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Morse theory for discrete magnetic operators and nodal count distribution for graphs

Lior Alon and Mark Goresky

Abstract. Given a discrete Schrödinger operator h on a finite connected graph G of n vertices, the nodal count $\phi(h,k)$ denotes the number of edges on which the k-th eigenvector changes sign. A *signing* h' of h is any real symmetric matrix constructed by changing the sign of some off-diagonal entries of h, and its nodal count is defined according to the signing. The set of signings of h lie in a naturally defined torus $\mathbb{T}_h$ of "magnetic perturbations" of h. G. Berkolaiko [Anal. PDE 6 (2013), 1213–1233] discovered that every signing h' of h is a critical point of every eigenvalue $\lambda_k\colon \mathbb{T}_h \to \mathbb{R}$, with Morse index equal to the nodal surplus. We add further Morse theoretic information to this result. We show if $h_\alpha \in \mathbb{T}_h$ is a critical point of λ_k and the eigenvector vanishes at a single vertex v of degree d, then the critical point lies in a non-degenerate critical submanifold of dimension $d+n-4$, closely related to the configuration space of a planar linkage. We compute its Morse index in terms of spectral data.

The *average nodal surplus distribution* is the distribution of values of $\phi(h',k)-(k-1)$, averaged over all signings h' of h. If all critical points correspond to simple eigenvalues with nowhere-vanishing eigenvectors, then the average nodal surplus distribution is binomial. In general, we conjecture that the nodal surplus distribution converges to a Gaussian in a CLT fashion as the first Betti number of G goes to infinity.

1. Introduction

In some ways, this paper is both an analogue of [2] for discrete graphs and a continuation and expansion of the papers [7, 13], although it is completely self-contained.

1.1. The setting

Let G be a simple graph on n ordered vertices labeled $1, 2, \dots, n$. Write $r \sim s$ if $r \neq s$ are vertices connected by an edge. A (real or complex) *function* on G is

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a function on the vertices of G, that is, a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$ and we denote the value of such a function $v = (v_1, v_2, \dots, v_n)$ by $v(r)$ or v_r. An $n \times n$ matrix h is *supported on* G if $h_{rs} \neq 0 \implies r \sim s$ or $r = s$. Let $\mathcal{S}(G)$ and $\mathcal{H}(G)$ denote the vector spaces of real symmetric matrices and complex Hermitian matrices supported on G. A *discrete Schrödinger operator* is a real symmetric matrix $h \in \mathcal{S}(G)$ with $h_{rs} < 0$ for $r \sim s$. The quadratic form associated with $h \in \mathcal{S}(G)$ may be expressed as the quadratic form of $\Delta + V$, that is

$$\langle f, hf \rangle = -\sum_{r \sim s} h_{rs}(f(r) - f(s))^2 + \sum_{r=1}^{n} V(r) f(r)^2 \tag{1.1}$$

where the "potential" is $V(r) = h_{rr} + \sum_{r \sim s} h_{rs}$ and Δ is a *weighted Laplace operator* on G.

A discrete Schrödinger operator h has real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Suppose λ_k is a simple (multiplicity one) eigenvalue of h with a nowhere-vanishing eigenvector v (meaning that $v_r \neq 0$ for all r). A basic problem in graph theory is to understand the behavior of the *nodal count* $\phi(h, k)$, that is, the number of edges $r \sim s$ for which v changes sign: $v(r)v(s) < 0$. It is known that

$$k - 1 \leq \phi(h, k) \leq k - 1 + \beta, \tag{1.2}$$

where β is the first Betti number of G. (See [16] for a review of the many works leading to the upper bound, an analogue of Courant's theorem,[1] and [6] for the lower bound.) This motivates the definition of the *nodal surplus*

$$\phi(h, k) - (k - 1) \in \{0, 1, \dots, \beta\}$$

and its probability distribution $P(h) = (P(h)_0, \dots, P(h)_\beta)$ over the n possible eigenvalues:

$$P(h)_s = \frac{1}{n} \#\{1 \leq k \leq n : \phi(h, k) - (k - 1) = s\}.$$

In numerical simulations for large graphs, this distribution seems to concentrate around $\frac{\beta}{2}$ with variance of the order of β, similar to the observations for metric graphs in [2].

[1] The Courant theorem states, for a domain Ω in Euclidean space with homogeneous boundary conditions, that the nodal set of the k-th eigenfunction of the Laplacian divides Ω into no more than k subdomains, see [15, Chapter 6, Section 6].

1.2. Nodal count for signed graphs

If $h \in \mathcal{S}(G)$ is a discrete Schrödinger operator we may consider other *signings* $h' \in \mathcal{S}(G)$ obtained from h by changing the sign of some collection of off-diagonal entries. Every symmetric matrix $h' \in \mathcal{S}(G)$ is a signing of a uniquely determined Schrödinger operator h. We may consider h' to be an analogue of the discrete Schrödinger operator on the corresponding *signed graph* G' obtained from G by attaching signs to the edges, as originally introduced in [20] and extensively studied, see [11, 28, 35]. In this case, taking the signing into account, the nodal count is defined to be the number of edges $r \sim s$ such that $v(r)h'_{rs}v(s) > 0$.

Denote by $\mathcal{S}(h)$ the collection of all possible signings of h (cf. Section 2.6). The inequality (1.2) continues to hold for any signing of h. The *average nodal surplus distribution* $P(\mathcal{S}(h))$ is the average of $P(h')$ over all signings $h' \in \mathcal{S}(h)$. In Theorem 3.2 we show that if the diagonal entries of h are all equal, then $P(\mathcal{S}(h))$ is symmetric around $\beta/2$. Numerical experiments lead to the following conjecture.

Conjecture. Given a simple connected graph G, there is a generic set (open, dense and full measure) of $h \in \mathcal{S}(G)$ for which the average nodal surplus distribution $P(\mathcal{S}(h))$ is symmetric around $\beta/2$ with variance σ_h^2 of order β. Moreover, the normalized distribution

$$\rho_{G,h} := \sum_{j=0}^{\beta} P(\mathcal{S}(h))_j \delta_{x_j} \quad \text{with } x_j = \frac{j - \beta/2}{\sigma_h},$$

converges in the weak topology to the normal Gaussian distribution $N(0,1)$ as $\beta \to \infty$, uniformly over all simple connected G with first Betti number β, and generic $h \in \mathcal{S}(G)$.

1.3. Gauge invariance

The *gauge group* $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ acts on the space $\mathcal{H}(G)$ where $(\theta_1, \theta_2, \dots, \theta_n)$ acts by conjugation with $\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$. This action preserves eigenvalues, nodal count, and most other graph properties that are studied in this paper. Elements $h, h' \in \mathcal{H}(G)$ that differ by a gauge transformation are said to be *gauge equivalent*. If $h \in \mathcal{S}(G)$ is a discrete Schrödinger operator, then the signings h' of h for which the corresponding signed graph G' is *balanced* (see [20]) are exactly those h' that are gauge equivalent to h.

1.4. Magnetic operators and nodal count

In [7, 9], G. Berkolaiko suggested that one might better understand the nodal count by considering its variation under *magnetic perturbations of* h. The discrete analogue for the Schrödinger operator associated to a particle in a magnetic field appears in [21,22]. See also [14, 26], [12, Section 2.1], and [13]. It is quickly reviewed in Appendix B.

Given a discrete Schrödinger operator $h \in \mathcal{S}(G)$, a magnetic potential α is a real anti-symmetric matrix supported on G and the associated *magnetic Schrödinger operator* $h_\alpha \in \mathcal{H}(G)$ is the Hermitian matrix $(h_\alpha)_{rs} = e^{i\alpha_{rs}} h_{rs}$. The manifold (2.7) of such magnetic perturbations, $\mathbb{T}_h \subset \mathcal{H}_n$, is a torus containing h, cf. Section 2.4 below. Its quotient, see equation (2.9), modulo gauge transformations, $\mathcal{M}_h$ is a torus of dimension β. In [13] and [7], G. Berkolaiko and Y. Colin de Verdière discovered a remarkable fact: for any real symmetric $h \in \mathcal{S}(G)$ with simple eigenvalue λ_k and nowhere vanishing eigenvector, the nodal surplus $\phi(h,k) - (k-1)$ is equal to the Morse index of λ_k, interpreted as a Morse function on the manifold $\mathcal{M}_h$.

1.5. Morse theory for magnetic perturbations modulo gauge transformations

We wish to apply Morse theory to the function $\lambda_k \colon \mathcal{H}_n \to \mathbb{R}$, restricted to the torus $\mathbb{T}_h$ or its quotient $\mathcal{M}_h$. In principle, Morse theory provides a prescription for building the homology of $\mathcal{M}_h$ from local data at the critical points of λ_k together with some homological information as to how these local data fit together. Since the homology of $\mathcal{M}_h$ is known, Morse theory should provide restrictions on the number and type of critical points of λ_k, and in turn, restrictions on the nodal surplus.

There are several difficulties with this plan, the first being that λ_k is continuous but not smooth: it is analytic on each stratum of a certain stratification of $\mathcal{H}_n$ (see Section 7) [25, 29]. If $\lambda_k(h)$ is simple, then λ_k is analytic near h and one may search for its critical points on $\mathbb{T}_h$. The torus $\mathbb{T}_h$ and its quotient $\mathcal{M}_h$ are preserved under complex conjugation, and the function λ_k is invariant under complex conjugation. The simplest critical points of λ_k are the *symmetry points* (Section 2.6): the points $h' \in \mathbb{T}_h$ (or $[h'] \in \mathcal{M}_h$) fixed by complex conjugation, i.e., the real symmetric matrices in $\mathbb{T}_h$.

The set of symmetry points of $\mathbb{T}_h$ is denoted $\mathcal{S}(h)$. If h is real symmetric, then $\mathcal{S}(h)$ consists precisely of the various signings of h. Following [5], we show the following result.

Theorem 3.2. *Each critical point $h' \in \mathbb{T}_h$ with simple eigenvalue $\lambda_k(h')$ and nowhere vanishing eigenvector is necessarily in the gauge equivalence class of a symmetry point. In other words, its image $[h'] \in \mathcal{M}_h$ is a symmetry point. Suppose that for each k $(1 \le k \le n)$ each critical point $h_\alpha \in \mathbb{T}_h$ of λ_k has $\lambda_k(h_\alpha)$ as a simple eigenvalue with nowhere vanishing eigenvector. Then the average nodal count distribution is a binomial distribution*[2] *with mean $\beta/2$ and variance $\beta/4$. Consequently, if the aver-*

[2] See Section 3.3.

age nodal distribution is not binomial, then there must exist critical points (of some eigenvalue) that are not symmetry points.

We give a homological characterization of symmetry points.

Theorem 2.7. *Let $h \in \mathcal{S}(G)$ and $\alpha \in \mathcal{A}(G)$ which we may identify as a 1-form on G. Then h_α is gauge equivalent to a symmetry point if and only if $\int_\xi \alpha \equiv 0 \pmod{\pi}$ for all cycles ξ, i.e., chains $\xi \in C_1(G, \mathbb{Z})$ with $\partial\xi = 0$.*

1.6. Classification of critical gauge-equivalence classes

In general, the nodal surplus distribution $P(\mathcal{S}(h))$ depends on Morse data from all critical points of λ_k (for all k), whether or not they are symmetry points. Following Theorem 3.2, there are two possible types of non-symmetry critical points $[h'] \in \mathcal{M}_h$ of λ_k.

(1) *Exceptional critical points*, for which $\lambda_k(h')$ is simple but its eigenvector vanishes on one or more vertices. In this case $[h']$ is (usually) a degenerate critical point (see Theorem 4.4): it is contained in a larger critical submanifold.

(2) *Incorrigible critical points*, for which the multiplicity of $\lambda_k(h')$ is greater than one. In this case, λ_k fails to be smooth and one must replace the usual Morse theory with stratified Morse theory ([19]).

Concerning the first case, suppose the eigenvector v vanishes only at a single vertex v_0 of the graph G. Suppose that v_0 has degree $\deg(v_0)$.

Theorem 4.4. *Assuming the critical point $[h'] \in \mathcal{M}_h$ is sufficiently generic,*[3] *then it lies in a non-degenerate (Morse–Bott) critical submanifold of $\mathcal{M}_h$, of dimension* $\deg(v_0) - 3$*, which is diffeomorphic to the configuration space of a particular planar linkage. Its Morse index may be expressed in terms of spectral data.*

The configuration spaces of planar linkages are fascinating objects. They have been extensively studied and their homology is completely known, cf. [17, 24, 33].

For the second case, when the multiplicity of $\lambda_k(h')$ is greater than one, G. Berkolaiko and I. Zelenko [10] have determined the *normal Morse data* for λ_k, and its Betti numbers, which forms the central ingredient required for stratified Morse theory. However, in order to apply stratified Morse theory to the mapping $\lambda_k\colon \mathbb{T}_h \to \mathbb{R}$ it is required that the manifold $\mathbb{T}_h \subset \mathcal{H}_n$ should be Whitney stratified. Its stratification comes by intersecting with the natural stratification of $\mathcal{H}_n$ (cf. Section 7), but this requires that $\mathbb{T}_h$ should be transverse to the strata of the stratification of $\mathcal{H}_n$. The challenge is to guarantee transversality of the torus $\mathbb{T}_h$ by a generic choice of the single

[3] Specific conditions on h' are given in Theorem 4.4 in Section 4.3.

element $h \in \mathcal{S}(G)$. The transversality lemma in [10] does not address this situation. The first non-trivial case concerns the stratum $S_2(k)$ where λ_k has multiplicity 2. Suppose h_α is a critical point of λ_k, an eigenvalue of multiplicity 2. In Section 7.8 we define the notion of a splitting of the graph G by the eigenspace of λ_k. (A related condition was considered by L. Lovász in [27, Section 10.5.2].)

