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Counting eigenvalues of Schrödinger operators using the landscape function

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Abstract. We prove an upper and a lower bound on the rank of the spectral projections of the Schrödinger operator $-\Delta + V$ in terms of the volume of the sublevel sets of an effective potential $\frac{1}{u}$. Here, u is the 'landscape function' of G. David, M. Filoche, and S. Mayboroda [Adv. Math. 390 (2021), article no. 107946], namely a solution of $(-\Delta + V)u = 1$ in $\mathbb{R}^d$. We prove the result for non-negative potentials satisfying a Kato-type and a doubling condition, in all spatial dimensions, in infinite volume, and show that no coarse-graining is required. Our result yields in particular a necessary and sufficient condition for discreteness of the spectrum. In the case of nonnegative polynomial potentials, we prove that the spectrum is discrete if and only if no directional derivative vanishes identically.

1. Introduction

In a celebrated body of work, Fefferman and Phong [12] carried out an extensive analysis of the spectrum of self-adjoint differential operators based on the uncertainty principle, namely the fact that there is lower bound on the localization of the Fourier transform of a function that is well localized in space. Among the far reaching consequences of this old observation, they show that the number of eigenvalues E_j of a Schrödinger operator with positive polynomial potential V, below an energy μ, is equivalent to a coarse-grained notion of the volume of the sublevel sets of the potential at μ. Precisely, they count the number of boxes of side length of order $\mu^{-1/2}$ inside which V is less than or equal to μ. This coarse-graining is shown to be a necessary feature that arises from the uncertainty principle: for a test function to fit into a very narrow box, its kinetic energy must be large.

The Fefferman–Phong result is a wide generalization of the classical Weyl law, which is an asymptotic result for the number of eigenvalues of $-\Delta - \lambda V$ as $\lambda \to \infty$, see [50–53], and [20] for many extensions and variations. It is also closely related to

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the Lieb–Thirring inequalities [26] (see also [15] for a recent overview)

$$\sum_j E_j^\gamma \le L_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma+d/2} dx.$$

The case $\gamma = 0$ of interest to us in this work was obtained in dimensions $d \ge 3$ independently and by very different techniques by Cwickl [8], Lieb [24], and Rozenblum [37, 38], and we shall refer to it as the CLR inequality. It is well known that the CLR inequality cannot hold in complete generality for dimensions $d = 1, 2$. A general result in one dimension is the classical Calogero inequality [5]. It can further be used to obtain CLR-type bounds in some two-dimensional cases, see e.g. [16, 21–23, 39]. For more general kinetic energies and simplified proofs of the CLR inequality see [14, 18, 19] and the references therein.

Shen generalized the results of Fefferman and Phong to potentials in some reverse Hölder class and also in the presence of a magnetic field [40, 41]. For this, he used previously established L^p estimates for such Schrödinger operators [43] (in the case of non-negative polynomials, see also Smith [46] and Zhong [54]). These results rely on estimates on the Green's function, see Davey, Hill, and Mayboroda [9], as well as Mayboroda and Poggi [34] for optimal kernel estimates for more general elliptic operators and in the presence of a magnetic field, in dimensions $d \ge 3$. Poggi [33] considered potentials of Kato-type defined in more general domains. In dimension $d = 2$, Christ [7] obtained kernel estimates under stronger assumptions. We point out that Otelbaev obtained similar two-sided estimates earlier [30, 31], see also [28, 29] for related bounds.

