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Tropical Methods in Geometry

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Abstract. The workshop *Tropical methods in geometry* was devoted to a wide discussion and exchange of ideas between the leading experts representing various points of view on the subject including tropical methods in symplectic and Lagrangian geometry, topology of real algebraic varieties and tropical homology, tropical methods in algebraic, Berkovich analytic and log geometries, refined tropical enumerative geometry and enriched counting, and algebraic geometry and matroids.

Mathematics Subject Classification (2020): 14Txx, 14-xx, 81Txx, 53Dxx, 57Mxx.

Introduction by the Organizers

The workshop *Tropical methods in geometry*, organized by Ilia Itenberg (Paris), Hannah Markwig (Tübingen), Kris Shaw (Oslo), and Ilya Tyomkin (Beer Sheva), took place at Oberwolfach during the week of May 14-19, 2023. Over 50 participants from all over the world attended the workshop, 45 of whom were on-site participants. The workshop included no Zoom talks, but the on-site talks have been broadcasted over Zoom and recorded for the sake of online participants. The workshop program consisted of 19 one-hour talks delivered by the leading experts and young participants, as well as 4 quarter-an-hour talks by the graduate students. Extended abstracts of the talks follow these introductory notes. The workshop concentrated on tropical methods in several areas of geometry. Below we briefly discuss the main topics that were covered.

Tropical methods in symplectic geometry. Tropical geometry is an important source of Lagrangian varieties. Abundant examples of exotic monotone Lagrangian tori in the complex projective plane, as well as in other toric del Pezzo surfaces, originated from the work of Galkin, also in collaboration with Usnich and Cruz-Morales. That work proved the Laurent phenomenon in a framework related to toric degeneration and conjectured that the corresponding Lagrangian tori are all symplectically different with a specific conjecture for their Floer potential. Combining some tropical and non-tropical methods, Vianna showed that the Lagrangian tori in question are indeed pairwise different by proving a weaker version of Galkin-Usnich conjecture (establishing that the Newton polygons of the Floer potentials are as predicted). A direct tropical translation of the original construction of Galkin was suggested by Galkin and Mikhalkin. The talk by Grigory Mikhalkin surveyed symplectic developments in the last decade and introduced more recent constructions. Diego Matessi's talk touched upon Lagrangian torus fibrations in the context of mirror symmetry and focused, in particular, on the topology of real Calabi-Yau manifolds.

Tropical methods in real geometry. Questions concerning the topology of real algebraic varieties (in particular, the question on isotopy classification of non-singular curves of given degree in the real projective plane) were already included by Hilbert in the first part of the 16th problem of his list but continue to be an active research direction in which tropical methods are particularly successful. The results on topology of real algebraic varieties can be (quite artificially) divided into two groups. The results of the first group are related to restrictions on the topology of real algebraic varieties. One of the goals is to obtain sharp upper bounds for individual Betti numbers (with $\mathbb{Z}/2\mathbb{Z}$-coefficients) of real algebraic varieties belonging to a given deformation family of complex varieties (e.g., in terms of Hodge numbers in the Kähler case). The second group deals with constructions of real algebraic varieties with prescribed topology. Viro's patchworking method and tropical geometry clearly belong to the most powerful approaches to these constructions. Johannes Rau's talk discussed patchworks of real algebraic varieties close to a smooth tropical limit. Lionel Lang discussed coordinates on the spaces of algebraic and tropical hypersurfaces with given boundaries constructed by measuring holes in the hypersurfaces.

Tropical methods in Berkovich analytic geometry. There is a tight relation between tropical and analytic geometries since tropical varieties appear as skeletons of analytic spaces, and the analytic spaces themselves can be represented as inverse limits of such skeletons. This relation allows one to apply tropical methods in analytic geometry and, in particular, define various analytic objects. Several years ago, Chambert-Loir and Ducros used this point of view to introduce Dolbeaut cohomology in the analytic setting. However, their cohomology theory behaved oddly compared to the classical one, as was observed by Jell. In his talk, Joe Rabinoff reported on recent advances in the development of the theory of weakly-smooth differential forms and Dolbeault cohomology on analytic curves that resolves some of the problems. On a slightly different note, Martin Ulirsch discussed the $P = W$ conjecture in the case of abelian varieties. In particular, he considered the analytic analog of the conjecture and explained the role of tropical vector bundles in its proof.

Tropical methods in algebraic geometry. Tropical geometry became a very useful and powerful tool in many problems in algebraic geometry and, in particular, in enumerative geometry. On the one hand, it provides a sort of linearization of algebra-geometric objects, whose combinatorial nature makes the treatment of such objects more accessible. On the other hand, it controls degenerations and deformations of the algebra-geometric objects, making it particularly useful in approaching questions about moduli spaces of curves and morphisms. In his talk, Bernd Siebert discussed recent advances in the logarithmic Gromov-Witten theory and Mirror Symmetry, where tropical geometry controls certain algebraic stacks playing a central role in the theory. Helge Ruddat concentrated on tropical geometry beyond the toric case, *i.e.*, for cases with less traditional log-structures. Dhruv Ranganathan presented results concerning the enumerative geometry of log double ramification cycles and its relations to tropical curve counting methods. Renzo Cavalieri introduced a new perspective on the traditional counts of covers satisfying fixed ramification data known as Hurwitz numbers and generalized to new enumerative invariants that show up in the context of spaces of log-differentials. Melody Chan discussed results about the cohomology of moduli spaces of tropical stable curves and their relations to the weight-zero part of the cohomology of classical moduli spaces of marked curves. Maria Angelica Cueto discussed applications of tropical methods to the theory of normal surface singularities.

Tropical methods in refined and quadratically enriched enumerative geometry. When we use tropical geometry to determine the number of plane curves satisfying certain constraints, we are dealing with exactly the same tropical curves, no matter whether we count complex or real curves. The tropical curves are only counted with a different lifting multiplicity, reflecting the number of complex, respectively, real curves that degenerate to a given tropical curve. Philosophically, one can say that tropical geometry is more or less independent of the field we work with. It is only the lifting behavior that differs. It is, therefore, not surprising that tropical geometry is an intriguing tool if one tries to combine counting results over various fields to obtain universal geometric counts. In his talk, Eugenii Shustin presented new results on real algebraic and refined tropical invariants for plane curves of higher genus. As a new impetus, the meeting showcased exciting new relations between tropical geometry and quadratically enriched counts of curves. The quest for such quadratically enriched counts is only about five years old, with influential, pioneering results by Kass-Wickelgren and Levine. Kirsten Wickelgren introduced quadratically enriched counts of nodal rational plane curves satisfying point conditions, and Sabrina Pauli discussed correspondence theorems relating these counts to the tropical world. Erwan Brugallé discussed complex and real versions of the Abramovich-Bertram formula relating the enumeration of curves on different surfaces and suggested possible generalizations to the case of quadratically enriched invariants. Thomas Blomme discussed refined counts of curves in abelian surfaces.

Tropical geometry and combinatorics. Matroids can be viewed as the building blocks of tropical varieties. They also play a role in the study of algebraic foundations of tropical geometry, presented in a talk by Diane Maclagan concerning tropical schemes and vector bundles. Finally, toric geometry and, in particular, the geometry of polytopes and their volume polynomials were at the center of attention of Karim Adiprasito's talk.

Talks by junior participants. The junior participants of the workshop, Edvard Aksnes, Aloïs Demory, Antoine Toussaint, and Uriel Sinichkin, all gave talks on their research on Wednesday evening.

We hope that the very intensive and substantial exchange of a broad spectrum of ideas during the workshop will stimulate further research in the variety of discussed problems, which still are far from being completely settled.

Workshop: Tropical Methods in Geometry

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Abstracts

Real algebraic and refined tropical invariants of positive genera in the plane

Eugenii Shustin

(joint work with Ilia Itenberg)

The talk was based on [2].

We review real rational enumerative invariants introduced by Mikhalkin [1] that enumerate real rational plane curves of degree d passing through $3d-1$ fixed real points on the coordinate axes, counted with Welschinger signs, equipped with complex orientations, and possessing a fixed Mikhalkin's quantum index. Their generating function yields tropical enumerative invariants directly related to the refined Block-Göttsche invariants.

Our goal was to extend these real algebraic invariants to positive genera and to study their tropical counterpart. To allow the existence of real enumerative invariants of positive genera, we impose even tangency conditions at real points on the boundary of the positive quadrant and adding extra fixed points outside the positive quadrant. In such a setting, we prove the existence of real algebraic enumerative invariants enumerating curves of any even degree and genus $g = 1$ or 2, counted with appropriately modified Welschinger-type signs, equipped with complex orientations, and possessing a given Mikhalkin's quantum index. The generating functions for these invariants yield refined tropical enumerative invariants which essintially differ from the Block-Göttsche invariants: namely, the weights of the counted tropical curves are no longer product of contributions of the vertices but contain factors related to the whole cycles of the tropical curve. For exmple, for curves of degree 4 having fixed quadratic tangency points along the boundary of the positive quadrant, the Block-Göttsche type invariant appears to be

$$(y^2 - y^{-2})^2(y - y^{-1})^4$$

while our invariant equals

$$3(y^2 - y^{-2})^2((y^4 + y^{-4}).$$

Furthermore, we show that there are no such real enumerative invariants for genera ≥ 3. However, we expect the existence of an infinite series of refined tropical enumerative invariants of arbitrary degree and genus that extend the above elliptic and genus two tropical invariants.

We also discuss a relation of the real enumerative invariants relative to the coordinate exes to the commutator formulas in the theory of quantum tropical vertex.

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Tropical curve counting and double ramification cycles

Dhruv Ranganathan

(joint work with Ajith Urundolil Kumaran)

In this talk, I described the relationship between tropical and logarithmic curve counting and recent developments concerning the double ramification cycle. The main result, joint with Ajith Urundolil Kumaran, is a complete solution to the logarithmic Gromov–Witten (GW) theory of toric varieties, relative to the full toric boundary. The solution is in terms of the intersection theory of logarithmic tautological classes, and is not practically implementable. However, in recent work of Kennedy-Hunt, Shafi, and Urundolil Kumaran, a simple tropical correspondence theorem is proved in special cases that gives a link to refined tropical curve counting.

1. The problem

We fix a toric variety X with toric boundary divisor D, itself a union of components $D_1, \ldots, D_k$. In logarithmic GW theory, one is interested in curves in X that meet D with fixed tangency data. Precisely, we study maps of pairs:

$$(C, p_1, \ldots, p_n) \xrightarrow{f} (X, D),$$

from smooth pointed curves to X, and fix:

- the genus g of C,
- the curve class β for $f_\star[C]$ in $H_2(X)$,
- matrix of tangency orders c_{ij} of p_i along D_j.

When (X, D) is a toric pair, there are a couple of simplifications. First, the curve class β is determined by the matrix of tangency orders; we will still keep the symbol β reserved for this curve class when we need it. Second, if we make an identification of the cocharacter lattice the dense torus with $\mathbb{Z}^r$, then at each marked point, we can record a point in this lattice. Precisely, if the point p_i has positive tangency order with some subset of divisors, it picks out a cone in the fan Σ, dual to the intersection of these. The multiplicities then give a lattice point in this cone.

Putting these vectors together, in this setting, we will denote this $r \times n$ matrix by the symbol Λ. Logarithmic GW theory gives rise to a proper Deligne–Mumford stack $\mathsf{M}_\Lambda(X|D)$ parameterizing such maps, and "logarithmic degenerations". We will not say too much here about the nature of these degenerate objects; the details can be found in the foundational papers of Abramovich–Chen and Gross–Siebert [1, 4, 5]. An important feature of the space is that every point in $\mathsf{M}_\Lambda(X|D)$ determines two things:

- A stable map from an n-pointed nodal curve $C \to X$, of class β and
- A *tropical map* $\Gamma \to \mathbb{R}^n$, i.e. a piecewise linear map from a metric graph to $\mathbb{R}^r$, enhancing the dual graph of C.

The data have to be compatible in various ways, which can be found in the original sources. For now, we encourage the reader to take $\mathsf{M}_\Lambda(X|D)$ to be a "good compactification" of the space of tangent curves described above.

Associated to the moduli space $\mathsf{M}_\Lambda(X|D)$ are certain tautological structures. First, at every marked point p_i, we can evaluate the stable map to obtain:

$$\mathsf{ev}_i : \mathsf{M}_\Lambda(X|D) \to X.$$

We can put these together to form

$$\mathsf{ev} : \mathsf{M}_\Lambda(X|D) \to \mathsf{Ev}_\Lambda(X).$$

The target space is, to first approximation, the product of n copies of X, though it is often useful to refine this. There is also a tautological map to the moduli space of curves:

$$\pi : \mathsf{M}_\Lambda(X|D) \to \overline{\mathsf{M}}_{g,n}.$$

Finally, the space $\mathsf{M}_\Lambda(X|D)$ has a "virtual class". It usually takes some technical machinery to say what this means, however this particular case we are lucky. For $(X|D)$ toric, it turns out that there is a canonical expression of $\mathsf{M}_\Lambda(X|D)$ as an intersection of two smooth schemes inside of a third scheme, each of predictable dimension see [11]. As a consequence, the space $\mathsf{M}_\Lambda(X|D)$ has a distinguished class in Chow homology. That is, if the expression is

$$\mathsf{M}_\Lambda(X|D) = \mathsf{M}_1 \cap \mathsf{M}_2 \text{ inside } \mathsf{B},$$

we can define $[\mathsf{M}_\Lambda(X|D)]^{\mathsf{vir}}$ to be the refined intersection class, in the sense of Fulton–Macpherson. The homology class lives in the "expected" or "virtual dimension":

$$\mathsf{vdim}\ \mathsf{M}_\Lambda(X|D) = (r-3)(1-g) + n.$$

The goal of logarithmic GW theory is, in some sense, to calculate the classes

$$\pi_\star\left(\mathsf{ev}^\star\gamma \cap [\mathsf{M}_\Lambda(X|D)]^{\mathsf{vir}}\right).$$

Special interest is paid to the intersection numbers of these classes with the ψ-classes of the moduli space $\overline{\mathsf{M}}_{g,n}$. The pushforwards are called *logarithmic Gromov–Witten classes*, while the numbers are called *logarithmic Gromov–Witten invariants*.

2. The main result and some specializations

The Chow ring of $\overline{\mathsf{M}}_{g,n}$ contains a subring known as the *tautological ring*. It includes two sets of classes in particular: substacks paramaterizing curves of fixed topological type (e.g. the singular curves) and Chern classes ψ_i of the cotangent line bundles.

The main theorem proved with Urundolil Kumaran is the following:

Main Theorem. *All logarithmic GW classes of (X, D) lie in the tautological ring of the moduli space of curves $\overline{\mathsf{M}}_{g,n}$.*

The proof of the theorem is effective: it actually produces an expression that calculates any such class in terms of the standard generators of the tautological ring. It also therefore gives the first complete method for calculating all logarithmic GW invariants of (X, D).

The main new input in the theorem is a method to reduce such calculations to a variant of the double ramification cycle; see [6] for an introduction. New methods in logarithmic intersection theory, developed with Molcho, play the key role [10].

3. Specializations, and refined curve counting

In the stated generality, it is not really practical to "do calculations" in this way. But let us conclude by explaining how the result comes alive when specializing to special geometries and special sectors.

Before doing this, we point out the terminology for two types of GW invariants. The *primary* invariants are those obtained by taking degrees of classes of the form $\pi_\star\left(\mathsf{ev}^\star\gamma \cap [\mathsf{M}_\Lambda(X|D)]^{\mathsf{vir}}\right)$.The *descendant GW invariants* are obtained by first capping $\pi_\star\left(\mathsf{ev}^\star\gamma \cap [\mathsf{M}_\Lambda(X|D)]^{\mathsf{vir}}\right)$ with a polynomial in the classes ψ_i on $\overline{\mathsf{M}}_{g,n}$, and then taking degree.

(1) When $r = 2$, the primary log GW invariants are computed by Mikhalkin's tropical correspondence theorem [9].
(2) When $g = 0$, the primary invariants are calculated by tropical correspondence theorems, by Nishinou–Siebert, and the descendants were treated by Mandel–Ruddat [8, 7].
(3) When $r = 3$, but the matrix Λ has rank 2, without descendants, Bousseau has shown that these invariants can be again computed by tropical correspondence using Block–Göttsche's refined multiplicities [3].

In recent work, Kennedy-Hunt, Shafi, and Urundolil Kumaran show that the main theorem rapidly specializes to all these results using intersection theory on $\overline{\mathsf{M}}_{g,n}$, quite different from the original proofs.

They also use the result to give a geometric interpretation of Blechman–Shustin's refined descendant tropical curve counting, which had earlier been defined purely combinatorially [2].

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Quadratically enriched Gromov–Witten invariants

Kirsten Wickelgren

(joint work with Jesse Leo Kass, Marc Levine, Jake Solomon)

Let S denote a projective variety over the complex numbers. Gromov–Witten invariants count curves on S in a given homology class through certain cycles. For example, the number of complex degree d rational plane curves passing through $3d-1$ points is such an invariant. It is independent of the generally chosen points over the complex numbers. (There is 1 line through 2 points, 1 conic through 5, 12 rational degree 3 curves through 8...)

Let $\bar{M}_{0,n}(S,D)$ denote the Kontsevich moduli space of stable maps $u : C \to S$ of degree D from an n-marked genus 0 curve C to S. These Gromov–Witten invariants (among others) can be interpreted as a degree of the evaluation map

$$ev : \bar{M}_{0,n}(S,D) \to S^n$$

from Kontsevich moduli space to S^n which sends $(u : C \to S, p_1, \ldots, p_n)$ to $\prod_i u(p_i)$. This implies the above claimed independence on the choice of the points.

