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Hypoelliptic Operators in Geometry

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Abstract. The workshop titled *Hypoelliptic Operators in Geometry* was coorganized by Davide Barilari (Padova), Xiaonan Ma (Paris), Nikhil Savale (Köln) and Yi Wang (Baltimore). It was well attended by 55 participants, with 45 of them being present in person and 10 being online. The participants came from several continents, age groups and included male as well as female researchers. Several interesting themes were discussed including: analysis around Kohn's Laplacian in CR geometry, analogous covariant operators arising in conformal geometry, the spectral theory of the sub-Riemannian Laplacian, pseudodifferential calculi in non-commutative geometry and the geometric applications of Bismut's hypoelliptic Laplacians.

Mathematics Subject Classification (2020): 32A25, 32V20, 35H10, 35H20, 35K08, 35R03, 53C17, 53C18, 58C40, 58J20, 58J40, 58J65.

Introduction by the Organizers

The workshop featured 22 one-hour-long talks, as well as 4 half-hour talks presented by junior researchers. The talks were of high quality and covered a wide range of topics, creating a highly interactive and stimulating environment at the workshop. The responsibility of organizing the talks was evenly shared among the coorganizers from different fields. The following provides a brief summary of the various presentations.

The opening talk of the workshop was given by Jean-Michel Bismut and was a comprehensive introduction to the theory of hypoelliptic Laplacian initiated by him approximately twenty years ago. In the talk, he described the precise construction of the hypoelliptic Laplacians on the cotangent bundle, drawing an analogy with the Witten deformation. Moreover, he highlighted its interpolation properties, showcasing its ability to connect the Hodge Laplacian and the geodesic flow.

Omar Mohsen described a joint work with Androulidakis and Yuncken on a higher order generalization of the classical Hörmander theorem regarding the hypoellipticity of second-order operators. This had previously been conjectured by Helffer-Nourrigat and the proof combines their ideas with recent advances in the non-commutative geometry of singular foliations.

Karen Habermann presented her study of the small-time fluctuations of hypoelliptic diffusion processes. Specifically, she discussed her results for sub-Riemannian diffusion loops, which extended the work by Bailleul, Mesnager, and Norris, which initially considered regions outside the sub-Riemannian cut locus.

Ursula Ludwig described her recent advances extending the theorem of Cheeger-Müller to manifolds with isolated conical singularities. The strategy based on the Witten deformation furthermore generalizes the work of Bismut-Zhang to the singular setting.

Jeffrey Case described the construction of conformally covariant poly-differential operators D^g corresponding to conformally variational scalar Riemannian invariants I^g. This construction generalizes the classical GJMS and σ_2 operators. Furthermore, he explained how the minimization problem for the associated Sobolev quotient relates to the uniformization problem for the corresponding scalar Riemannian invariant I^g.

Rémi Léandre first recalled his classical work on the small time asymptotics of subelliptic heat kernels, on and near-off diagonal. He then explained his recent work on how to extend the Varadhan estimate, the Wong-Zakai approximation and the classical flow theorem to big order generators by adopting to this setting the Malliavin calculus techniques of Bismut.

Andrea Malchiodi explained the latest developments in the continuing program around the CR Yamabe problem. Particularly its solvability for embeddable three dimensional CR structures for which the Yamabe constant, the infimum of the Einstein-Hilbert functional, is attained. As well as its unattainability for the non-standard Rossi spheres.

Chin-Yu Hsiao first explained his original results on the parametrix construction for the heat kernel of the Kohn Laplacian $\Box_b^{(q)}$ in degree q on a CR manifold whose Levi form is non-degenerate with signature $(q, n-q)$. Hypoellipticity for $\Box_b^{(q)}$ fails in this degree and the heat kernel is a complex FIO in the sense of Melin-Sjöstrand. He then explained more recent consequences towards the heat trace expansion for the semiclassical Kohn Laplacian acting on high tensor powers of a CR line bundle.

Shu Shen explained how a local version of the classical Riemann - Roch - Hirzebruch theorem can be obtained using a Bismut type hypoelliptic deformation of the Kodaira Laplacian.

Wolfram Bauer's talk surveyed his work on the spectral theory of the sub-Riemannian Laplacian. Particularly the construction of isospectral yet not diffeomorphic sub-Riemannian structures on the class of pseudo H-type nilmanifolds.

Xi-Nan Ma's talk explained the non-existence of non-trivial solutions to a semilinear subelliptic equation on the Heisenberg group. The equation arises in connection to the CR Yamabe problem and improves known results to exponents below the critical Sobolev embedding exponent.

Giulio Tralli explained how gauge balls in the Heisenberg group are characterized by prescribing their horizontal mean curvature.

George Marinescu described the most general leading asymptotic result for the smallest eigenvalue of the Bochner Laplacian on high tensor powers of a line bundle. This particularly leads to the Bergman kernel expansion for semi-positive line bundles on a Riemann surface. The proof exploits the relation of the Bochner Laplacian on tensor powers with the sub-Riemannian Laplacian on the unit circle bundle.

Ruobing Zhang's talk described his work on classifying the possible Gromov-Hausdorff limits of Ricci-flat metrics on K3 manifolds and its relation to the classification of Einstein manifolds.

Wei-Chuan Shen's talk described the parametrix construction as a complex FIO for functions of a Toeplitz operator on a CR manifold in the semi-classical limit.

Tania Bossio described an asymptotic formula of Steiner type for the volume of small tubular neighborhoods of a non-characteristic surface inside a 3D contact sub-Riemannian manifold.

Eric Chen described his results on the convergence properties for the Yamabe flow on asymptotically Euclidean (AE) manifolds contingent on the sign of the Yamabe constant.

Sundaram Thangavelu explained a generalization of the Hardy inequality to fractional powers of the sub-Laplacian on the Heisenberg group. This fractional sub-Laplacian arises in connection with the Dirichlet to Neumann map on the product of the Heisenberg group with the half-line.

Francis Nier showed how exponentially small eigenvalues of the Witten Laplacian can be described in terms of the persistent (co)-homology of the given Morse function. Furthermore, he outlined a program of answering analogous questions for Bismut's hypoelliptic Laplacian on the cotangent space in relation with Floer (co)-homology.

Dario Prandi described generalizations of the Hardy inequality to magnetic sub-Laplacians on the Heisenberg group.

Matthew Gursky explained an old conjecture of Singer stating that anti-self-dual (ASD) metrics on a four manifold with positive Yamabe constant are unobstructed, and he outlined a proof of it assuming a bound on the Yamabe constant in terms of the Euler characteristic and the signature of the manifold.

Bingxiao Liu's talk provided an overview of Bismut's hypoelliptic Laplacian on symmetric spaces and elucidated the ideas for computing explicitly the semisimple orbital integrals of the heat kernels through the hypoelliptic deformation.

Veronique Fischer's talk described her results concerning quantum limits for sub-elliptic operators and in particular the existence of micro-local defect and semi-classical measures on nilpotent Lie groups and nilmanifolds, as well as some of their properties.

Po-Lam Yung described the construction of four sub-elliptic pseudodifferential calculi associated to a smoothly varying quadratic form on the cotangent bundle of $\mathbb{R}^d$. The construction, inspired from several complex variables, generalizes some classical work of Nagel and Stein.

Tomasso Rossi's talk described the small time asymptotics of the relative heat content on sub-Riemannian manifolds with non-characteristic boundaries.

The final talk was from Guofang Wang and concerned geometric inequalities, especially ones of isoperimetric type, for capillary hypersurfaces. They are obtained by using suitable curvature flows in which conformal Killing vector fields play a crucial role.

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Workshop: Hypoelliptic Operators in Geometry

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Abstracts

The hypoelliptic Laplacian in geometry

Jean-Michel Bismut

In my talk, I gave an introduction to the hypoelliptic Laplacian in geometry. Our hypoelliptic Laplacian is used to improve our knowledge and understanding of classical Riemannian geometry and of the classical elliptic Laplacian.

Given a compact Riemannian manifold X, the hypoelliptic Laplacian acts on a larger space than X, the total space of a Euclidean vector bundle on X that contains the tangent bundle TX. Two threads appear in its construction:

(1) Index theory, which gives a formal model to explain the conserved spectral quantities that appear in the hypoelliptic deformation.
(2) Dynamical systems.

Each geometric Laplacian (real, complex, locally symmetric) has a specific hypoelliptic deformation in its category. Applications include:

(1) Hodge theory in de Rham theory and its hypoelliptic deformation, in connection with the Ray-Singer analytic torsion.
(2) Hodge theory for complex manifolds. Applications include holomorphic torsion, the Riemann-Roch-Grothendieck theorem for coherent sheaves.
(3) The geometric evaluation of orbital integrals in connection with Selberg's trace formula.

1. The form of the hypoelliptic Laplacian

Let X be a compact Riemannian manifold of dimension n, and let $\pi : \mathcal{X} \to X$ be the total space of its tangent bundle, with fiber $\widehat{TX}$. Let Y denote the canonical section of $\pi^*\widehat{TX}$ on $\mathcal{X}$. Let H be the harmonic oscillator along the fiber $\widehat{TX}$, given by

$$H = \frac{1}{2}\left(-\Delta^V + |Y|^2 - n\right).$$

Let $Y^{\mathcal{H}}$ denote the vector field on $\mathcal{X}$ which generates the geodesic flow. In geodesic coordinates centered at $x \in X$, then

$$Y^{\mathcal{H}}(x, Y) = \sum Y^i \frac{\partial}{\partial x^i}.$$

The hypoelliptic Laplacian is the family of operators $L_b^X|_{b>0}$ acting on $\mathcal{X}$ given by

$$L_b^X = \frac{H}{b^2} - \frac{Y^{\mathcal{H}}}{b} + \ldots,$$

where $\ldots$ denotes matrix terms of order 0. The operator L_b^X is a non-elliptic, non self-adjoint differential operator, that is hypoelliptic by Hörmander [2]. On $\mathbf{R}^3$, the scalar part of L_b^X is known in statistical physics as a Fokker-Planck operator.

2. The hypoelliptic Laplacian in de Rham theory

Let V be a finite dimensional Euclidean vector space, let H be the harmonic oscillator on V. We equip V with the twisted de Rham differential $\overline{d}^V = d^V + Y\wedge$. Then $\overline{d}^{V*} = d^V + i_Y$ is its L_2 adjoint. The corresponding Witten Laplacian is such that

$$\frac{1}{2}\left[d^V, \overline{d}^{V*}\right] = H + N^{\Lambda(V^*)},$$

where $N^{\Lambda(V^*)}$ is the number operator on $\Lambda(V^*)$.

Let (M,ω) be a symplectic manifold. Using the symplectic form, one can define a non-degenerate bilinear pairing ω^* on compactly supported differential forms. The corresponding symplectic Laplacian $\left[d^M, \overline{d}^M\right]$ vanishes. Let $\mathcal{H} : M \to \mathbf{R}$ be a smooth function. When scaling the symplectic volume by the factor $e^{-2\mathcal{H}}$, let $\overline{d}^M_{2\mathcal{H}}$ denote the associated adjoint of d^M. Let $Y^{\mathcal{H}}$ be the Hamiltonian vector field associated with $\mathcal{H}$, and let $L_{Y^{\mathcal{H}}}$ denote the associated Lie derivative operator. The corresponding Hodge Laplacian is given by

$$\left[d^M, \overline{d}^M_{2\mathcal{H}}\right] = -2L_{Y^{\mathcal{H}}}.$$

The hypoelliptic Laplacian in de Rham theory [3] is obtained by combining these two constructions. We take $X, \mathcal{X}$ as before. Let $\mathcal{X}^*$ be the total space of the cotangent bundle of X, which we identify to $\mathcal{X}$ by the metric. Then $\mathcal{X}^*$ is a symplectic manifold, whose symplectic form is denoted ω. Given $b > 0$, let η_b be the non-degenerate bilinear form on $T\mathcal{X}^*$,

$$\eta_b(U,V) = \langle \pi_* U, \pi_* V\rangle + b\omega(U,V).$$

Using the bilinear form η_b and the symplectic volume, we can define a non-degenerate bilinear form η_b^* on compactly supported forms. If $\mathcal{H} = \frac{1}{2}|Y|^2$, we correct the volume form by the factor $e^{-2\mathcal{H}}$. Let $d^{\mathcal{X}^*}$ be the de Rham operator on $\mathcal{X}^*$, and let $\overline{d}^{\mathcal{X}^*}_{2\mathcal{H}}$ denote the corresponding formal adjoint. By [3, Theorem 3.4], the corresponding Hodge Laplacian $\mathcal{L}^X_b = \left[d^{\mathcal{X}^*}, \overline{d}^{\mathcal{X}^*}_{2\mathcal{H}}\right]$ is a hypoelliptic Laplacian.

3. The limit of L^X_b as $b \to 0$

As $b \to 0$, the operator L^X_b blows up. I explained arguments developed in Bismut-Lebeau [4] showing that as $b \to 0$, the resolvent of $\mathcal{L}^X_b$ converges to the resolvent of $-\frac{1}{2}\Delta^X$. Let L^X_b be the restriction of $\mathcal{L}^X_b$ to smooth functions. Since $\ker H$ is spanned by $C^\infty(X,\mathbf{R}) \otimes \exp\left(-|Y|^2/2\right)$, the operator $\nabla_{Y^{\mathcal{H}}}$ maps $\ker H$ to its L_2 orthogonal $\ker H^\perp$. As $b \to 0$, the matrix of L^X_b with respect to the splitting $L_2(\mathcal{X}^*) = \ker H \oplus \ker H^\perp$ is of the form,

$$L^X_b \simeq \begin{bmatrix} 0 & -\frac{\nabla_{Y^{\mathcal{H}}}}{b} \\ -\frac{\nabla_{Y^{\mathcal{H}}}}{b} & \frac{H}{b^2} \end{bmatrix}.$$

Let P be the orthogonal projector on $\ker H$. From the above, if we pretend L_b^X to be a matrix in finite dimensions, as $b \to 0$,

$$\left(L_b^X - \lambda\right)^{-1} \to P\left(-\frac{1}{2}\Delta^X - \lambda\right)^{-1} P.$$

These arguments are developed rigorously in Bismut-Lebeau [4]. A simpler probabilistic proof was given in [5, Chapters 12–14].

4. The hypoelliptic Laplacian and dynamical systems

The geodesic flow corresponds to the equation $\ddot{x} = 0$. The trajectory $(x, \dot{x})$ integrates the vector field $Y^{\mathcal{H}}$ on $\mathcal{X}$. The differential equation $\dot{x} = \dot{w}$ gives a formal description of Brownian motion on X. Here $\dot{w}$ describes random impulses given to the trajectory. The standard heat semi-group $\exp\left(t\Delta^X/2\right)$ is the analytic counterpart to Brownian motion. The Langevin differential equation $b^2\ddot{x} + \dot{x} = \dot{w}$, introduced on $\mathbf{R}^3$ by Langevin [1], is the dynamical counterpart to the hypoelliptic Laplacian. Algebraically, it interpolates between Brownian motion for $b = 0$ and the geodesic flow for $b = +\infty$. This interpolation is the dynamical counterpart to the interpolation between the Laplacian and the vector field $Y^{\mathcal{H}}$ via the hypoelliptic Laplacian.

5. Hypoelliptic Laplacian and Poisson formula

When $X = S^1$, $L_b^{S^1}$ is conjugate to $\frac{H}{b^2} - \frac{1}{2}\Delta^{S^1}$ by an unbounded conjugation, from which one finds that $\mathrm{Sp} L_b^{S^1} = \frac{\mathbf{N}}{b^2} + \mathrm{Sp}\left(-\frac{1}{2}\Delta^{S^1}\right)$. If $\mathcal{L}_b^{S^1} = L_b^{S^1} + \frac{N^{\Lambda(\mathbf{R})}}{b^2}$, and if $\mathrm{Tr_s}$ is the supertrace, one deduces that for $t > 0, b > 0$,

$$\mathrm{Tr}\left[\exp\left(t\Delta^{S^1}/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^{S^1}\right)\right].$$

The above equation has an index theoretic quality. Making $b \to +\infty$, one gets Poisson's formula. In connection with Selberg's trace formula, we obtained this way an explicit geometric evaluation of semi-simple orbital integrals for reductive groups [5].

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Generalisation of the fundamental regularity theorem of elliptic operators in sub-Riemannian geometry

Omar Mohsen
(joint work with I. Androulidakis, R. Yuncken)

Elliptic linear differential operators are some of the most extensively studied differential operators in analysis. This is because of their wide applications in many areas of mathematics such as algebraic geometry, complex geometry, symplectic geometry and representation theory. These applications are based on the following fundamental regularity theorem, which is proved using the pseudodifferential calculus developed by Nirenberg, Kohn, Hörmander and others.

Theorem 1. *Let M be a smooth manifold, $D : C^\infty(M) \to C^\infty(M)$ a differential operator of order k. The following are equivalent*

(1) *The operator D is elliptic, i.e., for every $\xi \in T^*M\backslash\{0\}$, $\sigma(D, x, \xi) \neq 0$, where σ is the classical principal symbol of D.*
(2) *For every (or for some) $s \in \mathbb{R}$, and every distribution u on M, $Du \in H^s(M)$ implies $u \in H^{s+k}(M)$, where $H^\bullet$ are the local L^2 Sobolev spaces.*

Furthermore if M is compact, the previous statements are equivalent to the following

(4) *For every (or for some) $s \in \mathbb{R}$, $D : H^{s+k}(M) \to H^s(M)$ is Fredholm*

In a celebrated article, Hörmander proved that for some non-elliptic differential operators, now called Hörmander's sums of squares operators, one still has the regularity of solutions.

Theorem 2 ([7])**.** *Let $X_1, \cdots, X_{m+1}$ be vector fields satisfying Hörmander's Lie bracket generating condition, i.e., for every $x \in M$, T_xM is linearly spanned by $X_1(x), \cdots, X_{m+1}(x)$ and their higher Lie brackets $[X_i, X_j](x)$, $[X_i, [X_j, X_l]](x)$ etc. Then $D = \sum_{i=1}^m X_i^2 + X_{m+1}$ is hypoelliptic, i.e., if u is a distribution on M such that Du is smooth, then u is smooth.*

It is natural to try to extend Hörmander's theorem by finding sufficient conditions for the hypoellipticity of arbitrary polynomials in the vector fields X_i. Let P be a noncommutative polynomial in $m+1$ variables with coefficients in $C^\infty(M)$. In 1979, Helffer and Nourrigat [4] conjectured a generalization of both Theorem 1 and Theorem 2 which allows one to obtain hypoellipticity of operators of the form $P(X_1, \cdots, X_{m+1})$, and also generalises several regularity results in the literature.

Our main theorem is a resolution of the Helffer and Nourrigat's conjecture. For simplcity, we only consider here the restricted case $X_{m+1} = 0$. Thus, consider vector fields $X_1, \cdots, X_m$ satisfying the Lie bracket generating condition. This condition gives rise to the following notion of order for a differential operator. Every differential operator can then be written as $D = P(X_1, \cdots, X_m)$ where P is a noncommutative polynomial with coefficients in $C^\infty(M)$. The Hörmander order of D is the minimum of $\deg(P)$ for all possible Ps. The Hörmander order leads us to consider Sobolev spaces for $s \in \mathbb{N}$ defined by

$$\tilde{H}^s(M) := \{u \in L^2_{loc}M : Du \in L^2_{loc}M \text{ for all } D \text{ with Hörmander order} \leq s\}.$$

We extend these Sobolev spaces for any $s \in \mathbb{R}$ by interpolation for $s > 0$ and duality for $s < 0$. Trivially we have

$$\bigcap_{s\in\mathbb{R}} \tilde{H}^s(M) = \bigcap_{s\in\mathbb{N}} \tilde{H}^s(M) = C^\infty(M).$$

Our main theorem is the following.

Theorem 3 ([1]). *Let M be a smooth manifold, $X_1, \cdots, X_m$ are vector fields satisfying the Lie bracket generating condition, $D : C^\infty(M) \to C^\infty(M)$ a differential operator of Hörmander order k. The following are equivalent*

(1) *For every $x \in M$ and $\pi \in \mathcal{T}^*_x\backslash\{0\}$, $\tilde{\sigma}(D, x, \pi)$ is injective on $C^\infty(\pi)$.*
(2) *For every (or for some) $s \in \mathbb{R}$, and every distribution u on M, $Du \in \tilde{H}^s(M)$ implies $u \in \tilde{H}^{s+k}(M)$.*

Furthermore if M is compact, then the previous statements are equivalent to the following

(5) *For every (or for some) $s \in \mathbb{R}$, $D : \tilde{H}^{s+k}(M) \to \tilde{H}^s(M)$ is left invertible modulo compact operators*

We now explain the principal symbol $\tilde{\sigma}$ as well as the space of representations $\mathcal{T}^*_x$. Before we proceed, let us mention that if the vector fields satisfy Hörmander's Lie bracket generating condition of rank 1, i.e., $X_1(x), \cdots, X_m(x)$ span T_xM for all $x \in M$, then Theorem 3 is precisely Theorem 1. The Sobolev spaces $\tilde{H}^s(M)$ and $\tilde{\sigma}$ are equal to $H^s(M)$ and σ respectively. In [5], Helffer and Nourrigat proved Theorem 3 in the case of rank 2, i.e., $X_1(x), \cdots, X_m(x)$ and $[X_i, X_j](x)$ span T_xM for all $x \in M$. They also proved the implication $b \implies a$ in the general case with no assumptions on the rank. The main innovation in our work is combining their work [6, 5] with recent advances in noncommutative geometry by Debord and Skandalis [3] and van Erp and Yuncken [10] together with the C^*-algebra of singular foliations defined by Androulidakis and Skandalis [2] and their blowups defined by Mohsen [9]. This allows us to prove Theorem 3 with no hypothesis at all on the rank.

Suppose that $X_1, \cdots, X_m$ satisfy Hörmander's Lie bracket generating condition of rank $N \in \mathbb{N}$. Let G be the free nilpotent Lie group of rank N with one generator $\tilde{X}_1, \cdots, \tilde{X}_m$ for each vector field $X_1, \cdots, X_m$. Let π be an irreducible unitary representation of G on a Hilbert space $L^2\pi$. Then by taking the derivative of π, one obtains linear maps

$$d\pi(\tilde{X}_1), \cdots, d\pi(\tilde{X}_m) : C^\infty(\pi) \to C^\infty(\pi)$$

where $C^\infty(\pi) \subseteq L^2\pi$ is the space of smooth vectors.

We can now define $\tilde{\sigma}$. We write $D = P(X_1, \cdots, X_m)$ for some noncommutative polynomial P. This is the equivalent of taking local coordinates when defining the classical principal symbol. We then define

$$\tilde{\sigma}(D, x, \pi) : C^\infty(\pi) \to C^\infty(\pi), \quad \tilde{\sigma}(D, x, \pi) = P_{max,x}(d\pi(\tilde{X}_1), \cdots, d\pi(\tilde{X}_m)),$$

where $P_{max,x}$ is the maximal homogeneous part of P after replacing each coefficient $f \in C^\infty(M)$ by $f(x)$. Note that this definition may depend on P since if the operator D can be written $D = P(X_1, \cdots, X_m) = Q(X_1, \cdots, X_m)$ for two different polynomials P, Q, then in general $\tilde{\sigma}(D, x, \pi)$ depends on the choice of P or Q. But one of our main results is that this is not the case when π belongs to a certain naturally defined subset $\mathcal{T}_x^* \subseteq \hat{G}$. The set $\mathcal{T}_x^*$ can be thought of as a generalization of the cotangent space in sub-Riemannian geometry by using the Gromov-Hausdorff distance. We refer the reader to [8] for more details. The set $\mathcal{T}_x^*$ only depends on the vector fields $X_1, \cdots, X_m$ and not on D. This set was defined by Helffer and Nourrigat in [4] using Kirillov's orbit method. For this reason, we call it the **Helffer-Nourrigat cone.**

Theorem 4. *For each $x \in M$, for any representation $\pi \in \mathcal{T}_x^* \subseteq \hat{G}$, $\tilde{\sigma}(D, x, \pi)$ doesn't depend on the presentation of $D = P(X_1, \cdots, X_m)$.*

We end by the following remarking that one can extend Theorem 3 to allow the case $X_{m+1} \neq 0$ (and thus recover the full Hörmander's theorem). This is done by adding weights to each vector field. We refer the reader to [1] for more details.

