J. Spectr. Theory 13 (2023), 1261–1280
DOI 10.4171/JST/466

© 2023 European Mathematical Society
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Spectral analysis of an open q-difference Toda chain with two-sided boundary interactions on the finite integer lattice

Jan Felipe van Diejen

Abstract. A quantum n-particle model consisting of an open q-difference Toda chain with two-sided boundary interactions is placed on a finite integer lattice. The spectrum and eigenbasis are computed by establishing the equivalence with a previously studied q-boson model from which the quantum integrability is inherited. Specifically, the q-boson-Toda correspondence in question yields Bethe Ansatz eigenfunctions in terms of hyperoctahedral Hall–Littlewood polynomials and provides the pertinent solutions of the Bethe Ansatz equations via the global minima of corresponding Yang–Yang-type Morse functions.

1. Introduction

The relativistic Toda chain is an ubiquitous one-dimensional n-particle model introduced by Ruijsenaars that is integrable both at the level of classical and quantum mechanics [19]. In the case of an open chain, integrable perturbations at the boundary were implemented via the boundary Yang–Baxter equation [16, 22]. At the quantum level, the Hamiltonian of the relativistic Toda chain is given by a (q-)difference operator. Quantum groups connect the difference operator at issue to the quantum K-theory of flag manifolds [2, 11] and provide a natural representation-theoretical habitat for the construction of its eigenfunctions [7, 20].

When considering the quantum dynamics on an integer lattice the eigenvalue problem for the q-difference Toda chain can be solved in terms of q-Whittaker functions that arise as a parameter specialization of the Macdonald polynomials, both in the case of particles moving on an infinite lattice [10] and in the case of particles moving on a finite periodic lattice [6]. From the perspective of integrable probability, such particle models are of interest in connection with the q-Whittaker process [1].

Mathematics Subject Classification 2020: 33D52 (primary); 05E05, 81Q35, 81Q80, 81U15, 82B23 (secondary).
Keywords: Difference Toda chain, Bethe Ansatz, Hall–Littlewood polynomials.

Representation-theoretical constructions for the pertinent q-Whittaker functions can be found in [3, 4, 8].

This note addresses the spectral problem for an open n-particle q-difference Toda chain on the finite lattice $\{0, 1, 2, \ldots, m\}$ that is endowed with two-parameter boundary interactions on both ends. The model could be thought of as a finite discrete and q-deformed counterpart of Sklyanin's open quantum Toda chain with general two-sided boundary perturbations governed by Morse potentials [21]. In the limit $m \to \infty$, the pertinent q-difference Toda Hamiltonian was diagonalized in terms of hyperoctahedral q-Whittaker functions that arise in turn through a parameter specialization of the Macdonald–Koornwinder polynomials [27]. Here it will be shown that for finite m an explicit eigenbasis can be constructed from Bethe Ansatz wave functions given by Macdonald's hyperoctahedral Hall–Littlewood polynomials [17]. The main idea is to exploit an equivalence between q-difference Toda chains and q-boson models pointed out in [6]. By establishing a version of this equivalence in the current situation of an open chain with boundary perturbations, our q-difference Toda Hamiltonian is mapped to the Hamiltonian of a q-boson model previously diagonalized in [32]. The upshot is that the commuting quantum integrals and the Bethe Ansatz eigenfunctions for the q-difference Toda chain can in this approach be retrieved directly from those in [32] for the corresponding q-boson model.

The material is organized as follows. Section 2 describes the Hamiltonian of our q-difference Toda chain and verifies its self-adjointness. Section 3 establishes the equivalence with the q-boson model from [32] and therewith retrieves the corresponding Bethe Ansatz wave functions in terms of hyperoctahedral Hall–Littlewood polynomials. The Bethe Ansatz equations of interest are of a convex type studied in wider generality in [31], which entails an explicit description of the spectrum via the global minima of associated Yang–Yang-type Morse functions detailed in Section 4. The presentation closes in Section 5 with a description of the spectral analysis for the q-difference Toda chain in the degenerate limit $q \to 1$.

2. Open q-difference Toda chain with boundary interactions

2.1. Quantum Hamiltonian

Given $m, n \in \mathbb{N}$, the q-difference Toda chain under consideration describes the quantum dynamics of n interacting particles hopping over the finite integer lattice

$$\{0, 1, 2, \ldots, m\}.$$

The positions of these particles are encoded by a partition $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ in the configuration space

$$\Lambda^{(n,m)} = \{\mu \in \mathbb{Z}^n \mid m \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0\}.$$

The dynamics is governed in turn by the following quantum Hamiltonian

$$\begin{aligned} H = {} & \beta_+(1-q^{m-\mu_1}) + \beta_-(1-q^{\mu_n}) \\ & + \sum_{1\leq i\leq n}(1-\alpha_+ q^{m-\mu_1-1})^{\delta_{i-1}}(1-q^{\mu_{i-1}-\mu_i})T_i \\ & + \sum_{1\leq i\leq n}(1-\alpha_- q^{\mu_n-1})^{\delta_{n-i}}(1-q^{\mu_i-\mu_{i+1}})T_i^{-1}, \end{aligned} \tag{2.1}$$

with

$$\delta_i = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases} \qquad \begin{cases} \mu_0 \equiv m, \\ \mu_{n+1} \equiv 0. \end{cases}$$

Here T_i and T_i^{-1} denote hopping operators that act on n-particle wave functions ψ via a unit translation of the ith particle to the left and to the right, respectively:

$$(T_i^{\epsilon}\psi)(\mu_1, \dots, \mu_n) = \psi(\mu_1, \dots, \mu_{i-1}, \mu_i + \epsilon, \mu_{i+1}, \dots, \mu_n) \quad (\epsilon \in \{1, -1\}).$$

The action of H (2.1) on wave functions $\psi\colon \Lambda^{(n,m)} \to \mathbb{C}$ is well defined in the sense that the coefficient of $(T_i^{\epsilon}\psi)(\mu_1, \dots, \mu_n)$ in $(H\psi)(\mu_1, \dots, \mu_n)$ vanishes for any $(\mu_1, \dots, \mu_n) \in \Lambda^{(n,m)}$ such that $(\mu_1, \dots, \mu_{i-1}, \mu_i + \epsilon, \mu_{i+1}, \dots, \mu_n) \notin \Lambda^{(n,m)}$. Notice also that the convention in the second brace below eq. (2.1) can be interpreted as representing the positions of two additional particles fixed at the lattice end-points 0 and m, respectively. The parameter $q \in (-1, 1) \setminus \{0\}$ denotes a scale parameter of the model governing the nearest neighbor interaction between the particles whereas the parameters $\alpha_\pm \in (-1, 1)$ and $\beta_\pm \in \mathbb{R}$ represent coupling constants regulating additional interactions at the boundary of the chain.

Our main goal is to solve the spectral problem for the q-difference Toda Hamiltonian H (2.1) in the $\binom{n+m}{n}$-dimensional Hilbert space $\ell^2(\Lambda^{(n,m)}, \Delta)$ of functions $\psi\colon \Lambda^{(n,m)} \to \mathbb{C}$, endowed with an inner product

$$\langle \psi, \phi \rangle_\Delta = \sum_{\mu \in \Lambda^{(n,m)}} \psi(\mu)\overline{\phi(\mu)}\Delta_\mu \qquad (\psi, \phi \in \ell^2(\Lambda^{(n,m)}, \Delta))$$

determined by positive weights given by perturbed q-multinomials on $\Lambda^{(n,m)}$:

$$\Delta_\mu = \frac{(q;q)_m}{(\alpha_+;q)_{m-\mu_1}(\alpha_-;q)_{\mu_n}\prod_{0\leq i\leq n}(q;q)_{\mu_i-\mu_{i+1}}} \qquad (\mu \in \Lambda^{(n,m)}).$$

Here we have employed the standard q-shifted factorial

$$(a;q)_l = \begin{cases} 1 & \text{if } l = 0, \\ (1-a)(1-aq)\cdots(1-aq^{l-1}) & \text{if } l = 1,2,3,\ldots. \end{cases}$$

Proposition 2.1 (Self-adjointness). *For $\alpha_\pm \in (-1,1)$, $\beta_\pm \in \mathbb{R}$, and $q \in (-1,1)\setminus\{0\}$, the q-difference Toda Hamiltonian H (2.1) is self-adjoint in $\ell^2(\Lambda^{(n,m)},\Delta)$, i.e.*

$$\langle H\psi,\phi\rangle_\Delta = \langle \psi, H\phi\rangle_\Delta \quad \textit{for all } \psi,\phi \in \ell^2(\Lambda^{(n,m)},\Delta).$$

Remark 2.2. For $m\to\infty$, the q-difference Toda Hamiltonian H (2.1) was diagonalized in [27, Section 7] in terms of a unitary eigenfunction transform with a q-Whittaker kernel built from a parameter specialization of the Macdonald–Koornwinder polynomials.

