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A construction of the Shephard–Todd group G_{32} through the Weyl group of type E_6

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Abstract. It is well known that the quotient of the derived subgroup of the Shephard–Todd complex reflection group G_{32} (which has rank 4) by its center is isomorphic to the derived subgroup of the Weyl group of type E_6. We show that this isomorphism can be realized through the second exterior power, and take the opportunity to propose an alternative construction of the group G_{32}.

Let G_{32} denote the complex reflection group constructed by Shephard–Todd [5] and let E_6 be a Weyl group of type E_6 (which is denoted by G_{35} in the Shephard–Todd classification). Let $\mathbf{Sp}_4(\mathbb{F}_3)$ (resp. $\mathbf{SO}_5(\mathbb{F}_3)$) denote the symplectic (resp. orthogonal) group of dimension 4 (resp. 5) over the finite field with three éléments $\mathbb{F}_3$. Let $\Omega_5(\mathbb{F}_3)$ be the image of $\mathbf{Sp}_4(\mathbb{F}_3)$ in $\mathbf{SO}_5(\mathbb{F}_3)$ through the natural morphism $\mathbf{Sp}_4(\mathbb{F}_3) \to \mathbf{SO}_5(\mathbb{F}_3)$ induced by the second exterior power: this is the normal subgroup of index 2 of $\mathbf{SO}_5(\mathbb{F}_3)$. Finally, if G is a group, let $\mathrm{D}(G)$ and $\mathrm{Z}(G)$ denote respectively its derived subgroup and its center and, if d is a non-zero natural number, let $\boldsymbol{\mu}_d$ be the group of d-th roots of unity in $\mathbb{C}$.

It is shown in [4, Thms. 8.43 and 8.54] that $G_{32} \simeq \boldsymbol{\mu}_3 \times \mathbf{Sp}_4(\mathbb{F}_3)$ and that $E_6 \simeq \mathbf{SO}_5(\mathbb{F}_3)$. In particular,

$$\mathrm{D}(G_{32})/\boldsymbol{\mu}_2 \simeq \Omega_5(\mathbb{F}_3) \simeq \mathrm{D}(E_6). \tag{*}$$

The purpose of this note is to present a direct elementary explanation of the isomorphism $\mathrm{D}(G_{32})/\boldsymbol{\mu}_2 \simeq \mathrm{D}(E_6)$, which in fact allows us to construct the complex reflection group G_{32} from the rational reflection group E_6. This construction uses the classical morphism $\mathbf{SL}_4(\mathbb{C}) \to \mathbf{SO}_6(\mathbb{C})$, and follows the same lines as our previous paper [1] (in which we constructed the complex reflection group G_{31} from the Weyl group of type B_6).

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This note does not pretend to prove a deep result: it is just a nice example of the application of the classical theory of reflection groups (invariants, Springer theory, ...).

Remark. It is shown in [4, Thm. 8.53] that $G_{33} \simeq \boldsymbol{\mu}_2 \times \Omega_5(\mathbb{F}_3)$. In particular, this gives an indirect isomorphism $\mathrm{D}(G_{33}) \simeq \mathrm{D}(E_6)$. However, we do not know of any construction of this isomorphism which would be in the same spirit as above.

1. The morphism $\mathbf{SL}_4(\mathbb{C}) \to \mathbf{SO}_6(\mathbb{C})$

We recall here the construction of this morphism, using some notation from [1]. Let us fix a complex vector space V of dimension 4 and let

$$\Lambda : \mathbf{GL}(V) \to \mathbf{GL}(\wedge^2 V), \qquad g \mapsto \wedge^2 g$$

be the natural morphism of algebraic groups. Note that $\wedge^2 V$ has dimension 6 and that

$$\operatorname{Ker} \Lambda \simeq \boldsymbol{\mu}_2 = \{\pm \mathrm{Id}_V\}.$$

The lists of eigenvalues of elements of $\mathbf{GL}(V)$ or $\mathbf{GL}(\wedge^2 V)$ will always be given with multiplicities. If $g \in \mathbf{GL}(V)$ admits a, b, c and d as eigenvalues, then

$$\Lambda(g) \textit{ admits } ab, ac, ad, bc, bd \textit{ and } cd \textit{ as eigenvalues.} \tag{1.1}$$

