 of Lecture 8. In other words, establish the $P(1,0)$ case of the decomposition theorem. We first recall the statement:

Theorem 1 ([Sab05, Moc07]). *Let $\mathbb{X}$ be a smooth projective curve and let $a\colon \mathbb{X} \to \operatorname{Spec}\mathbb{C}$ denote the structure map for the curve. Let $(\mathcal{T}, \mathcal{S})$ be a polarized regular pure twistor D-module on $\mathbb{X}$ of weight 0.*

Then, the push-forward $(\oplus_{i=-1}^{1} a_+^i \mathcal{T}, \mathcal{L}, a_+\mathcal{S})$ is a polarized graded Lefschetz twistor structure.

We refer to the previous lectures for the precise definitions of polarized regular pure twistor D-modules and their pushforwards. However we do briefly recall the following correspondence (see e.g. [Moc07, Theorem 20.1]). For simplicity, we restrict to the weight 0 case.

Theorem 2. *There is a one-to-one correspondence between the variation of polarized pure twistor structures of weight 0 which are generically defined over $\mathbb{X}$ and the regular pure twistor D-modules of weight 0 whose strict support is $\mathbb{X}$.*

The correspondence goes via harmonic bundles since a variation of polarized pure twistor structures of weight 0 underlies a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X \coloneqq \mathbb{X} \setminus D$ for D a finite set of points (assume, only one point for simplicity). We let $\mathcal{X} \coloneqq X \times \mathbb{C}_\lambda$ and $\mathfrak{X} \coloneqq \mathbb{X} \times \mathbb{C}_\lambda$ and p the respective projections to X (or $\mathbb{X}$). From this data, we obtain an $\mathcal{R}_{\mathcal{X}}$-triple $(\mathcal{E}, \mathcal{E}, C_0)$. On its naïve algebraic extension bundle ${}^{\square}\mathcal{E}$ (i.e. roughly speaking the twistor incarnation of the algebraic extension of E over $\mathbb{X}$) we have a V-filtration $U_{\bullet}^{(\lambda_0)\square}\mathcal{E}$ defined on ${}^{\square}\mathcal{E}|_{\mathfrak{X}(\lambda_0,\epsilon_0)}$, where $\mathfrak{X}(\lambda_0, \epsilon_0) \coloneqq \mathbb{X} \times \Delta(\lambda_0, \epsilon_0)$ namely a slice of the product space over the disc $\Delta(\lambda_0, \epsilon_0)$ around $\lambda_0 \in \mathbb{C}_\lambda$. Define $\mathfrak{E}(\mathfrak{X}(\lambda_0, \epsilon_0)) \coloneqq$ the $\mathcal{R}_{\mathfrak{X}}$- submodule of ${}^{\square}\mathcal{E}$ generated by $U_{<0}^{(\lambda_0)\square}\mathcal{E}$. Then the glued $\mathcal{R}_{\mathfrak{X}}$-module gives rise to a polarized $\mathcal{R}_{\mathfrak{X}}$-triple $((\mathfrak{E}, \mathfrak{E}, \mathfrak{C}), (\mathrm{Id}, \mathrm{Id}))$ underlying a pure regular twistor D-module.

Conversely, given such a triple, generically on $\mathbb{X}$, namely for a Zariski open subset X the $\mathcal{R}_{\mathcal{X}}$-triple $\mathcal{T}|_{\mathcal{X}}$ is a λ-deformed bundle of the harmonic bundle $(E, \overline{\partial}_E, \theta_E, h)$. The regularity implies tameness of this harmonic bundle.

Proof Sketch. For the proof of 1 we follow [Moc07, §20.2.2] and it relies on the Dolbeault lemma for a singular Hermitian line bundle due to Zucker. A different proof can be found in [Sab05, §6.2.b–6.2.f].

The proof goes via a series of quasi-isomorphisms leading up-to

$$\mathcal{H}^{i+\dim X}(\mathbb{R}a_*(\mathfrak{E} \otimes \Omega^{\bullet}_{\mathfrak{X}}) \simeq \mathrm{Harm}^i \otimes \mathcal{O}_{\mathbb{C}_\lambda},$$

where Harm^i is a finite dimensional vector space. Thus by definition the pushforward is a twistor structure of weight i. The Lefschetz map in this case concerns only $i = 0$ and looks like

$$\mathcal{L} \coloneqq a_+^0 \mathfrak{E} \to a_+^0 \mathfrak{E} \otimes \mathbb{T}(0)$$

where $\mathbb{T}(0)$ is the Tate twistor structure of weight 0.

Roughly speaking, the reason why such vector spaces Harm^i are independent of λ is they are generated by the kernel of the Laplace operator

$$D^{\lambda^*} D^{\lambda} + D^{\lambda} D^{\lambda^*} = (1 + |\lambda|^2) \left((\overline{\partial}_E + \theta)^* (\overline{\partial}_E + \theta) + (\overline{\partial}_E + \theta)(\overline{\partial}_E + \theta)^* \right)$$

acting on certain finite dimensional space of global L^2-sections. Here D^λ is the connection associated to E^λ.