2.2. Proof of Proposition 2.1

The action of H (2.1) on $\psi \in \ell^2\big(\Lambda^{(n,m)},\Delta\big)$ is of the form

$$\begin{aligned}(H\psi)(\mu) = {} & (\beta_+(1-q^{m-\mu_1}) + \beta_-(1-q^{\mu_n}))\psi(\mu) \\ & + \sum_{\substack{1\le i\le n \\ \mu+e_i\in\Lambda^{(n,m)}}} (1-\alpha_+ q^{m-\mu_1-1})^{\delta_{i-1}}(1-q^{\mu_{i-1}-\mu_i})\psi(\mu+e_i) \\ & + \sum_{\substack{1\le i\le n \\ \mu-e_i\in\Lambda^{(n,m)}}} (1-\alpha_- q^{\mu_n-1})^{\delta_{n-i}}(1-q^{\mu_i-\mu_{i+1}})\psi(\mu-e_i),\end{aligned}$$

where the vectors $e_1,\ldots,e_n$ represent the standard unit basis for $\mathbb{Z}^n$.

Since all coefficients of the difference operator in question are real, the asserted symmetry $\langle H\psi,\phi\rangle_\Delta = \langle\psi,H\phi\rangle_\Delta$ is immediate from the following bilinear identity for all $\psi,\phi\in\ell^2(\Lambda^{(n,m)},\Delta)$:

$$\begin{aligned} & \sum_{\mu\in\Lambda^{(n,m)}} \Big(\sum_{\substack{1\le i\le n \\ \mu+e_i\in\Lambda^{(n,m)}}} (1-\alpha_+ q^{m-\mu_1-1})^{\delta_{i-1}}(1-q^{\mu_{i-1}-\mu_i})\psi(\mu+e_i)\Big)\phi(\mu)\Delta_\mu \\ & \overset{\text{(i)}}{=} \sum_{\tilde\mu\in\Lambda^{(n,m)}} \sum_{\substack{1\le i\le n \\ \tilde\mu-e_i\in\Lambda^{(n,m)}}} (1-\alpha_+ q^{m-\tilde\mu_1})^{\delta_{i-1}}(1-q^{\tilde\mu_{i-1}-\tilde\mu_i+1})\psi(\tilde\mu)\phi(\tilde\mu-e_i)\Delta_{\tilde\mu-e_i} \\ & \overset{\text{(ii)}}{=} \sum_{\tilde\mu\in\Lambda^{(n,m)}} \psi(\tilde\mu)\Big(\sum_{\substack{1\le i\le n \\ \tilde\mu-e_i\in\Lambda^{(n,m)}}} (1-\alpha_- q^{\tilde\mu_n-1})^{\delta_{n-i}}(1-q^{\tilde\mu_i-\tilde\mu_{i+1}})\phi(\tilde\mu-e_i)\Big)\Delta_{\tilde\mu}.\end{aligned}$$

Step (i) hinges on the substitution $\mu = \tilde{\mu} - e_i$, which for a given $i \in \{1, \ldots, n\}$ determines a bijection from the subset $\{\tilde{\mu} \in \Lambda^{(n,m)} \mid \tilde{\mu} - e_i \in \Lambda^{(n,m)}\}$ onto the subset $\{\mu \in \Lambda^{(n,m)} \mid \mu + e_i \in \Lambda^{(n,m)}\}$. Step (ii) uses the elementary recurrence

$$(1-\alpha_+ q^{m-\tilde{\mu}_1})^{\delta_{i-1}}(1-q^{\tilde{\mu}_{i-1}-\tilde{\mu}_i+1})\Delta_{\tilde{\mu}-e_i} = (1-\alpha_- q^{\tilde{\mu}_n-1})^{\delta_{n-i}}(1-q^{\tilde{\mu}_i-\tilde{\mu}_{i+1}})\Delta_{\tilde{\mu}}$$

for $\tilde{\mu} \in \Lambda^{(n,m)}$ such that $\tilde{\mu} - e_i \in \Lambda^{(n,m)}$ (and the convention $\tilde{\mu}_0 \equiv m$, $\tilde{\mu}_{n+1} \equiv 0$).

3. Eigenfunctions

3.1. Bethe Ansatz

While Proposition 2.1 implies that the existence of an orthogonal eigenbasis diagonalizing H (2.1) in $\ell^2(\Lambda^{(n,m)}, \Delta)$ is evident from the spectral theorem for self-adjoint operators in finite dimension, the aim here is to provide an *explicit* eigenbasis given by Bethe Ansatz wave functions in the spirit of [25] for $n = 1$.

To this end let us recall that for any $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda^{(m,n)}$ and $\xi = (\xi_1, \ldots, \xi_m)$ belonging to

$$\mathbb{R}^m_{\text{reg}} = \{\xi \in \mathbb{R}^m \mid 2\xi_j, \xi_j - \xi_k, \xi_j + \xi_k \notin 2\pi\mathbb{Z}, \text{ for all } 1 \leq j \neq k \leq m\}, \tag{3.1}$$

Macdonald's hyperoctahedral Hall–Littlewood polynomial (associated with the root system BC_m) is given by [17, §10]

$$R_\lambda(\xi_1, \ldots, \xi_m) = \sum_{\substack{\sigma \in S_m \\ \epsilon \in \{1,-1\}^m}} C(\epsilon_1 \xi_{\sigma(1)}, \ldots, \epsilon_m \xi_{\sigma(m)}) \exp(\mathrm{i}\epsilon_1 \xi_{\sigma(1)} \lambda_1 + \cdots + \mathrm{i}\epsilon_n \xi_{\sigma(m)} \lambda_m)$$

with

$$C(\xi_1, \ldots, \xi_m) = \prod_{1 \leq j \leq m} \frac{(1 - \beta_+ e^{-\mathrm{i}\xi_j} + \alpha_+ e^{-2\mathrm{i}\xi_j})}{1 - e^{-2\mathrm{i}\xi_j}} \times \prod_{1 \leq j < k \leq m} \Big(\frac{1 - q e^{-\mathrm{i}(\xi_j - \xi_k)}}{1 - e^{-\mathrm{i}(\xi_j - \xi_k)}}\Big)\Big(\frac{1 - q e^{-\mathrm{i}(\xi_j + \xi_k)}}{1 - e^{-\mathrm{i}(\xi_j + \xi_k)}}\Big),$$

where the summation is over all permutations $\sigma = \left(\begin{smallmatrix} 1 & 2 & \cdots & m \\ \sigma(1) & \sigma(2) & \cdots & \sigma(m) \end{smallmatrix}\right)$ of the symmetric group S_m and all sign configurations $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{1, -1\}^m$.

For any $\lambda \in \Lambda^{(m,n)}$ and $0 \leq i \leq n$ we denote the multiplicity of i in λ by

$$\mathrm{m}_i(\lambda) = |\{1 \leq j \leq m \mid \lambda_j = i\}|.$$

Additionally, for $\mu \in \Lambda^{(n,m)}$ we write

$$\mu' = (0^{m-\mu_1} 1^{\mu_1 - \mu_2} 2^{\mu_2 - \mu_3} \cdots (n-1)^{\mu_{n-1} - \mu_n} n^{\mu_n}) \tag{3.2}$$

for its conjugate partition $\mu' \in \Lambda^{(m,n)}$ (i.e., 'with the columns and rows swapped'). In other words, $\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_m)$ is the (unique) partition in $\Lambda^{(m,n)}$ such that $\mathrm{m}_i(\mu') = \mu_i - \mu_{i+1}$ for $i = 0, \ldots, n$ (where, recall, $\mu_0 \equiv m$ and $\mu_{n+1} \equiv 0$). Notice in this connection that $|\Lambda^{(n,m)}| = |\Lambda^{(m,n)}| = \frac{(n+m)!}{n!m!}$ and that the mapping $\mu \to \mu'$ (3.2) defines a bijection from $\Lambda^{(n,m)}$ onto $\Lambda^{(m,n)}$.