The Fefferman–Phong approach was also extended by David, Filoche, and Mayboroda [10], who introduced a new technique which is central to the present work. While the bounds of Fefferman and Phong only depend on the dimension and the degree of polynomial, the dependence on the potential in DFM is more subtle and relies on the so-called landscape function. One considers $-\Delta + V$ on finite boxes Λ_L of side length L and define the *landscape function* as the solution of

$$(-\Delta + V)u_L = 1 \tag{1.1}$$

with suitable boundary conditions. A formal computation shows that the operator $-\Delta + V$ on $L^2(\Lambda_L, dx)$ is unitarily equivalent to the elliptic operator

$$-\frac{1}{u_L^2}\operatorname{div} u_L^2 \nabla + \frac{1}{u_L}$$

on $L^2(\Lambda_L, u_L^2(x)dx)$. We shall henceforth call $1/u_L$ the *effective potential*. The landscape function encodes (via the effective potential) a substantial part of the spectral information of the original Schrödinger operator. Among the many rigorous results using the effective potential [2–4,6,10,13,47–49], we are interested in the following, see [10]: if $\mathcal{N}_L^V(\mu)$ is the number of eigenvalues of $-\Delta + V$, counted with multiplicity, on the box with sidelength L (with periodic boundary conditions) that are smaller than μ, then there are constants $c, C > 0$ such that

$$N(c\mu, L) \le \mathcal{N}_L^V(\mu) \le N(C\mu, L),$$

where $N(\mu, L)$ is the coarsed-grained volume corresponding to the effective potential, namely it is the number of boxes of sidelength $\mu^{-1/2}$ in which $1/u_L \le \mu$. While C only depends on the dimension, the constant c depends on the oscillation of u_L.

In this paper, we consider the setting of [42] where the potentials are Kato-class and satisfy a doubling condition (the precise assumptions are (2.1)–(2.2)). We first study the existence of the landscape function in the whole space $\mathbb{R}^d$, namely the existence and positivity of solutions of (1.1). This purely PDE question is set in an a priori inconvenient space since, unlike in the case of finite boxes considered in [10], the right-hand side belongs to no L^p space but for $p = \infty$. Not surprisingly, this can be addressed by considering a sequence of compactly supported functions converging pointwise to 1. A similar approach was used by Poggi [33, Theorem 1.18] in dimension $d \ge 3$. With the landscape function and therefore the effective potential $\frac{1}{u}$ in hand, we turn to the problem of the counting of eigenvalues. We extend the DFM result to the infinite volume setting and without coarse-graining. We show that the effective potential is confining if and only if the spectrum of the Schrödinger operator is discrete, in which case the measure of its sublevel sets is finite. We then prove that this measure controls the eigenvalue counting function, namely

$$(c\mu)^{d/2}\mathcal{V}(c\mu) \le \mathcal{N}^V(\mu) \le (C\mu)^{d/2}\mathcal{V}(C\mu),$$

where

$$\mathcal{V}(\mu) = \int_{\{x\in\mathbb{R}^d : \frac{1}{u(x)} \le \mu\}} dx.$$

Crucially, this CLR-type bound is valid in all spatial dimensions, and for all $\mu \in \mathbb{R}$. The latter is one of the advantages of working immediately in infinite volume, since otherwise the size of the domain Λ_L yields a lower bound on the energy levels μ that can be considered.

As already seen in other applications of the DFM landscape function, this bound where the volume is not coarse-grained reflects the fact that the transformation

$$-\Delta + V \mapsto -\frac{1}{u^2}\operatorname{div} u^2\nabla + \frac{1}{u}$$

'transfers' some of the kinetic energy to the effective potential.

One may wonder about the relationship of the effective potential with the semi-classical limit. Not surprisingly, one partial answer is provided by microlocal analysis. Indeed, the effective potential is given by the resolvent acting on the constant function. The inverse of $-\Delta + V$ is a pseudo-differential operator whose symbol is given to highest order by $\frac{1}{|\xi|^2+V(x)}$. In this approximation, we conclude that, formally,

$$u(x) \simeq \int_{\mathbb{R}^d} \frac{e^{i\xi\cdot x}}{|\xi|^2 + V(x)} \delta(\xi) d\xi = \frac{1}{V(x)}.$$

In other words, the effective potential $\frac{1}{u}$ is equal to the physical potential V up to lower order corrections in the sense of microlocal analysis. It is precisely these corrections however that remove the need for coarse-graining.