The crux of the proof lies in constructing this series of quasi-isomorphisms. It relies on the classical Dolbeault lemma for $\mathcal{C}^\infty$ rank 1 flat bundle (V, ∇) on the disc. This is due to Zucker [Zuc79] who established a quasi-isomorphism between the naïve algebraic extension ${}^{\square}V$ of V on $\mathbb{X}$ and the complex $\mathcal{L}^\bullet(V)_{(2)}$ of L^2-sections of $V \otimes A^p_X$ with L^2-derivatives [Zuc79, Theorem 6.2]. For all $\lambda \in \mathbb{C}_\lambda$, one can apply this construction to the bundles $\mathcal{E}^\lambda$ associated to $\mathfrak{E}|_{\mathcal{X}}$ and extend these ideas to construct a complex $S(\mathcal{E} \otimes \Omega^{\bullet;0}_{\mathfrak{X}})$ on $\mathfrak{X}$ whose fibres are certain λ-holomorphic sections of a sub-complex $\widetilde{\mathcal{L}}^\bullet(\mathcal{E}^\lambda)_{(2)} \subseteq \mathcal{L}^\bullet(\mathcal{E}^\lambda)_{(2)}$. This subcomplex is defined so that it is soft with respect to the global section functor and the i-th cohomology of the global section complex is a finite dimensional vector space Harm^i [Moc07, Lemma 20.23-24] and hence is independent of λ.

On the other hand, the relation between the complexes $a_*S(\mathcal{E} \otimes \Omega^{\bullet;0}_{\mathfrak{X}})$ to $Ra_*(\mathfrak{E} \otimes \Omega^{\bullet}_{\mathfrak{X}})$ is not so straightforward. To this end, one uses the V-filtration associated to ${}^{\square}\mathcal{E}$ defined over a neighbourhood $\Delta(\lambda_0, \epsilon_0)$ and the pieces of the weight filtrations whose sections are L^2 in order to construct $\mathcal{Q}^{(\lambda_0),\bullet}$ on $\mathfrak{X}(\lambda_0, \epsilon_0)$ for each $\lambda_0 \in$

$\mathbb{C}_\lambda$. Using Zucker's norm estimates one shows that that $\mathcal{Q}^{(\lambda),\bullet}\big|_{\mathbb{X}\times\{\lambda_0\}}$ is quasi-isomorphic to $S(\mathcal{E}\otimes\Omega_{\mathcal{X}}^{\bullet,0})\big|_{\mathbb{X}\times\{\lambda_0\}}$. For a discussion on how L^2-norm estimates on a harmonic bundle behaves with respect to the sections of parabolic filtrations, V-filtrations and the monodromy weight filtrations see Lecture 14. Since the cohomologies of the pushforwards of both complexes form coherent sheaves on the disc $\Delta(\lambda_0,\epsilon_0)$ and their fibres are already isomorphic, one can argue using the Nakayama lemma for graded rings to conclude

$$\mathcal{H}^i(\mathbb{R}a_*\mathcal{Q}^{(\lambda),\bullet}) \simeq \mathcal{H}^i(a_*S(\mathcal{E}\otimes\Omega_{\mathcal{X}}^{\bullet,0}))\big|_{\Delta(\lambda_0,\epsilon_0)} \simeq \mathrm{Harm}^i\otimes_{\mathbb{C}}\mathcal{O}_{\Delta(\lambda_0,\epsilon_0)}.$$

This is [Moc07, Lemma 20.38]. The quasi-isomorphism between the complexes $\mathcal{Q}^{(\lambda),\bullet}$ and $\mathfrak{E}\otimes\Omega_{\mathcal{X}}^{\bullet}$ follows from the properties of the V-filtration [Moc07, Lemma 20.35] completing the proof.

□

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Tame harmonic bundles in arbitrary dimension

Claus Hertling

The subject of the lecture were tame harmonic bundles in arbitrary dimension, namely their definition, basic properties and two theorems (though also the basic properties are theorems).

First, the local definition was given. In the case of the complex manifold $X = \Delta^n$ with $\Delta = \{z\in\mathbb{C}\,|\,|z|<1\}$ the unit disk and the normal crossing divisor $D=\bigcup_{i=1}^{l} D_i$ with $D_i=\{z_i=0\}\subset X$, a harmonic bundle $\mathcal{H}=(E,\overline{\partial}_E,\theta,h)$ on $X-D$ is *tame along* D if for each logarithmic vector field v the characteristic polynomial of the endomorphism θ_v of the holomorphic bundle E which comes from v and the Higgs field θ extends holomorphically over D.