Theorem 7.9. *If the eigenspace of $\lambda_k(h_\alpha)$ does not split the graph G, then the space $\mathcal{H}(G)$ is transverse to $S_2(k)$ at given point $h_\alpha \in \mathbb{T}_h$.*

Corollary 7.10. *As above, if the eigenspace of $\lambda_k(h_\alpha)$ does not split G, then for generic choice $h' \in \mathcal{S}(G)$ the torus $\mathbb{T}_{h'}$ is transverse to the stratum $S_2(k)$ near h_α.*

2. Notation and definitions

2.1. Symmetric and Hermitian forms

Let $\mathcal{S}_n$ denote the vector space of $n \times n$ real symmetric matrices, $\mathcal{A}_n$ the space of $n \times n$ real antisymmetric matrices, and $\mathcal{H}_n$ the space of $n \times n$ Hermitian matrices, that is, matrices of linear operators on $\mathbb{C}^n$ expressed in the standard basis and that are self-adjoint with respect to the standard Hermitian form $\langle x, y\rangle = \sum \bar{x}_i y_i$.

If $V \subset \mathbb{C}^n$ is a complex subspace, then the standard Hermitian form restricts to a Hermitian form on V and we denote by $\mathcal{H}(V)$ the self-adjoint linear operators $V \to V$. If $\xi \in \mathcal{H}_n$, then it may fail to preserve V however its "restriction" to V may be defined by expressing $\xi = \left(\begin{smallmatrix} A & B \\ B* & D\end{smallmatrix}\right)$ with respect to the decomposition $\mathbb{C}^n = V \oplus V^\perp$. The restriction $\xi|V$ is defined to be the operator $A \in \mathcal{H}(V)$. Equivalently, $\xi|V$ is the operator corresponding to the restriction to $x, y \in V$ of the *sesquilinear* form $(x, y)_\xi = \langle x, \xi y\rangle$.

2.2. Laplace and Schrödinger operators

Throughout this section, we fix a graph $G = G([n], E)$. The natural ordering on the set of vertices $[n] := \{1, 2, \dots, n\}$ determines an orientation for each edge. Write $r \sim s$ if $r \neq s$ and vertices r, s are joined by an edge. Write $r \simeq s$ if $r \sim s$ or $r = s$.

A (real or complex) *matrix supported on* G is an $n \times n$ matrix h such that

$$h_{rs} \neq 0 \implies r \simeq s.$$

Such a matrix is *properly supported* on G if, in addition,

$$r \sim s \implies h_{rs} \neq 0.$$

Symmetric, antisymmetric, and Hermitian matrices supported on G are denoted $\mathcal{S}(G)$, $\mathcal{A}(G)$, and $\mathcal{H}(G)$, respectively. Examples of matrices in $\mathcal{S}(G)$ include the *adjacency* *matrix* for G, (weighted) *Laplace operators* for G and discrete Schrödinger operators, see Section 1.1 above. More generally, any matrix $h \in \mathcal{H}(G)$ may be considered a *magnetic Schrödinger operator* for G (see Section 2.4 below and references [12,13]).

2.3. Graph homology

The space $C_0(G;\mathbb{Z}) \cong \mathbb{Z}^n$ of 0-chains is the vector space of formal linear combinations of vertices, $\sum_{r=1}^n c_r[r]$. Each edge rs with $r < s$ is orientated from r to s so that the group $C_1(G;\mathbb{Z})$ of 1-chains is the group of formal linear combinations

$$\xi = \sum_{\substack{r\sim s\\ r<s}} \xi_{rs}[rs], \quad \xi_{rs} \in \mathbb{Z}. \tag{2.1}$$

Then $H_1(G;\mathbb{Z}) = \ker(\partial)$, where $\partial: C_1(G;\mathbb{Z}) \to C_0(G;\mathbb{Z})$ with $\partial[rs] = [s] - [r]$. The first Betti number is

$$\beta = \operatorname{rank} H_1(G,\mathbb{Z}) = |E| - n + c,$$

where c is the number of connected components of G.

The vector space $\mathbb{R}^n$ may be viewed as the space of real-valued functions $\Omega^0(G)$ on the vertices of G. If $v = (v_1, v_2, \ldots, v_n)$, we sometimes write $v_r = v(r)$. The vector space $\mathcal{A}(G)$ of real, antisymmetric matrices supported on G may be viewed as the space of 1-forms $\Omega^1(G)$ on G with coboundary differential

$$d\colon \Omega^0(G) = \mathbb{R}^n \to \Omega^1(G) = \mathcal{A}(G); \quad (df)_{rs} = \begin{cases} f(s) - f(r) & \text{if } r \sim s, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

There are no 2-forms on a graph so $H^1(G;\mathbb{R}) = \Omega^1(G)/d\Omega^0(G)$ is canonically dual to the homology $H_1(G;\mathbb{R})$ under the natural pairing that is determined by integration $\Omega^1(G) \times C_1(G;\mathbb{R}) \to \mathbb{R}$. If $\alpha \in \mathcal{A}(G) = \Omega^1(G)$ and $\xi \in C_1(G;\mathbb{R})$ as in (2.1), then

$$\int_\xi \alpha = \sum_{\substack{r\sim s\\ r<s}} \xi_{rs}\alpha_{rs}$$

2.4. Action of $\mathcal{A}_n$

The vector space $\mathcal{A}_n = \mathcal{A}_n(\mathbb{R})$ of $n \times n$ real antisymmetric matrices acts on the vector space $\mathcal{H}_n$ of $n \times n$ Hermitian matrices by

$$(\alpha * h)_{rs} = e^{i\alpha_{rs}} h_{rs}$$

for all $\alpha \in \mathcal{A}_n(\mathbb{R})$ and $h \in \mathcal{H}_n$ with $(x + y) * h = x * (y * h)$ and with $0 * h = h$. Then $\mathcal{A}(G)$ acts on $\mathcal{H}(G)$.

If h is the discrete Schrödinger operator, then $\alpha * h$ may be interpreted as the corresponding magnetic Schrödinger operator in the presence of a *magnetic field* described by α, whose flux through a cycle ξ is $\int_\xi \alpha$, with a sesquilinear form

$$\langle f, (\alpha * h) f\rangle = -\sum_{r\sim s} h_{rs}|f(s) - e^{i\alpha_{rs}} f(r)|^2 + \sum_{r=1}^{n} V(r)|f(r)|^2, \tag{2.3}$$

instead of the quadratic form of h in (1.1). If $h = (h_{rs}) \in \mathcal{H}_n$ define $|h| \in \mathcal{S}_n$ by $|h|_{rs} = |h_{rs}|$ for $r \neq s$ and $|h|_{rr} = h_{rr}$ (diagonal entries of $|h|$ can be negative). Then there exists $\alpha \in \mathcal{A}_n$ so that $h = \alpha * (|h|)$.

2.5. Gauge invariance

The $*$ action factors through the torus $\mathcal{A}_n(\mathbb{R})/\mathcal{A}_n(2\pi\mathbb{Z})$. So, the subtorus *supported on G*

$$\mathbb{T}(G) := \{\alpha \in \mathcal{A}_n(\mathbb{R})/\mathcal{A}_n(2\pi\mathbb{Z})\colon \alpha_{rs} \neq 0 \implies r \sim s\},$$

acts on $\mathcal{H}(G)$ by the $*$ action. The differential (2.2) also factors

$$\begin{array}{ccc} \Omega^0(G) = \mathbb{R}^n & \xrightarrow{d} & \Omega^1(G) = \mathcal{A}(G) \\ {\scriptstyle (\mathrm{mod}\ 2\pi)}\downarrow & & \downarrow{\scriptstyle (\mathrm{mod}\ 2\pi)} \\ \mathbb{T}^n & \xrightarrow{\quad d \quad} & \mathbb{T}(G) \end{array} \tag{2.4}$$

through the *gauge group* $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. *Gauge invariance* is the statement that the $*$ action by coboundaries is simply given by conjugation: for any $\theta \in \mathbb{T}^n$ and any $h \in \mathcal{H}_n$, direct calculation gives

$$d\theta * h = e^{i\theta} h e^{-i\theta} \tag{2.5}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{T}^n$ and $e^{i\theta} = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$. The $*$ action by $d\theta$ preserves eigenvalues and preserves eigenvectors *up to phase*: if $E_\lambda(h) = \ker(h - \lambda I)$, then

$$E_\lambda(d\theta * h) = e^{i\theta} E_\lambda(h). \tag{2.6}$$

Elements $h, h' \in \mathcal{H}(G)$ that differ by a gauge transformation ($h' = d\theta * h$) are said to be *gauge equivalent*. Gauge equivalence determines an identification, cf. (2.10) of the *quotient torus* (the manifold of magnetic fields modulo gauge transformations) with cohomology

$$\mathbb{T}^{\mathcal{A}/d}(G) := \mathbb{T}(G)/d(\mathbb{T}^n) \cong H^1(G; \mathbb{R}/2\pi\mathbb{Z}).$$

2.6. The embedded torus and its symmetry points

Recall (2.2) that a matrix $h \in \mathcal{H}(G)$ is *properly supported* on G if $h_{rs} \neq 0$ whenever $r \sim s$. (Diagonal entries h_{rr} may vanish.) Such h defines a mapping $\mathbb{T}(G) \to \mathcal{H}_n$ by $\alpha \mapsto \alpha * h$, whose image is an embedding of $\mathbb{T}(G)$ into $\mathcal{H}(G)$,

$$\mathbb{T}_h := \mathbb{T}(G) * h = \{\alpha * h : \alpha \in \mathbb{T}(G)\} = \{\alpha * |h| : \alpha \in \mathbb{T}(G)\}. \tag{2.7}$$

We refer to $\mathbb{T}_h$ as *the embedded torus*. For $h \in \mathcal{H}(G)$ which is not properly supported on G, the dimension of the embedded torus $\mathbb{T}_h$ is the number of non-zero elements h_{rs} with $r < s$. The embedded torus is invariant under complex conjugation and we refer to the set of its fixed points (i.e. the real points)

$$\mathcal{S}(h) := \mathbb{T}_h \cap \mathcal{S}(G) = \{\alpha * |h| : \alpha \equiv 0 \ (\mathrm{mod}\ \pi)\}$$

as *symmetry points*. If $h \in \mathcal{S}_n$, then its symmetry points $\mathcal{S}(h)$ consist of symmetric matrices h' obtained from h by changing the signs in any subset of off-diagonal entries h_{rs} or equivalently

$$h' = \alpha * h \quad \text{where } \alpha \equiv 0 \ (\mathrm{mod}\ \pi).$$

The action of the *integral gauge group* $(\pi\mathbb{Z})^n \subset \mathbb{R}^n$ preserves the set of symmetry points and changes the signs of the components of the corresponding eigenvectors. The set $\mathcal{S}(h)$ decomposes into a union of orbits under the integral gauge group. If h is properly supported on G ($h_{rs} \neq 0$ whenever $r \sim s$), then $\mathcal{S}(h)$ has $2^{|E|}$ elements, partitioned into 2^β orbits (cf. Section 2.8). Each orbit corresponds to a choice of parity of the circulations around a choice of elementary cycles.

2.7 Theorem. *Suppose $h \in \mathcal{H}(G)$ is properly supported on G. Let $\alpha \in \mathcal{A}(G) = \Omega^1(G)$ so that $h = \alpha * |h|$. Then h is gauge-equivalent to a symmetry point $h' \in \mathcal{S}(h)$ if and only if*

$$\int_\xi \alpha \equiv 0 \ (\mathrm{mod}\ \pi) \tag{2.8}$$

for all cycles ξ, i.e. chains $\xi \in C_1(G, \mathbb{Z})$ with $\partial\xi = 0$.

Proof. Since h is properly supported, the element α is uniquely determined modulo $2\pi\mathbb{Z}$. If h is a symmetry point, then $\alpha \equiv 0 \ (\mathrm{mod}\ \pi)$ so (2.8) holds. If h changes by gauge-equivalence, the integral (2.8) is unchanged, by Stokes' theorem.

On the other hand, if (2.8) holds for all cycles, then by duality the cohomology class $[\alpha]$ vanishes in $H^1(G;\mathbb{R})/H^1(G;\pi\mathbb{Z})$, so it lies in $H^1(G;\pi\mathbb{Z}) \subset H^1(G;\mathbb{R})$ and it comes from a 1-form $\alpha' \in \Omega^1(\pi\mathbb{Z})$, that is, an antisymmetric matrix whose entries are multiples of π. Then the cohomology class $[\alpha' - \alpha] \in H^1(G;\mathbb{R})$ vanishes so there exists $\theta \in \Omega^0(G;\mathbb{R})$ with $\alpha' = \alpha + d\theta$. This proves that the symmetry point $\alpha' * |h|$ is gauge-equivalent to $h = \alpha * |h|$. ∎

2.8. Eigenvalues as Morse functions

Eigenvalues of elements $h \in \mathcal{H}_n$ are real and ordered, say

$$\lambda_1(h) \le \lambda_2(h) \le \cdots \le \lambda_n(h).$$

For each k $(1 \le k \le n)$ the mapping $\lambda_k\colon \mathcal{H}_n \to \mathbb{R}$ is well defined, continuous and piecewise real-analytic: there is a stratification of $\mathcal{H}_n$ by analytic subvarieties such that the restriction of λ_k to each stratum is analytic (cf. Section 7.1 and Lemma 7.3).

