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Spectral theory of Jacobi operators with increasing coefficients. The critical case

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Abstract. Spectral properties of Jacobi operators J are intimately related to an asymptotic behavior of the corresponding orthogonal polynomials $P_n(z)$ as $n \to \infty$. We study the case where the off-diagonal coefficients a_n and, eventually, diagonal coefficients b_n of J tend to infinity in such a way that the ratio $\gamma_n := 2^{-1} b_n (a_n a_{n-1})^{-1/2}$ has a finite limit γ. In the case $|\gamma| < 1$ asymptotic formulas for $P_n(z)$ generalize those for the Hermite polynomials and the corresponding Jacobi operators J have absolutely continuous spectra covering the whole real line. If $|\gamma| > 1$, then spectra of the operators J are discrete. Our goal is to investigate the critical case $|\gamma| = 1$ that occurs, for example, for the Laguerre polynomials. The formulas obtained depend crucially on the rate of growth of the coefficients a_n (or b_n) and are qualitatively different in the cases where $a_n \to \infty$ faster or slower than n. For the fast growth of a_n, we also have to distinguish the cases $|\gamma_n| \to 1 - 0$ and $|\gamma_n| \to 1 + 0$. Spectral properties of the corresponding Jacobi operators are quite different in all these cases. Our approach works for an arbitrary power growth of the Jacobi coefficients.

1. Introduction. Basic definitions

1.1. Jacobi operators

We consider Jacobi operators defined by three-diagonal matrices

$$\mathcal{J} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

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in the canonical basis of the space $\ell^2(\mathbb{Z}_+)$. Thus, if $u = (u_0, u_1, \dots)^\top =: (u_n)$ is a column, then

$$(\mathcal{J}u)_0 = b_0 u_0 + a_0 u_1 \quad \text{and} \quad (\mathcal{J}u)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} \quad \text{for } n \geq 1.$$

It is always supposed that $a_n > 0$, $b_n = \bar{b}_n$ so that the matrix $\mathcal{J}$ is symmetric and commutes with the complex conjugation. The minimal Jacobi operator $J_{\min}$ is defined by the equality $J_{\min} u = \mathcal{J}u$ on the set $\mathcal{D} \subset \ell^2(\mathbb{Z}_+)$ of vectors $u = (u_n)$ with only a finite number of non-zero components u_n. The operator $J_{\min}$ is symmetric in the space $\ell^2(\mathbb{Z}_+)$ and $J_{\min} : \mathcal{D} \to \mathcal{D}$. Its adjoint $J_{\min}^*$ coincides with the maximal operator $J_{\max}$ given by the same formula $J_{\max} u = \mathcal{J}u$ on the set $\mathcal{D}(J_{\max})$ of all vectors $u \in \ell^2(\mathbb{Z}_+)$ such that $\mathcal{J}u \in \ell^2(\mathbb{Z}_+)$.

The operator $J_{\min}$ is bounded if and only if both sequences a_n and b_n are in $\ell^\infty(\mathbb{Z}_+)$. In general, $J_{\min}$ may have deficiency indices $(0, 0)$ (that is, it is essentially self-adjoint) or $(1, 1)$. Its essential self-adjointness depends on a behavior of solutions to the difference equation

$$a_{n-1} F_{n-1}(z) + b_n F_n(z) + a_n F_{n+1}(z) = z F_n(z), \quad n \geq 1. \tag{1.1}$$

Recall that the Weyl theory developed by him for differential equations can be naturally adapted to equations (1.1) (see, e.g., [1, Section 3 of Chapter 1] and references therein). For $\operatorname{Im} z \neq 0$, equation (1.1) always has a non-trivial solution $F_n(z) \in \ell^2(\mathbb{Z}_+)$. This solution is either unique (up to a constant factor) or all solutions of equation (1.1) belong to $\ell^2(\mathbb{Z}_+)$. The first instance is known as the limit point case and the second one – as the limit circle case. It turns out that the operator $J_{\min}$ is essentially self-adjoint if and only if the limit point case occurs; then the closure $\operatorname{clos} J_{\min}$ of $J_{\min}$ equals $J_{\max}$. In the limit circle case, the operator $J_{\min}$ has deficiency indices $(1, 1)$.

It is well known that the limit point case occurs if $a_n \to \infty$ as $n \to \infty$ but not too rapidly. For example, the condition

$$\sum_{n=0}^{\infty} a_n^{-1} = \infty \tag{1.2}$$

(introduced by T. Carleman in his book [5]) is sufficient for the essential self-adjointness of the operator $J_{\min}$. Under this condition, no assumptions on the diagonal elements b_n are required. In general, the essential self-adjointness of $J_{\min}$ is determined by a competition between sequences a_n and b_n. For example, if b_n are much larger than a_n, then $J_{\min}$ is close to a diagonal operator so that it is essentially self-adjoint independently of the growth of a_n.

1.2. Orthogonal polynomials

Orthogonal polynomials $P_n(z)$ can be formally defined as "eigenvectors" of the Jacobi operators. This means that a column

$$P(z) = (P_0(z), P_1(z), \dots)^\top$$

satisfies the equation $\mathcal{J} P(z) = zP(z)$ with $z \in \mathbb{C}$ being an "eigenvalue." This equation is equivalent to the recurrence relation

$$a_{n-1} P_{n-1}(z) + b_n P_n(z) + a_n P_{n+1}(z) = zP_n(z), \quad n \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}, \tag{1.3}$$

complemented by boundary conditions $P_{-1}(z) = 0$, $P_0(z) = 1$. Determining $P_n(z)$, $n = 1, 2, \dots$, successively from (1.3), we see that $P_n(z)$ is a polynomial with real coefficients of degree n: $P_n(z) = p_n z^n + \cdots$ where $p_n = (a_0 a_1 \cdots a_{n-1})^{-1}$.

The spectra of all self-adjoint extensions J of the minimal operator $J_{\min}$ are simple with $e_0 = (1, 0, 0, \dots)^\top$ being a generating vector. Therefore, it is natural to define the spectral measure of J by the relation $d\,\Xi_J(\lambda) = d\langle E_J(\lambda) e_0, e_0\rangle$ where $E_J(\lambda)$ is the spectral family of the operator J and $\langle \cdot, \cdot \rangle$ is the scalar product in the space $\ell^2(\mathbb{Z}_+)$. For all extensions J of the operator $J_{\min}$, the polynomials $P_n(\lambda)$ are orthogonal and normalized in the spaces $L^2(\mathbb{R}; d\,\Xi_J)$:

$$\int_{-\infty}^{\infty} P_n(\lambda) P_m(\lambda) d\,\Xi_J(\lambda) = \delta_{n,m};$$

as usual, $\delta_{n,n} = 1$ and $\delta_{n,m} = 0$ for $n \neq m$. We always consider normalized polynomials $P_n(\lambda)$. They are often called orthonormal. If the operator $J_{\min}$ is essentially self-adjoint and $J = \operatorname{clos} J_{\min}$, we write $d\,\Xi(\lambda)$ instead of $d\,\Xi_J(\lambda)$.

It is useful to keep in mind the following elementary observation.

Proposition 1.1. *If a sequence $F_n(z)$ satisfies equation (1.1), then*

$$F_n^\sharp(z) = (-1)^n F_n(-z)$$

satisfies the same equation with the Jacobi coefficients $(a_n^\sharp, b_n^\sharp) = (a_n, -b_n)$. In particular, $P_n^\sharp(z) = (-1)^n P_n(-z)$ are the orthonormal polynomials for the coefficients $(a_n^\sharp, b_n^\sharp)$. In the limit point case, if $J^\sharp$ is the Jacobi operator in the space $\ell^2(\mathbb{Z}_+)$ with matrix elements $(a_n^\sharp, b_n^\sharp)$, then $J^\sharp = -\mathcal{U}^* J \mathcal{U}$ where the unitary operator $\mathcal{U}$ is defined by $(\mathcal{U}F)_n = (-1)^n F_n$ for $n \in \mathbb{Z}_+$. The corresponding spectral measures are linked by the relation $d\,\Xi^\sharp(\lambda) = d\,\Xi(-\lambda)$. In particular, if $b_n = 0$ for all n, then the operators J and $-J$ are unitarily equivalent.

The comprehensive presentation of the results described shortly above can be found in the books [1, 6, 22] and the surveys [14, 21, 23, 24].

1.3. Asymptotic results

We study the case $a_n \to \infty$ as $n \to \infty$ and are interested in the asymptotic behavior of the polynomials $P_n(z)$ as $n \to \infty$. The condition $a_n \to \infty$ is fulfilled for the Hermite polynomials where the Jacobi coefficients are

$$a_n = \sqrt{(n+1)/2} \quad \text{and} \quad b_n = 0 \tag{1.4}$$

and the Laguerre polynomials $L_n^{(p)}(z)$ where

$$a_n = \sqrt{(n+1)(n+1+p)} \quad \text{and} \quad b_n = 2n + p + 1, \quad p > -1. \tag{1.5}$$

In the general case there are two essentially different approaches to this problem. The first one derives asymptotic formulas for $P_n(z)$ from the spectral measure $d\Xi(\lambda)$, and the second proceeds directly from the coefficients a_n, b_n. The first method goes back to S. Bernstein (see his pioneering papers [3, 4] or G. Szegő's book [22, Theorem 12.1.4]), who obtained formulas generalizing those for the Jacobi polynomials. In terms of the coefficients a_n, b_n, the assumptions of [3, 4] correspond to the conditions

$$a_n \to a_\infty > 0, \quad b_n \to 0 \quad \text{as } n \to \infty. \tag{1.6}$$

Generalizations of the asymptotic formulas for the Hermite polynomials are known as the Plancherel–Rotach formulas.

A study of an asymptotic behavior of the orthonormal polynomials for given coefficients a_n, b_n was initiated by P. Nevai in his book [18]. He (see also the papers [15, 25]) investigated the case of stabilizing coefficients satisfying condition (1.6), but, in contrast to [3, 4], the results of [15, 18, 25] were stated directly in terms of the Jacobi coefficients. The case of the coefficients $a_n \to \infty$ was later studied in [11] by J. Janas and S. Naboko and in [2] by A. Aptekarev and J. Geronimo. It was assumed in these papers that there exists a finite limit

$$\frac{b_n}{2\sqrt{a_{n-1}a_n}} =: \gamma_n \to \gamma, \quad n \to \infty, \tag{1.7}$$

where $|\gamma| < 1$ so that b_n are relatively small compared to a_n. The Carleman condition (1.2) was also required. The famous example of this type is given by the Hermite coefficients (1.4). In the general case the results are qualitatively similar to this particular case. Asymptotics of $P_n(\lambda)$ are oscillating for $\lambda \in \mathbb{R}$ and $P_n(z)$ exponentially grow as $n \to \infty$ if $\operatorname{Im} z \neq 0$. Spectra of the operators J are absolutely continuous and fill the whole real axis. If (1.7) is satisfied with $|\gamma| > 1$, then diagonal elements b_n dominate off-diagonal elements a_n. This ensures that the spectra of such operators J are discrete. Note (see, e.g., [31]) that algebraic structures of asymptotic formulas for the orthonormal polynomials are quite similar in the cases $|\gamma| < 1$ and $|\gamma| > 1$, but in the second case $P_n(z)$ exponentially grow as $n \to \infty$ even for $z \in \mathbb{R}$ (unless z is an eigenvalue of J).

The case of rapidly increasing coefficients a_n when the Carleman condition (1.2) is violated, so that

$$\sum_{n=0}^{\infty} a_n^{-1} < \infty,$$

was investigated in a recent paper [27] where it was also assumed that $|\gamma| \neq 1$. Astonishingly, the asymptotics of the orthogonal polynomials in this *a priori* highly singular case is particularly simple and general.

1.4. Critical case

In the critical case $|\gamma| = 1$, the coefficients a_n and b_n are of the same order and asymptotic formulas for $P_n(z)$ are determined by details of their behavior as $n \to \infty$.

Thus, one has to require assumptions on the coefficients a_n and b_n more specific compared to (1.7). To make our presentation as simple as possible, we assume that, asymptotically,

$$a_n = n^\sigma(1 + \alpha n^{-1} + O(n^{-2})), \quad n \to \infty, \tag{1.8}$$

and

$$b_n = 2\gamma n^\sigma(1 + \beta n^{-1} + O(n^{-2})), \quad n \to \infty, \tag{1.9}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ and[1] $\sigma > 0$. Thus, the operators with periodically modulated coefficients (see, e.g., [7] and references therein) are out of the scope of this paper. The critical case is distinguished by the condition $|\gamma| = 1$. In view of Proposition 1.1, the results for $\gamma = 1$ and $\gamma = -1$ are equivalent. It turns out that the asymptotic formulas for $P_n(z)$ depend crucially on the parameter

$$\tau = 2\beta - 2\alpha + \sigma. \tag{1.10}$$

Roughly speaking, the cases $\tau < 0$ (or $\tau > 0$) correspond to dominating off-diagonal a_n (resp., diagonal b_n) Jacobi coefficients.

All the results of this paper can be extended to a more general situation where the terms αn^{-1} and βn^{-1} in (1.8), (1.9) are replaced by αn^{-p} and βn^{-p} for some $p \in (0, 2)$ and the error term $O(n^{-2})$ is replaced by $O(n^{-r})$ for $r > \max\{1, p\}$.

The classical example where the critical case occurs is given by the Laguerre coefficients (1.5). In this case, we have $\gamma = 1$, $\sigma = 1$ and $\alpha = 1 + p/2$, $\beta = (1 + p)/2$ so that $\tau = 0$. The corresponding Jacobi operators $J = J^{(p)}$ have absolutely continuous spectra coinciding with $[0, \infty)$. Another example is given by the Jacobi operators

[1] The case $\sigma > 3/2$ was considered earlier in [29]

describing birth and death processes investigated in [12] and [16]. The recurrence coefficients of such operators are rather close to (1.5) so that spectral and asymptotic results for these two classes of operators are similar.

Probably, a study of Jacobi operators in the critical case was initiatiated by J. Dombrowsi and S. Pedersen in the papers [8,9] where spectral properties of such operators were investigated under sufficiently general assumptions on the coefficients a_n and b_n. Asymptotics of the orthogonal polynomials in this situation was studied by J. Janas, S. Naboko, and E. Sheronova in the pioneering paper [13]. They accepted conditions (1.8), (1.9) with $\sigma \in (1/2, 2/3)$, $\alpha = \beta = 0$ and studied equation (1.1) for real $z = \lambda$. Both oscillating for $\lambda > 0$ (if $\gamma = 1$) and exponentially growing (or decaying) for $\lambda < 0$ (if $\gamma = 1$) asymptotics of solutions of equation (1.1) were investigated in [13]. The results of this paper imply that positive spectra of the operators J are absolutely continuous and negative spectra are discrete. Recently, the results of [13] were generalized and supplemented in [17] by some ideas of [2] – see Remark 7.9 below.

We note also the paper [20] by J. Sahbani where interesting spectral results were obtained avoiding a study of asymptotics of the orthogonal polynomials. The paper [20] relies on the Mourre method.

In the non-critical case $|\gamma| \neq 1$, asymptotic formulas are qualitatively different for $\sigma \leq 1$ when the Carleman condition is satisfied and for $\sigma > 1$ when the Carleman condition fails. In the critical case, the borderline is $\sigma = 3/2$. The case of rapidly increasing coefficients where $\sigma > 3/2$ was studied in [29]. For such σ, the limit circle case is realized (if $\tau < 0$) and the corresponding Jacobi operators have discrete spectra.

Our goal is to consistently study the regular critical case where $|\gamma| = 1$ and $\sigma \leq 3/2$. Then the Jacobi operator $J_{\min}$ is essentially self-adjoint, even if the Carleman condition (1.2) fails. Its spectral properties turn out to be qualitatively different in the cases $\sigma \in (0, 1)$, $\sigma = 1$ and $\sigma \in (1, 3/2]$. Moreover, for $\sigma \in (1, 3/2]$ the answers depend crucially on the sign of the parameter τ defined by (1.10). In all cases, our asymptotic formulas are constructed in terms of the sequence

$$t_n(z) = -\tau n^{-1} + z n^{-\sigma}. \tag{1.11}$$

Note that the critical situation studied here is morally similar to a threshold behavior of orthogonal polynomials for case (1.6). For such coefficients, the role of (1.7) is played (see [15, 18, 28]) by the relation

$$\lim_{n\to\infty} \frac{b_n - \lambda}{2a_n} = -\frac{\lambda}{2a_\infty}.$$

Since the essential spectrum of the operator J is now $[-2a_\infty, 2a_\infty]$, the values $\lambda = \pm 2a_\infty$ are the threshold values of the spectral parameter λ. The parameter $-\lambda/(2a_\infty)$ plays the role of γ so that the cases $|\gamma| < 1$ (resp., $|\gamma| > 1$) correspond to λ lying inside the essential spectrum of J (resp., outside of it).

1.5. Scheme of the approach

We use the traditional approach developed for differential equations

$$-(a(x)f'(x,z))' + b(x)f(x,z) = zf(x,z), \quad x > 0, \quad a(x) > 0. \tag{1.12}$$

To a large extent, x, $a(x)$, and $b(x)$ in (1.12) play the roles of the parameters n, a_n, and b_n in the Jacobi equation (1.1). The regular solution $\psi(x,z)$ of the differential equation (1.12) is distinguished by the conditions

$$\psi(0,z) = 0, \quad \psi'(0,z) = 1.$$

It plays the role of the polynomial solution $P_n(z)$ of equation (1.1) fixed by the conditions $P_{-1}(z) = 0$, $P_0(z) = 1$.

A study of an asymptotics of the regular solution $\psi(x,z)$ relies on a construction of special solutions of the differential equation (1.12) distinguished by their asymptotics as $x \to \infty$. For example, in the case $a(x) = 1$, $b \in L^1(\mathbb{R}_+)$, equation (1.12) has a solution $f(x,z)$, known as the *Jost solution,* behaving like $e^{i\sqrt{z}x}$, $\operatorname{Im}\sqrt{z} \geq 0$, as $x \to \infty$. Under fairly general assumptions, equation (1.12) has a solution $f(x,z)$ (we also call it the *Jost solution*) whose asymptotics is given by the classical Liouville–Green formula (see Chapter 6 of the book [19])

$$f(x,z) \sim \mathcal{G}(x,z)^{-1/2} \exp\Big(i \int_{x_0}^{x} \mathcal{G}(y,z)dy\Big) =: \mathcal{A}(x,z) \tag{1.13}$$

as $x \to \infty$. Here x_0 is some fixed number and

$$\mathcal{G}(x,z) = \sqrt{\frac{z - b(x)}{a(x)}}, \quad \operatorname{Im}\mathcal{G}(x,z) \geq 0.$$

Note that the function $\mathcal{A}(x,z)$ (the Ansatz for the Jost solution $f(x,z)$) satisfies equation (1.12) with a sufficiently good accuracy.