We conclude this introduction by commenting on the specific case $V(x, y) = x^2 y^2$ in two dimensions. In [45], Simon provided five proofs that this operator has discrete spectrum with strictly positive first eigenvalue, that the number of eigenvalues below any fixed energy μ follows the Fefferman–Phong estimate (in particular, the coarse-grained volume is finite), and established the precise asymptotics in [44]. These results are particularly remarkable given that the Lebesgue measure of $\{(x, y) \in \mathbb{R}^2 : x^2 y^2 \leq \mu\}$ is infinite for every $\mu > 0$. This shows in particular that the measure of the sublevel sets of the potential do not capture the spectral information, unlike the coarsed-grained volume associated with the potential $x^2 y^2$. Using pseudo-differential calculus [36] obtains similar results for more general degenerate polynomials. Our result removes the need for coarse-graining, provided V is replaced with the DFM effective potential $\frac{1}{u}$. Since it is simple to see that the effective potential is confining in this case (by Corollary 2.3), we obtain yet another proof of discreteness of the spectrum of $-\Delta + x^2 y^2$ in $\mathbb{R}^2$.

2. Main results

We denote by u a particular weak solution of

$$(-\Delta + V)u = 1$$

in $\mathbb{R}^d$, which can be realized as the pointwise limit of specific Lax–Milgram solutions. It will be constructed in details in Section 3. Our main result is that the volume of

$$\left\{x \in \mathbb{R}^d : \frac{1}{u(x)} \leq \mu\right\}$$

is comparable to the rank of the spectral projection of (the Friedrich's extension of) $-\Delta + V$ at energies less than or equal to μ.

Theorem 2.1. *Assume that $V \geq 0, V \not\equiv 0$ satisfies the following conditions.*

(1) Kato-type condition. *There exists $C_{\mathrm{K}}, \delta > 0$, such that*

$$\frac{1}{r^{d-2+\delta}} \int_{B(x,r)} V(y) dy \leq C_{\mathrm{K}} \frac{1}{R^{d-2+\delta}} \int_{B(x,R)} V(y) dy \tag{2.1}$$

for all $x \in \mathbb{R}^d$ and all r, R with $0 < r < R$.

(2) Doubling condition. *There exists $C_{\mathrm{D}} > 0$ such that*

$$\int_{B(x,2r)} V(y) dy \leq C_{\mathrm{D}} \Big(\int_{B(x,r)} V(y) dy + r^{d-2} \Big) \tag{2.2}$$

for all $x \in \mathbb{R}^d$ and all $r > 0$.

We denote by H the Friedrichs extension of the positive symmetric operator

$$-\Delta + V$$

defined on $C_c^\infty(\mathbb{R}^d)$. Let $\mathcal{N}^V(\mu)$ be the rank of the spectral projection $\mathbb{1}_{(\infty,\mu]}(H)$. Then there exist constants $c, C > 0$ such that for all $\mu \in \mathbb{R}$,

$$(c\mu)^{\frac{d}{2}} \mathcal{V}(c\mu) \leq \mathcal{N}^V(\mu) \leq (C\mu)^{\frac{d}{2}} \mathcal{V}(C\mu), \tag{2.3}$$

where

$$\mathcal{V}(\mu) = \int_{\{x \in \mathbb{R}^d : \frac{1}{u(x)} \leq \mu\}} dx.$$

The constants c, C depend only on $C_{\mathrm{K}}, C_{\mathrm{D}}, \delta$ and the spatial dimension d.

Assumption (2.1) is a scale-invariant variant of the standard Kato condition. For $d \geq 3$, one obtains via Fubini's theorem that the condition (2.1) is equivalent to

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{d-2}} dy \leq \frac{C}{R^{d-2}} \int_{B(x,R)} V(y) dy$$

for all $0 < R$, all $x \in \mathbb{R}^d$ and some C independent of x, R. Conditions (2.1)–(2.2) are satisfied by potentials in the reverse Hölder class $(RH)_{d/2}$ (see [43]). In particular, this include non-negative polynomials and fractional power functions $|x|^\alpha$ for $\alpha > -2$ for $d \geq 3$. On the other hand, potentials with compact support or exponential growth violate (2.1), respectively (2.2).