There is a very useful curve test for tameness. The harmonic bundle $\mathcal{H}$ on $X-D$ is tame along D if for any holomorphic curve C which intersects the smooth part of D transversely in one point Q, the restriction $\mathcal{H}|_{C-\{Q\}}$ of the harmonic bundle to $C-\{Q\}$ is tame along Q.

For each parameter $\lambda\in\mathbb{C}$ the bundle E comes equipped with a new holomorphic structure $\overline{\partial}_E+\lambda\theta^\dagger$ with which it is called $\mathcal{E}^\lambda$, and the bundle $\mathcal{E}^\lambda$ comes equipped with a holomorphic flat λ-connection $\mathbb{D}^\lambda=\lambda\partial_E+\theta$. These data glue to a holomorphic bundle $\mathcal{E}=\bigcup_{\lambda\in\mathbb{C}}\mathcal{E}^\lambda$ with a family $\mathbb{D}$ of λ-connections, with $\mathbb{D}^0=\theta$.

Consider a curve $C\subset X$ which intersects the smooth (with respect to D) part D_i° of a component D_i of D transversely in a point Q. For each $f\in\mathcal{E}^\lambda-\{0\}$, the order $\mathrm{ord}(f) \in \mathbb{R}$ is the biggest number $b \in \mathbb{R}$ with $|f|_h = \mathcal{O}(|z|^{b-\varepsilon})$ for any $\varepsilon > 0$. Simpson [Si90] proved that the sections $f \in \mathcal{E}^\lambda - \{0\}$ with $-\mathrm{ord}(f) \leq b$ for some $b \in \mathbb{R}$ give an extension of $\mathcal{E}^\lambda$ to a holomorphic vector bundle ${}_b\mathcal{E}^\lambda|_C$ on C and that $\mathbb{D}^\lambda$ has on this bundle a logarithmic pole at Q.

If ${}_b\mathcal{E}^\lambda / {}_{<b}\mathcal{E}^b \supsetneqq \{0\}$, then the residue endomorphism $\mathrm{Res}\,\theta|_C$ acts on this finite dimensional $\mathbb{C}$-vector space with some eigenvalues $\beta \in \mathbb{C}$. The set of all such pairs $(b, \beta) \in \mathbb{R} \times \mathbb{C}$ is the KMS-structure $KMS(\mathcal{E}^\lambda|_C) \subset \mathbb{R} \times \mathbb{C}$, a discrete subset which is invariant under addition of $\pm(1, -\lambda)$.

It turns out that this KMS-structure is independent of Q and C, so it gives a KMS-structure $KMS(\mathcal{E}^\lambda, i) \subset \mathbb{R} \times \mathbb{C}$ for the component D_i of the normal crossing divisor $D = \bigcup_{i=1}^l D_i$. Also the definition of ${}_b\mathcal{E}^\lambda|_C$ extends. For any $\mathbf{b} = (b_1, ..., b_l) \in \mathbb{R}^l$ an extension of $\mathcal{E}^\lambda$ to a holomorphic vector bundle ${}_\mathbf{b}\mathcal{E}^\lambda$ on X arises, and again $\mathbb{D}^\lambda$ has logarithmic poles along D. Especially, the Higgs field $\theta = \mathbb{D}^0$ has on the extension ${}_\mathbf{b}\mathcal{E}^0$ of $\mathcal{E}^0 = E$ a logarithmic pole along D.

Proofs of these statements can be found in [Mo07], which is the main source of this lecture.

Interesting is also the dependence of the KMS-structure on $\lambda \in \mathbb{C}$. For fixed λ, the map

$$(p(\lambda), e(\lambda)) : \mathbb{R} \times \mathbb{C} \to \mathbb{R} \times \mathbb{C}, \quad (a, \alpha) \mapsto (a + 2\mathrm{Re}(\lambda\overline{\alpha}), \alpha - \lambda a - \lambda^2\overline{\alpha}),$$

is a bijection from $\mathbb{R} \times \mathbb{C}$ to $\mathbb{R} \times \mathbb{C}$, and it restricts to a bijection from $KMS(\mathcal{E}^0, i)$ to $KMS(\mathcal{E}^\lambda, i)$ for any $i \in \{1, ..., l\}$.

A tame harmonic bundle as above is called *purely imaginary* if $KMS(\mathcal{E}^0, i) \subset \mathbb{R} \times \sqrt{-1}\mathbb{R}$ for any i. In this purely imaginary case one has for $\lambda = 1$

$$-\mathrm{Re}(e(\lambda)(a, \alpha)) = -\mathrm{Re}(\alpha - \lambda a - \lambda^2\overline{\alpha}) = a \overset{!}{=} a + 2\mathrm{Re}(\lambda\overline{\alpha}) = p(\lambda)(a, \alpha).$$

This implies that the filtration ${}_\bullet\mathcal{E}^1$ from the norm fits to the Kashiwara-Malgrange filtration which comes from the real parts of the eigenvalues of the residue endomorphisms of $\mathbb{D}^1$.