For real λ in the absolutely continuous spectrum of the operator

$$-\frac{d}{dx}\Big(a(x)\frac{d}{dx}\Big) + b(x),$$

the regular solution $\psi(x,\lambda)$ of (1.12) is a linear combination of the Jost solutions $f(x,\lambda + i0)$ and $f(x,\lambda - i0)$ which yields asymptotics of $\psi(x,\lambda)$ as $x \to \infty$. For example, in the case $a(x) = 1$, $b \in L^1(\mathbb{R}_+)$ and $\lambda > 0$, one has

$$\psi(x,\lambda) = \kappa(\lambda)\sin(\sqrt{\lambda}x + \eta(\lambda)) + o(1), \quad x \to \infty,$$

where $\kappa(\lambda)$ and $\eta(\lambda)$ are known as the scattering (or limit) amplitude and phase, respectively. If $\operatorname{Im} z \neq 0$, then one additionally constructs, by an explicit formula, a solution $g(x,z)$ of (1.12) exponentially growing as $x \to \infty$. This yields asymptotics of $\psi(x,z)$ for $\operatorname{Im} z \neq 0$.

An analogy between the equations (1.1) and (1.12) is of course very well known. However it seems to be never consistently exploited before. In particular, the papers cited above use also specific methods of difference equations. For example, the absolute continuity of the spectrum is often deduced from the subordinacy theory, the asymptotics of the orthonormal polynomials are calculated by studying infinite products of transfer matrices, etc. Some of these tools are quite ingenious, but, in the author's opinion, the standard approach of differential equations works perfectly well and allows one to study an asymptotic behavior of orthonormal polynomials in a very direct way. It permits an arbitrary growth of the coefficients a_n and b_n (all values of σ in formulas (1.8), (1.9)) and naturally leads to a variety of new results, for example, to a construction of the resolvents of Jacobi operators and to the limiting absorption principle. For Jacobi operators with increasing coefficients, this approach was already used in the non-critical case $|\gamma| \neq 1$ in [31].

We are applying the same scheme to the regular critical case when conditions (1.8) and (1.9) are satisfied with $\sigma \le 3/2$ and $|\gamma| = 1$ in (1.9). Under these assumptions the limit point case occurs although for $\sigma > 1$ the Carleman condition (1.2) is violated.

Let us briefly describe the main steps of our approach. In the non-critical case $|\gamma| \neq 1$, it was presented in [31].

(A) First, we distinguish solutions (the Jost solutions) $f_n(z)$ of the difference equation (1.1) by their asymptotics as $n \to \infty$. This requires a construction of an Ansatz $\mathcal{A}_n(z)$ for the Jost solutions such that the relative remainder

$$\mathbf{r}_n(z) := \big(\sqrt{a_{n-1}a_n}\mathcal{A}_n(z)\big)^{-1}\big(a_{n-1}\mathcal{A}_{n-1}(z) + (b_n - z)\mathcal{A}_n(z) + a_n\mathcal{A}_{n+1}(z)\big) \tag{1.14}$$

belongs at least to the space $\ell^1(\mathbb{Z}_+)$.

(B) We seek $\mathcal{A}_n(z)$ in the form

$$\mathcal{A}_n(z) = (-\gamma)^n n^{-\rho} e^{i\varphi_n(\gamma z)}, \quad \gamma = \pm 1, \tag{1.15}$$

where the power ρ in the amplitude and the phases φ_n are determined by the coefficients a_n, b_n. *Post factum,* $\mathcal{A}_n(z)$ turns out to be the leading term of the asymptotics of $f_n(z)$ as $n \to \infty$:

$$f_n(z) = \mathcal{A}_n(z)(1 + o(1)). \tag{1.16}$$

Actually, the Ansätzen we use are only distantly similar to the Liouville–Green Ansatz (1.13). On the other hand, for $\sigma = 1$, relation (1.15) is close to formulas of the Birkhoff–Adams method significantly polished in [26] (see also [10, Theorem 8.36]).

(C) Then we make a multiplicative change of variables

$$f_n(z) = \mathcal{A}_n(z)u_n(z) \tag{1.17}$$

which permits us to reduce the Jacobi equation (1.1) for $f_n(z)$ to a Volterra "integral" equation for the sequence $u_n(z)$. This equation depends of course on the parameters a_n, b_n. In particular, for $\sigma > 1$, it is qualitatively different in the cases $\tau < 0$ and $\tau > 0$. However in all cases the Volterra equation for $u_n(z)$ is standardly solved by iterations which allows us to prove that it has a solution such that $u_n(z) \to 1$ as $n \to \infty$. Then the Jost solutions $f_n(z)$ are defined by formula (1.17).

(D) To find an asymptotics of all solutions of the Jacobi equation (1.1) and, in particular, of the orthonormal polynomials $P_n(z)$, we have to construct a solution linearly independent with $f_n(z)$. If a real $z = \lambda$ belongs to the absolutely continuous spectrum of the operator J, then the solutions $f_n(\lambda + i0)$ and its complex conjugate $f_n(\lambda - i0)$ are linearly independent. For regular points z, a solution $g_n(z)$ of (1.1) linearly independent with $f_n(z)$ is constructed (see, e.g., Theorem 2.2 in [31]) by an explicit formula

$$g_n(z) = f_n(z) \sum_{m=n_0}^{n} (a_{m-1} f_{m-1}(z) f_m(z))^{-1}, \quad n \geq n_0, \tag{1.18}$$

where $n_0 = n_0(z)$ is a sufficiently large number. It follows from (1.15), (1.16) that this solution grows exponentially (for $\sigma < 3/2$) as $n \to \infty$:

$$g_n(z) = i\varkappa(z)(-\gamma)^{n+1} n^{-\rho} e^{-i\varphi_n(\gamma z)}(1 + o(1)); \tag{1.19}$$

the factor $\varkappa(z)$ here is given by equality (2.13), but it is inessential in (1.19). Since $g_n(z)$ is linearly independent with $f_n(z)$, the polynomials $P_n(z)$ are linear combinations of $f_n(z)$ and $g_n(z)$ which yields asymptotics of $P_n(z)$.

(E) Our results on the Jost solutions $f_n(z)$ allow us to determine the spectral structure of the operator J and to construct its resolvent $R(z)$. At the same time, we obtain the limiting absorption principle for the operator J stating that matrix elements of its resolvent $R(z)$, that is the scalar products $\langle R(z)u, v\rangle$, $\operatorname{Im} z \neq 0$, are continuous functions of z up to the absolutely continuous spectrum of the operator J if elements u and v belong to a suitable dense subset of $\ell^2(\mathbb{Z}_+)$.

All these steps, except possibly the construction of the exponentially growing solution $g_n(z)$, are rather standard. No more specific tools are required in the problem considered.

Actually, the scheme described above works virtually in all asymptotic problems in the limit point case, both for difference and differential operators. In the limit circle case, some modifications are required; see [27, 29]. The important differences are that, in the limit circle case, one has two natural Ansätzen $\mathcal{A}_n^{(\pm)} = n^{-\rho} e^{\pm i\varphi_n}$ where $\varphi_n = \bar{\varphi}_n$ does not depend on the spectral parameter $z \in \mathbb{C}$ and $\rho > 1/2$ so that $\mathcal{A}_n^{(\pm)} \in \ell^2(\mathbb{Z}_+)$.

To emphasize the analogy between differential and difference equations, we often use the "continuous" terminology (Volterra integral equations, integration by parts, etc.) for sequences labeled by the discrete variable n.

Our plan is the following. The main results of the paper are stated in Section 2. In Section 3, we define the number ρ and the phases φ_n in formula (1.15) for the Ansatz $\mathcal{A}_n(z)$ and check an estimate

$$\mathbf{r}_n(z) = O(n^{-\delta}), \quad n \to \infty, \tag{1.20}$$

with an appropriate $\delta = \delta(\rho) > 1$ for remainder (1.14). A Volterra integral equation for $u_n(z)$ is introduced and investigated in Section 4. This leads to a construction of the Jost solutions $f_n(z)$ in Section 5. In this section, the proofs of Theorems 2.1, 2.3, and 2.4 are concluded. Asymptotics of the orthonormal polynomials $P_n(z)$ are found in Section 6. The results for regular points z and for z in the absolutely continuous spectrum of the Jacobi operator J are stated in Theorems 6.6 and 6.11, respectively. The results on spectral properties of the Jacobi operators are collected in Theorem 2.11. Its proof is given in Section 7.

2. Main results

Our goal is to study the critical case when assumptions (1.8) and (1.9) are satisfied with $|\gamma| = 1$. In proofs, we may suppose that $\gamma = 1$. The results for $\gamma = -1$ then follow from Proposition 1.1.

The results stated below crucially depend on the values of σ and τ. In the cases $\sigma \in (1, 3/2]$ ($\sigma \in (0, 1)$) the first (resp., the second) term in (1.11) is dominating so that the asymptotic formulas are qualitatively different in these cases.

2.1. Jost solutions

Our approach relies on a study of solutions $f_n(z)$ of the Jacobi equation (1.1) distinguished by their behavior for $n \to \infty$. Actually, we determine the sequences $f_n(z)$ by their asymptotics

$$f_n(z) = (-\gamma)^n n^{-\rho} e^{i\varphi_n(\gamma z)}(1 + o(1)), \quad n \to \infty. \tag{2.1}$$

Here

$$\rho = \begin{cases} \sigma/2 - 1/4 & \text{for } \sigma \geq 1, \\ \sigma/4 & \text{for } \sigma \leq 1 \end{cases} \tag{2.2}$$

(observe that ρ takes the critical value $\rho = 1/2$ for the critical value $\sigma = 3/2$) and

$$\varphi_n(z) = \sum_{m=0}^{n} \theta_m(z). \tag{2.3}$$

The terms $\theta_n(z)$ will be defined by explicit formulas below in this section. Note that

$$\operatorname{Im} \theta_n(z) \geq 0. \tag{2.4}$$

By an analogy with differential equations, it is natural to use the term "Jost solutions" for $f_n(z)$. In the situation we consider, formula (2.1) plays the role of the Liouville–Green formula (1.13). Observe that, for an arbitrary constant $C(z)$, the sequence $C(z) f_n(z)$ can be also taken for the Jost solution. In particular, a finite number of terms in equality (2.3) is inessential.

We denote $\Pi = \mathbb{C} \setminus \mathbb{R}$ and $\Pi_0 = \mathbb{C} \setminus \mathbb{R}_+$. The sequence $t_n(z)$ is given by formula (1.11) where τ is number (1.10). The analytic function $\sqrt{t}$ is defined on Π_0 and $\operatorname{Im} \sqrt{t} > 0$ for $t \in \Pi_0$. Below C, sometimes with indices, and c are different positive constants whose precise values are of no importance.

We state the results about the Jost solutions $f_n(z)$ separately for the cases $\sigma \in (1, 3/2]$, $\sigma \in (0, 1)$ and $\sigma = 1$. Let us start with the case $\sigma > 1$.

Theorem 2.1. *Let assumptions (1.8), (1.9) with $|\gamma| = 1$ and $\sigma \in (1, 3/2]$ be satisfied. Set $\rho = \sigma/2 - 1/4$,*

$$\theta_n(z) = \sqrt{t_n(z)} \tag{2.5}$$

and let $\varphi_n(z)$ be sum (2.3).

If $\tau < 0$, then for every $z \in \operatorname{clos} \Pi$ equation (1.1) has a solution $f_n(z)$ with asymptotics (2.1). For all $n \in \mathbb{Z}_+$, the functions $f_n(z)$ are analytic in Π and are continuous up to the cut along the real axis.

If $\tau > 0$, then asymptotic formula (2.1) is true for all $z \in \mathbb{C}$. In this case the functions $f_n(z)$ are analytic in the whole complex plane $\mathbb{C}$.

For all $\tau \neq 0$, formula (2.1) is uniform in z from compact subsets of $\mathbb{C}$.

We emphasize that the asymptotic behavior of the solutions $f_n(z)$ as $n \to \infty$ is drastically different for small diagonal elements b_n when $\tau < 0$ and for large b_n when $\tau > 0$ – cf. formulas (2.17) and (2.18), below. This manifests itself in spectral properties of the corresponding Jacobi operators J – see Theorem 2.11 (1).

Remark 2.2. Formula (2.1) is true for all $\sigma > 3/2$, but in this case it can be simplified by setting $z = 0$ in the right-hand side of (2.1). Thus, the leading term of the asymptotics of $f_n(z)$ does not depend on $z \in \mathbb{C}$ and the power $\rho > 1/2$ so that $f_n(z) \in \ell^2(\mathbb{Z}_+)$. This leads to important spectral consequences: for $\sigma > 3/2$ the deficiency indices of the minimal Jacobi operator $J_{\min}$ are $(1, 1)$, and the spectra of all its self-adjoint extensions are discrete. The case $\sigma > 3/2$ was investigated in [29].

Let us pass to the case $\sigma < 1$. The phases $\theta_n(z)$ are again defined by formula (2.5) for $\sigma > 2/3$, but their construction is more complicated for $\sigma \leq 2/3$. Let us set

$$T_n(z) = t_n(z) + \sum_{l=2}^{L} p_l t_n^l(z) \tag{2.6}$$

where a sufficiently large L depends on σ and the real numbers p_l are defined in Lemma 3.5. In particular, $T_n(z) = t_n(z)$ for $\sigma > 2/3$. Given $T_n(z)$, the phases $\theta_n(z)$ are defined by the formula

$$\theta_n(z) = \sqrt{T_n(z)} \tag{2.7}$$

playing the role of (2.5). It is easy to show (see Remark 3.7, for details) that $T_n(z) \in \Pi_0$; thus, $\theta_n(z)$ are correctly defined.

Theorem 2.3. *Let assumptions* (1.8), (1.9) *with* $|\gamma| = 1$ *and* $\sigma \in (0, 1)$ *be satisfied. Set* $\rho = \sigma/4$ *and define the functions* $\theta_n(z)$ *by formulas* (2.6), (2.7). *Let* $\varphi_n(z)$ *be sum* (2.3). *Then for every* $z \neq 0$ *such that* $z \in \gamma \operatorname{clos} \Pi_0$, *equation* (1.1) *has a solution* $f_n(z)$ *with asymptotics* (2.1). *For all* $n \in \mathbb{Z}_+$, *the functions* $f_n(z)$ *are analytic in* $z \in \gamma\Pi_0$ *and are continuous up to the cut along the half-axis* $\gamma\mathbb{R}_+$, *with a possible exception of the boundary point* $z = 0$.

In the intermediary case $\sigma = 1$, the definition of the phases $\theta_n(z)$ is particularly explicit and the construction of the Jost solutions is simpler than for $\sigma \neq 1$.

Theorem 2.4. *Let assumptions* (1.8), (1.9) *with* $|\gamma| = 1$ *and* $\sigma = 1$ *be satisfied. Set* $\rho = 1/4$, *define the functions* $\theta_n(z)$ *by the formula*

$$\theta_n(z) = \sqrt{-\tau + \gamma z} n^{-1/2},$$

and let $\varphi_n(z)$ *be sum* (2.3). *Then for every* z *such that* $z \in \gamma(\tau + \operatorname{clos} \Pi_0)$, $z \neq \gamma\tau$, *equation* (1.1) *has a solution* $f_n(z)$ *with asymptotics* (2.1). *For all* $n \in \mathbb{Z}_+$, *the functions* $f_n(z)$ *are analytic in* $z \in \gamma(\tau + \Pi_0)$ *and are continuous up to the cut along the half-axis* $\gamma(\tau + \mathbb{R}_+)$, *with a possible exception of the boundary point* $\gamma\tau$.

We emphasize that in the case $\sigma \leq 1$ the condition $\tau \neq 0$ is not required.

It is convenient to introduce a notation

$$\mathcal{S} = \begin{cases} \mathbb{R} & \text{if } \sigma \in (1, 3/2],\ \tau < 0, \\ \emptyset & \text{if } \sigma \in (1, 3/2],\ \tau > 0, \\ \gamma(0, \infty) & \text{if } \sigma \in (0, 1), \\ \gamma(\tau, \infty) & \text{if } \sigma = 1. \end{cases} \tag{2.8}$$

We will see in Section 2.4 that the spectrum of the operator J is absolutely continuous on the closed interval $\operatorname{clos} \mathcal{S}$, and it may be only discrete on $\mathbb{R} \setminus \operatorname{clos} \mathcal{S}$. Note that

Theorems 2.1, 2.3 and 2.4 give asymptotic formulas for the Jost solutions $f_n(z)$ for all z in the complex plane with the cut along $\mathcal{S}$, except the thresholds in the absolutely continuous spectrum ($z = 0$ if $\sigma \in (0,1)$ and $z = \gamma\tau$ if $\sigma = 1$). For $\lambda \in \mathcal{S}$, equation (1.1) has two linearly independent solutions $f_n(\lambda + i0)$ and its complex conjugate

$$f_n(\lambda - i0) = \overline{f_n(\lambda + i0)}.$$

Under the assumptions of any of these theorems the solution $f_n(z)$ of equation (1.1) is determined essentially uniquely by its asymptotics (2.1). This is discussed in Section 5.5 (see Propositions 5.9, 5.11, and Remark 5.10).

Note that the values of u_{m-1} and u_m for some $m \in \mathbb{Z}_+$ determine the whole sequence u_n satisfying the difference equation (1.1). Therefore, it suffices to construct sequences $f_n(z)$ for sufficiently large n only. Then they are extended to all n as solutions of equation (1.1).

We also mention that $f_n^\sharp(z) = (-1)^n f_n(-z)$ is the Jost solution for the Jacobi equation (1.1) with the coefficients $(a_n^\sharp, b_n^\sharp) = (a_n, -b_n)$.