Combining the above theorem with the property that u varies slowly, we further derive an analogous result to [40, Corollary 0.11].

Corollary 2.2. *Let V be as in Theorem 2.1. Then the spectrum of H is discrete if and only if* $\lim_{R\to\infty} \|u\|_{L^\infty(\mathbb{R}^d\setminus B(0,R))} = 0$.

We will present the proofs in Section 4, while in Section 5 we concentrate on the case where V is a polynomial. For polynomial potentials one can further analyze the landscape function. In particular, one has the following.

Corollary 2.3. *Let V be a polynomial that is bounded from below. Then the spectrum of H is discrete if and only if none of the directional derivatives of V vanishes identically.*

If the condition of the last corollary is violated, then the corresponding operator has no eigenvalues. Indeed, after conjugating with a suitable rotation, we can assume without loss of generality that the polynomial does not depend on the last variable. We define $\widetilde{V}(x_1,\dots,x_{d-1}) = V(x_1,\dots,x_d)$. By taking a Fourier transform in the last variable, we get that the Friedrichs extension H of $-\Delta + V$ is unitarily equivalent to the direct integral

$$\int_{\mathbb{R}}^{\oplus} (p^2 + \widetilde{H})dp$$

where $\widetilde{H} = -\Delta_{\mathbb{R}^{d-1}} + \widetilde{V}$. It follows from [35, Theorem XIII.85] that $\sigma(H) = [\min\sigma(\widetilde{H}), \infty)$ and H admits no eigenvalue.

Finally, we point out that Theorem 2.1 together with (3.7) below recover the result of Shen [40, Theorem 0.9] in our class of potentials and in the absence of a magnetic potential.

3. Existence of the landscape function in infinite volume

In this section we show the existence of the landscape function in infinite volume and establish some estimates of the landscape function in terms of the Fefferman–Phong–Shen maximal function. In [33, Theorem 1.18, Theorem 1.31] Poggi proves this for $d \geq 3$. We briefly recall the construction and explain how to extend this to the case $d = 1, 2$. For this, we will rely on extensions of results for $d \geq 3$ due to Shen (see [42, Proposition 1.8]). One of the key objects is the Fefferman–Phong–Shen maximal function $m(\cdot, V)$, which is defined as

$$\frac{1}{m(x,V)} = \sup\Big\{r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)dy \leq C_{\mathrm{D}}\Big\}, \tag{3.1}$$

where C_{D} is the constant in (2.2). This maximal function satisfies the following properties.

Lemma 3.1. *Let V satisfy the conditions of Theorem 2.1.*

(1) $0 < m(x, V) < \infty$ *for every* $x \in \mathbb{R}^d$.

(2) *For every C', there exists C, depending only on $C_{\mathrm{K}}, C_{\mathrm{D}}, \delta$ and C', such that*

$$C^{-1} m(x, V) \leq m(y, V) \leq C m(x, V) \tag{3.2}$$

for all $x, y \in \mathbb{R}$ with $|x-y| \leq \frac{C'}{m(x,V)}$.

(3) *There exists $k_0, C > 0$, depending only on C_K, C_D, δ and d, such that for all $x, y \in \mathbb{R}^d$ we have*

$$m(x, V) \leq C m(y, V)(1 + |x-y| m(y, V))^{k_0}. \tag{3.3}$$

(4) *Let $d \leq 2$ and let $\widetilde{V}(x,t) = V(x)$ for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. Then for all $(x,t) \in \mathbb{R}^{d+1}$ and all $0 < r < R$,*

$$\frac{1}{r^{d-1+\delta}} \int\limits_{B((x,t),r)} \widetilde{V}(z) dz \leq C_{\mathrm{K}} \sqrt{2}^{d-1+\delta} \frac{1}{R^{d-1+\delta}} \int\limits_{B((x,t),R)} \widetilde{V}(z) dz.$$

Furthermore,

$$\int\limits_{B(x,2r)} \widetilde{V}(z) dz \leq 4C_{\mathrm{D}} \Big(\int\limits_{B((x,t),r)} \widetilde{V}(z) dz + r^{d-1} \Big)$$

for all $(x,t) \in \mathbb{R}^{d+1}$ and all $r > 0$.