In particular,

$$\det \Lambda(g) = (\det g)^3.$$

Let us fix now a generator ε of the one-dimensional vector space $\wedge^4 V$. The choice of this generator allows to identify $\mathbb{C}$ and $\wedge^4 V$ and to define a bilinear form

$$\beta_\wedge : \wedge^2 V \times \wedge^2 V \to \mathbb{C}, \qquad (x, y) \mapsto x \wedge y.$$

This bilinear form is symmetric and non-degenerate. By definition of the determinant,

$$\beta_\wedge(\Lambda(g)(x), \Lambda(g)(y)) = (\det g)\beta_\wedge(x, y)$$

for all $g \in \mathbf{GL}(V)$ and x, $y \in \wedge^2 V$. For dimension and connectedness reasons, the image of $\mathbf{GL}(V)$ through Λ is the neutral component $\mathbf{CO}(\wedge^2 V)^\circ$ of the conformal orthogonal group $\mathbf{CO}(\wedge^2 V) = \mathbf{CO}(\wedge^2 V, \beta_\wedge)$ and Λ induces an isomorphism of algebraic groups

$$\mathbf{SL}(V)/\boldsymbol{\mu}_2 \simeq \mathbf{SO}(\wedge^2 V).$$

The next elementary lemma will be useful (in this paper, we denote by j a primitive third root of unity).

Lemma 1.6. *Let $g \in \mathbf{SO}(\wedge^2 V)$ having j, j, j, j^2, j^2 and j^2 as eigenvalues. Then there exists a unique element $\tilde{g} \in \mathbf{SL}(V)$, having 1, j, j and j as eigenvalues, and such that*

$$\Lambda^{-1}(\{g, g^{-1}\}) = \{\pm\tilde{g}, \pm\tilde{g}^{-1}\}.$$

Proof. Let $h \in \mathbf{SL}(V)$ be such that $\Lambda(h) = g$. Then $\Lambda^{-1}(\{g, g^{-1}\}) = \{\pm h, \pm h^{-1}\}$. Let a, b, c and d be the eigenvalues of h. By (1.1), by reordering if necessary the eigenvalues of h, we may assume that $ab = ac = j$. In particular, $b = c$. So $bd = cd$, which implies that $bd = cd = j^2$ (for otherwise j would be an eigenvalue of g with multiplicity ≥ 4). Still by (1.1), $(ad, bc) = (j, j^2)$ or $(ad, bc) = (j^2, j)$.

If $(ad, bc) = (j, j^2)$, then $b = c = d$ and $b^2 = j^2$, so $b = c = d = \eta j$ and $a = \eta$, for some $\eta \in \{\pm 1\}$. By replacing h by $-h$ if necessary, the eigenvalues of h are then 1, j, j, j and so h is indeed the unique element of $\Lambda^{-1}(\{g, g^{-1}\})$ admitting this list of eigenvalues.

If $(ad, bc) = (j^2, j)$, then $d = jb = jc$ and $b^2 = j$, which implies that $a = b = c = \eta j^2$ and $d = \eta$, for some $\eta \in \{\pm 1\}$. By replacing h by $-h$ if necessary, the eigenvalues of h are then 1, j^2, j^2, j^2 and so h^{-1} is indeed the unique element of $\Lambda^{-1}(\{g, g^{-1}\})$ admitting 1, j, j, j as list of eigenvalues. ■

Corollary 1.7. *Let $g \in \mathbf{SO}(\wedge^2 V)$ having j, j, j, j^2, j^2 and j^2 as eigenvalues. Then g and g^{-1} are not conjugate in $\mathbf{SO}(\wedge^2 V)$.*

Proof. Assume that we have found $w \in \mathbf{SO}(\wedge^2 V)$ such that $wgw^{-1} = g^{-1}$. Let ω and γ be respective preimages of w and g in $\mathbf{SL}(V)$. Then $\Lambda(\omega\gamma\omega^{-1}) = g^{-1} = \Lambda(\gamma^{-1})$. This shows that γ^{-1} is conjugate to γ or to $-\gamma$, which is impossible by examining the possible lists of eigenvalues of γ obtained in the proof of Lemma 1.6. ■