2.2. Asymptotics at infinity

Here we find explicit asymptotic formulas for the phases $\theta_n(z)$ and then for their sums $\varphi_n(z)$ as $n \to \infty$. These formulas depend crucially on the values of the parameters σ and τ.

Suppose first that $\sigma \in (1, 3/2]$ and that $z \in \operatorname{clos}\Pi$ for $\tau < 0$ and $z \in \mathbb{C}$ for $\tau > 0$. Then the term $-\tau n^{-1}$ is dominating in (1.11) so that according to definition (2.5)

$$\theta_n(z) = n^{-1/2}\sqrt{|\tau| + zn^{1-\sigma}} = \pm\sqrt{|\tau|}n^{-1/2} \pm \frac{z}{2\sqrt{|\tau|}}n^{1/2-\sigma} + O(n^{3/2-2\sigma}) \quad (2.9)$$

for $\pm \operatorname{Im} z \geq 0$ if $\tau < 0$ and

$$\theta_n(z) = in^{-1/2}\sqrt{\tau - zn^{1-\sigma}} = i\sqrt{\tau}n^{-1/2} - i\frac{z}{2\sqrt{\tau}}n^{1/2-\sigma} + O(n^{3/2-2\sigma}) \quad (2.10)$$

for all $z \in \mathbb{C}$ if $\tau > 0$.

In the case $\sigma < 1$, the term $zn^{-\sigma}$ is dominating in (1.11). Moreover, for $\sigma \leq 2/3$, the phases $\theta_n(z)$ are given by formula (2.7) more general than (2.5). The last circumstance is however inessential because the terms t_n^l with $l > 1$ in (2.6) are negligible compared to t_n. This yields an asymptotics

$$\theta_n(z) = \sqrt{z}n^{-\sigma/2}(1 + O(n^{-\epsilon})), \quad \epsilon > 0. \quad (2.11)$$

In particular, these results imply the following assertion.

Proposition 2.5. *Set*

$$\nu = \begin{cases} 1/2 & \text{if } \sigma \geq 1, \\ \sigma/2 & \text{if } \sigma \leq 1, \end{cases} \tag{2.12}$$

and

$$\varkappa(z) = \begin{cases} \pm\sqrt{|\tau|} & \text{if } \sigma > 1,\ \tau < 0,\ \pm \operatorname{Im} z \geq 0, \\ i\sqrt{\tau} & \text{if } \sigma > 1,\ \tau > 0,\ z \in \mathbb{C}, \\ \sqrt{z} & \text{if } \sigma < 1,\ z \in \operatorname{clos} \Pi_0,\ z \neq 0, \\ \sqrt{z-\tau} & \text{if } \sigma = 1,\ z \in \tau + \operatorname{clos} \Pi_0,\ z \neq \tau. \end{cases} \tag{2.13}$$

Then

$$\theta_n(z) = \varkappa(z) n^{-\nu}(1 + o(1)). \tag{2.14}$$

To pass to asymptotics of sums (2.3), we use the Euler–Maclaurin formula

$$\sum_{m=1}^{n} F(m) = \int_1^n F(x)dx + \frac{F(n) + F(1)}{2} + \int_1^n F'(x)\Big(x - [x] - \frac{1}{2}\Big)dx, \tag{2.15}$$

where $[x]$ is the integer part of x. This formula is true for arbitrary functions $F \in C^1$.

Formula (2.15) allows one to deduce an asymptotics as $n \to \infty$ of sum (2.3) from that of the phases θ_n. For example, for $\sigma = 1$, we apply (2.15) to $F(x) = x^{-1/2}$ which yields

$$\varphi_n(z) = 2\sqrt{-\tau + z}\, n^{1/2} + C + o(1) \tag{2.16}$$

with some constant C. The remainder $C + o(1)$ here can be neglected in asymptotics (2.1) because the Jost solutions are defined up to a constant factor.

Next, we consider the case $\sigma \in (1, 3/2)$. If $\tau < 0$, it follows from (2.9) and the Euler–Maclaurin formula (2.15) that

$$\varphi_n(z) = \pm 2\sqrt{|\tau|} n \pm \frac{z}{\sqrt{|\tau|}(3 - 2\sigma)} n^{3/2-\sigma} + O(n^{5/2-2\sigma}) \quad \text{for } \pm \operatorname{Im} z \geq 0. \tag{2.17}$$

So, up to error terms, the functions $e^{i\varphi_n(z)}$ where $z = \lambda + i\varepsilon$ contain oscillating

$$\exp\Big(\pm 2i\sqrt{|\tau|} n \pm \frac{i\lambda}{\sqrt{|\tau|}(3 - 2\sigma)} n^{3/2-\sigma}\Big)$$

and exponentially decaying[2]

$$\exp\Big(-\frac{|\varepsilon|}{\sqrt{|\tau|}(3 - 2\sigma)} n^{3/2-\sigma}\Big)$$

[2] We say that a sequence x_n *tends to zero exponentially* if $x_n = O(e^{-n^a})$ for some $a > 0$.

factors. Note that the strongly oscillating factor $\exp(\pm 2i\sqrt{|\tau| n})$ in the asymptotics of $f_n(z)$ as $n \to \infty$ does not depend on z. In the case $\tau > 0$, we have

$$\varphi_n(z) = 2i\sqrt{\tau n} - \frac{iz}{\sqrt{\tau}(3-2\sigma)} n^{3/2-\sigma} + O(n^{5/2-2\sigma}) \quad \text{if } \sigma \in (1, 3/2). \tag{2.18}$$

Thus, the Jost solutions $f_n(z)$ contain an exponentially decaying factor $e^{-2\sqrt{\tau n}}$ for all $z \in \mathbb{C}$.

Formulas (2.17) and (2.18) remain true also for $\sigma = 3/2$ if $(3-2\sigma)^{-1} n^{3/2-\sigma}$ is replaced by $\ln n$. For example, for $\tau < 0$, we have

$$\varphi_n(z) = \pm 2\sqrt{|\tau| n} \pm \frac{z}{\sqrt{|\tau|}} \ln n + C + o(1) \quad \text{for } \pm \operatorname{Im} z \geq 0. \tag{2.19}$$

In the case $\sigma < 1$, asymptotics of the phases is given by relation (2.11). Therefore, using formula (2.15), we find that

$$\varphi_n(z) = 2\sqrt{z}(2-\sigma)^{-1} n^{1-\sigma/2} + O(n^{\sigma/2}). \tag{2.20}$$

So, $e^{i\varphi_n(z)}$ exponentially decays if $z \notin [0, \infty)$ and oscillates if $z = \lambda \pm i0$ for $\lambda > 0$. In the case $\sigma = 1$, relation (2.20) is true if $\sqrt{z}$ is replaced by $\sqrt{-\tau + z}$.

Note that explicit formulas for $\theta_n(z)$ allow one to find all power terms of asymptotic expansion of $\theta_n(z)$ as $n \to \infty$. In view of formula (2.15) this yields all growing terms of the phases $\varphi_n(z)$ as $n \to \infty$.

It follows from asymptotic formula (2.1) for the Jost solutions $f_n(z)$ and the results about the phases $\varphi_n(z)$ stated above that for all $\sigma \in (0, 3/2)$ (for $\sigma > 1$ it is also required that $\tau \neq 0$) and $\operatorname{Im} z \neq 0$, $f_n(z)$ tend to zero exponentially as $n \to \infty$. In the critical case $\sigma = 3/2$, the same is true if $\tau > 0$. If $\sigma = 3/2$ and $\tau < 0$, then relations (2.1), (2.19) show that

$$f_n(\lambda + i\varepsilon) = (-\gamma)^n e^{\pm 2i\sqrt{|\tau| n}} n^{\pm i\gamma\lambda_1} n^{-1/2-\varepsilon_1}(1 + o(1)) \quad \text{for } \pm \gamma\varepsilon > 0, \tag{2.21}$$

where $\lambda_1 = \lambda/\sqrt{|\tau|}$, $\varepsilon_1 = |\varepsilon|/\sqrt{|\tau|}$.

In particular, we have the following.

Proposition 2.6. *Under the assumptions of any of Theorems 2.1, 2.3, or 2.4 the inclusion*

$$f_n(z) \in \ell^2(\mathbb{Z}_+), \quad z \notin \operatorname{clos} \mathcal{S}, \tag{2.22}$$

holds. In particular, (2.22) is true for $\operatorname{Im} z \neq 0$.

Let us compare relation (2.21) with asymptotic formula (2.6) in [29] for the singular case $\sigma > 3/2$, $\tau < 0$. The formula in [29] is true for all $z \in \mathbb{C}$, the oscillating factor $e^{\pm 2i\sqrt{|\tau| n}}$ is the same as in (2.21), but the power of n is $1/4 - \sigma/2$. This coincides with expression (2.2), but $1/4 - \sigma/2 < -1/2$ for $\sigma > 3/2$. In this case all solutions of equation (1.1) are in $\ell^2(\mathbb{Z}_+)$ so that the deficiency indices of the operator $J_{\min}$ are $(1, 1)$.

Finally, we note that, on the absolutely continuous spectrum, formula (2.1) is consistent with a universal relation found in [30]. Indeed, let the assumptions of Theorems 2.1, 2.3, or 2.4 be satisfied. Using asymptotic formulas (2.16), (2.17) or (2.20) and calculating derivatives of the phases $\varphi_n(\lambda \pm i0)$ in λ we see that, with some constant factor $c_\pm(\lambda)$,

$$d\varphi_n(\lambda \pm i0)/d\lambda = c_\pm(\lambda) n^\varsigma (1 + o(1)) \tag{2.23}$$

where $\varsigma = 3/2 - \sigma$ for $\sigma \in [1, 3/2)$ and $\varsigma = 1 - \sigma/2$ for $\sigma \le 1$; if $\sigma = 3/2$, then n^ς in (2.23) should be replaced by $\ln n$. In view of definition (2.2), in all cases the powers of n in the amplitude and phase in formula (2.1) are linked by the equality

$$2\rho + \varsigma = 1. \tag{2.24}$$

This is one of the relations found in [30]; in the case $\sigma = 3/2$, this relation reduces to the equality $\rho = 1/2$.

For a comparison, we mention that, in the non-critical case $|\gamma| < 1$, we have $\rho = \sigma/2$ and $\varsigma = 1 - \sigma$ (see [31]) which is again consistent with equality (2.24).

2.3. Exponentially growing solutions

For regular points $z \in \mathbb{C}$, the solution $g_n(z)$ of equation (1.1) linearly independent with $f_n(z)$ is constructed by formula (1.18). Using the asymptotic formulas of Section 2.1 for the Jost solutions, we find a behavior of $g_n(z)$ as $n \to \infty$.

Theorem 2.7. *Let one of the following three assumptions be satisfied.*

1. *The conditions of Theorem 2.1 where either $\tau < 0$, $\sigma < 3/2$ and $\operatorname{Im} z \neq 0$ or $\tau > 0$ and $z \in \mathbb{C}$ is arbitrary.*
2. *The conditions of Theorem 2.3 where either $\gamma = 1$ and $z \notin [0, \infty)$ or $\gamma = -1$ and $z \notin (-\infty, 0]$.*
3. *the conditions of Theorem 2.4 where either $\gamma = 1$ and $z \notin [\tau, \infty)$ or $\gamma = -1$ and $z \notin (-\infty, -\tau]$*

Then the asymptotics of the solution $g_n(z)$ of equation (1.1) is given by formula (1.19). In particular,

$$g_n(z) \notin \ell^2(\mathbb{Z}_+) \quad \text{if} \quad z \notin \operatorname{clos} \mathcal{S}. \tag{2.25}$$

We emphasize that the definitions of the numbers ρ and of the sequences $\varphi_n(z)$ are different under assumptions (1), (2), and (3), but asymptotic formula (1.19) is true in all these cases.

In the critical case $\sigma = 3/2$ (and $\tau < 0$) the solution $g_n(z)$ of equation (1.1) behaves as a power of n as $n \to \infty$.

Proposition 2.8. *If $\sigma = 3/2$, $\tau < 0$ and $\pm\gamma\varepsilon > 0$, then*

$$g_n(\lambda + i\varepsilon) = (-\gamma)^n e^{\pm 2i\sqrt{|\tau| n}} n^{\pm i\gamma\lambda_1} n^{-1/2+\varepsilon_1}(1 + o(1)) \tag{2.26}$$

where $\lambda_1 = \lambda/\sqrt{|\tau|}$, $\varepsilon_1 = |\varepsilon|/\sqrt{|\tau|}$. In particular, relation (2.25) is preserved.

Theorem 2.7 and Proposition 2.8 will be proven in Section 6.1.

All solutions of equation (1.1) and, in particular, the orthonormal polynomials $P_n(z)$, are linear combinations of the solutions $f_n(z)$ and $g_n(z)$ for $z \notin \operatorname{clos} \mathcal{S}$ or of the solutions $f_n(\lambda + i0)$ and $f_n(\lambda - i0)$ for $z = \lambda \in \mathcal{S}$. Therefore, the results stated above yield an asymptotics of $P_n(z)$ as $n \to \infty$. This is discussed in Section 6 – see Theorem 6.6 and 6.11.

2.4. Spectral results

First, we discuss the essential self-adjointness of the minimal operator $J_{\min}$. According to the limit point/circle theory this is equivalent to the existence of solutions of equation (1.1) where $\operatorname{Im} z \neq 0$ not belonging to $\ell^2(\mathbb{Z}_+)$. Therefore, the following result is a direct consequence of Theorem 2.7 and Proposition 2.8.

Proposition 2.9. *Let assumptions (1.8), (1.9) with $|\gamma| = 1$ and some $\sigma \in (0, 3/2]$ be satisfied; for $\sigma > 1$ we additionally suppose that $\tau \neq 0$. Then the minimal operator $J_{\min}$ is essentially self-adjoint.*

Of course, for $\sigma \leq 1$ one can refer to the Carleman condition (1.2), but for $\sigma > 1$ the series in (1.2) is convergent.

The case $\sigma > 3/2$ was investigated in [29, Theorem 2.3]. According to [29, Theorem 2.3 2^0], for $\tau > 0$, the operator $J_{\min}$ remains essentially self-adjoint. The results for the case $\tau < 0$ are more interesting. Combining Proposition 2.9 with [29, Theorem 2.3 1^0], we can state the following result.

Proposition 2.10. *Suppose that assumptions (1.8), (1.9) with $|\gamma| = 1$, $\tau < 0$ and some $\sigma > 0$ are satisfied. Then the minimal operator $J_{\min}$ is essentially self-adjoint if and only if $\sigma \leq 3/2$.*

Note that Proposition 2.10 does not contradict Theorem 2.1 of [9] because the assumptions of [9] correspond to the case $\tau = 0$.

Below we always suppose that $\sigma \leq 3/2$ and denote by $J = \operatorname{clos} J_{\min}$ the closure of the essentially self-adjoint operator $J_{\min}$.

Spectral properties of Jacobi operators are determined by a behavior of solutions of equation (1.1) for real $z = \lambda$. In particular, oscillating solutions correspond to the absolutely continuous spectrum. On the contrary, for regular λ or eigenvalues of J, one solution of (1.1) exponentially decays and another one exponentially grows. On the heuristic level, the results of Section 2.1 imply that the absolutely continuous spectrum of a Jacobi operator J consists of λ where $-\tau n^{-1} + \gamma\lambda n^{-\sigma} \geq 0$ (for large n). On the contrary, the points λ where $-\tau n^{-1} + \gamma\lambda n^{-\sigma} < 0$ (again, for large n) are regular or, eventually, are eigenvalues of J. This intuitive picture turns out to be correct.

Theorem 2.11. *Suppose that assumptions* (1.8), (1.9) *with* $|\gamma| = 1$ *are satisfied.*

(1) *Let* $\sigma \in (1, 3/2]$. *If* $\tau < 0$, *then the spectrum of the operator* J *is absolutely continuous and covers the whole real line. If* $\tau > 0$, *then the spectrum of the operator* J *is discrete.*

(2) *Let* $\sigma \in (0, 1)$. *If* $\gamma = 1$, *then the absolutely continuous spectrum of the operator* J *coincides with the half-axis* $[0, \infty)$ *and its negative spectrum of* J *is discrete. If* $\gamma = -1$, *then the absolutely continuous spectrum of the operator* J *coincides with the half-axis* $(-\infty, 0]$ *and its positive spectrum is discrete.*

(3) *Let* $\sigma = 1$. *If* $\gamma = 1$, *then the absolutely continuous spectrum of the operator* J *coincides with the half-axis* $[\tau, \infty)$ *and its spectrum below the point* τ *is discrete. If* $\gamma = -1$, *then the absolutely continuous spectrum of the operator* J *coincides with the half-axis* $(-\infty, -\tau]$ *and its spectrum above the point* $-\tau$ *is discrete.*

Parts (2) and (3) of Theorem 2.11 can be considered as generalizations of the classical results about the Jacobi operators with the Laguerre coefficients (1.5). We emphasize that, in the case (1), the are no conditions on the parameter τ. The results of part (1) seem to be of a new nature.

The results stated above apply to Jacobi operators with the coefficients a_n, b_n growing as n^σ where σ is an arbitrary number in the interval $(0, 3/2]$. Together with the results of [29] where the case $\sigma > 3/2$ was considered, they cover an arbitrary power growth of the Jacobi coefficients.

Thus, our results show that, in the critical case $|\gamma| = 1$, there are two "phase transitions": for $\sigma = 1$ and for $\sigma = 3/2$. Indeed, the absolutely continuous spectrum of the Jacobi operator J coincides with a half-axis for $\sigma \leq 1$. In the case $\sigma \in (1, 3/2]$, the spectrum of J is either absolutely continuous and covers the whole real-axis for $\tau < 0$ or it is discrete for $\tau > 0$. If $\sigma > 3/2$, then the minimal Jacobi operator $J_{\min}$ has deficiency indices $(1, 1)$ and the spectra of all its self-adjoint extensions are discrete.