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Small-time fluctuations for hypoelliptic diffusion bridges

Karen Habermann

A diffusion process on a connected smooth manifold M is called hypoelliptic if its associated generator $\mathcal{L}$ is a hypoelliptic second order partial differential operator on M. We focus on those hypoelliptic diffusion processes whose generators can be written as

$$\mathcal{L} = X_0 + \frac{1}{2}\sum_{i=1}^{m} X_i^2 ,$$

for $m \in \mathbb{N}$ and smooth vector fields $X_0, X_1, \ldots, X_m$ on M.

1. Small-time fluctuations for sub-Riemannian diffusion bridges

Suppose that $X_1, \ldots, X_m$ satisfy the strong Hörmander condition, that is, for all $x \in M$, the vectors $X_1(x), \ldots, X_m(x)$ together with the collection of vectors

$$[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]](x) \quad \text{for} \quad k \geq 2 \text{ and } 1 \leq i_1, \ldots, i_k \leq m$$

span T_xM. Such vector fields induce a sub-Riemannian structure on M, and we call the hypoelliptic diffusion with generator $\mathcal{L}$ on M a sub-Riemannian diffusion.

For a sub-Riemannian diffusion with generator $\mathcal{L}$, for $\varepsilon > 0$ and for $x, y \in M$, let $\mu_\varepsilon^{x,y}$ denote the law of the sub-Riemannian diffusion with generator $\varepsilon\mathcal{L}$ conditioned to start at x at time 0 and to end in y at time 1. This corresponds to studying the sub-Riemannian diffusion with generator $\mathcal{L}$ conditioned to move from x to y in time ε. One expects that as $\varepsilon \to 0$, these diffusion bridges localise around the paths from x to y of minimal energy, for an appropriate energy functional defined in terms of the underlying sub-Riemannian structure. Bailleul, Mesnager and Norris prove in [1] that, subject to suitable conditions on the vector fields defining $\mathcal{L}$, if x and y are connected by a unique path γ of minimal energy then

$$\mu_\varepsilon^{x,y} \to \delta_\gamma$$

weakly on $\Omega^{x,y} = \{\omega \in C([0,1], M) \colon \omega_0 = x, \omega_1 = y\}$ in the limit $\varepsilon \to 0$. As a natural follow-up question, they further analyse the small-time fluctuations for sub-Riemannian diffusion bridges. Let $d = \dim M$. For a smooth map $\theta \colon M \to \mathbb{R}^d$ which restricts to a diffeomorphism on a domain in M containing the path γ of minimal energy and for $T_\gamma\Omega^{x,y}$ denoting the set of continuous paths $v \colon [0,1] \to TM$ over γ with $v_0 = v_1 = 0$, define a rescaling map $\sigma_\varepsilon \colon \Omega^{x,y} \to T_\gamma\Omega^{x,y}$ by

$$\sigma_\varepsilon(\omega)_t = \frac{\left(\mathrm{d}\theta_{\gamma_t}\right)^{-1}\left(\theta(\omega_t) - \theta(\gamma_t)\right)}{\sqrt{\varepsilon}} .$$

Bailleul, Mesnager and Norris [1] establish that if the pair (x, y) lies outside the sub-Riemannian cut locus, the rescaled diffusion bridge measures $\tilde{\mu}_\varepsilon^{x,y} = \mu_\varepsilon^{x,y} \circ \sigma_\varepsilon^{-1}$ converge weakly to a unique zero-mean Gaussian measure μ_γ on $T_\gamma\Omega^{x,y}$ as $\varepsilon \to 0$.

Small-time fluctuations for sub-Riemannian diffusion loops. The result by Bailleul, Mesnager and Norris generally cannot be used to study small-time fluctuations for sub-Riemannian diffusion loops since, unless the operator $\mathcal{L}$ is elliptic at the point x, the pair (x,x) lies inside the sub-Riemannian cut locus. In [3, 4], I adapt the approach by Bailleul, Mesnager and Norris, which is based on a method of Azencott, Bismut and Ben Arous, and relies on ideas from Malliavin calculus, to characterise suitably rescaled fluctuations of sub-Riemannian diffusion loops in terms of an underlying nilpotency structure. Working in an adapted chart at x, which was introduced by Bianchini and Stefani in [2], I define weights $w_1,\dots,w_d$ depending on a filtration induced by the vector fields $X_1,\dots,X_m$ around x, and consider the anisotropic dilation $\delta_\varepsilon\colon \mathbb{R}^d\to\mathbb{R}^d$ given by

$$\delta_\varepsilon\left(y^1,\dots,y^k,\dots,y^d\right)=\left(\varepsilon^{w_1/2}y^1,\dots,\varepsilon^{w_k/2}y^k,\dots,\varepsilon^{w_d/2}y^d\right).$$

For a smooth map $\theta\colon M\to\mathbb{R}^d$, which on a neighbourhood around x restricts to the chosen adapted chart at x, let $\tau_\varepsilon\colon \Omega^{x,x}\to T_x\Omega^{x,x}$ be the rescaling map

$$\tau_\varepsilon(\omega)_t=(\mathrm{d}\theta_x)^{-1}\left(\delta_\varepsilon^{-1}\left(\theta(\omega_t)-\theta(x)\right)\right),$$

and define the rescaled loop measures $\tilde{\nu}_\varepsilon^{x,x}=\mu_\varepsilon^{x,x}\circ\tau_\varepsilon^{-1}$. The nilpotent approximations $\tilde{X}_1,\dots,\tilde{X}_m$ of the vector fields $X_1,\dots,X_m$ around x are polynomial vector fields on $\mathbb{R}^d$ which satisfy that $(\delta_\varepsilon^{-1})_*\tilde{X}_i=\varepsilon^{-1/2}\tilde{X}_i$ and are obtained through the blow-up construction $\sqrt{\varepsilon}(\delta_\varepsilon^{-1})_*(\theta_*X_i)\to\tilde{X}_i$ as $\varepsilon\to 0$, assuming $\theta(x)=0$. They further inherit the strong Hörmander condition from the vector fields $X_1,\dots,X_m$. Let $\tilde{\nu}^{x,x,\mathbb{R}^d}$ be the law of the diffusion loop at $\theta(x)=0$ in $\mathbb{R}^d$ in time 1 associated with the generator $\tilde{\mathcal{L}}$ on $\mathbb{R}^d$ given by

$$\tilde{\mathcal{L}}=\frac{1}{2}\sum_{i=1}^m\tilde{X}_i^2,$$

and set $\tilde{\nu}^{x,x}=\tilde{\nu}^{x,x,\mathbb{R}^d}\circ\rho^{-1}$ for the map ρ defined by $\rho(\omega)_t=(\mathrm{d}\theta_x)^{-1}\omega_t$. I prove that if the drift vector field X_0 lies in the span of the vector fields $X_1,\dots,X_m$, the rescaled diffusion loop measures $\tilde{\nu}_\varepsilon^{x,x}$ converge weakly to the probability measure $\tilde{\nu}^{x,x}$ on $T_x\Omega^{x,x}$ as $\varepsilon\to 0$.

I further use my description of the suitably rescaled fluctuations to prove that the small-time fluctuations for sub-Riemannian diffusion loops which one obtains by using the scaling by Bailleul, Mesnager and Norris need no longer be Gaussian. Thus, depending on the properties of their endpoints, sub-Riemannian diffusion bridges can exhibit qualitatively different limit behaviours.

2. Small-time fluctuations for a model class of hypoelliptic diffusion bridges

In [5], I demonstrate that hypoelliptic diffusion bridges can exhibit qualitatively different small-time phenomena in comparison to sub-Riemannian diffusion bridges. For the model class discussed below, I establish that, unlike [1], hypoelliptic diffusion bridges need not localise around a path, but can show a blow-up behaviour in the small time limit. However, after compensating for this blow-up, I still make sense of suitably rescaled fluctuations for hypoelliptic diffusion bridges in that model class.

Let A be a $d \times d$ matrix and B be a $d \times m$ matrix such that A and B satisfy the Kalman rank condition, that is, there exists $N \in \mathbb{N}$ with

$$\operatorname{rank}\left[B, AB, A^2B, \ldots, A^{N-1}B\right] = d\,.$$

Let $X_0, X_1, \ldots, X_m$ be the vector fields on $\mathbb{R}^d$ defined by $X_0 = \sum_{j,k=1}^{d} A_{jk} x_k \frac{\partial}{\partial x_j}$ and $X_i = \sum_{j=1}^{d} B_{ji} \frac{\partial}{\partial x_j}$ for $i \in \{1, \ldots, m\}$, and let $\mathcal{L}$ be the second order partial differential operator on $\mathbb{R}^d$ given by

$$\mathcal{L} = X_0 + \frac{1}{2}\sum_{i=1}^{m} X_i^2 = \sum_{j=1}^{d} (Ax)_j \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{j,k=1}^{d} (BB^*)_{jk} \frac{\partial^2}{\partial x_j \partial x_k}\,.$$

Fix $x, y \in \mathbb{R}^d$. For $\varepsilon > 0$, there exists a unique diffusion $(x_t^\varepsilon)_{t\in[0,1]}$ starting from x and having generator $\varepsilon\mathcal{L}$. The Kalman rank condition ensures that the operator $\mathcal{L}$ is hypoelliptic, and thus, the diffusion $(x_t^\varepsilon)_{t\in[0,1]}$ is hypoelliptic. In addition, $(x_t^\varepsilon)_{t\in[0,1]}$ is a Gaussian process, and as a consequence, we always have an explicit expression for the bridge process $(z_t^\varepsilon)_{t\in[0,1]}$ obtained by conditioning $(x_t^\varepsilon)_{t\in[0,1]}$ on $x_1^\varepsilon = y$.

For hypoelliptic diffusions in the model class of consideration, we can exhibit a deterministic path $(\phi_t^\varepsilon)_{t\in[0,1]}$ in $\mathbb{R}^d$ that compensates for any blow-up occurring in the process $(z_t^\varepsilon)_{t\in[0,1]}$ in the limit $\varepsilon \to 0$. Let $(\phi_t^\varepsilon)_{t\in[0,1]}$ be defined by

$$\phi_t^\varepsilon = \mathrm{e}^{\varepsilon t A}\, x + \mathrm{e}^{\varepsilon t A}\, \Gamma_t^\varepsilon \left(\Gamma_1^\varepsilon\right)^{-1} \left(\mathrm{e}^{-\varepsilon A}\, y - x\right)\,,$$

where $\Gamma_t^\varepsilon = \int_0^t \mathrm{e}^{-\varepsilon s A}\, BB^*\, \mathrm{e}^{-\varepsilon s A^*}\, \mathrm{d}s$. The Kalman rank condition ensures that the $d \times d$ matrix Γ_1^ε is invertible. In [5], I show that the processes $(z_t^\varepsilon - \phi_t^\varepsilon)_{t\in[0,1]}$ converge weakly as $\varepsilon \to 0$ to the zero process on the set of continuous loops $\{\omega \in C([0,1], \mathbb{R}^d)\colon \omega_0 = 0, \omega_1 = 0\}$, and, working in a suitable basis of $\mathbb{R}^d$, that there exists a $d \times d$ diagonal matrix D_ε such that the rescaled processes $(\varepsilon^{-1/2} D_\varepsilon^{-1} (z_t^\varepsilon - \phi_t^\varepsilon))_{t\in[0,1]}$ converge weakly as $\varepsilon \to 0$ to a non-degenerate limit process. The matrix D_ε rescales the jth coordinate direction by ε^{w_j} for weights $w_1, \ldots, w_d$ characterised in terms of the matrices A and B. Furthermore, I explicitly determine the limit fluctuation process.

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Bismut-Zhang Theorem and Anomaly Formula for the Ray-Singer Metric on Singular Spaces

Ursula Ludwig

1. Introduction

An important comparison theorem in global analysis is the comparison of analytic (or Ray-Singer) and topological torsion, aka the Cheeger-Müller theorem, for smooth compact Riemannian manifolds equipped with a unitary flat vector bundle. It has been conjectured by Ray and Singer and has been independently proved by Cheeger [3] and Müller [8]. The most general version of the comparison of torsion for smooth compact manifolds, namely when the flat vector bundle is arbitrary, has been achieved by Bismut and Zhang in [2] by combining the Witten deformation and local index techniques. Bismut and Zhang compare the analytic torsion with the Milnor torsion, which is defined using Morse theory. If the flat vector bundle is not unitary or unimodular the two torsions are no longer equal and the difference between them can be expressed in terms of the Mathai-Quillen current. In this note we refer to this most general version of the comparison theorem of torsions as the Bismut-Zhang theorem.

In the last decade a lot of progress has been made in the study of the Cheeger-Müller theorem for spaces with singularities of different types. The question of a Cheeger-Müller theorem in the context of isolated conical singularities has first been raised in [4]. Following a strategy suggested by Lesch to study the question using gluing formulas, Vertman as well as Hartmann and Spreafico have studied mainly the analytic torsion of a truncated cone. In [1] Albin, Rochon and Sher prove a Cheeger-Müller theorem for wedge spaces with even-codimensional singular stratum under an acylicity assumption on the unimodular flat vector bundle.

The aim of this note is to present a Bismut-Zhang formula for singular spaces with isolated conical singularities and flat vector bundles, which are not necessarily unitary or unimodular. The strategy in this project is to generalise the approach of Bismut and Zhang [2] to this singular setting and relies on the generalisation of the Witten deformation to singular spaces with isolated conical singularities and anti-radial Morse functions achieved in [5].

2. Notation

In the following (X, g^{TX}) denotes a singular space with isolated conical singularities. The singular set of X will be denoted by $\mathrm{Sing}(X)$. Every point $p \in \mathrm{Sing}(X)$ has a neighbourhood, which can be identified with a cone cL_p, where L_p is a smooth compact manifold, called the link of the singularity. The Riemannian metric g^{TX} in a neighbourhood of singular points is conical, *i.e.* of the form $dr^2 + r^2 g^{TL_p}$, where r is the distance from the cone point and g^{TL_p} is a Riemannian metric on the link L_p. We denote by (F, ∇^F, g^F) a Hermitian flat vector bundle, *i.e.* F is a flat vector bundle on $X \setminus \mathrm{Sing}(X)$, ∇^F its flat connection, and g^F a Hermitian metric on F, which is not necessarily flat. We assume, that the restriction of (F,∇^F,g^F) to a neighbourhood of $p\in \operatorname{Sing}(X)$ can be written as a pull-back of a Hermitian flat vector bundle on the link L_p. An anti-radial Morse function $f: X\to\mathbb{R}$ is a continuous function, such that the restriction $f_{|X\setminus \operatorname{Sing}(X)}$ is a smooth Morse function on the smooth (non-compact) manifold $X\setminus\operatorname{Sing}(X)$ and near $p\in\operatorname{Sing}(X)$ we have the following normal form $f=f(p)-r^2/2$.

We denote by $\overline{m}$ (resp. by $\overline{n}$) the lower middle (resp. the upper middle) perversity. For $\overline{q}\in\{\overline{m},\overline{n}\}$, we denote by $IH^{\bullet}_{\overline{q}}(X,F)$ the intersection cohomology of X with perversity $\overline{q}$. The pair of metrics (g^{TX},g^F) on X and F induce an L^2-metric on the space of L^2-forms on $X\setminus\operatorname{Sing}(X)$ with values in F. We denote by $\Delta^{\overline{q}}$ the Laplacian associated to the maximal resp. the minimal L^2-complex on X, and by $\zeta_{\overline{q}}(s)$ its torsion zeta function. The restriction of the L^2-metric to $\ker(\Delta^{\overline{q}})$ induces via the Cheeger-Goresky-MacPherson theorem (*i.e.* the Hodge-de Rham theorem in this singular setting) a metric $|\ \ |^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$ on the determinant line $\det IH^{\bullet}_{\overline{q}}(X,F)$. By a result of A. Dar [4, Section 4], the function $\zeta_{\overline{q}}$ extends to a meromorphic function on the whole complex plane, which is holomorphic at $s=0$. The Ray-Singer metric on $\det IH^{\bullet}_{\overline{q}}(X,F)$ is defined by

$$\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)} := |\ \ |^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)} \exp\left(\frac{1}{2}\zeta'(0)\right). \tag{1}$$

We denote by $\Psi(TX,\nabla^{TX})$ the Mathai-Quillen current defined in [2, Definition 3.6]. We denote by $\theta(F,g^F):=\operatorname{Tr}[(g^F)^{-1}\nabla^F g^F]$, which is a smooth closed 1-form, measuring the obstruction to the existence of a flat volume form on F.

3. Bismut-Zhang Theorem

Using the singular Morse-Thom-Smale complex for the perversity $\overline{q}\in\{\overline{m},\overline{n}\}$ (defined in [5, Section 6], [7, Section 5.2]) and the model Witten Laplacian for the maximal resp. the minimal L^2-complex on the infinite cone cL_p, $p\in\operatorname{Sing}(X)$ (defined in [5, Section 4], [7, Section 4.2]), the so called Bismut-Zhang metric $\|\ \ \|^{Y,g^{TX},g^F}_{\det IH^{\bullet}_{\overline{q}}(X,F)}$ associated to a gradient vector field Y of the anti-radial Morse function f has been defined in [6, 7]. The Bismut-Zhang metric is a Milnor-type metric with an analytic correction term for each $p\in\operatorname{Sing}(X)$; this analytic correction is nothing else than the analytic torsion of the model Witten Laplacian. The following Bismut-Zhang formula has been proved in [7]:

Theorem 1. *Let (X,g^{TX}) be a space with isolated conical singularities, $\overline{q}\in\{\overline{m},\overline{n}\}$. Let (F,∇^F,g^F) be a Hermitian flat vector bundle over $X\setminus\operatorname{Sing}(X)$ as in Section 2. Let $f:X\to\mathbb{R}$ be an anti-radial Morse function, g_0^{TX} a Riemannian metric on X, coinciding with g^{TX} in a neighbourhood of $\operatorname{Sing}(X)$ and such that the pair (f,g_0^{TX}) is Morse-Smale. Set $Y:=\nabla_{g_0}f$. Then:*

$$\log\left(\frac{\|\ \ \|^{RS}_{\det IH^{\bullet}_{\overline{q}}(X,F)}}{\|\ \ \|^{Y,g^{TX},g^F}_{\det IH^{\bullet}_{\overline{q}}(X,F)}}\right)^2 = -\int_X \theta(F,g^F)Y^*\Psi(TX,\nabla^{TX}). \tag{2}$$

In the case of a smooth compact manifold, the statement of Theorem 1 reduces to the extension of the Cheeger-Müller theorem in [2]. For unitary flat vector bundles $\theta(F,\nabla^F)=0$; assuming the Witt and a spectral Witt condition, Theorem 1 has been proved already in [6]. Note that for non Witt spaces, Theorem 1 gives two formulas for the two middle perversities $\overline{q}\in\{\overline{m},\overline{n}\}$, they are compatible with Poincaré duality for intersection cohomology.

4. Anomaly formula for the Ray-Singer metric

In the present singular setting, the Ray-Singer torsion, unlike in the smooth situation, is no longer a topological invariant, even in the case of a unitary flat vector bundle. In [7] we also study variation formulas for all three terms in the Bismut-Zhang formula in Theorem 1 with respect to variations of both the conical Riemannian metric g^{TX} and the Hermitian metric g^F on the flat bundle F. In this note, we only present the anomaly formula for the Ray-Singer metric. In the following we use a subscript l to denote the different notions and operators associated to the pair of metrics (g_l^{TX},g_l^F).

Theorem 2. *Let $\mathbb{R}\ni l\to(g_l^{TX},g_l^F)$ be a family of metrics on TX,F such that a spectral gap condition is satisfied. We have*

$$\begin{aligned}\partial_l \log\left(\left(\|\quad\|^{RS}_{\det IH^\bullet_{\overline{q}}(X,F),l}\right)^2\right) &= \int_X \operatorname{Tr}\left[(g_l^F)^{-1}\frac{\partial g_l^F}{\partial l}\right] e(TX,\nabla_l^{TX}) \\ &\quad + \int_X \iota_{\partial_l} e(\rho^* TX,\nabla^{TX,\mathrm{tot}})\theta(F,g_l^F) + \sum_{p\in\operatorname{Sing}(X)} (c^{\overline{q}}_{p,l}+\widetilde{c}^{\overline{q}}_{p,l}),\end{aligned} \tag{3}$$

where the contributions of the singularities $c^{\overline{q}}_{p,l}$, $\widetilde{c}^{\overline{q}}_{p,l}$, $p\in\operatorname{Sing}(X)$, are Cheeger type invariants.

Note that the first two terms in the formula in Theorem 2 are the local contributions, well-known from the smooth anomaly formula for the Ray-Singer metric in [2]; they do not depend on the chosen extension of the Laplacian. The appearance of the local contributions $c^{\overline{q}}_{p,l}$, $\widetilde{c}^{\overline{q}}_{p,l}$, $p\in\operatorname{Sing}(X)$ is very similar to that of the Cheeger invariant in the Chern-Gauss-Bonnet formula for singular spaces.

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Conformally covariant polydifferential operators and curvature prescription problems

Jeffrey S. Case

(joint work with Yueh-Ju Lin, Yi Wang, Zetian Yan, and Wei Yuan)

A *conformally covariant* r*-differential operator* on (M^n,g) is a multilinear operator $D^g\colon \big(C^\infty(M)\big)^r \to C^\infty(M)$ such that fixing all but one input yields a differential operator and there are constants $a,b\in\mathbb{R}$ such that

$$D^{e^{2u}g}(v_1,\dots,v_r)=e^{-bu}D^g(e^{au}v_1,\dots,e^{au}v_r)$$

for all $u,v_1,\dots,v_r\in C^\infty(M)$. We say that D is *formally self-adjoint* if

$$(u_0,\dots,u_r)\mapsto\int_M u_0\,D^g(u_1,\dots,u_r)\,\mathrm{dv}_g$$

on $\big(C_0^\infty(M)\big)^{r+1}$ is symmetric, and we call $k:=b-ar$ the *total order* of D. If D is a formally self-adjoint polydifferential operator of total order k, then necessarily $a=\frac{n-2k}{r+1}$ and $b=\frac{rn+2k}{r+1}$. Similar definitions can be made on CR manifolds.

Familiar examples of conformally covariant polydifferential operators are the GJMS operators [10] and the CR GJMS operators [8]. Low-order examples of bi- and tri-differential operators are the *Ovsienko–Redou operators* [2, 3]

$$L_{\mathrm{OR}}(u):=\Delta^2u^2+2u\Delta^2u+\frac{2(n-4)}{n+2}\left((\Delta u)^2+\Delta(u\Delta u)\right)+\text{l.o.t.}$$

and the σ_2*-operator* [2, 5]

$$L_{\sigma_2}(u):=\frac12\delta\left(|\nabla u|^2\,du\right)-\frac{n-4}{16}\left(u\Delta|\nabla u|^2-\delta\left((\Delta u^2)\,du\right)\right)+\text{l.o.t.};$$

in both cases, the polydifferential operator is obtain by polarization.