From now on, it will moreover be assumed (unless explicitly stated otherwise) that the boundary parameters $\alpha_\pm$, $\beta_\pm$ have values such that the roots $p_\pm$, $q_\pm$ of the two quadratic polynomials $x^2 - \beta_\pm x + \alpha_\pm$ belong to the interval $(-1, 1) \setminus \{0\}$:

$$\boxed{\alpha_\pm = p_\pm q_\pm \quad \text{and} \quad \beta_\pm = p_\pm + q_\pm \quad \text{with } q_\pm, p_\pm \in (-1, 1) \setminus \{0\}} \tag{3.3}$$

or equivalently

$$\boxed{0 < \alpha_\pm^2 < 1 \quad \text{and} \quad 4\alpha_\pm \leq \beta_\pm^2 \leq (1 + \alpha_\pm)^2.} \tag{3.4}$$

Theorem 3.1 (Bethe Ansatz wave function). *Let $q \in (-1, 1) \setminus \{0\}$ and let the boundary parameters $\alpha_\pm$, $\beta_\pm$ belong to the domain specified in (3.3)–(3.4). Given $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_{\text{reg}}$ (3.1), we define the wave function $\psi_\xi \in \ell^2(\Lambda^{(n,m)}, \Delta)$ through its values on $\Lambda^{(n,m)}$ as follows:*

$$\psi_\xi(\mu) = R_{\mu'}(\xi_1, \ldots, \xi_m) \quad (\mu \in \Lambda^{(n,m)}). \tag{3.5}$$

The wave function ψ_ξ (3.5) solves the eigenvalue equation for the q-difference Toda Hamiltonian H (2.1)

$$H\psi_\xi = E(\xi)\psi_\xi \quad \textit{with} \quad E(\xi) = 2(1 - q) \sum_{1 \leq j \leq m} \cos(\xi_j), \tag{3.6}$$

provided the spectral parameter $\xi \in \mathbb{R}^m_{\text{reg}}$ satisfies the algebraic system of Bethe Ansatz equations

$$\begin{aligned} e^{2in\xi_j} &= \frac{(1 - \beta_+ e^{i\xi_j} + \alpha_+ e^{2i\xi_j})}{(e^{2i\xi_j} - \beta_+ e^{i\xi_j} + \alpha_+)} \frac{(1 - \beta_- e^{i\xi_j} + \alpha_- e^{2i\xi_j})}{(e^{2i\xi_j} - \beta_- e^{i\xi_j} + \alpha_-)} \\ &\quad \times \prod_{\substack{1 \leq k \leq m \\ k \neq j}} \frac{(1 - q e^{i(\xi_j - \xi_k)})(1 - q e^{i(\xi_j + \xi_k)})}{(e^{i(\xi_j - \xi_k)} - q)(e^{i(\xi_j + \xi_k)} - q)} \quad \textit{for } j = 1, \ldots, m. \end{aligned} \tag{3.7}$$

Remark 3.2. The hyperoctahedral Hall–Littlewood polynomial $R_\lambda(\xi_1, \ldots, \xi_m)$, $\lambda \in \Lambda^{(m,n)}$ is in fact a symmetric polynomial in $\cos(\xi_1), \ldots, \cos(\xi_m)$ of total degree $\lambda_1 + \lambda_2 + \cdots + \lambda_n$. Hence, it is clear that the Bethe Ansatz wave function ψ_ξ in Theorem 3.1 extends smoothly in the spectral parameter ξ from values in $\mathbb{R}^m_{\text{reg}}$ to values in $\mathbb{R}^m$.

3.2. Proof of Theorem 3.1

In [32], Propositions 8.4, 9.2, and (the proof of) Theorem 9.5 imply that the hyperoctahedral Hall–Littlewood polynomials satisfy the following recurrence for $\lambda \in \Lambda^{(m,n)}$:

$$\begin{aligned} E(\xi)R_\lambda(\xi) &= (\beta_+(1-q^{\mathrm{m}_0(\lambda)}) + \beta_-(1-q^{\mathrm{m}_n(\lambda)}))R_\lambda(\xi) \\ &\quad + \sum_{\substack{1\le j\le m \\ \lambda+e_j\in\Lambda^{(m,n)}}} (1-\alpha_+ q^{\mathrm{m}_0(\lambda)-1})^{\delta_{\lambda_j}} (1-q^{\mathrm{m}_{\lambda_j}(\lambda)}) R_{\lambda+e_j}(\xi) \\ &\quad + \sum_{\substack{1\le j\le m \\ \lambda-e_j\in\Lambda^{(m,n)}}} (1-\alpha_- q^{\mathrm{m}_n(\lambda)-1})^{\delta_{n-\lambda_j}} (1-q^{\mathrm{m}_{\lambda_j}(\lambda)}) R_{\lambda-e_j}(\xi), \end{aligned} \tag{3.8}$$

which is on-shell in the sense that the relation holds provided $\xi = (\xi_1, \ldots, \xi_m)$ satisfies the Bethe Ansatz equations (3.7). Substituting $\lambda = \mu'$ with $\mu \in \Lambda^{(n,m)}$ leads us via eq. (3.2) to the recurrence relation

$$\begin{aligned} E(\xi)R_{\mu'}(\xi) &= (\beta_+(1-q^{m-\mu_1}) + \beta_-(1-q^{\mu_n}))R_{\mu'}(\xi) \\ &\quad + \sum_{\substack{1\le j\le m \\ \mu'+e_j\in\Lambda^{(m,n)}}} (1-\alpha_+ q^{m-\mu_1-1})^{\delta_{\mu'_j}} (1-q^{\mathrm{m}_{\mu'_j}(\mu')}) R_{\mu'+e_j}(\xi) \\ &\quad + \sum_{\substack{1\le j\le m \\ \mu'-e_j\in\Lambda^{(m,n,)}}} (1-\alpha_- q^{\mu_n-1})^{\delta_{n-\mu'_j}} (1-q^{\mathrm{m}_{\mu'_j}(\mu')}) R_{\mu'-e_j}(\xi). \end{aligned}$$

We now observe that, for any $\mu \in \Lambda^{(n,m)}$ and $j \in \{1, \ldots, m\}$,

$$\mu' + e_j \in \Lambda^{(m,n)} \iff \mu' + e_j = (\mu + e_i)' \quad \text{with } i = \mu'_j + 1 \in \{1, \ldots, n\}$$

and

$$\mu' - e_j \in \Lambda^{(m,n)} \iff \mu' - e_j = (\mu - e_i)' \quad \text{with } i = \mu'_j \in \{1, \ldots, n\}.$$

The recurrence of interest can thus be rewritten in the form

$$\begin{aligned} E(\xi)R_{\mu'}(\xi) &= (\beta_+(1-q^{m-\mu_1}) + \beta_-(1-q^{\mu_n}))R_{\mu'}(\xi) \\ &\quad + \sum_{\substack{1\le i\le n \\ \mu+e_i\in\Lambda^{(n,m)}}} (1-\alpha_+ q^{m-\mu_1-1})^{\delta_{i-1}} (1-q^{\mu_{i-1}-\mu_i}) R_{(\mu+e_i)'}(\xi) \\ &\quad + \sum_{\substack{1\le i\le n \\ \mu-e_i\in\Lambda^{(n,m)}}} (1-\alpha_- q^{\mu_n-1})^{\delta_{n-i}} (1-q^{\mu_i-\mu_{i+1}}) R_{(\mu-e_i)'}(\xi), \end{aligned}$$

where we have employed once more eq. (3.2).

4. Spectral analysis

4.1. Solutions for the Bethe Ansatz equations

The following system of transcendental equations provides a logarithmic form of the Bethe Ansatz equations in Theorem 3.1:

$$\begin{aligned} 2n\xi_j &+ v_{p_+}(\xi_j) + v_{q_+}(\xi_j) + v_{p_-}(\xi_j) + v_{q_-}(\xi_j) \\ &+ \sum_{\substack{1\le k\le m \\ k\neq j}} \big(v_q(\xi_j+\xi_k) + v_q(\xi_j-\xi_k)\big) \\ &= 2\pi(m+1-j+\kappa_j), \end{aligned} \tag{4.1}$$

with $j = 1,\dots,m$, $\kappa \in \Lambda^{(m,n)}$ and

$$v_a(z) = \int_0^z \frac{(1-a^2)\,\mathrm{d}x}{1-2a\cos(x)+a^2} = \mathrm{i}\log\Big(\frac{1-ae^{\mathrm{i}z}}{e^{\mathrm{i}z}-a}\Big) \quad (-1<a<1). \tag{4.2}$$

Indeed, upon multiplying eq. (4.1) by i $(=\sqrt{-1})$, and applying the exponential function on both sides, it is readily seen that any of its solutions gives rise to a solution of the Bethe Ansatz equations (3.7) (where—recall—the boundary parameters $\alpha_\pm$, $\beta_\pm$ and $p_\pm$, $q_\pm$ are related via eq. (3.3)).

For any $\kappa \in \Lambda^{(m,n)}$, the system in (4.1)–(4.2) describes the critical point of a Yang–Yang-type Morse function:

$$\begin{aligned} V_\kappa(\xi_1,\dots,\xi_m) = &\sum_{1\le j<k\le m}\Big(\int_0^{\xi_j+\xi_k} v_q(x)\,\mathrm{d}x + \int_0^{\xi_j-\xi_k} v_q(x)\,\mathrm{d}x\Big) \\ &+ \sum_{1\le j\le m}\Big(n\xi_j^2 - 2\pi(m+1-j+\kappa_j)\xi_j \\ &\quad + \int_0^{\xi_j}\big(v_{p_+}(x)+v_{q_+}(x)+v_{p_-}(x)+v_{q_-}(x)\big)\,\mathrm{d}x\Big). \end{aligned} \tag{4.3}$$

The function $V_\kappa(\xi_1,\dots,\xi_m)$ belongs to a wider class of smooth, strictly convex and radially unbounded Morse functions studied in [31, Section 3]. The upshot is that via a Yang–Yang-type analysis, one arrives at $\binom{m+n}{n}$ solutions for the Bethe Ansatz equations given by the respective minima of $V_\kappa(\xi,\dots,\xi_m)$, $\kappa\in\Lambda^{(m,n)}$ (cf. [32, Remark 3.5]).