2. Construction of G_{32}

Let us see E_6 as a finite subgroup of $\mathbf{O}(\wedge^2 V)$ generated by reflections. Set

$$\mathbf{jSO}(\wedge^2 V) = \langle \mathbf{SO}(\wedge^2 V), j\,\mathrm{Id}_{\wedge^2 V} \rangle$$

and

$$\sqrt[3]{\mathbf{SL}}(V) = \{g \in \mathbf{GL}(V) \mid \det(g)^3 = 1\} = \langle \mathbf{SL}(V), j\,\mathrm{Id}_V \rangle.$$

Then

$$\Lambda^{-1}\big(\mathbf{jSO}(\wedge^2 V)\big) = \sqrt[3]{\mathbf{SL}}(V).$$

We then define

$$W = \Lambda^{-1}(\langle \mathrm{D}(E_6), j\,\mathrm{Id}_{\wedge^2 V}\rangle).$$

It is a subgroup of $\sqrt[3]{\mathbf{SL}}(V)$. The aim of this note is to show that W is isomorphic to the complex reflection group G_{32} of Shephard–Todd. Note first that

$$\boldsymbol{\mu}_6 \subset W.$$

2.A. Reflections in W

The list of degrees of E_6 is 2, 5, 6, 8, 9, 12 while its list of codegrees is 0, 3, 4, 6, 7, 10 (see [2, Table A.3]). In particular, exactly 3 of the degrees are divisible by 3, which shows [6, Thm. 3.4] that E_6 contains an element w_3 admitting the eigenvalue j with multiplicity 3. We will denote by C_3 the conjugacy class of w_3 in W. Since also exactly 3 of the codegrees are divisible by 3, this implies, for instance by [3, Thm. 1.2], that w_3 is regular in the sense of Springer [6, §4] (that is, admits an eigenvector for the eigenvalue j whose stabilizer in E_6 is trivial). Since E_6 is a rational group, w_3 also admits j^2 as an eigenvalue with multiplicity 3. Hence, the eigenvalues of w_3 are j, j, j, j^2, j^2 and j^2. In particular, $\det w_3 = 1$ and so

$$w_3 \in \mathrm{D}(E_6) = E_6 \cap \mathbf{SO}(\wedge^2 V).$$

By [6, Thm. 4.2 (iii)], the centralizer of w_3 in E_6 has order $6 \cdot 9 \cdot 12$, which shows that

$$|C_3| = 2 \cdot 5 \cdot 8 = 80. \tag{2.1}$$

Moreover [6, Thm. 4.2 (iv)], if $w \in E_6$, then

$$w \in C_3 \textit{ if and only if } \dim \mathrm{Ker}(w - j\,\mathrm{Id}_V) = 3. \tag{2.2}$$

Hence, if $w \in C_3$, then $w^{-1} \in C_3$ but w^{-1} is not conjugate to w in $\mathrm{D}(E_6)$ (by Corollary 1.7).

Now, let $\mathrm{Ref}(W)$ denote the set of reflections of W. If $s \in \mathrm{Ref}(W)$, the eigenvalues $\Lambda(s)$ are 1, 1, 1, $\det(s)$, $\det(s)$, $\det(s)$. Since $\det(s) \in \{j, j^2\}$, the eigenvalues of $\det(s)\Lambda(s)$ are then j, j, j, j^2, j^2, j^2. Since the characteristic polynomials of elements of $\mathrm{D}(E_6)$ have coefficients in $\mathbb{Q}$, this implies that $\Lambda(s)$ and $\det(s)^2\Lambda(s)$ do not belong to $\mathrm{D}(E_6)$. So $\det(s)\Lambda(s) \in \mathrm{D}(E_6)$ and it follows from (2.2) that $\det(s)\Lambda(s) \in C_3$. This defines a map

$$\begin{aligned} \lambda : \mathrm{Ref}(W) &\to C_3, \\ s &\mapsto \det(s)\Lambda(s). \end{aligned}$$

The next result will be very useful.

Lemma 2.5. *The map λ is bijective.*

Proof. First, if $\lambda(s) = \lambda(s')$, then there exists $\xi \in \mathbb{C}^\times$ such that $s' = \xi s$. Since s and s' are reflections, this is possible only if $\xi = 1$, and so $s = s'$. This shows that λ is injective.