Finally, there exists $C > 0$ depending on $C_{\mathrm{K}}, C_{\mathrm{D}}, \delta$ and d such that

$$C^{-1} m(x, V) \leq m((x,t), \widetilde{V}) \leq C m(x, V) \tag{3.4}$$

for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}$.

Note that in (3.4), the exponent in the maximal function involving V is d, while it is $d+1$ in the one involving $\widetilde{V}$.

Proof. First, we note that (2.1) yields for all $0 < r < R$

$$R^{\delta} \frac{1}{r^{d-2}} \int\limits_{B(x,r)} V(y) dy \leq C_{\mathrm{K}} r^{\delta} \frac{1}{R^{d-2}} \int\limits_{B(x,R)} V(y) dy.$$

Thus, $\lim_{r \to 0+} r^{2-d} \int_{B(x,r)} V(y) dy = 0$ and $\lim_{R \to \infty} R^{2-d} \int_{B(x,R)} V(y) dy = \infty$. This implies that $0 < m(x, V) < \infty$.

The validity of (2) *and* (3). for $d \geq 3$ is proved in [42, Proposition 1.8]. Hence, it suffices to prove (4) for (2). and (3) to hold for all $d \geq 1$.

The fact that $\widetilde{V}$ satisfies both the Kato-type and the doubling condition are simple computations. Since V satisfies the doubling condition in dimension d,

$$\begin{aligned}\int_{B((x,t),2r)} \widetilde{V}(z)dz &\leq \int_{-2r}^{2r} C_{\mathrm{D}}\Big(\int_{B(x,\sqrt{r^2-s^2/4})} V(y)dy + (r^2 - s^2/4)^{(d-2)/2}\Big)ds \\ &= 2C_{\mathrm{D}}\Big(\int_{B((x,t),r)} \widetilde{V}(z)dz + r^{d-1}\int_{-1}^{1}(1-\sigma^2)^{(d-2)/2}d\sigma\Big),\end{aligned}$$

which yields the claim upon noting that the last integral is bounded above by π. For the Kato-type condition, we first consider the case $0 < \sqrt{2}r < R$. Then we have

$$\begin{aligned}\int_{B((x,t),r)} \widetilde{V}(z)dz &\leq \int_{B(x,r)\times(t-r,t+r)} \widetilde{V}(z)dz = 2r\int_{B(x,r)} V(y)dy \\ &\leq C_{\mathrm{K}}(2r)\Big(\frac{r}{R/\sqrt{2}}\Big)^{d-2+\delta}\int_{B(x,R/\sqrt{2})} V(y)dy \\ &= C_{\mathrm{K}}\Big(\sqrt{2}\frac{r}{R}\Big)^{d-1+\delta}\int_{B(x,R/\sqrt{2})\times(t-R/\sqrt{2},t+R/\sqrt{2})} \widetilde{V}(z)dz \\ &\leq C_{\mathrm{K}}\Big(\sqrt{2}\frac{r}{R}\Big)^{d-1+\delta}\int_{B((x,t),R)} \widetilde{V}(z)dz.\end{aligned}$$

The bound is immediate if, on the other hand, $\frac{R}{\sqrt{2}} \leq r < R$, since

$$\int_{B((x,t),r)} \widetilde{V}(z)dz \leq \int_{B((x,t),R)} \widetilde{V}(z)dz \leq \Big(\sqrt{2}\frac{r}{R}\Big)^{d-1+\delta}\int_{B((x,t),R)} \widetilde{V}(z)dz.$$