We develop an $\mathbb{A}^1$-degree [1], following Morel's theorem on the $\mathbb{A}^1$-degree of a map between spheres over a field k. We assign a degree to a map $f : X \to Y$ of smooth k-schemes under appropriate hypotheses. This degree is valued in the Grothendieck–Witt group $GW(k)$ of k, defined to be the group completion of isomorphism classes of nondegenerate, symmetric bilinear forms over k. These hypotheses notably include appropriate orientation data on f and Y being $\mathbb{A}^1$-connected. (One can drop the hypothesis that Y is $\mathbb{A}^1$-connected. This results in the degree being valued in the Grothendieck–Witt sheaf applied to the $\mathbb{A}^1$-connected components of Y. Unfortunately, this group is likely large and difficult to compute for Y not $\mathbb{A}^1$-connected.) We construct the needed orientation data on ev for certain del Pezzo surfaces S and degrees D (see [2]) to obtain

$$N_{S,D} = \deg^{\mathbb{A}^1} ev \in GW(k).$$

This gives a count of genus 0 curves on S through the appropriate number of k-rational points. This count is valid for any perfect field k of characteristic not 2 or 3.

One can eliminate the restriction that the marked points be rational, instead choosing a list of extensions $\sigma = (L_1, \ldots, L_r)$ of k for their residue fields, provided $\sum [L_i : k] = -K_S \cdot D - 1$. The evaluation map is then twisted by an action associated to σ defining

$$ev_\sigma : \bar{M}_{0,n}(S, D) \to \prod_{i=1}^{n} Res_{L_i/k} S$$

We show there is a degree

$$N_{S,D,\sigma} = \deg^{\mathbb{A}^1} ev_\sigma \in GW(k),$$

resulting in the count of genus 0 curves on S through generally chosen points $p_1, \ldots, p_r$ with residue fields $k(p_i) = L_i$. (Generally chosen means this holds for rational points of an open subset U of $\prod_{i=1}^n Res_{L_i/k} S$. Although there may not be any rational points, the construction is stable under pullback and one obtains a modified but similar count for any closed point of U, giving infinitely many elementary counts equal to a pullback of $N_{S,D,\sigma}$. Alternatively, one may simply view the degree as a virtual count.)

For example, let $S_0 = \{x^2y + y^2z + z^2w + w^2x = 0\} \subset \mathbb{P}^3$. Then

$$N_{S_0, -K_{S_0}, \sigma} = \langle 5 \rangle + \langle 1 \rangle + 4(\langle 1 \rangle + \langle -1 \rangle) + \sum_{i=1}^{r} Tr_{L_i/k} \langle 1 \rangle.$$

Here $Tr_{L_i/k}\langle 1 \rangle$ denotes the class in $GW(k)$ of the trace form of the étale algebra $k \subset L_i$.

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Topology of real Calabi-Yau manifolds via mirror symmetry

Diego Matessi

We present joint work in progress with A. Renaudineau where we expect to refine results in [5], [1], [3] and [4]. Given a real Calabi-Yau manifold X with a mirror $\check{X}$, the goal is to compute the $\mathbb{Z}/2$ cohomology of the real part $X_{\mathbb{R}}$ in terms of the $\mathbb{Z}/2$-cup product in the mirror $\check{X}$. We consider two versions of mirror symmetry. In the first one, mirror Calabi-Yau varieties arise from dual reflexive polytopes, where the real Calabi-Yau is constructed via Viro's patchworking. In the second, a mirror pair is described as a pair of dual Lagrangian torus fibrations (from the SYZ conjecture) and the real Calabi-Yau is the fixed point set of a fibre preserving anti-symplectic involution.

Let P and $\check{P}$ be two dual reflexive $n+1$ dimensional polytopes. Assume that the boundaries of both polytopes admit unimodular subdivisions. These induce central subdivisions of both polytopes with respect to their unique interior points. By taking cones, these subdivisions also describe fans Σ and $\check{\Sigma}$ giving toric varieties Y and $\check{Y}$. Inside of them let X and $\check{X}$ respectively be anticanonical hypersurfaces. These give a pair of mirror Calabi-Yau varieties. Notice that $\check{P}$ is the Newton polytope of X. Take one of them, e.g. X, to be real and constructed using Viro's patchworking with respect to the given subdivision of $\check{P}$. This amounts to choosing signs for each vertex. Notice that boundary vertices correspond to rays of the fan $\check{\Sigma}$, and therefore to toric divisors in $\check{Y}$. Let D be the divisor, with coefficients in $\mathbb{Z}/2$, corresponding to the rays with the same sign as the interior vertex.

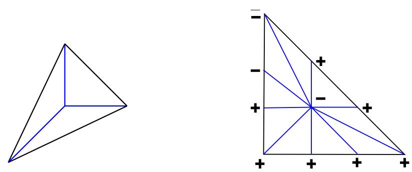

Figure 1. Reflexive polytopes

As a toy example consider the two polytopes in Figure 1, where P is on the left and $\check{P}$ on the right. In this case X and $\check{X}$ are both elliptic curves. The divisor corresponding to the choice of signs is $D = D_1 + D_2$, where D_1 and D_2 correspond to the two rays labeled by a minus sign. We claim that the real part $X_{\mathbb{R}}$ is connected if and only if the divisor class of D has non zero intersection with the mirror $\check{X}$. Figure 2 shows $X_{\mathbb{R}}$ on the left and the intersection of the divisor with $\check{X}$ on the right. The divisor is represented by the dashed edges in the moment polytope of $\check{Y}$. Indeed the real part is connected and the divisor has intersection 1 with $\check{X}$. We conjecture and hope to prove, that this is true in all dimensions.

Let us now describe mirror symmetry via Lagrangian torus fibrations. Most of the known results hold for complex dimension $n = 3$, so we restrict to this case. Given a symplectic manifold X, a Lagrangian fibration $f : X \to B$ is a map onto a manifold B whose generic fibre is a Lagrangian submanifold. Let X_0 be the

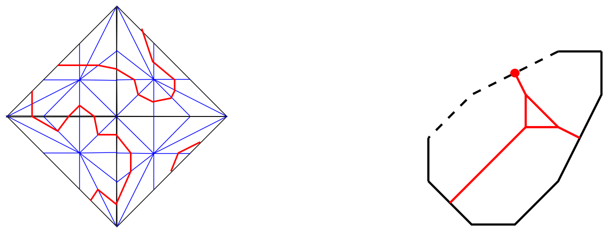

Figure 2. Connectedness and intersection with divisors

union of all smooth fibres, with immage $B_0 \subset B$. It follows from the Arnold-Liouville theorem that if f is proper and admits a Lagrangian section, then X_0 is symplectomorphic to T^*B_0/Λ, where $\Lambda \subset T^*B_0$ is a lattice. In particular, all fibres are tori. In the SYZ description of mirror symmetry, a pair of mirror Calabi-Yau manifolds should admit dual Lagrangian torus fibrations. Duality means that the mirror $\check{X}$ can be obtained by taking the dual fibration $TB_0/\check{\Lambda}$ on the tangent bundle and then compactifying it by gluing singular fibres. Such fibrations have been constructed topologically by Gross [6] for many examples of mirror Calabi-Yau threefolds. A real structure is given by an involution on T^*B_0/Λ which, on each fibre, acts as $[y] \mapsto [-y]$ and extending it to singular fibres [2]. This particular involution fixes the zero section and therefore the real part $X_{\mathbb{R}}$ has a connected component diffeomorphic to B. There are twisted versions of this involution which do not fix a section [2]. These involve the choice of a Lagrangian section τ which is not twice another section. The space parametrizing the twists is $H^1(B, \Lambda \otimes \mathbb{Z}/2)$. The papers [2], [1] and [4] study the cohomology of $X_{\mathbb{R}}$ by linking it with the cohomology of X. The idea is to compare the Leray spectral sequence of $f : X \to B$, with its restriction to the real part. The former sequence was studied by Gross and it involves the following sheaves on B

$$\mathcal{R}^q = j_* \check{\Lambda}^{\wedge q} \otimes \mathbb{Z}/2$$

where $\check{\Lambda}$ is the dual lattice and $j : B_0 \to B$ is the inclusion. When comparing with the restriction to $X_{\mathbb{R}}$ it turns out that there are sheaves $\mathcal{L}^1$ and $\mathcal{L}^2$ on B inducing the following long-exact sequence in cohomology

$$(1) \qquad ... \to H^k(B, \mathcal{L}^1) \longrightarrow H^k(X_{\mathbb{R}}, \mathbb{Z}/2) \longrightarrow H^k(B, \mathcal{L}^2) \xrightarrow{\beta} H^{k+1}(B, \mathcal{L}^1)...$$

and homomorphisms

$$H^k(B, \mathcal{L}^1) \to H^k(B, \mathcal{R}^1) \quad \text{and} \quad H^k(B, \mathcal{R}^2) \to H^k(B, \mathcal{L}^2).$$

Hence we get a morphism $\beta' : H^1(B, \mathcal{R}^2) \to H^2(B, \mathcal{R}^1)$ which determines the cohomology of $X_{\mathbb{R}}$. We now apply mirror symmetry. Indeed we can consider the dual sheaves

$$\check{\mathcal{R}}^q = j_* \Lambda^{\wedge q} \otimes \mathbb{Z}/2$$

which determine the cohomology of the mirror $\check{X}$. It was shown by Gross that $\check{\mathcal{R}}^q \cong \mathcal{R}^{n-q}$. In particular the map β' becomes

$$\beta' : H^1(B, \check{\mathcal{R}}^1) \to H^2(B, \check{\mathcal{R}}^2).$$

When both X and $\check{X}$ are simply connected and B is a sphere, Gross shows that $H^q(B, \check{\mathcal{R}}^q) \cong H^{2q}(\check{X}, \mathbb{Z}/2)$. So that the cohomology of $X_{\mathbb{R}}$ is determined by the map β' on the even cohomology of $\check{X}$. Arguz and Prince [1] proved that in the untwisted case this map is the cup product on divisor classes, i.e. $\beta' : D \mapsto D^2$. In [4] we extend this to the twisted case. Indeed the twist τ can be seen as a divisor class $L_\tau \in H^1(B, \check{\mathcal{R}}^1)$ in the mirror. Then, in this case, $\beta' : D \mapsto D^2 + DL_\tau$.

As an application, in [4] we find an example of a connected real quintic $X_{\mathbb{R}}$ of type $M-2$, i.e. such that the Smith-Thom inequality becomes

$$\sum_k b_k(X_{\mathbb{R}}) = \sum_k b_k(X) - 4.$$

This is given by a divisor L in the mirror such that $D^2 + DL = 0$ for all divisor classes D. This particular divisor is hinted at in Figure 3.

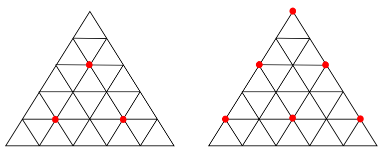

FIGURE 3. A divisor giving an $(M-2)$ real quintic (a hint).

If we go back to the patchworking construction, we can take the Newton polytope of a quintic in $\mathbb{P}^4$. The red dots indicate the rays on two dimensional faces with the same sign as the interior point. Does patchworking give the same real quintic? To prove this we plan to extend the above results by working directly on the patchworking with reflexive polytopes. One of the major steps is to replace the sheaves $\mathcal{R}^q$ with tropical homology and the sequence (1) with the Renaudineau-Shaw spectral sequence. The latter has been defined and used successfully in [5] to bound the Betti numbers of $X_{\mathbb{R}}$ with the Hodge numbers of X.

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Toric and tropical method in symplectic geometry

Grigory Mikhalkin

The talk is a survey of symplectic developments in the last decade inspired by the seminal paper of Galkin and Usnich [1] where the pioneering idea of using toric degenerations from [2] in symplectic geometry was introduced. Namely, in their 2010 paper, Galkin and Usnich have described mutations of toric fans resulting from mutations of Markov triples $(a, b, c) \in \mathbb{Z}^3$ with $a^2+b^2+c^2 = 3abc$. They have conjectured that these triples produce non-equivalent Lagrangian tori in $\mathbb{CP}^2$ (the proof was readily provided independently by several groups of authors shortly after [1] appeared), they have also conjecturally described the corresponding Maslov 2 disk potential, but perhaps even more importantly, they have revealed the strength of toric symplectic constructions based on non-Delzant polygons. More recent constructions based on these ideas include the constructions of knotted symplectic cube embeddings by Brendel, Mikhalkin and Schlenk, as well as very recent examples of symplectically equivalent, but Hamiltonianly non-isotopic Lagrangian tori in $S^2 \times S^2$ by Hind, Mikhalkin and Schlenk.

It turns out that the underlying planar constructions have intrinsic meaning in terms of *tropical planimetry*, i.e. the 2-dimensional geometric structure, invariant with respect to invertible linear transformations over $\mathbb{Z}$ as well as all real translations. From the point of view of algebraic geometry, this structure can be thought as the one coming from tropical addition and multiplication. Namely, we have a distinguished class of lines, those that have rational slopes and distances between points within such lines. The tropical angles are more interesting, as they can no longer be characterized by a single number. The corresponding tropical trigonometry is non-trivial, and turns out to be responsible for exotic symplectic behavior in $\mathbb{CP}^2$, $S^2 \times S^2$, as well as their blowups.

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A quadratically enriched correspondence theorem

Sabrina Pauli

(joint work with Andrés Jaramillo Puentes)

In this talk I presented work in progress joint with Andrés Jaramillo Puentes on a quadratically enriched correspondence theorem.

Let N_d be the number of rational degree d complex curves in $\mathbb{P}^2$ passing through a generic configuration of $3d-1$ points. This number is independent of the choice of points. However, this invariance breaks down if one counts curves which are defined over a non-algebraically closed field k. For example, there can be 8, 10 or

12 plane rational degree 3 real curves. Welschinger observed that a signed count of the real curves restores the invariance, cf. [7]. The signs in this signed count depend on the types of nodes of the curve. There are two *types* of real nodes:

(1) the *hyperbolic node* locally defined by the equation $x^2-y^2=(x-y)(x+y)$
(2) and the *elliptic node* locally defined by the equation x^2+y^2.

Welschinger defined the *Welschinger sign*

$$\mathrm{Wel}_{\mathbb{R}}(C) := (-1)^e = \prod_{z \text{ a node of } C} \mathrm{sign}\,(-\det \mathrm{Hessian}(f)(z))$$

of a plane rational real curve C, where e is the number of its real elliptic nodes and f is a defining equation for C. Then he showed that the sum

$$W_d := \sum_C \mathrm{Wel}_{\mathbf{R}}(C)$$

running over all plane rational degree d curves C defined over $\mathbf{R}$ passing through a generic configuration of $3d-1$ real points is invariant, i.e., it is independent of the generic configuration of points.

Methods from $\mathbb{A}^1$-homotopy theory allow to get invariant answers to questions in enumerative geometry over an arbitrary base field k. We call these answers *quadratic enrichments* since they live in the *Grothendieck-Witt ring* $\mathrm{GW}(k)$ *of* k of quadratic forms over k. More precisely, let k be a field of characteristic not equal to 2. Then $\mathrm{GW}(k)$ is the group completion of the set of isometry classes of non-degenerate quadratic forms over k with respect to the direct sum $\oplus$. This becomes a ring with multiplication given by the tensor product $\otimes$. The ring $\mathrm{GW}(k)$ is generated by symbols $\langle a\rangle$ for each $a \in k^\times/(k^\times)^2$. Here, $\langle a\rangle$ represents the class of the quadratic form ax^2. Note that for an algebraically closed field k there is only one generator, namely $\langle 1\rangle$. $\mathrm{GW}(\mathbf{R})$ has two generators, namely $\langle 1\rangle$ and $\langle -1\rangle$. Also, observe that when we count curves over an algebraically closed field, we count them with a 1 and that when we count real curves we count them with a $+1$ or a -1 sign. In both cases this corresponds to the generators of $\mathrm{GW}(k)$. So a natural generalization is to count curves defined over a field k with a *quadratic weight* $\langle a\rangle \in \mathrm{GW}(k)$. Instead of remembering the sign of $-\det \mathrm{Hessian}(f)(z)$ in the definition of $\mathrm{Wel}_{\mathbf{R}}(C)$, we want to remember its class in $k^\times/(k^\times)^2$ as in the following definition.

Definition (Levine [3])**.** *Let C be a plane curve defined over k. Then its* quadratic weight *is*

$$\mathrm{Wel}_k^{\mathbb{A}^1}(C) := \langle \prod_{\textit{nodes } z} N_{\kappa(z)/k}(-\det \mathrm{Hessian}\, f(z))\rangle \in \mathrm{GW}(k)$$

where $\kappa(z)$ is the residue field of z and $N_{\kappa(z)/k}\colon \kappa(z) \to k$ the field norm.

Note that $\langle \mathrm{Wel}_{\mathbf{R}}(C)\rangle = \mathrm{Wel}_{\mathbf{R}}^{\mathbb{A}^1}(C)$ for a real curve C.

Let

$$N_d^{\mathbb{A}^1} := \sum_C \mathrm{Tr}_{\kappa(C)/k}\left(\mathrm{Wel}_{\kappa(C)}^{\mathbb{A}^1}(C)\right) \in \mathrm{GW}(k)$$

be the quadratically weighted count of plane rational degree d curves C through a generic configuration of $3d-1$ points defined over k. Here, $\kappa(C)$ is the field of definition of C and $\mathrm{Tr}_{\kappa(C)/k}\colon \mathrm{GW}(\kappa(C)) \to \mathrm{GW}(k)$ is the *trace map* defined by composing the quadratic form with the field trace $\mathrm{Tr}_{\kappa(C)/k}$. By works of Levine [3] and Kass-Levine-Solomon-Wickelgren [1, 2], the sum $N_d^{\mathbb{A}^1}$ is an invariant, i.e., it is independent of the generic configuration of k-points.