Proposition 4.1 (Solutions for the Bethe Ansatz equations). *Let $q \in (-1,1) \setminus \{0\}$ and let the boundary parameters $\alpha_\pm$, $\beta_\pm$ belong to the domain specified in (3.3)–(3.4).*

(i) *For any $\kappa \in \Lambda^{(m,n)}$, the logarithmic form of the Bethe Ansatz equations in (4.1)–(4.2) has a unique solution $\xi_\kappa \in \mathbb{R}^m$ given by the global minimum of the strictly convex radially unbounded Morse function $V_\kappa(\xi_1,\dots,\xi_m)$ (see (4.3)).*

(ii) *The global minima ξ_κ, $\kappa \in \Lambda^{(m,n)}$ in part* (i) *are all distinct and located within the open alcove*

$$\mathbb{A}^m = \{(\xi_1,\xi_2,\dots,\xi_m) \in \mathbb{R}^m \mid \pi > \xi_1 > \xi_2 > \cdots > \xi_m > 0\} \subset \mathbb{R}^m_{\mathrm{reg}}.$$

Moreover, at a global minimum $\xi = \xi_\kappa$ the following estimates are fulfilled:

$$\frac{\pi(m+1-j+\kappa_j)}{n+\mathrm{K}_+} \le \xi_j \le \frac{\pi(m+1-j+\kappa_j)}{n+\mathrm{K}_-} \tag{4.4}$$

(for $1 \le j \le m$), and

$$\frac{\pi(k-j+\kappa_j-\kappa_k)}{n+\mathrm{K}_+} \le \xi_j - \xi_k \le \frac{\pi(k-j+\kappa_j-\kappa_k)}{n+\mathrm{K}_-} \tag{4.5}$$

(for $1 \le j < k \le m$), where

$$\begin{aligned}\mathrm{K}_\pm = (m-1)\Big(\frac{1+|q|}{1-|q|}\Big)^{\pm 1} + \frac{1}{2}\Big(\Big(\frac{1+|p_+|}{1-|p_+|}\Big)^{\pm 1} + \Big(\frac{1+|q_+|}{1-|q_+|}\Big)^{\pm 1} \\ + \Big(\frac{1+|p_-|}{1-|p_-|}\Big)^{\pm 1} + \Big(\frac{1-|q_-|}{1-|q_-|}\Big)^{\pm 1}\Big).\end{aligned}$$

Proof. The assertions of this proposition follow by applying [31, Propositions 3.1 and 3.2] to the Bethe Ansatz equations of Theorem 3.1 (cf. [32, Remark 3.5]). ∎

Remark 4.2. From a mostly academic perspective, Proposition 4.1 invites us to compute ξ_κ via the gradient flow of the pertinent Morse function

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} + \partial_{\xi_j} V_\kappa(\xi_1,\dots,\xi_m) = 0, \quad j = 1,\dots,m,$$

which gives rise to the following system of differential equations (cf. eq. (4.1))

$$\begin{aligned}&\frac{\mathrm{d}\xi_j}{\mathrm{d}t} + 2n\xi_j + v_{p_+}(\xi_j) + v_{q_+}(\xi_j) + v_{p_-}(\xi_j) + v_{q_-}(\xi_j) \\ &\quad + \sum_{\substack{1\le k\le m \\ k\neq j}} (v_q(\xi_j+\xi_k) + v_q(\xi_j-\xi_k)) \\ &= 2\pi(m+1-j+\kappa_j),\end{aligned} \tag{4.6}$$

$j = 1, \ldots, m$. For $n = 0$ and $\kappa = (0^m)$, the corresponding gradient flow was analyzed in [24] and seen to converge exponentially fast to the roots of the Askey–Wilson polynomial $p_m(\cos\vartheta; p_+, q_+ p_-, q_- |\mathrm{q})$ [15, equation (14.1.1)] located within the interval of orthogonality $0 < \vartheta < \pi$. A minor variation of [24, Theorem 2] reveals that in our present setting the equilibrium ξ_κ of the gradient system in eq. (4.6) remains globally exponentially stable, i.e., for any initial condition $\xi_\kappa(0)$ the unique solution $\xi_\kappa(t)$, $t \geq 0$ of the gradient system converges exponentially fast to the equilibrium ξ_κ. More specifically, by slightly adapting the analysis in [24, Section 4] one readily deduces that for any $0 < \varepsilon < 2(n + \mathrm{K}_-)$ ($=$ a lower bound for the eigenvalues of the hessian of $V_\kappa(\xi_1, \ldots, \xi_m)$), there exists a constant $C_\varepsilon > 0$ such that

$$\|\xi_\kappa(t) - \xi_\kappa\|_\infty \leq C_\varepsilon e^{-\varepsilon t} \quad \text{for all } t \geq 0,$$

where $\|\xi\|_\infty \equiv \max_{1 \leq j \leq m} |\xi_j|$. Apart from ε, the actual value of the constant C_ε in the uniform estimate of the error term will depend on the choice of the initial condition $\xi_\kappa(0) \in \mathbb{R}^m$, as well as on $\kappa \in \Lambda^{(m,n)}$ and $q, p_\pm, q_\pm \in (-1, 1)$ (cf. Remark 4.4 below). Notice that the q-difference Toda Hamiltonian H (2.1) degenerates to a discrete Laplacian on $\Lambda^{(n,m)}$ in the symplectic Schur limit $\alpha_\pm, \beta_\pm, q \to 0$. At this elementary point in the parameters space, one has that (cf. eq. (4.4))

$$\xi_\kappa \to \Bigl(\frac{\pi(m + \kappa_1)}{m + n + 1}, \frac{\pi(m - 1 + \kappa_2)}{m + n + 1}, \ldots, \frac{\pi(m + 1 - j + \kappa_j)}{m + n + 1}, \ldots, \frac{\pi(1 + \kappa_m)}{m + n + 1}\Bigr),$$

which serves as a convenient initial condition for the gradient flow (4.6). Indeed, at this particular value of the spectral parameter the bounds in (4.4) and (4.5) are fulfilled for any $p_\pm, q_\pm, q \in (-1, 1)$.

4.2. Spectrum and eigenbasis

By combining Theorem 3.1 and Proposition 4.1, an eigenbasis of Bethe Ansatz wave functions for the q-difference Toda Hamiltonian H (2.1) is found in the Hilbert space $\ell^2(\Lambda^{(n,m)}, \Delta)$ together with the corresponding eigenvalues.

Theorem 4.3 (Spectrum and eigenbasis). *Let $q \in (-1, 1) \setminus \{0\}$ and let the boundary parameters $\alpha_\pm$, $\beta_\pm$ belong to the domain specified in (3.3)–(3.4). For any $\xi \in \mathbb{R}^m_{\mathrm{reg}}$ and $\kappa \in \Lambda^{(m,n)}$, the function $\psi_\xi \colon \Lambda^{(n,m)} \to \mathbb{R}$ refers to the Hall–Littlewood Bethe Ansatz wave function from Theorem 3.1 and $\xi_\kappa \in \mathbb{A}^m \subset \mathbb{R}^m_{\mathrm{reg}}$ denotes the unique global minimum of $V_\kappa(\xi_1, \ldots, \xi_m)$ (4.3) detailed in Proposition 4.1.*

(i) *The spectrum of the q-difference Toda Hamiltonian H (2.1) in the Hilbert space $\ell^2(\Lambda^{(n,m)}, \Delta)$ consists of the eigenvalues $E(\xi_\kappa)$, $\kappa \in \Lambda^{(m,n)}$, where $E(\xi)$ is given by eq. (3.6).*

(ii) *The corresponding Bethe Ansatz wave functions* ψ_{ξ_κ}, $\kappa \in \Lambda^{(m,n)}$ *constitute an eigenbasis for* H (2.1) *in* $\ell^2(\Lambda^{(n,m)}, \Delta)$ *such that*

$$H\psi_{\xi_\kappa} = E(\xi_\kappa)\psi_{\xi_\kappa} \quad (\kappa \in \Lambda^{(m,n)}). \tag{4.7}$$

Proof. It is clear from Theorem 3.1 and Proposition 4.1 that for any $\kappa \in \Lambda^{(m,n)}$ the Bethe Ansatz wave function ψ_{ξ_κ} solves the eigenvalue equation (4.7). Moreover, the value of the Bethe Ansatz wave functions at the origin is given by Macdonald's three-parameter Poincaré series for the root system BC_m [17, §10]:

$$\begin{aligned}\psi_\xi(0^n) &= R_{(0^m)}(\xi_1, \ldots, \xi_m)\\ &= \sum_{\substack{\sigma\in S_m\\ \epsilon\in\{1,-1\}^m}} C(\epsilon_1\xi_{\sigma(1)}, \ldots, \epsilon_m\xi_{\sigma(m)}) = \frac{(\alpha_+;q)_m(q;q)_m}{(1-q)^m} \neq 0\end{aligned}$$