Conformally covariant polydifferential operators are in one-to-one correspondence with conformally variational scalar Riemannian invariants. This is made precise in the following way: A scalar Riemannian invariant I^g is *conformally variational* if $I^{cg} = c^{-k} I^g$ for some $k \in \mathbb{N}$ and if there is a functional $\mathcal{F}\colon [g] \to \mathbb{R}$ defined on conformal classes such that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(e^{2tu} g) = \int_M u I^g \,\mathrm{dv}_g$$

for all $g \in [g]$ and all $u \in C^\infty(M)$. Building on a result of Gover and Ørsted [9], Yueh-Ju Lin, Wei Yuan and I proved [2] the following result:

Theorem 1. *Suppose that I^g is a conformally variational scalar Riemannian invariant on an n-dimensional manifold such that $I^{cg} = c^{-k} I^g$ for all $c > 0$ and some positive integer $k < n/2$. Then there is a formally self-adjoint conformally covariant polydifferential operator $D\colon (C^\infty(M))^r \to C^\infty(M)$, $r \leq k-1$, such that*

$$D^g(1, \ldots, 1) = \left(\frac{n-2k}{r+1}\right)^r I^g.$$

There is a corresponding version of this statement when $n = 2k$. Applying Theorem 1 to the Q-curvature of order $2k$ (resp. the σ_2-curvature recovers the GJMS operator P_{2k} (resp. the σ_2-operator).

Let $D^g\colon \left(C^\infty(M^n)\right)^r \to C^\infty(M)$ is a conformally covariant polydifferential operator of total order $2k < n$. Given an open cone $U \subseteq C^\infty(M)$, set

$$Y_D(M,[g]) := \inf \left\{ \int_M u\, D^g(u, \ldots, u) \,\mathrm{dvol}_g : u \in U, \int_M |u|^{\frac{(r+1)n}{n-2k}} \,\mathrm{dvol}_g = 1 \right\}. \tag{1}$$

Our assumptions imply that $Y_D(M,[g])$ is conformally invariant and that if $u > 0$ realizes $Y_D(M,[g])$, then $I^{u^{\frac{2(r+1)}{n-2k}} g}$ is constant. I proved [1] the following abstraction of ideas of Frank and Lieb [6, 7]:

Theorem 2. *Let $D\colon (C^\infty(S^n))^r \to C^\infty(S^n)$ be a formally self-adjoint conformally covariant polydifferential operator of total order $2k$. Suppose that*

$$\mathcal{D} := D - \frac{r(n-2k)}{2(r+1)k} \sum_{j=0}^{n} x^j [D, x^j] \tag{2}$$

is such that $\mathcal{D} \geq 0$ with $\ker \mathcal{D} = \mathbb{R}$, where

$$[D, x](u_1, \ldots, u_r) := D(xu_1, u_2, \ldots, u_r) - xD(u_1, \ldots, u_r).$$

Let $U \subseteq C^\infty(S^n)$ be closed under the action $\Phi \cdot u := |J_\Phi|^{\frac{n-2k}{n(r+1)}} \Phi^* u$ of the conformal group $\mathrm{Conf}(S^n)$ on $C^\infty(M)$. If $u \in U$ is a local minimizer of

$$u \mapsto \left(\int_{S^n} u\, D(u, \ldots, u) \,\mathrm{dv} \right) \left(\int_{S^n} |u|^{\frac{n(r+1)}{n-2k}} \,\mathrm{dv} \right)^{-\frac{n-2k}{n}},$$

then $u = c\Phi \cdot 1$ for some nonzero constant $c \in \mathbb{R}$ and some $\Phi \in \mathrm{Conf}(S^n)$.

The assumption that $\mathcal{D} \geq 0$ with $\ker \mathcal{D} = \mathbb{R}$ means that

$$\int_M u\,\mathcal{D}(u,\dots,u)\,\mathrm{dv} \geq 0$$

with equality if and only if u is constant.

Combining Theorem 2 with a density argument allows one to prove regularity of minimizers of (1). This has been applied in many settings; a sharp Sobolev inequality refers to the fact that equality is achieved only by the standard bubbles:

(1) By applying Theorem 2 with D equal to the GJMS operator of order $2k$, I gave [1] a direct, rearrangement-free proof of the sharp Sobolev inequality

$$\int_{\mathbb{R}^n} |\Delta^{k/2}u|^2\,dx \geq C\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2k}}\,dx\right)^{\frac{n-2k}{n}}.$$

(Frank and Lieb [6] prove the dual Hardy–Littlewood–Sobolev inequality.)

(2) Using existence results of Guan and Wang [11], I re-proved [1] the fully nonlinear Sobolev inequality

$$\int_{S^n} \sigma_2^g\,\mathrm{dvol}_g \geq C\,\mathrm{Vol}_g(S^n)^{\frac{n-4}{n}} \tag{3}$$

for all $g \in [g_{\mathrm{rd}}]$ with $R^g \geq 0$.

(3) My Ph.D. student Zetian Yan [12, 13] gave a direct, rearrangement-free proof of the sharp Sobolev inequality

$$\int_{\mathbb{H}^n} \overline{u}\,P_{2k}u\,\mathrm{dv}_{\theta_0} \geq C\left(\int_{\mathbb{H}^n} |u|^{\frac{2(n+1)}{n+1-k}}\,\mathrm{dv}_{\theta_0}\right)^{\frac{n+1-k}{n+1}} \tag{4}$$

on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$, where P_{2k} is the CR GJMS operator of order $2k$. His argument significantly simplifies the original proof of Frank and Lieb [7].

(4) Yueh-Ju Lin, Wei Yuan, and I proved [4] the sharp fully nonlinear sharp Sobolev inequality

$$\int_{\mathbb{R}^n} \left(u|\nabla^2 u|^2 + \frac{2n-5}{n+2}u(\Delta u)^2\right)dx \geq C\left(\int_{\mathbb{R}^n} u^{\frac{3n}{n-4}}\right)^{\frac{n-4}{n}} \tag{5}$$

for all sufficiently integrable nonnegative $u \in C^\infty(\mathbb{R}^n)$.

These examples, especially the last two points, suggest that there is wide scope to apply conformally covariant polydifferential operators to proving sharp fully nonlinear Sobolev inequalities and sharp fully nonlinear Sobolev trace inequalities on Euclidean space and the Heisenberg group.

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Last developments for stochastic analysis for Non-Markovian semi-groups

Rémi Léandre

This talk belongs to the beautiful project of Bismut of probabilistic index theory: by path integrals you can see formulas which are simpler to check by analysis whenever you have them.

In the first part we establish a Varadhan estimate for big order generators by adapting to this context our proof of Varadhan estimates for subelliptic operators. This involves a mixture of large deviation estimates and the Malliavin Calculus following Bismut's Book [1]. Let X_i be some vectors fields without divergence on a compact Riemannian manifold endowed with the Riemannian measure which span at all points the tangent space of M. Consider the elliptic operator

$$L = (-1)^k \sum_{i=1}^{m} X_i^{2k} \tag{1}$$

and P_t^L the generator spanned by it. We introduce the Hamiltonian

$$H(x,\xi) = \sum < X_i, \xi >^{2k} \tag{2}$$

$x \in M$, $\xi \in T_x^*(M)$ and the Lagrangian

$$L(x,p) = \sup_{\xi}(< p,\xi > - H(x,\xi)) \tag{3}$$

where p belongs to $T_x(M)$. If we consider a curve $\gamma(t)$, we introduce its action

$$S(\gamma) = \int_0^1 L(\gamma(t), d/dt\gamma(t))dt \tag{4}$$

and we introduce the variational problem

$$l(x,y) = \inf_{\gamma(0)=x;\gamma(1)=y} S(\gamma) \tag{5}$$

By Malliavin Calculus of Bismut type for big order generators, we again find the classical theorem of analysis that P_t^L has a heat-kernel $p_t^L(x,y)$ which can change sign. We show the theorem:

$$\overline{\lim}_{t\to 0^+} t^{1/(2k-1)} Log|p_t^L(x,y)| \le -l(x,y). \tag{6}$$

In a second part, we extend the Wong-Zakai approximation for big order generators. We introduce the generator on $\mathbb{R}^m$

$$L^k = (-1)^k \sum_{i=1}^m \partial^{2k}/\partial w_i^{2k} \tag{7}$$

and we consider the semi-group P_t^k associated to it. It has a heat-kernel $p_t^k(0,w)$. We introduce the ordinary differential equation starting from x on the manifold

$$dx_t(w)(x) = \sum X_i(x_t(w)(x))w_i dt \tag{8}$$

and the Wong-Zakai kernel

$$Q_t^k f(x) = \int f(x_t(w)(x)) p_t^k(0,w) dw \tag{9}$$

We show: if the vector fields commute

$$(Q_{t/n}^k)^n f \to P_t^k f \tag{10}$$

in $L^2(M)$.

In this case, the Wong-Zakai approximation is exact, and is a generalisation of the classic Doss-Sussmann representation of a stochastic differential equation in Stratonovitch sense.

In the second case, we work on a compact Lie group and we consider the set of eigenspaces E_λ of L. We consider the spectral decomposition of $f = \sum a_\lambda f_\lambda$ where f_λ belongs to E_λ. We suppose that the vector fields belong to the Lie algebra of G. And we show that (10) holds if $\sum |a_\lambda|^2 C^\lambda < \infty$ for all C.

In a third part of the talk, we extend the classical flow theorem of Malliavin to big order generators. We suppose that we work on $\mathbb{R}^d$ and that the vector fields have bounded derivatives at each order. Let $\mathbb{G}_d$ the space of invertible matrices on this linear space endowed with the Haar measure dg. On $\mathbb{R}^d \times \mathbb{G}_d$, we introduce the vector field

$$X_i^{tot} = (X_i, \partial X_i/\partial x\, g) \tag{11}$$

$g \in \mathbb{G}_d$ and the generator

$$L^{tot} = (-1)^k \sum (X_i^{tot})^{2k}. \tag{12}$$

L^{tot} is an essentially self-adjoint generator on $L^2(dx \otimes dg)$ which generates a semi-group P_t^{tot} on $L^2(dx \otimes dg)$.

If h is a one form on $\mathbb{R}^d$, we consider the function on $\mathbb{R}^d \times \mathbb{G}_d$ defined via

$$h^{tot}(x,g) = h(x).g \tag{13}$$

We show the following result:

$$(dP_t f)^{tot} = P_t^{tot}(df^{tot}) \tag{14}$$

The algebra in the proof of this theorem is very simple. The difficulty is that in df^{tot}, g is not bounded. We solve this problem by using a Volterra expansion.

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On the role of embeddability in conformal sub-Riemannian geometry

Andrea Malchiodi

(joint work with Claudio Afeltra, Jih-Hsin Cheng and Paul Yang)

We consider closed *CR manifolds* M, which are smooth manifolds endowed with a $2n$-dimensional subbundle $\mathscr{H} = T_{1,0}M$ of the complexified tangent bundle of M, $T^{\mathbb{C}}M$, such that $\mathscr{H} \cap \overline{\mathscr{H}} = \{0\}$ and $[\mathscr{H}, \mathscr{H}] \subseteq \mathscr{H}$. Let $H(M)$ denote the space $\mathfrak{Re}(\mathscr{H} \oplus \overline{\mathscr{H}})$. Then there exists a natural complex structure on $H(M)$ given by

$$J(Z + \overline{Z}) = i(Z - \overline{Z}).$$

The CR structure is uniquely determined by $H(M)$ and J. There exists a non-zero real differential form θ (the *contact form*) whose kernel at every point coincides with $H(M)$; it is unique up to scalar multiplication by a non-zero function. A triple (M, J, θ) as above is called a *pseudohermitian structure*. Assuming *pseudoconvexity*, i.e. that $\theta \wedge (d\theta)^n \neq 0$ everywhere, such manifolds carry the Tanaka-Webster connection, which gives rise to a counterpart $W = W_{J,\theta}$ of the scalar curvature, called *Webster curvature*.

In this setting we consider the CR Yamabe problem, initiated in [7], and consisting in finding a contact form θ with constant Webster curvature. As for the Riemannian case, it is possible to find solutions to this problem as critical points w.r.t. θ of the normalized (scaling-invariant) *Einstein-Hilbert functional*

$$\tilde{\mathscr{W}}(J,\theta) = \left(\int_M \theta \wedge (d\theta)^n \right)^{-\frac{Q-2}{Q}} \mathscr{W}(J,\theta); \qquad \mathscr{W}(J,\theta) = \int_M W_{J,\theta}\, \theta \wedge (d\theta)^n.$$

Here $Q = 2n + 2$ is the *homogeneous dimension* of the manifold.

It is relevant to understand the infimum $\mathcal{Y}$ with respect to θ of the above quantity. In [8] the extremal contact forms for the standard $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ were characterized, while in [7] it was proved that if $\mathcal{Y}$ is less than the corresponding quantity $\mathcal{Y}_n$ for the standard sphere, then the infimum is attained. In this respect, we have the following results, the first of which is also a consequence of [10].

Theorem 1. [5] *Suppose (M, J, θ) is a three-dimensional embeddable CR structure with $\mathcal{Y} > 0$. Then $\mathcal{Y}$ is attained, and the CR Yamabe problem is solvable.*

The requirement of embeddability is peculiar of CR geometry and it is not seen in the Riemannian case. It is always true in higher dimensions (see [4]), but might fail in dimension three. We show with the next result that it cannot be removed in general: the standard CR structure J_0 of S^3, embedded in $\mathbb{C}^2 = \{(z_1, z_2)\}$, satisfies

$$J_0 Z_1 = i Z_1; \qquad Z_1 = Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

Rossi spheres have the same contact plane, but complex rotation $J_{(s)}$ for which

$$J_{(s)}(Z_1^{S^3} + s\bar{Z}_1^{S^3}) = i\left(Z_1^{S^3} + s\bar{Z}_1^{S^3}\right).$$

These are homogeneous manifolds, and they are a well-known example of non-embeddable structures.

Theorem 2. [6] *Consider the Rossi spheres $(S^3, J_{(s)})$. Then for $s \simeq 0$, $s \neq 0$, $\mathcal{Y}$ is not attained.*

We do not have a full understanding on the attainment of $\mathcal{Y}$ for general perturbations of the CR structure of the standard S^3. However, thanks to a result in [2], we can prove the following theorem.

Theorem 3. [1] *Consider on S^3 the standard pseudo-Hermitian structure (J_0, θ_0). Then $\tilde{\mathscr{W}}$ is a local minimum with respect to the variations in θ and is a local maximum with respect to the variations in J that are tangent to the family of embeddable structures. It is instead a local mimimum with with respect to the variations in J that are L^2-orthogonal to the latter ones.*

Concerning instead the higher-dimensional CR structures of dimension $2n + 1$ with $n > 1$, which are always embeddable, thanks to the results in [3] we can show still in [1] that the situation for perturbations the standard spheres is more similar to the Riemannian case (described for example in [9]), i.e. one has minimality in θ and maximality in J.

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Microlocal and Semi-classical heat kernel asymptotics in CR geometry

Chin-Yu Hsiao

Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n+1$, $n \geq 1$. Let $\Box_b^{(q)} : \mathrm{Dom}\,\Box_b^{(q)} \subset L^2_{(0,q)}(X) \to L^2_{(0,q)}(X)$ be the Gaffney extension of Kohn Laplacian for $(0,q)$ forms. The study of $\Box_b^{(q)}$ plays an important role in CR geometry, P.D.E. and microlocal analysis. The Kohn Laplacian is not elliptic. It is not difficult to see that principal symbol of $\Box_b^{(q)}$ vanishes at $\Sigma = \{(x, \lambda\omega_0(x)) \in T^*X;\, \lambda \neq 0\}$, where ω_0 is the fixed unique global non-vanishing one form so that $\langle\, \omega_0(x)\,,\, u\,\rangle = 0$, for every $u \in T^{1,0}_xX$ and every $x \in X$. The principal symbol of $\Box_b^{(q)}$ vanishes exactly to second order at Σ. Thus, $\Box_b^{(q)}$ is a P.D.E. with double characteristic manifold. The study of double characteristic P.D.E. is an important subject in microlocal analysis, see [1, 9]. The characteristic manifold Σ is a symplectic submanifold in T^*X with respect to the canonical two form on T^*X if and only if the Levi from is non-degenerate. From now on, we assume that the Levi form is non-degenerate of constant signature (n_-, n_+) on X.

In the first part of my talk, we introduce the heat operator method of Menikoff-Sjöstrand [8] to study $\Box_b^{(q)}$. Let $D \subset X$ be an open local coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$. We say that $a \in \mathcal{C}^\infty(\overline{\mathbb{R}}_+ \times D \times \dot{\mathbb{R}}^{2n+1})$ is quasi-homogeneous of degree j if $a(t, x, \lambda\eta) = \lambda^j a(\lambda t, x, \eta)$ for all $\lambda > 0$. By using Menikoff-Sjöstrand's heat operator construction, we can find

$$A(t,x,y) = A(t) = \frac{1}{(2\pi)^{2n+1}} \int_{\mathbb{R}^{2n+1}} e^{i(\psi(t,x,\eta) - <y,\eta>)} a(t,x,\eta) d\eta \tag{1}$$

so that

$$A'(t) + \Box_b^{(q)} A(t) = \frac{1}{(2\pi)^{2n+1}} \int_{\mathbb{R}^{2n+1}} e^{i(\psi(t,x,\eta) - <y,\eta>)} r(t,x,\eta) d\eta, \tag{2}$$

where $\psi(t,x,\eta)$ is quasi-homogeneous of degree one such that $\psi(0,x,\eta)=\langle x,\eta\rangle$, $\operatorname{Im}\psi(t,x,\eta)\asymp\frac{t|\eta|^2}{1+|t\eta|}\operatorname{dist}((x,\frac{\eta}{|\eta|}),\Sigma)^2$ on $t\in\mathbb{R}_+$, $a(t,x,\eta)\sim\sum_{j=0}^{\infty}a_j(t,x,\eta)$, $a_j(t,x,\eta)$ is a matrix-valued quasi-homogeneous function of degree $-j$, and

$$|r(t,x,\eta)|\leq C_N e^{-\varepsilon t|\eta|}(1+|\eta|)^{2-N}(\operatorname{Im}\psi)^N,\quad \forall N\geq 1,\quad \text{if } q\notin\{n_-,n_+\},$$
$$|r(t,x,\eta)|\leq C_N(1+|\eta|)^{2-N}(\operatorname{Im}\psi)^N,\quad \forall N\geq 1,\quad \text{if } q\in\{n_-,n_+\}, \tag{3}$$

and similar for the derivatives, where $C_N>0$ and $\varepsilon>0$ are constants. Moreover, $\psi(t,x,\eta)\to\psi(\infty,x,\eta)\in\mathcal{C}^\infty(D\times\dot{\mathbb{R}}^{2n+1})$ exponentially fast as $t\to+\infty$, $\operatorname{Im}\psi(\infty,x,\eta)\asymp|\eta|\operatorname{dist}((x,\frac{\eta}{|\eta|}),\Sigma)^2$ and if $q\notin\{n_-,n_+\}$, then $a_j(t,x,\eta)$ converges exponentially fast to zero as $t\to+\infty$, for all $j=0,1,2,\ldots$, if $q\in\{n_-,n_+\}$, there exist $a_j(\infty,x,\eta)\in\mathcal{C}^\infty(D\times\dot{\mathbb{R}}^{2n+1},T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$ such that $a_j(t,x,\eta)$ converges exponentially fast to $a_j(\infty,x,\eta)$ as $t\to+\infty$, for all $j=0,1,2,\ldots$, $a_0(\infty,x,\eta)\neq 0$. We set

$$\begin{aligned}G&=\frac{1}{(2\pi)^{2n+1}}\int\Big(\int_0^\infty\Big(e^{i(\psi(t,x,\eta)-<y,\eta>)}a(t,x,\eta)\\&\qquad\qquad-e^{i(\psi(\infty,x,\eta)-<y,\eta>)}a(\infty,x,\eta)\Big)dt\Big)d\eta,\\ \hat{S}&=\frac{1}{(2\pi)^{2n+1}}\int\big(e^{i(\psi(\infty,x,\eta)-<y,\eta>)}a(\infty,x,\eta)\big)d\eta.\end{aligned}\tag{4}$$

It was shown in [3, Chapter 5, Part I] that G is a pseudodifferential operator of order -1 type $(\frac{1}{2},\frac{1}{2})$ and $\hat{S}$ is a pseudodifferential operator of order 0 type $(\frac{1}{2},\frac{1}{2})$. From (2), (3) and by integrating over t, we get

$$\begin{aligned}&\hat{S}+\Box_b^{(q)}\circ G\equiv I,\\&\Box_b^{(q)}\circ\hat{S}\equiv 0,\\&\hat{S}\equiv 0\ \text{ if } q\notin\{n_-,n_+\},\\&\hat{S}\text{ is a complex F.I.O. if } q\in\{n_-,n_+\}.\end{aligned}\tag{5}$$

From (5), we get parametrix for $\Box_b^{(q)}$ if $q\notin\{n_-,n_+\}$. If $q\in\{n_-,n_+\}$ and assume that $\Box_b^{(q)}$ has closed range, we can show that $\hat{S}$ is the Szegő projection up to some smoothing operator (see [3, Chapter 7, Part I], for the details) and when $q=0$, by using complex stationary phase formula of Melin-Sjöstrand [7], we get the classical result of Boutet de Monvel and Sjöstrand about Szegő kernels asymptotics on strongly pseudoconvex CR manifolds [2].

In the second part of my talk, we introduce semi-classical analysis for hypoelliptic operators. Assume that $q\notin\{n_-,n_+\}$. Then, $\Box_b^{(q)}$ is hypoelliptic and $\operatorname{Ker}\Box_b^{(q)}$ could be trivial. From now on, assume that X admits a CR line bundle L. Let $\Box_{b,k}^{(q)}$ be the Kohn Laplacian with values in L^k. We have the following heat kernel asymptotics for $\Box_{b,k}^{(q)}$ (see [6]):

Theorem 1. *For every $x \in X$, we have*

$$\lim_{k\to\infty} k^{-(n+1)} e^{-\frac{t}{k}\Box^{(q)}_{b,k}}(x,x) = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}} \frac{\det(\mathcal{R}^L_x - 2\eta\mathcal{L}_x)}{\det\left(1-e^{-t(\mathcal{R}^L_x-2\eta\mathcal{L}_x)}\right)} e^{-t\omega^\eta_x} d\eta, \tag{6}$$

where $\mathcal{R}^L$ is the curvature of the line bundle L, $\mathcal{L}_x$ is the Levi form of X at $x \in X$, $\omega^\eta_x = \sum_{j,l=1}^n (\mathcal{R}^L_x - 2\eta\mathcal{L}_x)(U_l, \overline{U}_j)\overline{\omega}^j \wedge (\overline{\omega}^l\wedge)^* : T^{*0,q}_x X \to T^{*0,q}_x X$, $\{U_j\}_{j=1}^n$ is an orthonormal frame of $T^{1,0}_x X$ with dual frame $\{\omega^j\}_{j=1}^n \subset T^{*1,0}_x X$.

From Theorem 1, we obtain CR Demailly Morse inequalities of Hsiao and Marinescu [5].

We now consider $q = 0$. For $\mu \geq 0$, $H^0_{b,\leq\mu}(X, L^k)$ be the eigenspace of $\Box^{(0)}_{b,k}$ corresponding to the eigenvalues of $\Box^{(0)}_{b,k}$ less or equal to μ. Let $S_{\leq\mu,k} : L^2(X, L^k) \to H^0_{b,\leq\mu}(X, L^k)$ be the orthogonal projection. Fix $D \subset X$ be an open set and fix a local weight ϕ of the Hermitian metric of L. The semi-classical characteristic manifold Σ of $\Box^{(0)}_{b,k}$ on D is:

$$\Sigma = \{(x, \lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)) \in T^*D;\ \lambda \in \mathbb{R}\}.$$

Now, assume L is positive and set $\Sigma' = \{(x, \lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)) \in T^*D;\ \mathcal{R}^L_x - 2\lambda\mathcal{L}_x > 0\}$. We have the following semi-classical aymptotic results for $S_{\leq\mu,k}$ (see [4]).