One elegant and efficient way to compute the numbers N_d and W_d is given by Mikhalkin's correspondence theorem [4]. This theorem says that

$$N_d = N_d^{\mathrm{trop}} \coloneqq \sum_A \mathrm{mult}_{\mathbb{C}}(A)$$

and

$$W_d = W_d^{\mathrm{trop}} \coloneqq \sum_A \mathrm{mult}_{\mathbf{R}}(A),$$

where the sum runs over all rational degree d tropical curves A passing through a generic configuration of $3d-1$ points in $\mathbf{R}^2$. Here, the complex and real multiplicities of A are defined as follows:

$$\mathrm{mult}_{\mathbb{C}}(A) \coloneqq \prod_v |\Delta_v|$$

and

$$\mathrm{mult}_{\mathbb{R}}(A) \coloneqq \begin{cases} \prod_v (-1)^{\mathrm{int}(\Delta_v)} & \text{if all edges of } A \text{ have odd weight,} \\ 0 & \text{else;} \end{cases}$$

where both products run over the 3-valent vertices v of A, the triangle Δ_v denotes the dual to v in the dual subdivision of A, the normalized area $|\Delta_v|$ is its double Euclidean area and $\mathrm{int}(\Delta_v)$ is the number of its interior lattice points. So by Mikhalkin's correspondence theorem one can count algebraic curves by counting tropical curves which can be done using combinatorial methods.

The goal of the talk was to present a quadratic enrichment of Mikhalkin's correspondence theorem. For this, we start by defining a quadratic enrichment of $\mathrm{mult}_{\mathbb{C}}(A)$ and $\mathrm{mult}_{\mathbf{R}}(A)$:

$$\mathrm{mult}^{\mathbb{A}^1}(A) \coloneqq \begin{cases} \frac{\prod_v |\Delta_v| - 1}{2} \cdot h + \langle \prod_v (-1)^{\mathrm{int}(\Delta_v)} \rangle & \text{if all edges of } A \text{ have odd weights,} \\ \frac{\prod_v |\Delta_v|}{2} \cdot h & \text{else;} \end{cases}$$

where $h = \langle 1 \rangle + \langle -1 \rangle \in \mathrm{GW}(k)$ is the hyperbolic form, and where analogously, the products run over all 3-valent vertices of A. Note that $\mathrm{mult}^{\mathbb{A}^1}(A)$ only has summands of the form $\langle 1 \rangle$ and $\langle -1 \rangle$ and therefore it is completely determined by $\mathrm{mult}_{\mathbb{C}}(A)$ and $\mathrm{mult}_{\mathbf{R}}(A)$:

$$\mathrm{mult}^{\mathbb{A}^1}(A) = \frac{\mathrm{mult}_{\mathbb{C}}(A) - \mathrm{mult}_{\mathbf{R}}(A)}{2} \cdot h + \langle \mathrm{mult}_{\mathbf{R}}(A) \rangle.$$

Now, we can state our quadratically enriched correspondence theorem.

Theorem (Jaramillo Puentes–Pauli).

$$N_d^{\mathbb{A}^1} = N_d^{\mathbb{A}^1,\mathrm{trop}} := \sum_A \mathrm{mult}^{\mathbb{A}^1}(A) \in \mathrm{GW}(k)$$

where the sum runs over all rational degree d tropical curves passing through a generic configuration of $3d-1$ points in $\mathbf{R}^2$, and where k is an arbitrary field with $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > d$.

A direct consequence is the following formula for the computation of $N_d^{\mathbb{A}^1}$.

Corollary.

$$N_d^{\mathbb{A}^1} = \frac{N_d - W_d}{2} \cdot h + W_d \cdot \langle 1 \rangle$$

The proof of the quadratically enriched correspondence theorem uses degeneration techniques similar to the ones in [6] and [5] and the computations generalize the computations in [6].

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Weakly Smooth Forms and Dolbeault Cohomology of Curves

Joseph Rabinoff

(joint work with Walter Gubler, Philipp Jell)

1. Orientation

This is a report on completed and continuing work on real-valued smooth differential forms on non-Archimedean analytic spaces. The study of such forms is largely motivated by Arakelov theory. In Arakelov geometry, one often starts with a Diophantine problem over the rational numbers, represented by a variety X over $\mathrm{Spec}(\mathbf{Q})$. One then "spreads out" this variety to a scheme $\mathfrak{X}$ over $\mathrm{Spec}(\mathbf{Z})$, and "compactifies" the situation by including the place at ∞. Intersecting divisors in this situation consists of performing arithmetic intersection theory on

$\mathfrak{X} \to \mathrm{Spec}(\mathbf{Z})$, and computing various analytic quantities on $X(\mathbf{C})$ (using Green's functions, smooth forms, pluripotential theory, etc).

Arakelov theory has been applied with great success to questions about curves and abelian varieties: for instance, Vojta's and Bombieri's simplifications of Faltings' proof of the Mordell Conjecture, and a full proof of the Bogomolov Conjecture due to Ullmo–Zhang, Gubler, Yamaki, and Xie–Yuan.

An obstacle to extending these computations to other high-dimensional situations is the nonexistence of a suitable model $\mathfrak{X}$ of X over $\mathrm{Spec}(\mathbf{Z})$. One idea is to replace the arithmetic theory over the finite places of $\mathrm{Spec}(\mathbf{Z})$ with an *analytic* theory at the corresponding non-Archimedean places. This involves developing a theory of real-valued differential forms on non-Archimedean analytic spaces.

2. Berkovich Analytic Spaces

At this point we fix some standard notation that will remain in effect for the rest of this report.

- K is non-Archimedean field: that is, it is complete with respect to a nontrivial, non-Archimedean valuation.
- $\mathrm{val}\colon K \to \mathbf{R} \cup \{\infty\}$ is the fixed valuation on K.
- $|\cdot| = \exp(-\mathrm{val}(\cdot))$ is an associated absolute value.
- $\overset{\circ}{K} = \{x \in K\colon |x| \le 1\}$ is the valuation ring.
- $\overset{\circ\circ}{K} = \{x \in \overset{\circ}{K}\colon |x| < 1\}$ is the maximal ideal.
- $\widetilde{K} = \overset{\circ}{K}/\overset{\circ\circ}{K}$ is the residue field.

The most important non-Archimedean fields for us will be $K = \mathbf{Q}_p$ and $K = \mathbf{C}_p$, the completion of an algebraic closure of $\mathbf{Q}_p$.

There is a functor $X \mapsto X^{\mathrm{an}}$ from finite-type K-schemes to analytic spaces over K in the sense of Berkovich [1]. The topological space underlying the analytification of an affine scheme $X = \mathrm{Spec}(A)$ is the space

$$X^{\mathrm{an}} = \big\{\|\cdot\|\colon A \to \mathbf{R}_{\ge 0} \text{ seminorm extending } |\cdot|\big\},$$

equipped with the topology generated by the sets $\{\|\cdot\|\colon r_1 < \|f\| < r_2\}$ for $f \in A$ and $0 < r_1 < r_2$. In general, the topological space X^{an} is locally compact, it is Hausdorff when X is separated, it is locally contractible when X is smooth, and it enjoys a number of other properties of a reasonable topological space.

Aside. It is not easy to find a justification for the definition of X^{an} in the literature, so we attempt to provide one here. To begin, consider the Zariski topology on $\mathrm{Spec}(A)$. One would like to define this topology to have a base of sets of the form $D(f) = \{x\colon f(x) \ne 0\}$. The expression "$f(x) = 0$" should satisfy some formal properties: for instance, if $(fg)(x) = 0$ then either $f(x) = 0$ or $g(x) = 0$ (or both). In other words, it is very natural to take the set underlying $\mathrm{Spec}(A)$ to be the set of prime ideals $\mathfrak{p}$ of A, as the data of a prime ideal is equivalent to specifying the set of f such that "$f(x) = 0$" for a given "point" x.

In the analytic topology on $\mathrm{Spec}(A)^{\mathrm{an}}$, a set of the form $\{x\colon r_1 < |f(x)| < r_2\}$ should be an open set. In this language, we need the expression "$|f(x)|$" to be defined, and to satisfy some formal properties, such as $|(fg)(x)| = |f(x)|\,|g(x)|$. As above, it is natural to consider the space of all multiplicative seminorms $\|\cdot\|$ on A, with the notational convention $|f(x)| = \|f\|$ when $\|\cdot\|$ corresponds to the point x. However, now there are at least two natural choices of where the quantity $|f(x)|$ should live:

- If we demand that $|f(x)|$ be contained in $\mathbf{R}$ then we arrive at Berkovich's theory.
- If instead we allow $|f(x)|$ to take values in any totally ordered abelian group, then we arrive at Huber's theory of adic spaces.

In this report, we will restrict our attention to Berkovich analytic spaces.

Berkovich spaces are natural from the point of view of tropicalization. Given units $f_1, \ldots, f_n \in A^\times$, we get a map trop: $X^{\mathrm{an}} \to \mathbf{R}^n$ defined by

$$\mathrm{trop}(x) = \big(-\log|f_1(x)|, \ldots, -\log|f_n(x)|\big).$$

We call this a *tropicalization map* or a *moment map*. The set $\mathrm{Trop}(X) = \mathrm{trop}(X^{\mathrm{an}})$ is the classical tropicalization; in particular, it is a balanced weighted polyhedral complex.

In order to motivate the definition of smooth forms on Berkovich spaces, we recall the following theorem of Payne [7]:

Theorem. *If X is a quasiprojective variety, then X^{an} is naturally homeomorphic to $\varprojlim \mathrm{Trop}(X)$, where the limit is taken over all extended tropicalizations into toric varieties.*

In other words, at least as a topological space, one can recover X^{an} by successive approximations by tropicalizations. As $\mathrm{Trop}(X)$ is a polyhedral subset of a Euclidean space, it is possible to define a bigraded sheaf of $\mathcal{A}^{p,q}(\mathrm{Trop}(X))$ differential forms on $\mathrm{Trop}(X)$ (see below). Very roughly, the idea of Chambert-Loir–Ducros [2] is to define smooth forms on X^{an} by

$$\mathcal{A}^{p,q}(X^{\mathrm{an}}) = \varinjlim \mathcal{A}^{p,q}(\mathrm{Trop}(X)).$$

3. Smooth Forms

Lagerberg [6] has defined a bigraded sheaf of smooth forms on $\mathbf{R}^n$, in which a (p,q) form is locally given in coordinates by a sum of expressions of the form

$$\eta = \alpha(x)\,\mathrm{d}'x_{i_1} \wedge \cdots \wedge \mathrm{d}'x_{i_p} \wedge \mathrm{d}''x_{j_1} \wedge \cdots \wedge \mathrm{d}''x_{j_q},$$

where α is a smooth function. Such a form can be restricted to a polyhedral complex in a natural way.

Given an open set $U \subset X^{\mathrm{an}}$, invertible functions $f_1, \ldots, f_n \in \mathcal{O}(U)^\times$ and associated moment map trop: $U \to \mathbf{R}^n$, and a (p,q)-form η defined on $\mathrm{Trop}(U)$, Chambert-Loir and Ducros [2] associate the "pullback form" $\mathrm{trop}^*\eta$ on U, which essentially consists of the data (U, trop, η) up to equivalence. After sheafifying, they obtain a bigraded sheaf $(\mathcal{A}^{p,q}_{\mathrm{sm}}, \mathrm{d}', \mathrm{d}'', \wedge)$ of differential algebras on X, known as the sheaf of *smooth forms*. This sheaf enjoys many of the basic properties as its Archimedean counterpart: for instance, there exist partitions of unity, an integration theory, Stokes' theorem, a Poincaré–Lelong formula, etc. In particular, one can define Dolbeault cohomology groups

$$H^{p,q}_{\mathrm{sm}}(X^{\mathrm{an}}) = \ker\big(\mathrm{d}''\colon \mathcal{A}^{p,q}_{\mathrm{sm}}(X^{\mathrm{an}}) \longrightarrow \mathcal{A}^{p,q+1}_{\mathrm{sm}}(X^{\mathrm{an}})\big)/\mathrm{d}''\mathcal{A}^{p,q-1}_{\mathrm{sm}}(X^{\mathrm{an}}).$$

At this point it is natural to try to compute an example: what is the Hodge diamond of a smooth curve? This was accomplished by Jell [5].

Theorem (Jell, 2019)**.** *Suppose that $K = \bar{K}$ and that $\widetilde{K}$ has characteristic zero. Let X be a smooth, projective, connected K-curve of genus g.*

(1) *If X is a Mumford curve then $h^{0,0}(X^{\mathrm{an}}) = h^{1,1}(X^{\mathrm{an}}) = 1$ and $h^{1,0}(X^{\mathrm{an}}) = h^{0,1}(X^{\mathrm{an}}) = g$. Furthermore, $H^{\bullet,\bullet}(X^{\mathrm{an}})$ satisfies Poincaré duality.*
(2) *If X is not a Mumford curve then $h^{1,1}(X^{\mathrm{an}}) = \infty$.*

Let us sketch what happens in case (2). Define $\mathcal{K} = \mathbf{R}\log|\mathcal{O}^\times|$, the sheaf generated by real linear combinations of valuations of invertible functions. Jell proves that there is a short exact sequence

$$0 \longrightarrow \underline{\mathbf{R}} \longrightarrow \mathcal{K} \xrightarrow{\mathrm{d}'} \mathcal{A}^{1,0}_{\mathrm{sm,closed}} \longrightarrow 0,$$

which should be regarded as a non-Archimedean analogue of the exponential exact sequence. This gives rise to a long exact sequence

$$\cdots \longrightarrow H^{0,1}_{\mathrm{sm}}(X^{\mathrm{an}}) \longrightarrow H^1(X^{\mathrm{an}}, \mathcal{K}) \longrightarrow H^{1,1}(X^{\mathrm{an}}) \longrightarrow 0.$$

In the Archimedean situation, the sheaf $\mathcal{K}$ is equal to the sheaf $\mathcal{H}$ of *harmonic* functions. A sheaf of harmonic functions was defined in the non-Archimedean situation by Thuillier [9], and it is a standard fact (in both the Archimedean and non-Archimedean contexts) that $h^1(X^{\mathrm{an}}, \mathcal{H}) = 1$, which implies $h^{1,1}(X^{\mathrm{an}}) = 1$. However, Jell proves that there is a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{H} \longrightarrow \mathcal{S} \longrightarrow 0,$$

where $\mathcal{S}$ is a skyscraper sheaf which vanishes if and only if X is a Mumford curve.

In short, when X is not a Mumford curve, then $h^{1,1}(X) = \infty$ because *harmonic functions are not smooth* in the theory of Chambert-Loir–Ducros.

Example. The simplest example of a curve that is not a Mumford curve is an elliptic curve X with good reduction. In this case (assuming $K = \bar{K}$), there exists a smooth model $\mathfrak{X}$ for X over $\overset{\circ}{K}$. The "order of vanishing along the special fiber $\mathfrak{X}_s$" defines a point $\xi \in X^{\mathrm{an}}$, and there is a bijective correspondence between tangent directions at ξ and points of $\mathfrak{X}(\widetilde{K})$. A rational function $f \in K(X)^\times$ with $|f(\xi)| = 1$ reduces to a rational function $\tilde{f} \in \widetilde{K}(\mathfrak{X}_s)$, and the slope of $-\log|f|$ along a tangent direction at ξ is equal to the order of the zero or pole of $\tilde{f}$ at the corresponding closed point of $\mathfrak{X}_s$. In particular, one obtains a principal divisor $\mathrm{div}(\tilde{f}) \in \mathrm{Prin}(\mathfrak{X}_s)$. See [9] for details.

On the other hand, a piecewise linear function g on X^{an} is by definition harmonic at ξ when the sum of the slopes of g along the tangent directions at ξ is equal to zero. In other words, such a function gives rise to a real divisior of degree zero on $\mathfrak{X}_s$, i.e., an element of $\mathrm{Div}^0_{\mathbf{R}}(\mathfrak{X}_s)$. In fact, one can show that a divisor

$D \in \mathrm{Div}^0_{\mathbf{R}}(\mathfrak{X}_s)$ lifts to a function $g \in \mathbf{R}\log|\mathcal{O}^\times|$ if and only if D is principal. But $\mathrm{Div}^0(\mathfrak{X}_s)/\,\mathrm{Prin}(\mathfrak{X}_s) \cong \mathfrak{X}_s(\widetilde{K})$ because $\mathfrak{X}_s$ is an elliptic curve, so

$$\mathrm{Div}^0_{\mathbf{R}}(\mathfrak{X}_s)/\,\mathrm{Prin}_{\mathbf{R}}(\mathfrak{X}_s) \cong \mathfrak{X}_s(\widetilde{K}) \otimes_{\mathbf{Z}} \mathbf{R}.$$

This is an infinite-dimensional real vector space if $\widetilde{K}$ is a field of characteristic zero.

4. Weakly Smooth Forms

In [3], we define a sheaf $\mathcal{A}^{p,q} \supset \mathcal{A}^{p,q}_{\mathrm{sm}}$ of *weakly smooth forms*, in which harmonic functions are smooth by definition. This is difficult because it is not evident what "harmonic" should mean in higher dimensions: Thuillier's definition for curves does not obviously generalize. We proceed as follows:

(1) Using Temkin's theory of reduction of germs [8], to any $x \in X^{\mathrm{an}}$ we can associate a ringed space (X^{an}, x), which is basically a Riemann–Zariski space.
(2) To any piecewise linear function h at x, Chambert-Loir–Ducros [2] canonically associate a line bundle $L_h \in \mathrm{Pic}(X^{\mathrm{an}}, x)$.
(3) We say that h is *harmonic* at x if L_h is numerically trivial.

This is a direct generalization of Thuillier's definition: in the case of curves, the line bundle L_h corresponds to the divisor D of outgoing slopes of h, and a line bundle on a projective curve is numerically trivial if and only if it has degree zero.