The restriction of each λ_k to the embedded torus $\mathbb{T}_h$ is invariant under gauge transformations, so it determines a function on the quotient,

$$\mathcal{M}_h = \mathbb{T}_h /\!\!/ \mathbb{T}^n, \tag{2.9}$$

where we use the notation $/\!\!/\mathbb{T}^n$ to denote dividing by gauge equivalence. The torus $\mathcal{M}_h$ has dimension β, and is referred to in [13] as the *manifold of magnetic perturbations modulo gauge transformations*:

$$\begin{array}{ccccccc} \mathbb{T}(G) & \xrightarrow{*h} & \mathbb{T}_h & \hookrightarrow & \mathcal{H}_n & \xrightarrow{\lambda_k} & \mathbb{R} \\ \big\downarrow{\scriptstyle d(\mathbb{T}^n)} & & \big\downarrow{\scriptstyle /\!\!/\mathbb{T}^n} & & & \nearrow{\scriptstyle \lambda_k} & \\ \mathbb{T}^{\mathcal{A}/d}(G)\mathcal{M}_h & \longrightarrow & & & & & \end{array} \tag{2.10}$$

If $\alpha \in \mathbb{T}(G)$, then the equivalence class of $\alpha * h$ in $\mathcal{M}_h$ is denoted $[\alpha * h]$ or $[h_\alpha]$.

If $\theta \in \mathbb{R}^n$, then $\overline{(d\theta) * h} = d(-\theta) * \bar{h}$ so complex conjugation passes to an involution on $\mathcal{M}_h$. *Every fixed point of this involution comes from a symmetry point in* $\mathbb{T}_h$: for if $h \in \mathcal{H}_n$ and $[h] \in \mathcal{M}_h$ is fixed, this means $\bar{h} = (d\theta) * h$ for some $\theta \in \mathbb{R}^n$, so $(d\frac{\theta}{2}) * h$ is a symmetry point. It is therefore reasonable to refer to these fixed points of $\mathcal{M}_h$ as *symmetry points of* $\mathcal{M}_h$.

2.9 Lemma. *Let G be a simple graph with c connected components and let $h \in \mathcal{H}_n(G)$, properly supported on G. Then, each symmetry point $h' \in \mathcal{S}(h)$ has exactly 2^{n-c} gauge-equivalent symmetry points. Thus, the number of symmetry points in $\mathcal{M}_h$ is $2^{|E|-(n-c)} = 2^\beta$.*

Proof. It is enough to consider the case of real symmetric $h \in \mathcal{S}_n$, in which case its gauge-equivalent symmetry points are

$$[h] \cap \mathcal{S}(h) = \{df * h\colon f(r) \in \{0, \pi\} \text{ for all } r\}.$$

There are 2^n choices for f among which 2^c are in the kernel of d (those which are constant on connected components of G). So, there are 2^{n-c} distinct values for df, and therefore 2^{n-c} distinct values of $df * h$ since h is properly supported on G.

Hence, $[h]$ contains exactly 2^{n-c} gauge-equivalent symmetry points. Repeating this argument for any other $h' \in S(h)$ leaves $2^{|E|-(n-c)} = 2^\beta$ equivalence classes of symmetry points in $\mathcal{M}_h$. ∎

2.10. Nodal surplus

Generalizing the notions described in the introduction, let h be a Hermitian matrix supported on G, suppose $\lambda_k(h)$ is a simple eigenvalue with nowhere vanishing eigenvector $v = (v_1, v_2, \dots, v_n)$. Further, assume that $\bar{v}_r h_{rs} v_s \in \mathbb{R}$ for all $r \sim s$ (which is equivalent to h being a critical point of λ_k, see Theorem 3.2 part (3)). Define the *nodal count* $\phi(h,k)$ to be the number of edges $r \sim s$ such that

$$\bar{v}_r h_{rs} v_s > 0. \tag{2.11}$$

The *nodal surplus* is the number $\phi(h,k) - (k-1)$. This number does not change under gauge transformation and it is known (see Theorem 3.2 below) that the nodal surplus is between 0 and β, the first Betti number of G. The *nodal surplus distribution* $P(h) = (P(h)_0, P(h)_1, \dots, P(h)_\beta)$ is the vector representing the probability distribution of these numbers over the n possible eigenvalues:

$$P(h)_s = \frac{1}{n}\#\{1 \le k \le n : \phi(h,k) - (k-1) = s\}.$$

Assuming that $h \in \mathcal{S}_n$ and all its signings $h' \in \mathcal{S}(h)$ have all eigenvalues simple with nowhere-vanishing eigenvectors, the distribution can be averaged over signings to give the *average nodal distribution*

$$P(\mathcal{S}(h)) = 2^{-|E|} \sum_{h' \in \mathcal{S}(h)} P(h').$$

3. Morse theory

3.1. Critical points

Throughout this section, we fix a graph G with vertices $1, \dots, n$ and edges $r \sim s$. Let $h \in \mathcal{S}(G)$ be a real symmetric matrix properly supported on G, cf. Section 2.2. For $\alpha \in \mathcal{A}(G)$, denote by $h_\alpha = \alpha * h$ the magnetic perturbation of h. Fix k and write $\lambda_k(\alpha) = \lambda_k(h_\alpha)$ for the k-th eigenvalue. Let $\mathcal{M}_h$ be the manifold (2.9) of magnetic perturbations of h modulo gauge transformations. It is a torus of dimension β, the first Betti number of the graphs G. By equation (2.6), the eigenvalue $\lambda_k(\alpha)$ of an element $[h_\alpha] \in \mathcal{M}_h$, and its multiplicity are well defined; and whether or not an eigenvector vanishes at a given vertex is well defined.

We consider $\lambda_k\colon \mathcal{M}_h \to \mathbb{R}$ to be a sort of generalized Morse function. If λ_k is smooth at a point $x = [h_\alpha] \in \mathcal{M}_h$ (in which case it is also analytic), we say that x is a *smooth point* of λ_k. A *critical point* of λ_k is either a non-smooth point or a smooth point where $\nabla\lambda_k(x) = 0$. Consider the following possibilities:

(0) x may be a *smooth, regular* (i.e., not critical) point of λ_k;

(1) x may be a *symmetry* point of $\mathcal{M}_h$;

(2) x may be a *non-symmetry, smooth*, (possibly degenerate) *critical* point of λ_k;

(3) x may be a *non-smooth* point of λ_k.

3.2 Theorem. *Fix properly supported $h \in \mathcal{S}(G)$. Consider $\lambda_k\colon \mathcal{M}_h \to \mathbb{R}$ as above.*

(1) *Every symmetry point of $\mathcal{M}_h$ is a critical point of λ_k.*

(2) *If the only critical points of λ_k on $\mathcal{M}_h$ are the symmetry points and if they are non-degenerate, then the number of such critical points of index s is $\binom{\beta}{s}$.*

(3) *Suppose $h_\alpha \in \mathbb{T}_h$ has a simple eigenvalue $\lambda_k(h_\alpha)$ with eigenvector v. Then $(h_\alpha)_{rs}\bar{v}_r v_s$ is real for all $r \sim s$ if and only if h_α is a critical point of λ_k as a function on $\mathbb{T}_h$, in which case h_α is gauge equivalent to a matrix h' such that $h'_{rs} \notin \mathbb{R} \implies \bar{v}_r v_s = 0$.*

(4) *In particular, if $h_\alpha \in \mathbb{T}_h$ is a critical point of λ_k and $\lambda_k(h_\alpha)$ is* simple *with nowhere vanishing eigenvector, then $[h_\alpha] \in \mathcal{M}_h$ is a symmetry point.*
(Equivalently, there exists $\theta \in \mathbb{T}^n$ such that $h_{\alpha+d\theta} \in \mathcal{S}(h)$.)

(5) *A critical point $[h_\alpha] \in \mathcal{M}_h$ as in* (4) *is non-degenerate and its Morse index is the nodal surplus, $\phi(h_\alpha, k) - (k-1)$.*

(6) *If the diagonal entries of h are all equal, then the average nodal count distribution is symmetric,*

$$P(\mathcal{S}(h))_s = P(\mathcal{S}(h))_{\beta-s}, \quad s \in \{0, 1, \dots, \beta\}.$$

(7) *Suppose that for each k ($1 \le k \le n$) each critical point $h_\alpha \in \mathbb{T}_h$ of λ_k has $\lambda_k(h_\alpha)$ as a simple eigenvalue with nowhere vanishing eigenvector. Then the average nodal count distribution is binomial:*

$$P(\mathcal{S}(h))_s = 2^{-\beta}\binom{\beta}{s}.$$

Parts (1) and (5) of Theorem 3.2 are due to Berkolaiko and Colin de Verdière[4] [7, 13]. Part (2) is an immediate consequence, also known to both of these authors.

[4]In both works [7, 13] the matrix h was assumed to be real symmetric, but essentially the same proof works in general.

Part (3) was already observed in [5, Theorem A.1 and Lemma A.2]. Part (4) is an immediate consequence known to the authors of [5]. It says that the only simple critical points of $\lambda_k|\mathbb{T}_h$ with non-vanishing eigenvector occur along the intersection of $\mathbb{T}_h$ with the conjugacy classes of symmetry points: the $2^{|E|}$ real elements $\mathcal{S}(h)$. Proofs for Theorems 3.2 and 4.4 below will appear in Section 5 and Section 6.

3.3. Example of matrices with binomial nodal count distribution

Let $h = h_0 + \eta V$ be a Schrödinger operator on the complete graph, with h_0 properly supported (i.e., $(h_0)_{rs} \neq 0$ for all $r \neq s$), $V = \operatorname{diag}(V_1, \dots, V_n)$ with distinct entries, and $\eta \in \mathbb{R}$. If η is sufficiently large, then all matrices $\alpha * h \in \mathbb{T}_h$ will have simple eigenvalues and nowhere vanishing eigenvectors, so $P(S(h))$ is binomial.

To see that, set $\varepsilon = \frac{1}{\eta}$ and let $h_\varepsilon = \varepsilon h = V + \varepsilon h_0$. We treat $\alpha * h_\varepsilon = V + \varepsilon(\alpha * h_0)$ as a small perturbation of V whose distinct eigenvalues are V_j with eigenvectors e_j for $j = 1, \dots, n$. The min-max principle gives $|\lambda_j(\alpha * h_\varepsilon) - V_j| \leq \max_{rs} |\varepsilon(\alpha * h_0)_{rs}| = \max_{rs} |\varepsilon(h_0)_{rs}|$, so there is a uniform constant $C > 0$ such that when $0 < \varepsilon < C$, the eigenvalues of $\alpha * h_\varepsilon$ are distinct, for every α. Suppose $0 < \varepsilon < C$ and let v_ε be the j-th eigenvector of $\alpha * h_\varepsilon$. Comparing v_ε to e_j, perturbation theory gives $v_\varepsilon(j) = 1 + O(\varepsilon^2) \neq 0$, and for $i \neq j$,

$$|v_\varepsilon(i)| = \varepsilon \left| \frac{(\alpha * h_0)_{ij}}{V_i - V_j} \right| + O(\varepsilon^2) \geq \varepsilon \min_{r<s} \left| \frac{(h_0)_{rs}}{V_r - V_s} \right| + O(\varepsilon^2) \neq 0,$$

for sufficiently small ε, uniformly in α.

4. Exceptional critical points and the linkage equation

In this section we consider the case where $[h_\alpha] \in \mathcal{M}_h$ is an exceptional critical point of λ_k (cf. Section 1.6). That is, $[h_\alpha]$ is a non-symmetry, smooth, critical point with $\lambda_k(h_\alpha)$ simple. According to Theorem 3.2, the eigenvector v corresponding to $\lambda_k(h_\alpha)$ vanishes somewhere. (By generic choice of h, we can guarantee that every eigenvector of h is nowhere vanishing, cf. [32], but we cannot guarantee the same holds for all $h_\alpha \in \mathbb{T}_h$.) We address the simple case of eigenvector v that vanishes at a single vertex. By possibly replacing h_α with a gauge equivalent $h_{\alpha+d\theta}$ and v with $e^{i\theta}v$, we may assume that v is real with non-negative entries. The setting for Theorem 4.4 is described next.

4.1. The setting

To simplify the notation we assume the graph G has $n + 1$ vertices labeled $0, 1, 2, \dots, n$, with corresponding properly supported real symmetric matrix $h \in \mathcal{S}(G)$.

Suppose $h_\alpha = \alpha * h$ is a critical point of λ_k with a simple eigenvalue $\lambda := \lambda_k(h_\alpha)$ and a normalized eigenvector $v = (v_0, v_1, \ldots, v_n) = (0, v')$ and $v_0 = 0$, $v_r > 0$ for $1 \le r \le n$. Writing h and h_α as block matrices in the $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$ decomposition gives

$$h = \begin{pmatrix} a & b \\ b^* & D \end{pmatrix} \quad \text{and} \quad h_\alpha = \begin{pmatrix} a & b_\alpha \\ b_\alpha^* & D_\alpha \end{pmatrix} \quad \text{with} \begin{pmatrix} a & b_\alpha \\ b_\alpha^* & D_\alpha \end{pmatrix}\begin{pmatrix} 0 \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda v' \end{pmatrix}. \tag{4.1}$$

Let E_0 be the edges connected to vertex 0. For convenience, write $r \in E_0$ if $0r \in E_0$. Let H be the induced subgraph of G on the vertices $r \ge 1$. Thus, H is obtained from G by removing vertex 0 and its edges E_0. Then, $a \in \mathbb{R}$, $b \in \mathbb{R}^{E_0}$, $b_\alpha \in \mathbb{C}^{E_0}$, $D \in \mathcal{S}(H)$, and $D_\alpha \in \mathcal{H}(H)$. In fact, since h_α is critical and v_r is real and non-zero for $r \ge 1$, then D_α is real by part (3) of Theorem 3.2. Hence, $D_\alpha \in \mathcal{S}(H)$ is a signing of D.

The vector b_α has the form $(b_\alpha)_r = e^{i\alpha_{0r}} b_r$ for $r \in E_0$. Let $M_r := |b_r v_r| > 0$ and $\theta_r \in \mathbb{R}/2\pi\mathbb{Z}$ be the polar coordinates of $(b_\alpha)_r v_r = M_r e^{i\theta_r}$ for every $r \in E_0$.