(cf. also [32, Remark 3.4]). Hence, for any $\kappa \in \Lambda^{(m,n)}$ the wave function ψ_{ξ_κ} constitutes a proper (i.e., nontrivial) eigenfunction of the q-difference Toda Hamiltonian H with eigenvalue $E(\xi_\kappa)$. To confirm the completeness of the Bethe Ansatz, it remains to check that the wave functions in question indeed form a basis for $\ell^2(\Lambda^{(n,m)}, \Delta)$, or equivalently, that the hyperoctahedral Hall–Littlewood polynomials $R_{\mu'}(\xi_1, \ldots, \xi_m)$, $\mu \in \Lambda^{(n,m)}$ are linearly independent as functions on the Bethe spectrum $\{\xi_\kappa \mid \kappa \in \Lambda^{(m,n)}\}$. This independence is immediate from the second part of [32, Theorem 3.1]. ∎

Remark 4.4. The q-difference Toda Hamiltonian H, the positive weights Δ_μ, and the Bethe Ansatz wave function ψ_ξ clearly extend smoothly in the parameters to the domain $q, p_\pm, q_\pm \in (-1, 1)$. With the aid of the implicit function theorem, it is seen that the same is true for the solutions ξ_κ, $\kappa \in \Lambda^{(n,m)}$ of the (logarithmic) Bethe Ansatz equation in Proposition 4.1 (cf. [32, Remark 3.6]). Indeed, the Morse function $V_\kappa(\xi_1, \ldots, \xi_m)$ extends smoothly in the parameters and remains strictly convex. Hence, for $\kappa \in \Lambda^{(m,n)}$ the Bethe Ansatz wave function ψ_{ξ_κ} constitutes in fact an eigenfunction of H (2.1) in $\ell^2(\Lambda^{(n,m)}, \Delta)$ with eigenvalue $E(\xi_\kappa)$ (3.6) for any $q, p_\pm, q_\pm \in (-1, 1)$ (i.e., even if one or more of the parameters in question vanish).

Remark 4.5. The self-adjointness of H (2.1) in Proposition 2.1 implies that

$$\langle \psi_{\xi_\kappa}, \psi_{\xi_\nu}\rangle_\Delta = 0 \quad \text{if } \kappa \neq \nu \quad \text{for all } \kappa, \nu \in \Lambda^{(m,n)},$$

provided $E(\xi_\kappa) \neq E(\xi_\nu)$. Rewritten in terms of the (real-valued) hyperoctahedral Hall–Littlewood polynomials $R_\lambda(\xi) = R_\lambda(\xi_1, \ldots, \xi_m)$, one obtains that in this situation:

$$\sum_{\lambda\in\Lambda^{(m,n)}} R_\lambda(\xi_\kappa)R_\lambda(\xi_\nu)\Delta'_\lambda = 0 \quad \text{if } \kappa \neq \nu, \tag{4.8}$$

with

$$\Delta'_\lambda = \frac{(q;q)_m}{(\alpha_+;q)_{\mathrm{m}_0(\lambda)}(\alpha_-;q)_{\mathrm{m}_n(\lambda)}\prod_{0\le i\le n}(q;q)_{\mathrm{m}_i(\lambda)}} \tag{4.9}$$

(so $\Delta'_{\mu'} = \Delta_\mu$ for $\mu \in \Lambda^{(n,m)}$). To date, the latter orthogonality relation has been checked directly without the proviso regarding the nondegeneracy of the corresponding eigenvalues of H in the following four cases:

(a) if $q = 0$ and $p_\pm, q_\pm \in (-1, 1)$, cf. [29, Theorem 3.1];

(b) if $0 < q < 1$, $\alpha_\pm = 0$ and $\beta_\pm \in (-1, 1)$, cf. [28, Section 11.4];

(c) if $q, p_\pm, q_\pm \in (-1, 1)$ and $n \ge 2m$, cf. [26, Theorem 4.2];

(d) if $q, p_\pm, q_\pm \in (-1, 1)$ and $n = 1$, cf. [26, Section 5.7].

In view of Remark 4.4, within these four subdomains the statements concerning the spectrum and completeness formulated in parts (i) and (ii) of Theorem 4.3 therefore persist with the corresponding Bethe Ansatz eigenbasis being orthogonal in the Hilbert space $\ell^2(\Lambda^{(n,m)}, \Delta)$ (as expected).

Remark 4.6. In [32], eq. (3.8) is interpreted as the eigenvalue equation for a Hamiltonian of an m-particle q-boson model on the lattice $\{0, 1, \ldots, n\}$. The quantum integrability of this m-particle q-boson Hamiltonian, which is thus given explicitly by the difference operator acting at the right-hand side of eq. (3.8), was established for $\alpha_+ = \alpha_- = 0$ in [28] (using the quantum inverse scattering method) and for general boundary parameters in [32, Section 8] (using representations of the double affine Hecke algebra of type $C^\vee C$ at the critical level $\mathrm{q} = 0$). As detailed explicitly for the Hamiltonian in the proof of Theorem 3.1, the mapping $\mu \to \mu'$ from $\Lambda^{(n,m)}$ onto $\Lambda^{(m,n)}$ allows us to pull back the commuting quantum integrals for the q-boson model from $\ell^2(\Lambda^{(m,n)}, \Delta')$ to $\ell^2(\Lambda^{(n,m)}, \Delta)$. This maps the commuting quantum integrals in question to an algebra of commuting difference operators in $\ell^2(\Lambda^{(n,m)}, \Delta)$ containing H, see (2.1). Since [32, Theorem 9.5] guarantees that the latter algebra of commuting difference operators is (Harish–Chandra-)isomorphic to the algebra of complex functions on the joint spectrum $\{\xi_\kappa \mid \kappa \in \Lambda^{(m,n)}\} \subset \mathbb{A}^m$, this establishes the quantum integrability of our q-difference Toda Hamiltonian H (2.1).

5. The limit $q \to 1$

5.1. Quantum Hamiltonian

Upon dividing out an overall scaling factor $1 - q$, the q-difference Toda Hamiltonian H (2.1) degenerates for $q \to 1$ to an elementary difference operator $\widetilde{H}$ with linear coefficients:

$$\begin{aligned}\widetilde{H} &= \beta_+(m-\mu_1) + \beta_-(\mu_n) \\ &\quad + \sum_{1\le i\le n}\big((1-\alpha_+)^{\delta_{i-1}}(\mu_{i-1}-\mu_i)T_i + (1-\alpha_-)^{\delta_{n-i}}(\mu_i-\mu_{i+1})T_i^{-1}\big). \end{aligned} \tag{5.1}$$

From Proposition 2.1, it follows that – assuming $\alpha_\pm \in (-1,1)$ and $\beta_\pm \in \mathbb{R}$ – this limiting quantum Hamiltonian is self-adjoint in a Hilbert space $\ell^2(\Lambda^{(n,m)}, \widetilde{\Delta})$ governed by the weights of a two-parameter multinomial distribution on the partitions $\Lambda^{(n,m)}$:

$$\begin{aligned}\widetilde{\Delta}_\mu = \lim_{q\to 1}\Delta_\mu &= \frac{m!}{(1-\alpha_+)^{m-\mu_1}(1-\alpha_-)^{\mu_n}\prod_{0\le i\le n}(\mu_i-\mu_{i+1})!} \\ &= \mathcal{N}\cdot\frac{m!\prod_{0\le i\le n}\rho_i^{\mu_i-\mu_{i+1}}}{\prod_{0\le i\le n}(\mu_i-\mu_{i+1})!},\end{aligned} \tag{5.2}$$

with

$$\rho_i = \begin{cases} \frac{(1-\alpha_+)^{-1}}{(n-1)+(1-\alpha_+)^{-1}+(1-\alpha_-)^{-1}} & \text{if } i=0, \\ \frac{1}{(n-1)+(1-\alpha_+)^{-1}+(1-\alpha_-)^{-1}} & \text{if } 0<i<n, \\ \frac{(1-\alpha_-)^{-1}}{(n-1)+(1-\alpha_+)^{-1}+(1-\alpha_-)^{-1}} & \text{if } i=n, \end{cases}$$

and

$$\mathcal{N} = \sum_{\mu\in\Lambda^{(n,m)}}\widetilde{\Delta}_\mu = \big((n-1)+(1-\alpha_+)^{-1}+(1-\alpha_-)^{-1}\big)^m. \tag{5.3}$$

Notice in particular that $\rho_0 = \rho_2 = \cdots = \rho_n = \frac{1}{n+1}$ if $\alpha_+ = \alpha_- = 0$.