Definition. A *harmonic tropicalization* on an open set $U \subset X^{\mathrm{an}}$ is a function

$$h = (h_1, \ldots, h_n)\colon U \longrightarrow \mathbf{R}^n$$

for harmonic functions $h_1, \ldots, h_n$ on U.

We define a natural structure of weighted polyhedral complex on $h(U)$ and proceed as in [2], defining a bigraded sheaf of differential algebras $\mathcal{A}^{p,q}$ by locally pulling back Lagerberg forms under *harmonic* tropicalizations. This sheaf also admits an integration theory, Stokes' theorem, a Poincaré–Lelong formula, etc.

A key fact is that $h(U)$ satisfies the balancing condition: this is roughly equivalent to Stokes' theorem.

Theorem. *If $h\colon U \to \mathbf{R}^n$ is a harmonic tropicalization then $h(U)$ is a balanced polyhedral complex.*

5. Weakly Smooth Dolbeault Cohomology of Curves

With this definition and underlying theory in place, in [4] we proceed to compute the Hodge diamond of a smooth curve for the Dolbeault cohomology associated to the sheaf of weakly smooth forms.

Theorem. *Let X be a smooth, proper, geometrically connected K-curve, and let g be the first Betti number of $|X^{\mathrm{an}}|$. Then $h^{0,0}(X^{\mathrm{an}}) = h^{1,1}(X^{\mathrm{an}}) = 1$ and $h^{1,0}(X^{\mathrm{an}}) = h^{0,1}(X^{\mathrm{an}}) = g$.*

It is important to note here that g is not the genus of X, but rather the toric rank. For instance, if X has good reduction then $g = 0$.

We prove this theorem by relating smooth forms on X^{an} with a combinatorial theory of smooth forms on metric graphs, via a notion of "pullback from the skeleton."

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Configuration spaces on graphs and a Serre spectral sequence for $M_{g,n}^{\mathrm{trop}}$

Melody Chan

(joint work with Christin Bibby, Nir Gadish, Claudia He Yun)

Consider the Serre spectral sequence, on compactly supported cohomology with rational coefficients, associated to the fibration $\mathcal{M}_{g,n} \to \mathcal{M}_g$ of complex moduli spaces of curves. It is

$$E_2^{p,q} = H_c^p(\mathcal{M}_g, H_c^q(\mathrm{Conf}_n(S_g); \mathbb{Q})) \Rightarrow H_c^{p+q}(\mathcal{M}_{g,n}). \tag{1}$$

Our goal is to present a tropical analogue: a Serre spectral sequence for $M_{g,n}^{\mathrm{trop}}$. Moreover, this spectral sequence shall furnish calculations of compactly supported cohomology with rational coefficients of $\mathcal{M}_{g,n}$ in weight 0, in the sense of Deligne's mixed Hodge theory [7, 8].

Let us define the space $M_{g,n}^{\mathrm{trop}}$ [3, 10]. See, e.g., the survey article [6] for details. For $g, n \geq 0$ such that $2g - 2 + n > 0$, let $\mathbf{\Gamma}_{g,n}$ denote the category whose objects are triples

$$(G, m, w)$$

where

- G is a connected multigraph, with vertex and edge sets denoted $V(G)$ and $E(G)$,
- $m\colon \{1,\dots,n\} \to V(G)$ is a function,
- $w\colon V(G) \to \mathbb{Z}_{\geq 0}$ is a function,

such that the triple (G, m, w) is isomorphic, as a vertex-weighted, marked graph, to the dual graph of a *stable* curve, i.e., a point in Deligne-Mumford-Knudsen's moduli space $\overline{\mathcal{M}}_{g,n}$. The morphisms are compositions of isomorphisms and vertex-weighted edge contractions; see [6] for what that means precisely. Then a morphism $(G, m, w) \to (G', m', w')$ induces an injective map $E(G') \to E(G)$, and hence, writing

$$\sigma_{(G,m,w)} = \mathbb{R}^{E(G)},$$

a morphism

$$\sigma_{(G',m',w')} \to \sigma_{(G,m,w)}$$

which is an isomorphism onto a face, possibly non-proper, of the target. In this way, $\sigma\colon \mathbf{\Gamma}_{g,n}^{\mathrm{op}} \to \mathsf{Top}$ is a functor. Then the *tropical moduli space of curves* is the colimit

$$M_{g,n}^{\mathrm{trop}} = \varinjlim_{(G,m,w)\in\mathbf{\Gamma}_{g,n}} \sigma_{(G,m,w)}$$

of this functor.

Theorem 1. [2] *For any $g \geq 2$ and $n \geq 0$, there is a spectral sequence of rational S_n-representations*

$$E_1^{p,q} = \bigoplus_{\substack{G\in \mathrm{Iso}(\Gamma_g^{(2)}) \\ |E(G)|=p}} (H_c^q(\mathrm{Conf}_n(G);\mathbb{Q}) \otimes \det E(G))^{\mathrm{Aut}(G)} \Longrightarrow H_c^{p+q}(M_{g,n}^{\mathrm{trop}};\mathbb{Q}). \tag{2}$$

The spectral sequence degenerates at E_1 when $g = 2$; at E_2 when $g = 3$; and at E_3 when $g > 3$.

Using this theorem, and some further techniques not mentioned here, we compute, S_n-equivariantly, $H_c^*(M_{g,n}^{\mathrm{trop}})$ in the range $g = 2$ and all $n \leq 11$ [2], and $g = 3$ and all $n \leq 8$ [2].

The interest from the algebro-geometric point of view in computing $H_c^*(M_{g,n}^{\mathrm{trop}})$ comes from the fact that there is a canonical isomorphism

$$W_0 H_c^*(\mathcal{M}_{g,n};\mathbb{Q}) \cong H_c^*(M_{g,n}^{\mathrm{trop}};\mathbb{Q})$$

which in fact is induced by a continuous, proper morphism

$$\mathcal{M}_{g,n} \to M_{g,n}^{\mathrm{trop}}. \tag{3}$$

Namely, the map on compactly supported cohomology

$$H_c^*(M_{g,n}^{\mathrm{trop}};\mathbb{Q}) \to H_c^*(\mathcal{M}_{g,n};\mathbb{Q})$$

induced by (3) is an injection onto the weight zero subspace of the latter [4, 9].

Very recently, Hainaut and Petersen have made an intriguing conjecture about the spectral sequence (2). They conjecture that not only does it abut to the weight

0 part of (1) (by [4, 5]), but that the spectral sequence itself is isomorphic to the weight 0 part of (1). See [9, §5].

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Tropical geometry rules: Tropical base schemes in logarithmic Gromov-Witten theory and canonical wall structures in mirror symmetry

Bernd Siebert

(joint work with Dan Abramovich, Qile Chen, Mark Gross, Yixian Wu)

The purpose of the talk was to showcase the power of tropical techniques in recent advances in Gromov-Witten theory and mirror symmetry. At the center of this development are certain algebraic stacks associated to tropical spaces that replace and refine the stack of pre-stable curves in ordinary Gromov-Witten theory.

The relevant version of tropical geometry works with complexes of rational polyhedral cones, generalizing the notion of a fan in toric geometry. There is no embedding into an ambient vector space, hence also no notion of balancing. Given a rational polyhedral cone $\sigma \subset \mathbb{R}^n$, one has the associated affine toric variety $A_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$ and its stacky quotient

$$\mathcal{A}_\sigma = \left[A_\sigma/\mathbb{G}_m^n\right].$$

For example, for $\sigma \subset \mathbb{R}^2$ the positive quadrant, there are four toric orbits on $A_\sigma = \mathbb{A}^2$. The free orbit gives rise to an open embedding of a point $\operatorname{Spec} \mathbb{C} \to \mathcal{A}_\sigma$, while the toric fixed point provides a closed embedding of $B\mathbb{G}_m^2$ in $\mathcal{A}_\sigma$.

If $\sigma \subseteq \tau$ is a face, we have the usual toric open embedding $A_\sigma \to A_\tau$, and in turn an open embedding of Artin stacks $\mathcal{A}_\sigma \to \mathcal{A}_\tau$. Given a complex Σ of rational polyhedral cones, one then defines the *Artin fan of* Σ by the colimit over all such open embeddings:

$$\mathcal{A}_\Sigma = \varinjlim_{\sigma \in \Sigma} \mathcal{A}_\sigma.$$

Under mild assumptions this stack is algebraic [1, 4]. It also carries a logarithmic structure in the sense of Fontaine-Illusie-Kato. Recall that a log structure on a space X in this sense is a homomorphism of sheaves of multiplicative monoids

$$\alpha : \mathcal{M} \longrightarrow \mathcal{O}_X$$

with the property that $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. The paradigmical example is a smooth variety X with a simple normal crossings divisor $D = \bigcup_i D_i$ with D_i irreducible. Then the sheaf

$$\mathcal{M} = \mathcal{O}_X \cap \mathcal{O}_{X \setminus D}^\times$$

of regular functions with zeros contained in D, together with its inclusion into $\mathcal{O}_X$, is a logarithmic structure. For this example, the quotient $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{O}^\times$ is isomorphic to $\bigoplus_{i=1}^r \mathbb{N}_{D_i}$, a subsheaf of a constructible sheaf. The preimage of the section of $\bigoplus_i \mathbb{N}_{D_i}$ given by $(a_i) \in \mathbb{N}^r$ under the quotient map $\mathcal{M} \to \overline{\mathcal{M}}$ is the $\mathcal{O}_X^\times$-torsor associated to the invertible sheaf $\mathcal{O}_X(-\sum_i a_i D_i)$. Thus a log structure has some discrete aspects captured by $\overline{\mathcal{M}}$ and an algebraic-geometric one given by a system of line bundles with cosections, compatible in some way according to the monoid structure of $\overline{\mathcal{M}}$.

The construction generalizes to toric varieties with D the toric divisor. Then $\overline{\mathcal{M}}$ has stalks that reflect local Cartier divisors supported on D. For example, if $\sigma \subset \mathbb{R}^n$ is strongly convex and x is the 0-dimensional torus orbit of $X = \mathrm{Spec}[\sigma^\vee \cap \mathbb{Z}^n]$, then $\overline{\mathcal{M}}_x = \sigma^\vee \cap \mathbb{Z}^n$.

Now given a (fine, saturated) log structure $\mathcal{M}$ on a scheme X, for each $x \in X$ one has the rational polyhedral cone $\sigma_x = \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0})$. With face inclusions induced by generization maps, these cones define a complex

$$\Sigma_X = \varinjlim_{x \in X} \sigma_x$$

of rational polyhedral cones, the *tropicalization of* $(X, \mathcal{M}_X)$. Properly interpreted, this notion of tropicalization refines the notion of tropicalization for subvarieties of $(K^*)^n$ when K is, say, the Puiseux series field.

Moreover, there is a unique morphism

$$X \longrightarrow \mathcal{A}_{\Sigma_X}$$

such that $\mathcal{M}_X$ is the pull-back of the log structure on $\mathcal{A}_{\Sigma_X}$.

This statement is an incarnation of the following amazing fact, a special case of [2, Prop.2.10].

Proposition 1. *Let Σ be a cone complex and $(X, \mathcal{M})$ a (fine, saturated) logarithmic space. Then tropicalization defines a canonical bijection*

$$\mathrm{Hom}_{\log}\big((X, \mathcal{M}), \mathcal{A}_\Sigma\big) \longrightarrow \mathrm{Hom}_{\text{cones}}\big(\Sigma_X, \Sigma\big).$$

Thus while the stack $\mathcal{A}_\Sigma$ is algebraic, it is really a completely tropical object.

Now the natural base stack for logarithmic Gromov-Witten theory on $(X, \mathcal{M})$ alluded to in the title of the talk is the algebraic stack $\mathfrak{M}(\mathcal{X})$ of (basic) logarithmic maps from (log-) curves to the Artin fan $\mathcal{X} = \mathcal{A}_{\Sigma_X}$ of X. In fact, if $\mathscr{M}(X)$ is the Deligne-Mumford stack of (basic) stable logarithmic maps to X, then the composition of stable maps to X with $X \to \mathcal{X}$ defines a virtually smooth morphism

$$\pi : \mathscr{M}(X, \boldsymbol{\tau}) \longrightarrow \mathfrak{M}(\mathcal{X}, \tau).$$

Here $\boldsymbol{\tau}$, τ represent genus, curve class, and tropical type data to fix dimensions. Virtual smoothness means that π has a perfect relative obstruction theory $\mathcal{E}$ in the sense of Behrend-Fantechi. Thus for any k-dimensional algebraic cycle α on $\mathfrak{M}(\mathcal{X}, \tau)$, we have a $(k+d)$-dimensional virtual pull-back class $\pi^!_{\mathcal{E}}(\alpha)$ on $\mathscr{M}(X, \boldsymbol{\tau})$ [7] that provides logarithmic Gromov-Witten invariants. Here k is the relative virtual dimension.

Note that Proposition 1 says that $\mathfrak{M}(\mathcal{X})$ really is a discrete, tropical refinement of the moduli stack of pre-stable curves used as base stack in ordinary Gromov-Witten theory. Indeed, by the proposition, a geometric point in this stack consists of a pre-stable curve C along with a family of stable tropical maps to Σ_X with domain the dual intersection graph of C!

One immediate benefit of this point of view is that each type τ of tropical curves defines a *pure-dimensional* closed subspace $\mathfrak{M}(\mathcal{X}, \tau) \subseteq \mathfrak{M}(\mathcal{X})$, and hence by virtual pull-back a virtual fundamental class $[\mathscr{M}(X, \boldsymbol{\tau})]_{\text{virt}}$. The tropicalizations of stable logarithmic maps in $\mathfrak{M}(X, \tau)$, and in turn in $\mathscr{M}(X, \boldsymbol{\tau})$, degenerate to tropical stable maps of type τ, so are tropically more generic/less degenerate. Thus studying the moduli space of tropical curves in Σ_X tells us everything about the stratified structure of $\mathfrak{M}(\mathcal{X})$.

This picture has been developed and applied in almost a decade of joint work with Abramovich, Chen and Gross to formulate a gluing formula for logarithmic Gromov-Witten theory [2, 3]. With the present notation, the gluing problem asks to compute the logarithmic Gromov-Witten invariant for type $\boldsymbol{\tau}$ from logarithmic Gromov-Witten theory for the types $\boldsymbol{\tau_i}$ obtained from $\boldsymbol{\tau}$ after splitting a set of edges. Splitting edges means restricting to certain subcurves of the domain of a stable map. Unfortunately, restricting a logarithmically smooth structure on a nodal curve C to a union C' of components may produce a *negative contact order* at the marked points obtained from the nodes after splitting. This reflects the fact that the equation $xy = t$ in the log structure of a node of C leads to the section $y = x^{-1} \cdot t$ on the component defined by $y = 0$. The negative power of x gives room for negative contact orders of the restriction to $C' \subset C$ of a logarithmic map $(C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$. The paper [3] provides the necessary generalization of stable logarithmic maps to *punctured logarithmic maps*. The gluing problem is then reduced to a problem of intersection theory on $\mathfrak{M}(\mathcal{X},\tau)$ by the following crucial result.

Theorem 1. [3, Thm.C] *There is a cartesian diagram*

$$\begin{array}{ccc} \mathscr{M}(X,\tau) & \xrightarrow{\ \delta\ } & \prod_{i=1}^r \mathscr{M}(X,\tau_i) \\ \varepsilon\downarrow & & \downarrow \hat{\varepsilon}=\prod_i \varepsilon_i \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau) & \xrightarrow{\ \delta^{\mathrm{ev}}\ } & \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau_i) \end{array}$$

with the horizontal arrows $\delta^{\mathrm{ev}},\delta$ finite morphisms.

The superscript "ev" stands for "evaluation" and indicates an important partial rigidification of the gluing problem by choosing a lift of each gluing point from $\mathcal{X}$ to X. Without the rigidification, the splitting map on the level of $\mathfrak{M}$-spaces has torus fibers, hence is non-proper and cannot be used to push-forward cycles.

For the gluing problem, virtual pull-back expresses $\delta_*[\mathscr{M}(X,\tau)]_{\mathrm{virt}}$ in terms of virtual pull-back by $\hat{\varepsilon}$ of $\delta_*^{\mathrm{ev}}[\mathfrak{M}(\mathcal{X},\tau)]$. Thus the decomposition of logarithmic Gromov-Witten invariants for τ in terms of punctured invariants of τ_i amounts to find a Künneth decomposition of $\delta_*^{\mathrm{ev}}[\mathfrak{M}(\mathcal{X},\tau)]$ in $\prod_i \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau_i)$. Note this is a problem in classical intersection theory, without any virtual aspects.

While a general solution to the Künneth decomposition problem should require a yet to be found version of tropical intersection theory decorated by homology classes (of possibly odd degree), my student Yixian Wu in her thesis found a very clean algebraic solution under the asumption that each gluing stratum is toric in terms of a *tropical displacement rule* [9]. The displacement rule is a version of the fan displacement method in toric intersection theory [5]. Namely, assuming the gluing strata are toric, the tropical stable map has an embedding into a vector space with integral structure locally near any edge to split. Now choose a general integral vector in each such vector space and look at the moduli space of tropical curves of types τ_i that match up to displacement vectors. This typically forces the tropical subcurves indexed by i to move into tropically more general position, defining a deformed type ρ_i. Such tuples (ρ_i) of minimal tropical dimension defines a finite set Δ of types matching up to the chosen displacement vector.

Schematically, her decomposition result is then as follows.

Theorem 2. [9] $\delta_*^{\mathrm{ev}}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)] = \sum_{(\rho_i)\in\Delta} m_{(\rho_i)} \cdot [\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\rho_i)]$.

The $m_{(\rho_i)} \in \mathbb{N}$ are certain lattice indices defined by the tropical gluing maps. See also [6, App.A] for a self-contained comprehensive explanation of the formula.