Let us now show the surjectivity. Let $w \in C_3$. By Corollary 1.7, there exists a unique $\widetilde{w} \in \Lambda^{-1}(\{w, w^{-1}\})$ admitting $1, j, j, j$ as eigenvalues. Then $j^2\widetilde{w}$ and $j\widetilde{w}^{-1}$ are reflections satisfying

$$\lambda(j^2\widetilde{w}) = \det(j^2\widetilde{w})\Lambda(j^2\widetilde{w}) = j^8 \cdot j^4 \Lambda(\widetilde{w}) = \Lambda(\widetilde{w})$$

and

$$\lambda(j\widetilde{w}^{-1}) = \det(j\widetilde{w}^{-1})\Lambda(j\widetilde{w}^{-1}) = j^4 \cdot j^2 \Lambda(\widetilde{w}^{-1}) = \Lambda(\widetilde{w})^{-1}.$$

So $w \in \{\lambda(j^2\widetilde{w}), \lambda(j\widetilde{w}^{-1})\}$, which shows that λ is surjective. ■

We then deduce from (2.1) and Lemma 2.5 that

$$|\mathrm{Ref}(W)| = 80. \tag{2.6}$$

2.B. Structure of W

Our main result is Theorem 2.7 below: the proof we propose here uses neither known properties of the group G_{32} nor the classification of complex reflection groups and so might be viewed as an alternative construction of G_{32} starting from E_6 (however, note that we use properties of E_6).

Theorem 2.7. *The group W*

(a) *has order* 155 520*;*

(b) *is generated by reflections of order* 3*;*

(c) *is irreducible and primitive;*

(d) *admits* 12, 18, 24, 30 *as list of degrees.*

Proof. Set $E_6^\# = \langle \mathrm{D}(E_6), j\,\mathrm{Id}_{\wedge^2 V}\rangle$ and $W^+ = W \cap \mathbf{SL}(V)$. Recall that $|E_6| = 51\,840$. So

$$|\mathrm{D}(E_6)| = 25\,920, \quad |E_6^\#| = 77\,760, \quad |W| = 155\,520 \quad \text{and} \quad |W^+| = 51\,840.$$

This shows (a). Moreover,

$$\mathrm{Z}(W) = \boldsymbol{\mu}_6 \quad \text{and} \quad W/\boldsymbol{\mu}_6 \simeq \mathrm{D}(E_6).$$

Let $\mathcal{R} = \{\det(s)^{-1}s \mid s \in \mathrm{Ref}(W)\}$. Set

$$G = \langle \mathrm{Ref}(W)\rangle \quad \text{and} \quad H = \langle \mathcal{R}\rangle.$$

Statement (b) is equivalent to the following one:

$$W = G. \tag{\#}$$

First, $\Lambda(\mathcal{R}) = C_3$ and so $\Lambda(H) = \mathrm{D}(E_6)$ (as this last group is simple and C_3 is a conjugacy class). In particular, $W = H \cdot \boldsymbol{\mu}_6$ and so, since $H \subset G \cdot \boldsymbol{\mu}_3$, we obtain

$$W = G \cdot \boldsymbol{\mu}_6,$$

which is almost the expected result (#). Before showing (#), note that, since $\Lambda(H) = \mathrm{D}(E_6)$, we have that H (and so G) acts irreducibly on V (indeed, if H did not act irreducibly on V, then the representation of $\mathrm{D}(E_6)$ on $\wedge^2 V$ would split as a direct sum of four submodules, which is impossible since $\mathrm{D}(E_6)$ has index 2 in E_6). Moreover, if G is not primitive, then G (and so H) would be monomial [4, Lem. 2.12], which would imply that $\Lambda(H) = \mathrm{D}(E_6)$ is monomial, which is false. So

$$G \textit{ is irreducible and primitive.} \tag{$\clubsuit$}$$

Statement (c) follows.