5.2. Bethe Ansatz

The Bethe Ansatz wave function ψ_ξ (3.5) degenerates in the limit $q \to 1$ to a wave function $\tilde{\psi}_\xi \colon \Lambda^{(n,m)} \to \mathbb{C}$ with values

$$\tilde{\psi}_\xi(\mu) = \widetilde{R}_{\mu'}(\xi_1,\ldots,\xi_m) \tag{5.4}$$

that separate in terms of univariate (BC_1-type) Hall–Littlewood polynomials. Specifically, for any $\lambda \in \Lambda^{(m,n)}$ and $\xi \in \mathbb{R}^m_{\text{reg}}$ one has that

$$\begin{aligned}\widetilde{R}_\lambda(\xi_1,\ldots,\xi_m) &= \lim_{q\to 1} R_\lambda(\xi_1,\ldots,\xi_m) \\ &= \sum_{\sigma\in S_m} R_{\lambda_1}(\xi_{\sigma(1)})R_{\lambda_2}(\xi_{\sigma(2)})\cdots R_{\lambda_m}(\xi_{\sigma(m)})\end{aligned} \tag{5.5}$$

with

$$R_l(\vartheta) = \frac{(1-\beta_+ e^{-i\vartheta} + \alpha_+ e^{-2i\vartheta})}{1-e^{-2i\vartheta}} \exp(il\vartheta) + \frac{(1-\beta_+ e^{i\vartheta} + \alpha_+ e^{2i\vartheta})}{1-e^{2i\vartheta}} \exp(-il\vartheta) \tag{5.6}$$

($l \in \{0,\ldots,n\}$, $\vartheta \notin \pi\mathbb{Z}$).

Heuristically, for the Bethe Ansatz wave function $\tilde{\psi}_\xi$ (5.4)–(5.6) to solve the $q \to 1$ eigenvalue equation

$$\widetilde{H}\tilde{\psi}_\xi = \widetilde{E}(\xi)\tilde{\psi}_\xi \quad \text{with } \widetilde{E}(\xi) = 2\sum_{1\le j\le m}\cos(\xi_j), \tag{5.7}$$

one expects the spectral parameter ξ to be required to satisfy the following decoupled system of Bethe Ansatz equations arising from eq. (3.7) in the limit $q \to 1$:

$$e^{2in\xi_j} = \frac{(1-\beta_+ e^{i\xi_j} + \alpha_+ e^{2i\xi_j})}{(e^{2i\xi_j} - \beta_+ e^{i\xi_j} + \alpha_+)} \frac{(1-\beta_- e^{i\xi_j} + \alpha_- e^{2i\xi_j})}{(e^{2i\xi_j} - \beta_- e^{i\xi_j} + \alpha_-)}, \tag{5.8}$$

$j = 1,\ldots,m$.

5.3. Solutions for the Bethe Ansatz equations

It is illuminating to emphasize that for $p_+, q_+, p_-, q_- \in (-1,1)$ the decoupled Bethe Ansatz equations in eq. (5.8) can be conveniently solved in terms of the roots of the Askey–Wilson polynomial $p_{n+1}(\cos\vartheta; p_+, q_+ p_-, q_-|\mathrm{q})$ [15, eq. (14.1.1)] at $\mathrm{q}=0$. Indeed, it is clear from the orthogonality relation [15, eq. (14.1.2)] that at $\mathrm{q}=0$ the Askey–Wilson polynomials fall within a well-known class of orthogonal polynomials studied by Bernstein and Szegő [23, Chapter 2.6]. The classical theory of Bernstein and Szegő tells us, moreover, that the polynomials in question can be written explicitly as follows (cf. e.g. [30, Section 4.3]):

$$\frac{p_{n+1}(\cos\vartheta; p_+, q_+ p_-, q_-|0)}{(p_+q_+p_-q_-q^n;q)_{n+1}} = \frac{\prod_{\epsilon=\pm}(1-p_\epsilon e^{-i\vartheta})(1-q_\epsilon e^{-i\vartheta})}{1-e^{-2i\vartheta}}\exp(i(n+1)\vartheta) + \frac{\prod_{\epsilon=\pm}(1-p_\epsilon e^{i\vartheta})(1-q_\epsilon e^{ix\vartheta})}{1-e^{2i\vartheta}}\exp(-i(n+1)\vartheta)$$

($\vartheta \notin \pi\mathbb{Z}$). This explicit formula reveals in particular that the roots

$$0 < \vartheta_0 < \vartheta_1 < \cdots < \vartheta_n < \pi \tag{5.9}$$

of $p_{n+1}(\cos\vartheta; p_+, q_+ p_-, q_- | 0)$ solve the Bethe Ansatz equation in eq. (5.8) (with ξ_j replaced by ϑ). More specifically, the root $\vartheta_{\Bbbk}$ corresponds to the solution of the associated logarithmic Bethe Ansatz equation (cf. Remark 4.2 above)

$$2n\vartheta + v_{p_+}(\vartheta) + v_{q_+}(\vartheta) + v_{p_-}(\vartheta) + v_{q_-}(\vartheta) = 2\pi(\Bbbk + 1),$$

with $\Bbbk \in \{0, 1, \ldots, n\}$.

The upshot is that to any $\kappa = (\kappa_1, \ldots, \kappa_m) \in \Lambda^{(m,n)}$, we can now attach a solution $\tilde{\xi}_\kappa$ of the decoupled system of Bethe Ansatz equations in eq. (5.8) by forming the following vector of q = 0 Askey–Wilson roots:

$$\tilde{\xi}_\kappa = (\vartheta_{\kappa_1}, \vartheta_{\kappa_2}, \ldots, \vartheta_{\kappa_m}) \in \{\xi \in \mathbb{R}^m \mid \pi > \xi_1 \geq \xi_2 \geq \cdots \geq \xi_m > 0\}. \qquad (5.10)$$

Notice that $\tilde{\xi}_\kappa$ encodes the unique global minimum of the decoupled Morse function

$$\widetilde{V}_\kappa(\xi_1, \ldots, \xi_m) = \sum_{1 \leq j \leq m} \left(n\xi_j^2 - 2\pi(\kappa_j + 1)\xi_j + \int_0^{\xi_j} (v_{p_+}(x) + v_{q_+}(x) + v_{p_-}(x) + v_{q_-}(x))\, \mathrm{d}x \right).$$

5.4. Spectrum and eigenbasis for $q \to 1$

When tying the above observations together, one is led to the following $q \to 1$ counterpart of Theorem 4.3.

Theorem 5.1 (Spectrum and eigenbasis for $q \to 1$). *Let the boundary parameters $\alpha_\pm$, $\beta_\pm$ be of the form in eq. (3.3) with $p_\pm, q_\pm \in (-1, 1)$. For any $\xi \in \{\xi \in \mathbb{R}^m \mid \xi_j \notin \pi\mathbb{Z}$, for all $1 \leq j \leq m\}$ and $\kappa \in \Lambda^{(m,n)}$, the function $\tilde{\psi}_\xi : \Lambda^{(n,m)} \to \mathbb{R}$ refers to the $q = 1$ Hall–Littlewood Bethe Ansatz wave function in eq. (5.4)–(5.6) and $\tilde{\xi}_\kappa$ denotes the solution in eq. (5.10) of the decoupled Bethe Ansatz equations (5.8).*

(i) *The spectrum of $\widetilde{H}$ (5.1) in the Hilbert space $\ell^2(\Lambda^{(n,m)}, \widetilde{\Delta})$ consists of the eigenvalues $\widetilde{E}(\tilde{\xi}_\kappa)$, $\kappa \in \Lambda^{(m,n)}$, where $\widetilde{E}(\xi)$ is given by eq. (5.7).*

(ii) *The corresponding Bethe Ansatz wave functions $\tilde{\psi}_{\tilde{\xi}_\kappa}$, $\kappa \in \Lambda^{(m,n)}$ constitute an orthogonal eigenbasis for $\widetilde{H}$ (5.1) in $\ell^2(\Lambda^{(n,m)}, \widetilde{\Delta})$ such that*

$$\widetilde{H}\tilde{\psi}_{\tilde{\xi}_\kappa} = \widetilde{E}(\tilde{\xi}_\kappa)\tilde{\psi}_{\tilde{\xi}_\kappa} \quad (\kappa \in \Lambda^{(m,n)}).$$

Remark 5.2. Systematic studies of orthogonal polynomials associated with the multinomial distribution give rise to multivariate generalizations of the Krawtchouk polynomials [5,9,12–14,18]. For the particular instance of the two-parameter multinomial distribution in (5.2)–(5.3), Theorem 5.1 suggests an intriguing link to the $q = 1$ Hall–Littlewood Bethe Ansatz wave function $\tilde{\psi}_{\tilde{\xi}_\kappa}$. When $n = 1$, this link is actually well understood in the literature as a relation between classical univariate Krawtchouk polynomials and elementary symmetric polynomials, cf. e.g. [25, Equation (5.14)].