The local definition of the tameness of a harmonic bundle along a normal crossing divisor gives rise in an easy way to the notion of a *generically defined tame harmonic bundle* on an irreducible (reduced complex analytic) subvariety Z in a complex manifold X. One just needs a Zariski open smooth subset $U \subset Z$, a harmonic bundle $\mathcal{H} = (E, \overline{\partial}_E, \theta, h)$ on U, and the existence of a resolution $\varphi : \widetilde{Z} \to Z$ such that (i) $\widetilde{D} := \varphi^{-1}(Z - U)$ is a normal crossing divisor in $\widetilde{Z}$, (ii) $\varphi : \widetilde{Z} - \widetilde{D} \to U$ is an isomorphism, and (iii) the harmonic bundle $\varphi^{-1}\mathcal{H}$ is tame along $\widetilde{D}$.

Two such generically defined tame harmonic bundles $(U, \mathcal{H})$ and $(U', \mathcal{H}')$ on Z are equivalent if a third one $(U'', \mathcal{H}'')$ with $U'' \subset U \cap U'$ and $\mathcal{H}|_{U''} \cong \mathcal{H}'' \cong \mathcal{H}'|_{U''}$ exists (the next theorem implies that then $\mathcal{H}|_{U \cap U'} \cong \mathcal{H}'|_{U \cap U'}$).

Now the first of the two theorems can be formulated.

Theorem 1. [Mo07, Theorem 19.6 and Theorem 19.42] *Let X be a complex manifold and $Z \subset X$ an irreducible (reduced complex analytic) subvariety.*

(a) There is a natural correspondence between the set

$$\{\text{generically defined tame harmonic bundles on } Z\}/\text{equivalence}$$

and the set $MPT_Z^{(r)}(X,0)$ of regular polarized pure twistor D-modules of weight 0 with strict support Z.

(b) The correspondence restricts to a correspondence between the purely imaginary objects on both sides.

The definition of a purely imaginary generically defined tame harmonic bundle on Z is obvious from the definitions above. The definition of a purely imaginary element of $MPT_Z^{(r)}(X,0)$ had been given in an earlier lecture.

The proof of the theorem is given in [Mo07] in the subsections 19.2 (tameness of the harmonic bundle which one obtains by restricting an element of $MPT_Z^{(r)}(X,0)$ to a suitable Zariski open subset of Z), 19.3 and 19.4 (existence and uniqueness of an extension of a generically defined tame harmonic bundle on Z to an object of $MPT_Z^{(r)}(X,0)$) and 19.5 (part (b) of the theorem).

Perhaps most interesting is the first step in 19.3, which starts with the local situation above: the complex manifold $X=\Delta^n$, the normal crossing divisor $D=\bigcup_{i=1}^l D_i$, the harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ on $X-D$ which is tame along D. Then for any $\lambda_0\in\mathbb{C}$ one finds a small $\varepsilon>0$ and a disk $\Delta(\lambda_0,\eta_0)\subset\mathbb{C}$ around λ_0 such that for any $i\in\{1,...,l\}$

$$\begin{aligned} \{1-\varepsilon\}\times\mathbb{C}\cap KMS(\mathcal{E}^\lambda,i) &= \emptyset \quad \text{for } \lambda\in\Delta(\lambda_0,\eta_0)-\{\lambda_0\}, \\ [1-\varepsilon,1[\times\mathbb{C}\cap KMS(\mathcal{E}^{\lambda_0},i) &= \emptyset. \end{aligned}$$

The bundle

$$_{(<\mathbf{1})}\mathcal{E}^{(\lambda_0)} := \bigcup_{\lambda\in\Delta(\lambda_0,\eta_0)} {}_{(1-\varepsilon,\dots,1-\varepsilon)}\mathcal{E}^\lambda$$

on $(X-D)\times\Delta(\lambda_0,\eta_0)$ turns out to be holomorphic, and the $\mathcal{R}$-module which it generates is the correct extension from $(X-D)\times\Delta(\lambda_0,\eta_0)$ to $X\times\Delta(\lambda_0,\eta_0)$.

If $\lambda_1\neq\lambda_0$ with $\Delta(\lambda_0,\eta_0)\neq\emptyset$, the bundles $_{(<\mathbf{1})}\mathcal{E}^{(\lambda_1)}$ and $_{(<\mathbf{1})}\mathcal{E}^{(\lambda_0)}$ may differ on the intersection $\Delta(\lambda_0,\eta_0)\cap\Delta(\lambda_1,\eta_1)$, but they generate the same $\mathcal{R}$-module on this intersection. Thus these (in λ) local extensions glue to the right object on $X\times\mathbb{C}$ in $MPT^{(r)}(X,0)$.

The second theorem of this lecture is as follows.