Let us conclude this proof by showing simultaneously (b) and (d). Let d_1, d_2, d_3, d_4 be the degrees of G. It follows from (2.6) and for instance from [2, Theo. 4.1] that

$$\begin{cases} d_1 d_2 d_3 d_4 = |G|, \\ d_1 + d_2 + d_3 + d_4 = 84. \end{cases} \tag{$\diamondsuit$}$$

The morphism $\det : G \to \boldsymbol{\mu}_3$ is surjective (since $\det(s) \in \{j, j^2\}$ for any $s \in \mathrm{Ref}(W)$), and this implies that $\boldsymbol{\mu}_3 \subset G$ (because $G/(G \cap \boldsymbol{\mu}_6) \simeq \mathrm{D}(E_6)$ is simple). It remains to show that $|G| \neq |W|/2 = 77\,760$.

So assume that $|G| = 77\,760$. Since $\boldsymbol{\mu}_3 \subset G$, all the d_i's are divisible by 3 and, since $\boldsymbol{\mu}_6 \not\subset G$, at least one of them (say d_4) is not divisible by 6. Write $e_i = d_i/3$. Then

$$\begin{cases} e_1 e_2 e_3 e_4 = 960 = 2^6 \cdot 3 \cdot 5, \\ e_1 + e_2 + e_3 + e_4 = 28, \\ e_4 \text{ is odd.} \end{cases}$$

From the second equality, we deduce that at least one more of the e_i's (say e_3) is odd. From the first equality, we deduce that at least one of the e_i's (say e_2) is even and so e_1 is also even. The first equality shows that e_1 or e_2 (say e_2) is divisible by 8. So $e_2 \in \{8, 16, 24\}$. A quick inspection of the possibilities shows that $e_1 = 4$, $e_2 = 16$ and $\{e_3, e_4\} = \{3, 5\}$. The degrees of G are then 9, 12, 15 and 48. Since 16 divides one of the degrees, it then follows from [6, Thm. 3.4 (i)] that $G = H \cdot \boldsymbol{\mu}_3$ contains an element of order 16 and so H contains an element of order 16. Therefore,

$D(E_6) = \Lambda(H)$ contains an element of order 8, which is impossible (see Remark 2.11 below for a proof of this fact, based only on Springer theory). This contradicts the fact that $|G| = 77\,760$. So we have shown (#), that is,

$$W = \langle \mathrm{Ref}(W) \rangle = G.$$

This is statement (b). In particular, $\boldsymbol{\mu}_6 \subset W$ and so all the d_i's are divisibles by 6. Set $a_i = d_i/6$. Then ($\diamondsuit$) implies that

$$\begin{cases} a_1 a_2 a_3 a_4 = 120 = 2^3 \cdot 3 \cdot 5, \\ a_1 + a_2 + a_3 + a_4 = 14. \end{cases}$$

By the same argument as before, since $\boldsymbol{\mu}_{12} \not\subset W$, we may assume that a_3 and a_4 are odd and that a_1 and a_2 are even. A quick inspection of the possibilities shows that $\{a_1, a_2\} = \{2, 4\}$ and $\{e_3, e_4\} = \{3, 5\}$. This concludes the proof of (d). ■

Corollary 2.10. *The group W is isomorphic to the reflection group G_{32} of Shephard–Todd.*

Proof. This follows from Theorem 2.7 and from the classification of complex reflection groups [5]. ■

Hence, Corollary 2.10 gives an explanation for the fact, mentioned in the introduction, that $D(G_{32})/\boldsymbol{\mu}_2 \simeq D(E_6)$: the isomorphism is realized by Λ.

Remark 2.11. In the proof of Theorem 2.7, we have used the fact that $D(E_6)$ does not contain any element of order 8. This fact can be easily obtained by a computer calculation for instance, but we propose here a proof using only Springer theory. Let $w \in E_6$ be an element of order 8. Then w necessarily admits an eigenvalue which is a primitive 8-th root of unity ζ. As only one of the degrees of E_6 and only one of the codegrees of E_6 is divisible by 8, this implies that w is a regular element in the sense of Springer [3, Thm. 1.2]. Then, by [6, Thm. 4.2 (v)] the list of eigenvalues of w is $\zeta^{-1}, \zeta^{-4}, \zeta^{-5}, \zeta^{-7}, \zeta^{-8}, \zeta^{-11}$, and so $\det(w) = \zeta^{-36} = -1$. So $w \notin D(E_6)$.

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