4.2. Configuration space of a planar linkage

Equation (4.1) implies that the following *planar linkage equation* ([17,24,33]) holds:

$$b_\alpha . v' = \sum_{r \in E_0} e^{i\theta_r} M_r = 0. \tag{4.2}$$

This equation (4.2) describes a collection of vectors $M_r e^{i\theta_r} \in \mathbb{C} = \mathbb{R}^2$ in the plane, placed end to tail, that starts and ends at the origin, that is, a planar linkage, depending on a collection of lengths $L = \{M_r\}_{r \in E_0}$. Let $S^1 \subset \mathbb{C}$ be the unit circle. The *configuration space* Θ_L (see [17]) of the planar linkage defined by (4.2) is the set of solutions modulo rotations, that is,

$$\Theta_L = \Big\{(e^{i\theta_r})_{r \in E_0} \colon \sum_{r \in E_0} e^{i\theta_r} M_r = 0\Big\} / S^1 \subset (S^1)^{E_0} / S^1,$$

where the unit circle acts diagonally on $(S^1)^{E_0}$ by multiplication. The planar linkage is said to be *generic* if for any $\varepsilon \in \{-1, 1\}^{E_0}$,

$$\sum_{r \in E_0} \varepsilon_r M_r \ne 0. \tag{4.3}$$

Let M_s be the maximal length, $M_s = \max(M_r)_{r \in E_0}$. If $M_s > \sum_{r \ne s} M_r$, then there are no solutions, $\Theta_L = \emptyset$. If the planar linkage is generic and $M_s < \sum_{r \ne s} M_r$, then Θ_L is a smooth manifold of dimension $|E_0| - 3$ ([17,24,33]) whose Betti numbers have been computed in [17,23]. Let M_t be the second largest length. If $M_s + M_t \le \frac{1}{2} \sum_r M_r$, then Θ_L is connected, otherwise it has two connected components, exchanged by complex conjugation, each diffeomorphic to the torus of dimension $|E_0| - 3$.

4.3. Exceptional points

In the notation of Section 4.1, suppose $h_\alpha = \alpha * h$ is an exceptional critical point of λ_k with real eigenvector $v = (0, v')$ and simple eigenvalue $\lambda = \lambda_k(h_\alpha)$. The complex conjugate point $\bar{h}_\alpha = (-\alpha) * h$ is also a critical point of λ_k, with the same eigenvalue λ and eigenvector v. Moreover, λ is also an eigenvalue of D_α, say $\lambda = \lambda_{k'}(D_\alpha)$ is its k'-th eigenvalue.

Let F be the connected component of the critical set in $\mathcal{M}_h$ of λ_k that contains h_α, union with the connected component of the critical set of λ_k that contains $\bar{h}_\alpha$, noting that these two sets may be the same.[5]

4.4 Theorem. *Assume the following.*

(1) *The eigenvalue $\lambda = \lambda_{k'}(D_\alpha)$ is simple.*

(2) *The collection $\{M_r = |b_r v_r|\}_{r \in E_0}$ is generic (4.3).*

(3) *For any $[h'] \in F$ the eigenvalue $\lambda = \lambda_k(h')$ is simple, and*

$$c(h') = \sum_{j \neq k} \frac{|\psi_j(0)|^2}{\lambda_k(h') - \lambda_j(h')} \neq 0, \tag{4.4}$$

where $(\psi_j)_{j=1}^{n+1}$ are a choice of orthonormal eigenvectors of h' corresponding to the ordered eigenvalues.

Then the critical set F coincides with the explicitly defined set

$$F' = \{[h'] \in \mathcal{M}_h : h'v = \lambda v \text{ and there exists } \alpha'_0 \in \mathbb{T}(E_0) \text{ such that } h' = \alpha'_0 * h_\alpha\}.$$

It is a non-degenerate (Morse–Bott) critical submanifold of dimension $|E_0| - 3$ which is diffeomorphic to the configuration space Θ_L. Moreover, the Morse index of this critical submanifold is equal to

$$\mathrm{ind}(F) = \phi(D_\alpha, k') - (k' - 1) + \begin{cases} 2 & \text{if } c(h') < 0, \\ 0 & \text{if } c(h') > 0. \end{cases}$$

4.5 Remarks. Recall that the pseudo-inverse B^+ of a Hermitian matrix B with kernel V has the same kernel V and acts as $B|_{V^\perp}^{-1}$ on $V^\perp$. If we define the resolvent $(h' - z)^{-1}$ at $z = \lambda_k(h')$ using the pseudo-inverse $A = (h' - \lambda_k(h'))^+$, then $c(h') = A_{0,0}$.

A related observation for periodic metric (quantum) graphs appears in [8, Section 3.4] , where certain graphs are constructed, so that the maximum of their first spectral band is obtained on a critical manifold which is a planar linkage configuration space.

[5] Thus, the set $F \subset \mathcal{M}_h$ has either one or two connected components.

5. Proof of Theorem 3.2

5.1. For part (1), suppose $h' \in \mathbb{T}_h$ is a symmetry point (of $\mathbb{T}_h$), namely $h' = \bar{h}' \in \mathcal{S}(h)$. If h' is not a smooth point of λ_k, then it is a critical point. Suppose h' is smooth, then the directional derivative of λ_k in the direction $\alpha \in \mathcal{A}(G)$ is $\frac{d}{dt}\lambda_k(t\alpha * h')|_{t=0} = 0$ because

$$\lambda_k(t\alpha * h') = \lambda_k(\overline{t\alpha * h'}) = \lambda_k(-t\alpha * h').$$

If $h'' \in [h']$, then it is conjugate to h'. Conjugation takes a neighborhood of h' in $\mathcal{H}(G)$ to a neighborhood of h'', preserving the eigenvalue λ_k, so it also preserves the derivative of λ_k.

Part (2) follows immediately from the Morse inequalities, $C_i(\mathcal{M}_h) \geq b_i(\mathcal{M}_h)$ where C_i denotes the number of critical points of index i and where b_i is the i-th Betti number of $\mathcal{M}_h$. There are 2^β critical points by Lemma 2.9, and the sum of the Betti numbers of $\mathcal{M}_h$, a β-dimensional torus, is also 2^β. So, $C_i = b_i = \binom{\beta}{i}$ for all i.

Assuming parts (4) and (5), the proof of part (7) is a simple computation. In part (7) we assume all the critical points of λ_k correspond to simple eigenvalues with nowhere-vanishing eigenvectors, which means that all critical points are non-degenerate and are symmetry points, by part (4). Part (5) says that in such cases the nodal surplus equals the Morse index. Therefore, the average nodal surplus is

$$\begin{aligned} P(\mathcal{S}(h))_s &= 2^{-|E|} \sum_{h' \in \mathcal{S}(h)} P(h')_s \\ &= \frac{2^{-|E|}}{n} \sum_{h' \in \mathcal{S}(h)} \#\{k \leq n : \text{index } (\lambda_k(h')) = s\} \\ &= \frac{2^{-|E|}}{n} \sum_{k=1}^{n} \#\{h' \in \mathcal{S}(h) : \text{index } (\lambda_k(h')) = s\}. \end{aligned}$$

Using Lemma 2.9, the number inside the parenthesis can be expressed on the quotient $\mathcal{M}_h$

$$P(\mathcal{S}(h))_s = \frac{2^{-\beta}}{n} \sum_{k=1}^{n} \#\{[h'] \in [\mathcal{S}(h)] : \text{index } (\lambda_k([h'])) = s\} = 2^{-\beta}\binom{\beta}{s}$$

because, by part (2), the number in the parentheses is independent of k.

For part (6), by subtracting a multiple of the identity we may assume the diagonal entries of h are all zero. Let $\alpha_\pi \in \mathcal{A}(G)$ be properly supported on G, with $\pm\pi$ on the non-zero entries. For any $h_\alpha = \alpha * h \in \mathbb{T}_h$, the element $-h_\alpha = (\alpha + \alpha_\pi) * h \in \mathbb{T}_h$ is also in the same torus but the order of the eigenvalues is reversed, $\lambda_k(h_\alpha) = \lambda_{n-k}(-h_\alpha)$. This results in an inversion that sends every critical point of λ_k with index s, to a critical point of λ_{n-k} with index $\beta - s$. When averaged it gives the needed symmetry around $\beta/2$.

5.2. In this paragraph we prove parts (3) and (4) of Theorem 3.2. Let $\tilde{h} \in \mathbb{T}_h$ and suppose that $\lambda_k(\tilde{h})$ has multiplicity one. λ_k is analytic in a $\mathbb{T}_h$ neighborhood of $\tilde{h}$, and we ask when is it a critical point. To ease notation, for this paragraph only, we replace $\tilde{h}$ by h so that h is now Hermitian rather than real symmetric. Fix a direction $\alpha \in T_0(\mathbb{T}(G)) = \mathcal{A}(G)$ and consider the one-parameter perturbation of h in that direction $h_t = (t\alpha_0) * h$ for small $t \in (-\varepsilon, \varepsilon)$ so that $\dot{h}_{rs} = i\alpha_{rs}h_{rs}$.

Since λ_k is simple, we get analytic functions $v(t) \in \mathbb{C}^n$ and $\lambda_k(t) \in \mathbb{C}$, such that for all $t \in (-\varepsilon, \varepsilon)$, the vector $v(t)$ is normalized and satisfies $h_t v(t) = \lambda_k(t)v(t)$. Using the Leibniz "dot" notation for derivative with respect to t at $t = 0$ we have

$$\dot{h}v + h\dot{v} = \dot{\lambda}v + \lambda\dot{v}. \tag{5.1}$$

Taking the inner product with $v = v(0)$, using that h is self-adjoint, gives

$$\langle \nabla\lambda(h), \alpha\rangle := \dot{\lambda} = \langle v, \dot{h}v\rangle = \sum_{r\sim s} i\alpha_{rs}(h_{rs}\bar{v}_r v_s - \bar{h}_{rs}v_r\bar{v}_s),$$

so $(\nabla\lambda(h))_{rs} = i(h_{rs}\bar{v}_r v_s - \bar{h}_{rs}v_r\bar{v}_s) = 2\Im(\bar{h}_{rs}v_r\bar{v}_s)$ for all $r \sim s$. Therefore, h is a critical point if and only if $h_{rs}\bar{v}_r v_s \in \mathbb{R}$.

Assume $\nabla\lambda(h) = 0$. If $v_r \neq 0$, set $v_r = R_r e^{i\theta_r}$ with $R_r > 0$; otherwise set $\theta_r = 0$. Then $h_{rs}R_r R_s e^{i(\theta_s - \theta_r)}$ is real for all r, s. Set $e^{i\theta} = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$. Then $h' = e^{-i\theta}he^{i\theta}$ is gauge equivalent to h, and $h'_{rs} = h_{rs}e^{i(\theta_s-\theta_r)}$ is real whenever $\bar{v}_r v_s \neq 0$. In particular, if v is nowhere vanishing then h' is a symmetry point. If $h'_{rs} \notin \mathbb{R}$, then either $v_r = 0$ or $v_s = 0$. This completes the proof.

5.3. In the following paragraphs we prove part (5) of Theorem 3.2. The result was proven by Berkolaiko [7] and Colin de Verdière [13] for real symmetric h with non-positive off-diagonal entries. Both proofs extend to any real symmetric h, as the authors noted, if one defines the nodal count as in equation (2.11). We reorganize the proof of [13] and present it here for completeness and for later use. Now, let $h_\alpha \in \mathbb{T}_h \subset \mathcal{H}(G)$ be an element whose equivalence class is a symmetry point, i.e., h_α is gauge equivalent to a real symmetric matrix. For convenience, we change the notation slightly, using h instead of h_α, so suppose $h \in \mathcal{H}(G)$ is Hermitian properly supported on G, which is a critical point of λ_k, with a simple eigenvalue $\lambda := \lambda_k(h)$ and a nowhere-vanishing eigenvector v. By Theorem 3.2,

$$h_{rs}\bar{v}_r v_s \in \mathbb{R} \quad \text{for all } r \sim s. \tag{5.2}$$

Let $\mathrm{ind}(Q)$ denote the number of negative eigenvalues of a quadratic form Q and use $\mathrm{Hess}(F)$ for the Hessian of a function $F\colon \mathbb{T}(G) \to \mathbb{R}$, evaluated at $\alpha = 0$. It is a quadratic form on the tangent space $T_0\mathbb{T}(G) = \mathcal{A}(G)$.