In the talk I also briefly explained how punctured invariants and this gluing formula provide the key inputs for a very transparent approach [6] to intrinsic mirror symmetry for a logarithmic Calabi-Yau pair (X,D) via a *canonical wall structure* defined by enumerative geometry. The walls are polyhedral cones of codimension one in the tropicalization Σ of (X,D). They carry the information of Laurent polynomials whose coefficients are defined by 1-point invariants, with the single contact order defined by an integral tangent vector on the wall. One can then define the coordinate ring of the mirror of (X, D) in terms of *broken lines*, a piecewise straight path carrying a monomial on each straight line segment with exponent defined by an integral vector tangent to the line segment. When passing through a wall, a broken line interacts with the Laurent polynomials, leading to a bend. A central result is then to show that these algebraically defined broken lines agree with (the spines of) the tropicalization of a 2-punctured logarithmic map, with one positive puncture giving the incoming direction, and a possibly negative puncture the tangent vector at the endpoint of the broken line. We call these the *logarithmic broken lines*.

The comparison of the algebraic with the enumerative geometry bending of broken lines is made possible since it happens tropically in codimension 0 or 1, which by [8] corresponds to gluing strata a point or a toric $\mathbb{P}^1$. Thus Wu's gluing formula applies. We split the tropicalization of a 2-punctured map with outgoing vertex on a wall at all edges adjacent to this vertex, and show that there is a displacement vector leading to only one summand in the right-hand side of Theorem 2, with all $\rho_i = \tau_i$ except the one for the outgoing vertex. The ρ_i for the outgoing vertex leads to a simple punctured Gromov-Witten computation for maps to a point or to $\mathbb{P}^1$. Hence the right-hand side expresses the coefficient of the broken line after crossing the wall in terms of a sum of products of terms of the Laurent polynomial on the wall and the incoming monomial, up to an explicitly computable lattice index.

Thus tropical geometry techniques are also instrumental in the following result we have dreamed of proving for many years.

Theorem 3. [6, Thm.A] *The broken lines for the canonical wall structure are exactly the logarithmic broken lines.*

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Tropical curves in Abelian surfaces and multiple cover formulas

Thomas Blomme

The enumeration of curves of given genus and degree satisfying a number of point constraints is a well-known problem, the most famous and easiest case being the counting of line passing through two points in the plane. Still in the plane, a complete answer is given by the Caporaso-Harris formula. In this talk, we address the case of the Abelian surfaces.

Given an Abelian surface, *i.e.* a complex torus $\mathbb{C}A$ of dimension two along with a class of curves β called polarization (a generic complex torus not containing any curves), we count the stable maps $\varphi : C \to \mathbb{C}A$ with the source being of genus g, realizing the class β, and that pass through g points in generic position inside $\mathbb{C}A$. This number is finite and does not depend on the choice of $\mathbb{C}A$ nor the points inside it if these choices are generic. Moreover, this number only depends on β through its divisibility d as an element of $H_2(\mathbb{C}A, \mathbb{Z})$, and its square by the intersection form: $\beta^2 = 2d^2n$. We write the invariant $N_{g,(d,dn)}$.

The values for the *primitive classes*, *i.e.* $d = 1$, were computed by J. Bryan and N. Leung [2]. They give the following formula:

$$N_{g,(1,n)} = g \sum_{a_1+\cdots+a_{g-1}=n} \prod_{i=1}^{g-1} a_i \sigma_1(a_i),$$

where σ_1 is the sum of divisors. For non-primitive classes, the enumeration of curves turns out to be significantly more complicated. Yet, they obey the following formula, called *multiple cover formula* [1]:

$$N_{g,(d,dn)} = \sum_{k|d} k^{4g-3} N_{g,(1,(d/k)^2n)}.$$

In particular, the formula reduces the computation for non-primitive classes to the primitive ones. This formula is in fact a particular case of a conjecture by G. Oberdieck which concerns all (reduced) Gromov-Witten invariants of Abelian surfaces. The above case deals with the case of point insertions.

To prove such a formula, we use the tropical geometry approach. This approach was originally implemented by G. Mikhalkin [3] to count curves passing through points in toric surfaces. In our case, we consider tropical Abelian surfaces, which are some quotient $\mathbb{T}A = \mathbb{R}^2/\Lambda$, *i.e.* real two-dimensional tori endowed with a lattice structure. Tropical curves are some graphs with integer slope on $\mathbb{T}A$ satisfying a balancing condition. A correspondence theorem in the Abelian setting is proven by T. Nishinou [4] and allows one to relate the complex count, known to be invariant, to a tropical count, which is thus also invariant. Due to the combinatorial nature of the tropical problem, it is possible to tackle the tropical problem with floor diagram techniques which lead to the formula.

The same techniques also allow one to get results for the enumeration of curves in fixed linear systems on an Abelian surface, and also yield multiple cover formulas for tropical refined invariants.

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Complex, real (and quadratically enriched?) Abramovich-Bertram formula

Erwan Brugallé

In this talk I present complex and real versions of Abramovich-Bertram formula, and suggest that a potential quadratically enriched version may exist.

A non-singular projective algebraic variety is always implicitly assumed to be equipped with some Kähler form. Unless otherwise stated, all algebraic surfaces considered here are assumed to be projective and non-singular.

1. The complex case

Let (X, ω) be a compact symplectic rational manifold[1] of dimension 4. Choose a class $d \in H_2(X; \mathbf{Z})$, and a configuration $\underline{x}$ of $c_1(X) \cdot d - 1$ points in X. Given an almost complex structure J tamed by ω, we denote by $\mathcal{C}(d, \underline{x}, J)$ the set of rational J-holomorphic curves in X realizing the class d, and passing through $\underline{x}$. The integer

$$GW_{(X,\omega)}(d) = Card(\mathcal{C}(d, \underline{x}, J))$$

is finite, and is known not to depend neither on $\underline{x}$, the deformation class of (X, ω), nor on the choice of a generic J [12]. We call these numbers the *(absolute) Gromov-Witten invariants of* (X, ω).

Now suppose that J is chosen to be mildly non-generically, that is X contains a unique[2] smooth (-2)-rational J-holomorphic curve E. The cardinal of $\mathcal{C}(d, \underline{x}, J)$, denoted by $GW^E_{X,\omega}(d)$, is still finite and does not depend on $\underline{x}$ nor on the choice of a generic choice of J among ω-tamed almost complex structures on (X, ω) for which E is J-holomorphic [6]. We call these numbers the *Gromov-Witten invariants of* (X, ω) *relative to* E.

Example. The standard complex structure on $\mathbf{CP}^1 \times \mathbf{CP}^1$ is generic. Recall that $\mathbf{F}_2$, the second Hirzebruch surface, has $S^2 \times S^2$ as underlying smooth manifold, but is equipped with a mildly non-generic, in the above sense, complex structure for which there exists a unique non-singular (-2)-rational curve.

[1] One may think in terms of algebraic surfaces, the general picture is still correct.

[2] In fact, finitely many, but disjoint, such curves are allowed.

Absolute and relative invariants are not equal, but are related via the following formula.

Theorem 1 (Abramovich-Bertram Formula). *One has*

$$GW_{(X,\omega)}(d) = \sum_{k\geq 0} \binom{d\cdot E + 2k}{k} GW^E_{(X,\omega)}(d-kE).$$

This formula has been first proved by Abramovich and Bertram [1] in the case of $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{F}_2$, and has known several generalizations since then, see for example [14, 13]. The symplectic version has been proved in [6]. Far beyond an equality between two series of numbers, Abramovich-Bertram Formula has a very clear geometric interpretation that in best expressed using degeneration formulas in Gromov-Witten theory [8, 11]. We restricted to the case of rational curves with a view toward enumeration of curves over non-algebraically closed fields, nevertheless Theorem 1 holds true for curves of any genus. Theorem 1 has several applications, for example it allows the computation of Gromov-Witten invariants in any genus of del Pezzo surfaces [14, 13, 3].

Abramovich-Bertram Formula being linear and upper triangular, it is clearly invertible. Unfortunately the general expression of $GW^E_{(X,\omega)}$ in terms of $GW_{(X,\omega)}$ is not as sympathetic as the one from Theorem 1 (see for example [4]). There is nevertheless one particular instance when one obtains a nice expression.

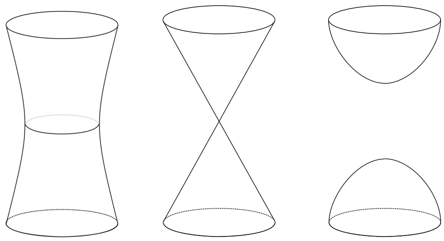
Corollary 1. *If $d\cdot E = 0$, then*

$$GW^E_{(X,\omega)}(d) = GW_{(X,\omega)}(d) + 2\sum_{k\geq 1}(-1)^k GW_{(X,\omega)}(d-kE).$$

2. The real case

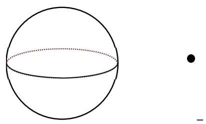
A *real symplectic manifold* $X_{\mathbf{R}} = (X,\omega,\tau)$ is a symplectic manifold (X,ω) equipped with an anti-symplectic involution τ. The *real part* of (X,ω,τ), denoted by $\mathbf{R}X$, is by definition the fixed point set of τ. An almost complex structure J on X is called *τ-compatible* if it is tamed by ω, and if τ is J-anti-holomorphic. Here we always assume that $\mathbf{R}X \neq \emptyset$. We denote by $H_2^{-\tau}(X;\mathbf{Z})$ the space of τ-anti-invariant classes. A non-singular projective real algebraic variety is always implicitly assumed to be equipped with some Kähler form which turns it into a real symplectic manifold.

2.1. (-2)**-rational curves in dimension 4.** The (-2)-rational curves in real symplectic manifolds of dimension 4 play a special role: they provide *walls* that separates different real structures on the same (up to deformation) underlying symplectic manifold. Indeed, contracting a smooth (-2)-rational curve E in an algebraic surface $\widetilde{X}$ produces a nodal surface X_{sing}, and there are two ways to smooth a node over $\mathbf{R}$ (see Figure 1). The homology class of E (up to orientation) is the vanishing cycle of the degeneration to X_{sing}. Note that E is no more symplectic but Lagrangian in the smoothings of X_{sing}. The right and left hand-sides in Figure 1 can be distinguished by the Euler characteristic of the real part



$\mathbf{R}X$ $\mathbf{R}Y$

$\mathbf{R}E \neq \emptyset$: smoothing with local equation $x^2 + y^2 - z^2 = t$



$\mathbf{R}X$ $\mathbf{R}Y$

$\mathbf{R}E = \emptyset$: smoothing with local equation $x^2 + y^2 + z^2 = t$

FIGURE 1. Smoothing of a real node.

($\chi(\mathbf{R}X) = \chi(\mathbf{R}Y) - 2$) or the action of the real structure on E (it is τ_X-anti-invariant but τ_Y-invariant).

Following [4], two real symplectic manifolds are said to differ by a *surgery along a real Lagrangian sphere* if they can be obtained one from the other by a local surgery in a neighborhood of a real (-2)-symplectic sphere as depicted in Figure 1.

Example. The quadric hyperboloid and ellipsoid differ by a surgery along a real Lagrangian sphere realizing the class $(-1, 1) \in H_2(\mathbf{CP}^1 \times \mathbf{CP}^1; \mathbf{Z}) = \mathbf{Z}^2$.

The following classification of real rational algebraic surfaces and of real rational symplectic 4-manifolds are due to Kharlamov-Degtyarev [7] and Kharlamov-Schevchishin [10], respectively.

Theorem 2. *Each deformation class of real rational symplectic* 4*-manifolds contains a real rational algebraic surface. Furthermore any such deformation class can be obtained from* $\mathbf{CP}^2$ *by a finite sequence of blow-up, blow-down, and surgeries along real Lagrangian spheres.*

The following example shows that the real part of a real rational algebraic surface may not be connected.

Example (Minimal real conic bundles). These constitute the infinite family of minimal real rational algebraic surfaces up to deformation. Let $L_1, \cdots, L_n$ be n real lines in $\mathbf{CP}^2$ passing through a common real point p, see Figure 2. Choose

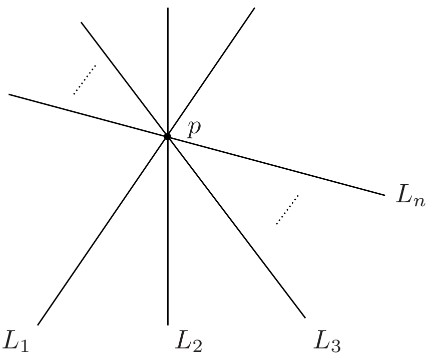


FIGURE 2. Construction of minimal real conic bundles

a pair of complex conjugated points $\{p_{2i-1}, p_{2i}\}$ on each line L_i, and denote by $\widetilde{X}_n$ the real blow-up of $\mathbf{CP}^2$ at the points $p, p_1, \cdots, p_{2n}$. The strict transform in $\widetilde{X}_n$ of the lines $L_1, \cdots, L_n$ provide n disjoint real (-2)-symplectic spheres, and we denote by X_n the surgery of $\widetilde{X}_n$ along all of them. The real part $\mathbf{R}X_n$ consists in n spheres S^2. Denote by E and E_i the exceptional curve corresponding to p and p_i respectively, by l the class of a line, and by τ_n the real structure on X_n. If $n = 1$, then the exceptional curves realizing the classes $[E]$ and $l - [E_1] - [E_2]$ are disjoint and τ_1-conjugated. Hence they can be contracted, that is X_1 is quadric ellipsoid blown-up in two τ_1-conjugated points. If $n \geq 2$, a simple homological computation shows that the real algebraic surface X_n is minimal.

2.2. **Welschinger invariants.** Let $X_{\mathbf{R}} = (X, \omega, \tau)$ be a real rational compact symplectic manifold of dimension 4. Choose a class $d \in H_2^{-\tau}(X; \mathbf{Z})$, and a two non-negative integers r, s such that

$$r + 2s = c_1(X) \cdot d - 1.$$

Choose a configuration $\underline{x}$ made of r points in $\mathbf{R}X$ and s pairs of τ-conjugated points in $X \setminus \mathbf{R}X$. Given a τ-compatible almost complex structure J, we denote by $\mathcal{C}_{X_{\mathbf{R}}}(d, \underline{x}, J)$ the set of real rational J-holomorphic curves in X realizing the class d, and passing through $\underline{x}$. Then we define the integer

$$W_{X_{\mathbf{R}}}(d; s) = \sum_{C \in \mathcal{C}_{X_{\mathbf{R}}}(d, \underline{x}, J)} (-1)^{m(C)},$$

where $m(C)$ is the number of elliptic nodes of C in $\mathbf{R}X$ (i.e. with two τ-conjugated branches). For a generic choice of J, the set $\mathcal{C}_{X_{\mathbf{R}}}(d, \underline{x}, J)$ is finite.

Theorem 3 (Welschinger, Brugallé)**.** *The number $W_{X_{\mathbf{R}}}(d; s)$ only depends on d, s, and the deformation class of $X_{\mathbf{R}}$.*

We call these numbers the *Welschinger invariants of* $X_{\mathbf{R}}$. The proof of Theorem 3 that I know has two steps:

(1) proof of the independence on d, s, the deformation class of $X_{\mathbf{R}}$, but with a fixed number of real points on each connected component of $\mathbf{R}X$ ([15, 16]).
(2) proof of the Independence[3] on the distribution of the r real points in $\mathbf{R}X$ [5].

In its turn Step (2) is an immediate consequence of Step (1), Theorem 2, and the real version of Abramovich-Bertram Formula given below. It would be very interesting to have a direct and one-step proof of Theorem 3.

The following immediate corollary of Theorem 4 has been first proved by Brugallé-Puignau [6].

Corollary 2. *If $\mathbf{R}X$ is not connected and $r \geq 2$, then $W_{X_{\mathbf{R}}}(d; s) = 0$.*

Let us now turn to the real version of Abramovich-Bertram Formula.

Theorem 4. *(Abramovich-Bertram Formula – real version [5]) Let $X_{\mathbf{R}}$ and $Y_{\mathbf{R}}$ be two real rational symplectic manifolds of dimension 4 that differ by a surgery along a real Lagrangian sphere. Suppose that $\chi(\mathbf{R}X) = \chi(\mathbf{R}Y) - 2$. Then for any class $d \in H^{-\tau_Y}(Y; \mathbf{Z})$ (in particular $d \cdot E = 0$), one has*

$$W_{Y_{\mathbf{R}}}(d; s) = W_{X_{\mathbf{R}}}(d; s) + 2 \sum_{k \geq 1} (-1)^k W_{X_{\mathbf{R}}}(d - kE; s).$$

Theorem 4 partially generalizes both [9, Corollary 4.2] and [4, Theorem 1.1, Remark 1.3]. As in the complex case, the proof goes by using a real version of degeneration formulas. In particular the geometric framework is quite clear. Nevertheless the proof relies on the enumeration of real rational curves that are J-holomorphic for a mildly non-generic almost complex structure, enumeration which heavily depends on the choice of J when working over $\mathbf{R}$. In particular contrary to the complex situation, it is not obvious à priori that such universal formula should exist, relating Welschinger invariants of both sides of the wall defined by a real (-2)-symplectic sphere. I am still puzzled by the spectacular cancellations occurring in the technical computations of the proof of Theorem 4 from [5].

The following Welschinger Formula [15] can be obtained by applying Theorem 4 to the blow-up of two infinitely close real points.