Theorem 2. *Assume that L is positive. Let $F_k : L^2(X, L^k) \to L^2(X, L^k)$ be a global classical semi-classical pseudodifferential operator on X of order 0 such that F_k is semi-classically supported in Σ'. For every $N_0 \geq 1$, $\ell \in \mathbb{N}\bigcup\{0\}$, there is a constant $C_{\ell,N_0} > 0$ independent of k, such that*

$$\begin{aligned} &|F^*_k S_{\leq k^{-N_0},k} F_k(x) - a(x,k)|_{\mathcal{C}^\ell(X)} \leqslant C_{\ell,N_0}\, k^{3n+2\ell-N_0}, \\ &a(x,k) \sim \textstyle\sum_{j=0}^\infty k^{n+1-j} a_j(x) \ \text{in } \mathcal{C}^\infty(X), \\ &a_j(x) \in \mathcal{C}^\infty(X), \quad j = 0,1,2,\ldots, \end{aligned} \tag{7}$$

$a(x,k)$ is independent of N.

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The hypoelliptic Laplacian in complex geometry

Shu Shen

1. Introduction

Given a compact complex manifold, Hodge theory asserts that the kernel of the Kodaira Laplacian is canonically isomorphic to the Dolbeault cohomology of the underlying manifold. However, as the time parameter goes to zero, the heat kernel on the diagonal associated with the Kodaira Laplacian generally does not satisfy the so-called "fantastic cancellations" of McKean-Singer, which is a fundamental property in local index theory.

The hypoelliptic Laplacian, or more precisely its exotic version constructed by Bismut [2, 3], possesses the aforementioned two remarkable properties. This operator plays an essential role in the recent proof of the Riemann-Roch-Grothendieck formula for coherent sheaves given by Bismut-Shen-Wei [4].

2. The Kodaira Laplacian

2.1. **Chern connection.** Let X be a compact complex manifold of dimension n. Denote by $T_{\mathbf{R}}X$ and TX the real and holomorphic tangent vector bundles of X. Then, $T_{\mathbf{R}}X \otimes_{\mathbf{R}} \mathbf{C} = TX \oplus \overline{TX}$. Let E be a holomorphic vector bundle on X. For $0 \leq p, q \leq n$, let $\Omega^{p,q}(X,E)$ be the space of smooth E-valued (p,q)-forms on X. Denote by $\nabla^{E\prime\prime} : \Omega^{0,\bullet}(X,E) \to \Omega^{0,\bullet+1}(X,E)$ the holomorphic structure of E.

Let $\alpha \in \Lambda^\bullet(T^*_{\mathbf{C}}X) \to \widetilde{\alpha} \in \Lambda^\bullet(T^*_{\mathbf{C}}X)$ be the involution such that if $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_p$ with $\deg\alpha_i = 1$, then $\widetilde{\alpha} = (-\alpha_p) \wedge \ldots (-\alpha_1)$. If $s_1, s_2 \in \Omega^{\bullet,\bullet}(X,\mathbf{C})$, define

$$\theta(s_1,s_2) = \left(\frac{i}{2\pi}\right)^n \int_X \widetilde{s}_1 \wedge \overline{s}_2. \tag{1}$$

It is easy to see that θ is a non-degenerate Hermitian form on $\Omega^{\bullet,\bullet}(X,\mathbf{C})$. If $d = \overline{\partial} + \partial$ is the de Rham operator, then ∂ is the adjoint of $\overline{\partial}$ with respect to θ.

If g^E is a Hermitian metric on E, we can associate a corresponding non-degenerate Hermitian form θ_{g^E} on $\Omega^{\bullet,\bullet}(X,E)$. If $\nabla^E = \nabla^{E\prime\prime} + \nabla^{E\prime}$ is the Chern connection on (E, g^E), then $\nabla^{E\prime}$ is the θ_{g^E}-adjoint of $\nabla^{E\prime\prime}$. We have the classical relations

$$(2) \qquad \left(\nabla^{E''}\right)^2 = 0, \qquad \left(\nabla^{E'}\right)^2 = 0, \qquad (\nabla^E)^2 = \left[\nabla^{E''}, \nabla^{E'}\right].$$

The curvature $(\nabla^E)^2 \in \Omega^{1,1}(X, \mathrm{End}(E))$ is a nilpotent operator.

2.2. **Kodaira Laplacian.** Put $\overline{\partial}^E = \nabla^{E''}$. If g^{TX}, g^E are Hermitian metrics on TX, E, let $\langle , \rangle_{\Omega^{0,\bullet}(X,E), g^{TX}, g^E}$ be the associated L^2-metric on $\Omega^{0,\bullet}(X, E)$.

Let $\overline{\partial}^{E*} : \Omega^{0,\bullet}(X, E) \to \Omega^{0,\bullet-1}(X, E)$ be the formal adjoint of $\overline{\partial}^E$ with respect to $\langle , \rangle_{\Omega^{0,\bullet}(X,E), g^{TX}, g^E}$. Set $D^E = \overline{\partial}^E + \overline{\partial}^{E*}$. We have the classical relations

$$(3) \qquad \left(\overline{\partial}^E\right)^2 = 0, \qquad \left(\overline{\partial}^{E*}\right)^2 = 0, \qquad \left(D^E\right)^2 = \left[\overline{\partial}^E, \overline{\partial}^{E*}\right].$$

The operator $(D^E)^2$ is the Kodaira Laplacian. It is a second order self-adjoint non negative elliptic operator. By Hodge theory, $\ker(D^E)^2 \simeq H^{0,\bullet}(X, E)$.

2.3. **Elliptic local index theory.** By the McKean-Singer formula, for $t > 0$, the holomorphic Euler characteristic of E is given by $\mathrm{Tr_s}\left[\exp\left(-t(D^E)^2\right)\right]$.

Let ω^X be the fundamental $(1,1)$-form associated with g^{TX}. If $\overline{\partial}\partial\omega^X = 0$ (which includes the case when (X, g^{TX}) is Kähler), by the local index theorem in [1], as $t \to 0$, we have

$$(4) \qquad \mathrm{Tr_s}\left[\exp\left(-t(D^E)^2\right)(x,x)\right] \to [\alpha_0]^{\max},$$

where $\alpha_0 \in \oplus_p \Omega^{p,p}(X, \mathbf{R})$ is a closed form whose Bott-Chern cohomology class is $\mathrm{Td_{BC}}(TX)\mathrm{ch_{BC}}(E)$. This corresponds to the fantastic cancellations of McKean-Singer. From (4), we get the Riemann-Roch-Hirzebruch Theorem.

3. Hypoelliptic Laplacians

3.1. **The Dolbeault-Koszul complex.** Let $\widehat{TX}$ be another copy of TX. Let $\pi : \mathcal{X} \to X$ be the total space of $\widehat{TX}$. Let y (resp. Y) be the holomorphic (resp. smooth) tautological section on $\mathcal{X}$ of π^*TX (resp. $\pi^*T_{\mathbf{R}}X$). Then, $Y = y + \overline{y}$.

The degree of $\Omega^{0,p}(\mathcal{X}, \pi^*\Lambda^q(T^*X) \otimes \pi^*E)$ is defined to be $p - q$. The Dolbeault-Koszul operator $A''_Y = \overline{\partial}^{\pi^*\Lambda^\bullet(T^*X)\otimes\pi^*E} + i_y$ acts on $\Omega^{0,\bullet}(\mathcal{X}, \pi^*\Lambda^\bullet(T^*X) \otimes \pi^*E)$ and has degree 1. Moreover, $(A''_Y)^2 = 0$ and the cohomology of $(\Omega^{0,\bullet}(\mathcal{X}, \pi^*\Lambda^\bullet(T^*X) \otimes \pi^*E), A''_Y)$ coincides with $H^{0,\bullet}(X, E)$.

3.2. **Constructions of the classical hypoelliptic Laplacian.** The classical hypoelliptic Laplacian is constructed using a non-degenerate Hermitian form ϵ_X on $\Omega^{0,\bullet}(\mathcal{X}, \pi^*\Lambda^\bullet(T^*X) \otimes \pi^*E)$. It is obtained by combining a twisted Hermitian form similar to θ on X and an L^2-metric on the fibre $\widehat{TX}$.

Recall that ω^X is the fundamental $(1,1)$-form associated to g^{TX} and that g^E is the Hermitian metric on E. We consider the non-degenerate Hermitian form $\theta_{g^E}(\cdot, e^{-i\omega^X}\cdot)$ on $\Omega^{\bullet,\bullet}(X, E)$. Let $g^{\widehat{TX}}$ be a Hermitian metric on $\widehat{TX}$. Let $\sigma : \widehat{y} \to -\widehat{y}$ be the involution of $\widehat{TX}$, and let σ^* be the induced morphism on $\Omega^{0,\bullet}(\widehat{TX}, \mathbf{C})$. We have a non-degenerate Hermitian form $\langle \cdot, \sigma^* \cdot \rangle_{\Omega^{0,\bullet}(\widehat{TX}, \mathbf{C}), g^{\widehat{TX}}}$ on $\Omega^{0,\bullet}(\widehat{TX}, \mathbf{C})$.

The Chern connection on $(\widehat{TX}, g^{\widehat{TX}})$ induces a smooth splitting $T\mathcal{X} = \pi^*(TX \oplus \widehat{TX})$. The above two non-degenerate Hermitian forms induce a non-degenerate Hermitian form ϵ_X on $\Omega^{0,\bullet}(\mathcal{X}, \pi^*\Lambda^\bullet(T^*X) \otimes \pi^*E)$. Define A'_Y to be the ϵ_X-adjoint of A''_Y. Set $A_Y = A''_Y + A'_Y$. Then,

$$(5) \qquad \big(A''_Y\big)^2 = 0, \qquad \big(A'_Y\big)^2 = 0, \qquad A_Y^2 = \big[A''_Y, A'_Y\big].$$

The Hodge-like Laplacian A_Y^2 has the form

$$(6) \qquad A_Y^2 = \frac{1}{2}\big(- \Delta^V_{g^{\widehat{TX}}} + |Y|^2_{g^{TX}}\big) + \nabla^H_Y - \overline{\partial}\partial i\omega^X + \ldots,$$

where $\Delta^V_{g^{\widehat{TX}}}$ denotes the vertical Laplacian, ∇^H is the horizontal derivation, and $\ldots$ are matrix terms. By Hörmander [5], A_Y^2 is hypoelliptic.

For $b > 0$, if we consider instead the metrics $(\omega^X, b^4 g^{\widehat{TX}}, g^E)$, up to conjugation, the hypoelliptic Laplacian takes the form

$$(7) \qquad \frac{1}{2b^2}\big(- \Delta^V_{g^{\widehat{TX}}} + |Y|^2_{g^{TX}}\big) + \frac{1}{b}\nabla^H_Y - \overline{\partial}\partial i\omega^X + \ldots$$

If $g^{\widehat{TX}} = g^{TX}$, when $b \to 0$, the above operator converges in the proper sense to the Kodaira Laplacian $(D^E)^2$. We refer to Bismut's talk of the conference for the precise statement. By a deformation argument, the original holomorphic Euler characteristic can be calculated as the supertrace of the heat operator associated with the hypoelliptic Laplacian.

3.3. **Hypoelliptic local index theory.** Consider the hypoelliptic Laplacian associated to $(\omega^X/t, g^{\widehat{TX}}/t^3, g^E)$ and $b = 1$. Up to conjugation, this operator has the form

$$(8) \qquad \mathfrak{M}_t = \frac{1}{2}\big(- \Delta^V_{g^{\widehat{TX}}} + t^2|Y|^2_{g^{TX}}\big) + t^{3/2}\nabla^H_Y - \frac{1}{t}\overline{\partial}\partial i\omega^X + \ldots$$

If $\overline{\partial}\partial\omega^X = 0$, the hypoelliptic local index theorem [2] asserts that, when $t \to 0$, up to rescaling by 2π, we have

$$(9) \qquad \int_{\widehat{TX}} \mathrm{Tr_s}\big[\exp(-\mathfrak{M}_t)\big((x,\widehat{Y}),(x,\widehat{Y})\big)\big]d\widehat{Y} \to \big[\mathrm{Td}\big(TX, g^{\widehat{TX}}\big)\mathrm{ch}\big(E, g^E\big)\big]^{\max},$$

where $\mathrm{Td}(TX, g^{\widehat{TX}}), \mathrm{ch}(E, g^E)$ are the Chern-Weil characteristic forms associated to the Chern connections. This hypoelliptic theory is still not adequate for arbitrary metric g^{TX}.

3.4. **An exotic Hypoelliptic Laplacian.** When replacing ω^X/t by $\frac{1}{2}|Y|^2_{g^{\widehat{TX}}}\omega^X$, we can construct an exotic hypoelliptic Laplacian. After a t-rescaling, we get a new operator

$$(10) \quad \mathcal{M}_t = \frac{1}{2}\big(- \Delta^V_{g^{\widehat{TX}}} + \frac{1}{2}t^3|Y|^2_{g^{\widehat{TX}}}|Y|^2_{g^{TX}}\big) + t^{3/2}\nabla^H_Y - \frac{1}{2}|Y|^2_{g^{\widehat{TX}}}\overline{\partial}\partial i\omega^X + \ldots$$

Note that the last term is independent of t. Therefore, when $\mathfrak{M}_t$ is replaced by $\mathcal{M}_t$, it is shown in [3] that the limit (9) converges without any assumption on the metric g^{TX}. This way, we get the local index theorem for arbitrary metric

g^{TX}. This method is used to establish the Riemann-Roch-Grothendieck theorem for coherent sheaves [4].

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Sub-Riemannian geometry and spectral analysis

Wolfram Bauer

(joint work with K. Furutani, C. Iwasaki, A. Laaroussi, I. Markina and G. Vega-Molino)

A regular sub-Riemannian (SR) manifold M carries a geometric hypoelliptic operator which often is referred to as sublaplacian. Due to the specific degeneracy of its symbol, interesting geometric and analytic effects can be observed in the study of this operator, which have no counterpart in the area of Riemannian geometry. In particular, during the last decades an inverse spectral problem which aims to extract geometric information from the operator spectrum has been considered by various authors. Typical approaches are based on the analysis of the induced sub-Riemannian heat or wave equation.

In this talk we survey some results on the spectral theory of the sublaplacian in the case of certain compact nilmanifolds and so-called H-type foliations. Moreover, we address some seemingly open problems in the area.

We start the presentation with a short introduction into sub-Riemannian geometry, its numerous realizations and applications (s. the monographs [1, 12]). The local model to M at a point $x \in M$ is a stratified nilpotent Lie group $G = G_x$ which itself carries a SR structure. The intrinsic hypoelliptic sublaplacian on G is a sum of squares of globally defined left-invariant vector fields.

In the first part of the talk we consider a specific class of step-two nilpotent Lie groups $G = G_{r,s}$ with $(r, s) \in \mathbb{N}_0^2$. These groups have been introduce by P. Ciatti in [8] and are defined through a $C\ell_{r,s}$-(Clifford)-module representation. It has been shown that $G_{r,s}$ admits a (standard) lattice $\Gamma_{r,s}$ and the compact left-coset space $M_{r,s} := \Gamma_{r,s}\backslash G_{r,s}$ is called a pseudo-H-type nilmanifold. As is well-known, the induced intrinsic sublaplacian on $M_{r,s}$ has discrete spectrum consisting of eigenvalues with finite multiplicities. Our aim is to detect isospectral (w.r.t. the sublaplacian) non-homeomorphic pseudo-H-type nilmanifolds $M_{r,s}$. A classification of these examples has been given in [6] and the arguments are based on an explicit heat trace formula for the sublaplacian in [4] combined with an algebraic classification of (non)-isomorphic pseudo-H-type Lie algebras in [10].

In the second part of the talk we consider the inverse spectral problem for the sublaplacian on compact nilmanifolds and H-type foliations which - very roughly speaking - are curved versions of H-type nilmanifolds. We aim to geometrically interpret the second coefficient in the heat trace asymptotic expansion for small times. In the flat case (in the sense of SR geometry) of a compact nilmanifold (not necessarily of step two) this coefficient vanishes by not obvious reasons (see [5, 9]) and not much can be said. However, in the more general case of an H-type foliation we detect a horizontal and a vertical invariant of the geometric structure defined via the Bott connection. A suitable combination of these quantities gives a spectral invariant which is shown to coincides with the second heat invariant of the intrinsinc sublaplacian (see [7] and [11] in the case of a quaternionic contact manifold).

Finally we mention that pseudo-H-type nilmanifolds and their compact quotients can as well be considered a pseudo-SR manifolds. Instead of the sublaplacian this geometric structure induces an ultra-hyperbolic operator and we study the problem of local solvability and the existence of fundamental solutions. In all cases in which this ultra-hyperbolic operator is (locally) solvable we construct an infinite family of fundamental solutions, see [3].

This presentation is based on joint work with K. Furutani, C. Iwasaki and A. Laaroussi, I. Markina and G. Vega-Molino.

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A Liouville theorem for a class semilinear elliptic equations on the Heisenberg group

Xi-Nan Ma

(joint work with Qianzhong Ou)

In this talk, we study the following equation

$$-\triangle_{\mathbb{H}^n} u = 2n^2 u^q \quad \text{in} \quad H^n, \tag{1}$$

where $\mathbb{H}^n$ denotes the Heisenberg group, and u is a real, nonnegative and smooth function while $\triangle_{\mathbb{H}^n} u = \sum_{\alpha=1}^n (u_{\alpha\overline{\alpha}} + u_{\overline{\alpha}\alpha})$ is the Heisenberg Laplacian of u. We shall first give a brief introduction to the Heisenberg group $\mathbb{H}^n$ and some notations. We consider $\mathbb{H}^n$ as the set $\mathbb{C}^n \times \mathbb{R}$ with coordinates $(z,\, t)$ and group law $\circ$:

$$(z,t)\circ(w,s) = \big(z+w,\, t+s+2\mathbf{Im} z^\alpha \overline{w}^\alpha\big) \quad \text{for } \xi=(z,t),\, \zeta=(w,s)\in\mathbb{C}^n\times\mathbb{R},$$

where and in the sequel, we shall use the Einstein sum with the convention: the Greek indices $1 \le \alpha, \beta, \gamma, \delta \le n$. For $\xi = (z,t) = (z_1, z_2, ..., z_n, t) \in \mathbb{C}^n \times \mathbb{R}$ as an element of $\mathbb{H}^n$, the norm $|\xi|$ is defined by $|\xi|^4 = |(z,t)|^4 = |z|^4 + t^2$, with associated distance function $d(\xi,\zeta) = |\zeta^{-1}\xi|$. We will use the notation $B(\xi, r)$ for the metric ball centered at $\xi = (z,t)$ with the radius $r > 0$. The Heisenberg group is a dilation group and the associated homogeneous dimension $Q = 2n+2$ such that the volume $|B(\xi, r)| \approx r^Q$.

The CR structure of $\mathbb{H}^n$ is given by the bundle $\mathcal{H}$ spanned by the left-invariant vector fields $Z_\alpha = \partial/\partial z^\alpha + \sqrt{-1}\overline{z}^\alpha \partial/\partial t$ and $Z_{\bar{\alpha}} = \partial/\partial \bar{z}^\alpha - \sqrt{-1} z^\alpha \partial/\partial t$, $\alpha = 1, \cdots, n$. Then the standard (left-invariant) contact form on $\mathbb{H}^n$ is $\mathbf{\Theta} = dt + \sqrt{-1}(z^\alpha d\overline{z}^\alpha - \overline{z}^\alpha dz^\alpha)$. With respect to the standard holomorphic frame $\{Z_\alpha\}$ and dual admissible coframe $\{dz^\alpha\}$, the *Levi forms* $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$. Accordingly, for a smooth function f on $\mathbb{H}^n$, denote its derivatives by $f_\alpha = Z_\alpha f$, $f_{\alpha\overline{\beta}} = Z_{\overline{\beta}}(Z_\alpha f)$, $f_0 = \frac{\partial f}{\partial t}$, $f_{0\alpha} = Z_\alpha(\frac{\partial f}{\partial t})$, etc. We would also indicate the derivatives of functions or vector fields with indices preceded by a comma, to avoid confusion. Then as in [6] we have the following commutation formulae:

$$f_{\alpha\beta} - f_{\beta\alpha} = 0, \qquad f_{\alpha\overline{\beta}} - f_{\overline{\beta}\alpha} = 2\sqrt{-1}\delta_{\alpha\overline{\beta}} f_0, \qquad f_{0\alpha} - f_{\alpha 0} = 0,$$
$$f_{\alpha\beta 0} - f_{\alpha 0\beta} = 0, \qquad f_{\alpha\beta\overline{\gamma}} - f_{\alpha\overline{\gamma}\beta} = 2\sqrt{-1}\delta_{\beta\overline{\gamma}} f_{\alpha 0}, \cdots .$$

Let $Q = 2n+2$ be the homogeneous dimension of $\mathbb{H}^n$. Denote $q^* = \frac{Q+2}{Q-2}$. Our main purpose in this paper is to present an entire Liouville type theorem and a pointwise estimate near the isolated singularity for solutions to (1) for the subcritical case $1 < q < q^*$.

The equation (1) that is already studied intensively by many authors in decades is connected to the CR Yamabe problem on $\mathbb{H}^n$. Let $\mathbf{\Theta}$ be the standard contact form on $\mathbb{H}^n$, we consider another smooth contact form $\theta = u^{\frac{2}{n}}\mathbf{\Theta}$, where u is a smooth positive function in $\mathbb{H}^n$. Then the pseudo-Hermitian scalar curvature associated to the new contact form $(\mathbb{H}^n,\, \theta)$ is $R = 4n(n+1)u^{q-q^*}$ while u satisfies the equation (1). The CR Yamabe problem is to find such a contact form θ so that the pseudo-Hermitian scalar curvature R is a constant(i.e. $q = q^*$). The number

$q^* + 1 = \frac{2Q}{Q-2}$ is the CR Sobolev embedding exponent [2]. For the equation (1) with $q = q^* = \frac{Q+2}{Q-2}$, by the splendid work [6] of D. Jerison and J.M. Lee , there are nontrivial solutions as follows

$$u(z,t) = C\left|t + \sqrt{-1}z \cdot \overline{z} + z \cdot \mu + \lambda\right|^{-n} \tag{2}$$

for some $C > 0$, $\lambda \in \mathbb{C}$, Im(λ)> $|\mu|^2/4$, and $\mu \in \mathbb{C}^n$ (where $\sqrt{-1}$ denote the imaginary unit in the complex space $\mathbb{C}$), which are the only extremals of the CR Sobolev inequality on $\mathbb{H}^n$. The CR Yamabe problem had been studied by D. Jerison and J.M. Lee in a series of fundamental works (see [5]-[7]). For compact, strictly pseudoconvex CR manifold, the CR Yamabe problem had been solved in case of not locally CR equivalent to sphere $\mathbf{S}^{2n+1}$ by Jerison-Lee [7] for $n \geq 2$ and Gamara [3] for $n = 1$, and in case of locally CR equivalent to $\mathbf{S}^{2n+1}$ by Gamara-Yacoub [4] for all $n \geq 1$. One can see the more recent progress in Cheng-Malchiodi-Yang [1] for the CR Yamabe problem. Using the Jerison-Lee's identity [6], Wang [9] obtained related result for a closed Einstein pseudohermitian manifold.

In fact, for the equation (1) with $q = \frac{Q+2}{Q-2}$, Jerison-Lee [6] obtained the uniqueness of the solutions in case of finite volume, i.e., $u \in L^{\frac{2Q}{Q-2}}(\mathbb{H}^n)$.

In this talk we presented the following theorem from Ma-Ou [8].

Theorem 1. *Let $\Omega = \mathbb{H}^n$ be the whole space and $1 < q < q^*$, then the equation (1) has no positive solution, namely, any nonnegative entire solution of (1) must be the trivial one.*

For our proof, we used a generalization of the Jerison-Lee's divergence identity. There are still many interesting questions such as:

1. Could we delete the condition on finite volume in Jerison-Lee [6] theorem?
2. How about the fourth order equations which is related to CR Q-curvature?
3. How about the p-Laplace equation in H^n?