Define $\mu\colon \mathbb{T}(G) \to \mathbb{R}$ by $\mu(\alpha) = \lambda_k(\alpha * h)$. Since $\mu(\alpha + d\theta) = \mu(\alpha)$ for all $\theta \in \mathbb{R}^n$, it follows that the Morse index of λ_k at the point $h \in \mathbb{T}(G)$ is

$$\operatorname{ind}(\lambda_k)(h) = \operatorname{ind}(\operatorname{Hess}(\mu)) = \operatorname{ind}(\operatorname{Hess}(\mu)|V)$$

for *any* complement $V \oplus d\mathbb{R}^n = \mathcal{A}(G)$. The trick ([13]) is to define $F\colon \mathbb{T}(G) \to \mathbb{R}$ by

$$F(\alpha) = \langle v, (\alpha * h - \lambda)v \rangle = \sum_{r\sim s} \bar{v}_r e^{i\alpha_{rs}} h_{rs} v_s + \sum_r |v_r|^2 h_{rr} - \lambda$$

where $\lambda = \lambda_k(h)$ and v are constant, and show that

(1) $\alpha = 0$ is a non-degenerate critical point of F with $\operatorname{ind}(\operatorname{Hess}(F)) = \phi(h,k)$;

(2) $\operatorname{ind}(\operatorname{Hess}(F)|d\mathbb{R}^n) = k - 1$;

(3) $\operatorname{ind}(\operatorname{Hess}(\mu)|V) = \operatorname{ind}(\operatorname{Hess}(F)|V)$ where V is now chosen to be the orthogonal complement of $d\mathbb{R}^n$ with respect to $\operatorname{Hess}(F)$:

$$\alpha \in V \iff \langle \alpha, \operatorname{Hess}(F)(d\theta)\rangle = 0 \quad \text{for all } \theta \in \mathbb{R}^n.$$

These three steps complete the proof of Theorem 3.2 (5) because they give

$$\begin{aligned}\operatorname{ind}(\lambda_k)(h) = \operatorname{ind}(\operatorname{Hess}(F)|V) &= \operatorname{ind}(\operatorname{Hess}(F)) - \operatorname{ind}(\operatorname{Hess}(F)|d\mathbb{R}^n)\\ &= \phi(h,k) - (k-1).\end{aligned}$$

5.4. Step 0

To compute $\operatorname{Hess}(\mu)$, namely the Hessian of $\lambda_k(h_\alpha)$ at $\alpha = 0$, let $\gamma, \delta \in \mathcal{A}_n(G)$, set $h(s,t) = (s\gamma + t\delta) * h$ and set $\mu(s,t) = \lambda_k(h(s,t))$ with corresponding normalized eigenvector $v(s,t)$. Using dot and prime to denote derivatives in s, t at $s = 0, t = 0$ respectively, we claim that

$$\langle \gamma, \operatorname{Hess}(\mu)\delta\rangle = 2\Re(\langle v', \dot{h} v\rangle) + \langle \gamma, \operatorname{Hess}(F)\delta\rangle. \tag{5.3}$$

Differentiating $\mu(s,t) - \lambda I = \langle v(s,t), (h(s,t) - \lambda)v(s,t)\rangle$ gives

$$\begin{aligned}\dot{\mu}' &= \langle \dot{v}', (h - \lambda I)v\rangle + \langle v', \dot{h} v\rangle + \langle v', (h - \lambda I)\dot{v}\rangle\\ &\quad + \langle \dot{v}, h' v\rangle + \langle v, \dot{h}' v\rangle + \langle v, h'\dot{v}\rangle\\ &\quad + \langle \dot{v}, (h - \lambda I)v'\rangle + \langle v, \dot{h} v'\rangle + \langle v, (h - \lambda I)\dot{v}'\rangle\\ &= 2\Re[\langle v', \dot{h} v\rangle + \langle \dot{v}, (h - \lambda I)v'\rangle + \langle \dot{v}, h' v\rangle] + \langle v, \dot{h}' v\rangle,\end{aligned}$$

where $\langle \dot{v}', (h - \lambda I)v\rangle$ and $\langle v, (h - \lambda I)\dot{v}'\rangle$ vanish because $v \in \ker(h - \lambda I)$. Furthermore, the criticality condition $\mu' = 0$ applied to (5.1) gives

$$(h - \lambda I)v' = -h' v, \tag{5.4}$$

so the term $\langle \dot{v}, (h-\lambda I)v' \rangle + \langle \dot{v}, h'v \rangle$ vanishes and we are left with

$$\dot{\mu}' = 2\Re(\langle v', \dot{h} v \rangle) + \langle v, \dot{h}' v \rangle.$$

We are done, since $\dot{\mu}' = \langle \gamma, \mathrm{Hess}(\mu)\delta \rangle$ and $\langle v, \dot{h}' v \rangle = \langle \gamma, \mathrm{Hess}(F)\delta \rangle$.

5.5. Step 1

Calculate

$$\frac{\partial F}{\partial \alpha_{rs}} = i(\bar{v}_r e^{i\alpha_{rs}} h_{rs} v_s - \bar{v}_s e^{-i\alpha_{rs}} \bar{h}_{rs} v_r),$$

which vanishes at $\alpha = 0$ by (5.2), and

$$\frac{\partial^2 F}{\partial^2 \alpha_{rs}}(0) = -(\bar{v}_r h_{rs} v_s + \bar{v}_s \bar{h}_{rs} v_r) = -2\bar{v}_r h_{rs} v_s,$$

which is real and non-zero for $r \sim s$, and all other second derivatives vanish. Therefore, $\mathrm{Hess}(F)$ is non-degenerate with index

$$\mathrm{ind}(\mathrm{Hess}(F)) = \#\{r \sim s, r < s \colon \bar{v}_r h_{rs} v_s > 0\} = \phi(h,k).$$

5.6. Step 2

For (small) $t \in \mathbb{R}$ and $\theta \in \mathbb{R}^n$, we will find the second derivative of

$$F(d(t\theta)) = \langle v, (d(t\theta) * h - \lambda) v \rangle = \langle e^{-it\Theta} v, (h-\lambda) e^{-it\Theta} v \rangle$$

by equation (2.5), where $\Theta = \mathrm{diag}(\theta_1, \theta_2, \dots, \theta_n)$ so $e^{i\Theta} = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$ (which was formerly denoted $e^{i\theta}$). Then

$$\frac{d}{dt} F(d(t\theta)) = i\langle \Theta e^{-it\Theta} v, (h-\lambda) e^{-it\Theta} v \rangle - i\langle e^{-it\Theta} v, (h-\lambda)\Theta e^{-it\Theta} v \rangle,$$

$$\frac{d^2}{dt^2} F(d(t\theta))\Big|_{t=0} = -\langle \Theta^2 v, (h-\lambda) v \rangle - \langle v, (h-\lambda)\Theta^2 v \rangle + 2\langle \Theta v, (h-\lambda)\Theta v \rangle.$$

The first two terms vanish. Using $M_v = \mathrm{diag}(v_1, v_2, \dots, v_n)$, we get

$$\langle d\theta, \mathrm{Hess}(F)(d\theta) \rangle = 2\langle \Theta v, (h-\lambda)\Theta v \rangle = 2\langle \theta, M_v^*(h-\lambda) M_v \theta \rangle,$$

for all $\theta \in \mathbb{R}^n$. According to (5.2), $2M_v^*(h-\lambda)M_v$ is real and is therefore equal to $\mathrm{Hess}(F)|d\mathbb{R}^n$ as these are real symmetric matrices with equal quadratic forms. The matrix M_v is invertible since v is nowhere-vanishing. Two conclusions follow.

(1) Assume $\alpha = d\theta \in V \cap d\mathbb{R}^n$. Then $M_v\theta \in \ker(h-\lambda)$ and so $M_v\theta \propto v$ since λ is simple. Then θ is constant, so $d\theta = 0$. We conclude that

$$V \oplus d\mathbb{R}^n = \mathcal{A}(G).$$

(2) $\mathrm{Hess}(F)|d\mathbb{R}^n$ and $h-\lambda$ have the same number of negative eigenvalues, so

$$\mathrm{ind}(\mathrm{Hess}(F)|d\mathbb{R}^n) = k-1.$$

5.7. Step 3

For any $\theta \in \mathbb{R}^n$, the derivative of v in direction $d\theta$ is $v' = i\Theta v$ and $\mathrm{Hess}(\mu)d\theta = 0$ due to gauge invariance. Let $\partial_\alpha h$ stand for the derivative of h in direction $\alpha \in \mathcal{A}(G)$, so that for any $\delta = d\theta$ equation (5.3) gives

$$\langle \alpha, \mathrm{Hess}(F)d\theta\rangle = -2\Re(\langle v', \partial_\alpha h v\rangle) = 2\Im(\langle \Theta v, \partial_\alpha h v\rangle).$$

It follows from (5.2) that $\langle \Theta v, \partial_\alpha h v\rangle$ is purely imaginary, so $\alpha \in V$ if and only if $\langle \Theta v, \partial_\alpha h v\rangle$ vanish for all real diagonal Θ. Since v is nowhere-vanishing,

$$\alpha \in V \iff \partial_\alpha h v = 0.$$

Consequently, equation (5.3) shows that $\mathrm{Hess}(\mu)$ and $\mathrm{Hess}(F)$ agree on V. ■

6. Proof of Theorem 4.4

6.1. The critical set F'

Recalling the notations of Section 4.1, the graph G has $n+1$ vertices labeled $0, 1, 2, \ldots, n$. The set E_0 is the set of edges connected to 0. The graph H is the induced graph on the non-zero vertices. The torus of perturbations and its tangent space decompose as

$$\mathcal{A}(G) = \mathcal{A}(E_0) \oplus \mathcal{A}(H), \quad \mathbb{T}(G) = \mathbb{T}(E_0) \oplus \mathbb{T}(H),$$

and $h_\alpha = \alpha * h = \begin{pmatrix} a & b_\alpha \\ b_\alpha^* & D_\alpha \end{pmatrix}$ is an exceptional critical point of λ_k with simple eigenvalue $\lambda = \lambda_k(h_\alpha)$ and real eigenvector $v = (0, v')$ with $v_0 = 0$ and $v_r > 0$ for $r > 0$. As discussed in Section 4.1, D_α must be real, so it is a signing of D. By replacing $h \in \mathcal{S}(G)$ with a signing of h (if necessary), we may assume $D_\alpha = D$ and $\alpha \in \mathcal{A}(E_0)$, so $h_\alpha v = \lambda v$ becomes $Dv' = \lambda v'$ and

$$b_\alpha . v' = \sum_{r\in E_0} b_r v_r e^{i\alpha_{0r}} = 0.$$

Recall that F denotes the union of the connected components of the critical set of λ_k in $\mathcal{M}_h$ that contain $[h_\alpha]$ and $[\bar{h}_\alpha]$. In Section 6.2 and Section 6.8, we will prove that $F = F'$ where

$$F' := \{[h'] \in \mathcal{M}_h : h'v = \lambda v \text{ and there exists } \gamma \in \mathcal{A}(E_0) \text{ such that } h' = \gamma * h_\alpha\}.$$

Observe that *the set F' is closed under complex conjugation* because λ and v are real, and $\overline{\gamma * h_\alpha} = (-\gamma - 2\alpha) * h_\alpha$ for any $\gamma \in \mathcal{A}(E_0)$. Moreover, *the set F' consists of critical points of λ_k*: since h_α is critical and λ_k is simple, Theorem 3.2 (3) implies $v_s(h_\alpha)_{rs}v_r$ is real for every $r \sim s$. Since $\lambda = \lambda_k(h')$ is simple for any $h' = \gamma * h_\alpha \in F'$, then $v_s h'_{rs} v_r = v_s(h_\alpha)_{rs}v_r$ is real for every $r \sim s$ (since $v_0 = 0$) hence $[h']$ is also a critical point.

6.2. The diffeomorphism between F' and Θ_L

Define $\Phi\colon \mathbb{T}(E_0) * h_\alpha \xrightarrow{\cong} (S^1)^{E_0}$ by

$$\Phi(\gamma * h_\alpha)_r = \begin{cases} e^{i\gamma_{0r}} e^{i\alpha_{0r}} & \text{if } b_r > 0, \\ -e^{i\gamma_{0r}} e^{i\alpha_{0r}} & \text{if } b_r < 0, \end{cases}$$

for any $\gamma \in \mathcal{A}(E_0)$. An element $h' \in \mathbb{T}(E_0) * h_\alpha$ satisfies $h'v = \lambda v$ (with the same λ and $v = (0, v')$) if and only if $\Phi(h') = (e^{i\theta_r})_{r\in E_0}$ is a solution to the planar linkage equation $\sum_{r\in E_0} e^{i\theta_r} M_r = 0$ with $M_r := |b_r v_r|$. The normalizer of $\mathbb{T}(E_0) * h_\alpha$ in the gauge group is the zeroth coordinate $\mathbb{T}^{(0)} := \{(x, 0, 0, \ldots, 0) \in \mathbb{T}^{n+1} : x \in S^1\}$, cf. equation (2.4). Therefore, the diffeomorphism Φ passes to the quotient,

$$\Phi\colon (\mathbb{T}(E_0) * h_\alpha) \mathbin{/\!/} \mathbb{T}^{(0)} \xrightarrow{\cong} (S^1)^{E_0}/S^1$$

with $\Phi(F') = \Theta_L$. By [17, p. 78], the set F' is a smooth manifold, closed under complex conjugation, and either it is connected or it has two connected components that are exchanged by complex conjugation. Moreover, it consists of critical points, so $F' \subset F$. (The reverse inclusion is proven in Section 6.8.)

6.3. Gauge transformations on G, H and E_0

Decompose the space of functions on G, $\mathbb{R}^G = \mathbb{R}^{n+1}$, into $\mathbb{R}^{n+1} = \mathbb{R}^{(0)} \oplus \mathbb{R}^H$, where $\mathbb{R}^H := \{(0, x) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\} \cong \mathbb{R}^n$. The coboundary differential on H is denoted

$$d_H\colon \mathbb{R}^H \to \mathcal{A}(H) \subset \mathcal{A}(G); \quad (d_H f)_{rs} = \begin{cases} f(s) - f(r) & \text{if } r \sim s \text{ and } r, s \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The image $d_H\mathbb{R}^H$ is the projection of $d\mathbb{R}^H$ into $\mathcal{A}(H)$ and in fact, for any $f \in \mathbb{R}^H$,

$$df = d_H f + \sum_{r\in E_0} f(r) J(r, 0) \tag{6.1}$$

where $J(r,0) \in \mathcal{A}(E_0)$ is the antisymmetric matrix with $J(r,0)_{r0} = 1$, $J(r,0)_{0r} = -1$ and all other entries are 0. Let $\mathbf{1}_H = (0,1,1,\dots,1) \in \mathbb{R}^H$ denote the constant vector on H and let $\mathbf{1}_{E_0} = \sum_{r\in E_0} J(r,0) \in \mathcal{A}(E_0)$. It is easy to verify that

$$\operatorname{span}_{\mathbb{R}}(\mathbf{1}_{E_0}) = d\mathbb{R}^{(0)} = d(\operatorname{span}_{\mathbb{R}}(\mathbf{1}_H)) = (d\mathbb{R}^H) \cap \mathcal{A}(E_0) = (d\mathbb{R}^{n+1}) \cap \mathcal{A}(E_0). \tag{6.2}$$

Since D is properly supported on H and $\lambda = \lambda'_k(D)$ is a simple eigenvalue of D with a nowhere-vanishing eigenvector v', then H is connected. Theorem 3.2 gives a decomposition

$$\mathcal{A}(H) = V_H \oplus d_H\mathbb{R}^H \tag{6.3}$$

and V_H can be described in terms of directional derivatives, according to Section 5.6,

$$V_H = \{\alpha_H \in \mathcal{A}(H) \colon (\partial_{\alpha_H} h_\alpha)v = 0\}, \quad \partial_{\alpha_H} h_\alpha := \frac{d}{dt}(t\alpha_H * h_\alpha)\Big|_{t=0}. \tag{6.4}$$

Let $\mathcal{A}_0(E_0)$ denote the orthogonal complement to $\operatorname{span}_{\mathbb{R}}(\mathbf{1}_{E_0})$,

$$\mathcal{A}_0(E_0) := \Big\{\gamma \in \mathcal{A}(E_0) \colon \sum_{r\in E_0} \gamma_{0r} = 0\Big\}.$$

Let

$$\pi \colon \mathcal{A}(G) \to \mathcal{A}(G)/d\mathbb{R}^{n+1} \cong H^1(G;\mathbb{R})$$

denote the quotient by

$$d\Omega^0(G) = d\mathbb{R}^{n+1},$$

cf. equation (2.2).