5.5. Proof of Theorem 5.1

In order to establish the claims of the theorem in full rigor avoiding tricky formal limits, let us first check that the Bethe Ansatz wave functions $\tilde{\psi}_{\tilde{\xi}_\kappa}$, $\kappa \in \Lambda^{(m,n)}$ are indeed orthogonal in the Hilbert space $\ell^2(\Lambda^{(n,m)}, \tilde{\Delta})$ (i.e., with the weights $\tilde{\Delta}_\mu$ replacing Δ_μ):

$$\langle \tilde{\psi}_{\tilde{\xi}_\kappa}, \tilde{\psi}_{\tilde{\xi}_\nu} \rangle_{\tilde{\Delta}} = 0 \quad \text{if } \kappa \neq \nu \quad \text{for all } \kappa, \nu \in \Lambda^{(m,n)}. \tag{5.11}$$

Rewritten in terms of the $q = 1$ hyperoctahedral Hall–Littlewood polynomials, the inner product on the left-hand side of eq. (5.11) becomes (cf. (4.8)–(4.9)):

$$\sum_{\lambda \in \Lambda^{(m,n)}} \frac{\tilde{R}_\lambda(\tilde{\xi}_\kappa)\tilde{R}_\lambda(\tilde{\xi}_\nu)\, m!}{(1-\alpha_+)^{\mathrm{m}_0(\lambda)}(1-\alpha_-)^{\mathrm{m}_n(\lambda)} \prod_{0 \le i \le n} \mathrm{m}_i(\lambda)!}, \tag{5.12}$$

with $\tilde{R}_\lambda(\xi)$ and $\tilde{\xi}_\kappa$ taken from (5.5) and (5.10), respectively. In particular, if $m = 1$ then the sum in eq. (5.12) is of the form

$$\sum_{0 \le \lambda \le n} \frac{R_\lambda(\vartheta_\kappa) R_\lambda(\vartheta_\nu)}{(1-\alpha_+)^{\delta_\lambda}(1-\alpha_-)^{\delta_{n-\lambda}}} \tag{5.13}$$

with $0 \le \kappa \neq \nu \le n$, where $\vartheta_0, \ldots, \vartheta_n$ refer to the q $= 0$ Askey–Wilson roots from eq. (5.9). Since the sum in eq. (5.13) coincides with that of the inner product in case (a) of Remark 4.5 (specialized to $m = 1$), in this simplest situation the asserted orthogonality is immediate from the remark in question.

If on the other hand $m > 1$, then the inner product in eq. (5.12) decomposes into a sum of contributions of the form

$$\begin{aligned}
&\sum_{\lambda \in \Lambda^{(m,n)}} \frac{R_{\lambda_1}(\vartheta_{\mathtt{k}_1}) \cdots R_{\lambda_m}(\vartheta_{\mathtt{k}_m}) R_{\lambda_1}(\vartheta_{\mathtt{n}_1}) \cdots R_{\lambda_m}(\vartheta_{\mathtt{n}_m}) m!}{(1-\alpha_+)^{\mathrm{m}_0(\lambda)}(1-\alpha_-)^{\mathrm{m}_n(\lambda)} \prod_{0 \le i \le n} \mathrm{m}_i(\lambda)!} \\
&\quad = \sum_{0 \le \lambda_1, \ldots, \lambda_m \le n} \frac{R_{\lambda_1}(\vartheta_{\mathtt{k}_1}) \cdots R_{\lambda_m}(\vartheta_{\mathtt{k}_m}) R_{\lambda_1}(\vartheta_{\mathtt{n}_1}) \cdots R_{\lambda_m}(\vartheta_{\mathtt{n}_m})}{\prod_{1 \le j \le m} (1-\alpha_+)^{\delta_{\lambda_j}} (1-\alpha_-)^{\delta_{n-\lambda_j}}} \\
&\quad = \prod_{1 \le j \le m} \sum_{0 \le \lambda_j \le n} \frac{R_{\lambda_j}(\vartheta_{\mathtt{k}_j}) R_{\lambda_j}(\vartheta_{\mathtt{n}_j})}{(1-\alpha_+)^{\delta_{\lambda_j}} (1-\alpha_-)^{\delta_{n-\lambda_j}}},
\end{aligned}$$

where the m-tuple $(\mathtt{k}_1, \mathtt{k}_2, \ldots, \mathtt{k}_m)$ denotes a reordering $(\kappa_{\sigma(1)}, \kappa_{\sigma(2)}, \ldots, \kappa_{\sigma(m)})$ of $\kappa \in \Lambda^{(m,n)}$ and the m-tuple $(\mathtt{n}_1, \mathtt{n}_2, \ldots, \mathtt{n}_m)$ denotes a reordering of $\nu \in \Lambda^{(m,n)}$ (not necessarily stemming from the same permutation $\sigma \in S_m$). From the orthogonality for $m = 1$, it is now clear that all such contributions vanish, unless $\mathtt{k}_j = \mathtt{n}_j$ for $j = 1, \ldots, m$, i.e., except when $\kappa = \nu$ and both reorderings coincide.

It remains to verify that the eigenvalue equation in eq. (5.7) is satisfied at $\xi = \tilde{\xi}_\kappa$ for $\kappa \in \Lambda^{(m,n)}$. Rewritten in terms of $\lambda = \mu' \in \Lambda^{(m,n)}$, this eigenvalue equation reads explicitly (cf. eq. (3.8)):

$$\begin{aligned}\widetilde{E}(\tilde{\xi}_\kappa)\widetilde{R}_\lambda(\tilde{\xi}_\kappa) &= (\beta_+ \mathrm{m}_0(\lambda) + \beta_- \mathrm{m}_n(\lambda))\widetilde{R}_\lambda(\tilde{\xi}_\kappa) \\ &\quad + \sum_{\substack{1\le j\le m \\ \lambda+e_j\in\Lambda^{(m,n)}}} (1-\alpha_+)^{\delta_{\lambda_j}} \mathrm{m}_{\lambda_j}(\lambda)\widetilde{R}_{\lambda+e_j}(\tilde{\xi}_\kappa) \\ &\quad + \sum_{\substack{1\le j\le m \\ \lambda-e_j\in\Lambda^{(m,n)}}} (1-\alpha_-)^{\delta_{n-\lambda_j}} m_{\lambda_j}(\lambda)\widetilde{R}_{\lambda-e_j}(\tilde{\xi}_\kappa). \end{aligned} \tag{5.14}$$

For instance, if $m = 1$ then eq. (5.14) simplifies to

$$\begin{aligned} 2\cos(\vartheta_\kappa)R_\lambda(\vartheta_\kappa) &= (\beta_+\delta_\lambda + \beta_-\delta_{n-\lambda})R_\lambda(\vartheta_\kappa) + (1-\alpha_+)^{\delta_\lambda}(1-\delta_{n-\lambda})R_{\lambda+1}(\vartheta_\kappa) \\ &\quad + (1-\alpha_-)^{\delta_{n-\lambda}}(1-\delta_\lambda)R_{\lambda-1}(\vartheta_\kappa), \end{aligned} \tag{5.15}$$

where $0 \le \kappa, \lambda \le n$. Apart from a missing overall factor $(1-q)$ on both sides (which was actually divided out at the start of Subsection 5.1), eq. (5.15) coincides precisely with the $m = 1$ specialization of the eigenvalue equation from Theorem 4.3 at $\mu = \lambda'$ (which agrees with the observation that the dependence on q drops out when $m = 1$). Upon recalling Remark 4.4, this settles the validity of eq. (5.15) for the full parameter regime $p_\pm, q_\pm \in (-1, 1)$.

Moreover, by virtue of eq. (5.15) one has more generally that for $m > 1$ and any $\sigma \in S_m$:

$$\begin{aligned} &2\sum_{1\le j\le m}\cos(\vartheta_{\kappa_j})\prod_{1\le l\le m} R_{\lambda_l}(\vartheta_{\kappa_{\sigma(l)}}) \\ &\quad = \sum_{1\le j\le m}(\beta_+\delta_{\lambda_j} + \beta_-\delta_{n-\lambda_j})\prod_{1\le l\le m} R_{\lambda_l}(\vartheta_{\kappa_{\sigma(l)}}) \\ &\qquad + \sum_{1\le j\le m}(1-\alpha_+)^{\delta_{\lambda_j}}(1-\delta_{n-\lambda_j})R_{\lambda_j+1}(\vartheta_{\kappa_{\sigma(j)}})\prod_{\substack{1\le l\le m \\ l\ne j}} R_{\lambda_l}(\vartheta_{\kappa_{\sigma(l)}}) \\ &\qquad + \sum_{1\le j\le m}(1-\alpha_-)^{\delta_{n-\lambda_j}}(1-\delta_{\lambda_j})R_{\lambda_j-1}(\vartheta_{\kappa_{\sigma(j)}})\prod_{\substack{1\le l\le m \\ l\ne j}} R_{\lambda_l}(\vartheta_{\kappa_{\sigma(l)}}) \\ &\quad = (\beta_+\mathrm{m}_0(\lambda) + \beta_-\mathrm{m}_n(\lambda))\prod_{1\le l\le m} R_{\lambda_l}(\vartheta_{\kappa_{\sigma(l)}}) \\ &\qquad + \sum_{\substack{1\le j\le m \\ \lambda+e_j\in\Lambda^{(m,n)}}}(1-\alpha_+)^{\delta_{\lambda_j}}\mathrm{m}_{\lambda_j}(\lambda)R_{\lambda_j+1}(\vartheta_{\kappa_{\sigma(j)}})\prod_{\substack{1\le l\le m \\ l\ne j}} R_{\lambda_l}(\vartheta_{\kappa_{\sigma(l)}}) \\ &\qquad + \sum_{\substack{1\le j\le m \\ \lambda-e_j\in\Lambda^{(m,n)}}}(1-\alpha_-)^{\delta_{n-\lambda_j}} m_{\lambda_j}(\lambda)R_{\lambda_j-1}(\vartheta_{\kappa_{\sigma(j)}})\prod_{\substack{1\le l\le m \\ l\ne j}} R_{\lambda_l}(\vartheta_{\kappa_{\sigma(l)}}), \end{aligned}$$

which entails eq. (5.14) through symmetrization by summing over all $\sigma \in S_m$ on both sides.