To show (3.4), we introduce the following maximal function

$$\frac{1}{m_Q(x,V)} = \sup\Big\{r > 0 : \frac{1}{r^{d-2}}\int_{Q(x,r)} V(y)dy \leq C_{\mathrm{D}}\Big\},$$

which is defined over cubes $Q(x,r)$ centered at x and of sidelength r, rather than over balls. Clearly, (3.4) holds true with $C = 1$ for m replaced by m_Q. Thus, we only need to show that m and m_Q are equivalent. For $d \leq 2$, the inclusion $Q(x,r) \subset B(x,r)$

and the positivity of V yield immediately $\frac{1}{m(x,V)} \leq \frac{1}{m_Q(x,V)}$, and hence $m_Q(x,V) \leq m(x,V)$. Reciprocally, let $x \in \mathbb{R}^d$ and let $r = \frac{2}{m(x,V)}$. Then for any $R > C_{\mathrm{K}}^{1/\delta} r$,

$$\begin{aligned} C_{\mathrm{D}} &= \frac{1}{(r/2)^{d-2}} \int\limits_{B(x,r/2)} V(y)dy \leq C_{\mathrm{K}} \Big(\frac{r}{R}\Big)^{\delta} \frac{1}{(R/2)^{d-2}} \int\limits_{B(x,R/2)} V(y)dy \\ &\leq \frac{1}{R^{d-2}} \int\limits_{Q(x,R)} V(y)dy \end{aligned}$$

where we used (2.1) in the second inequality, and the fact that $B(x,R/2) \subseteq Q(x,R)$ in the third. It follows that $\frac{1}{m_Q(x,V)} \leq C_{\mathrm{K}}^{1/\delta} r$ and so $m_Q(x,V) \geq \frac{m(x,V)}{2C_{\mathrm{K}}^{1/\delta}}$. ∎

Let $f \in L^\infty(\mathbb{R}^d)$ be compactly supported. We call u_f a *Lax–Milgram solution* of

$$(-\Delta + V)u = f$$

if u_f is in the form domain $\mathcal{H}$ of H and

$$\int\limits_{\mathbb{R}^d} (\nabla u_f(y) \cdot \nabla v(y) + V(y)u_f(y)v(y))dy = \int\limits_{\mathbb{R}^d} f(y)v(y)dy$$

for all $v \in \mathcal{H}$. Note that

$$\mathcal{H} = \Big\{ v \in H^1(\mathbb{R}^d) : \int\limits_{\mathbb{R}^d} V(y)|v(y)|^2 dy < \infty \Big\},$$

see [11, Theorem 8.2.1]. The following proposition yields estimates for Lax–Milgram solutions.

Proposition 3.2 ([42, Theorem 0.8, Theorem 2.16]). *Let $d \geq 3$ and assume V satisfies the conditions of Theorem 2.1. For every $x \in \mathbb{R}^d$ there exists a function $\Gamma_V(x,\cdot) \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, for $1 < p < d/(d-2)$, such that for all $f \in L^\infty(\mathbb{R}^d)$ with compact support and $f \geq 0$, the unique Lax–Milgram solution u_f of $(-\Delta + V)u = f$ can be written as*

$$u_f(x) = \int\limits_{\mathbb{R}^d} \Gamma_V(x,y) f(y)dy \tag{3.5}$$

for almost every $x \in \mathbb{R}^d$. Furthermore, one has the kernel estimate

$$\frac{c e^{-\varepsilon(1+|x-y|m(x,V))^{k_0+1}}}{|x-y|^{d-2}} \leq \Gamma_V(x,y) \leq \frac{C e^{-\varepsilon(1+|x-y|m(x,V))^{1/(k_0+1)}}}{|x-y|^{d-2}}. \tag{3.6}$$