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Darboux-type results in the Heisenberg group: the case of gauge balls

Giulio Tralli
(joint work with C. Guidi and V. Martino)

We report on some recent advances in the characterization of hypersurfaces of spherical type via the prescription of mean curvatures related to hypoelliptic operators.

We mainly focus on the problem of characterizing gauge balls in the Heisenberg group by prescribing their (non-constant) horizontal mean curvature. In this direction we discuss two uniqueness results obtained in [1]: in the lowest dimensional case under an assumption on the location of the singular set, and in higher dimensions in the proper class of horizontally umbilical hypersurfaces. A special role is played by the degenerate ellipticity of the underlying operators in such a sub-Riemannian setting, and especially by the presence/absence of Hörmander-type properties.

We then address the uniqueness problem of Alexandrov-type in complex spaces related to the so-called Levi curvature (trace of the Levi form). We discuss partial results in the literature where the standard CR sphere is characterized as the only closed real hypersurface with constant Levi curvature in the class of hypersurfaces enjoying suitable a-priori symmetry assumptions (see, e.g., [2]).

Finally we highlight similarities and differences between these two rigidity problems.

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Bochner-Laplacian and Bergman kernel expansion

George Marinescu
(joint work with Xiaonan Ma and Nikhil Savale)

The Bergman kernel expansion for high powers of a line bundle has many applications in several research areas, namely in Tian-Yau-Donaldson program (existence and uniqueness of constant scalar curvature Kähler metrics), Berezin-Toeplitz quantization, arithmetic geometry (asymptotics of the analytic torsion) quantization of Chern-Simons theory, random Kähler geometry, quantum chaos and quantum Hall effect, see [9] for a comprehensive study.

Let (X, Θ) be a complex Hermitian manifold of dimension n equipped with a holomorphic, Hermitian vector bundle (E, h^E). The Bergman projector is the orthogonal projector $P : L^2(X,E) \to H^0_{(2)}(X,E)$ from the space of square integrable holomorphic sections of E onto the square integrable holomorphic ones. The Schwartz kernel $P(x,x')$ of the Bergman projector is called the Bergman kernel. Its on-diagonal restriction $P(x,x)$ is called Bergman kernel function.

In his thesis, Tian [14] found the leading asymptotics of the Bergman kernel function $P_p(x,x)$, associated to tensor powers $L^p := L^{\otimes p}$ of a positive line bundle in the semiclassical limit as $p \to \infty$. This was extended to a full asymptotic expansion in [4, 15] as an application of the parametrix of Boutet de Monvel-Sjöstrand [3] for the Szegő kernel of a strongly pseudoconvex CR manifold, in this case the circle bundle of the dual bundle L^*.

Theorem 1 ([4, 5, 9, 15]). *Let (X,Θ) be a compact Hermitian manifold of dimension n and $(L,h^L) \to X$ be a positive line bundle. Let $(F,h^F) \to X$ be another Hermitian holomorphic vector bundle. Then the Bergman kernel function has the pointwise asymptotic expansion*

$$P_p(x,x) = \sum_{j=0}^{N} c_j(x) p^{n-j} + O\left(p^{-N}\right), \quad \text{for all } N \in \mathbb{N}. \tag{1}$$

Here c_j are sections of $\mathrm{End}\,(F)$, with the leading term $c_0 = (c_1(L,h^L)^n/\Theta^n)\,\mathrm{Id}_F$, where $c_1(L,h^L)$ is the Chern curvature form of (L,h^L).

Later, a new method to prove the asymptotic expansion was developed by Dai-Liu-Ma [5] and Ma-Marinescu [9, 10]. It is based on the analytic localization technique of Bismut-Lebeau [2]. A key step in the latter method is the study of the asymptotics of the smallest positive eigenvalue of the associated Bochner and Kodaira Laplacians.

Let X be a compact Riemannian manifold endowed with a complex Hermitian line bundle (L,h^L) and a vector bundle (F,h^F). We equip them with unitary connections ∇^L, ∇^F to obtain the Bochner Laplacian

$$\Delta_p := \left(\nabla^{F\otimes L^p}\right)^* \nabla^{F\otimes L^p} : C^\infty(X, F\otimes L^p) \to C^\infty(X, F\otimes L^p), \quad p \in \mathbb{N}, \tag{2}$$

on tensor powers $F \otimes L^p$, where the adjoint is taken with respect to the natural L^2 metric. As Δ_p is elliptic, self-adjoint and positive, one has a complete orthonormal basis $\{\psi_j^p\}_{j=1}^{\infty}$ of $L^2(X, F\otimes L^p)$ consisting of its eigenvectors $\Delta_p \psi_j^p = \lambda_j(p)\psi_j^p$, with eigenvalues $0 \le \lambda_0 \le \lambda_1 \ldots$. Denote by R^L the curvature form of the connection ∇^L. The order of vanishing of R^L at a point $x \in X$ is defined by [1]

$$r_x - 2 := \mathrm{ord}_x\left(R^L\right) := \min\left\{\ell : J^\ell\left(\Lambda^2 T^*Y\right) \ni j_x^\ell R^L \neq 0\right\}, \quad r_x \ge 2, \tag{3}$$

where $j^\ell R^L$ denotes the ℓth jet of the curvature. We shall assume that the order of vanishing is finite at any point of the manifold i. e. $r := \max_{x\in X} r_x < \infty$. Since the function $x \mapsto r_x$ is upper semi-continuous, we have a decomposition $X = \bigcup_{j=2}^{r} X_j$ of the manifold via $X_j = \{x \in X : r_x = j\}$ with $X_{\le j} := \bigcup_{j'=2}^{j} X_{j'}$ being open.

[1]The reason for this normalization, besides a simplification of resulting formulas, is the significance of r_x as the degree of nonholonomy of a relevant sR distribution.

Theorem 2 ([13]). *Let $(L,h^L),(F,h^F)\to(X,g^{TX})$ be Hermitian line and vector bundles on a compact Riemannian manifold with unitary connections ∇^L, ∇^F. Assuming that the curvature R^L vanishes to finite, and maximally order r at any point, then there exists $C>0$ such that the first eigenvalue $\lambda_0(p)$ of the Bochner Laplacian satisfies*

$$\lambda_0(p)\sim Cp^{2/r},\quad \text{as } p\to\infty, \tag{4}$$

and the first eigenfunction concentrates on X_r, $|\psi_0^p(x)|=O(p^{-\infty})$, $x\in X_{\le r-1}$.

Theorem 2 is the most general leading asymptotic for the first Bochner eigenvalue, the only assumption being finite order of vanishing of the curvature R^L. The proof uses the relation of the Bochner Laplacian with the hypoelliptic sub-Riemannian (sR) Laplacian on the unit circle bundle of L^* and we exploit a pointwise heat kernel expansion for the sR Laplacian [1, 8] on the circle bundle.

By combining Theorem 2 with the method of [5, 9] we obtain an $O\left(p^{2/r}\right)$ spectral gap for the Kodaira Laplacian,

$$\Box_p:=\left(\bar\partial^{F\otimes L^p}\right)^*\bar\partial^{F\otimes L^p}:C^\infty(X,F\otimes L^p)\to C^\infty(X,F\otimes L^p),\quad p\in\mathbb{N}, \tag{5}$$

on a Riemann surface, thus generalizing the spectral gap property [9, Theorem 1.5.5] for positive line bundles (where $r=2$).

Corollary 3 ([13]). *Let X be a compact Riemann surface, $(L,h^L)\to X$ a semipositive line bundle whose curvature R^L vanishes to finite order at any point. Let $(F,h^F)\to X$ be a Hermitian holomorphic vector bundle. Then there exist $c_1,c_2>0$, such that* $\mathrm{Spec}(\Box_p)\subset\{0\}\cup\left[c_1p^{2/r}-c_2,\infty\right)$ *for each p.*

Donnelly has earlier shown in [6] that the spectral gap does not hold in higher dimensions as a counterexample to Siu's eigenvalue conjecture. Using Corollary 3 and the methods of [5, 9, 10] we obtain the following generalization of Theorem 1.

Theorem 4 ([13]). *Under the hypotheses of Corollary 3 the Bergman kernel function has the pointwise asymptotic expansion*

$$P_p(x,x)=\sum_{j=0}^{N}c_j(x)p^{(2-2j)/r_x}+O\left(p^{-2N/r_x}\right),\quad \text{for all } N\in\mathbb{N}. \tag{6}$$

Here c_j are sections of $\mathrm{End}(F)$, *and the leading term $c_0(x)>0$ is given in terms of the Bergman kernel of a model Kodaira Laplacian on the tangent space at x.*

The problem of the expansion for semipositive line bundles is well known and largely unsolved, see [9, Problem 4.8]. Theorem 4 is the first instance where the expansion has been proved at vanishing points of the curvature for surfaces, and this is yet unresolved in higher dimensions. Previously, Berndtsson proved an asymptotic estimate for the Bergman kernel of semipositive line bundles. Berman showed the expansion on the positive part, and away from the augmented base locus, assuming the line bundle to be ample. Finally, Hsiao-Marinescu proved the expansion on the positive part when one twists by the canonical bundle ($F=K_X$).

The analogous problem of the boundary expansion of the Bergman kernel of weakly pseudoconvex domains in $\mathbb{C}^2$ has been recently solved in [7].

In a forthcoming paper [11] we generalize the results of Marinescu-Savale [13] to families of Bochner Laplacians. In particular, this leads to the fiberwise expansion for families Bergman kernels of horizontally semipositive index bundles. The proof uses Ma-Zhang's description [12] of the curvature of the index bundle as a fiberwise Toeplitz operator.

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Metric geometry of Einstein 4-manifolds: recent progress and open questions

Ruobing Zhang

Einstein manifold is a fundamental object in differential geometry. It has been a central and active area in Riemannian geometry and geometric analysis to explore the delicate properties of Einstein metrics and the structure of their moduli. In low dimensions, i.e., dimension 2 and dimension 3, Einstein metrics have constant Gaußian and sectional curvatures, respectively, so that both existence and moduli space problems have been widely investigated and well understood so far. In higher dimensions, such problems become significantly harder. A fundamental reason is that Einstein manifolds even in dimension 4 have highly non-trivial moduli spaces. That is, most Einstein families exhibit very complicated degeneration behaviors in both analytic and geometric senses. One hopes to understand the properties of a natural compactification of Einstein moduli which is analogous to the Deligne-Mumford compactification theory in Riemann surfaces. In more concrete terms, one wishes to describe how Einstein metrics can degenerate.

The simplest example of Einstein manifold whose moduli space has substantial complexity is the K3 manifold. A complex 2-dimensional manifold $\mathcal{K}$ is called a K3 manifold if its first Chern class $c_1(\mathcal{K})$ and fundamental group $\pi_1(\mathcal{K})$ are both trivial. By [2], the condition $c_1(\mathcal{K}) = 0$ implies that there exist Ricci-flat metrics on $\mathcal{K}$. On the other hand, as a corollary of the Bieberbach Theorem, none of the Ricci-flat metrics on $\mathcal{K}$ is flat since the condition $\pi_1(\mathcal{K}) = \{e\}$ implies that $\mathcal{K}$ cannot be finitely covered by a flat torus. All these show that the moduli space $\mathfrak{M}(\mathcal{K})$ of Ricci-flat metrics on $\mathcal{K}$ is non-trivial. In complex geometry, the Global Torelli Theorem plays a fundamental role in describing the structure of $\mathfrak{M}(\mathcal{K})$, and it is well known that the moduli space $\mathfrak{M}(\mathcal{K})$ can be identified with a non-compact locally homogeneous space. To obtain more quantitative geometric profiles of the Einstein metrics on $\mathcal{K}$, one is interested in the compactification of the moduli space $\mathfrak{M}(\mathcal{K})$ in the Gromov-Hausdorff topology, that is, one wishes to study how the elements in $\mathfrak{M}(\mathcal{K})$ can degenerate when they are approaching to the boundary compactified space.

Let $g_j \in \mathfrak{M}(\mathcal{K})$ be a sequence of Ricci-flat metrics with unit diameter. From the metric point of view, characterizing the degeneration of g_j consists of two questions:

(1) identify the boundary elements in $\overline{\mathfrak{M}(\mathcal{K})}^{GH} \backslash \mathfrak{M}(\mathcal{K})$, i.e., classify all possible Gromov-Hausdorff limits of g_j;
(2) describe the structure a neighborhood of $\overline{\mathfrak{M}(\mathcal{K})}^{GH} \setminus \mathfrak{M}(\mathcal{K})$ in $\overline{\mathfrak{M}(\mathcal{K})}^{GH}$, i.e., classify all possible bubble limits of g_j, namely all possible rescaling limits.

To address the first question, joint with Song Sun, we proved the following classification result.

Theorem 1 ([1]). *Let* $(\mathcal{K}, g_j) \xrightarrow{GH} (X_\infty, d_\infty)$ *be a sequence of Ricci-flat metrics with unit diameter. Then* (X_∞, d_∞) *is isometric to one of the following:*

(a) *a Calabi-Yau orbifold with ADE singularities;*
(b) *a flat orbifold* $\mathbb{T}^3/\mathbb{Z}_2$*;*
(c) *a singular special Kähler metric on* S^2*;*
(d) *the unit interval* $[0, 1]$.

At the next stage, a full understanding of the degenerations of Ricci-flat metrics on $\mathcal{K}$ demands the classification of all bubbles among which the gravitational instantons capture the essential information regarding the singularity formation. In our context, such metrics are Calabi-Yau metrics. A complete Calabi-Yau manifold

(X, g) of complex dimension 2 is called a *gravitational instanton* if its curvature is quadratically integrable. Together with Song Sun, for any gravitational instanton, we managed to construct and classify all possible model geometries on large scales. This enables us to obtain a complete classfication of such metrics.

Theorem 2 ([1]). *Any gravitational instanton (X, g) has a unique asymptotic cone (Y, d_C) which is a flat metric cone. Furthermore, there exists a compact subset $K \subset X$ such that $X \setminus K$ admits an ALX model geometry with $X \in \{E, F, G, H, G^*, H^*\}$ and the dimension equality:*

$$\textit{dimension of collapse} = \dim X - \dim Y = \textit{difference of } X \textit{ and } E.$$

Moreover, the asymptotics yields at least certain polynomial order.

Roughly speaking, ALX model geometry (X_K, g_{ALX}) gives a *canonical nilpotent structure* on a gravitational instanton in the following sense:

(i) there exists a fiber bundle $\mathcal{N} \to X_K \xrightarrow{\pi} Y \setminus K'$ with $K' \subset Y$ compact and $\mathcal{N}$ nilpotent;
(ii) g_{ALX} is Calabi-Yau and invariant under the nilpotent action given by $\mathcal{N}$;
(iii) the blow-down metric $r^2 g_{ALX}$ is collapsing along the nilpotent fiber $\mathcal{N}$ as $r \to 0$.

Classification of model geometries has an immediately application.

Corollary 3. *Any gravitational instanton is biholomorphic to a Zariski closed and dense subset of some compact algebraic surface.*

The above corollary confirms a compactification conjecture of Yau in the context of graviational instanton. Notice that, the original statement for all Calabi-Yau metrics has counterexamples.

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Semi-classical spectral asymptotics of Toeplitz operators on CR manifolds

Wei-Chuan Shen

(joint work with Herndrik Herrmann, Chin-Yu Hsiao and George Marinescu)

In this talk I will present some results obtained in the recent preprint [11]. Let X be a compact strictly pseudoconvex embeddable CR manifold of $\dim_{\mathbb{R}} X = 2n+1$, $n \geq 1$, and Π be the Szegő projection on X. We denote by $\Pi(x, y) \in \mathcal{D}'(X \times X)$ the Schwartz kernel of Π. From the theory of Kohn (cf. [6]), $\Pi : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ is continuous and $\Pi(x, y) \in \mathcal{C}^\infty(X \times X \setminus \operatorname{diag}(X \times X))$. Moreover, we have the following fundamental theorem of Boutet de Monvel–Sjöstrand [2] (also cf. [8] and [9]): For any coordinate patch (Ω, x) of X, there is a smooth function $\varphi : \Omega \times \Omega \to \mathbb{C}$ satisfying $\operatorname{Im} \varphi(x,y) \geq 0$, $\varphi(x,y) = 0$ if and only if $y = x$, and $d_x\varphi(x,x) = -d_y\varphi(x,x) = \xi(x)$, where ξ is the given contact form inducing pseudoconvexity on X, such that the Szegő projector can be approximated by a Fourier integral operator

$$\Pi(x,y) = \int_0^{+\infty} e^{it\varphi(x,y)} s(x,y,t)dt + F(x,y) \text{ on } \Omega \times \Omega.$$

Here F is a smoothing operator, $s(x,y,t) \sim \sum_{j=0}^{+\infty} s_j(x,y)t^{n-j}$ in $S^n_{1,0}(\Omega \times \Omega \times \mathbb{R}_+)$ is a classical Hörmander symbol with $s_0(x,x) = (2\pi^{n+1})^{-1}\frac{dV_\xi}{dV}(x)$, dV is the given volume form on X and dV_ξ is the one induced by ξ, and $\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$. Through Boutet de Monvel–Guillemin [3], such description of Szegő kernel had a profound impact in many research areas, see [1], [4], [5], [15]. These ideas also partly motivated the introduction of alternative approach in [13].

Let $T_P := \Pi \circ P \circ \Pi$ be the Toeplitz operator on X associated with a *first order* and *formally self-adjoint* classical pseudodifferential operator P. The theory of Toeplitz operators can be regarded as an analogue of the theory of pseudodifferential operators [3]. From now on, we consider *elliptic* Toeplitz operators. A Toeplitz operator T_P is called elliptic if the principal symbol σ_P of P satisfies $\sigma_P(\xi) > 0$. We are interested in *functional calculus of elliptic Toeplitz operators.* In fact, given any function $\chi \in \mathcal{C}^\infty(\mathbb{R}_+)$, for the operator $\chi(T_P)$ defined by functional calculus Boutet de Monvel and Guillemin prove that $\chi(T_P) = \Pi \circ \chi(Q) \circ \Pi$ for some elliptic first order pseudodifferential operator Q such that $T_P = \Pi \circ Q \circ \Pi$ and $\Pi \circ Q = Q \circ \Pi$. This implies that $\chi(T_P)$ is also a Toeplitz operator. Especially, when χ has compact support, $\chi(T_P)$ becomes a smoothing operator. By introducing a semi-classical parameter $k > 0$ we obtain the following semi-classical limit $k \to \infty$ of $\chi(k^{-1}T_P)$.

Theorem 1 ([11]). *Under the same assumptions and notations above, for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $\chi \not\equiv 0$, the Schwartz kernel of $\chi_k(T_P)$, $\chi_k(\lambda) := \chi(k^{-1}\lambda)$, can be represented for k large by*

$$\chi_k(T_P)(x,y) = \int_0^{+\infty} e^{ikt\varphi(x,y)} A(x,y,t,k)dt + O\left(k^{-\infty}\right) \text{ on } \Omega \times \Omega,$$

where $A(x,y,t,k) \sim \sum_{j=0}^{\infty} A_j(x,y,t)k^{n+1-j}$ in $S^{n+1}_{\mathrm{loc}}(1; \Omega \times \Omega \times \mathbb{R}_+)$, $A_j(x,y,t) \in \mathcal{C}^\infty(\Omega \times \Omega \times \mathbb{R}_+)$, $j = 0,1,2,\ldots$, $A_0(x,x,t) = s_0(x,x)\chi(\sigma_P(\xi_x)t)t^n \not\equiv 0$, and for some compact interval $I \subset \mathbb{R}_+$, $\operatorname{supp}_t A(x,y,t,k)$, $\operatorname{supp}_t A_j(x,y,t) \subset I$, $j = 0,1,2,\cdots$. Moreover, for any $\tau_1, \tau_2 \in \mathcal{C}^\infty(X)$ such that $\operatorname{supp}(\tau_1) \cap \operatorname{supp}(\tau_2) = \emptyset$, we have $\tau_1\chi_k(T_P)\tau_2 = O\left(k^{-\infty}\right)$.

Let us explain the strategy of our proof in the case $P = -iT$, where T is the Reeb vector field with respect to ξ. Here we also take $L^2(X, dV_\xi)$ be the one induced by the Levi metric such that $P = P^*$. We remark that this Toeplitz operator is important in CR geometry and give a bridge to complex geometry by the circle bundle framework [3, 4, 15]. Our approach relies on Helffer–Sjöstrand formula [7] and Melin–Sjöstrand theory of complex-valued phase Fourier integral operators [14]. A key formula is then

$$\chi_k(T_P) = \chi_k(T_P) \circ \Pi = \int_{\mathbb{C}} \frac{\partial \tilde{\chi}_k}{\partial \overline{z}} (z - T_P)^{-1} \circ \Pi \frac{dz \wedge d\overline{z}}{2\pi i}.$$

This formula was already applied in [10] for order zero Toeplitz operators. However, in our case T_P is a Fourier integral operator and a pseudodifferential operator of type $(\frac{1}{2}, \frac{1}{2})$. It is hard to get the formula of $(z - T_P)^{-1}$ directly from classical results and even the term $(z - T_P)^{-1} \circ \Pi$ is very complicated. We then proceed as follows. For all $N \in \mathbb{N}_0$, we construct z-dependent Fourier integral operators $B_z^{(j)}$ with the same phase function as Π given by

$$B_z^{(j)}(x, y) \equiv \int_0^{+\infty} e^{it\varphi(x,y)} \frac{\sum_{h=0}^{p} b_h^{(j)}(x, y, t) z^h}{(z - t)^{\ell}} t^{n-j-1} dt \mod \mathcal{C}_z^{\infty},$$

where $p \leq \ell$ and $b_h^{(j)}(x, y, t) \in S_{\mathrm{cl}}^{\ell-h}(\Omega \times \Omega \times \mathbb{R}_+)$, such that $\Pi B_z^{(j)} \equiv B_z^{(j)} \Pi \equiv B_z^{(j)}$ mod $\mathcal{C}_z^{\infty}$, $(z - T_P) \sum_{j=0}^{N} B_z^{(j)} \equiv \Pi + R_z^{(N+1)} \mod \mathcal{C}_z^{\infty}$ and

$$R_z^{(N+1)}(x, y) = \int_0^{+\infty} e^{it\varphi(x,y)} \frac{\sum_{h=0}^{p'} r_h^{(N+1)}(x, y, t) z^h}{(z - t)^{\ell'}} t^{n-N-1} dt.$$

Here $\mathcal{C}_z^{\infty}$ is a class of z-dependent smoothing operators, and $B_z^{(0)}(x, y)$ satisfies $\ell = 1$, $p = 0$ and $b_0^{(0)}(x, y, t) \sim \sum_{h=0}^{\infty} b_h^{(0)}(x, y) t^{1-h}$ in $S_{1,0}^{1}(\Omega \times \Omega \times \mathbb{R}_+)$ with $b_0^{(0)}(x, x) = s_0(x, x)$.

Although $\int_{\mathbb{C}} \frac{\partial \tilde{\chi}}{\partial \overline{z}} B_z^{(j)}(x, y) \frac{dz \wedge d\overline{z}}{2\pi i}$ can always be calculated as a smoothing operator by Cauchy–Pompeiu formula, it is difficult to define a suitable z-dependent symbol space to control the remainder $\int_{\mathbb{C}} \frac{\partial \tilde{\chi}}{\partial \overline{z}} (z - T_P)^{-1} R_z^{(N+1)} \frac{dz \wedge d\overline{z}}{2\pi i}$ in the approximation of $\chi(T_P)$ just by Helffer–Sjöstrand formula. Surprisingly, for k large enough, after a suitable truncation and using the fact that T_P has discrete spectrum, we can estimate $\int_{\mathbb{C}} \frac{\partial \tilde{\chi}_k}{\partial \overline{z}} (z - T_P)^{-1} R_z^{(N+1)} \frac{dz \wedge d\overline{z}}{2\pi i}$, and our asymptotic expansion theorem follows.