Our spectral results can be summarized in the following table where $\mathbf{\Sigma}_{\rm ac}$ and $\mathbf{\Sigma}_{\rm ess}$ are the absolutely continuous and essential spectra of the operator J. For definiteness, we choose $\gamma = 1$:

$$\begin{aligned}
&\sigma \in (0,1) &&\Longrightarrow \boldsymbol{\Sigma}_{\rm ac} = \boldsymbol{\Sigma}_{\rm ess} = [0,\infty),\\
&\sigma = 1 &&\Longrightarrow \boldsymbol{\Sigma}_{\rm ac} = \boldsymbol{\Sigma}_{\rm ess} = [\tau,\infty),\\
&\sigma \in [1,3/2],\ \tau < 0 &&\Longrightarrow \boldsymbol{\Sigma}_{\rm ac} = \mathbb{R},\\
&\sigma \in [1,3/2],\ \tau > 0 &&\Longrightarrow \boldsymbol{\Sigma}_{\rm ess} = \emptyset,\\
&\sigma > 3/2 &&\Longrightarrow \boldsymbol{\Sigma}_{\rm ess} = \emptyset.
\end{aligned}$$

3. Ansatz

As usual, we suppose that the recurrence coefficients a_n, b_n obey conditions (1.8), (1.9) with $|\gamma| = 1$. We define the Ansatz $\mathcal{A}_n = \mathcal{A}_n(z)$ by formula (1.15) where the power ρ and the phases $\varphi_n = \varphi_n(\gamma z)$ will be found in this section.

3.1. Construction

Our goal here is to determine ρ and φ_n in such a way that remainder (1.14) satisfies condition (1.20) for

$$\delta = 1/2 + \sigma \quad \text{if } \sigma > 1 \quad \text{and some}^{3} \quad \delta > 1 + \sigma/2 \quad \text{if } \sigma < 1. \tag{3.1}$$

If $\sigma = 1$, then $\delta = 2$, so that the estimate of the remainder is more precise in this particular case. We emphasize that estimate (1.20) with $\delta > 1$ used in the non-critical case $|\gamma| \neq 1$ in [31] is not sufficient now.

Put

$$\mathcal{B}_n = \frac{\mathcal{A}_{n+1}}{\mathcal{A}_n}. \tag{3.2}$$

Then expression (1.14) for the remainder can be rewritten as

$$\mathbf{r}_n(z) = \sqrt{\frac{a_{n-1}}{a_n}}\mathcal{B}_{n-1}^{-1} + \sqrt{\frac{a_n}{a_{n-1}}}\mathcal{B}_n + 2\gamma_n - \frac{z}{\sqrt{a_{n-1}a_n}}. \tag{3.3}$$

Assumption (1.8) on a_n implies that

$$\sqrt{\frac{a_n}{a_{n-1}}} = (n+1)^{\sigma/2} n^{-\sigma/2}(1 + O(n^{-2})) = 1 + \frac{\sigma/2}{n} + O(n^{-2}) \tag{3.4}$$

and

$$(a_n a_{n-1})^{-1/2} = n^{-\sigma}(1 + O(n^{-1})).$$

[3] The precise value of $\delta = \delta(\sigma)$ for $\sigma \in (2/3, 1)$ is indicated in Propositions 3.4. For $\sigma \le 2/3$, it can be deduced from the proof of Proposition 3.6, but we do not need it.

Using also assumption (1.9) on b_n, we see that sequence (1.7) satisfies a relation

$$\gamma_n = 1 + (\tau/2)n^{-1} + O(n^{-2}), \tag{3.5}$$

where τ is defined by equality (1.10).

We seek $\mathcal{A}_n$ in form (1.15) where the phases φ_n are defined as sums (2.3). The power ρ and the differences

$$\theta_n = \varphi_{n+1} - \varphi_n$$

will be determined by condition (1.20). The sequences θ_n constructed below tend to zero as $n \to \infty$ and satisfy condition (2.4). It follows from (1.15) and (3.2) that

$$\mathcal{B}_n = -(n+1)^{-\rho} n^{\rho} e^{i\theta_n} = -(1 - \rho n^{-1} + O(n^{-2}))e^{i\theta_n}. \tag{3.6}$$

According to relations (3.4), (3.5), and (3.6) the following intermediary assertion is a direct consequence of expression (3.3).

Lemma 3.1. *Relative remainder* (1.14) *admits a representation*

$$\mathbf{r}_n = -(1-(\nu/2)n^{-1})e^{-i\theta_{n-1}} - (1+(\nu/2)n^{-1})e^{i\theta_n} + 2 + \tau n^{-1} - z n^{-\sigma} + O(n^{-\delta}) \tag{3.7}$$

where

$$\nu = \sigma - 2\rho \tag{3.8}$$

and $\delta = \min\{2, 1+\sigma\}$.

Note that in view of (2.2) expressions (2.12) and (3.8) for ν are equivalent.

For all $\sigma \in (2/3, 3/2]$, the phases θ_n are defined by the same formulas (1.11) and (2.5), that is,

$$\theta_n = \theta_n(z) = \sqrt{-\tau n^{-1} + z n^{-\sigma}}, \quad \operatorname{Im} \theta_n(z) \geq 0, \tag{3.9}$$

although the estimates of the remainder $\mathbf{r}_n$ are rather different in the cases $\sigma > 1$, $\sigma = 1$ and $\sigma < 1$. For $\sigma \leq 2/3$, expression (3.9) requires some corrections.

3.2. The case $\sigma > 1$

For such σ, we suppose that $\tau \neq 0$. We treat the cases $\tau < 0$ and $\tau > 0$ parallelly putting $\sqrt{-\tau} > 0$ if $\tau < 0$ and $\sqrt{-\tau} = i\sqrt{|\tau|}$ if $\tau > 0$.

It follows from definition (3.9) that $\theta_n = O(n^{-1/2})$, whence

$$e^{i\theta_n} = \sum_{k=0}^{3} \frac{i^k}{k!}\theta_n^k + O(n^{-2}). \tag{3.10}$$

Substituting (3.10) into representation (3.7), we see that

$$\mathbf{r}_n = \sum_{k=0}^{3} r_n^{(k)} + O(n^{-2}) \tag{3.11}$$

where

$$r_n^{(0)} = -(1+(\nu/2)n^{-1}) - (1-(\nu/2)n^{-1}) + 2 + \tau n^{-1} - zn^{-\sigma}$$
$$= \tau n^{-1} - zn^{-\sigma} = -t_n, \tag{3.12}$$

by definition (1.11), and

$$r_n^{(1)} = i(1-(\nu/2)n^{-1})\theta_{n-1} - i(1+(\nu/2)n^{-1})\theta_n, \tag{3.13}$$
$$2r_n^{(2)} = (1-(\nu/2)n^{-1})\theta_{n-1}^2 + (1+(\nu/2)n^{-1})\theta_n^2, \tag{3.14}$$
$$6r_n^{(3)} = -i(1-(\nu/2)n^{-1})\theta_{n-1}^3 + i(1+(\nu/2)n^{-1})\theta_n^3. \tag{3.15}$$

Since $\theta_n^2 = t_n$, it follows from (3.14) that

$$\begin{aligned} 2r_n^{(2)} &= (1-(\nu/2)n^{-1})t_{n-1} + (1+(\nu/2)n^{-1})t_n \\ &= 2t_n + (1-(\nu/2)n^{-1})(t_{n-1}-t_n). \end{aligned} \tag{3.16}$$

Comparing this equality with (3.12), we find that

$$r_n^{(0)} + r_n^{(2)} = 2^{-1}(1-(\nu/2)n^{-1})(t_{n-1}-t_n) = O(n^{-2}). \tag{3.17}$$

The power ρ in (1.15) is determined by linear term (3.13) which we write as

$$r_n^{(1)} = i(\theta_{n-1}-\theta_n) - i(\nu/2)n^{-1}(\theta_n + \theta_{n-1}). \tag{3.18}$$

Let us distinguish the leading term in (3.9) setting

$$\theta_n = \sqrt{-\tau n^{-1}} + \tilde\theta_n \tag{3.19}$$

where

$$\tilde\theta_n = \sqrt{-\tau n^{-1} + zn^{-\sigma}} - \sqrt{-\tau n^{-1}} = \frac{zn^{1/2-\sigma}}{\sqrt{-\tau + zn^{1-\sigma}} + \sqrt{-\tau}} = O(n^{1/2-\sigma}). \tag{3.20}$$

Let us substitute (3.19) into (3.18) and observe that

$$\begin{aligned} &(\sqrt{(n-1)^{-1}} - \sqrt{n^{-1}}) - (\nu/2)n^{-1}(\sqrt{(n-1)^{-1}} + \sqrt{n^{-1}}) \\ &= (2^{-1}-\nu)n^{-3/2} + O(n^{-5/2}). \end{aligned}$$

According to (3.20) we have

$$\tilde{\theta}_n - \tilde{\theta}_{n-1} = O(n^{-\sigma-1/2}). \tag{3.21}$$

Thus, it follows from (3.18) that

$$r_n^{(1)} = i\sqrt{-\tau}(2^{-1} - \nu)n^{-3/2} + O(n^{-\sigma-1/2}).$$

The coefficient of $n^{-3/2}$ here is zero if $\nu = 1/2$ which, by (3.8), yields $\rho = \sigma/2 - 1/4$; in this case $r_n^{(1)} = O(n^{-\sigma-1/2})$.

It remains to consider the term $r_n^{(3)}$. In view of (3.15), it equals

$$6r_n^{(3)} = i(\theta_n^3 - \theta_{n-1}^3) + i(\nu/2)n^{-1}(\theta_n^3 + \theta_{n-1}^3). \tag{3.22}$$

Observe that

$$\theta_n^3 - \theta_{n-1}^3 = (\theta_n - \theta_{n-1})(\theta_n^2 + \theta_n\theta_{n-1} + \theta_{n-1}^2). \tag{3.23}$$

It follows from relations (3.19) and (3.21) that the first factor here is $O(n^{-3/2})$. The second factor is $O(n^{-1})$ because $\theta_n = O(n^{-1/2})$. Therefore, expression (3.23) is $O(n^{-5/2})$. Obviously, the second term in the right-hand side of (3.22) satisfies the same estimate.

Let us state the result obtained.

Proposition 3.2. *Let the assumptions of Theorem 2.1 be satisfied, and let the phases $\theta_n(z)$ be given by formula (3.9). Define the Ansatz $\mathcal{A}_n(z)$ by formula (1.15) where $\rho = \sigma/2 - 1/4$. Then remainder (1.14) satisfies estimate (1.20) where $\delta = \sigma + 1/2$.*

3.3. The intermediary case $\sigma = 1$

The results of this section are a particular case of Proposition 3.2, but the construction of the phases is now simpler:

$$t_n = (z - \tau)n^{-1} \quad \text{and} \quad \theta_n = \sqrt{z - \tau}n^{-1/2}. \tag{3.24}$$

The estimate of the remainder $\mathbf{r}_n$ is also simpler and more precise than in the general case. Indeed, according to (3.24), we now have

$$\theta_{n-1} - \theta_n = 2^{-1}\sqrt{z - \tau}n^{-3/2} + O(n^{-5/2}).$$

Therefore, it follows from (3.13) where $\nu = 1/2$ that $r_n^{(1)} = O(n^{-5/2})$. The same estimate for $r_n^{(3)}$ is a direct consequence of (3.23). Estimate (3.17) remains of course true. Thus, using equality (3.11) we can state the limit case of Proposition 3.2.

Proposition 3.3. *Let the assumptions of Theorem 2.4 be satisfied, and let the phases $\theta_n(z)$ be given by formula (3.24). Define the Ansatz $\mathcal{A}_n(z)$ by formula (1.15) where $\rho = 1/4$. Then remainder (1.14) satisfies estimate (1.20) where $\delta = 2$.*

3.4. The case $\sigma \in (2/3, 1)$

We again define the phases θ_n by formula (3.9), but now the term $zn^{-\sigma}$ is dominating so that, instead of (3.19), (3.20), we have a relation

$$\theta_n = \sqrt{t_n} = \sqrt{z} n^{-\sigma/2}(1 + O(n^{\sigma-1})). \tag{3.25}$$

Therefore, the scheme exposed in Section 3.2 for the case $\sigma > 1$ requires some modifications.

It again suffices to keep 4 terms in expansion of $e^{i\theta_n}$, but the remainders in formulas (3.10) and (3.11) are now $O(n^{-2\sigma})$. Estimates of $r_n^{(k)}$ where $k = 0, 1, 2, 3$ are the same as in Section 3.2 if the roles of the terms $-\tau n^{-1}$ and $zn^{-\sigma}$ are interchanged. Relations (3.12) and (3.16) are preserved, but the remainder $O(n^{-2})$ in (3.17) is replaced by $O(n^{-1-\sigma})$. It directly follows from definition (1.11) that

$$t_{n-1} - t_n = z\sigma n^{-1-\sigma}(1 + O(n^{-1+\sigma})). \tag{3.26}$$

Similarly to (3.21), it follows from (3.25), (3.26) that

$$\theta_{n-1} - \theta_n = 2^{-1}\sqrt{z}\sigma n^{-1-\sigma/2}(1 + O(n^{-1+\sigma})). \tag{3.27}$$

Therefore, expression (3.18) equals

$$r_n^{(1)} = i\sqrt{z}(\sigma/2 - \nu)n^{-1-\sigma/2} + O(n^{-2+\sigma/2}). \tag{3.28}$$

The coefficient at $n^{-1-\sigma/2}$ is zero if $\nu = \sigma/2$ which yields $2\rho = \sigma - \nu = \sigma/2$; in this case $r_n^{(1)} = O(n^{-2+\sigma/2})$. Putting together equality (3.17) and estimate (3.26) , we see that $r_n^{(0)} + r_n^{(2)} = O(n^{-1-\sigma})$ which is $O(n^{-2\sigma})$ because $\sigma < 1$. According to (3.25) and (3.27), expression (3.23) is estimated by $Cn^{-1-3\sigma/2}$. In view of (3.22), the same bound is true for $r_n^{(3)}$.

Thus, we arrive at the following result.

Proposition 3.4. *Let the assumptions of Theorem 2.3 be satisfied with $\sigma \in (2/3, 1)$, and let the phases $\theta_n(z)$ be given by formula (3.9). Define the Ansatz $\mathcal{A}_n(z)$ by formula (1.15) where $\rho = \sigma/4$. Then remainder (1.14) satisfies estimate (1.20) with $\delta = \min\{2\sigma, 2 - \sigma/2\} > 1 + \sigma/2$.*

We emphasize that for all $\sigma \in (2/3, 3/2]$ the phases θ_n are given by the same formula (3.9). However asymptotics of θ_n are different for $\sigma > 1$ and for $\sigma < 1$ – cf. (3.19), (3.20) with (3.25).

3.5. The case $\sigma \leq 2/3$. Eikonal equation

The leading term of the asymptotics of the phases θ_n is again given by formula (3.25), but, additionally, lower order terms appear. Now, we need to keep more terms in expansion (3.10) setting

$$e^{i\theta_n} = \sum_{k=0}^{K} \frac{i^k}{k!}\theta_n^k + O(n^{-(K+1)\sigma/2}). \tag{3.29}$$

Substituting (3.29) into representation (3.7), we see that

$$\mathbf{r}_n = \sum_{k=0}^{K} r_n^{(k)} + O(n^{-(K+1)\sigma/2}), \tag{3.30}$$

where $r_n^{(0)}$ are again given by equality (3.12) and

$$\begin{aligned} -i^k k! r_n^{(k)} =& (1-(\nu/2)n^{-1})\theta_{n-1}^k + (-1)^k(1+(\nu/2)n^{-1})\theta_n^k \\ =& \theta_{n-1}^k + (-1)^k\theta_n^k - (\nu/2)n^{-1}(\theta_{n-1}^k - (-1)^k\theta_n^k), \quad k \geq 1. \end{aligned} \tag{3.31}$$

Of course, for $k = 1, 2, 3$, this expression coincides with (3.13), (3.14), (3.15), respectively. It is convenient to choose an even $K = 2L$ with a sufficiently large L. We suppose that

$$(L + 1/2)\sigma > 1. \tag{3.32}$$

Let us distinguish the terms corresponding to $k = 0$ and $k = 1$ in sum (3.30) and then split it into the sums over even and odd k:

$$\mathbf{r}_n = r_n^{(0)} + r_n^{(1)} + \mathbf{r}_n^{(ev)} + \mathbf{r}_n^{(odd)} + O(n^{-(L+1/2)\sigma}),$$

where $r_n^{(0)}$, $r_n^{(1)}$ are given by formulas (3.12), (3.18) and

$$\mathbf{r}_n^{(ev)} = \sum_{l=1}^{L} r_n^{(2l)}, \quad \mathbf{r}_n^{(odd)} = \sum_{l=1}^{L-1} r_n^{(2l+1)}. \tag{3.33}$$

To satisfy estimate (1.20) with a suitable δ, we now have to take the even terms $r_n^{(2l)}$ for all $l \leq L$ into account. The odd terms $r_n^{(2l+1)}$ turn out to be negligible. To be precise, we define the phases θ_n by formula (2.7) where T_n is sum (2.6). The coefficients p_l will be found from the relation

$$r_n^{(0)} + \mathbf{r}_n^{(ev)} = O(n^{-1-\sigma}) \tag{3.34}$$

generalizing (3.17). To satisfy this relation, we use that the differences between θ_n and θ_{n-1} in the expression

$$(-1)^{l+1}(2l)! r_n^{(2l)} = \theta_{n-1}^{2l} + \theta_n^{2l} - (\nu/2)n^{-1}(\theta_{n-1}^{2l} - \theta_n^{2l})$$

(it is a particular case of (3.31)) can be neglected. Thus, we set

$$\Theta_n = 2\sum_{l=1}^{L} \frac{(-1)^{l+1}}{(2l)!}\theta_n^{2l}.$$

Since $r_n^{(0)} = -t_n$, we find that

$$r_n^{(0)} + \mathbf{r}_n^{(ev)} = (-t_n + \Theta_n) + (1 - (\nu/2)n^{-1})\sum_{l=1}^{L}(-1)^l(2l)!^{-1}(\theta_n^{2l} - \theta_{n-1}^{2l}). \quad (3.35)$$

As we will see the sum here is negligible, and hence we can replace (3.34) by the (approximate) eikonal equation

$$\Theta_n = t_n + O(n^{-1-\sigma}). \quad (3.36)$$

Our goal is to solve this equation with respect to θ_n^2. Note that $\Theta_n = \theta_n^2$ if $L = 1$ so that (3.36) again yields expression $\theta_n^2 = t_n$. The following elementary assertion shows that equation (3.36) can be efficiently solved for all $L \geq 1$. It is convenient to consider this problem in a somewhat more general setting. Denote by $\mathcal{P}$ the set of all polynomials (of the variable t), and let $\mathcal{P}_L = t^{L+1}\mathcal{P}$, that is, $\mathcal{P}_L \subset \mathcal{P}$ consists of polynomials with zero coefficients at powers t^k for all $k = 0, 1, \dots, L$.