Theorem 2. [Mo07, Proposition 22.15, Theorem 25.21, Theorem 25.28] *Let X be complex projective manifold and $D\subset X$ a normal crossing divisor.*

(a) Let $(E,\overline{\partial}_E,\theta,h)$ be a harmonic bundle on $X-D$ which is tame and purely imaginary along D. Then the flat vector bundle $(\mathcal{E}^1,\mathbb{D}^1)$ is semisimple.

(b) Let $(\widetilde{E},\widetilde{\nabla})$ be a semisimple flat complex vector bundle on $X-D$. Then a harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ on $X-D$ exists which is tame and purely imaginary along D and which satisfies $(\mathcal{E}^1,\mathbb{D}^1)\cong(\widetilde{E},\widetilde{\nabla})$. The pluriharmonic metric h is

unique up to automorphisms of the flat bundle $(\mathcal{E}^1, \mathbb{D}^1)$. *In other words: If one writes*

$$(\mathcal{E}^1, \mathbb{D}^1) = \bigoplus_{\rho \in \mathrm{Irred}(X-D)} (E(\rho), \nabla(\rho)) \otimes \mathbb{C}^{m(\rho)}$$

with $(E(\rho), \nabla(\rho)$ *a simple flat bundle associated to* ρ *and* $m(\rho) \in \mathbb{Z}_{\geq 0}$, *then the metric splits as* $h = \bigoplus_{\rho \in \mathrm{Irred}(X-D)} h(\rho) \otimes g_\rho$ *with* g_ρ *a metric on (the vector space)* $\mathbb{C}^{m(\rho)}$.

There was no time in the lecture to discuss the proof of this theorem. Though the following remarks were made. If a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$ is tame, but not purely imaginary along D, then the associated flat bundle $(\mathcal{E}^1, \mathbb{D}^1)$ is not necessarily semisimple. On the other hand, there exist already for $\dim X = 1$ harmonic bundles on $X - D$ which are tame, but not purely imaginary, and whose associated flat bundles $(\mathcal{E}^1, \mathbb{D}^1)$ are nonetheless semisimple.

In the case $\dim X = 1$ this theorem is due to Simpson [Si90]. In the case $\dim X = 2$ Mochizuki's proof of part (b) in [Mo07, 25.1-25.4] follows an argument of Jost and Zuo [JZ97], and like them he assumes only that X is a compact Kähler surface with a normal crossing divisor. But the elegant inductive argument for part (b) in the cases $\dim X \geq 3$ in [Mo07, 25.5] requires that X is a complex projective manifold.

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Irregular singularities in dimension one

Alex Takeda

The goal of this lecture was to recall the classical theory of meromorphic flat bundles with irregular singularities, in order to later relate it sto the analogous developments in the setting of twistor D-modules. The lecture mostly followed the exposition in chapters III and IV of Malgrange's book [Ma91], and was roughly divided in two parts: the first half devoted to the formal classification of irregular singularities, presenting the ideas behind the proofs of the decomposition result of Levelt, Turrittin, Hukuhara [Le75, Tu55], and the second half passing to the analytic theory of irregular singularities over the punctured disk, which builds on the formal classification by adding the notion of Stokes structures, as developed by Deligne, Birkhoff, Malgrange [De06, Bi13, Ma91], among others, and commonly known as the '(irregular) Riemann-Hilbert-Birkhoff correspondence'. A third section, which was planned but not presented due to time constraints, deals with an extension of this RHB correspondence by Mochizuki [Mo11, Sec.4.5.2] to certain types of families of irregular connections, where the irregular part varies under multiplication by positive factors, and that plays a role in the study of wild twistor D-modules. These results were partly presented in other later talks.

The formal classification. Let $\widehat{\Delta}$ be the formal disk, with structure sheaf $\widehat{\mathcal{O}} = \mathbb{C}[[x]]$, and $\widehat{K} = \widehat{\mathcal{O}}[x^{-1}]$ the ring of meromorphic formal functions. We study flat connections with a meromorphic singularity at zero; these are $\widehat{K}\langle\partial\rangle$-modules that are flat connections away from zero, with a pole of any order in x.

The Hukuhara-Levelt-Turrittin decomposition. The formal classification starts with the observation that every such connection M admits a cyclic vector (this also works over the actual, non-formal disk) [Co36, De06, Ka87]. In other words, there is an element $e \in M$ such that $\{e, \partial e, \ldots, \partial^{m-1}e\}$ is a basis of M. From the equation expressing $\partial^m e$ in this basis, one obtains the *Newton polygon* of M, which one proves to be independent of M. This polygon gives a one-parameter family of valuations on $\widehat{K}\langle\partial\rangle$; using these one can decompose M as a direct sum of submodules, each summand corresponding to a slope of the polygon. The submodule with slope zero is the *regular part* of M; using an iterative argument for the other components leads to decomposition theorem of Levelt, Hukuhara, Turrittin, stating that (possibly after a finite ramification) there is a decomposition $M \cong \bigoplus_\omega L_\omega \otimes M_\omega$, where each L_ω is a certain rank-one connection with irregular value ω, and M_ω is regular.