Proof. Existence and uniqueness of Lax–Milgram solution follows directly from the Lax–Milgram theorem on the form domain $\mathcal{H}$ equipped with its standard inner product $\langle v, w\rangle_{\mathcal{H}} = \langle \nabla v, \nabla w\rangle_{L^2(\mathbb{R}^d)} + \langle \sqrt{V} v, \sqrt{V} w\rangle_{L^2(\mathbb{R}^d)}$. The representation of the Lax–Milgram solution in terms of the integral kernel Γ_V was shown in [42, Theorem 2.16] and the estimate in terms of the Fefferman–Phong–Shen maximal function follow from [42, Theorem 3.11, Remark 3.21, Theorem 4.15]. ∎

In what follows, we will also consider *weak solutions* of $(-\Delta + V)u = f$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, namely a function u_f such that

$$\int_{\mathbb{R}^d} u_f(y)(-\Delta\varphi(y) + V(y)\varphi(y))dy = \int_{\mathbb{R}^d} f(y)\varphi(y)dy$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

We shall now construct the landscape function in infinite volume. This is an alternative and simpler approach, valid in the present setting, than that of [33, Theorem 1.18].

Proposition 3.3. *Let V be as in Theorem 2.1. Then there exist constants $c, C > 0$ depending only on $C_{\mathrm{K}}, C_{\mathrm{D}}, \delta$ and d, and a weak solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap C^0(\mathbb{R}^d)$ of $(-\Delta + V)u = 1$ such that*

$$\frac{c}{m(x,V)^2} \le u(x) \le \frac{C}{m(x,V)^2}, \tag{3.7}$$

for almost every $x \in \mathbb{R}^d$.

For later purposes, we immediately note that the proof of the proposition yields the following 'finite volume' result. If $d \geq 3$ we denote by u_L the Lax–Milgram solutions of

$$(-\Delta + V)u_L = \mathbb{1}_{B(0,L)}. \tag{3.8}$$

Then

$$\frac{c}{m(x,V)^2} \le u_L(x) \le \frac{C}{m(x,V)^2}, \tag{3.9}$$

for almost every $x \in \mathbb{R}^d$. We remark that all the results work equally well if we replace the indicator function over balls by indicator function over other compact sets $\{\Omega_L : L \in \mathbb{N}\}$ such that $\Omega_L \subseteq \Omega_{\tilde{L}}$ for $L \le \tilde{L}$ and $\bigcup_{L\ge 1} \Omega_L = \mathbb{R}^d$.

Proof. First we consider the case $d \geq 3$. Denote by u_L the Lax–Milgram solution (3.8) given by Proposition 3.2. As $\mathbb{1}_{B(0,L_2)} - \mathbb{1}_{B(0,L_1)} \geq 0$ for $L_2 \geq L_1$, we get from (3.5) that $(u_L)_{L\geq 1}$ is monotone increasing almost everywhere. On the other hand, it is essentially bounded since

$$0 \leq u_L(x) \leq \frac{C}{m(x,V)^2} \int_{\mathbb{R}^d} \frac{e^{-\varepsilon(1+|y|)^{1/(k_0+1)}}}{|y|^{d-2}} dy = \frac{\widetilde{C}}{m(x,V)^2}$$

for almost every $x \in \mathbb{R}^d$ by (3.5)–(3.6). Thus, we can define

$$u(x) = \lim_{L\to\infty} u_L(x).$$

As u_L are Lax–Milgram solutions of $(-\Delta + V)u_L = \mathbb{1}_{B(0,L)}$, one easily checks that u is a weak solution of $(-\Delta + V)u = 1$. The lower bound for u follows from the lower bound in (3.6).

We show now that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ for $d \geq 3$. Fix any ball $B \subseteq \mathbb{R}^d$ and a smooth cut-off function $\chi_B \in C_c^\infty(\mathbb{R}^d)$ such that $\chi_B \equiv 1$ on B. As $\chi_B u_L \in \operatorname{dom}(H^{1/2})$ and u_L is a Lax–Milgram solution of (3.8), we get

$$\int_{\mathbb{R}^d} \nabla u_L \cdot \nabla(\chi_B u_L) + \int_{\mathbb{R}^d} V u_L(\chi_B u_L) = \int_{\mathbb{R}^d} u_L \chi_B \mathbb{1}_{B(0,L)}.$$