Lemma 3.5. *Let $L \geq 2$ and $a_2, \dots, a_L$ be arbitrary given numbers. Then there exists a polynomial*

$$P_L(t) = \sum_{l=2}^{L} p_l t^l$$

such that the polynomial

$$Q_L(t) := P_L(t) + \sum_{k=2}^{L} a_k (P_L(t) + t)^k \in \mathcal{P}_L. \quad (3.37)$$

Proof. For arbitrary $p_2, \dots, p_L$, the polynomial $Q_L(t)$ defined by (3.37) has degree L^2 and it does not contain terms with zero and first powers of t. We have to choose the numbers $p_2, \dots, p_L$ in such a way that the coefficients of $Q_L(t)$ at t^l are zeros for all $l = 2, \dots, L$. This assertion is obvious for $L = 2$ because

$$Q_2(t) = P_2(t) + a_2(P_2(t) + t)^2 = (p_2 + a_2)t^2 + 2a_2 p_2 t^3 + a_2 p_2^2 t^4,$$

and, so, $Q_2(t) \in \mathcal{P}_2$ if $p_2 = -a_2$.

Let us pass to the general case. Suppose that (3.37) is satisfied. Then there exists a number q_{L+1} such that

$$Q_L(t) - q_{L+1}t^{L+1} \in \mathcal{P}_{L+1}. \quad (3.38)$$

We will find a number p_{L+1} such that the polynomial

$$P_{L+1}(t) = P_L(t) + p_{L+1} t^{L+1} \tag{3.39}$$

satisfies (3.37) for $L+1$, that is,

$$Q_{L+1}(t) := P_{L+1}(t) + \sum_{k=2}^{L+1} a_k (P_{L+1}(t) + t)^k \in \mathcal{P}_{L+1}. \tag{3.40}$$

Let us calculate the polynomial $Q_{L+1}(t)$ neglecting terms in $\mathcal{P}_{L+1}$. First, we observe that, for all $k = 2, \dots, L, L+1$, the difference

$$(P_{L+1}(t) + t)^k - (P_L(t) + t)^k = \sum_{n=1}^{k} \binom{k}{n} p_{L+1}^n t^{(L+1)n} (P_L(t) + t)^{k-n} \in \mathcal{P}_{L+1}.$$

Using also (3.39), we see that, up to terms in $\mathcal{P}_{L+1}$, polynomial (3.40) equals

$$Q_{L+1}(t) = P_L(t) + p_{L+1} t^{L+1} + \sum_{k=2}^{L} a_k (P_L(t) + t)^k + a_{L+1} (P_L(t) + t)^{L+1}$$

whence, by assumption (3.37),

$$Q_{L+1}(t) = Q_L(t) + (p_{L+1} + a_{L+1}) t^{L+1} \in \mathcal{P}_{L+1}.$$

It follows from (3.38) that this relation is equivalent to

$$Q_{L+1}(t) - (p_{L+1} + q_{L+1} + a_{L+1}) t^{L+1} \in \mathcal{P}_{L+1}.$$

Thus, inclusion $Q_{L+1}(t) \in \mathcal{P}_{L+1}$ is true if $p_{L+1} = -a_{L+1} - q_{L+1}$. This proves (3.37) for $L+1$. ■

Note particular cases

$$p_2 = -a_2, \quad p_3 = 2a_2^2 - a_3.$$

Let us come back to relation (3.36). Let us use Lemma 3.5 with the coefficients $a_l = 2(-1)^{l+1}/(2l)!$, $t = t_n$ defined by equality (1.11), and let p_l be the coefficients constructed in this lemma. It follows from equality (3.37) that the phases

$$\theta_n^2 = t_n + \sum_{l=2}^{L} p_l t_n^l =: T_n \tag{3.41}$$

satisfy, for some coefficients q_l, the equation

$$2 \sum_{l=1}^{L} \frac{(-1)^{l+1}}{(2l)!} \theta_n^{2l} - t_n = \sum_{l=L+1}^{L^2} q_l t_n^l = O(t_n^{-L-1}). \tag{3.42}$$

Since $t_n = O(n^{-\sigma})$, the right-hand side here is $O(n^{-(L+1)\sigma})$ which is $O(n^{-\delta})$ with $\delta > 1 + \sigma/2$ if condition (3.32) is satisfied.

The definition of the phases by formula (3.41) coincides of course with their definition by relations (2.6), (2.7). The asymptotics of θ_n as $n \to \infty$ is given by formula (2.11) generalizing (3.25). Next, we estimate the differences

$$\theta_{n-1} - \theta_n = \frac{T_{n-1} - T_n}{\theta_{n-1} + \theta_n}.$$

According to (2.6) we have

$$T_{n-1} - T_n = t_{n-1} - t_n + \sum_{l=2}^{L} p_l (t_{n-1}^l - t_n^l)$$

so that it satisfies the same relation (3.26) as t_n:

$$T_{n-1} - T_n = z\sigma n^{-1-\sigma}(1 + O(n^{-1+\sigma})).$$

Combining this relation with (2.11), we see that

$$\theta_{n-1} - \theta_n = 2^{-1}\sqrt{z}\sigma n^{-1-\sigma/2}(1 + O(n^{-\epsilon})) \tag{3.43}$$

for some $\epsilon > 0$ (compared with (3.27) only the estimate of the remainder is changed).

It easily follows from (2.11) and (3.43) that

$$|\theta_{n-1}^k - \theta_n^k| \leq C_k n^{-1-k\sigma/2} \tag{3.44}$$

for all $k = 1, 2, \ldots$.

Let us come back to Ansatz (1.15). Similarly to Section 3.4, the power ρ in (1.15) is determined by the linear term $r_n^{(1)}$ given by equality (3.18). It again satisfies relation (3.28) (with the remainder $O(n^{-2+\sigma/2})$ replaced by $O(n^{-\delta})$ for some $\delta > 1 + \sigma/2$). The coefficient at $n^{-1-\sigma/2}$ is zero if $\nu = \sigma/2$ which yields $\rho = \sigma/4$; in this case $r_n^{(1)} = O(n^{-\delta})$.

Given inequalities (2.11) and (3.44), we can estimate the remainder $\mathbf{r}_n$ essentially similarly to Proposition 3.4. The only differences are that estimates of the remainders are slightly weaker and that we have to take into account higher powers of θ_n. First, we consider term (3.35) with even powers of θ_n. Both the first term $-t_n + \Theta_n$ and the sum on the right are $O(n^{-1-\sigma})$ by virtue of relations (3.42) and (3.44), respectively. The term $\mathbf{r}_n^{(odd)}$ is also negligible. Indeed, according to (3.31) and (3.33) it equals

$$\mathbf{r}_n^{(odd)} = i \sum_{l=1}^{L-1} \frac{(-1)^l}{(2l+1)!}\big((\theta_{n-1}^{2l+1} - \theta_n^{2l+1}) - (\nu/2)n^{-1}(\theta_{n-1}^{2l+1} + \theta_n^{2l+1})\big).$$

Relations (2.11) and (3.44) allow us to estimate all terms here by $n^{-1-3\sigma/2}$.

Thus, we arrive at the following assertion generalizing Proposition 3.4.

Proposition 3.6. *Let the assumptions of Theorem 2.3 be satisfied, and let the phases* $\theta_n(z)$ *be given by formulas* (2.6)*,* (2.7) *with* $(L+1/2)\sigma > 1$ *and the coefficients* $p_2, \dots, p_L$ *constructed in Lemma 3.5. Let the Ansatz* $\mathcal{A}_n(z)$ *be defined by formula* (1.15) *where* $\rho = \sigma/4$*. Then remainder* (1.14) *satisfies estimate* (1.20) *with some* $\delta > 1 + \sigma/2$*.*

Remark 3.7. In all estimates, we suppose that $z \in \operatorname{clos}\Pi_0$, $0 < r \leq |z| \leq R < \infty$ for some r and R and $n \geq N = N(r, R)$. Then it follows from equality (2.6) that $\pm \operatorname{Im} T_n(z) > 0$ as long as $\pm \operatorname{Im} t_n(z) > 0$, that is, $\pm \operatorname{Im} z > 0$. Therefore, $T_n(z) \in \operatorname{clos}\Pi_0$, and hence condition (2.4) is satisfied.

Note two particular cases. If $\sigma > 2/3$, then we can take $L = 1$; this is the case considered in Proposition 3.4. If $\sigma > 2/5$, then $L = 2$ so that the formula for θ_n contains only one additional (compared with (2.5)) term:

$$\theta_n = \sqrt{t_n + t_n^2/6}.$$

We, finally, note that constructions of Ansätzen were important steps also in the papers [13, 17]. However, the form of the Ansatz $\mathcal{A}_n(z)$ suggested in this section is different from [13, 17]; in particular, the phases $\varphi_n(z)$ in (1.15) are simplest in the case $\sigma > 2/3$ while this case was excluded in [13]. They are also different from [17] – see Remark 7.9.

4. Difference and Volterra equations

Here we reduce a construction of the Jost solutions $f_n(z)$ of the Jacobi equation (1.1) to a Volterra "integral" equation which is then solved by iterations. In this section, we do not make any specific assumptions about the recurrence coefficients a_n, b_n and the Ansatz $\mathcal{A}_n(z)$ except of course that $\mathcal{A}_n(z) \neq 0$; for definiteness, we set $\mathcal{A}_{-1} = 1$. We present a general scheme of investigation and then, in Section 5, apply it to Jacobi operators with coefficients a_n and b_n satisfying conditions (1.8) and (1.9) with $|\gamma| = 1$.

4.1. Multiplicative change of variables

For a construction of $f_n(z)$, we will reformulate the problem introducing a sequence

$$u_n(z) = \mathcal{A}_n(z)^{-1} f_n(z), \quad n \in \mathbb{Z}_+. \tag{4.1}$$

In proofs, we usually omit the dependence on z in notation; for example, we write f_n, u_n, $\mathbf{r}_n$.

First, we derive a difference equation for $u_n(z)$.

Lemma 4.1. *Let the remainder $\mathbf{r}_n(z)$ be defined by formula (1.14). Set*

$$\Lambda_n(z) = \frac{a_n}{a_{n-1}} \frac{\mathcal{A}_{n+1}(z)}{\mathcal{A}_{n-1}(z)} \tag{4.2}$$

and

$$\mathcal{R}_n(z) = -\sqrt{\frac{a_n}{a_{n-1}} \frac{\mathcal{A}_n(z)}{\mathcal{A}_{n-1}(z)}}\,\mathbf{r}_n(z). \tag{4.3}$$

Then equation (1.1) for a sequence $f_n(z)$ is equivalent to the equation

$$\Lambda_n(z)(u_{n+1}(z) - u_n(z)) - (u_n(z) - u_{n-1}(z)) = \mathcal{R}_n(z)u_n(z), \quad n \in \mathbb{Z}_+, \tag{4.4}$$

for sequence (4.1).

Proof. Substituting expression $f_n = \mathcal{A}_n u_n$ into (1.1) and using definition (1.14), we see that

$$\begin{aligned}
(\sqrt{a_{n-1}a_n}\mathcal{A}_n)^{-1}\big(a_{n-1}f_{n-1} + (b_n - z)f_n + a_n f_{n+1}\big)&\\
= \sqrt{\frac{a_{n-1}}{a_n}\frac{\mathcal{A}_{n-1}}{\mathcal{A}_n}}u_{n-1} + \frac{b_n - z}{\sqrt{a_{n-1}a_n}}u_n + \sqrt{\frac{a_n}{a_{n-1}}\frac{\mathcal{A}_{n+1}}{\mathcal{A}_n}}u_{n+1}&\\
= \sqrt{\frac{a_{n-1}}{a_n}\frac{\mathcal{A}_{n-1}}{\mathcal{A}_n}}(u_{n-1} - u_n) + \sqrt{\frac{a_n}{a_{n-1}}\frac{\mathcal{A}_{n+1}}{\mathcal{A}_n}}(u_{n+1} - u_n) + \mathbf{r}_n u_n&\\
= \sqrt{\frac{a_{n-1}}{a_n}\frac{\mathcal{A}_{n-1}}{\mathcal{A}_n}}\big((u_{n-1} - u_n) + \Lambda_n(u_{n+1} - u_n) - \mathcal{R}_n u_n\big)&
\end{aligned}$$

where the coefficients Λ_n and $\mathcal{R}_n$ are defined by equalities (4.2) and (4.3), respectively. Therefore, equations (1.1) and (4.4) are equivalent. ■

Our next goal is to construct a solution of difference equation (4.4) such that

$$\lim_{n\to\infty} u_n(z) = 1. \tag{4.5}$$

To that end, we will reduce equation (4.4) to a Volterra "integral" equation which can be standardly solved by successive approximations.

4.2. Volterra equation

It is convenient to consider this problem in a more general setting. We now do not make any specific assumptions about the sequences Λ_n and $\mathcal{R}_n$ in (4.4) except that $\Lambda_n \neq 0$. Denote

$$X_n = \Lambda_1\Lambda_2\cdots\Lambda_n \tag{4.6}$$

and

$$G_{n,m}=X_{m-1}\sum_{p=n}^{m-1}X_p^{-1},\quad m\geq n+1. \tag{4.7}$$

The sequence u_n will be constructed as a solution of a discrete Volterra integral equation

$$u_n=1+\sum_{m=n+1}^{\infty}G_{n,m}\mathcal{R}_m u_m. \tag{4.8}$$

Under natural assumptions, this equation can be standardly solved by successive approximations. First, we estimate its iterations.

Lemma 4.2. *Let us set*

$$h_m=\sup_{n\leq m-1}|G_{n,m}\mathcal{R}_m| \tag{4.9}$$

and suppose that

$$(h_m)\in\ell^1(\mathbb{Z}_+). \tag{4.10}$$

Put $u_n^{(0)}=1$ *and*

$$u_n^{(k+1)}=\sum_{m=n+1}^{\infty}G_{n,m}\mathcal{R}_m u_m^{(k)},\quad k\geq 0, \tag{4.11}$$

for all $n\in\mathbb{Z}_+$*. Then estimates*

$$|u_n^{(k)}|\leq\frac{H_n^k}{k!},\quad \text{for all } k\in\mathbb{Z}_+, \tag{4.12}$$

where

$$H_n=\sum_{p=n+1}^{\infty}h_p, \tag{4.13}$$

are true.

Proof. Suppose that (4.12) is satisfied for some $k\in\mathbb{Z}_+$. We have to check the same estimate (with k replaced by $k+1$ in the right-hand side) for $u_n^{(k+1)}$. According to definitions (4.9) and (4.11), it follows from estimate (4.12) that

$$|u_n^{(k+1)}|\leq\sum_{m=n+1}^{\infty}h_m|u_m^{(k)}|\leq\frac{1}{k!}\sum_{m=n+1}^{\infty}h_m H_m^k. \tag{4.14}$$

Since $H_{m-1}=H_m+h_m$, we have an inequality

$$H_m^{k+1}+(k+1)h_m H_m^k\leq(H_m+h_m)^{k+1}=H_{m-1}^{k+1},$$

and hence, for all $N \in \mathbb{Z}_+$,

$$(k+1)\sum_{m=n+1}^{N} h_m H_m^k \le \sum_{m=n+1}^{N}(H_{m-1}^{k+1} - H_m^{k+1}) = H_n^{k+1} - H_N^{k+1} \le H_n^{k+1}.$$

Substituting this bound into (4.14), we obtain estimate (4.12) for $u_n^{(k+1)}$. ∎

Now, we are in a position to solve equation (4.8) by iterations.

Theorem 4.3. *Let assumption* (4.10) *be satisfied. Then equation* (4.8) *has a bounded solution* u_n. *This solution satisfies an estimate*

$$|u_n - 1| \le e^{H_n} - 1 \le CH_n \tag{4.15}$$

where H_n *is sum* (4.13). *In particular, condition* (4.5) *holds.*

Proof. Set

$$u_n = \sum_{k=0}^{\infty} u_n^{(k)} \tag{4.16}$$

where $u_n^{(k)}$ are defined by recurrence relations (4.11). Estimate (4.12) shows that this series is absolutely convergent. Using the Fubini theorem to interchange the order of summations in m and k, we see that

$$\begin{aligned}\sum_{m=n+1}^{\infty} G_{n,m}\mathcal{R}_m u_m &= \sum_{k=0}^{\infty}\sum_{m=n+1}^{\infty} G_{n,m}\mathcal{R}_m u_m^{(k)} = \sum_{k=0}^{\infty} u_n^{(k+1)} \\ &= -1 + \sum_{k=0}^{\infty} u_n^{(k)} = -1 + u_n.\end{aligned}$$

This is equation (4.8) for sequence (4.16). Estimate (4.15) also follows from (4.12) and (4.16). ∎

Remark 4.4. A bounded solution u_n of (4.8) is of course unique. Indeed, suppose that $(v_n) \in \ell^\infty(\mathbb{Z}_+)$ satisfies homogeneous equation (4.8), that is,

$$v_n = \sum_{m=n+1}^{\infty} G_{n,m}\mathcal{R}_m v_m.$$

Then, by assumption (4.10), we have

$$|v_n| \le \sum_{m=n+1}^{\infty} h_m |v_m|.$$

Iterating this estimate, we find that

$$|v_n| \le \frac{1}{k!} H_n^k \max_{n\in\mathbb{Z}_+} \{|v_n|\}, \quad \text{for all } k \in \mathbb{Z}_+.$$

Taking the limit $k \to \infty$, we see that $v_n = 0$. Note however that we do not use the unicity in our construction.

4.3. Back to the difference equation

It turns out that the construction above yields a solution of difference equation (4.4).