Good Deligne-Malgrange lattices. We consider now a disk Δ of some radius R and denote $\mathcal{O}$ for its structure sheaf (and analogously for $K, K\langle\partial\rangle$). Given any meromorphic connection on the punctured disk, that is, a $K\langle\partial\rangle$-module M that restricts to a flat connection away from zero, we can take its formalization $\widehat{M}$ and get a formal connection of the type we described above.

We begin describing an analytic connection M by noting that due to the results above for formal connections, one can produce a *good Deligne-Malgrange lattice* [Mo09] for the connection M (again, possibly after a finite ramification): such a lattice is a locally free $\mathcal{O}$-submodule E of M such that $KE \cong M$, and the property of 'goodness' means its monodromy satisfies a certain condition. This lattice E is obtained by pulling back a lattice for $\widehat{M}$, constructed using the Hukuhara-Levelt-Turrittin decomposition. Note that this existence result is specific to dimension one, and fails in higher dimensions due to the existence of turning points.

The analytic classification. This formal information, however, does not suffice to classify meromorphic connections over Δ; the mapping $M \mapsto \widehat{M}$ forgets some information. This data turns out to be captured by a notion called a 'Stokes structure' associated to M. This appears in different forms in many works; the formulation in the book is attributed to Deligne and measures roughly how the formal models for the flat sections of $\widehat{M}$ in angular sectors around zero glue together to local flat sections of M.

Flat sections on angles. We consider a real blow-up of the disc $\tilde{D} \to D$ which replaces zero with a circle S of directions, and then define a sheaf $\mathcal{A}$ on $\tilde{D}$ which, close to each small open in S, has as sections germs of meromorphic functions that admit an asymptotic Laurent series in that small angular sector. This sheaf naturally has a subsheaf $\mathcal{A}^{<0}$ of functions with zero Laurent series.

Using elementary complex analysis, one can prove that the natural map $H^1(S, \mathcal{A}^{<0}) \to H^1(S, \mathcal{A})$ is zero [RS89], and using asymptotic analysis, that the endomorphism ∂ is surjective on $\mathcal{A}^{<0}(M) = \mathcal{A}^{<0} \otimes_{\mathcal{O}} M$. Now, for each angle $\theta \in S$, we take the localization $\mathcal{A}(M)_\theta$; the kernel of ∂ on this space is the germs of flat sections of M around that angle. It follows from these results then that the natural map $\ker(\partial, \mathcal{A}(M)_\theta) \to \ker(\partial, \widehat{M})$ is surjective; that is, every formal solution can be locally extended to a small interval around any angle.

Stokes structures. One concludes from this result above that locally around every angle, there is an identification between the flat sections of M and the flat sections of some connection of the form $\bigoplus_\omega L_\omega \otimes M_\omega$; these are all of the form $\exp(\int \omega) f_\omega$ for some f_ω solving a differential equation with a regular singularity.

Thus every solution has a certain asymptotic behavior $\sim \exp(\int \omega)(\dots)$, well-defined up to terms of smaller growth; this gives a filtration on the sheaf of flat sections. Putting this all together, each irregular connection produces (possibly after finite ramification) an Ω-filtered local system of solutions, where Ω is a family of local systems indexed over the irregular values ω; this is what is called a *Stokes structure.*

The (irregular) Riemann-Hilbert-Birkhoff theorem is the statement that this map, from meromorphic connections on the disk, to Stokes-filtered local systems, is an equivalence of categories. Malgrange attributes this statement to Deligne, and provides references to many proofs [Bi13, Ju06, Ma79]. It was noted during the talk at Oberwolfach that the statement given in the book [Ma91] is slightly inaccurate; one must be more careful in specifying which Ω-filtered local systems are allowed in the category of Stokes-filtered local systems. Roughly, one must require that these filtrations be, angular-locally, isomorphic to filtrations coming from standard meromorphic connections; for more details, the work of Sabbah on Stokes structures ([Sa12] and references within) has more precise descriptions of this correspondence.

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Wild harmonic bundles on curves

Szilárd Szabó

We define the notions of (unramifiedly, purely imaginary) wild Higgs bundles and wild harmonic bundles on the complement of a finite number of points on a curve, first locally then globally by requiring the corresponding conditions on coordinate charts. We give examples in arbitrary rank, with arbitrary irregular part, parabolic weights and graded residues. We define the metric prolongment of the holomorphic bundle associated to the λ-connection of a wild harmonic bundle on the punctured disc. We state that it is a filtered bundle. We define the graded residue maps, and for a frame compatible with the irregular parts, the parabolic filtration and the weight filtration on the associated graded pieces of the parabolic filtration, we define a model metric. We state the norm estimate due to T. Mochizuki: the model metric so constructed is mutually bounded with the harmonic metric, and we sketch its proof in the case $\lambda = 0$. For the proof we invoke C. Simpson's Corollary saying that if two metrics H, K both have curvature in L^p for some $p > 1$ and they induce the same filtration both on the bundle and its dual, then they are mutually bounded. We state T. Mochizuki's result saying that the components of a harmonic 1-form corresponding to irregular part $\mathfrak{a} \neq 0$ decay exponentially in $r^{\mathrm{ord}(\mathfrak{a})}$, where $\mathrm{ord}(\mathfrak{a}) < 0$ is the degree of $\mathfrak{a}$ with respect to z. We indicate a proof using energy estimates based on the Weitzenböck formula.