We mention some applications of our result. The first is a Szegő type limit theorem following from the asymptotic expansion of

$$\mathrm{Tr} \chi_k(T_P) = \int_X \chi_k(T_P)(x, x) dV(x).$$

Namely, we prove that the counting measure $k^{-n-1} \sum_{j=1}^{\infty} \delta\left(t - k^{-1} \lambda_j\right)$ converges weakly to $\mathcal{C}_P\, t^n dt$ as $k \to +\infty$, where $\mathcal{C}_P := (2\pi^{n+1})^{-1} \int_X \sigma_P(\xi)^{-n-1} dV_\xi$.

For the second application we take advantage of the fact that $\chi_k(T_P)$ exhausts the space of CR functions when $k \to +\infty$, so we can obtain several CR analogues of results concerning high powers of line bundles in complex geometry. We denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the positive eigenvalues counting multiplicities of T_P and let $f_1, f_2 \ldots$ be a orthonormal set in the space of CR functions on X such that $T_P f_j = \lambda_j f_j$ for all $j \in \mathbb{N}$. Let $0 < \delta_1 < \delta_2$ and $\chi \in \mathcal{C}_c^{\infty}((\delta_1, \delta_2))$. Let $N_k = \#\{j \in$

$\mathbb{N} : 0 < \lambda_j \leq k\delta_2\}$. Then for all sufficiently large $k \in \mathbb{N}$, the Kodaira map

$$G_k \colon X \to \mathbb{C}^{N_k}, \quad G_k(x) = \big(\chi(k^{-1}\lambda_1)f_1, \ldots, \chi(k^{-1}\lambda_{N_k})f_{N_k}\big)$$

is a CR embedding, and for the CR embedding

$$F_k \colon X \to \mathbb{C}^{N_k}, \quad F_k(x) := \sqrt{\frac{2\pi^{n+1}}{k^{n+1}}}G_k(x)$$

we also prove a Tian type theorem that $F_k^*\omega_{\lambda(k)} = \sigma_P(\xi)^{-1}\xi + O(k^{-1})$ in the $\mathcal{C}^\infty$-topology, where $\omega_{\lambda(k)} := (2i\sum_{j=1}^{N_k}\lambda_j|z_j|^2)^{-1}\sum_{j=1}^{N_k}(\overline{z}_j dz_j - z_j d\overline{z}_j)$ is a contact form when restricted to the sphere $S^{2N_k-1} \subset \mathbb{C}^{N_k}$. Moreover, we show that X can be CR embedded into an arbitrary small perturbation of a sphere (cf. [12] for the Sasakian case).

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On the Steiner formula in sub-Riemannian geometry

Tania Bossio
(joint work with Davide Barilari)

Let S be a compact surface bounding a closed regular region $\Omega \subset \mathbb{R}^3$. The *half-tubular neighborhood* of S is defined as the set of points S_ε that are not in Ω and that are at distance at most ε from S. Steiner's formula states that the Lebesgue volume of S_ε is the following polynomial in ε:

$$\text{Vol}^{\mathbb{R}^3}(S_\varepsilon) = \varepsilon \text{Area}(S) - \varepsilon^2 \int_S H\, dA + \frac{\varepsilon^3}{3} \int_S K\, dA, \tag{1}$$

where H is the mean curvature of S, computed with respect to the orientation of the surface given by the outward pointing normal of Ω, and K is the Gaussian curvature of S, while dA denotes the induced surface measure.

Our goal is to compute the volume of the half-tubular neighborhood of a compact surface S bounding a closed regular region Ω in a three-dimensional sub-Riemannian contact manifold $(M, \mathcal{D}, g)$ and defined with respect to the sub-Riemannian distance. In other words, we compute the volume of the set

$$S_\varepsilon = \{x \in M \setminus \Omega \mid 0 < \delta(x) < \varepsilon\},$$

where $\delta : M \to \mathbb{R}$ is the sub-Riemannian distance from the surface. Moreover, we provide a geometrical interpretation of the coefficients appearing in the expansion as $\varepsilon \to 0$, generalizing some previous results obtained for surfaces which are embedded in the Heisenberg group.

We recall that a *3D sub-Riemannian contact manifold* $(M, \mathcal{D}, g)$ is a triple where M is a three-dimensional smooth manifold, $\mathcal{D}$ is a rank two distribution in TM, locally defined as the kernel of a smooth one-form ω such that $\omega \wedge \mathrm{d}\omega \neq 0$, and g is a smooth metric defined on $\mathcal{D}$ such that $\text{vol}_g = \mathrm{d}\omega|_{\mathcal{D}}$. Moreover it is defined the *Reeb vector field* X_0 is the unique vector field such that $\iota_{X_0}\omega = 1$, and $\iota_{X_0}\mathrm{d}\omega = 0$, where ι_{X_0} denotes the interior product.

Let S be a surface in $(M, \mathcal{D}, g)$. A point $x \in S$ is said to be *characteristic* if $\mathcal{D}_x = T_xS$. The set of characteristic points is closed and of zero measure in S. Let us consider a surface S that does not contain characteristic points. There exists $\varepsilon > 0$ such that $\delta : S_\varepsilon \to \mathbb{R}$, the sub-Riemannian distance function from the surface, is smooth and in addition S_ε is diffeomorphic to $S \times (0, \varepsilon)$.

Let $x \in S$ not characteristic, there are two relevant vector fields of unit norm with respect to g that are uniquely defined up to a sign:

- The *characteristic vector field* X_S generates $\mathcal{D}_x \cap T_xS$;
- The *horizontal normal* N is a vector field in $\mathcal{D}$ ortogonal to X_S.

We obtain the following formula for the measure ν of the half-tubular neighborhood S_ε, that is the measure associated to the volume form $\omega \wedge \mathrm{d}\omega$.

Theorem 1. *Let S be a smooth compact surface embedded in $(M, \mathcal{D}, g)$, a 3D sub-Riemannian contact manifold, bounding a closed region Ω, and such that it does* *not contain characteristic points. The volume of the half-tubular neighborhood S_ε, is smooth with respect to ε and satisfies for $\varepsilon \to 0$:*

$$\nu\left(S_\varepsilon\right) = \varepsilon \int_S dA - \frac{\varepsilon^2}{2}\int_S \mathcal{H}\, dA + \frac{\varepsilon^3}{6}\int_S \left(\mathcal{H}^2 - N\left(\mathcal{H}\right)\right)\, dA + o(\varepsilon^3), \tag{2}$$

where $dA = \iota_N(\omega \wedge \mathrm{d}\omega)$ is the sub-Riemannian area measure, $\mathcal{H}$ is the sub-Riemannian mean curvature of the surface S defined on S_ε as

$$\mathcal{H} = -\mathrm{div}\left(\nabla_H \delta\right) = -X_1X_1\delta - X_2X_2\delta - c^2(X_1\delta) + c^1(X_2\delta),$$

where X_1, X_2 is a orthonormal frame in $\mathcal{D}$ such that $[X_2, X_1] = X_0 + c^1X_1 + c^2X_2$.

In contrast to the euclidean case, the formula obtained here is not a polynomial in ε. In fact previous results state that in the Heisenberg group the volume of S_ε is analytic and all coefficients of the expansion are explicitly computed.

Furthermore, the second order coefficients in (1) and (2) are both the integral of a mean curvature. The sub-Riemannian mean curvature $\mathcal{H}$ is the limit of the mean curvatures of the surface S in the Riemannian approximation of the space. We notice that the same does not hold for the third order coefficient in (2). The integral of the limit of the Riemannian Gaussian curvatures of S does not coincide with this last coefficient.

Finally, to compute the coefficients of expansion (2) one a priori needs the knowledge of the explicit expression of δ. We provide a formula which permits to compute those coefficients only in terms of a function f locally defining S. We notice that the formula we state is an expansion of order three in ε. For higher order coefficients it is not possible to obtain an expression independent of δ already when considering the Heisenberg group. We recall that the horizontal gradient $\nabla_H f$ of $f : M \to \mathbb{R}$ differentiable is defined as the unique vector field in $\mathcal{D}$ such that $\mathrm{d}f|_{\mathcal{D}}(\cdot) = g(\nabla_H f, \cdot)$.

Proposition 2. *Under the assumptions of the Theorem, let us suppose that S is locally defined as the zero level set of a smooth function $f : M \to \mathbb{R}$ such that $\nabla_H f \neq 0$ and $g\left(\nabla_H f, \nabla_H \delta\right)|_S > 0$. The following formula is equivalent to (2):*

$$\begin{aligned}\nu(S_\varepsilon) = \varepsilon \int_S dA &- \frac{\varepsilon^2}{2}\int_S \mathrm{div}\left(\frac{\nabla_H f}{\|\nabla_H f\|}\right) dA\\ &+ \frac{\varepsilon^3}{6}\int_S \Bigg[2X_S\left(\frac{X_0 f}{\|\nabla_H f\|}\right) - \left(\frac{X_0 f}{\|\nabla_H f\|}\right)^2\\ &\qquad - \kappa - g\left(\mathrm{Tor}\left(X_0, X_S\right), \frac{\nabla_H f}{\|\nabla_H f\|}\right)\Bigg] dA + o(\varepsilon^3),\end{aligned}$$

with $\kappa = g(R(X_S, N)N, X_S)$. Here R and Tor *are the curvature and the torsion operators associated with the Tanno connection that is the natural linear and metric connection defined on the contact structure.*

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The Yamabe flow on asymptotically Euclidean manifolds

Eric Chen
(joint work with Gilles Carron, Yi Wang)

A Yamabe flow starting from a Riemannian manifold (M^n, g_0) consists of a family of conformal metrics $g(t)$ on M^n for $t \in [0,T)$ and some $T > 0$ satisfying $\frac{\partial}{\partial t} g = -Rg$ and $g(0) = g_0$, where R denotes the scalar curvature of g. This flow was introduced by Hamilton [11] as an alternative approach to the Yamabe problem, which asks for a constant scalar curvature metric in any conformal class of Riemannian metrics on a compact manifold.

By writing $g(t) = u^{\frac{4}{n-2}} g_0$ for a time-dependent function u, the flow can be viewed as a parabolic evolution equation associated with the conformal Laplacian $L_{g_0} := -4\frac{n-1}{n-2}\triangle_{g_0} + R_{g_0}$, which for $n \geq 3$ takes the form

$$\frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} = -\frac{n+2}{4} L_{g_0} u.$$

We always have short-time existence of the flow when starting from a compact manifold, or more generally, a complete manifold with bounded scalar curvature [6]. Since the flow is formally the gradient flow of the Einstein–Hilbert functional within the conformal class $[g_0]$, we might hope that the flow exists for all positive times and converges to a metric of constant scalar curvature. This is indeed true in most cases for compact manifolds (after renormalizing the flow to fix the volume) [8, 14, 13, 2, 3], but on noncompact manifolds it is possible for the flow to develop a finite-time singularity, or to fail to converge as $t \to \infty$ [7].

Our results characterize the behavior of the Yamabe flow on noncompact, asymptotically Euclidean (AE) manifolds. A Riemannian manifold (M^n, g_0) is said to be a $C^{k+\alpha}_{-\tau}$ AE manifold if the complement of some compact subset can be identified with the complement of a closed ball in $\mathbb{R}^n$ such that in the $\mathbb{R}^n$ coordinates we have $|D^\gamma(g_{ij} - \delta_{ij})|(x) \leq C|x|^{-|\gamma|-\tau}$ for any $|\gamma| \leq k$, with an analogous requirement for the α-Hölder seminorms of order k derivatives of $g_{ij} - \delta_{ij}$. The $C^{k+\alpha}_{-\tau}$ function spaces are defined in a similar fashion.

With Yi Wang, we show that on such spaces the Yamabe flow solution defined in [9] never encounters a finite-time singularity, and describe the class of initial data for which the flow converges [5].

Theorem 1. *Let (M^n, g_0) be a $C^{k+\alpha}_{-\tau}$ AE manifold with $k \geq 3$ and $\tau > 0$. Then the Yamabe flow $g(t)$ starting from g_0 exists for all $t > 0$.*

(1) *If $Y > 0$, the flow converges uniformly in $C^{k+\alpha}_0$ to the unique scalar-flat $C^{k+\alpha}_{-\tau}$ AE metric $g_\infty \in [g_0]$ as $t \to \infty$.*

(2) *If* $Y \leq 0$*, then* $\sup_{M^n} u(x,t) \xrightarrow{t\to\infty} \infty$ *and the flow does not converge as* $t \to \infty$.

Above, Y denotes the Yamabe constant of an AE manifold, in analogy with the analytic definition of Y on compact manifolds,

$$Y = Y(M^n, [g_0]) := \inf_{u \in C_0^\infty} \frac{\int u \cdot L_{g_0} u \, dV_{g_0}}{\|u\|^2_{\frac{2n}{n-2}}}.$$

The positivity of this conformally invariant quantity is equivalent to the existence of a scalar-flat AE metric in the conformal class of g_0 [10].

If in the first case above we also assume that $R_{g_0} \geq 0$, $\int R_{g_0} \, dV_{g_0} \in L^1$, and $\tau \in \left(\frac{n-2}{2}, n-2\right)$, so that the ADM mass $m(g_0)$ is well-defined, then $m(g(t))$ is monotonically nonincreasing under the Yamabe flow [9] and $g(t)$ has stronger, weighted convergence in $C^{k+\alpha}_{-\tau'}$ to the scalar flat AE metric $g_\infty \in [g_0]$, for any $\tau' < \tau$. Consequently, using lower semicontinuity properties of the ADM mass under such convergence [12], we can characterize the difference between $\lim_{t\to\infty} m(g(t))$ and $m(g_\infty)$.

Corollary 2. *For* $n \geq 3$*, let* (M^n, g_0) *be a* $C^{k+\alpha}_{-\tau}$ *AF manifold with non-negative scalar curvature and* $k \geq 3$*,* $\tau > \frac{n-2}{2}$*, along with* $R_{g_0} \in L^1(M^n, g_0)$*. Then along the Yamabe flow* $(M^n, g(t))$ *starting from* (M^n, g_0)*,*

$$\lim_{t\to\infty} m(g(t)) - m(g_\infty) = \lim_{t\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int R_{g(t)} \, dV_t.$$

When $Y(M^n, [g_0]) \leq 0$ and the flow fails to converge, by comparing with the solution of the Yamabe problem on the one-point compactification $\overline{M^n} := M^n \cup \{q\}$ with metric $\overline{g_0} := \phi^{\frac{4}{n-2}} g_0$, which extends as a $W^{2,p}$ metric to q for suitable $p > n/2$ and smooth ϕ on M^n [10, 1], we give in joint work with Gilles Carron and Yi Wang a more precise description of the blowup profile at time infinity [4].

Theorem 3. *Let* (M^n, g_0) *be a* $C^{k+\alpha}_{-\tau}$ *AF manifold with* $k \geq 3$ *and* $\tau > 0$*. Consider the Yamabe flow* $g(t)$ *starting from* g_0*.*

(1) *If* $Y < 0$*, then* $t^{-1} g(t)$ *converges to the unique metric* $\overline{g_\infty} \in [\overline{g_0}]$ *with constant scalar curvature* $Y < 0$*.*
(2) *If* $Y = 0$*, then there exists a function* $\lambda(t) = o(t)$ *with* $\lim_{t\to\infty} \lambda(t) = \infty$ *such that* $\lambda(t)^{-1} g(t)$ *converges to the unique metric* $\overline{g_\infty} \in [\overline{g_0}]$ *with constant scalar curvature* $Y = 0$*.*

The convergence as $t \to \infty$ *above holds in* $C^{k,\alpha'}_{loc}$ *on* M^n *for any* $\alpha' < \alpha$*.*

A Yamabe flow $g(t)$ of maximal existence interval $[0, T)$ is said to be of type I if either $T < \infty$ and $\sup_{M^n} (T-t)|\mathrm{Rm}| < \infty$, or $T = \infty$ and $\sup_{M^n} |\mathrm{Rm}| < \infty$. Otherwise, it is said to be of type II. On compact manifolds, the Yamabe flow exhibits type I behavior for generic initial data, and it is expected this should indeed hold for all initial data.

Our blowup analysis above implies that when $Y \leq 0$, on any compact set $K \subset M^n$ we have $\sup_K |\mathrm{Rm}| \to 0$ as $t \to \infty$. It would be interesting to see whether, unlike in the general noncompact case [7], all (necessarily infinite-time) singularities of the Yamabe flow on AE manifolds must be of type I.

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Hardy's inequality for conformally invariant fractional powers of the sublaplacian on Heisenberg groups

Sundaram Thangavelu

(joint work with Luz Roncal)

In this talk we plan to discuss Hardy's inequality for conformally invariant fractional powers $\mathcal{L}_s$ of the sublaplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}^n$. We consider two kinds of inequalities involving weighted L^2 norms, the difference being whether the weight function is homogeneous or not. In the Euclidean setting, for fractional powers of the standard Laplacian Δ on $\mathbb{R}^n$, the non-homogeneous Hardy's inequality reads as

$$\langle(-\Delta)^s f, f\rangle \geq 4^s \rho^{2s} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})} \int_{\mathbb{R}^n} \frac{(f(x))^2}{(\rho^2+|x|^2)^{2s}}\, dx \tag{1}$$

where $0 < s < 1$ and $\rho > 0$. Here the constant is sharp and is achieved when

$$f(x) = \psi_{-s,\rho}(x) =: \pi^{-n/2} \frac{\Gamma(\frac{n-2s}{2})}{|\Gamma(-s)|} (\rho^2 + |x|^2)^{-\frac{n-2s}{2}}.$$

The inequality of second kind, which is more difficult to prove, is given by

$$\langle (-\Delta)^s f, f \rangle \geq 4^s \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2} \int_{\mathbb{R}^n} \frac{(f(x))^2}{|x|^{2s}} dx. \tag{2}$$

Here too the inequality is sharp but the constant is never achieved. There are several ways of proving these inequalities, one method that is well suited for proving similar inequalities on the Heisenberg groups is via extension problem and trace Hardy inequality.

On the Heisenberg group we can consider two kinds of fractional powers, denoted by $\mathcal{L}^s$ and $\mathcal{L}_s$, of the sublaplacian. The first one, though natural, is not the right one in view of the CR geometry, owing to the fact that it is not conformally invariant. The other one, which has this property, also has the added advantage that the kernel of $\mathcal{L}_s$ can be computed explicitly. This fractional power has already appeared in earlier works, e.g. in [4] in connection with Hardy-Littlewood-Sobolev inequality on the Heisenberg group. In view of the work by Caffarelli and Silvestre [1] the operator $(-\Delta)^s$ occurs as the Dirichlet-Neumann map associated to the extension problem

$$\left(\Delta + \partial_\rho^2 + \frac{1-2s}{\rho} \partial_\rho \right) u(x, \rho) = 0, \;\; u(x, 0) = f(x) \tag{3}$$

on the upper half space $\mathbb{R}^n \times \mathbb{R}^+$. If we want to obtain $\mathcal{L}_s$ as the Dirichlet-Neumann map of an extension problem on $\mathbb{H}^n \times \mathbb{R}^+$, we need to modify the above equation.

The modified extension problem, already studied in [3], is given by the equation

$$\left(-\mathcal{L} + \partial_\rho^2 + \frac{1-2s}{\rho} \partial_\rho + \frac{1}{4} \rho^2 \partial_a^2 \right) u(z, a, \rho) = 0, \;\; u(z, a, 0) = f(z, t) \tag{4}$$

where $(z, t) \in \mathbb{H}^n =: \mathbb{C}^n \times \mathbb{R}$. A solution to this extension problem is written down explicitly as $u = \rho^{2s} \, f * \Phi_{s,\rho}$ where

$$\Phi_{s,\rho}(z, a) = \frac{2^{-(n+1+s)}}{\pi^{n+1} \Gamma(s)} \Gamma(\frac{n+1+s}{2})^2 \varphi_{s,\rho^2/4}(z, a). \tag{5}$$

Here the function $\varphi_{s,\delta}$ is defined by the expression

$$\varphi_{s,\delta}(z, a) = ((\delta + \frac{1}{4}|z|^2)^2 + a^2)^{-\frac{n+1+s}{2}}. \tag{6}$$

A formula attributed to Cowling and Haagerup [2] states that $\Phi_{s,\rho}$ and $\Phi_{-s,\rho}$ are related via

$$\mathcal{L}_s \Phi_{-s,\rho}(z, a) = (2\rho)^{2s} \frac{\Gamma(s)}{\Gamma(-s)} \Phi_{s,\rho}(z, a). \tag{7}$$

This is the analogue of the identity

$$(-\Delta)^s\psi_{-s,\rho}(x) = \rho^{2s}4^s\frac{\Gamma(1+s)}{\Gamma(1-s)}\psi_{s,\rho}(x)$$

in the Euclidean setting. Once we have a trace Hardy inequality, the Cowling-Haagerup formula can be used to deduce the inequality

$$\langle \mathcal{L}_s f, f\rangle \geq (4\delta)^s\frac{\Gamma(\frac{Q+2s}{4})^2}{\Gamma(\frac{Q-2s}{4})^2}\int_{\mathbb{H}^n}\frac{(f(z,a))^2}{((\delta+\frac{1}{4}|z|^2)^2+a^2)^s}dzda \tag{8}$$

valid for $0 < s < (n+1)/2, \delta > 0$. The constant is sharp and is achieved when $f(x) = \varphi_{-s,\delta}$.

In order to state the trace Hardy inequality for the Heisenberg group, let

$$\nabla u(z,a,\rho) = (X_1u, ..., X_nu, Y_1u, ...Y_nu, \frac{1}{2}\rho Tu, \partial_\rho u)$$

where $X_j, Y_j, T, j = 1, 2, ...n$ are left invariant vector fields on $\mathbb{H}^n$ forming a basis for the Heisenberg Lie algebra. Then for $0 < s < 1$, we have the inequality

$$\int_0^\infty\int_{\mathbb{H}^n}\Big|\nabla u(z,a,\rho)\Big|^2\rho^{1-2s}\,dz\,da\,d\rho \geq 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}\int_{\mathbb{H}^n}(u(z,a,0))^2\frac{\mathcal{L}_s\varphi(z,a)}{\varphi(z,a)}\,dz\,da$$

valid for suitable functions u and φ. From this inequality, making use of Cowling-Haagerup formula, we can prove the following Hardy's inequality for $\mathcal{L}_s$: For any $0 < s < 1$, and real valued f from the Sobolev space $\mathcal{H}^s(\mathbb{H}^n)$,

$$(\mathcal{L}_s f, f) \geq 4^s\frac{\Gamma(\frac{n+1+s}{2})^2}{\Gamma(\frac{n+1-s}{2})^2}\int_{\mathbb{H}^n}\frac{(f(z,a))^2}{w_s(z,a)}\,dz\,da. \tag{9}$$

Here $w_s(z,a)$ is homogeneous of degree $2s$ with respect to the non-isotropic dilations on the Heisenberg group. Once again, the constant is sharp but equality is never achieved.

We conjecture that in the above inequality we can replace $w_s(z,a)$ with $|(z,a)|^{2s}$. Though $w_s(z,a)$ is given as an integral over $\mathbb{H}^n$ we are not able to compute or estimate it from above sharply.

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Exponentially small eigenvalues of Witten Laplacians and persistent (co)homology

Francis Nier
(joint work with Dorian Le Peutrec, Claude Viterbo)

After recalling some motivations and issues about the accurate computation of exponentially small eigenvalues of semiclassical Witten Laplacians, I explained why Morse theory is not the most appropriate point of view for this problem. The bar code of persistent homology (see e.g. [2][3][6] and references therein) actually provides a stable topological framework and our result with D. Le Peutrec and C. Viterbo in [4] states this precisely. The bar code of persistent homology is firstly presented in the earlier approach of Barannikov for Morse theory and which was used in [5]. Forgetting the Morse property of the potential function for the asymptotic analysis of Witten Laplacians requires to consider from the beginning exponentially small eigenvalues, that is the ones which are $O(e^{-\frac{c}{h}})$ as $h \to 0^+$ for some fixed $c > 0$. Handling carefully the exponential deay of eigenvectors associated with such eigenvalues and more generally of some well chosen solutions to $d_{f,h}\omega = 0$ in some increasing domain Ω, allows to express the asymptotic behaviour as $h \to 0^+$ of all the exponentially small eigenvalues in terms of the lengthes of the bar code.