Corollary 3. *Let $X_{\mathbf{R}}$ be a compact real symplectic manifold of dimension 4, and let $\widetilde{X}_{\mathbf{R}}$ be the blow up of $X_{\mathbf{R}}$ at one real ball. We denote by E the exceptional divisor. Then for any $d \in H^{-\tau_X}(X; \mathbf{Z})$ and $s \in \mathbf{Z}_{\geq 0}$ such that $c_1(X) \cdot d - 2s \geq 3$, one has*

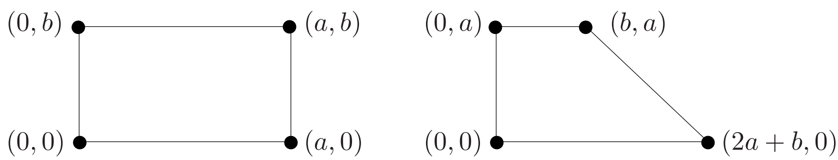
$$W_{X_{\mathbf{R}}}(d; s+1) = W_{X_{\mathbf{R}}}(d; s) - 2W_{\widetilde{X}_{\mathbf{R}}}(d - 2E; s).$$

[3] The Independence statement proved in [5] is even stronger: $W_{X_{\mathbf{R}}}(d; s)$ only depends on s, the deformation class of a minimal model of $X_{\mathbf{R}}$, and the number of blown-up real points on this minimal model.

3. The quadratically enriched case

There is a striking similarity between Corollary 1 and Theorem 4. In particular it is reasonable to wonder, given an invariant interpolating between Gromov-Witten and Welschinger invariants, whether it satisfies the same recursion. There exist at least two examples of such interpolating invariants: the refined tropical invariants, and the quadratically enriched refined invariants.

The case of $\mathbf{CP}^1 \times \mathbf{CP}^1$ and $\mathbf{F}_2$, originally considered by Abramovich and Bertram, provides a first situation where to test this interpolating formula. According to the talk by Sabrina Pauli reporting on a recent work with Andrés Puentes Jaramillo, the following conjecture seems easy to check with currently available computations. We denote by $\square_{a,b}$ (resp. $\triangle_{a,b}$) convex quadrangle in $\mathbf{R}^2$ with vertices $(0,0)$, $(a,0)$, $(0,b)$, and (a,b) (resp. $(0,0)$, $(2a+b,0)$, $(0,a)$, and (b,a)), see Figure 3. Given such a convex quadrangle Δ, there exists a quadrat-



a) $\square_{a,b}$ b) $\triangle_{a,b}$

Figure 3

ically enriched tropical invariant for the corresponding toric surface and ample divisor, that we denote by $N_\Delta^{\mathbb{A}_1}$.

Conjecture 1. *For any integers $a, b \geq 0$, the quadratically enriched tropical invariants $N_\square^{\mathbb{A}_1}$ and $N_\triangle^{\mathbb{A}_1}$ satisfy the following relations:*

$$N^{\mathbb{A}_1}_{\square_{a,a+b}} = \sum_{k \geq 0} \binom{b+2k}{k} N^{\mathbb{A}_1}_{\triangle_{a-k,b+2k}}.$$

An analogous conjecture regarding tropical refined invariants has been formulated in [5] and proved by Bousseau in [2].

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Logarithmic, tropical, leaky double Hurwitz numbers. And friends...

Renzo Cavalieri

(joint work with Hannah Makrwig, Dhruv Ranganathan, Johannes Schmitt)

This talk is aimed at presenting some recent developments in the story that initiated over a decade ago, when double Hurwitz numbers where for the first time tied with tropical geometry.

Let $\mathbf{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}$ be a vector of integers adding to zero. The *double Hurwitz number* $H_g(\mathbf{x})$ counts the number of covers of $\mathbb{CP}^1$ by a curve of genus g, with ramification profiles

$$\mathbf{x}^- = \{x_i | x_i < 0\} \text{ over } \infty, \quad \text{and} \quad \mathbf{x}^+ = \{x_i | x_i > 0\} \text{ over } 0,$$

and simple ramification over $r = 2g - 2 + n$ fixed points of $\mathbb{P}^1$.

It is standard to weight every cover by the size of its automorphism group; the ramification data $\mathbf{x}$ is a vector, rather than a multi-set. Note in particular that the inverse images of 0 and ∞ in the covering curve are labelled, and cannot be interchanged by automorphisms. Double Hurwitz numbers have incarnations in many different mathematical areas. For example, they may be viewed as counting factorizations of the identity in the symmetric group or as the result of a multiplication problem in the class algebra of the symmetric group. The piecewise polynomial structure of double Hurwitz numbers was first obtained by interpreting them as counts of decorated ribbon graphs, which naturally led to a translation to the count of lattice points inside appropriate polytopes.

In joint work with Johnson and Markwig, double Hurwitz numbers are computed as a sum over weighted graphs, called monodromy graphs. Such combinatorial computation may also be viewed as the degree of a tropical branch morphisms between a space of tropical stable maps and a space of tropical branch divisors, thus giving rise to a correspondence theorem.

Definition 1 (Monodromy graphs)**.** For fixed g and $\mathbf{x} = (x_1, \dots, x_n)$, a graph Γ is a *monodromy graph of type* $(g, \mathbf{x})$ if:

(1) Γ is a connected, directed graph,with first betti number equal to g.
(2) Γ has n ends which are directed inward, and labeled by the expansion factors $x_1, \dots, x_n$. If $x_i > 0$, we say it is an *in-end*, otherwise it is an *out-end*.
(3) Vertices of Γ of valence greater than 2 are exactly 3-valent.
(4) After reversing the orientation of the out-ends, Γ does not have sinks or sources. The vertices are ordered compatibly with the partial ordering now induced by the directions of the edges.
(5) Every bounded edge e of the graph is equipped with an expansion factor $w(e) \in \mathbb{N}$. Each integer x_i should be thought as an expansion factor for the corresponding end of Γ, and its sign should be switched when reversing the orientation of the end. These satisfy the *balancing condition* at each 3-valent vertex: the sum of all expansion factors of incoming edges equals the sum of the expansion factors of all outgoing edges.

Monodromy graphs remember the combinatorial information that is needed to compute the double Hurwitz numbers.

Theorem 1. *The double Hurwitz number $H_g(\mathbf{x})$ equals the sum over all monodromy graphs Γ of type $(g, \mathbf{x})$, where each is given multiplicity m_Γ equal the product of the expansion factors of its bounded edges:*

$$H_g(\mathbf{x}) = \sum_{\Gamma} \frac{1}{|Aut(\Gamma)|} \prod_{e} \omega(e).$$

In recent work with Markwig and Ranganathan we introduced a new perspective on double Hurwitz numbers as intersection numbers in the logarithmic Chow ring of the moduli spaces of curves.

Given a stack X with a normal crossings boundary divisor, a *simple toroidal blowup* is a blowup $\mathsf{X}' \to \mathsf{X}$ along a smooth stratum; the preimage of the boundary is again normal crossings. A *toroidal blowup* is a morphism $\mathsf{Y} \to \mathsf{X}$ obtained by a sequence of simple toroidal blowups. The *logarithmic Chow ring of* X is the colimit of the Chow rings $\mathsf{CH}^\star(\mathsf{X}')$ under pullback. A fruitful source of logarithmic cohomology classes in this Chow ring are piecewise polynomial functions on the cone complex of X. A distinguished, and intensely studied class in the logarithmic Chow ring of the moduli spaces of curves is the logarithmic double ramification cycle $\mathsf{DR}_g^{\mathsf{log}}(\mathbf{x}, 0)$.

Theorem 2. *The intersection number of* $\mathsf{DR}_g^{\mathsf{log}}(\mathbf{x},0)$ *with the branch polynomial is equal to the double Hurwitz number:*

$$H_g(\mathbf{x}) = \deg\left([br(\mathbf{x},0)] \cap \mathsf{DR}_g^{\mathsf{log}}(\mathbf{x},0)\right).$$

The branch polynomial may be explicitly represented by a cycle consisting of a collection of strata in a toroidal blowup of $\overline{M}_{g,n}$.

Once the double Hurwitz numbers have been expressed as above, the double ramification cycle can be replaced with its pluricanonical variants, with the intersection number coming from the same piecewise polynomial function as before. We show that these intersection numbers can be calculated by tropical geometry, and import statements from double Hurwitz theory into the pluricanonical context via their tropical interpretations.

This most recent work is collaborative work with Hannah Markwig, Dhruv Ranganathan and Johannes Schmitt.

Fourier decompositions of the volume polynomial, and applications

Karim Adiprasito

The volume polynomial, or degree map, is the canonical way to write the fundamental class in a standard Gorenstein ring, such as the semigroup algebra of a reflexive IDP lattice polytope, a complete toric variety or a complete intersection. As such, it is a rational function over the torus action, or alternatively, a rational function over the Artinian reduction of the ring.

We take this perspective, and describe this rational function in terms of Parseval identities in the first two cases. This implies Lefschetz properties for the semigroup algebras of reflexive lattice polytopes, and hence implies the unimodality of their h^*-vectors.

Measuring holes of hypersurfaces

Lionel Lang

In 2000, Mikhalkin introduced a class of real algebraic planar curves now known as simple Harnack curves, see [1]. Among their many nice properties, these curves appear as spectral curves of planar dimers. In this context, Kenyon and Okounkov showed in 2003 that any simple Harnack curve is determined by the logarithmic area of some well chosen membranes bounded on the curve (plus some boundary conditions), see [2]. In this talk, we report on a work in progress whose aim is to generalise these results and discuss potential applications.

Consider a bivariate polynomial $f(z,w)$ of degree d together with its zero locus $C := \{f = 0\}$ in $(\mathbb{C}^\star)^2$. In toric and tropical geometry, it is customary to consider the amoeba $\mathcal{A}(C) \subset \mathbb{R}^2$ of C where

$$\begin{array}{rcl} \mathcal{A} : (\mathbb{C}^\star)^2 & \to & \mathbb{R}^2 \\ (z,w) & \mapsto & \big(\log|z|, \log|w|\big) \end{array}.$$

The curve C is simple Harnack if its defining polynomial f is real and if the restriction of $\mathcal{A}$ to C is at most 2-to-1 on $\mathcal{A}(C)$. In this case, the restriction $\mathcal{A}_{|C}$ is the quotient of C under complex conjugation and the complement of the closed subset $\mathcal{A}(C) \subset \mathbb{R}^2$ contains $g := \binom{d-1}{2}$ open discs, the *holes* of C. Denote by $\Phi(C) \in \big(\mathbb{R}_{>0}\big)^g$ the vector whose coordinates are the Euclidean area of the latter holes. We denote by $\overline{C}$ the closure of C in $\mathbb{C}P^2$ and refer to the set $\overline{C} \setminus (\mathbb{C}^\star)^2$ as the boundary of C. Kenyon and Okounkov showed that Φ is a global diffeomorphism from the space of simple Harnack curves of degree d and with given boundary onto $\big(\mathbb{R}_{>0}\big)^g$.

We can actually measure holes of arbitrary planar curves. Consider a collection of g discs $M_1, \cdots, M_g \subset (\mathbb{C}^\star)^2$ with respective boundary $\gamma_1, \cdots, \gamma_g \subset C$. Given the form $\omega := \frac{dz\wedge dw}{zw}$, define now $\Phi(C) := \big(\int_{M_j} \omega\big)_{1\leq j\leq g} \in \mathbb{C}^g$. We can even make sense of $\Phi(\widetilde{C})$ for curves $\widetilde{C}$ is any simply connected neighbourhood of C. Then, we have the following statement.

Theorem 1. Assume that the subspace of $H_1(\overline{C}, \mathbb{Z})$ generated by $\gamma_1, \cdots, \gamma_g$ is Lagrangian with respect to the intersection form on H_1. For any simply connected neighbourhood U of C consisting of curves with given boundary, then $\Phi : U \to \mathbb{C}^g$ is locally biholomorphic.

We refer to [4] for a proof and [3] for a similar statement in the context of K3 surfaces.

The above result can be generalised in two directions: first, to higher dimensions and second, to the tropical setting. Consider a hypersurface $Y \subset (\mathbb{C}^\star)^{n+1}$ of some degree d. Given discs $M_1, \cdots, M_g \subset (\mathbb{C}^\star)^{n+1}$ of real dimension $n+1$ with respective boundary $\gamma_1, \cdots, \gamma_g \subset Y$, where $g := h^{n,0}(Y)$, we can consider the vector $\Phi(Y) := \big(\int_{M_j} \omega\big)_{1\leq j\leq g} \in \mathbb{C}^g$, where $\omega := \frac{dz_1\wedge\cdots\wedge dz_{n+1}}{z_1\cdots z_{n+1}}$. Denote by V the subspace of $H_n(\overline{Y}, \mathbb{C})$ generated by $\gamma_1, \cdots, \gamma_g \subset Y$. Then, we have the following statement.

Theorem 2. Assume that the pairing $V \times H^{n,0}(\overline{Y}, \mathbb{C}) \to \mathbb{C},\ (\gamma, \xi) \mapsto \int_\gamma \xi$ is nondegenerate. For any simply connected neighbourhood U of Y consisting of hypersurfaces with given boundary, then $\Phi : U \to \mathbb{C}^g$ is locally biholomorphic.

In the tropical setting, we have a stronger statement. For any tropical hypersurface $Z \subset \mathbb{R}^{n+1}$ with Newton polytope Δ, the set of compact connected components of $\mathbb{R}^{n+1} \setminus Z$ is canonically indexed by a subset of the set A of lattice points in the interior of Δ. We can define $\Phi^T(Z) \in (\mathbb{R}_{\geq 0})^A$ to be the vector that records the Euclidean volume of these components.

Theorem 3. The map Φ^T is a homeomorphism from the space of tropical hypersurfaces of degree Δ with fixed boundary to $(\mathbb{R}_{\geq 0})^A$.

We have two potential applications in mind. On the one hand, we can use these tools to understand tropicalisation of hypersurfaces and abstract varieties. A family $Y_t \subset (\mathbb{C}^\star)^{n+1}$ of hypersurfaces is said to tropicalise to a tropical hypersurface $Z \subset \mathbb{R}^{n+1}$ if the latter is the Hausdorff limit of the family of rescaled amoebas

$\frac{1}{\log(t)}\mathcal{A}(Y_t) \subset \mathbb{R}^{n+1}$. This can also be understood in term of the convergence of $\frac{1}{\log(t)}\mathrm{Re}\big(\Phi(Y_t)\big)$ towards $\Phi^T(Z)$.

On the other hand, we hope to generalise the description of simple Harnack curves and their deformations to other families of real algebraic hypersurfaces with the help of the map Φ. Using combinatorial patchworking, we can construct real projective hypersurfaces Y of any degree in any dimension whose real part contains g components $M_1, \cdots, M_g$ satisfying the assumptions of Theorem 2. The challenge is to exhibit large neighbourhoods of Y on which Φ is a global biholomorphism in order to describe possible deformations of Y in the real setting.

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The Hitchin fibration and tropical geometry

Martin Ulirsch

(joint work with Barbara Bolognese and Alex Küronya, Andreas Gross and Dmitry Zakharov as well as with Andreas Gross, Inder Kaur, and Annette Werner)

Let X be a smooth and projective complex variety. The non-abelian Hodge correspondence tells us that there is a one-to-one correspondence between semisimple representations of $\pi_1(X)$ and topologically trivial polystable Higgs bundles on X. It induces a real analytic isomorphism between the Betti moduli space $M^r_{\mathrm{Betti}}(X)$ of characters of $\pi_1(X)$ and the Dolbeault moduli space $M^r_{\mathrm{Dol}}(X)$ of topologically trivial semistable Higgs bundles on X. This tells us that these moduli spaces have the same topological invariants, although the extra structures on cohomology, e.g. coming from Hodge theory, are known to be different.

In [1] we prove a version of the $P = W$ conjecture when X is abelian variety. This generalizes the original case when X is a compact Riemann surface, that was originally stated by Cataldo–Hausel–Migliorini and which has seen two recent proofs in [4] and [3]. On the one hand, associated to the spectral data morphism $M^r_{\mathrm{Dol}}(X) \to \mathrm{Sym}^r(\mathbb{C}^r)$, that naturally factors the Hitchin morphism, there is a *perverse fitlration* P_k on the cohomology $H^*\big(M^r_{\mathrm{Dol}}(X), \mathbb{Q}\big)$ on the Dolbeault moduli space. On the other hand, on Deligne's theory of weigths provides us with a weight filtration W_k on the cohomology $H^*\big(M^r_{\mathrm{Betti}}(X), \mathbb{Q}\big)$ of the Betti moduli space.

A priori these two filtrations might be quite different. In [1] we show that under the isomorphism $H^*\big(M^r_{\mathrm{Dol}}(X),\mathbb{Q}\big) \simeq H^*\big(M^r_{\mathrm{Betti}}(X),\mathbb{Q}\big)$ induced from the non-abelian Hodge correspondence, we have

$$P_k H^*\big(M^r_{\mathrm{Dol}}(X),\mathbb{Q}\big) = W_{2k} H^*\big(M^r_{\mathrm{Betti}}(X),\mathbb{Q}\big) = W_{2k+1} H^*\big(M^r_{\mathrm{Betti}}(X),\mathbb{Q}\big)$$

In upcoming joint work with I. Kaur, A. Gross, and A. Werner, we consider a non-Archimedean analogue of this situation. Let K be an algebraically closed complete non-Archimedean field of characteristic zero. Fix a prime ℓ different from the residue characteristic of K. Let X be an abelian variety over K with completely degenerate reduction. In this case, we may write $X^{an} = \mathbb{G}_m^g/\Lambda$ for a lattice $\Lambda \simeq \mathbb{Z}^g$. There is a natural surjective analytic morphism $M^r_{\mathrm{Betti}}(\Lambda)^{an} \to M_{r,0}(X)^{an}$ from the Betti moduli space of characters of Λ to the moduli space of topologically semistable vector bundles on X. We observe that the ℓ-adic étale cohomology $H^*\big(M_{r,0}(X)^{an},\mathbb{Q}_\ell\big)$ is naturally isomorphic to the ℓ-adic étale cohomology of the Dolbeault moduli space $M^r_{\mathrm{Dol}}(X)$.