Polarizable wild twistor $\mathcal{D}$-modules

Qianyu Chen

The goal of my talk is to define wild twistor $\mathcal{D}$-modules and present the decomposition theorem for them, as developed by Sabbah and Mochizuki. To begin, I provided the definition of Deligne's irregular nearby cycle for $\mathcal{D}$-modules and $\mathcal{R}$-modules, which is necessary for the wild case as the usual nearby cycle may not provide enough information for irregular $\mathcal{D}$-modules. The irregular nearby cycle takes into account all exponential and ramification twists.

Moving on to the definition of wild twistor $\mathcal{D}$-modules, one simply mimics the regular case but replaces the regular nearby cycles with Deligne's irregular nearby cycles. The properties of wild twistor $\mathcal{D}$-modules are similar to those of the regular case; for example, the category of polarizable wild twistor $\mathcal{D}$-modules is abelian and semisimple.

I then introduced the decomposition theorem, which states that (1) the cohomology modules of the pushforward of a polarizable wild twistor $\mathcal{D}$-module along a projective morphism are again a polarizable wild twistor $\mathcal{D}$-module (with weight shifted by the cohomological degree), and (2) the Hard Lefschetz holds on the cohomology modules of the pushforward.

Finally, I noted that the first step of the proof is to reduce to the case of curves, and that the reduction is similar to the regular case. However, the case of curves will be analyzed further in later lectures.

Proof of wild statement 1

Benedict Morrissey

In the case of curves, we give an overview of T. Mochizuki's proof of "wild" version of statement 1. If (τ, S) is a polarized wild pure twistor D-module of weight w on an algebraic curve X, then its pushfoward to a point is a polarized graded Lefschetz twistor structure of weight w. We will focus on how the proof in this case differs from the tame case, which was proven earlier in this Arbeitsgemeinschaft. This statement can be seen as a generalization of the Hodge decomposition and Hard Lefschetz theorem in the compact curve case.

Deligne-Malgrange lattices and purely imaginary wild harmonic bundles

Mauro Porta

In this talk we approach the subtleties arising from passing from dimension 1 to higher dimensions. In a nutshell, when passing from meromorphic connections with isolated singularities to meromorphic connections with singularities concentrated along a simple normal crossing divisor, Hukuhara-Levelt-Turrittin formal composition theorem will only hold outside of a codimension 1 open on the divisor (so, outside of codimension 2 in X). Roughly speaking, the closed subset of points where we don't have a good formal decomposition is called the set of turning points, and it is a major obstruction in establishing the Riemann-Hilbert correspondence and in studying the existence of pluriharmonic metrics on the meromorphic connection. Through the lenses of an explicit example, I will introduce the former Sabbah's conjecture, stating that there exists a resolution of turning points, which is analogous to resolution of singularities. This conjecture has been proven independently by Mochizuki and Kedlaya, and if time permits I will briefly survey the broad ideas involved.

Wild harmonic bundles and semi-simple meromorphic flat bundles

Jean-Babtiste Teyssier

As proved by Mochizuki, wild harmonic bundles canonically extend at infinity as meromorphic flat bundles. The goal of this talk is to explain how semi-simplicity of the extended flat bundle translates in terms of the original harmonic bundle. The emphasis is put on the Kobayashi-Hitchin correspondence as the main player of this interplay, giving necessary and suffisant conditions for a meromorphic flat bundle to come from a wild harmonic bundle. The roles of Mochizuki's resolution of turning points and Deligne-Malgrange's parabolic structures are underlied. A special emphasis is put on how the DM parabolic structure appears as the most natural candidate satisfying the numerical conditions of the Kobayashi-Hitchin correspondence. As application of this circle of results, we give a detailed proof of Mochizuki's theorem stating that semi-simplicity for algebraic flat bundles is stable under pull-back.