6.4 Lemma. *The space $\mathcal{A}(G)$ decomposes as a direct sum*

$$\mathcal{A}(G) = \mathcal{A}_0(E_0) \oplus V_H \oplus d\mathbb{R}^{n+1}. \tag{6.5}$$

In particular, $\mathcal{A}_0(E_0) \oplus V_H \xrightarrow{\cong} \pi(\mathcal{A}(G))$ and $\mathcal{A}_0(E_0) \xrightarrow{\cong} \pi(\mathcal{A}(E_0))$.

Proof. It follows from (6.1) that $d_H\mathbb{R}^n \subset d\mathbb{R}^{n+1} + \mathcal{A}(E_0)$. Using (6.2) and (6.3),

$$\begin{aligned}\mathcal{A}_0(E_0) + V_H + d\mathbb{R}^{n+1} &= \mathcal{A}(E_0) + V_H + d\mathbb{R}^{n+1}\\ &\supset \mathcal{A}(E_0) + (V_H + d_H\mathbb{R}^n) = \mathcal{A}(G)\end{aligned}$$

so $\mathcal{A}(G)$ is spanned by the sum on the left side. On the other hand, the sum on the left-hand side is a direct sum because the sum of the dimensions of the vector spaces is

$$\begin{aligned}(|E_0| - 1) + (\beta_H) + n &= (|E_0| - 1) + (|E_H| - n + 1) + n\\ &= |E_G| = \dim(\mathcal{A}(G)).\end{aligned}$$

∎

6.5. The tangent space to F'

Consider the preimage of F' in $\mathbb{T}(E_0) * h_\alpha$,

$$\begin{aligned}\widehat{F}' &:= \{\gamma * h_\alpha : \gamma \in \mathbb{T}(E_0),\ [\gamma * h_\alpha] \in F'\} \\ &= \Big\{\gamma * h_\alpha : \gamma \in \mathbb{T}(E_0) \text{ and } \sum_{r \in E_0} e^{i\gamma_{0r}} (b_\alpha)_r v_r = 0\Big\}.\end{aligned}$$

Differentiate and use the identification $T_{h_\alpha}\mathbb{T}_h = \mathcal{A}(G)$ to obtain the tangent space

$$T_{h_\alpha}\widehat{F}' \cong \Big\{\gamma \in \mathcal{A}(E_0): \sum_{r \in E_0} \gamma_{0r}(b_\alpha)_r v_r = 0\Big\} \subset \mathcal{A}(E_0).$$

By Lemma 6.4, the quotient projection π takes $\mathcal{L} := T_{h_\alpha}\widehat{F}' \cap \mathcal{A}_0(E_0)$ isomorphically to $T_{[h_\alpha]}F'$, that is,

$$\mathcal{L} = \Big\{\gamma \in \mathcal{A}_0(E_0): \sum_{r \in E_0} \gamma_{0r}(b_\alpha)_r v_r = 0\Big\} \cong \pi(\mathcal{L}) = T_{[h_\alpha]}F'. \tag{6.6}$$

Let $\mathbb{R}_0^{E_0}$ be the space of mean zero elements of $\mathbb{R}^{E_0}$, which we identify with $\mathcal{A}_0(E_0)$. Let $\mathbf{x} \in \mathbb{R}_0^{E_0}$ and $\mathbf{y} \in \mathbb{R}_0^{E_0}$ such that $(b_\alpha)_r v_r = \mathbf{x}_r + i\mathbf{y}_r$ for all $r \in E_0$. Then

$$\mathcal{L} = \{\gamma \in \mathcal{A}_0(E_0): \gamma \cdot \mathbf{x} = 0 \text{ and } \gamma \cdot \mathbf{y} = 0\}.$$

6.6. The Hessian of λ

Since $d\mathbb{R}^{n+1}$ acts by gauge transformations, the quadratic form $\mathrm{Hess}(\lambda_k)$ at the critical point h_α, expressed with respect to the decomposition (6.5) has the following form:

$$\mathrm{Hess}\lambda = \begin{pmatrix} A & C & 0 \\ C^* & B & 0 \\ 0 & 0 & 0. \end{pmatrix}.$$

We will show that $C = 0$ and $\det(B) \neq 0$ with $\mathrm{ind}(B) = \Phi(D, k') - (k' - 1)$.

Using the notation ∂_γ for the directional derivative in direction γ and $\partial^2_{\delta,\gamma}$ for the second derivative in direction γ and then in direction δ, equation (5.3) states that

$$\langle \gamma, \mathrm{Hess}\lambda\,\delta\rangle = 2\Re[\langle \partial_\gamma v, (\partial_\delta h_\alpha)v\rangle] + \langle v, (\partial^2_{\gamma,\delta} h_\alpha)v\rangle.$$

If γ is supported on E_0 and δ is supported on H, then $\partial^2_{\gamma,\delta} h_\alpha = 0$. If $\delta \in V_H$, then $(\partial_\delta h_\alpha)v = 0$ according to (6.4). We conclude that $\langle \gamma, \mathrm{Hess}\lambda\,\delta\rangle = 0$ when $\delta \in V_h$ and $\gamma \in \mathcal{A}_0(E_0)$. Namely, $C = 0$.

Now, consider the block $B = \mathrm{Hess}\lambda|_{V_H}$. Let $\mathrm{Hess}F$ be the Hessian of the function $F(\delta) := \langle v', (\delta * D)v'\rangle$ for $\delta \in \mathbb{T}(H)$ evaluated at $\delta = 0$. Since D is real, then it is a critical point of $\lambda_{k'}$. Since $\lambda = \lambda_{k'}(D)$ is simple with a nowhere-vanishing eigenvector v', Theorem 3.2 implies the restriction $\mathrm{Hess}F|_{V_H}$ is non-degenerate and has $\Phi(D,k') - (k'-1)$ negative eigenvalues. Suppose $\gamma, \delta \in V_H$, then $(\partial_\delta h_\alpha)v = 0$ and $(\partial_\gamma h_\alpha)v = 0$ due to (6.4), and equation (5.3) gives

$$\langle \gamma, B\delta \rangle = \langle v, (\partial^2_{\gamma,\delta} h_\alpha)v \rangle = \langle \gamma, \mathrm{Hess}F\delta \rangle.$$

Therefore, $B = \mathrm{Hess}F|_{V_H}$. That is,

$$\det(B) \neq 0 \quad \text{and} \quad \mathrm{ind}(B) = \Phi(D,k') - (k'-1).$$

6.7. The block $A = \mathrm{Hess}\lambda|_{\mathcal{A}_0(E_0)}$

In this case, for $\gamma, \delta \in \mathcal{A}_0(E_0)$ the matrix $\partial^2_{\gamma,\delta} h_\alpha$ is supported on E_0 so the second term in equation (5.3) vanishes and we get

$$\langle \gamma, \mathrm{Hess}\lambda\delta \rangle = 2\Re[\langle \partial_\gamma v, (\partial_\delta h_\alpha)v \rangle].$$

The vector $(\partial_\delta h_\alpha)v$ is only non-zero at the first coordinate,

$$((\partial_\delta h_\alpha)v)_0 = i \sum_{r \in E_0} \delta_{0r}(b_\alpha)_r v_r = i\delta \cdot \mathbf{x} - \delta \cdot \mathbf{y}.$$

To calculate $\partial_\gamma v$, use (5.4), which states

$$(h_\alpha - \lambda I)\partial_\gamma v = -(\partial_\gamma h_\alpha)v. \tag{6.7}$$

Let ψ_j for $j = 1, 2, \ldots, n+1$ be a choice of orthonormal eigenvectors of h_α corresponding to the ordered eigenvalues. The Moore–Penrose Pseudo-inverse of $(h_\alpha - \lambda I)$ is the matrix

$$(h_\alpha - \lambda I)^+ := \sum_{j:\lambda_j(h_\alpha) \neq \lambda} \frac{1}{\lambda_j(h_\alpha) - \lambda} \psi_j \psi_j^* = \sum_{j \neq k} \frac{1}{\lambda_j(h_\alpha) - \lambda} \psi_j \psi_j^*,$$

where in the last equality we used that $\lambda = \lambda_k(h_\alpha)$ is simple. By left multiplying (6.7) with the matrix $(h_\alpha - \lambda I)^+$ (whose kernel is spanned by v), we get

$$\partial_\gamma v = -(h_\alpha - \lambda I)^+ (\partial_\gamma h_\alpha)v + \tilde{c}v,$$

for some constant $\tilde{c}$. Having $v_0 = 0$ yields

$$\begin{aligned}\langle \partial_\gamma v, (\partial_\delta h_\alpha)v \rangle &= -\overline{(h_\alpha - \lambda I)^+_{00}((\partial_\gamma h_\alpha)v)_0}((\partial_\delta h_\alpha)v)_0 \\ &= c(h_\alpha)\overline{(i\gamma \cdot \mathbf{x} - \gamma \cdot \mathbf{y})}(i\delta \cdot \mathbf{x} - \delta \cdot \mathbf{y}),\end{aligned}$$

where, from (4.4),

$$c(h_\alpha) = \sum_{j \neq k} \frac{|\psi_j(0)|^2}{\lambda - \lambda_j(h_\alpha)} = -(h_\alpha - \lambda I)^+_{00}.$$

We conclude that

$$\langle \gamma, A\delta \rangle = 2c(h_\alpha)((\gamma \cdot \mathbf{x})^2 + (\delta \cdot \mathbf{y})^2) = \langle \gamma, 2c(h_\alpha)(\mathbf{x}\mathbf{x}^* + \mathbf{y}\mathbf{y}^*)\delta \rangle.$$

Since $c(h_\alpha) \neq 0$ by assumption, then A has rank two over $\mathcal{A}_0(E_0)$. In particular,

$$\ker(A) = \mathcal{L}, \quad \text{and} \quad \operatorname{ind}(A) = \begin{cases} 2 & \text{if } c(h_\alpha) < 0, \\ 0 & \text{if } c(h_\alpha) > 0. \end{cases}$$

Since $\mathrm{Hess}\lambda|_{\mathcal{A}_0(E_0)\oplus V_H} = A \oplus B$, then we conclude that $\ker(\mathrm{Hess}\lambda) = \mathcal{L} \oplus d\mathbb{R}^{n+1}$ and

$$\operatorname{ind}(\mathrm{Hess}\lambda) = \phi(D_\alpha, k') - (k'-1) + \begin{cases} 2 & \text{if } c(h_\alpha) < 0, \\ 0 & \text{if } c(h_\alpha) > 0. \end{cases}$$

6.8. F is Morse–Bott

Recall that the submanifold of critical points $F' \subset \mathcal{M}_h$ is a *Morse–Bott* critical submanifold of λ_k if, at every point $[h'] \in F'$, the kernel of $\mathrm{Hess}\lambda_k([h'])$ is exactly the tangent space $T_{[h']}F'$ and the number of negative eigenvalues of $\mathrm{Hess}\lambda_k([h'])$ is constant for all $[h'] \in F'$. By (6.6) the kernel condition holds at $[h_\alpha]$. Since $c(h_\alpha)$ is non-zero and continuous, it does not change sign, so the Morse–Bott condition holds at every point $[h'] \in F'$. As the kernel of the Hessian at a point $[h'] \in F' \subset F$ is $T_{[h']}F'$ and λ_k is constant on F, the tangent spaces agree, $T_{[h']}F' = T_{[h']}F$. Since F' is closed it is a union of connected components of F. It contains both $[h_\alpha]$ and its complex conjugate, so $F = F'$ is Morse–Bott and

$$\operatorname{ind}(F) = \phi(D_\alpha, k') - (k'-1) + \begin{cases} 2 & \text{if } c(h_\alpha) < 0, \\ 0 & \text{if } c(h_\alpha) > 0. \end{cases}$$

7. Transversality to the strata of $\mathcal{H}_n$

7.1. The strata

The vector space $\mathcal{H}_n$ of Hermitian $n \times n$ matrices is stratified according to the multiplicities of the eigenvalues, as described in [3]. (See also [1, 4, 30].) Suppose $h \in \mathcal{H}_n$ has k *distinct* eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_k$. Specifying a multiplicity $r(i)$ for the eigenvalue μ_i determines a stratum $T(r)$, consisting of Hermitian matrices with eigenvalues μ_i and multiplicities $r(i)$. The multiplicity vector r is an ordered partition of n, meaning that $n = \sum_{i=1}^{k} r(i)$, and every ordered partition of n determines a stratum. The set of possible eigenvalues for h forms an open set

$$\mathbb{R}^k_< = \{x \in \mathbb{R}^k : x_1 < x_2 < \cdots < x_k\}$$

in $\mathbb{R}^k$. The eigenspaces V_i determine a partial flag $V_1 \subset V_1 \oplus V_2 \subset \cdots \subset \mathbb{C}^n$. Therefore, the stratum $T(r)$ may be canonically identified with the product

$$P(r) = \mathcal{F}l(r) \times \mathbb{R}^k_<$$

where $\mathcal{F}l(r)$ denotes the partial flag manifold of subspaces $0 \subset W_1 \subset W_2 \subset \cdots \subset \mathbb{C}^n$ with $\dim(W_k) = \sum_{i=1}^{k} r(i)$. This identification endows the stratum $T(r)$ with the canonical structure of an analytic manifold, and each eigenvalue $\mu_i : T(r) \to \mathbb{R}$ is an analytic function.