Funding. This work was partially supported by the *Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT)* Grant #1210015.

References

[1] A. Borodin and I. Corwin, Macdonald processes. *Probab. Theory Related Fields* **158** (2014), no. 1-2, 225–400 Zbl 1291.82077 MR 3152785

[2] A. Braverman and M. Finkelberg, Finite difference quantum Toda lattice via equivariant K-theory. *Transform. Groups* **10** (2005), no. 3-4, 363–386 Zbl 1122.17008 MR 2183117

[3] P. Di Francesco and R. Kedem, Difference equations for graded characters from quantum cluster algebra. *Transform. Groups* **23** (2018), no. 2, 391–424 Zbl 1437.13034 MR 3805210

[4] P. Di Francesco, R. Kedem, and B. Turmunkh, A path model for Whittaker vectors. *J. Phys. A* **50** (2017), no. 25, article no. 255201 Zbl 1370.81090 MR 3659140

[5] P. Diaconis and R. Griffiths, An introduction to multivariate Krawtchouk polynomials and their applications. *J. Statist. Plann. Inference* **154** (2014), 39–53 Zbl 1306.60003 MR 3258404

[6] A. Duval and V. Pasquier, q-bosons, Toda lattice, Pieri rules and Baxter q-operator. *J. Phys. A* **49** (2016), no. 15, article no. 154006 Zbl 1351.37257 MR 3479118

[7] P. Etingof, Whittaker functions on quantum groups and q-deformed Toda operators. In *Differential topology, infinite-dimensional Lie algebras, and applications*, pp. 9–25, Amer. Math. Soc. Transl. Ser. 2 194, American Mathematical Society, Providence, RI, 1999 Zbl 1157.33327 MR 1729357

[8] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, Fermionic formulas for eigenfunctions of the difference Toda Hamiltonian. *Lett. Math. Phys.* **88** (2009), no. 1-3, 39–77 Zbl 1180.37091 MR 2512140

[9] V. X. Genest, L. Vinet, and A. Zhedanov, The multivariate Krawtchouk polynomials as matrix elements of the rotation group representations on oscillator states. *J. Phys. A* **46** (2013), no. 50, article no. 505203 Zbl 1282.22008 MR 3146038

[10] A. Gerasimov, D. Lebedev, and S. Oblezin, On q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function III. *Lett. Math. Phys.* **97** (2011), no. 1, 1–24 Zbl 1239.17009 MR 2802312

[11] A. Givental and Y.-P. Lee, Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups. *Invent. Math.* **151** (2003), no. 1, 193–219 Zbl 1051.14063 MR 1943747

[12] F. A. Grünbaum and M. Rahman, A system of multivariable Krawtchouk polynomials and a probabilistic application. *SIGMA Symmetry Integrability Geom. Methods Appl.* **7** (2011), article no. 119 Zbl 1244.33003 MR 2861222

[13] P. Iliev, A Lie-theoretic interpretation of multivariate hypergeometric polynomials. *Compos. Math.* **148** (2012), no. 3, 991–1002 Zbl 1248.33028 MR 2925407

[14] P. Iliev and Y. Xu, Hahn polynomials on polyhedra and quantum integrability. *Adv. Math.* **364** (2020), article no. 107032 Zbl 1448.33012 MR 4062926

[15] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Monographs in Mathematics, Springer, Berlin, 2010 Zbl 1200.33012 MR 2656096

[16] V. B. Kuznetsov and A. V. Tsyganov, Quantum relativistic Toda lattices. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **205** (1993), no. Differentsial'naya Geom. Gruppy Li i Mekh. 13, 71–84, 180; English transl., *J. Math. Sci. (N.Y.)* **80** (1996), no. 3, 1802–1810 Zbl 0860.17049 MR 1255304

[17] I. G. Macdonald, Orthogonal polynomials associated with root systems. *Sém. Lothar. Combin.* **45** (2000/01), article no. B45a Zbl 1032.33010 MR 1817334

[18] H. Mizukawa and H. Tanaka, $(n+1, m+1)$-hypergeometric functions associated to character algebras. *Proc. Amer. Math. Soc.* **132** (2004), no. 9, 2613–2618 Zbl 1059.33020 MR 2054786

[19] S. N. M. Ruijsenaars, Relativistic Toda systems. *Comm. Math. Phys.* **133** (1990), no. 2, 217–247 Zbl 0719.58019 MR 1090424

[20] A. Sevostyanov, Quantum deformation of Whittaker modules and the Toda lattice. *Duke Math. J.* **105** (2000), no. 2, 211–238 Zbl 1058.17009 MR 1793611

[21] E. K. Sklyanin, Boundary conditions for integrable quantum systems. *J. Phys. A* **21** (1988), no. 10, 2375–2389 Zbl 0685.58058 MR 953215

[22] Y. B. Suris, Discrete time generalized Toda lattices: complete integrability and relation with relativistic Toda lattices. *Phys. Lett. A* **145** (1990), no. 2-3, 113–119 MR 1047621

[23] G. Szegő, *Orthogonal polynomials*. Fourth edn., Amer. Math. Soc. Colloq. Publ. XXIII, American Mathematical Society, Providence, RI, 1975 Zbl 0305.42011 MR 372517

[24] J. F. van Diejen, Gradient system for the roots of the Askey-Wilson polynomial. *Proc. Amer. Math. Soc.* **147** (2019), no. 12, 5239–5249 Zbl 1429.33031 MR 4021083

[25] J. F. van Diejen, q-deformation of the Kac-Sylvester tridiagonal matrix. *Proc. Amer. Math. Soc.* **149** (2021), no. 6, 2291–2304 Zbl 1465.15013 MR 4246783

[26] J. F. van Diejen, Harmonic analysis of boxed hyperoctahedral Hall–Littlewood polynomials. *J. Funct. Anal.* **282** (2022), no. 1, article no. 109256 Zbl 07423996 MR 4324286

[27] J. F. van Diejen and E. Emsiz, Integrable boundary interactions for Ruijsenaars' difference Toda chain. *Comm. Math. Phys.* **337** (2015), no. 1, 171–189 Zbl 1361.37058 MR 3324160

[28] J. F. van Diejen and E. Emsiz, Orthogonality of Bethe ansatz eigenfunctions for the Laplacian on a hyperoctahedral Weyl alcove. *Comm. Math. Phys.* **350** (2017), no. 3, 1017–1067 Zbl 1360.82029 MR 3607469

[29] J. F. van Diejen and E. Emsiz, Discrete Fourier transform associated with generalized Schur polynomials. *Proc. Amer. Math. Soc.* **146** (2018), no. 8, 3459–3472 Zbl 1478.65148 MR 3803671

[30] J. F. van Diejen and E. Emsiz, Quadrature rules from finite orthogonality relations for Bernstein-Szegö polynomials. *Proc. Amer. Math. Soc.* **146** (2018), no. 12, 5333–5347 Zbl 1451.65021 MR 3866872

[31] J. F. van Diejen and E. Emsiz, Solutions of convex Bethe ansatz equations and the zeros of (basic) hypergeometric orthogonal polynomials. *Lett. Math. Phys.* **109** (2019), no. 1, 89–112 Zbl 1408.33033 MR 3897593

[32] J. F. van Diejen, E. Emsiz, and I. N. Zurrián, Completeness of the Bethe Ansatz for an open q-boson system with integrable boundary interactions. *Ann. Henri Poincaré* **19** (2018), no. 5, 1349–1384 Zbl 1388.81240 MR 3784914

Received 6 January 2023.

Jan Felipe van Diejen
Instituto de Matemáticas, Universidad de Talca, Casilla 747, 3460000 Talca, Chile;
diejen@inst-mat.utalca.cl