Using the notion of tropical vector bundles introduced in [2], we show that the essential skeleton of $M_{r,0}(X)^{an}$ is naturally a moduli space $M_{r,0}(X^{trop})$ of homogeneous tropical vector bundles on the tropicalization $X^{trop} = \mathbb{R}^g/\Lambda$ of X. The tropicalization map $M_{r,0}(X)^{an} \to M_{r,0}(X^{trop})$ may be used to pull back the reduced cohomology of $M_{r,0}(X^{trop})$ and it turns out there is a natural isomorphism

$$H^*_{et}\big(M_{r,0}(X),\mathbb{Q}_\ell\big)/\big\langle \widetilde{H}^*(M_{r,0}(X^{trop},\mathbb{Q}_\ell)\big\rangle \simeq H^*_{et}\big(M^r_{\mathrm{Betti}}(\Lambda),\mathbb{Q}_\ell\big)$$

induced by the analytic morphism $M^r_{\mathrm{Betti}}(\Lambda)^{an} \to M_{r,0}(X)^{an}$. Here the term $\big\langle \widetilde{H}^*(M_{r,0}(X^{trop},\mathbb{Q}_\ell)\big\rangle$ denotes the ideal generated by the pullback of the reduced cohomology $\widetilde{H}^*\big(M_{r,0}(X^{trop}),\mathbb{Q}_\ell\big)$.

Our main result is that under this isomorphism we again have an equality $P_k = W_{2k} = W_{2k+1}$ for a perverse filtration P_k defined on the left via the spectral data morphism and a weight filtration on the right defined using Deligne's theory of weights. Our result is the first known case of a $P = W$ phenomenon over non-Archimedean fields and the first that involves a tropical correction term.

This raises a natural question whether a similar $P = W$ phenomenon with a tropical correction occurs in other non-Archimedean situations, e.g. on a Mumford curve.

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Local tropicalization: a combinatorial approach to singularity links

Maria Angelica Cueto
(joint work with Patrick Popescu-Pampu, Dmitry Stepanov)

Local tropicalizations are combinatorial shadows of singularities (i.e., germs of complex analytic spaces). First introduced by Popescu-Pampu and Stepanov [8], they were constructed by mimicking the tropicalization of ideals of polynomial rings. More precisely,

Definition 1. Given an equidimensional singularity $(X, 0)$ of $(\mathbb{C}^n, 0)$ defined by an ideal $I = \langle f_1, \ldots, f_s\rangle \subset \mathcal{O} := \mathbb{C}\{z_1, \ldots, z_n\}$ of convergent power series near the origin, the *local tropicalization* of I (or X) equals

$$\mathrm{Trop}_{\geq 0}(I) = \{w \in \mathbb{R}^n_{\geq 0} : \mathrm{in}_w(I) \subset \mathcal{O} \text{ has no monomials}\}.$$

These object share many of the nice properties satisfied by embedded tropical varieties: they support weighted balanced rational polyhedral fans of the same dimension as the input germ. These local tropical fans can be used to recover topological information of singularity links, provide partial resolutions of singularity germs in toroidal varieties ("à la Tevelev" [9]), simplify proofs of classical statements [1], and establish some yet open conjectures in singularity theory [2].

The *link* of a complex isolated normal surface singularity $(X, 0)$ is a compact, connected and oriented real threefold. Via the plumbing calculus of negative-definite connected plumbing graphs, Neumann [5] showed that the oriented topological type of the link determines the oriented topological type of the minimal good resolution of $(X, 0)$. However, the zoology of such links is still a mystery. In particular, it is not known which links arise as hypersurface links in $\mathbb{C}^3$ or local complete intersection surfaces.

The goal of this talk was to illustrate how these techniques can be used to shed new light into splice type surface singularities, a large class whose links include all known integral homology sphere links. Such germs were introduced by Neumann and Wahl in [6, 7] as a generalization of the class of Pham-Brieskorn-Hamm complete intersections [4] of dimension two. Their construction depends on a weighted tree (i.e., a graph with no loops) called a splice diagram.

Definition 2. A *splice diagram* Γ is a tree with no bivalent vertices, such that for each node v (i.e., a vertex of valency at least three), every incident edge e is decorated by a positive integer $d_{v,e}$ in the neighborhood of v. We let $\partial\Gamma = \{\lambda_1, \ldots, \lambda_n\}$ denote the set of leaves of Γ (i.e., valency one vertices), and identify each λ_i with a variable z_i. We say Γ is *coprime* if the weights around each fixed node v are pairwise coprime.

In order to produce a system of $n-2$ equations in the variables $z_1, \ldots, z_n$, the weights on Γ are required to satisfy two arithmetic properties: the edge determinant and semigroup conditions. If u and v are two adjacent nodes of Γ, the *determinant* of the edge $[u, v]$ is the number obtained by subtracting from the product of the two decorations on $[u, v]$ the product of the remaining decorations in the neighborhoods of u and v. We say that Γ verifies the *edge determinant condition* if the determinant of every internal edge of Γ is positive.

The *linking number* $\ell_{u,v}$ between two vertices u, v of Γ is the product of all weights adjacent to but not on the unique geodesic $[u, v]$ joining u and v. These numbers determine an integer weight vector for each node v of Γ, namely $w_v := (\ell_{v,\lambda_1}, \ldots, \ell_{v,\lambda_n})$. Given an edge e of Γ and a node v on it, the pair (v, e) satisfies the *semigroup condition* if $\ell_{v,v} \in \mathbb{N}_0\langle \ell_{v,\lambda} : \lambda \in \partial\Gamma, e \subset [v, \lambda]\rangle$. We require this condition to hold for all such pairs (v, e). The scalars in the semigroup condition for (v, e) give a monomial $\underline{z}^{m_{v,e}}$ in the variables seen from v in the direction of e.

A *strictly splice-type system* associated to Γ is a finite family of $(n-2)$ series

$$\{f_{v,i} := \sum_{e \text{ edge}, v \in e} c_{v,e,i}\, \underline{z}^{m_{v,e}} : v \text{ node of } \Gamma, i = 1, \ldots, \mathrm{val}(v) - 2\},$$

where all maximal minors of each $(\mathrm{val}(v)-2)\times\mathrm{val}(v)$ matrix of coefficients $(c_{v,e,i})_{i,e}$ are non-zero. Pham-Brieskorn-Hamm systems arise from splice diagrams with a single node, and include all A_n, E_6 and E_8 singularities. A *splice type* system $\mathcal{S}(\Gamma)$ for Γ is obtained by replacing each series $f_{v,i}$ by one of the form $F_{v,i} := f_{v,i} + g_{v,i}$ where the w_v-weight of each monomial $\underline{z}^m$ in the support of $g_{v,i} \in \mathcal{O}$ is strictly larger than $\ell_{v,v}$. This condition ensures that $\mathrm{in}_{w_v}(F_{v,i}) = \mathrm{in}_{w_v}(f_{v,i})$ for all v, i.

Definition 3. A *splice type surface singularity* $(X, 0)$ is any germ defined by a splice-type system $\mathcal{S}(\Gamma)$.

Local tropicalization recovers the following central result of Neumann and Wahl:

Theorem 1 ([1, 6, 7])**.** *Splice type systems define isolated complete intersection normal complex surface singularities.*

The rich convexity properties of the collection of weight $\{w_v\}_v$ defined by the nodes of Γ allow us to embed $\Gamma \hookrightarrow \Delta_{n-1}$ via $v \mapsto w_v/|w_v|$ for each node v of Γ and $\lambda_i \mapsto e_i$ for each leaf λ_i, and extending linearly along edges. Here, e_i is the ith. standard basis element of $\mathbb{R}^n$ and $|\cdot|$ denotes the 1-norm in $\mathbb{R}^n$. We let $\mathcal{F} := \mathbb{R}_{\geq 0}\Gamma$ be the collection of cones induced by this this embedded graph. Theorem 1 becomes a direct consequence of the following result:

Theorem 2 ([1])**.** *The set $\mathcal{F}$ is a fan, and it is supported on the local tropicalization* $\mathrm{Trop}_{\geq 0}\langle\mathcal{S}(\Gamma)\rangle$*. The weights of Γ yield explicit combinatorial formulas for the tropical multiplicities on $\mathcal{F}$. Furthermore, the collection of initial ideals* $(\mathrm{in}_w(\langle\mathcal{S}(\Gamma)\rangle))_w$ *in $\mathcal{O}$ is constant along relative interiors of cones of $\mathcal{F}$.*

Remark 1. In particular, when Γ is coprime, all tropical multiplicities equal one and all weight vectors w_v are primitive. Furthermore, Γ can be uniquely recovered from $\mathrm{Trop}_{\geq 0}\langle\mathcal{S}(\Gamma)\rangle$. Both statements fail for general Γ [1].

Theorem 3. *Splice type systems determine Newton non-degenerate complete intersections in the sense of Khovanskii. More precisely, the set $\{F_{v,i}\}_{v,i}$ defining the system $\mathcal{S}(\Gamma)$ is a regular sequence in the local ring $\mathcal{O}$ and for all $w \in \mathbb{R}^n_{>0}$, the gradients $\{\nabla_p \,\mathrm{in}_w(F_{v,i})\}_{v,i}$ are linearly independent for each point $p \in (\mathbb{C}^*)^n$ in the zero-locus of $\langle\mathcal{S}(\Gamma)\rangle$.*

Surprisingly, not many examples of Newton non-degenerate complete intersection systems are known in codimension two or higher. Theorem 3 contributes a large class of examples of such systems.

The previous results can be used to obtain embedded resolutions of complex plane curve singularities by composing re-embeddings of $\mathbb{C}^2$ into higher-dimensional smooth spaces $\mathbb{C}^n$ with toric modifications of $\mathbb{C}^n$. More precisely, we recover the following statement due to de Felipe, González Pérez and Mourtada [3]:

Corollary 1. *Let* $(Y, 0) \hookrightarrow \mathbb{C}^2$ *be the germ of a reduced complex analytic plane curve. Then, the ambient germ* $(\mathbb{C}^2, 0)$ *can be holomorphically re-embedded into a suitable higher-dimensional germ* $(\mathbb{C}^n, 0)$ *in such a way that the induced germ* $(Y, 0) \hookrightarrow \mathbb{C}^n$ *can be resolved by a single toric modification of* $\mathbb{C}^n$.

The talk concluded with a partial resolution statement for splice type surface singularities, which is a direct consequence from the local version of Tevelev's construction of tropical compactifications [9]. It provides the starting point to perform a full resolution of splice type surface singularities. We will address the latter in future work.

Corollary 2. *Let* $(Y, 0)$ *be the germ defined by a splice type system* $\mathcal{S}(\Gamma)$. *Consider the birational map* $\pi_{\mathcal{F}} \colon X_{\mathcal{F}} \to \mathbb{C}^n$ *of toric varieties induced by the fan* $\mathcal{F}$ *from Theorem 2 and let* $\tilde{Y}$ *be the strict transform of* Y *under* $\pi_{\mathcal{F}}$. *Then, the pair* $(\tilde{Y}, \tilde{Y} \cap \partial(X_{\mathcal{F}}))$ *is toroidal and boundary transversal.*

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Tropical Geometry beyond the toric case

Helge Ruddat
(joint work with Michel van Garrel, Tim Gräfnitz, Bernd Siebert, Eric Zaslow, Benjamin Zhou)

Meta-theorem 1. *Counting tropical curves in $\mathbb{R}^n$ yields in the same non-negative number as counting algebraic curves in a projective toric variety (when matching constraints, genus and degree).*

Proof: [B17, CFPU16, CJMR17, GM07, GPS10, G10, G18, GS22, IKS03, LR18, M05, M07, MR09, MR20, NS06, O15, R17] and more. □

A counterexample:

$$^{\mathrm{trop}}\langle \ell\psi^2\rangle_{0,\ell}^{\mathbb{P}^2} = 0 \qquad \text{versus} \qquad \langle \ell\psi^2\rangle_{0,\ell}^{\mathbb{P}^2} = -3$$

for ℓ the class of a line in $\mathbb{P}^2$; in words: the count of genus zero tropical curves satisfying a double ψ-class condition at a line in $\mathbb{R}^2$ is zero whereas the similar count for algebraic curves in $\mathbb{P}^2$ is strictly negative. It is straightforward to see the tropical count is zero because a double psi class condition means the tropical line must have a valency higher than 3, however a tropical line has only a single vertex and this one is of valency three. Gross gave a peculiar way of combining counts of tropical curves for several different constraints with intricate multiplicities to account for this difference in [G10]. On the other hand, it was shown in [MR20] that

$$^{\mathrm{trop}}\langle \ell\psi^2\rangle_{0,\ell}^{\mathbb{P}^2} = \langle \ell\psi^2\rangle_{0,\ell}^{\mathbb{P}^2(\log D)}$$

(and a similar more general correspondence of tropical counts and log Gromov-Witten counts was proved). $\mathbb{P}^2(\log D)$ is the log pair of $\mathbb{P}^2$ together with the toric boundary divisor D which is the union of the three coordinate lines. A log stable map is tacitly equipped with additional markings that map to D, so there are three such extra markings when looking at a line. When removing these markings via the divisor axiom, one does indeed arrive at the ordinary curve count of -3:

$$\langle \ell\psi^2\rangle_{0,\ell}^{\mathbb{P}^2(\log D)} = \langle \ell\psi^2\rangle_{0,\ell}^{\mathbb{P}^2} + 3\langle [\mathrm{pt}]\psi\rangle_{0,\ell}^{\mathbb{P}^2}$$

where [pt] is the class of a point and it is well known that $\langle [\mathrm{pt}]\psi\rangle_{0,\ell}^{\mathbb{P}^2} = 1$, for instance because it matches the tropical curve counts of a tropical line that meets a fixed given point with its vertex ([MR09]).

We see that it needed the study of ψ-class curve counts in order to notice that there is a difference between ordinary curve counts and log curve counts. The upshot is that the tropical count is very generally a log curve count but not as generally an ordinary curve count.

What if we change the log divisor? The log space $\mathbb{P}^2(\log D)$ is neither isomorphic to $\mathbb{P}^2(\log E)$ nor is it deformation equivalent when E is any smooth cubic plane curve even though D is of course deformation equivalent to E inside $\mathbb{P}^2$. So if counting log curves in $\mathbb{P}^2(\log D)$ matches counting tropical curves in the $\mathbb{R}^2$, is

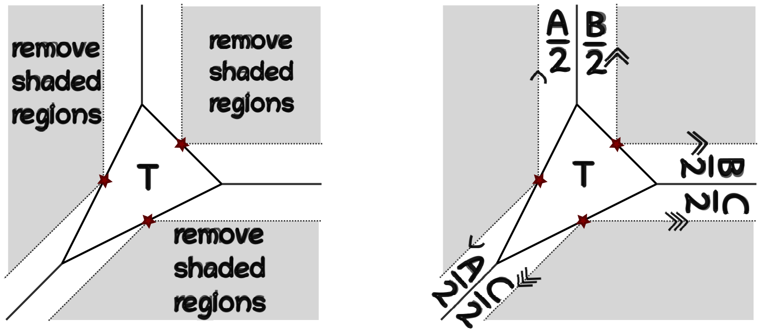


FIGURE 1. Solution 1: glue the three pairs of boundary rays of the removed regions by the unique affine transformation respectively that preserves the adjacent edge of T.

there a tropical space that captures the geometry of $\mathbb{P}^2(\log E)$ and can we count curves in it whose count matches a log Gromov-Witten invariant for $\mathbb{P}^2(\log E)$?

The answer is "yes" but we have to accept a new reality where we replace the non-singular integral affine manifold $\mathbb{R}^2$ by a singular integral affine manifold B that I next want to describe. The do-it-yourself instructions for making this singular manifold is

Task of making B.

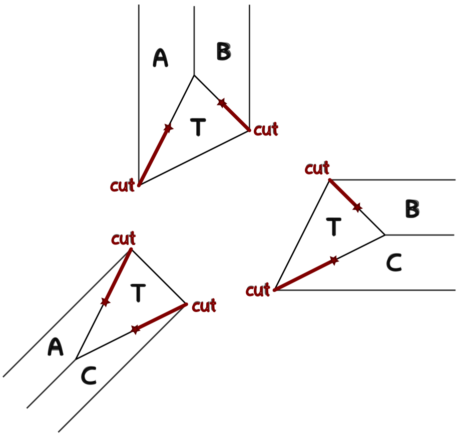
(1) Take three copies of the strip $[0,1] \times \mathbb{R}_{\geq 0}$ (let us call these A, B, C) and one copy of the affine triangle T given by the convex hull of the points $(1,0)$, $(0,1)$ and $(-1,-1)$;
(2) Glue these four pieces along their boundaries to produce a planar object without boundary.

I am going to describe three different solutions to this task, but first let me point out the typical mistake that students who work on this problem will do: they will want to take the boundary of $T \times \mathbb{R}_{\geq 0}$ inside $\mathbb{R}^2 \times \mathbb{R}$ as their solution but this is not a *planar* object. After all, we are trying to make a tropical surface because $\mathbb{P}^2(\log E)$ is also a surface.

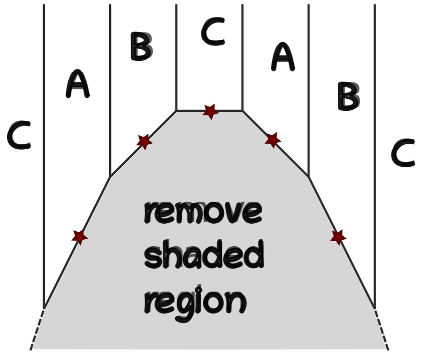
You find the solutions in Figures 1, 2 and 3 respectively. The results are the same: an integral affine manifold B with three "focus-focus" singularities whose underlying topological manifold is homeomorphic to $\mathbb{R}^2$.