It is well known [34] that $\mathcal{F}l(r)$ is isomorphic to the quotient $U(n)/\prod_{i=1}^{k} U(r(i))$ of unitary groups, so it has dimension $n^2 - \sum_{i=1}^{k} r(i)^2$ from which it follows that the stratum $T(r)$ has codimension $\sum_{i=1}^{k}(r(i)^2 - 1)$ in $\mathcal{H}_n$.

7.2. The manifold $S_m(k)$

For any $h \in \mathcal{H}_n$, we may label the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Fix m, k. Let $V_k(h) = \ker(h - \lambda_k(h).I)$ be the eigenspace with eigenvalue λ_k. Define[6]

$$S_m(k) = \{h \in \mathcal{H}_n : \dim(V_k(h)) = m \text{ and } \lambda_{k-1}(h) < \lambda_k(h)\}. \tag{7.1}$$

Each $h \in S_m(k)$ has exactly $k-1$ eigenvalues less than λ_k and $n-m-k+1$ eigenvalues greater than λ_k. It is foliated with leaves indexed by $\lambda \in \mathbb{R}$,

$$S_m(k,\lambda) = \{h \in S_m(k) : \lambda_k(h) = \lambda\}. \tag{7.2}$$

7.3 Lemma. *The set $S_m(k)$ (resp. $S_m(k,\lambda)$) is an analytic manifold of codimension m^2-1 (resp. codimension m^2) in $\mathcal{H}_n$. The eigenvalue $\lambda_k : S_m(k) \to \mathbb{R}$ is analytic. If r is an ordered partition of n, then stratum $T(r)$ is the transverse intersection*

$$S_{r(1)}(1) \cap S_{r(2)}(1 + r(1)) \cap S_{r(3)}(1 + r(1) + r(2)) \cap \cdots \cap S_{r(k)}(n - r(k) + 1). \tag{7.3}$$

[6]We are grateful to the referee for pointing out an error in our earlier definition of $S_m(k)$.

Proof. If $h \in S_m(k)$, then the eigenspace $V = V_k(h)$ is an element of the Grassmann manifold $G_m(\mathbb{C}^n)$ of m-dimensional subspaces of $\mathbb{C}^n$. Set $t = \lambda_k(h)$. The restriction $h|V^\perp$ determines an orthogonal decomposition $V_k(h)^\perp = W_- \oplus W_+$ as the sum of the $< t$ (resp. $> t$) eigenspaces of $h|V_k(h)^\perp$ of dimension $k-1$ and dimension $n-m-k+1$ respectively. The further restriction $h|W_-$ lies in the set $\mathcal{H}(W_-)^{<t}$ of Hermitian operators all of whose eigenvalues are $< t$, and similarly for $h|W_+$. Therefore, we have parametrized $S_m(k)$ by a double fibration

$$\begin{array}{ccccc}
\mathcal{H}(W_-)^{<t} \times \mathcal{H}(W_+)^{>t} & & \hookrightarrow & & S_m(k) \\
& \searrow & & & \downarrow \\
& & W_- \in G_{k-1}(V^\perp) & \hookrightarrow & E \\
& & & \searrow & \downarrow \\
& & & & (V,t) \in G_m(\mathbb{C}^n) \times \mathbb{R}
\end{array}$$

where E is the bundle whose fiber over (V,t) is the Grassmannian of $(k-1)$-dimensional complex subspaces $W_- \subset V^\perp$. From this, we see that $S_m(k)$ is an analytic manifold and $t = \lambda_k$ is an analytic function on $S_m(k)$, as a coordinate in the parametrization.

The dimension of $S_m(k)$ may be calculated from the above diagram,

$$\dim(S_m(k)) = \dim_{\mathbb{R}}(G_m(\mathbb{C}^n)) + \dim_{\mathbb{R}}(G_{k-1}(V^\perp)) + \dim(\mathcal{H}(W_-)^{<t} \times \mathcal{H}(W_+)^{>t}),$$

and a miraculous cancellation of terms gives $\operatorname{codim}(S_m(k)) = m^2 - 1$. It is a direct consequence of the definitions that $T(r)$ is the intersection (7.3). The intersection is transversal because the different factors in (7.3) involve independent conditions. ∎

7.4 Proposition. *Fix $h \in \mathcal{H}_n$ with eigenvalue $\lambda = \lambda_k(h)$ and eigenspace $V = V_k$ of dimension m.*

(A) *The tangent space $T_h S_m(k,\lambda)$ (resp. $T_h S_m(k)$) consists of all tangent vectors $\xi \in T_h \mathcal{H}_n = \mathcal{H}_n$ such that, as sesquilinear forms,*[7] *the restriction $\xi|V = 0$ (resp. such that $\xi|V$ is a scalar*[8]*). With respect to the decomposition*

$$\mathbb{C}^n = V_k \oplus V_k^\perp,$$

[7] Cf. Section 2.1.

[8] In fact, it is multiplication by the directional derivative $\partial_\xi(\lambda_k)$.

it is the subspace of matrices

$$\xi = \begin{pmatrix} 0 & B \\ B^* & D \end{pmatrix}, \quad \textit{resp. } \xi = \begin{pmatrix} c.I_V & B \\ B^* & D \end{pmatrix}, \tag{7.4}$$

where $D \in \mathcal{H}_{n-k}$, $B \in M_{k\times(n-k)}(\mathbb{C})$, $c \in \mathbb{R}$, *and* I_V *the identity on* V.

(B) *A submanifold* $Q \subset \mathcal{H}_n$ *is transverse to* $S_m(k,\lambda)$ *(resp.* $S_m(k)$*) at* $h \in Q$ *if and only if the elements* $\xi|V$ *(resp. the elements* $c.I_V + \xi|V$*) account for all the Hermitian operators in* $\mathcal{H}(V)$*, as* ξ *varies within* T_hQ *and* c *within* $\mathbb{R}$.

(C) *The tangent space* $T_hS_m(k,\lambda)$ *(respectively,* $T_hS_m(k)$*) can also be expressed as the set of all* $\xi \in \mathcal{H}_n$ *of the form*

$$\xi = (h-\lambda.I)U + U^*(h-\lambda.I) \quad \textit{with } U \in M_{n\times n}(\mathbb{C})$$

(respectively, $\xi = (h-\lambda.I)U + U^*.(h-\lambda.I) + c.I_V$ *with* $U \in M_{n\times n}(\mathbb{C})$ *and* $c \in \mathbb{R}$*).*

Proof. Let ξ be a tangent vector to $S_m(k,\lambda)$ at the point $h = h_0$. Let $h_t \in S_m(k,\lambda)$ be a smooth one parameter family with $\xi = \dot{h} = \frac{d}{dt}h(0)$. Suppose $u_t \in V \subset \mathbb{C}^n$ is an eigenvector of h_t with eigenvalue λ. Differentiating the eigenvalue equation $hu = \lambda u$ gives $\dot{h}u + h\dot{u} = \lambda\dot{u}$. Taking the inner product with any $w \in V$ gives $\langle w, \xi u\rangle = 0$ which shows that ξ has the form of equation (7.4) above. On the other hand, the codimension of the space of matrices (7.4) is m^2 which equals the codimension of $S_m(k,\lambda)$ so (7.4) describes the full tangent space. A similar procedure works for the tangent space to $S_m(k)$.

Part (B) of the proposition is an immediate consequence.

For part (C), using the decomposition $\mathbb{C}^n = V_k \oplus V_k^\perp$, the matrix of $T = h - \lambda.I$ is

$$h - \lambda.I = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \tag{7.5}$$

where A is non-singular. Given $U \in M_{n\times n}(\mathbb{C})$, we have

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \implies TU + U^*T = \begin{pmatrix} 0 & U_3^*A \\ AU_3 & AU_4 + U_4^*A \end{pmatrix}.$$

Hence, $TU + U^*T \in T_hR_m(\lambda)$ as it has the form (7.4). Conversely, since A is invertible, AU_3 and AU_4 account for all matrices in $M_{(n-k)\times k}(\mathbb{C})$ and $M_{k\times k}(\mathbb{C})$ as U varies within $M_{n\times n}(\mathbb{C})$. This proves the case $\xi = \left(\begin{smallmatrix} 0 & B \\ B^* & D \end{smallmatrix}\right)$, and the case of $\xi = \left(\begin{smallmatrix} c.I_V & B \\ B^* & D \end{smallmatrix}\right)$ follows. ■

A closely related result, concerning transversality with respect to the manifold of matrices with constant rank, appears in [27, Chapter 10.5]. Although Lovász considers only real symmetric matrices, his proof applies also to Hermitian matrices.

7.5. Application to graphs

Fix $k, m, n \geq 1$ and let $\lambda \in \mathbb{R}$. Recall $S_m(k), S_m(k,\lambda) \subset \mathcal{H}_n$ from (7.1), (7.2). Let G be a graph on n vertices with a set of edges E and associated spaces $\mathcal{S}(G) \subset \mathcal{H}(G) \subset \mathcal{H}_n$. In this section we determine when the inclusion $\mathcal{H}(G) \to \mathcal{H}_n$ is transverse to the manifold $S_m(k)$ at a point h of their intersection. The results are used in Corollary 7.10, the case of multiplicity 2, to provide sufficient conditions which guarantee that the mapping $\mathbb{T}_h \to \mathcal{H}_n$ is transverse to $S_2(k)$ locally near h.

Let $\mathcal{H}(\overline{G})$ denote those Hermitian matrices that are supported on the complement of E. That is $h \in \mathcal{H}(\overline{G})$ if $h^* = h$, $h_{rr} = 0$, and $h_{rs} = 0$ for all r and for any edge rs. It is the orthogonal complement[9] in $\mathcal{H}_n$ to $\mathcal{H}(G)$.

7.6 Proposition. *Fix $h \in \mathcal{H}_n$. Suppose $\lambda_{k-1}(h) < \lambda_k(h)$ and the eigenvalue $\lambda := \lambda_k(h)$ has multiplicity m and eigenspace $V = V_k$. The following statements are equivalent.*

(1) *The inclusion $\mathcal{H}(G) \to \mathcal{H}_n$ is transverse to $S_m(k)$ at h.*

(2) *The inclusion $\mathcal{H}(G) \to \mathcal{H}_n$ is transverse to $S_m(k,\lambda)$ at h.*

(3) *$(h - \lambda.I)X \neq 0$ for every non-zero $X \in \mathcal{H}(\overline{G})$.*

(4) *There exist $\xi_1, \xi_2, \dots, \xi_N \in \mathcal{H}(G)$ whose restrictions $\{\xi_1|V, \xi_2|V, \dots, \xi_N|V\}$ span (over $\mathbb{R}$) the space $\mathcal{H}(V)$ of Hermitian operators on V.*

Proof. Let V denote the m-dimensional eigenspace of h with eigenvalue λ. Parts (1) and (2) are equivalent by Proposition 7.4 because the tangent space $T_h\mathcal{H}(G) = \mathcal{H}(G)$ contains the identity matrix $\xi = I$, and $\xi|V = I_V$.

For part (3), as in [27, Section 10.5.2], the manifolds $\mathcal{H}(G)$ and $S_m(k,\lambda)$ are transverse at h if and only if the orthogonal complements of their tangent spaces intersect trivially. The orthogonal complement to $T_h S_m(k,\lambda)$ is

$$T_h S_m(k,\lambda)^{\perp} = \{X \in \mathcal{H}_n : (h - \lambda.I)X = 0\}$$

by equations (7.5) and (7.4). The orthogonal complement of $T_h\mathcal{H}(G) = \mathcal{H}(G)$ is $\mathcal{H}(\overline{G})$, so transversality to $S_m(k,\lambda)$ fails if and only if there exists $0 \neq X \in \mathcal{H}(\overline{G})$ such that $(h - \lambda.I)X = 0$.

Part (4) is a restatement of part (B) of Proposition 7.4. ∎

7.7. Example – Graph splitting

The following example provides some intuition for the definitions in Section 7.8. Given graphs H_1, H_2 of size n_1, n_2 respectively. Suppose that both $h_1 \in \mathcal{H}(H_1)$

[9] $\mathcal{H}_n$ is equipped with the standard inner product $\langle A, B\rangle := \operatorname{trace}(A^* B) = \operatorname{trace}(AB)$.

and $h_2 \in \mathcal{H}(H_2)$ have the same simple eigenvalue λ. Let ϕ_1, ϕ_2 be corresponding eigenfunctions. Suppose there is a vertex v_1 of H_1 with $\phi_1(v_1) = 0$ and a vertex v_2 of H_2 with $\phi_2(v_2) = 0$. Let G be the graph of size $n = n_1 + n_2 - 1$ obtained from joining H_1, H_2 by identifying the vertices v_1 and v_2. Define $h \in \mathcal{H}(G)$ such that its restrictions to H_1, H_2 agree with h_1, h_2. Then λ is an eigenvalue of h with 2-dimensional eigenspace V and eigenfunctions $\Phi_1 = \phi_1 \times \{0\}$ and $\Phi_2 = \{0\} \times \phi_2$ such that $\langle \Phi_1, \Phi_2 \rangle = 0$. For any $\xi \in \mathcal{H}(G)$, we have $\langle \Phi_1, \xi \Phi_2 \rangle = 0$ so the elements $\xi|V$ fail to account for all quadratic forms on V, that is, $\mathcal{H}(G)$ is not transverse to $S_2(k)$ at the point h. In the graph G, both eigenfunctions vanish at the vertex $v_1 = v_2$ so the graph G is "split" into two pieces by this eigenvalue λ.