I explained some ideas of the analysis. In particular the introduction of boundary Witten Laplacian, with Neumann boundary conditions on the upper level and Dirichlet boundary conditions on the lower level, allows to localize the analysis along the energy axis. For a potential $f(x)$ accurate enough exponential decay estimates can be formulated in terms of the pseudodistance $|f(x) - f(y)|$ and this makes this problem essentially one dimensional along the energy axis $\mathbb{R} \supset f(M)$. The final step is a long and technical induction proof for any finite number of critical values for f, which relies on a mixture of spectral an functional properties, exponential decay propagation and tunneling effect analysis, and finally Mayer-Vietoris sequences.

After this, I presented some consequences and a sketch of the program in progress for similar questions about Bismut's hypoelliptic Laplacian of which the two first steps were [7] and [8].

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Horizontal magnetic fields and improved Hardy inequalities in the Heisenberg group

Dario Prandi
(joint work with Biagio Cassano, Valentina Franceschi, David Krejčiřík)

In order to introduce magnetic fields in the Heisenberg group [2], and study their influence on spectral properties of the corresponding magnetic sub-Laplacian, we start by some preliminaries on magnetic fields in Riemannian geometry.

1. Magnetic fields in Riemannian geometry

A magnetic field on a Riemannian manifold M is given by a closed 2-form $B \in \Omega^2(M)$. In order to define the corresponding magnetic Laplacian, we need to introduce a magnetic vector potential A, which is a 1-form such that $dA = B$. The magnetic Laplacian $-\Delta_A$ is then the operator on $L^2(M)$ given by the closure of the magnetic Dirichlet energy :

$$\mathfrak{q}_A(u) = \int_M H((d+iA)u)\, d\mu, \qquad u \in C_c^\infty(M).$$

Here, $d\mu$ denotes the Riemannian volume and $H : T^*M \to \mathbb{R}$ is the Hamiltonian associated with the Riemannian metric. Observe that we considering the Friedrichs extension of $-\Delta_A$.

The presence of the magnetic vector potential introduces a seemingly non-physical quantity (i.e., it is possible to have $A \neq A'$ but $B = dA = dA'$) and one would expect its choice, as in the classical case, to be irrelevant for the particle evolution. This is the case if the manifold M is simply connected, where vector potentials yielding the same magnetic field must differ by an exact form and thus $-\Delta_A$ and $-\Delta_{A'}$ are unitarily equivalent. If the manifold is not simply connected, however, the choice of the vector potential can deeply affect the properties of the magnetic Laplacian. This is the case of the celebrated Aharonov-Bohm effect [1]

Let us detail some spectral consequences due to magnetic fields:

(R1) In the 2D case a magnetic field can always be expressed as $B = b(x)d\mu$ for some smooth function b. Then, $\inf \sigma(-\Delta_A) \geq \|b\|_\infty$ for any choice of the vector potential A, see e.g., [7]. In particular, constant magnetic fields on $\mathbb{R}^2$ uplift the bottom of the spectrum (indeed, $\sigma(-\Delta_0) = [0, +\infty)$ in the unperturbed case).

(R2) In the compact non-simply connected case (where $\inf \sigma(-\Delta_0) = 0$), for vector potentials such that $dA = 0$ we have that $\inf \sigma(-\Delta_A) > 0$ if and only if $\int_\gamma A \notin 2\pi\mathbb{Z}$ for some closed curve γ [9, Theorem 4.2]. Again, the spectrum is uplifted.

(R3) In the non-compact case, more finer effect appear. Let $M = \mathbb{R}^n$ and consider a magnetic field B that vanishes at infinity. Then, $\sigma(-\Delta_A) = \sigma(-\Delta_0) = [0, +\infty)$ but if $B \not\equiv 0$ we have the following improvement on the classical Hardy inequality for some constant $c = c(n, B)$ (see, e.g., [3]):

$$-\Delta_A - \left(\frac{n-2}{2}\right)^2 \frac{1}{\rho^2} \geq \frac{c}{1 + \rho^2 \log^2 \rho}, \qquad \rho^2 = x_1^2 + \ldots + x_n^2.$$

In particular, the operator on the l.h.s. is sub-critical. This is in contrast with the non-magnetic case, where $-\Delta_0 - (\frac{n-2}{2})^2 \rho^{-2}$ is critical.

(R4) In $M = \mathbb{R}^n \setminus \{r = 0\}$, where we consider hypercylindrical coordinates $(r, \theta, x_3, \ldots, x_n)$ we consider the magnetic field $A = \alpha\, d\theta$ for $\alpha \in \mathbb{R}$. Then, $dA = 0$, and again $\sigma(-\Delta_A) = \sigma(-\Delta_0) = [0, +\infty)$, but the following sharp improved Hardy inequality holds (see, [6, 4])

$$-\Delta_A - \left(\frac{n-2}{2}\right)^2 \frac{1}{\rho^2} \geq \frac{d(\alpha, \mathbb{Z})^2}{r^2}.$$

In particular, if A has non-integer flux and sub-critical otherwise.

We can now turn to the counterparts of the above in the Heisenberg group.

2. Horizontal magnetic fields in the Heisenberg group

Let us consider the Heisenber group, i.e., the sub-Riemannian structure on $\mathbb{R}^3$ with hortonormal frame

$$X = \partial_x - \frac{y}{2}\partial_z \qquad \text{and} \qquad Y = \partial_y + \frac{x}{2}\partial_z.$$

The only non-trivial commutator is $[X, Y] = Z =: \partial_z$, and thus the family $\{X, Y\}$ is bracket-generating. The dual basis of the cotangent bundle associated with $\{X, Y, Z\}$ is $\{dx, dy, \omega\}$, where $\omega = dz + \frac{y\,dx - x\,dy}{2}$.

An horizontal magnetic potential A is a 1-form, and the magnetic sub-Laplacian $-\Delta_A$ is then defined as the operator on $L^2(\mathbb{R}^3)$ associated with the energy:

$$\int_{\mathbb{R}^3} H((d_H + iA)u)\, dp, \qquad u \in C_c^\infty(\mathbb{R}^3). \tag{1}$$

Here, $d_H u = Xu\, dx + Yu\, dy$ is the horizontal exterior differential, $H(\lambda) = \langle \lambda, X\rangle^2 + \langle \lambda, Y\rangle^2$ is the sub-Riemannian Hamiltonian, and dp denotes the Lesbegue measure.

Observe that, since H is degenerate in the ω direction, A needs to be determined up to its component w.r.t. ω. This yields to consider the Rumin complex [8], letting $A \in \Omega^1_H = \Omega^1/\operatorname{span}\{\omega\}$, and considering the associated magnetic field $B = DA \in \Omega^2_H = \operatorname{span}\{dx \wedge \omega, dy \wedge \omega\}$, where D is defined as $DA = d\tilde{A}$ where $\tilde{A}$ is the representative of A such that $d\tilde{A} \wedge \omega = 0$. Observe that this is a second order operator. Nevertheless, the cohomology $\ker D/\operatorname{im} d_H$ coincides with the standard de Rham cohomology H^1_{dR}.

The spectral consequences we obtain in [2] are the following (enumerated relative to their Riemannian counterparts):

(sR1) For a constant magnetic field $B = b_1\, dx \wedge \omega + b_2\, dy \wedge \omega$, we have

$$\inf \sigma(-\Delta_A) = c|b|^{2/3},$$

where $c > 0$ is a universal constant and $|b| = \sqrt{b_1^2 + b_2^2}$. As in the Riemannian case, we have an uplift of the bottom of the spectrum but with a different scaling relative to the strength of the magnetic field.

(sR3) For a magnetic field $B = DA$ non-vanishing on some open set $\Omega \subset \mathbb{R}^3$, we improve the Garofalo-Lanconelli Hardy inequality [5]. Namely, there exists a constant $c = c(\Omega, B) > 0$ such that :

$$-\Delta_A - \frac{H(d_H\rho)}{\rho^2} \geq c\chi_\Omega, \qquad \rho^4 = (x^2+y^2)^2 + 16z^2.$$

As in the Riemannian case, the operator on the r.h.s. becomes sub-critical although for $A = 0$ it is critical. Here, ρ is the Koranyi gauge.

(sR4) If we consider $M = \mathbb{R}^3 \setminus \{r = 0\}$, where we consider cylindrical coordinates (r, θ, z), we can focus on Aharonov-Bohm magnetic fields (i.e., such that $A \neq 0$ but $DA = 0$). These are all gauge equivalent to $A = \alpha\, d\theta$ for some $\alpha \neq 0$ and the operators $-\Delta_A - H(d_H\rho)/\rho^2$ are sub-critical as soon as $\alpha \notin \mathbb{Z}$. Moreover, on a class of functions with standard symmetry assumptions (e.g., invariant by rotations around the z-axis or by the reflection $(x, y, z) \mapsto (x, y, -z)$) we have

$$-\Delta_A - \frac{H(d_H\rho)}{\rho^2} \geq (1 - H(d_H\rho)^2)\frac{d(\alpha, \mathbb{Z})^2}{r^2}.$$

Finally, we mention that in an ongoing work with R. Bonalli we are studying the generalization of these concepts to 3D contact manifolds, where the Rumin complex is well understood and the sub-Laplacian can be defined as in (1). In this context we have the following generalization of (R2):

(sR2) Let M be a 3D contact sub-Riemannian manifold, which is compact and endowed with an horizontal magnetic field A such that $DA = 0$. Then, $\inf \sigma(-\Delta_A) > 0$ if and only if $\int_\gamma A \notin 2\pi\mathbb{Z}$ for some closed curve γ.

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The Anti-Self-Dual Deformation Complex and a conjecture of Singer

Matthew J. Gursky

(joint work with A. Rod Gover)

Let M^4 be a smooth, closed, oriented four-manifold. Given a Riemannian metric g on M^4, the bundle of two-forms $\Lambda^2 = \Lambda^2(M^4)$ splits into the sub-bundles of self-dual and anti-self=dual two-forms under the action of the Hodge $\star$=operator:

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-.$$

If W denotes the Weyl tensor, then

$$W^\pm : \Lambda^2_\pm \to \Lambda^2_\pm$$

(see [10]).

Definition 1. *We say that* (M^4, g) *is* anti-self-dual *(ASD) if* $W^+_g \equiv 0$.

The notion of (anti-)self-duality is conformally invariant: if $W^+_g = 0$ for a metric g and $\tilde{g} = e^f g$, then $W^+_{\tilde{g}} = 0$.

There are topological obstructions to the existence of ASD metrics. By the Hirzebruch signature formula,

$$48\pi^2 \tau(M^4) = \int \left(|W^+_g|^2 - |W^-_g|^2 \right) dv_g, \tag{1}$$

where $\tau(M^4)$ is the signature of the intersection form on $H^2(M^4)$. In particular, we see that if (M^4, g) is ASD then $\tau(M^4) \leq 0$, with equality if and only if g is LCF. If (M^4, g) is ASD with positive scalar curvature, then the intersection form is positive definite (see Proposition 1 of [6]).

Examples of ASD manifolds include locally conformally flat (LCF) manifolds, since in dimensions greater than three LCF is equivalent to the vanishing of the Weyl tensor. In particular, S^4 endowed with the round metric g_c is ASD. A non-LCF example is given by complex projective space with the Fubini-Study metric, and we take the opposite of the its natural orientation as a complex manifold; i.e., $(-\mathbb{CP}^2, g_{FS})$. There are many constructions ASD manifolds in the literature; see for example [8], [9], [7], [2], [3], [11], and the references in Lecture 6 of [12].

Let $\mathcal{M}(M^4)$ be the space of smooth Riemannian metrics on M^4, and $\mathcal{R}(M^4)$ the bundle of algebraic curvature tensors. We can view W^+ as a mapping

$$W^+ : \mathcal{M}(M^4) \to \mathcal{R}(M^4).$$

Let $g \in \mathcal{M}(M^4)$ be an ASD metric. We can identify the formal tangent space of $\mathcal{M}$ at g with sections of the bundle of symmetric two-tensors, $S^2(T^*M^4)$. Let

$$\mathcal{D} : \Gamma(S^2(T^*M^4)) \to \Gamma(\mathcal{R}(M^4)) \tag{2}$$

denote the linearization of W^+ at g; i.e., for $h \in \Gamma(S^2(T^*M^4))$,

$$\mathcal{D}_g h = \frac{d}{ds} W^+(g + sh)\big|_{s=0}.$$

The choice of a conformal class of metrics $[g]$ determines the bundle of algebraic Weyl tensors, $\mathcal{W} = \mathcal{W}(M^4, [g])$, and the sub-bundles $\mathcal{W}^{\pm} \subset \mathcal{W} \subset S^2_0(\Lambda^2_+)$, where $S^2_0(\Lambda^2_+)$ is the bundle of symmetric, trace-free endomorphisms of Λ^2_+. Note that

$$\mathcal{D}_g : \Gamma(S^2(T^*M^4)) \to \Gamma(\mathcal{W}^+) \subset \Gamma(S^2_0(\Lambda^2_+)). \tag{3}$$

In fact, since g is ASD, by conformal invariance $\mathcal{D}_g(fg) = 0$ for any $f \in C^\infty(M^4)$, hence

$$\mathcal{D}_g : \Gamma(S^2_0(T^*M^4)) \to \Gamma(\mathcal{W}^+). \tag{4}$$

We also let

$$\mathcal{D}^*_g : \Gamma(\mathcal{W}^+) \to \Gamma(S^2_0(T^*M^4)) \tag{5}$$

denote the L^2-formal adjoint of $\mathcal{D}_g$. Finally, let $\mathcal{K}_g : \Gamma(T^*M^4) \to S^2_0(T^*M^4)$ denote the Killing operator, whose kernel consists of conformal Killing vector fields.

The *ASD deformation complex* is given by

$$\Gamma(T^*M^4) \xrightarrow{\mathcal{K}} \Gamma(S^2_0(T^*M^4)) \xrightarrow{\mathcal{D}} \Gamma(S^2_0(\Lambda^2_+)). \tag{6}$$

This complex is elliptic; see [5], Section 2.

Definition 2. *Let (M^4, g) be ASD. We say that (M^4, g) is* unobstructed *if*

$$\ker \mathcal{D}^* = \{0\}.$$

The condition that the kernel of $\mathcal{D}^*$ is trivial is equivalent to the vanishing of the second cohomology group of the deformation complex. By the work of Floer [3] and Donaldson-Friedman [2], if (M_1, g_1) and (M_2, g_2) are unobstructed ASD manifolds, then the connected sum $M_1 \sharp M_2$ admits an ASD metric. Thus, we are lead to the following question: under what condition is an ASD manifold unobstructed? The following conjecture is attributed to Singer:

Conjecture 1. *Let (M^4, g) be ASD. If the Yamabe invariant of (M^4, g) is positive, then (M^4, g) is unobstructed.*

The main result described in the talk was the following:

Theorem 2. *Suppose (M^4, g) is ASD with Yamabe invariant $Y(M^4, [g]) > 0$. If*

$$2\chi(M^4) + 3\tau(M^4) \geq -\frac{1}{24\pi^2} Y(M^4, [g])^2, \tag{7}$$

then (M^4, g) is unobstructed.

The proof of this theorem relies on two important ideas. First, elements in the kernel of $\mathcal{D}^*$ can be identified with self-dual two-forms that are harmonic with respect to a twisted version of the Hodge Laplacian. More precisely, to any kernel element $U \in \ker \mathcal{D}^*$ we can associate a self-dual two-form z taking its values in the tractor bundle [4]. This 'tractor' interpretation allows us to derive a Weitzenböck formula for z.

Second, by conformal invariance of the ASD condition we have the freedom to choose any metric in the conformal class of an ASD metric when applying the Weitzenböck formula. By suitably modifying the work of Chang-Yang [1], we show that the inequality (7) implies the existence of a conformal metric g whose scalar and Ricci curvaures satisfy the (differential) inequality

$$\Delta R \leq -\frac{3}{2}|Ric|^2 + R^2, \tag{8}$$

where R is the scalar curvature and Ric is the Ricci tensor of g. This curvature condition implies the vanishing of any element in the kernel of $\mathcal{D}^*$.

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Hypoelliptic Laplacian of Bismut on symmetric spaces

Bingxiao Liu

The theory of hypoelliptic Laplacian, introduced by Bismut, presents an innovative approach to exploring the relationship between elliptic Laplacians and the dynamics of geodesic flows for Riemannian manifolds [1]. Bismut's geometric formula [2] for semisimple orbital integrals stands out as a remarkable accomplishment in this field. In my presentation, I provided an overview of Bismut's hypoelliptic Laplacians on symmetric spaces and elucidated his ideas for computing semisimple orbital integrals through the hypoelliptic deformation. I also recommended referring to Ma's Bourbaki talk [6] for a comprehensive introduction on this topic.

We start with the geometric setting. Let G be a connected real reductive Lie group, and let K be a maximal compact subgroup. Let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution that fixes K. The Cartan decomposition of the Lie algebra of G is $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Let B be an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$, which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Take $X = G/K$, then $TX = G \times_K \mathfrak{p}$. Moreover, B induces a Riemannian metric g^{TX} on X, so that (X, g^{TX}) is a Riemannian symmetric space of nonpositive sectional curvature. Let $d(\cdot,\cdot)$ denote the Riemannian distance in X. Given a homogeneous Hermitian bundle $F := G \times_K E \to X$ via a K-representation $\rho^E : K \to \mathrm{Aut}(E)$, it is canonically equipped with a unitary connection ∇^F. Let $\Delta^{X,F}$ denote the (negative) Bochner Laplacian. For $t > 0$, let $\exp(t\Delta^{X,F}/2)$ denote the heat operator of $-\Delta^{X,F}/2$ with the smooth integral kernel $p_t^{X,F}(x, x')$ (w.r.t. the Riemannian volume form).

Selberg's trace formula and orbital integrals. Let's consider a cocompact, torsion-free, discrete subgroup $\Gamma \subset G$. The resulting quotient, $M := \Gamma\backslash X$, is a compact locally symmetric space equipped with the induced Riemannian metric g^{TM}. The bundle F also descends to M, and we denote the corresponding Bochner Laplacian as $\Delta^{M,F}$. Selberg's trace formula allows us to compute the heat trace $\mathrm{Tr}[\exp(t\Delta^{M,F}/2)]$ using the orbital integrals $\mathrm{Tr}^{[\gamma]}[\exp(t\Delta^{X,F}/2)]$ associated with the conjugacy classes $[\gamma] \in [\Gamma]$. The main objective at hand is to calculate the orbital integrals $\mathrm{Tr}^{[\gamma]}[\exp\big(t\Delta^{X,F}/2\big)]$.

Geometric orbital integrals. An element $\gamma \in G$ is said to be semisimple if the convex function $d_\gamma(x) := d(x, \gamma\sigma(x))$ can reach its infimum m_γ in X. In this case, after conjugation, we may and we will always assume that $\gamma = e^a k^{-1}$, $a \in \mathfrak{p}$, $k \in K$, $\mathrm{Ad}(k^{-1})a = a$. Note that the elements in a cocompact lattice Γ are always semisimple. Let $Z(\gamma) \subset G$ be the centralizer of γ. Then θ acts on $Z(\gamma)$ with the Lie algebra $\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$. Let $\mathfrak{p}^\perp(\gamma)$ be the orthogonal space of $\mathfrak{p}(\gamma)$ in $\mathfrak{p}$. Let $X(\gamma)$ be the minimizing set of d_γ so that

$$X(\gamma) = Z(\gamma)/K(\gamma),\ K(\gamma) = Z(\gamma) \cap K. \tag{1}$$

The orthogonal normal bundle of $X(\gamma)$ is given by $N_{X(\gamma)/X} = Z(\gamma) \times_{K(\gamma)} \mathfrak{p}^\perp(\gamma)$. As a consequence, up to the factor K, the integrations on the quotient $Z(\gamma)\backslash G$ can be viewed as integrations along $\mathfrak{p}^\perp(\gamma)$, the fibre of $N_{X(\gamma)/X}$. Based on these constructions, we can give the definition of $\mathrm{Tr}^{[\gamma]}[\exp\big(t\Delta^{X,F}/2\big)]$,

$$\mathrm{Tr}^{[\gamma]}[\exp(t\Delta^{X,F}/2)] = \int_{\mathfrak{p}^\perp(\gamma)} \mathrm{Tr}^F[\gamma p_t^{X,F}(e^f p1, \gamma e^f p1)] r(f) df. \tag{2}$$

Here $r(f)$ is a Jacobian term relating the unimodular measure on $Z(\gamma)\backslash G$ and the Euclidean measure df on $\mathfrak{p}^\perp(\gamma)$. Note that the function $d(e^f p1, \gamma e^f p1)$ grows at least linearly along $f \in \mathfrak{p}^\perp(\gamma)$, together with the Gaussian estimates on heat kernels, the integral in (2) is always well-defined.

Bismut's hypoelliptic Laplacians. Bismut's construction begins with the deformation of a specific Dirac operator, which serves as a square root of $-\Delta^{X,F}$. The cubic Dirac operator introduced by Kostant is our candidate for this purpose. Another crucial component in the construction is the harmonic oscillator.

Let $U\mathfrak{g}$ denote the algebra of left-invariant differential operators on G, and let $c(\mathfrak{g})$, $\widehat{c}(\mathfrak{g})$ be the Clifford algebras of $(\mathfrak{g}, B)$ acting on $\Lambda^\bullet(\mathfrak{g}^*)$. Let $C^\mathfrak{g} \in U\mathfrak{g}$ be the Casimir associated to B. The Dirac operator of Kostant $\widehat{D}^\mathfrak{g}$ is an element in $\widehat{c}(\mathfrak{g}) \otimes U\mathfrak{g}$ so that there exists an explicitly defined real constant $c_\mathfrak{g}$ such that $\widehat{D}^{\mathfrak{g},2} = -C^\mathfrak{g} - c_\mathfrak{g}$. Note that when acting on $\mathscr{C}^\infty(X, F) = \mathscr{C}^\infty_K(G, E)$, we have the following relation of operators, $-\Delta^{X,F} = C^\mathfrak{g} - \rho^E(C^\mathfrak{k})$.

Let $Y = (Y^\mathfrak{p}, Y^\mathfrak{k})$ denote the tautological vector field of $\mathfrak{g}$. For $b > 0$, set the operator acting on $\mathscr{C}^\infty(G \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*) \otimes E)$,

$$\mathfrak{D}_b = \widehat{D}^\mathfrak{g} + \mathbf{i}c([Y^\mathfrak{k}, Y^\mathfrak{p}]) + \frac{1}{b}\big(d^\mathfrak{p} + Y^{\mathfrak{p}*}\wedge + d^{\mathfrak{p},*} + i_{Y^\mathfrak{p}}\big) + \frac{\mathbf{i}}{b}\big(-d^\mathfrak{k} - Y^{\mathfrak{k}*}\wedge + d^{\mathfrak{k},*} + i_{Y^\mathfrak{k}}\big), \tag{3}$$

where the last two big terms are the Witten Dirac operators as square roots of the harmonic oscillators. Put $N = G \times_K \mathfrak{k}$. Let $\widehat{\mathcal{X}}$ be the total space of $\widehat{\pi} : TX \oplus N \to X$, so that $\widehat{\mathcal{X}} \simeq X \times \mathfrak{g}$. After taking the K-quotients, we get the differential operators $\widehat{D}^{\mathfrak{g},X}$, $\mathfrak{D}_b^X$ acting on $\mathscr{C}^\infty(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$. Bismut's hypoelliptic Laplacian is defined as follows

$$\mathcal{L}_b^X = -\frac{1}{2}\widehat{D}^{\mathfrak{g},X,2} - \frac{1}{2}\rho^E(C^\mathfrak{k}) - \frac{1}{2}c_\mathfrak{g} + \frac{1}{2}\mathfrak{D}_b^{X,2}. \tag{4}$$

The main terms in $\mathcal{L}_b^X$ are the harmonic oscillator along the fibre of $TX \oplus N$ and the generator of geodesic flow on $TX \subset \widehat{\mathcal{X}}$. By Hörmander's result, $\mathcal{L}_b^X$ is hypoelliptic, and let $q_{b,t}^X((x,Y),(x',Y'))$ denote its heat kernel. Let $\mathbf{P}$ be the projection from $\Lambda^\cdot(T^*X \oplus N^*) \otimes F$ onto $\Lambda^0(T^*X \oplus N^*) \otimes F$. Then as $b \to 0$, we have [2, Theorem 4.5.2]

$$q_{b,t}^X((x,Y),(x',Y')) \to \mathbf{P}p_t^{X,F}(x,x')\pi^{-\dim\mathfrak{g}/2}\exp(-\frac{1}{2}\big(|Y|^2 + |Y'|^2\big))\mathbf{P}. \tag{5}$$

Moreover, we have the Bianchi identity $[\mathfrak{D}_b^X, \mathcal{L}_b^X] = 0$.