Wild harmonic bundles and wild pure twistor D-module

Bruno Klingler

Let X be a smooth complex projective variety and X^{an} its associated compact complex manifold. Non-abelian Hodge theory relates topological, algebraic differential, metric and algebraic $\mathcal{O}_X$-linear objects on X (the Betti, de Rham, harmonic and Dolbeault objects). The simplest ("smooth") instance is the equivalence of categories between semi-simple (complex) local systems on X^{an}, semi-simple flat algebraic connections on X, harmonic bundles on X^{an} and semi-stable Higgs bundles on X with vanishing Chern classes. Allowing tame singular objects, this generalizes to an equivalence between semi-simple perverse sheaves on X^{an}, semi-simple regular holonomic D-modules on X and $\sqrt{-1}\mathbb{R}$ polarizable tame twistor D-modules of weight 0 on X^{an} (the Higgs picture is already unclear at this level of tame singular objects). Passing to general singular objects, the main purpose of this talk was to sketch the proof the following result (where now both the Betti and Dolbeault pictures are unclear):

Theorem 1 (T. Mochizuki)**.** *The functor "fiber at* 1*"*

$$\Xi_{dR} : \mathrm{MT}^{\mathrm{wild}}(X, 0, \sqrt{-1}\mathbb{R})^{(p)} \to \mathrm{Hol}_{ss}(X)$$

between the category of $\sqrt{-1}\mathbb{R}$ *polarizable wild twistor* D*-modules of weight* 0 *on* X^{an} *and the category of general semi-simple holonomic* D*-modules on* X *is an equivalence of category.*

Kashiwara's conjecture (existence of the "Lefschetz package" for semi-simple holonomic D-modules on X) then follows immediately from the corresponding statement for polarizable pure wild twistor D-modules on X^{an} (proven in Lecture 12).

The proof of Theorem 1 reduces easily to simple objects with strict support:

Theorem 2. [Moc11, (19.4.1)] *Let* $Z \subset X$ *be a closed irreducible subvariety of* X*. The functor "fiber at* 1*"*

$$\Xi_{dR} : \mathrm{MT}^{\mathrm{wild}}_{Z,s}(X, 0, \sqrt{-1}\mathbb{R})^{(p)} \to \mathrm{Hol}_{Z,s}(X)$$

between the category of $\sqrt{-1}\mathbb{R}$ *polarizable wild twistor simple* D*-modules of weight* 0 *on* X^{an} *with strict support* Z *and the category of simple holonomic* D*-modules on* X *with support contained in* Z *is an equivalence of category.*

The proof of Theorem 2 consists in two steps: (A): prove that Ξ_{dR} maps simple objects to simple objects, so that the statement actually makes sense. (B): prove the equivalence of category. Here the main difficulty is the essential surjectivity of Ξ_{dR}.

Using the Kedlaya-Mochizuki resolution of turning points (Lectures 16 and 17), one reduces the proof of Theorem 2 to showing:

Theorem 3. [Moc11, (19.1.2)] *Let* $Z \subset X$ *be a closed irreducible subvariety of* X*. The functor*

$$\Phi : \mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, 0, \sqrt{-1}\mathbb{R}) \to \mathrm{VPT}^{\mathrm{wild}}(Z, U, 0, \sqrt{-1}\mathbb{R})$$

which, to a polarized wild pure twistor D*-module with strict support* Z *whose restriction to an open subset* $U \subset Z$ *is a variation of polarized twistor structure of weight* 0*, associates the unique pluriharmonic bundle* $(E, \overline{\partial}_E, \theta, h)$ *on* U *whose metric* h *is adapted to the Deligne-Malgrange lattice, is an equivalence of category. Here the morphisms on both sides are isomorphisms.*

Again, the proof of Theorem 3 consists in two steps: (A): prove that Φ maps simple objects to simple objects, so that the statement actually makes sense. (B): prove the equivalence of category. And again the main difficulty is the essential surjectivity of Φ, equivalently the construction of the minimal extension of a given wild harmonic bundle $(E, \overline{\partial}_E, \theta, h) \in \mathrm{VPT}^{\mathrm{wild}}(Z, U, 0, \sqrt{-1}\mathbb{R})$ as an object of $\mathrm{MPT}^{\mathrm{wild}}_{\mathrm{strict}}(Z, U, 0, \sqrt{-1}\mathbb{R})$.

The main step consists in dealing with the local situation: $(E, \overline{\partial}_E, \theta, h)$ is an unramifiedly good wild harmonic bundle on $X - D = \Delta^n - \bigcup_{i=1}^{l} D_i$, $D_i = \{z_i = 0\}$. In that case, one extracts the minimal extension from the "meromorphic extension" ($\mathcal{Q}\mathcal{E}$ on $(X, D) \times \mathcal{C}_\lambda, \mathbb{D}$) introduced in Lecture 15. This is done in [Moc11, (19.2.1)] by induction on the dimension of X: the case of curves is rather easy, while the inductive step is carried out in [Moc11, Chap.12].

It remains to prove the full faithfulness of Φ in Theorem 3. The faithfulness is elementary. The fullness on the other hand is non trivial, because of the non-uniqueness of the meromorphic extension in the wild case (existence of Stokes structures). When X is a curve the key ingredient is [Moc11, (12.6.1)].

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