7.8. Graph theoretic conditions

Maintain the notation of Section 7.1 and Section 7.5. If $u \in \mathbb{R}^n$ the *support* of u is the set $\mathrm{spt}(u)$ of vertices j such that $u_j \neq 0$. If $V \subset \mathbb{R}^n$ is a subspace, its support is

$$\mathrm{spt}(V) = \bigcup_{u \in V} \mathrm{spt}(u).$$

If λ is an eigenvalue of $h \in \mathcal{H}(G)$ with eigenspace $V = \ker(h - \lambda I)$, we say that *the eigenspace V splits G* if the induced subgraph $G|\mathrm{spt}(V)$ of G on the vertices in $\mathrm{spt}(V)$ is *not* connected. Given an edge (rs), we say that V *projects surjectively onto* (rs) if $\{(u_r, u_s) \colon u \in V\} = \mathbb{C}^2$.

7.9 Theorem. *Suppose $h \in \mathcal{H}(G)$ has eigenvalue $\lambda = \lambda_k(h)$ and eigenspace V.*

(A) *Suppose the multiplicity of λ_k is 2 and either*

- (a) *there exists an edge (rs) on which V projects surjectively, or*
- (b) *the eigenspace V does not split G.*

Then $\mathcal{H}(G)$ is transverse to $S_2(k)$ at the point h.

(B) *For arbitrary multiplicity m, suppose there exist nonzero vectors $u, v \in V$ whose supports are edge-separated, meaning that* $\mathrm{spt}(u) \cap \mathrm{spt}(v) = \emptyset$ *and there are no edges between* $\mathrm{spt}(u)$ *and* $\mathrm{spt}(v)$. *Then $\mathcal{H}(G)$ is not transverse to $S_m(k)$ at h.*

Proof. Assume there exists an edge (rs) on which V projects surjectively. In this case, we may choose u and v in V such that $(u_r, u_s) = (1, 0)$ and $(v_r, v_s) = (0, 1)$, so that in the (not necessarily orthonormal) basis $\{u, v\}$ of V, for any $\xi \in \mathcal{H}(G)$ with $\xi_{ij} = 0$ for all $ij \notin \{rr, rs, sr, ss\}$,

$$\xi|_V := \begin{pmatrix} \langle u, \xi u \rangle & \langle u, \xi v \rangle \\ \langle v, \xi u \rangle & \langle v, \xi v \rangle \end{pmatrix} = \begin{pmatrix} \xi_{rr} & \xi_{rs} \\ \xi_{sr} & \xi_{ss} \end{pmatrix}.$$

So, these vectors span $\mathcal{H}(V)$ verifying part (4) of Proposition 7.4.

For condition (b), we will show that if V does not split G, then there must be an edge (rs) on which V projects surjectively. Let H be the induced subgraph on $\mathrm{spt}(V)$ and assume it is connected. Choose a generic basis $\{u, v\}$ of V, so that $u(r) \neq 0$ and $v(r) \neq 0$ for all $r \in H$. Assume by contradiction that V does not project surjectively on any edge (rs). That is, $\frac{u(s)}{u(r)} = \frac{v(s)}{v(r)}$ for every edge (rs) in H. Fix an initial vertex $r_0 \in H$. Any other vertex $s \in H$ is connected to r_0 by a path in H, say $(r_0, r_1, r_2, \dots, r_m, s)$, and so

$$\frac{v(s)}{v(r_0)} = \frac{v(r_1)}{v(r_0)} \cdot \frac{v(r_2)}{v(r_1)} \cdots \frac{v(s)}{v(r_m)} = \frac{u(r_1)}{u(r_0)} \cdot \frac{u(r_2)}{u(r_1)} \cdots \frac{u(s)}{u(r_m)} = \frac{u(s)}{u(r_0)}.$$

Thus, u and v are linearly dependent.

For part (B), given $u, v \in V$ as described in part (B), the matrix $X = uv^* + vu^*$ is in $\mathcal{H}_n(\overline{G})$ and satisfies $(h - \lambda.I)X = 0$, which contradicts Proposition 7.6 (3). ∎

7.10 Corollary. *Let $h \in \mathcal{S}(G)$ be properly supported, and let $\alpha \in \mathcal{A}(G)$. Let $\mathbb{T}_h = \mathbb{T}(G) * h \subset \mathcal{H}(G)$ be the embedded torus. Suppose the eigenvalue λ_k of $h_\alpha = \alpha * h$ has multiplicity 2 with eigenspace that does not split G. Then there is a neighborhood $V \subset \mathcal{A}(G)$ of α and a neighborhood $U \subset \mathcal{S}(G)$ of h such that for a generic*[10] *set of $h' \in U$, the embedding map $\mathbb{T}_{h'} \hookrightarrow \mathcal{H}_n$ takes the open subset*

$$V * h' = \{\alpha' * h' : \alpha' \in V\} \subset \mathbb{T}_{h'}$$

transversally to the stratum $S_2(k)$.

Proof. Consider the composition

$$\Phi: \mathbb{T}(G) \times \mathcal{S}(G) \xrightarrow{*} \mathcal{H}(G) \overset{j}{\hookrightarrow} \mathcal{H}_n$$

given by $\Phi(\alpha', h') = \Phi_{h'}(\alpha') = \alpha' * h'$. The map $*$ above is surjective, since h is properly supported, with finite fibers, it is an open mapping and a submersion. The non-splitting assumption implies the embedding map j takes $\mathcal{H}(G)$ transversally to $S_2(k)$ at the point h_α, so it takes a neighborhood $W \subset \mathcal{H}(G)$ of h_α transversally to $S_2(k)$. Choose the neighborhoods $V \subset \mathcal{A}(G)$ and $U \subset \mathcal{S}(G)$ so that $V * U \subset W$. Then $\Phi: V \times U \to \mathcal{H}_n$ is transverse to $S_2(k)$. Lemma A.1 implies there exists a dense set of values $h' \in U$ so that the resulting map $\Phi_{h'}: V \to \mathcal{H}_n$ is transverse to $S_2(k)$. But this map is the composition

$$V \xrightarrow{\cong} V * h' \xrightarrow{j} \mathcal{H}_n.$$

∎

[10] An open, dense and full measure set.

7.11. Example – Graphs for which $\mathbb{T}_h$ is generically transverse to S_2

Suppose G is a graph obtained by removing a set of disjoint edges from the complete graph, say (r_j, s_j) for $j = 1, \ldots, m$ such that the vertices $\{r_1, s_1, r_2, s_2, \ldots\}$ are all distinct. If $h \in \mathcal{H}(G)$ has distinct diagonal elements (a generic assumption), then the embedded torus $\mathbb{T}_h$ intersects $S_2(k)$ transversally for every k. To prove this, it suffices by Corollary 7.10 to show, for any $h_\alpha \in \mathbb{T}_h$, that no multiplicity-two eigenvalue of h_α splits G.

Assume by contradiction that some $h_\alpha \in \mathbb{T}_h$ has a multiplicity two eigenvalue $\lambda = \lambda_k(h_\alpha)$ with eigenspace V such that the induced graph $G|\text{spt}(V)$ is disconnected. By the construction of G, this means that $\text{spt}(V) = \{r_j, s_j\}$ for one of the missing edges (r_j, s_j). So, λ is a multiplicity-two eigenvalue of the restriction

$$h_\alpha|\text{spt}(V) = \begin{pmatrix} h_{r_j,r_j} & 0 \\ 0 & h_{s_j,s_j} \end{pmatrix}.$$

This contradicts the assumption that diagonal elements of h are distinct.

A. Transversality

A.1 Transversality Lemma. *Let $\Phi\colon \mathbb{T} \times B \to \mathcal{H}$ be a smooth map between smooth manifolds and suppose this map is transverse to a submanifold $S \subset \mathcal{H}$. Then there is a dense set of values $b \in B$ such that the partial map*

$$\phi_b\colon \mathbb{T} \to \mathcal{H} \quad \text{given by } \phi_b(x) = \Phi(x, b)$$

is transverse to S. If Φ is proper and $S \subset \mathcal{H}$ is closed, then this set of values is open in B. If $\Phi, \mathbb{T}, B, \mathcal{H}$ and S are analytic then the set of values $b \in B$ for which transversality of ϕ_b fails is a subanalytic subset of B of positive codimension.

Remarks. Here, $\mathbb{T}$ is any finite-dimensional smooth manifold. The symbol $\mathbb{T}$ is being used to indicate that for our application, $\mathbb{T}$ is an open subset of the torus $\mathbb{T}(G)$.

This result says, for example, that two submanifolds of Euclidean space may be made transverse by an arbitrarily small *translation*. The transversality lemma is due originally to R. Thom ([31]). The proof described here may be found in ([18]).

Proof. It suffices to consider the case when B is open in some Euclidean space. By assumption, the set $P = \Phi^{-1}(S)$ is a smooth submanifold of $\mathbb{T} \times B$ and it is easy to check that $b \in B$ is a regular value of the projection $\pi\colon P \to B$ if and only if the partial map $\phi_b\colon \mathbb{T} \to \mathcal{H}$ is transverse to S. But Sard's theorem says that the set of non-regular values of π has Lebesgue measure zero.

Now, assume S is closed and Φ is proper (i.e., the preimage of a compact set is compact). To show the set of "transversal" elements $b \in B$ is open, we show its complement is closed. Let $b_i \in B$ be a convergent sequence of points, say $b_i \to b \in B$ for which there exists points $t_i \in \mathbb{T}$ such that ϕ_{b_i} fails to take the tangent space $T_{t_i}\mathbb{T}$ transversally to $T_{s_i}S$ where $s_i = \Phi(t_i, b_i)$. Since Φ is proper, by taking a subsequence if necessary we may assume the sequence converge, say $t_i \to t \in \mathbb{T}$ and therefore $s_i \to s$ for some $s \in \mathcal{H}$. Since S is closed, we also have $s \in S$. The failure of transversality is a closed condition so ϕ_b fails to take $T_t\mathbb{T}$ transversally to T_sS.

Finally, if $\Phi, \mathbb{T}, B, \mathcal{H}, S$ are analytic then the set of points $(t,b) \in \mathbb{T} \times B$ for which ϕ_b fails to be transverse at t is again analytic so its image $Z \subset B$ is a subanalytic subset of B. It has positive codimension, for if Z contains an open set in B, then this contradicts the assumption that Φ is transverse to S. ∎

B. Heuristics for discretization of magnetic Schrödinger operators

The definition of discrete magnetic operators can be found in [14, 26] for example, however, we will give here a heuristic explanation for why this is the right discretization for magnetic Schrödinger operators. For simplicity, we consider domains in $\mathbb{R}^3$, so that magnetism can be described using vector fields: a magnetic field B and magnetic potential A such that $B = \nabla \times A$. (The modern approach would consider A and B as a 1-form and 2-forms).

The quadratic form of a Schrödinger operator $H = \Delta + V$ on a domain $\Omega \subset \mathbb{R}^n$ is

$$\langle f, Hf\rangle = \int\limits_{\Omega} \sum_{j=1}^{n} \Big(\frac{\partial f(x)}{\partial x_j}\Big)^2 + V(x) f(x)^2 \, dx,$$

for the relevant class of functions f on Ω. If we approximate the quotient $\frac{\partial f(x)}{\partial x_j}$ with $\frac{f(x+\varepsilon e_j)-f(x)}{\varepsilon}$, the quadratic form can be written as in (1.1)

$$\sum_{x,y\in\Lambda_\varepsilon} h_{xy}(f(y)-f(x))^2 + V(x)f(x)^2 dx, \qquad h_{xy} = \begin{cases} \frac{1}{\varepsilon^2} & \text{if } x \sim y, \\ 0 & \text{otherwise}, \end{cases}$$

where $\Lambda_\varepsilon \subset \Omega$ is a grid of side length ε. Introducing a magnetic field B, the operator H is changed to a magnetic Schrödinger operator H_A by the rule $\frac{\partial f(x)}{\partial x_j} \mapsto \frac{\partial f(x)}{\partial x_j} + iA_j(x)f(x)$, where $A = (A_1, A_2, A_3)$ is the *magnetic potential* defined (uniquely up to gauge transformations $A \sim A' = A + \nabla g$) by the relation $\nabla \times A = B$. Notice that

$$\frac{\partial f(x)}{\partial x_j} + iA_j(x)f(x) = \lim_{t\to 0} \frac{(e^{i\int_x^{x+te_j} A(s)ds} f(x+te_j)) - f(x)}{t},$$

and the ε discretization of the quadratic form can be written as in (2.3)

$$\int_\Omega \sum_{j=1}^{3} \Big| \frac{\partial f(x)}{\partial x_j} + iA_j(x) f(x) \Big|^2 + V(x)|f(x)|^2 \, dx$$

$$\approx \sum_{x,y\in\Lambda_\varepsilon} h_{xy} \Big| f(y) - e^{i\alpha_{xy}} f(x) \Big|^2 + V(x)|f(x)|^2 \, dx,$$

using $\alpha_{xy} = \int_x^{x+\varepsilon e_j} A(s)ds$ when $y = x + \varepsilon e_j$ and extending it antisymmetrically.

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Lior Alon
Department of Mathematics, Massachusetts Institute of Technology, Simons Building (Building 2), 77 Massachusetts Avenue, Cambridge, MA 02139, USA; lioralon@mit.edu

Mark Goresky
School of Mathematics, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, USA; goresky@ias.edu