Meta-theorem 2. *Counting tropical curves in B yields the same number as counting algebraic curves in $\mathbb{P}^2(\log D)$.*

Proof: [CPS23, Gr20, Gr22, GRZ22, GRS23] and more to come! □



(A) Solution 2: Glue three affine charts, pairwise by a piecewise affine transformation that identifies the pieces with the same labels.



(B) Solution 3: After removing the convex hull of the discrete parabola, take the quotient by the free cyclic group generated by the third power of the affine transformation $\binom{x}{y} \mapsto \begin{pmatrix}1 & 0\\ -3 & 1\end{pmatrix} \cdot \binom{x}{y} + \binom{1}{0}$. The triangle T can then be glued to the boundary of the quotient.

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Patchworks of real algebraic varieties close to a smooth tropical limit

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(joint work with Arthur Renaudineau, Kris Shaw)

In my talk, I will present generalizations to higher codimensions of constructions related to Viro's primitive patchworking method for hypersurfaces [6]. In the following, I give a brief summary of our basic definitions and statements.

Let X be a polyhedral complex in $\mathbf{R}^n$. Here, by abuse of notation, X denotes a collection of polyhedra (a fixed polyhedral subdivision of X) as well as the support of X. We denote by $X^{(k)}$ the subset of polyhedra in X of dimension k. Throughout the following we assume that X is of pure dimension d and rational. The latter means that all polyhedra in X have rational tangent spaces: For $\sigma \in X$, let $T_{\mathbf{R}}(\sigma)$ denote the real vector space spanned by differences of vectors in σ. Set $T_{\mathbf{Z}}(\sigma) = T_{\mathbf{R}}(\sigma) \cap \mathbf{Z}^n$. Then $T_{\mathbf{R}}(\sigma) = T_{\mathbf{Z}}(\sigma) \otimes \mathbf{R}$. We are mostly interested in the mod 2 reduction of the tangent space,

$$T(\sigma) := T_{\mathbf{Z}_2}(\sigma) = T_{\mathbf{Z}}(\sigma) \otimes \mathbf{Z}_2.$$

Here, $\mathbf{Z}_2 = \{0, 1\}$ denotes the field with two elements.

Definition 1. A *real phase structure* on X is a map

$$\mathcal{E} \colon X^{(d)} \to \mathrm{Pow}(\mathbf{Z}_2^n)$$

satisfying

(1) for every facet $\sigma \in X^{(d)}$, the set $\mathcal{E}(\sigma)$ is an affine subspace of $\mathbf{Z}_2^n$ parallel to $T(\sigma)$, that is, $\mathcal{E}(\sigma) = \epsilon_0 + T(\sigma)$ for some $\epsilon_0 \in \mathbf{Z}_2^n$,
(2) for every codimension one face $\tau \in X^{(d-1)}$ with adjacent facets $\sigma_1, \dots, \sigma_k$, we have

$$\mathcal{E}(\sigma_1) \triangle \dots \triangle \mathcal{E}(\sigma_k) = \emptyset.$$

Here, $S \triangle T := (S \cup T) \setminus (S \cap T)$ denotes the symmetric difference.

We note that in general the symmetric difference $\mathcal{E}(\sigma_1)\triangle\dots\triangle\mathcal{E}(\sigma_k)$ is the set of elements that is contained in an odd number of $\mathcal{E}(\sigma_i)$, $i = 1,\dots,k$. Hence, our condition requires that every element is contained in an even number of $\mathcal{E}(\sigma_i)$ (possibly in none). We call such a collection of subsets an *even covering*.

From now on, we fix a unimodular pointed fan Σ in $\mathbf{R}^n$. We denote by $\mathbf{R}\Sigma$ and $\mathbf{T}\Sigma$ the associated real and tropical toric varieties. Via the extended logarithm map, we can think of $\mathbf{R}\Sigma$ as a space obtained by gluing 2^n copies of $\mathbf{T}\Sigma$, labelled by $\epsilon \in \mathbf{Z}_2^n$. Given to a polyhedral complex X with real phase structure $\mathcal{E}$, we define its *patchwork* $\mathbb{R}(X,\mathcal{E})$ as the subset in $\mathbf{R}\Sigma$ whose restriction to the copy of $\mathbf{T}\Sigma$ labelled by $\epsilon \in \mathbf{Z}_2^n$ is (the closure of) the union of all facets $\sigma \in X^{(d)}$ for which $\epsilon \in \mathcal{E}(\sigma)$. In other words, the sets $\mathcal{E}(\sigma)$ label the copies of $\mathbf{T}\Sigma$ in which we put a copy of σ.

Remark 2. We note $\mathbb{R}(\mathbf{X},\mathcal{E})$ defines a (cellular) $\mathbf{Z}_2$-chain in $\mathbf{R}\Sigma$. Moreover, this chain is closed: Indeed, condition (2) is equivalent to $\mathbb{R}(\mathbf{X},\mathcal{E})$ being closed in the big open torus $(\mathbf{R}^*)^n \subset \mathbf{R}\Sigma$ while condition (1) implies (and is close to being equivalent to) $\mathbb{R}(X,\mathcal{E})$ being closed at the toric boundary. We recall that the standard balancing condition in tropical geometry can be expressed using the closedness of an associated $\mathcal{F}_d$-chain in tropical homology. We hence may think of conditions (1) and (2) as the real version of the tropical balancing condition.

We now restrict our attention tropically *non-singular* polyhedral complexes X. For brevity, we refrain from giving a detailed definition here. We just state that X is called *non-singular* if for any $p \in X$ the star fan $\operatorname{Star}_p X$ is of degree 1, up to a $\mathrm{GL}(n,\mathbf{Z})$ coordinate change. Equivalently, up to a $\mathrm{GL}(n,\mathbf{Z})$ coordinate change $\operatorname{Star}_p X$ is (the support of) the matroid fan Σ_M associated to a loopless matroid M. First, let us explain the relation to Viro's primitive patchworking.

Remark 3. Let $\mathcal{T}$ be a convex subdivision of an integer polytope Δ, and let X be a tropical hypersurface dual to $\mathcal{T}$ (in particular, $d = n-1$). The condition for X to be non-singular is equivalent to $\mathcal{T}$ being a primitive triangulation. Under this condition, a *sign distribution* $S\colon \Delta\cap\mathbf{Z}_2^n$ induces a real phase structure on X by the following rule. Let $v, w \in \Delta\cap\mathbf{Z}_2^n$ be two vertices connected by an edge in $\mathcal{T}$. Let σ be the facet in X dual to this edge. By condition (1), we only have two choices for $\mathcal{E}(\sigma)$: $T(\sigma)$ or the translation of $T(\sigma)$ not containing 0. We set $\mathcal{E}(\sigma) = T(\sigma)$ if and only if the $S(v) \neq S(w)$ (if the two vertices have different signs). It is not hard to show that this rule defines a real phase structures (satisfying condition (2)) and moreover yields a bijection between sign distributions (up to inverting all signs, $S \mapsto -S$) and real phase structures on X. In this case, $\mathbb{R}(X,\mathcal{E})$ is a well-known tropical description of the Viro's original patchwork space associated to $(\mathcal{T}, S)$.

We prove the following statements.

Theorem 1. *Let X be a non-singular polyhedral complex and $\mathcal{E}$ a real phase structure on X. Then the following holds true.*

(1) *The space $\mathbb{R}(X,\mathcal{E})$ is a topological manifold.*

(2) *The* $\mathbf{Z}_2$*-Betti numbers of* $\mathbb{R}(X, \mathcal{E})$ *are bounded by the tropical* $\mathbf{Z}_2$*-homology numbers of* X*,*

$$b_q(\mathbb{R}(X, \mathcal{E}); \mathbf{Z}_2) \leq \sum_{p=0}^{d} h_q(X; \mathcal{F}_p^{\mathbf{Z}_2}).$$

The same is true for the closed support versions of these numbers.

(3) *The Euler characteristic of* $\mathbb{R}(X, \mathcal{E})$ *equals the signature of* X*,*

$$\chi(\mathbb{R}(X, \mathcal{E})) = \sum_{p,q} (-1)^q h_q(X; \mathcal{F}_p).$$

The same is true for the closed support versions of these numbers.

The proof of (1) is based on the relation of real phase structures to oriented matroids established in [3]. The proof of (2) and (3) is analogous to the hypersurface case treated in [4].

Our main theorem, however, is the fact that $\mathbb{R}(X, \mathcal{E})$ describes the locus of real points of a real family of algebraic varieties closed to the tropical limit $(X, \mathcal{E})$. Let us be more precise:

Let $\mathcal{D}^* \subset \mathbf{C}$ be the punctured unit disc and let $\mathbf{X} \subset (\mathbf{C}^*)^n \times \mathcal{D}^*$ a real meromorphic family of algebraic varieties. Here, $\mathbf{X}$ being real means it is invariant under coordinate-wise conjugation in $(\mathbf{C}^*)^n \times \mathcal{D}^*$. Assume that the associated tropicalisation $\mathrm{Trop}(\mathbf{X}) = X \subset \mathbf{R}^n$ is non-singular. Using the real structure of $\mathbf{X}$, we can equip X with a real phase structure $\mathcal{E}$: For each facet $\sigma \in X^{(d)}$, let p be a point in the relative interior of σ and define $\mathcal{E}(\sigma)$ as the collection of sign vectors coming from points in $\mathbf{R}\mathrm{in}_p\mathbf{X}$. Here, $\mathrm{in}_p\mathbf{X} \subset (\mathbf{C}^*)^n$ denotes the initial variety of $\mathbf{X}$ with respect to the weight vector p, and $\mathbf{R}\mathrm{in}_p\mathbf{X} \subset (\mathbf{R}^*)^n$ is the locus of real points. For $t \in \mathcal{D}^*$, we denote by $\mathbf{X}_t$ the fibre of $\mathbf{X}$ over t, and by $\overline{\mathbf{X}}_t$ its closure in $\mathbf{C}\Sigma$. Our main theorem is as follows.

Theorem 2. *Let* $\mathbf{X} \subset (\mathbf{C}^*)^n \times \mathcal{D}^*$ *be a real meromorphic family with non-singular tropical limit* $X = \mathrm{Trop}(\mathbf{X})$ *and associated real phase structure* $\mathcal{E}$*. Assume that* X *admits a subdivision such that such that* $\mathrm{RecCon}(X) \cup \Sigma$ *is a fan. Here,* $\mathrm{RecCon}(X)$ *denotes the collection of recession cones of all faces of* X*.*

Then for sufficiently small $t \in (0, 1) \subset \mathcal{D}^*$ *the pairs*

$$\mathbf{R}\overline{\mathbf{X}}_t \subset \mathbf{R}\Sigma \qquad \text{and} \qquad \mathbb{R}(X, \mathcal{E}) \subset \mathbf{R}\Sigma$$

are homeomorphic. Moreover, the homeomorphism can be chosen to respect the stratification of $\mathbf{R}\Sigma$ *by torus orbits.*

For $d = n - 1$, using the translation from Remark 3, this is just the tropical version of Viro's primitive patchworking [5].

Combining Theorems 1 and 2, we can conclude that a real algebraic variety close to a non-singular tropical limit has the Euler characteristic equal signature property. cf. [1, 2].

Corollary 1. *Under the conditions of the previous statement (and assuming* $|\Sigma| = \mathbf{R}^n$ *for simplicity), we have*

$$\chi(\mathbf{R}\overline{\mathbf{X}}_t) = \sum_{p,q}(-1)^q h_{p,q}(\overline{\mathcal{X}}_t)$$

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Recent adventures in tropical schemes

Diane Maclagan

The goal of this talk was to give two updates on my program, largely joint with Felipe Rincón, on developing a scheme theory in tropical geometry.

Tropical subschemes of tropical toric varieties were introduced in [3], based on ideas in [1], [4]. We present here the case that the ambient toric variety is $\text{trop}(\mathbb{A}^n) = \overline{\mathbb{R}}^n$.

We write $\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus = \min, \odot = +)$ for the tropical semiring, and $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ for the semiring of tropical polynomials. For a polynomial $f = \sum c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ in $K[x_1, \ldots, x_n]$ we write $\text{trop}(f)$ for the polynomial $\oplus$ $\text{val}(c_{\mathbf{u}}) \odot \mathbf{x}^{\mathbf{u}}$ in $\overline{\mathbb{R}}[x_1, \ldots, x_n]$. The ideal $\text{trop}(I) \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ is the ideal generated by $\{\text{trop}(f) : f \in I\}$. Classically, a subscheme Z of $\mathbb{A}^n$ is given by an ideal I in a polynomial ring $K[x_1, \ldots, x_n]$:

$$Z = \text{Spec}(K[x_1, \ldots, x_n]/I).$$

In [1] the Giansiracusas proposed the following tropicalization:

$$\text{trop}(Z) = \text{Spec}(\overline{\mathbb{R}}[x_1, \ldots, x_n]/\mathcal{B}(\text{trop}(I))). \tag{1}$$

Here Spec should be taken in the sense of $\mathbb{F}_1$-geometry, and $\mathcal{B}(\text{trop}(I))$ is the *bend congruence*. The bend congruence is the congruence generated by $f \sim f_{\hat{\mathbf{u}}}$ for all $f \in I$ and monomials $\mathbf{x}^{\mathbf{u}}$ occuring in f, where $f_{\hat{\mathbf{u}}}$ is the tropical sum of all terms of f except the one containing $\mathbf{x}^{\mathbf{u}}$.

This gives a theory of *realizable* tropical schemes. To extend this to an arbitrary subscheme of $\text{trop}(\mathbb{A}^n)$, without the condition that it be the tropicalization of a subscheme of $\mathbb{A}^n$, a natural generalization would be to replace $\text{trop}(I)$ in (1) by an arbitrary ideal $J \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$. However varieties of arbitrary ideals in

$\overline{\mathbb{R}}[x_1, \ldots, x_n]$ can be fairly arbitrary, and in particular, are often not the supports of finite polyhedral complexes as is expected in tropical geometry.

In [3], with Rincón we defined the following special class of ideals, which contains the class of tropicalizations of ideals in a polynomial ring, and proposed these as the correct class of ideals to define tropical schemes, replacing $\operatorname{trop}(I)$ in (1).

Definition 1. A *tropical ideal* is an ideal $J \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ with the property that for all $f, g \in J$ with the coefficient $[f]_{\mathbf{u}}$ of $\mathbf{x}^{\mathbf{u}}$ equal to that of g, there is $h \in J$ with $[h]_{\mathbf{u}} = \infty$, and $[h]_{\mathbf{v}} \geq \min([f]_{\mathbf{v}}, [g]_{\mathbf{v}})$ for all other $\mathbf{x}^{\mathbf{v}}$, with equality when they differ.

Definition 1 can be reinterpreted as imposing some valuated matroid structure on J. This has the following consequences, which appear in [3] and [5].

Theorem 1. *Let $J \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ be a tropical ideal. Then*

(1) *The variety $V(J) \subseteq \overline{\mathbb{R}}^n$ is the support of a finite polyhedral complex. The top dimensional cells of this complex are balanced with respect to an intrinsic balancing condition.*
(2) *The ideal J satisfies the* weak *Nullstellensatz: $V(J) = \emptyset$ if and only if $J = \overline{\mathbb{R}}[x_1, \ldots, x_n]$.*

In forthcoming work with Rincón, we prove the following version of the *strong* Nullstellensatz. Recall that the classical version of the strong Nullstellensatz states that, for a polynomial f and ideal I in $K[x_1, \ldots, x_n]$, we have $V(I)$ contained in $V(f)$ if and only if $f^m \in I$ for some $m > 0$. For a polynomial $f \in \overline{\mathbb{R}}[x_1, \ldots, x_n]$ we write $\operatorname{conv}(f)$ for the polynomial with the smallest possible coefficients that determines the same function as f. For example, $\operatorname{conv}(x^2 \oplus 0) = x^2 \oplus x \oplus 0$. The name comes from the fact that $\operatorname{conv}(f)$ has all terms on the boundary of the extended Newton polytope $\operatorname{conv}((\mathbf{u}, c_{\mathbf{u}}) : c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ is a term of $f)$.

Theorem 2. *Let $J \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ be a tropical ideal. If $f \in \overline{\mathbb{R}}[x_1, \ldots, x_n]$ satisfies $V(J) \subseteq V(f)$ then there exists $m \in \mathbb{Z}_{>0}$ such that* $\operatorname{conv}(f)^m \in J$.

When $J = \operatorname{trop}(I)$ for some $I \subseteq K[x_1, \ldots, x_n]$, we can replace $\operatorname{conv}(f)$ by f in Theorem 2.

One corollary of Theorem 2, which can also be proved directly, is that if X is a finite set of points in $\mathbb{A}^n$, then any rational polytope arises (up to translation and scaling) as the Newton polytope of a polynomial in $I(X)$.

In other forthcoming work, with Bivas Khan, we begin to define tropical vector bundles in this setting. Recall that a vector bundle on a scheme X is given by a locally free coherent sheaf, and thus on an affine chart $\operatorname{Spec}(R)$ by a locally free R-module M. A natural tropical generalization is to replace R by a semiring, and M by a semimodule. However in [2] Jun, Mincheva, and Tolliver show that when X is the tropicalization of a toric variety all tropical vector bundles would then be the direct sum of line bundles, which is drastically different from the classical case.

Our approach replaces the condition that there be a local chart where the semimodule is free with the requirement that it be the quotient of a free module by linear equations. This should be compared to requiring a smooth tropical manifold to be locally the Bergman fan of a matroid, rather than locally $\mathbb{R}^n$. With this approach we get indecomposable vector bundles on tropical toric varieties of higher rank, and a notion of stability. A tropical vector bundle on a subscheme of a tropical toric variety can then be defined to be the restriction of a tropical vector bundle on the tropical toric variety to the subscheme. This allows us to tropicalize those vector bundles on a scheme X that can be realized as the restriction to X of a toric vector bundle on a toric variety into which X is embedded.

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