Compute the orbital integrals: hypoelliptic deformations. We extend (2) to the hypoelliptic orbital integrals $\mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X))]$ by the following formula:

$$\int_{\mathfrak{p}^\perp(\gamma)} \Big[\int_{TX\oplus N} \mathrm{Tr_s}^{\Lambda^\cdot(T^*X\oplus N^*)\otimes F}[\gamma q_{b,t}^X((e^f p1, Y), \gamma(e^f p1, Y))]dY\Big] r(f) df. \tag{6}$$

Then by (5) and the Bianchi identity, we obtain that for $t > 0$, $b > 0$,

$$\mathrm{Tr}^{[\gamma]}[\exp(t\Delta^{X,F}/2)] = \mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X)] \tag{7}$$

This identity is an analogy with the McKean-Singer theorem in index theory, where the vanishing of the b-derivative of $\mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X)]$ is ensured by the Bianchi identity. The subsequent step involves computing the limit of the right-hand side of (7) as $b \to +\infty$. As explained in Bismut's talk, $\exp(-t\mathcal{L}_b^X)$ concentrates towards the geodesic flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on TX. Consequently, the integrand in (6) concentrates around points $(x, Y^{TX}) \in TX$ for which $\varphi_t(x, Y^{TX}) = \gamma(x, Y^{TX})$, leading to $x \in X(\gamma)$. Then Bismut employed techniques from local index theory near such points to derive an explicit formula for $\mathrm{Tr}^{[\gamma]}[\exp(t\Delta^{X,F}/2)]$, which can be described as follows.

Theorem 1. [2, Theorem 6.1.1] *Set $p = \dim \mathfrak{p}(\gamma)$, $q = \dim \mathfrak{k}(\gamma)$. For any $t > 0$,*

$$\begin{aligned}&\mathrm{Tr}^{[\gamma]}[\exp(t\Delta^{X,F}/2)] = \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}}\times \\ &\int_{\mathfrak{k}(\gamma)} J_\gamma(Y_0^{\mathfrak{k}})\mathrm{Tr}^E[\rho^E(k^{-1})\exp(-\mathbf{i}\rho^E(Y_0^{\mathfrak{k}})+tA)]e^{-|Y_0^{\mathfrak{k}}|^2/2t}\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}},\end{aligned} \tag{8}$$

where $A = \frac{1}{2}(\rho^E(C^{\mathfrak{k}}) + c_{\mathfrak{g}})$, and J_γ is an analytic function on given by

$$\begin{aligned}J_\gamma(Y_0^{\mathfrak{k}}) = &\frac{1}{|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}|^{1/2}}\frac{\widehat{A}(\mathbf{i}\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma)})}{\widehat{A}(\mathbf{i}\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(\gamma)})} \\ &\left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{z}_0^\perp(\gamma)}}\frac{\det(1-e^{-\mathbf{i}\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}))|_{\mathfrak{k}_0^\perp(\gamma)}}{\det(1-e^{-\mathbf{i}\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}))|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{\frac{1}{2}}.\end{aligned}$$

Building upon Bismut's seminal work on orbital integrals, there have been several extensions developed in various contexts: [3](orbital integrals for eta invariants), [5] (twisted orbital integrals), [4](actions of center elements of $U\mathfrak{g}$), etc.

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Quantum Limits for Sub-Elliptic Operators

Veronique Fischer

The talk reports on the progress on the project with the same title [10] funded by The Leverhulme Trust. It is a joint work with Clotilde Fermanian-Kammerer (Co-I), and the research associates Steven Flynn and Søren Mikkelsen. It aims at developing the semiclassical and microlocal analysis for subelliptic operators such as sub-Laplacians.

One aspect of the project is to show the existence of (the analogues of) microlocal defect measures and semiclassical measures in sub-Riemannian or sub-elliptic contexts. This was obtained on graded nilpotent Lie groups and on nilmanifolds in the microlocal case [2] and in the semiclassical case [1, 9]. It has required a deeper understanding of the Euclidean / Riemannian / abelian cases as well as a viewpoint from C^*-algebras and representation theory to tackle the non-commutativity. This has led to a re-definition of quantum limits as states of C^*-algbebras of spaces of symbols [7].

This approach has already allowed us to tackle many problems on nilmanifolds such as observability [3], the study of zeta functions for sub-Laplacians [8] and more general semi-classical questions [9]. It also gives a framework where the invariance properties of quantum limits for sub-Laplacians can be analysed. So far, they have been obtained for the semi-classical Schrödinger equation on H-type groups [4] and nilmanifolds of step 2 [6].

To date, the published works have been set on nilpotent Lie groups and nilmanifolds, apart from the preliminary work [5] on geometric invariance. We are turning our efforts towards generalising the pseudo-differential calculus and the subsequent notion of quantum limits to filtered manifolds.

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Four subelliptic pseudodifferential calculi

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(joint work with Elias M. Stein)

Two pseudodifferential calculi were introduced in [7] when a smooth distribution of tangent subspaces was prescribed on $\mathbb{R}^d$. A typical such distribution is given by the contact distribution on the Heisenberg group, but the theory allows also for other distributions; in particular, no curvature assumption on the distributions is necessary.

One of the pseudodifferential calculi in [7] is a purely non-isotropic algebra, corresponding to the parametrices of purely non-isotropic differential operators such as the sum of squares of degree 1 left-invariant vector fields on the Heisenberg group. Another is an algebra consisting of operators with mixed homogeneities, obtained by composing a purely non-isotropic operator with an isotropic operator that respects Euclidean (isotropic) dilations.

In more recent work, Stein and I studied generalizations of these two calculi, to a situation where instead of being given a smooth tangent distribution, one is given a smoothly varying quadratic form on the cotangent bundle of $\mathbb{R}^d$. This allows one to study projections of pseudodifferential operators in our previous calculi, and understand for instance operators associated to the Grushin vector fields $\{\partial_x, x\partial_t\}$ on $\mathbb{R}^2$. The purely non-isotropic algebra actually goes all the way back to classical work of Nagel and Stein [2]; we offer a slight reformulation which paves the way to the algebra with mixed homogeneities in this context. The latter is a variable coefficient generalization of the singular integrals in [4]. We show that our algebra with mixed homogeneities is closed under composition, are bounded on L^p for $1 < p < \infty$, and exhibit underlying geometric structures that allows us to describe the Schwartz kernels for such pseudodifferential operators. The details will appear in a forthcoming article (in preparation). Some earlier related work include [1, 3, 5, 6].

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Relative heat content asymptotics for sub-Riemannian manifolds

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(joint work with Andrei Agrachev, Luca Rizzi)

We study the small-time asymptotics of the relative heat content in sub-Riemannian manifolds. A sub-Riemannian manifold is a triple $(M, \mathcal{D}, g)$ where M is a smooth connected manifold, $\mathcal{D}$ is a generalized distribution and g is a smoothly-varying inner product on $\mathcal{D}$. By generalized distribution, we mean that there exists a *generating frame*, namely a family of N smooth global vector fields $\{X_1, \dots, X_N\}$ such that

$$\mathcal{D}_x = \mathrm{span}\{X_1(x), \dots, X_N(x)\} \subseteq T_xM, \qquad \forall\, x \in M. \tag{1}$$

Moreover, we assume that the distribution satisfies the *Hörmander condition*, i.e. the Lie algebra of smooth vector fields generated by $\mathcal{D}$, evaluated at the point x, coincides with T_xM, for all $x \in M$.

In this setting, letting ω be a smooth measure on M, we can define the *sub-Laplacian* as the operator $\Delta := \mathrm{div}_\omega \circ \nabla$, acting on $C^\infty(M)$. We may write its expression with respect to a generating frame (1), obtaining

$$\Delta f = \sum_{i=1}^{N} \left\{ X_i^2(f) + X_i(f)\mathrm{div}_\omega(X_i) \right\}, \qquad \forall\, f \in C^\infty(M). \tag{2}$$

Since $\mathcal{D}$ satisfies the Hörmander condition, Δ is hypoelliptic.

Definition 1 (Relative heat content)**.** *Let $\Omega \subset M$ be an open relatively compact set with smooth boundary and let $u(t,x)$ be the solution to the Cauchy problem for the heat equation on Ω, that is*

$$\begin{aligned} (\partial_t - \Delta)\, u(t,x) = 0, &\qquad \forall (t,x) \in (0,\infty) \times M, \\ u(0,\cdot) = \chi_\Omega, &\qquad \text{in } L^2(M,\omega). \end{aligned} \tag{3}$$

We define the relative heat content*, associated with Ω, as*

$$H_\Omega(t) = \int_\Omega u(t,x)\, d\omega(x), \qquad \forall\, t > 0. \tag{4}$$

The relative heat content represents the total amount of heat contained in Ω at time t, assuming that the heat can flow also outside the set Ω.

This quantity has been studied in connection with geometric properties of subsets of $\mathbb{R}^n$, starting from the seminal work of De Giorgi [2], where he introduced the notion of perimeter of a set in $\mathbb{R}^n$ and proved a characterization of sets of finite perimeter in terms of the first-order relative heat content small-time asympotics. In [4], the authors extended De Giorgi's result, obtaining an asymptotic expansion of order 3 in $\sqrt{t}$, assuming the boundary of $\Omega \subset \mathbb{R}^n$ to be a $C^{1,1}$ set. For simplicity, we state here the result of [4, Thm. 1.1] assuming $\partial\Omega$ is smooth:

$$\begin{aligned} (5) \quad H_\Omega(t) = |\Omega| - \frac{1}{\sqrt{\pi}}\mathrm{Per}(\Omega)t^{1/2} \\ + \frac{(n-1)^2}{12\sqrt{\pi}}\int_{\partial\Omega}\left(H^2_{\partial\Omega}(x) + \frac{2}{(n-1)^2}c_{\partial\Omega}(x)\right)d\mathrm{Per}(x)t^{3/2} + o(t^{3/2}), \end{aligned}$$

where $|\cdot|$ is the Lebesgue measure and Per is the perimeter measure in $\mathbb{R}^n$. Here $H = \mathrm{Tr}(II)$ and $C = \mathrm{Tr}(II^2)$, having denoted by II the second fundamental form of $\partial\Omega$. In the Riemannian setting, Van den Berg and Gilkey in [3] proved the existence of a complete asymptotic expansion for $H_\Omega(t)$, generalizing (5), when $\partial\Omega$ is smooth. Moreover, they were able to compute explicitly the coefficients of the expansion up to order 4 in $\sqrt{t}$.

In this talk, we discuss how we can extend (5) to the sub-Riemannian setting. Under the additional assumption of not having characteristic points, we prove the existence of the asymptotic expansion of $H_\Omega(t)$, up to order 4 in $\sqrt{t}$, as $t \to 0$ (including also the rank-varying case). Recall that, for a subset $\Omega \subset M$ with smooth boundary, $x \in \partial\Omega$ is a characteristic point if $\mathcal{D}_x \subset T_x(\partial\Omega)$.

Theorem 1. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset whose boundary is smooth and has no characteristic points. Then, as $t \to 0$,*

$$(6)\quad H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2} - \frac{1}{12\sqrt{\pi}}\int_{\partial\Omega}\left(N(\Delta\delta) - 2(\Delta\delta)^2\right)d\sigma\, t^{3/2} + o(t^2),$$

where σ denotes the sub-Riemannian perimeter measure, $\delta\colon M \to \mathbb{R}$ is the sub-Riemannian signed distance function from $\partial\Omega$ and, for $\phi \in C^\infty(M)$ with compact support close to $\partial\Omega$,

$$(7)\qquad N\phi := 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta,$$

Remark 1. The compactness assumption in Theorem 1 is technical and can be relaxed by requiring, instead, global doubling of the measure and a global Poincaré inequality. Some notable examples satisfying these assumptions are:

- M is a Lie group with polynomial volume growth, the distribution is generated by a family of left-invariant vector fields satisfying the Hörmander condition and ω is the Haar measure. This family includes also Carnot groups.
- $M = \mathbb{R}^n$, equipped with a sub-Riemannian structure induced by a family of vector fields $\{Y_1, \dots, Y_N\}$ with bounded coefficients together with their derivatives, and satisfying the Hörmander condition.

- M is a complete Riemannian manifold, equipped with the Riemannian measure, and with non-negative Ricci curvature.

In all these examples, Theorem 1 holds.

The strategy of the proof of Theorem 1 follows a similar strategy of [6], inspired by the method introduced in [5], used for the classical heat content. However, new technical difficulties arise, the main one being related to the fact that the small-time behavior of $u(t,\cdot)|_{\partial\Omega}$ is not known. At order zero, we obtain the following result, which is of independent interest.

Theorem 2. *Let M be a sub-Riemannian manifold, equipped with a smooth measure ω and let $\Omega \subset M$ be an open relatively compact subset, whose boundary is smooth and has no characteristic points. Let $x \in \partial\Omega$ and consider a chart of privileged coordinates $\psi\colon U \to V \subset \mathbb{R}^n$ centered at x, such that $\psi(U\cap\Omega) = V\cap\{z_1 > 0\}$. Then,*

$$\lim_{t\to 0} u(t,x) = \int_{\{z_1>0\}} \hat{p}_1^x(0,z)d\hat{\omega}^x(z) = \frac{1}{2}, \qquad \forall\, x \in \partial\Omega, \tag{8}$$

where $\hat{\omega}^x$ denotes the nilpotentization of ω at x and $\hat{p}_t^x$ denotes the heat kernel associated with the nilpotent approximation of M at x and measure $\hat{\omega}^x$.

We conclude by mentioning that, in Carnot groups, a characterization of sets of finite horizontal perimeter in terms of the relative heat content asymptotics has been recently proved in [1], in the same spirit of [2]. In particular, such characterization is independent of the presence of characteristic points, suggesting that an asymptotic expansion such as (6) may still hold, dropping the non-characteristic assumption.

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Optimal geometric inequalities for capillary hypersurfaces

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(joint work with Liangjun Weng, Chao Xia)

We are interested in the study of hypersurfaces in a space form with boundary supported on a given umbilical hypersurface and intersecting at a constant angle. Such a hypersurface is called a capillary hypersurface. If the contact angle is $\pi/2$ we call it a free boundary hypersurface.

In my previous work joint with Chao Xia [6], we used the following conformal Killing vector

$$X_a = \langle x, a\rangle x - \frac{1}{2}(|x|^2+1)a, \qquad \text{for } a \in \mathbb{R}^{n+1}$$

to obtain a New Minkowski formula

$$\int_\Sigma \{\langle x, a\rangle - H_1\langle X_a, \nu\rangle\} dA = 0 \tag{1}$$

where H_1 is the *normalized* mean curvature and ν is the normal vector of Σ. With it we solved completely the uniqueness of stable CMC free boundary hypersurfaces

Any immersed stable free boundary CMC hypersurface is a spherical cap.

When $n = 2$ it was proved by Ros-Vergasta [4, 2]. The uniqueness holds true for capillary hypersurfaces in any space form with boundary supported on a given umbilical hypersurface. The new Minkowski formula (1) helps us to define a suitable curvature flow, a Guan-Li type flow [1], to study optimal geometric inequalities in [3], which are counterparts of classical Alexendrov-Fenchel inequalities for closed hypersurfaces.

More recently we are interested in capillary in the half-space. More precisely, we consider hypersurfaces in $\mathbb{R}^{n+1}_+$ with boundary supported on the hyperplane $\partial\mathbb{R}^{n+1}_+$. Let Σ be a compact manifold with boundary $\partial\Sigma$, which is properly embedded hypersurface into $\mathbb{R}^{n+1}_+$. In particular, $\text{int}(\Sigma) \subset \mathbb{R}^{n+1}_+$ and $\partial\Sigma \subset \partial\mathbb{R}^{n+1}_+$. Let $\widehat{\Sigma}$ be the bounded domain enclosed by Σ and the hyperplane $\partial\overline{\mathbb{R}}^{n+1}_+$. It is clear that the following relative isoperimetric inequality follows from the classical isoperimetric inequality

$$\frac{|\Sigma|}{|\mathbb{S}^n_+|} \geq \left(\frac{|\widehat{\Sigma}|}{|\mathbb{B}^{n+1}_+|}\right)^{\frac{n}{n+1}}, \tag{2}$$

where $\mathbb{B}^{n+1}_+$ is the upper half unit ball and $\mathbb{S}^n_+$ is the upper half unit sphere. As a domain in $\mathbb{R}^{n+1}$, $\widehat{\Sigma}$ has a boundary, which consists of two parts: one is Σ and the other, which will be denoted by $\widehat{\partial\Sigma}$, lies on $\partial\overline{\mathbb{R}}^{n+1}_+$. Both have a common boundary, namely $\partial\Sigma$. Instead of just considering the area of Σ, it is interesting to consider the following free energy functional

$$|\Sigma| - \cos\theta|\widehat{\partial\Sigma}| \tag{3}$$

for a fixed angle constant $\theta \in (0,\pi)$. The second term $\cos\theta|\widehat{\partial\Sigma}|$ is the so-called wetting energy in the theory of capillarity. If we consider to minimize this functional under the constraint that the volume $|\widehat{\Sigma}|$ is fixed, then we have the following optimal inequality, which is called the capillary isoperimetric inequality,

$$\frac{|\Sigma| - \cos\theta|\widehat{\partial\Sigma}|}{|\mathbb{S}^n_\theta| - \cos\theta|\widehat{\partial\mathbb{S}^n_\theta}|} \geq \left(\frac{|\widehat{\Sigma}|}{|\mathbb{B}^{n+1}_\theta|}\right)^{\frac{n}{n+1}}, \tag{4}$$

with equality holding if and only if Σ is homothetic to $\mathbb{S}^n_\theta$, namely a spherical cap with contact angle θ. Here $\mathbb{S}_\theta$, $\widehat{\partial\mathbb{S}_\theta}$, $\mathbb{B}^{n+1}_\theta$ are defined by

$$\mathbb{S}^n_\theta = \{x \in \mathbb{S}^n \mid \langle x, e_{n+1}\rangle > \cos\theta\}, \quad \mathbb{B}^{n+1}_\theta = \{x \in \mathbb{B}^{n+1} \mid \langle x, e_{n+1}\rangle > \cos\theta\},$$

$$\widehat{\partial\mathbb{S}^n_\theta} = \{x \in \mathbb{B}^{n+1} \mid \langle x, e_{n+1}\rangle = \cos\theta\},$$

where e_{n+1} the $(n+1)$-th standard basis in $\mathbb{R}^{n+1}_+$. For simplicity we denote

$$\mathbf{b}_\theta := \mathbf{b}^\theta_{n+1} := |\mathbb{B}^{n+1}_\theta|, \qquad \omega_\theta := \omega_{n,\theta} := |\mathbb{S}^n_\theta| - \cos\theta|\widehat{\partial\mathbb{S}^n}_\theta|. \tag{5}$$

In particular, it is easy to check that $(n+1)\mathbf{b}_\theta = \omega_\theta$. The proof of (4) is not trivial, which uses the spherical symmetrization.

The main objectives of this talk are considering the following problems

(1) To find suitable generalizations of the quermassintergals V_k for hypersurfaces with boundary supported on $\partial\overline{\mathbb{R}}^{n+1}_+$, which are closely related to the free energy (3).
(2) To establish the Alexandrov-Fenchel inequality for these new quermassintegrals.

To answer the first question, we have introduced the following new geometric functionals.

$$V_{0,\theta}(\widehat{\Sigma}) := |\widehat{\Sigma}|, \qquad V_{1,\theta}(\widehat{\Sigma}) := \frac{1}{n+1}(|\Sigma| - \cos\theta|\widehat{\partial\Sigma}|),$$

and for $1 \leq k \leq n$,

$$V_{k+1,\theta}(\widehat{\Sigma}) \;:=\; \frac{1}{n+1}\left(\int_{\S} H_k dA - \frac{\cos\theta\sin^k\theta}{n}\int_{\partial\Sigma} H^{\partial\Sigma}_{k-1} ds\right), \tag{6}$$

where $H^{\partial\Sigma}_{k-1}$ is the normalized $(k-1)$-th mean curvature of $\partial\Sigma \subset \mathbb{R}^n$.

Let $\Sigma_t \subset \overline{\mathbb{R}}^{n+1}_+$ be a family of smooth, embedded capillary hypersurfaces with a constant contact angle $\theta \in (0,\pi)$, which are given by the embedding $x(\cdot,t) : M \to \overline{\mathbb{R}}^{n+1}_+$ and satisfy

$$(\partial_t x)^\perp = f\nu,$$

for some speed function f. Then for $0 \leq k \leq n$,

$$\frac{d}{dt}V_{k,\theta}(\widehat{\Sigma_t}) = \frac{n+1-k}{n+1}\int_{\S_t} fH_k dA_t. \tag{7}$$

For $n \geq 2$, let $\Sigma \subset \overline{\mathbb{R}}^{n+1}_+$ be a convex capillary hypersurface with a constant contact angle $\theta \in (0, \frac{\pi}{2}]$, then there holds

$$\frac{V_{n,\theta}(\widehat{\Sigma})}{\boldsymbol{b}_\theta} \geq \left(\frac{V_{k,\theta}(\widehat{\Sigma})}{\boldsymbol{b}_\theta}\right)^{\frac{1}{n+1-k}}, \quad \forall\, 0 \leq k < n, \tag{8}$$

with equality if and only if Σ is $\mathbb{S}^n_\theta$.

When $n = 2$, (8) implies a Willmore inequality for capillary hypersurfaces with contact angle θ.

Willmore type inequality. Let $\Sigma \subset \overline{\mathbb{R}}^3_+$ be a convex capillary surface with a constant contact angle $\theta \in (0, \frac{\pi}{2}]$, then

$$\int_\Sigma H^2 dA \geq 4|\mathbb{S}^2_\theta|, \tag{9}$$

with equality if and only if Σ is a spherical cap.

Here $H = 2H_1$ is the ordinary mean curvature for surfaces and it is obvious that $H^2 \geq 4H_2$. When $n = 2$, (8) implies a Minkowski type inequality for convex capillary surfaces with boundary in $\overline{\mathbb{R}}^3_+$.

Minkowski inequality. Let $\Sigma \subset \overline{\mathbb{R}}^3_+$ be a convex capillary surface with a constant contact angle $\theta \in (0, \frac{\pi}{2}]$, then

$$\int_\Sigma H dA \geq 2\sqrt{\omega_{2,\theta}} \cdot (|\Sigma| - \cos\theta|\widehat{\partial\Sigma}|)^{\frac{1}{2}} + \sin\theta\cos\theta|\partial\Sigma|, \tag{10}$$

where $\omega_{2,\theta} = 3\boldsymbol{b}_\theta = \left(2 - 3\cos\theta + \cos^3\theta\right)\pi$. Moreover, equality holds if and only if Σ is a spherical cap.

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