Lemma 4.5. *Under assumption* (4.10) *a solution u_n of integral equation* (4.8) *satisfies an identity*

$$u_{n+1} - u_n = -X_n^{-1} \sum_{m=n+1}^{\infty} X_{m-1} \mathcal{R}_m u_m \tag{4.17}$$

and difference equation (4.4).

Proof. It follows from (4.8) that

$$u_{n+1} - u_n = \sum_{m=n+2}^{\infty} (G_{n+1,m} - G_{n,m}) \mathcal{R}_m u_m - G_{n,n+1} \mathcal{R}_{n+1} u_{n+1}. \tag{4.18}$$

According to (4.7), we have

$$G_{n+1,m} - G_{n,m} = -X_n^{-1} X_{m-1} \quad \text{and} \quad G_{n,n+1} = 1.$$

Therefore, relation (4.18) can be rewritten as (4.17).

Putting together equality (4.17) with the same equality where $n+1$ is replaced by n, we see that

$$\begin{aligned} &\Lambda_n (u_{n+1} - u_n) - (u_n - u_{n-1}) \\ &= -\Lambda_n X_n^{-1} \sum_{m=n+1}^{\infty} X_{m-1} \mathcal{R}_m u_m + X_{n-1}^{-1} \sum_{m=n}^{\infty} X_{m-1} \mathcal{R}_m u_m. \end{aligned}$$

Since $X_n = \Lambda_n X_{n-1}$, the right-hand side here equals $\mathcal{R}_n u_n$, and hence the equation obtained coincides with (4.4). ■

Corollary 4.6. *It follows from* (4.17) *that*

$$|u_{n+1} - u_n| \le \max_{n\in\mathbb{Z}_+} \{|u_n|\} \, |X_n|^{-1} \sum_{m=n}^{\infty} |X_m \mathcal{R}_{m+1}|. \tag{4.19}$$

Lemma 4.5 allows us to reformulate Theorem 4.3 in terms of solutions of equation (4.4).

Theorem 4.7. *Let assumption* (4.10) *be satisfied. Then difference equation* (4.4) *has a solution $u_n(z)$ satisfying estimates* (4.15) *and* (4.19)*. In particular, condition* (4.5) *holds.*

Let us now discuss the dependence on the spectral parameter z. Suppose that the coefficients $\Lambda_n(z)$ and $\mathcal{R}_n(z)$ in equation (4.8) are functions of $z \in \Omega$ on some open set $\Omega \subset \mathbb{C}$.

Lemma 4.8. *Let the coefficients $\Lambda_n(z)$ and $\mathcal{R}_n(z)$ be analytic functions of $z \in \Omega$. Suppose that assumption* (4.10) *is satisfied uniformly in z on compact subsets of Ω. Then the solutions $u_n(z)$ of integral equation* (4.8) *are also analytic in $z \in \Omega$. Moreover, if $\Lambda_n(z)$ and $\mathcal{R}_n(z)$ are continuous up to the boundary of Ω and assumption* (4.10) *is satisfied uniformly on Ω, then the same is true for the functions $u_n(z)$.*

Proof. Consider series (4.16) for a solution $u_n(z)$ of integral equation (4.8). Observe that if the functions $u_m^{(k)}(z)$ in (4.11) depend analytically (continuously) on z, then the function $u_n^{(k+1)}(z)$ is also analytic (continuous). Since series (4.16) converges uniformly, its sums $u_n(z)$ are also analytic (continuous) functions. ∎

In view of Lemma 4.5 this result applies also to solutions of difference equation (4.4).

5. Jost solutions

Here we use the results of the previous section to construct the Jost solutions $f_n(z)$ of the Jacobi equation (1.1) with the coefficients a_n and b_n satisfying conditions (1.8) and (1.9) where $|\gamma| = 1$. This leads to Theorems 2.1, 2.3 and 2.4.

First, in Sections 5.1 and 5.2, we state some necessary technical results.

5.1. Discrete derivatives

Let us collect standard formulas for "derivatives"

$$x_n' = x_{n+1} - x_n$$

of various sequences x_n:

$$(x_n^{-1})' = - x_n^{-1} x_{n+1}^{-1} x_n', \tag{5.1a}$$

$$(e^{x_n})' = (e^{x_n'} - 1) e^{x_n}, \tag{5.1b}$$

$$(\sqrt{x_n})' = x_n' (\sqrt{x_n} + \sqrt{x_{n+1}})^{-1}, \tag{5.1c}$$

and

$$(x_n y_n)' = x_{n+1} y_n' + x_n' y_n. \tag{5.2}$$

Note the Abel summation formula ("integration by parts"):

$$\sum_{p=n}^{m} x_p y_p' = x_m y_{m+1} - x_{n-1} y_n - \sum_{p=n}^{m} x_{p-1}' y_p; \tag{5.3}$$

here $m \geq n \geq 0$ are arbitrary (we set $x_{-1} = 0$ so that $x_{-1}' = x_0$).

We mention also an obvious estimate

$$|f(x_{n+1}) - f(x_n)| \leq (\max_{|x|\leq 1} |f'(x)|)|x_n'| \tag{5.4}$$

valid for an arbitrary function $f \in C^1$, an arbitrary sequence $x_n \to 0$ as $n \to \infty$ and sufficiently large n.

Let us now consider equation (1.1). A direct calculation shows that, for two $f = (f_n)_{n=-1}^{\infty}$ and $g = (g_n)_{n=-1}^{\infty}$ solutions of this equation, their Wronskian

$$W[f, g] := a_n(f_n g_{n+1} - f_{n+1} g_n) \tag{5.5}$$

does not depend on $n = -1, 0, 1, \ldots$. In particular, for $n = -1$ and $n = 0$, we have

$$W[f, g] = a_{-1}(f_{-1} g_0 - f_0 g_{-1}) \quad \text{and} \quad W[f, g] = a_0(f_0 g_1 - f_1 g_0)$$

(the number $a_{-1} \neq 0$ is arbitrary, but the products $a_{-1} f_{-1}$ do not depend on its choice). Clearly, the Wronskian $W[f, g] = 0$ if and only if the solutions f and g are proportional.

5.2. Oscillating sums

Below we need to estimate sums of oscillating or exponentially growing terms. First, we note an integration-by-parts formula. The following elementary assertion does not require specific assumptions about amplitudes $\boldsymbol{\kappa}_n$ and phases $\boldsymbol{\varphi}_n$.

Lemma 5.1. *Set* $\boldsymbol{\theta}_n = \boldsymbol{\varphi}_{n+1} - \boldsymbol{\varphi}_n$ *and*

$$\boldsymbol{\zeta}_n = \boldsymbol{\kappa}_n (e^{-i\boldsymbol{\theta}_n} - 1)^{-1}. \tag{5.6}$$

Then

$$\sum_{p=n}^{m} \boldsymbol{\kappa}_p e^{-i\boldsymbol{\varphi}_p} = \boldsymbol{\zeta}_m e^{-i\boldsymbol{\varphi}_{m+1}} - \boldsymbol{\zeta}_{n-1} e^{-i\boldsymbol{\varphi}_n} - \sum_{p=n}^{m} \boldsymbol{\zeta}_{p-1}' e^{-i\boldsymbol{\varphi}_p} \tag{5.7}$$

for all n *and* m.

Proof. According to (5.1) the left-hand side of (5.7) can be rewritten as

$$\sum_{p=n}^{m} \zeta_p (e^{-i\varphi_p})'.$$

It follows from formula (5.3) that this sum equals the right-hand side of (5.7). ∎

Corollary 5.2. *Suppose that*

$$\zeta_n' \in \ell^1(\mathbb{Z}_+) \tag{5.8}$$

and $\operatorname{Im} \theta_n \geq 0$. *Then*

$$\Big| \sum_{p=n}^{m} \kappa_p e^{-i\varphi_p} \Big| \leq C e^{\operatorname{Im} \varphi_{m+1}} \tag{5.9}$$

where the constant C does not depend on n and m.

Remark 5.3. If

$$\theta_n' \in \ell^1(\mathbb{Z}_+), \tag{5.10}$$

then condition (5.8) can be replaced by more convenient ones:

$$\frac{\kappa_n}{\theta_n} \in \ell^\infty(\mathbb{Z}_+) \quad \text{and} \quad \Big(\frac{\kappa_n}{\theta_n}\Big)' \in \ell^1(\mathbb{Z}_+). \tag{5.11}$$

Proof. It follows from (5.2) that

$$\zeta_n' = \Big(\frac{\kappa_n}{\theta_n}\Big)' \frac{\theta_{n+1}}{e^{-i\theta_{n+1}} - 1} + \frac{\kappa_n}{\theta_n} \Big(\frac{\theta_n}{e^{-i\theta_n} - 1}\Big)'. \tag{5.12}$$

Note that the function $f(\theta) = \theta(e^{-i\theta} - 1)^{-1}$ is C^1 in a neighborhood of the point $\theta = 0$. Therefore, the sequence $f(\theta_n)$ is bounded as $n \to \infty$ and $f'(\theta_n) \in \ell^1(\mathbb{Z}_+)$ according to estimate (5.4) and condition (5.10). Thus, conditions (5.11) imply that both terms in the right-hand side of (5.12) are in $\ell^1(\mathbb{Z}_+)$. ∎

5.3. Estimate of the "integral" kernel

Recall that the sequences $\mathcal{A}_n = \mathcal{A}_n(z)$ and $\Lambda_n = \Lambda_n(z)$ are given by relations (1.15) and (4.2), respectively. Our goal is to estimate the matrix elements $G_{n,m}$ defined by equalities (4.6) and (4.7) and to prove inclusion (4.10). Our estimates apply to all values of σ.

Putting together formulas (4.2) and (4.6), we see that

$$X_n = c a_n \mathcal{A}_{n+1} \mathcal{A}_n$$

where the constant $c = (a_0 \mathcal{A}_1 \mathcal{A}_0)^{-1}$. According to definition (1.15) this yields equality

$$X_n^{-1} = -c \kappa_n e^{-i\varphi_n} \tag{5.13}$$

where

$$\varkappa_n = n^{\rho}(n+1)^{\rho} a_n^{-1} \tag{5.14}$$

and

$$\boldsymbol{\varphi}_n = \varphi_n + \varphi_{n+1}. \tag{5.15}$$

It follows from condition (1.8) that

$$\varkappa_n = n^{-\nu}\big(1 + (\rho - \alpha)n^{-1} + O(n^{-2})\big), \tag{5.16}$$

where $\nu = \sigma - 2\rho$ satisfies (2.12).

First, we reformulate Lemma 5.1 and its consequences in a particular form adapted to our problem.

Lemma 5.4. *Let the assumptions of one of Theorems* 2.1, 2.3, *or* 2.4 *be satisfied. Define the sequences* $\varkappa_n$ *and* $\boldsymbol{\varphi}_n$ *by equalities* (5.14) *and* (5.15). *Then estimate* (5.9) *holds.*

Proof. Set

$$\boldsymbol{\theta}_n = \boldsymbol{\varphi}_{n+1} - \boldsymbol{\varphi}_n = \theta_n + \theta_{n+1}. \tag{5.17}$$

It follows from relations (3.21) or (3.43) that inclusion (5.10) holds. Therefore, in view of Remark 5.3, it suffices to check inclusions (5.11). By definition (2.7), we have

$$\varkappa_n \boldsymbol{\theta}_n^{-1} = (n^{\nu}\varkappa_n)(n^{\nu} S_n)^{-1}, \quad S_n = \sqrt{T_n} + \sqrt{T_{n+1}}, \tag{5.18}$$

where T_n is defined by equality (2.6) (in particular, $T_n = t_n$ if $\sigma > 2/3$). Inclusions $(n^{\nu}\varkappa_n) \in \ell^{\infty}(\mathbb{Z}_+)$ and $(n^{\nu}\varkappa_n)' \in \ell^1(\mathbb{Z}_+)$ are direct consequences of formula (5.16). It follows from relations (2.9), (2.10), or (2.11) that the product $n^{\nu} S_n$ has a finite non-zero limit (it is used here that $z \neq 0$ under the assumptions of Theorem 2.3 and that $z \neq \gamma\tau$ under the assumptions of Theorem 2.4). The inclusion $(n^{\nu} S_n)' \in \ell^1(\mathbb{Z}_+)$ is again a consequence of (3.21) or (3.43). Therefore, (5.18) implies inclusions (5.11) which yields (5.9). ■

Now, we are in a position to estimate the matrix elements $G_{n,m}$. First, we note that

$$C_1 m^{\nu} e^{-\operatorname{Im} \boldsymbol{\varphi}_m} \leq |X_m| \leq C_2 m^{\nu} e^{-\operatorname{Im} \boldsymbol{\varphi}_m} \tag{5.19}$$

according to definition (5.13) and relation (5.16). Next, we apply inequality (5.9) to elements (5.13) which yields

$$\Big|\sum_{p=n}^{m-1} X_p^{-1}\Big| \leq C e^{\operatorname{Im} \boldsymbol{\varphi}_m}. \tag{5.20}$$

Combining (5.19) and (5.20), we obtain a convenient estimate on product (4.7).

Lemma 5.5. *Under the assumptions of any of Theorems 2.1, 2.3, or 2.4, we have an estimate*

$$|G_{n,m}| \leq Cm^{\nu} \tag{5.21}$$

where ν is given by (2.12) and the constant C does not depend on n and m.

5.4. Solutions of the integral equation

Next, we consider integral equation (4.8). Observe that remainder (4.3) obeys the same estimate (1.20) as $\mathbf{r}_n$. Thus, according to the results of Section 3 (see Propositions 3.2, 3.3, 3.4, and 3.6)

$$|R_m| \leq Cm^{-\delta} \tag{5.22}$$

where δ satisfies conditions (3.1).

Putting together (5.21) and (5.22), we obtain an estimate on sequence (4.9):

$$h_m \leq Cm^{\nu-\delta}.$$

Comparing (2.12) and (3.1), we see that $\nu - \delta < -1$. It follows that sum (4.13) satisfies an estimate

$$H_n \leq Cn^{\nu-\delta+1}.$$

Therefore, condition (4.10) holds, and Theorem 4.7 applies in our case. This yields estimates (4.15) and (4.19). Moreover, the right-hand side of (4.19) can be estimated explicitly. Indeed, note that $\operatorname{Im} \boldsymbol{\varphi}_n \leq \operatorname{Im} \boldsymbol{\varphi}_m$ for $m \geq n$ according to (2.4). Thus, it follows from (5.13) and (5.16) that

$$|X_n^{-1} X_m| \leq |k_n k_m^{-1}| e^{\operatorname{Im}(\boldsymbol{\varphi}_n - \boldsymbol{\varphi}_m)} \leq Cn^{-\nu} m^{\nu}, \quad m \geq n,$$

so that inequality (4.19) yields an estimate

$$|u_{n+1} - u_n| \leq Cn^{-\nu} \sum_{m=n}^{\infty} m^{\nu-\delta} \leq C_1 n^{-\delta+1}.$$

We see that, under the assumptions of any of Theorems 2.1, 2.3 or 2.4, condition (4.10) is satisfied. Hence, the following three results are direct consequences of Theorem 4.7 (see also Lemma 4.8). Recall that the number ν is defined by relations (2.12) and δ satisfies conditions (3.1).

Theorem 5.6. *Let the assumptions of Theorem 2.1 be satisfied.*

If $\tau < 0$, then for every $z \in \operatorname{clos} \Pi$ equation (4.8) has a solution $u_n(z)$ satisfying asymptotic relations

$$u_n(z) = 1 + O(n^{\nu-\delta+1}) \tag{5.23}$$

and

$$u_n'(z) = O(n^{-\delta+1}) \tag{5.24}$$

where $\nu = 1/2$ *and* $\delta > 1/2 + \sigma$. *For all* $n \in \mathbb{Z}_+$, *the functions* $u_n(z)$ *are analytic in* Π *and are continuous up to the cut along the real axis.*

If $\tau > 0$, *then relations* (5.23) *and* (5.24) *are true for all* $z \in \mathbb{C}$. *In this case the functions* $u_n(z)$ *are analytic in the whole complex plane* $\mathbb{C}$.

For all $\tau \neq 0$, *asymptotic formula* (4.5) *is uniform in* z *from compact subsets of* $\mathbb{C}$.

Theorem 5.7. *Let the assumptions of Theorem* 2.3 *be satisfied. Then for every* $z \neq 0$ *such that* $z \in \gamma \operatorname{clos} \Pi_0$, *equation* (4.8) *has a solution* $u_n(z)$ *with asymptotics* (5.23), (5.24) *where* $\nu = \sigma/2$ *and* $\delta > 1 + \sigma/2$. *For all* $n \in \mathbb{Z}_+$, *the functions* $u_n(z)$ *are analytic in* $z \in \gamma\Pi_0$ *and are continuous up to the cut along the half-axis* $\gamma\mathbb{R}_+$, *with a possible exception of the point* $z = 0$.

Theorem 5.8. *Let the assumptions of Theorem* 2.4 *be satisfied. Then for every* z *such that* $z \in \gamma(\tau + \operatorname{clos} \Pi_0)$, $z \neq \gamma\tau$, *equation* (4.8) *has a solution* $u_n(z)$ *with asymptotics* (5.23), (5.24) *where* $\nu = 1/2$ *and* $\delta = 2$. *For all* $n \in \mathbb{Z}_+$, *the functions* $u_n(z)$ *are analytic in* $z \in \gamma(\tau + \Pi_0)$ *and are continuous up to the cut along the half-axis* $\gamma(\tau + \mathbb{R}_+)$, *with a possible exception of the point* $\gamma\tau$.

5.5. The Jost solutions

Now, it is easy to construct solutions of the Jacobi equation (1.1) with asymptotics (2.1) as $n \to \infty$. We call them the Jost solutions.

According to Lemma 4.1 equation (4.4) for the sequence $u_n(z)$ and equation (1.1) for the sequence

$$f_n(z) = (-\gamma)^n n^{-\rho} e^{i\varphi_n(\gamma z)} u_n(z) \tag{5.25}$$

are equivalent. Therefore, Theorems 2.1, 2.3, and 2.4 are direct consequences of Theorems 5.6, 5.7, and 5.8, respectively.

Finally, we show that the Jost solutions are determined uniquely by their asymptotics (2.1). This is quite simple for regular z. Recall that the set $\mathcal{S}$ was defined by relations (2.8).

Proposition 5.9. *Let the assumptions of one of Theorems* 2.1, 2.3, *or* 2.4 *be satisfied. If* $\sigma = 3/2$, *we also assume that* $\tau > 0$. *Suppose that* $z \notin \operatorname{clos} \mathcal{S}$. *Then the solution of* $f_n(z)$ *of equation* (1.1) *satisfying condition* (2.1) *is unique.*

Proof. Suppose that solutions f_n and $\tilde{f}_n$ of equation (1.1) are given by equality (5.25) where u_n and $\tilde{u}_n$ obey condition (4.5). Then their Wronskian (5.5) equals

$$W[f, \tilde{f}] = -\gamma a_n n^{-\rho} (n+1)^{-\rho} e^{i\varphi_n} e^{i\varphi_{n+1}} (u_n \tilde{u}_{n+1} - u_{n+1} \tilde{u}_n). \tag{5.26}$$

As explained in Section 2.2, under the assumptions of Proposition 5.9 the sequence $e^{i\varphi_n}$ tends to zero exponentially as $n \to \infty$ whence $W[f, \tilde{f}] = 0$ and consequently $\tilde{f}_n = C f_n$ for some constant C. It now follows from (5.25) that $\tilde{u}_n = C u_n$ where $C = 1$ by (4.5). ■

Remark 5.10. If $\sigma = 3/2$ and $\tau < 0$, then instead of (4.5) we have to require a stronger condition

$$u_n = 1 + O(n^{-1/2}). \tag{5.27}$$

Note that in view of (2.21) this condition is satisfied for the Jost solution $f_n(\lambda + i\varepsilon)$ of equation (1.1) constructed in Theorem 2.1. Suppose that two solutions f_n and $\tilde{f}_n$ are given by formula (5.25) where u_n and $\tilde{u}_n$ satisfy (5.27) whence $u_n\tilde{u}_{n+1} - u_{n+1}\tilde{u}_n = O(n^{-1/2})$. Since $\rho = 1/2$ now, it follows from asymptotic formula (2.21) and relation (5.26) that

$$|W[f, \tilde{f}]| = O(a_n n^{-1-2|\varepsilon|/\sqrt{|\tau|}}(u_n\tilde{u}_{n+1} - u_{n+1}\tilde{u}_n)) = O(n^{-2|\varepsilon|/\sqrt{|\tau|}}) = 0$$

because $\varepsilon \neq 0$. This implies that $\tilde{f}_n = f_n$.

The results for z in the spectrum of the operator J are slightly weaker.

Proposition 5.11. *Let the assumptions of one of Theorems 2.1, 2.3, or 2.4 be satisfied. Suppose that $z = \lambda \pm i0$ where $\lambda \in \mathcal{S}$. Then the solution $f_n(z)$ of equation (1.1) satisfying relation (5.25) with u_n obeying conditions (4.5) and (5.24) is unique.*

Proof. Suppose that two solutions f_n and $\tilde{f}_n$ of equation (1.1) satisfy these conditions. Their Wronskian is given by equality (5.26) where

$$u_n\tilde{u}_{n+1} - u_{n+1}\tilde{u}_n = u_n(\tilde{u}_{n+1} - \tilde{u}_n) + (u_n - u_{n+1})\tilde{u}_n = O(n^{-\delta+1}).$$

It follows that

$$W[f, \tilde{f}] = O(n^{\nu-\delta+1}), \quad \nu = \sigma - 2\rho, \quad n \to \infty.$$

Putting together relations (2.12) and (3.1), we see that $\nu - \delta + 1 < 0$ for all $\sigma \in (0, 3/2]$. Therefore, $W[f, \tilde{f}] = 0$ and, consequently, $\tilde{f} = f$. ■

6. Orthogonal polynomials

Here we describe an asymptotic behavior as $n \to \infty$ of all solutions $F_n(z)$ of equation (1.1). In particular, these results apply to the orthonormal polynomials $P_n(z)$. We have to distinguish values of $z = \lambda \in \mathcal{S}$ (this set was defined by relations (2.8)) in the absolutely continuous spectrum of a Jacobi operator and regular points $z \in \mathbb{C} \setminus \operatorname{clos} \mathcal{S}$.

6.1. Regular points

Our goal in this section is to prove Theorem 2.7 and Proposition 2.8. Let us proceed from the following assertion.

Proposition 6.1 ([31, Theorem 2.2]). *Let $f(z) = (f_n(z))$ be an arbitrary solution of the Jacobi equation (1.1) such that $f_n(z) \neq 0$ for sufficiently large n, say $n \geq n_0$. Then sequence $g(z) = (g_n(z))$ defined by (1.18) also satisfies equation (1.1), and the Wronskian*

$$W[f(z), g(z)] = 1,$$

so that the solutions $f(z)$ and $g(z)$ are linearly independent.

In this section, we suppose that $z \in \mathbb{C} \setminus \operatorname{clos} \mathcal{S}$ and $f_n = f_n(z)$ is the Jost solution of equation (1.1). Its asymptotics is given by formulas (1.15), (1.16). Our aim is to find an asymptotic behavior of the solution $g_n = g_n(z)$ as $n \to \infty$. The dependence on z will be omitted in notation. Let us set

$$\Sigma_n = \sum_{m=n_0}^{n} (a_{m-1} f_{m-1} f_m)^{-1}, \quad n \geq n_0; \tag{6.1}$$

then (1.18) reads as

$$g_n = f_n \Sigma_n. \tag{6.2}$$

Using equalities (1.15), (1.17) and notation (5.14), (5.15), we can rewrite sum (6.1) as

$$\Sigma_n = -\gamma \sum_{m=n_0-1}^{n-1} \boldsymbol{\kappa}_m \mathbf{u}_m e^{-i\boldsymbol{\varphi}_m} \quad \text{where} \quad \mathbf{u}_m = (u_m u_{m+1})^{-1}. \tag{6.3}$$

In view of identity (5.1), we have

$$e^{-i\boldsymbol{\varphi}_m} = (e^{-i\boldsymbol{\theta}_m} - 1)^{-1} (e^{-i\boldsymbol{\varphi}_m})',$$

with $\boldsymbol{\theta}_m$ given by (5.17). This allows us to integrate by parts in (6.3). Indeed, using formula (5.3), we find that

$$-\gamma \Sigma_n e^{i\boldsymbol{\varphi}_n} = \boldsymbol{\zeta}_{n-1} \mathbf{u}_{n-1} - \boldsymbol{\zeta}_{n_0-2} \mathbf{u}_{n_0-2} e^{-i\boldsymbol{\varphi}_{n_0-1}} e^{i\boldsymbol{\varphi}_n} + \widetilde{\Sigma}_n e^{i\boldsymbol{\varphi}_n} \tag{6.4}$$

where $\boldsymbol{\zeta}_n$ is defined by equality (5.6) and

$$\widetilde{\Sigma}_n = -\sum_{m=n_0-1}^{n-1} (\boldsymbol{\zeta}_{m-1} \mathbf{u}_{m-1})' e^{-i\boldsymbol{\varphi}_m}. \tag{6.5}$$

We will see that asymptotics of Σ_n as $n \to \infty$ is determined by the first term in the right-hand side of expression (6.4). Let us calculate it. Recall that $\mathbf{u}_n \to 1$ as $n \to \infty$

according to Theorem 4.3. Therefore, putting together asymptotic formulas (2.14) for $\boldsymbol{\theta}_n$ and (5.16) for $\boldsymbol{\kappa}_n$, we find that

$$\lim_{n\to\infty} \boldsymbol{\zeta}_n \mathbf{u}_n = \lim_{n\to\infty} \boldsymbol{\zeta}_n = i\varkappa \tag{6.6}$$

with the coefficient $\varkappa = \varkappa(z)$ given by (2.13).

The second term in the right-hand side of (6.4) tends to zero as $n \to \infty$ due to the factor $e^{i\boldsymbol{\varphi}_n}$. The same is true for the third term. To show this, we need to estimate the derivatives in (6.5).

Lemma 6.2. *Let the sequence $\boldsymbol{\zeta}_n$ be defined by equality* (5.6). *Then*

$$\boldsymbol{\zeta}_n' = O(n^{-1-\varepsilon}) \tag{6.7}$$

for some $\varepsilon > 0$.

Proof. Let us write $\boldsymbol{\zeta}_n$ as a product

$$\boldsymbol{\zeta}_n = (\boldsymbol{\kappa}_n n^\nu)(n^\nu \boldsymbol{\theta}_n)^{-1}(\boldsymbol{\theta}_n(e^{-i\boldsymbol{\theta}_n} - 1)^{-1}), \quad \nu = \sigma - 2\rho, \tag{6.8}$$

and estimate all factors separately. It follows from relation (5.16) that the product $\boldsymbol{\kappa}_n n^\nu$ tends to 1 and its derivative is $O(n^{-2})$ as $n \to \infty$. Next, we consider $(n^\nu \boldsymbol{\theta}_n)^{-1}$. According to definitions (2.6) and (2.7) we have

$$n^\nu \theta_n = (n^\nu \sqrt{t_n})\sqrt{1 + \sum_{l=1}^{L-1} p_{l+1} t_n^l}. \tag{6.9}$$

By definition (1.11), the factor $n^\nu \sqrt{t_n}$ has a finite non-zero limit as $n \to \infty$. Moreover, its derivative is $O(n^{-\sigma})$ for $\sigma > 1$ and $O(n^{\sigma-2})$ for $\sigma < 1$ (it is zero if $\sigma = 1$). Similarly, the derivative of the second factor in (6.9) is $O(n^{-2})$ for $\sigma > 1$ and $O(n^{-1-\sigma})$ for $\sigma < 1$. These arguments also show that $\theta_n' = O(n^{-3/2})$ for $\sigma > 1$ and $\theta_n' = O(n^{-\sigma/2-1})$ for $\sigma < 1$. Therefore, the derivative of the third factor in (6.8) is also $O(n^{-1-\varepsilon})$, $\varepsilon > 0$. This proves estimate (6.7) on product (6.8). ∎

To estimate sum (6.5), we use the following elementary assertion of a general nature.

Lemma 6.3. *Suppose that a sequence $x_n \in \ell^1(\mathbb{Z}_+)$ and a sequence $\vartheta_n \geq 0$. Set*

$$\phi_n = \sum_{m=0}^{n} \vartheta_m \tag{6.10}$$

and assume that

$$\lim_{n\to\infty} \phi_n = \infty. \tag{6.11}$$

Then

$$\lim_{n\to\infty} e^{-\phi_n}\sum_{m=0}^{n} x_m e^{\phi_m} = 0.$$

Proof. By definition (6.10), we have

$$e^{-\phi_n}\sum_{m=0}^{n} x_m e^{\phi_m} = \sum_{m=0}^{\infty} X_m(n) \tag{6.12}$$

where

$$X_m(n) = x_m \exp\Big(-\sum_{p=m}^{n}\vartheta_p\Big) \quad \text{if } m \le n$$

and $X_m(n) = 0$ if $m > n$. Clearly, $X_m(n) \le x_m$ because $\vartheta_n \ge 0$ and $X_m(n) \to 0$ as $n \to \infty$ for fixed m by virtue of condition (6.11). Therefore, by the dominated convergence theorem, sum (6.12) tends to zero as $n \to \infty$. ■

Now, we are in a position to estimate the third term in (6.4).

Lemma 6.4. *Sum* (6.5) *satisfies the condition*

$$\lim_{n\to\infty} \widetilde{\Sigma}_n e^{i\varphi_n} = 0. \tag{6.13}$$

Proof. It follows from estimates (5.24) and (6.7) that

$$|(\boldsymbol{\zeta}_n \mathbf{u}_n)'| \le |\boldsymbol{\zeta}_n'||\mathbf{u}_n| + |\boldsymbol{\zeta}_{n+1}||\mathbf{u}_n'| \le C n^{-\delta+1} \tag{6.14}$$

where the value of δ is indicated in Theorems 5.6, 5.7 and 5.8. Therefore, by definition (6.5) and the differentiation formula (5.1), we have

$$|\widetilde{\Sigma}_n| \le C\sum_{m=n_0-1}^{n-1} m^{-\delta+1}e^{\phi_m} = C\sum_{m=n_0-1}^{n-1} y_m (e^{\phi_m})', \quad \phi_m = \operatorname{Im}\varphi_m, \tag{6.15}$$

where

$$y_m = m^{-\delta+1}(e^{\vartheta_m}-1)^{-1}, \quad \vartheta_m = \phi_{m+1}-\phi_m. \tag{6.16}$$

Using relation (5.3) and integrating in the right-hand side of (6.15) by parts, we find that

$$|\widetilde{\Sigma}_n| \le C\Big(y_{n-1}e^{\phi_n} - y_{n_0-2}e^{\phi_{n_0-1}} - \sum_{m=n_0-1}^{n-1} y_{m-1}' e^{\phi_m}\Big). \tag{6.17}$$

Let us estimate expression (6.16). It follows from relations (2.9), (2.10), and (2.11) that

$$\phi_n = cn^{-\mu}(1+o(1))$$

for some $c = c_{\sigma,\tau} > 0$. Here $\mu = \sigma/2$ if $\sigma \leq 1$, $\mu = 1/2$ if $\sigma \in [1, 3/2]$, $\tau > 0$ and $\mu = \sigma - 1/2$ if $\sigma \in [1, 3/2]$, $\tau < 0$. Therefore, product (6.16) is estimated as

$$|y_n| \leq C n^{-\delta+1+\mu}. \tag{6.18}$$

Note that $-\delta + 1 + \mu < 0$ for all values of σ and τ. Moreover, estimate (6.18) can be differentiated which yields

$$|y_n| \leq C n^{-\varepsilon}, \quad |y_n'| \leq C n^{-1-\varepsilon}$$

for some $\varepsilon > 0$.

Thus, it follows from inequality (6.17) that

$$e^{-\phi_n} |\tilde{\Sigma}_n| \leq C \Big(n^{-\varepsilon} + \sum_{m=n_0-1}^{n-1} m^{-1-\varepsilon} e^{-\phi_n + \phi_m} \Big)$$

which in view of Lemma 6.3 implies relation (6.13). ■

Let us now recall equality (6.4) and put relations (6.6) and (6.13) together. This leads to the following result.

Lemma 6.5. *Sum* (6.1) *satisfies the condition*

$$\lim_{n \to \infty} \Sigma_n e^{i\varphi_n} = -i\gamma\varkappa. \tag{6.19}$$

Proof of Theorem 2.7 *and Proposition* 2.8. Using equality (6.2) we can now conclude the proofs of Theorem 2.7 and Proposition 2.8. Indeed, combining asymptotics (2.1) and (6.19), we obtain relation (1.19). This implies both formulas (2.21) and (2.26). ■

Recall (see Section 1.1, for more details) that equation (1.1) is in the limit point case if, for $\operatorname{Im} z \neq 0$, it has a unique, up to a constant factor, non-trivial solution $f_n(z)$ such that inclusion (2.22) is satisfied. This is equivalent to the essential self-adjointness of the minimal Jacobi operator $J_{\min}$ in the space $\ell^2(\mathbb{Z}_+)$. In this case we set $\operatorname{clos} J_{\min} = J_{\max} =: J$.

According to Theorem 2.7 for $\operatorname{Im} z \neq 0$, the sequences $g_n(z)$ tend to infinity exponentially as $n \to \infty$ and according to Proposition 2.8 they tend to infinity as a power of n (or to zero but slower than $n^{-1/2}$). In all cases, relation (2.25) is satisfied. Therefore, it follows from the limit point/circle theory that under our assumptions the operators $J_{\min}$ are essentially self-adjoint. This proves Proposition 2.9.

Now, it is easy find an asymptotics of all solutions $F = (F_n)$ of equation (1.1). Indeed, using Proposition 6.1, we see that

$$F_n = -W[F, f] g_n + c f_n$$

for some constant c. The asymptotics of the solutions g_n and f_n are given by formulas (1.19) and (2.1). Obviously, f_n makes no contribution to the asymptotics of F_n. This leads to the following result.

Theorem 6.6. *Let one of the following three assumptions be satisfied:*

(1) *the conditions of Theorem 2.1 where either $\tau < 0$ and $\operatorname{Im} z \neq 0$ or $\tau > 0$ and $z \in \mathbb{C}$ is arbitrary;*

(2) *the conditions of Theorem 2.3 where either $\gamma > 0$ and $z \notin [0, \infty)$ or $\gamma < 0$ and $z \notin (-\infty, 0]$;*

(3) *the conditions of Theorem 2.4 where either $\gamma > 0$ and $z \notin [\tau, \infty)$ or $\gamma < 0$ and $z \notin (-\infty, -\tau]$.*

Then an arbitrary solution $F(z) = (F_n(z))$ has an asymptotics, as $n \to \infty$,

$$F_n(z) = -iW[F(z), f(z)]\varkappa(z)(-\gamma)^{n+1} n^{-\rho} e^{-i\varphi_n(\gamma z)}(1 + o(1)), \quad z \notin \operatorname{clos} \mathcal{S},$$

where the coefficient $\varkappa(z)$ is given by formula (2.13).

In particular, Theorem 6.6 applies to the orthonormal polynomials $P_n(z)$. Apparently, in the critical case $|\gamma| = 1$, an asymptotic behavior of the orthonormal polynomials $P_n(z)$ for regular points $z \in \mathbb{C}$ was never investigated before (except of the Laguerre polynomials). This is technically the most difficult part of this paper.

6.2. Continuous spectrum

First, we check that, on the continuous spectrum of the operator J, the Jost solutions $f_n(\lambda + i0)$ and $f_n(\lambda - i0) = \overline{f_n(\lambda + i0)}$ of equation (1.1) are linearly independent. Recall that the Wronskian of two solutions of this equation is given by formula (5.5), the number ρ is defined by equalities (2.2) and the sequences $\theta_n(\lambda)$, $\varphi_n(\lambda)$ are constructed in Theorems 2.1, 2.3 and 2.4. Observe that boundary values of the coefficient $\varkappa(z)$ defined by formula (2.13) are given by the equalities

$$\varkappa(\lambda + i0) = \begin{cases} \sqrt{|\tau|} & \text{if } \sigma > 1,\ \tau < 0,\ \lambda \in \mathbb{R}, \\ \sqrt{\lambda} & \text{if } \sigma < 1,\ \lambda > 0, \\ \sqrt{\lambda - \tau} & \text{if } \sigma = 1,\ \lambda > \tau, \end{cases} \tag{6.20}$$

and $\varkappa(\lambda - i0) = -\varkappa(\lambda + i0)$.

Lemma 6.7. *Let one of the following three assumptions be satisfied:*

(1) *the conditions of Theorem 2.1 with $\tau < 0$ and $\lambda \in \mathbb{R}$;*

(2) *the conditions of Theorem 2.3 with $\gamma\lambda > 0$;*

(3) *the conditions of Theorem 2.4 with $\gamma\lambda > \tau$.*

Then the Wronskian

$$w(\lambda) := \frac{1}{2i} W[f(\lambda + i0), f(\lambda - i0)] = \gamma \varkappa(\gamma(\lambda + i0)) > 0. \tag{6.21}$$

Proof. Set $\varphi_n = \varphi_n(\gamma(\lambda + i0))$, $u_n = u_n(\gamma(\lambda + i0))$. It follows from formulas (1.15) and (1.17) that

$$2i w(\lambda) = -\gamma a_n n^{-\rho}(n+1)^{-\rho}(e^{i\varphi_n} e^{-i\varphi_{n+1}} u_n \bar{u}_{n+1} - e^{-i\varphi_n} e^{i\varphi_{n+1}} \bar{u}_n u_{n+1}).$$

Using condition (1.8), we see that

$$w(\lambda) = -\gamma n^{\nu} \operatorname{Im}(e^{-i\theta_{n+1}} u_n \bar{u}_{n+1})(1 + o(1)) \tag{6.22}$$

where $\theta_n = \varphi_{n+1} - \varphi_n$ and $\nu = \sigma - 2\rho$. Observe that

$$\operatorname{Im}(e^{-i\theta_{n+1}} u_n \bar{u}_{n+1}) = \operatorname{Im}((u_n - u_{n+1})\bar{u}_{n+1}) - \theta_{n+1} \operatorname{Re}(u_n \bar{u}_{n+1}) + O(\theta_{n+1}^2). \tag{6.23}$$

According to Theorems 5.6, 5.7, or 5.8 the first term in the right-hand side of (6.23) is $O(n^{-\delta+1})$ where $\delta - 1 > \nu$. It follows from (2.14) that the second term is

$$-\varkappa(\gamma(\lambda + i0)) n^{-\nu}(1 + o(1)).$$

Finally, the contribution of $O(\theta_{n+1}^2)$ to (6.22) is zero. Therefore, equality (6.21) is a direct consequence of (6.22) and (6.23). ■

Let us introduce the Wronskian of the solutions $P(z) = (P_n(z))$ and $f(z) = (f_n(z))$ of equation (1.1):

$$\begin{aligned}\Omega(z) := W[P(z), f(z)] &= a_{-1}(P_{-1}(z) f_0(z) - P_0(z) f_{-1}(z)) \\ &= -a_{-1} f_{-1}(z), \quad z \notin \mathcal{S}.\end{aligned} \tag{6.24}$$

Lemma 6.8. *The function* $\Omega(z)$ *is analytic in* $\mathbb{C} \setminus \operatorname{clos} \mathcal{S}$ *and* $\Omega(z) = 0$ *if and only if* z *is an eigenvalue of the operator* J*. In particular,* $\Omega(z) \neq 0$ *for* $\operatorname{Im} z \neq 0$.

Proof. The analyticity of $\Omega(z)$ is a direct consequence of definition (6.24) because $f_{-1}(z)$, as well as all functions $f_n(z)$, is analytic. If $\Omega(z) = 0$, then $P(z)$ and $f(z)$ are proportional whence $P(z) \in \ell^2(\mathbb{Z}_+)$ by virtue of Proposition 2.6. Since $P_{-1}(z) = 0$, it follows that $JP(z) = zP(z)$ so that z is an eigenvalue of the operator J. For $\operatorname{Im} z \neq 0$, this is impossible because J is a self-adjoint operator. Conversely, if z is an eigenvalue of J, then $P(z) \in \ell^2(\mathbb{Z}_+)$, and hence $f(z)$ and $P(z)$ are proportional. ■

Now, we are in a position to find an asymptotic behavior of the polynomials $P_n(\lambda)$ for λ in the absolutely continuous spectrum (except thresholds) of the Jacobi operator J. Since the Jost solutions $f_n(\lambda \pm i0)$ are linearly independent and $\overline{P_n(\lambda)} = P_n(\lambda)$, we see that

$$P_n(\lambda) = \overline{c(\lambda)} f_n(\lambda + i0) + c(\lambda) f_n(\lambda - i0) \tag{6.25}$$

for some complex constant $c(\lambda)$. Taking the Wronskian of this equation with $f(\lambda + i0)$, we can express $c(\lambda)$ via Wronskian (6.24):

$$-c(\lambda) W[f(\lambda + i0), f(\lambda - i0)] = W[P(\lambda), f(\lambda + i0)] = \Omega(\lambda + i0)$$

whence

$$c(\lambda) = -\frac{\Omega(\lambda + i0)}{2iw(\lambda)}.$$

In view of formula (6.25), this yields the following result.

Lemma 6.9. *For all $\lambda \in \mathcal{S}$, we have the representation*

$$P_n(\lambda) = \frac{\Omega(\lambda - i0) f_n(\lambda + i0) - \Omega(\lambda + i0) f_n(\lambda - i0)}{2iw(\lambda)}, \quad n \in \mathbb{Z}_+. \tag{6.26}$$

Properties of the Wronskians $\Omega(\lambda \pm i0)$ are summarized in the following statement.

Theorem 6.10. *Let the assumptions of Lemma 6.7 be satisfied. Then the Wronskians $\Omega(\lambda + i0)$ and $\Omega(\lambda - i0) = \overline{\Omega(\lambda + i0)}$ are continuous functions of $\lambda \in \mathcal{S}$ and*

$$\Omega(\lambda \pm i0) \neq 0, \quad \lambda \in \mathcal{S}. \tag{6.27}$$

Proof. The functions $\Omega(\lambda \pm i0)$ are continuous in the same region as the Jost solutions. If $\Omega(\lambda \pm i0) = 0$, then, according to (6.26), $P_n(\lambda) = 0$ for all $n \in \mathbb{Z}_+$. However, $P_0(\lambda) = 1$ for all λ. ∎

Let us set

$$\kappa(\lambda) = |\Omega(\lambda + i0)|, \quad \Omega(\lambda \pm i0) = \kappa(\lambda) e^{\pm i\eta(\lambda)}. \tag{6.28}$$

In the theory of short-range perturbations of the Schrödinger operator, the functions $\kappa(\lambda)$ and $\eta(\lambda)$ are known as the *limit amplitude* and the *limit phase,* respectively; the function $\eta(\lambda)$ is also called the *scattering phase* or the *phase shift.* Definition (6.28) fixes the phase $\eta(\lambda)$ only up to a term $2\pi m$ where $m \in \mathbb{Z}$. We emphasize that the amplitude $\kappa(\lambda)$ and the phase $\eta(\lambda)$ depend on the values of the coefficients a_n and b_n for all n, and hence they are not determined by an asymptotic behavior of a_n, b_n as $n \to \infty$.

Combined together, relations (2.1) and (6.26) yield asymptotics of the orthonormal polynomials $P_n(\lambda)$.

Theorem 6.11. *Let one of the following three assumptions be satisfied:*

(1) *the conditions of Theorem 2.1 with* $\tau < 0$ *and* $\lambda \in \mathbb{R}$*;*

(2) *the conditions of Theorem 2.3 with* $\gamma\lambda > 0$*;*

(3) *the conditions of Theorem 2.4 with* $\gamma\lambda > \tau$*.*

Let the number ρ *be defined by equalities (2.2), and let* $\Phi_n(\lambda) = \varphi_n(\gamma(\lambda + i0))$ *where the sequences* $\varphi_n(\lambda)$ *are constructed in Theorems 2.1, 2.3, and 2.4. Then, for* $\lambda \in \mathcal{S}$*,*

$$P_n(\lambda) = \kappa(\lambda) w(\lambda)^{-1} (-\gamma)^n n^{-\rho} \sin(\Phi_n(\lambda) - \eta(\lambda))(1 + o(1)), \quad n \to \infty, \quad (6.29)$$

where the Wronskian $w(\lambda)$ *is given by equalities (6.20), (6.21) and the amplitude* $\kappa(\lambda)$ *and the phase* $\eta(\lambda)$ *are defined by relations (6.28).*

We emphasize that the definitions of the numbers ρ and $\Phi_n(\lambda)$ are different under assumptions (1), (2), and (3), but relation (6.29) is true in all these cases. Under the assumptions of Theorem 6.11 the functions $\Phi_n(\lambda)$ are real and $\Phi_n(\lambda) \to \infty$ so that $P_n(\lambda)$ are oscillating as $n \to \infty$.

A formula completely similar to (6.29) is true for all real solutions of equation (1.1). Only the coefficients $\kappa(\lambda)$ and $\eta(\lambda)$ are changed.

7. Spectral results

7.1. Resolvent. Discrete spectrum

If the minimal Jacobi operator $J_{\min}$ is essentially self-adjoint in the space $\ell^2(\mathbb{Z}_+)$, then, for $\operatorname{Im} z \neq 0$, equation (1.1) has a unique (up to a constant factor) solution $f_n(z) \in \ell^2(\mathbb{Z}_+)$. Let I be the identity operator in the space $\ell^2(\mathbb{Z}_+)$, and let $R(z) = (J - zI)^{-1}$ be the resolvent of the operator $J = \operatorname{clos} J_{\min}$. Recall that the Wronskian $\Omega(z)$ of the solutions $P_n(z)$ and $f_n(z)$ of equation (1.1) was defined by formula (6.24). The following statement is very close to the corresponding result for differential operators.

Proposition 7.1 ([31, Proposition 2.1]). *In the limit point case, for all* $h = (h_n) \in \ell^2(\mathbb{Z}_+)$*, we have*

$$(R(z)h)_n = \Omega(z)^{-1} \Big(f_n(z) \sum_{m=0}^{n} P_m(z) h_m + P_n(z) \sum_{m=n+1}^{\infty} f_m(z) h_m \Big), \quad \operatorname{Im} z \neq 0. \quad (7.1)$$

Remark 7.2. Let $e_0, e_1, \dots, e_n, \dots$ be the canonical basis in the space $\ell^2(\mathbb{Z}_+)$. Then representation (7.1) can be equivalently rewritten as

$$\langle R(z)e_n, e_m\rangle = \Omega(z)^{-1} P_n(z) f_m(z) \text{ if } n \leq m \text{ and } \langle R(z)e_n, e_m\rangle = \langle R(z)e_m, e_n\rangle. \tag{7.2}$$

According to Theorem 2.7 and Proposition 2.8, under our assumptions the operator $J_{\min}$ is essentially self-adjoint in the space $\ell^2(\mathbb{Z}_+)$. In view of Proposition 2.6, in this case $f_n(z)$ is the Jost solution. Thus, the resolvent of the Jacobi operator J admits representation (7.1) where $f_n(z)$ is the Jost solution.

Spectral results about the Jacobi operators J are direct consequences of representation (7.1). As far as the discrete spectrum is concerned, we use that according to Theorems 2.1, 2.3, and 2.4, the functions $f_n(z)$, $n = -1, 0, 1, \dots,$ and, in particular, $\Omega(z)$ are analytic functions of $z \in \mathbb{C} \setminus \operatorname{clos} \mathcal{S}$. In view of Lemma 6.8 this yields the part of Theorem 2.11 concerning the discrete spectrum. Let us state it explicitly.

Theorem 7.3. *Let assumptions* (1.8), (1.9) *with* $|\gamma| = 1$ *be satisfied.*

(1) *If* $\sigma \in (1, 3/2]$ *and* $\tau > 0$, *then the spectrum of the operator* J *is discrete.*

(2) *If* $\sigma \in (0, 1)$, *then the spectrum of the operator* J *is discrete on the half-axis* $(-\infty, 0)$ *for* $\gamma = 1$, *and it is discrete on* $(0, \infty)$ *for* $\gamma = -1$.

(3) *If* $\sigma = 1$, *then the spectrum of the operator* J *is discrete on the half-axis* $(-\infty, \tau)$ *for* $\gamma = 1$, *and it is discrete on* $(-\tau, \infty)$ *for* $\gamma = -1$.

7.2. Limiting absorption principle. Continuous spectrum

Next, we consider the absolutely continuous spectrum. According to Theorems 2.1, 2.3, and 2.4, the functions $f_n(z)$, $n = -1, 0, 1, \dots,$ and, in particular, $\Omega(z)$ are continuous up to the cut along the interval $\mathcal{S}$. Therefore, the following result is a direct consequence of relation (6.27) and representation (7.1). Recall that the set $\mathcal{D} \subset \ell^2(\mathbb{Z}_+)$ consists of finite linear combinations of the basis vectors $e_0, e_1, \dots$.

Theorem 7.4. *Let the assumptions of Theorems 2.1 for* $\tau < 0$, *2.3, or 2.4 be satisfied. Then for all* $u, v \in \mathcal{D}$, *the functions* $\langle R(z)u, v\rangle$ *are continuous in* z *up to the cut along the interval* $\mathcal{S}$ *as* z *approaches* $\mathcal{S}$ *from upper or lower half-planes.*

This result is known as the limiting absorption principle. It implies

Corollary 7.5. *The spectrum of the operator* J *is absolutely continuous on the closed interval* $\operatorname{clos} \mathcal{S}$, *except, possibly, eigenvalues at its endpoints. In particular, it is absolutely continuous and coincides with the whole real axis* $\mathbb{R}$ *if* $\sigma \in (1, 3/2]$ *and* $\tau < 0$.

Let us now consider the spectral projector $E(\lambda)$ of the operator J. By the Cauchy–Stieltjes–Privalov formula for $u, v \in \mathcal{D}$, its matrix elements satisfy the identity

$$2\pi i \frac{d\langle E(\lambda)u, v\rangle}{d\lambda} = \langle R(\lambda + i0)u, v\rangle - \langle R(\lambda - i0)u, v\rangle, \quad \lambda \in \mathcal{S}. \tag{7.3}$$

Therefore, the following assertion is a direct consequence of Theorem 7.4.

Corollary 7.6. *For all $u, v \in \mathcal{D}$, the functions $\langle E(\lambda)u, v\rangle$ are continuously differentiable in $\lambda \in \mathcal{S}$.*

Formulas (7.2) and (7.3) allow us to calculate the spectral family $dE(\lambda)$ in terms of the orthonormal polynomials and the Jost function. Indeed, substituting the expression

$$\langle R(\lambda \pm i0)e_n, e_m\rangle = \Omega(\lambda \pm i0)^{-1} P_n(\lambda) f_m(\lambda \pm i0), \quad n \le m,\ \lambda \in \mathcal{S},$$

into (7.3) and using the identity $\Omega(\lambda - i0) = \overline{\Omega(\lambda + i0)}$, we find that

$$2\pi i \frac{d\langle E(\lambda)e_n, e_m\rangle}{d\lambda} = P_n(\lambda) \frac{\Omega(\lambda - i0) f_m(\lambda + i0) - \Omega(\lambda + i0) f_m(\lambda - i0)}{|\Omega(\lambda \pm i0)|^2}.$$

Combining this representation with formula (6.26) for $P_m(\lambda)$, we obtain the following result.

Theorem 7.7. *Let the assumptions of Theorems 2.1 for $\tau < 0$, 2.3 or 2.4 be satisfied. Then for all $n, m \in \mathbb{Z}_+$, we have the representation*

$$\frac{d\langle E(\lambda)e_n, e_m\rangle}{d\lambda} = (2\pi)^{-1} w(\lambda) |\Omega(\lambda \pm i0)|^{-2} P_n(\lambda) P_m(\lambda), \quad \lambda \in \mathcal{S}, \tag{7.4}$$

where $w(\lambda)$ and $\Omega(z)$ are the Wronskians (6.21) and (6.24), respectively. In particular, the spectral measure of the operator J equals

$$d\Xi(\lambda) := d\langle E(\lambda)e_0, e_0\rangle = \xi(\lambda) d\lambda, \quad \lambda \in \mathcal{S},$$

where the weight $\xi(\lambda)$ is given by the formula

$$\xi(\lambda) = (2\pi)^{-1} w(\lambda) |\Omega(\lambda \pm i0)|^{-2}. \tag{7.5}$$

Remark 7.8. Formulas (7.4), (7.5) are also true (see [31]) in the non-critical case $|\gamma| < 1$ with $w = \sqrt{1 - \gamma^2}$ and $\mathcal{S} = \mathbb{R}$ as well as (see [28]) for stabilizing coefficients satisfying (1.6) with $w(\lambda) = 2^{-1}\sqrt{1 - \lambda^2}$ and $\mathcal{S} = (-1, 1)$ (if $a_\infty = 1/2$).

Remark 7.9. For the case $\sigma \in (0, 1)$, another representation for the weight $\xi(\lambda)$ was obtained in [17] – see formula [17, (4.12)]. It is difficult to compare these two representations because the Jost solutions were defined in [17] in terms of infinite products and formula [17, (4.12)] contains an implicit factor [17, (4.8)].

Putting together Theorem 6.10 and formula (7.5), we obtain the following.

Theorem 7.10. *Under the assumptions of Theorem 7.7 the weight $\xi(\lambda)$ is a continuous strictly positive function of $\lambda \in \mathcal{S}$.*

Note that this result was deduced in [13] from the subordinacy theory. The assumptions of [13] are more restrictive compared to Theorem 7.7; in particular, it was required in [13] that $\sigma \in (1/2, 2/3)$.

In view of (7.5) the scattering amplitude $\kappa(\lambda)$ defined by (6.28) can be expressed via the weight $\xi(\lambda)$:

$$\kappa(\lambda) = (2\pi)^{-1/2} w(\lambda)^{1/2} \xi(\lambda)^{-1/2}.$$

Hence, asymptotic formula (6.29) can be rewritten as

$$P_n(\lambda) = (2\pi w(\lambda)\xi(\lambda))^{-1/2} (-\gamma)^n n^{-\rho} \big(\sin(\Phi_n(\lambda) - \eta(\lambda)) + o(1)\big)$$

as $n \to \infty$. This form seems to be more common for the orthogonal polynomials literature.

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