The product rule for Sobolev functions yields

$$\int_{\mathbb{R}^d} \nabla u_L \cdot \nabla(\chi_B u_L) = \int_{\mathbb{R}^d} |\nabla u_L|^2 \chi_B + \int_{\mathbb{R}^d} \nabla u_L \cdot (\nabla \chi_B) u_L.$$

Using integration by parts for the second term on the right-hand side yields

$$\int_{\mathbb{R}^d} \nabla u_L \cdot (\nabla \chi_B) u_L = -\int_{\mathbb{R}^d} ((\Delta \chi_B) u_L^2 + u_L (\nabla \chi_B) \cdot \nabla u_L).$$

Thus, we get

$$\int_B |\nabla u_L|^2 \leq \int_{\mathbb{R}^d} \chi_B |\nabla u_L|^2 = \int_{\mathbb{R}^d} u_L \chi_B \mathbb{1}_{B(0,L)} - \int_{\mathbb{R}^d} V \chi_B u_L^2 + \frac{1}{2} \int_{\mathbb{R}^d} (\Delta \chi_B) u_L^2.$$

Hence, there exists a constant $C > 0$ depending only on the dimension such that

$$\int_B (|\nabla u_L|^2 + V u_L^2) \leq \int_{2B} u_L + C \int_{2B} u_L^2$$

for all balls B and all $L > 0$. By (3.9) and (3.3), we get that $(\nabla u_L)_{L\geq 1}$ is uniformly bounded in $L^2(B)$ for fixed B. Therefore, by Banach–Alaoglu, there exists a subsequence u_{L_k} converging weakly to some $g_B \in L^2(\mathbb{R}^d)$. One readily checks that g_B is the weak derivative of u and hence $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Next, we consider the case $d = 2$. For this, we use Hadamard's method of descent. Recall that $\widetilde{V}(x,t) = V(x)$ for $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$. By Lemma 3.1, the function $\widetilde{V}$ satisfies (2.1)–(2.2), and therefore the first part yields a weak solution $\tilde{u}$ of $(-\Delta + \widetilde{V})\tilde{u} = 1$ on $\mathbb{R}^3$. Let $\alpha \in \mathbb{R}$. One readily checks that $\widetilde{v_L}(x,t) = \widetilde{u_L}(x,t+\alpha)$ is a Lax–Milgram solution of $(-\Delta + \widetilde{V})\widetilde{v_L} = \mathbb{1}_{B((0,0,\alpha),L)}$. Thus, by (3.5), we have

$$\widetilde{u_L}(x,t+\alpha) = \int_{\mathbb{R}^3} \Gamma_{\widetilde{V}}((x,t),y)\mathbb{1}_{B((0,0,\alpha),L)}(y)dy.$$

As $\Gamma_{\widetilde{V}}((x,t),\cdot) \in L^1(\mathbb{R}^3)$ by (3.6), we get by dominated convergence

$$\tilde{u}(x,t+\alpha) = \lim_{L\to\infty} \widetilde{u_L}(x,t+\alpha) = \int_{\mathbb{R}^3} \Gamma_{\widetilde{V}}((x,t),y)dy = \tilde{u}(x,t).$$

Hence, for almost every $x \in \mathbb{R}^2$ there exists C_x such that for almost every $t \in \mathbb{R}$ we have $\tilde{u}(x,t) = C_x$ and we define u on $\mathbb{R}^2$ by $u(x) = C_x$. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ and $\psi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \psi(t)dt = 1$. Then, as $\int_{\mathbb{R}} \psi''(t)dt = 0$ and $\tilde{u}$ is a weak solution of $(-\Delta + \widetilde{V})\tilde{u} = 1$ on $\mathbb{R}^3$, we get

$$\begin{aligned}\int_{\mathbb{R}^2} u(x)(-\Delta + V(x))\varphi(x)dx &= \int_{\mathbb{R}^3} \tilde{u}(x,t)(-\Delta + \widetilde{V}(x,t))(\varphi(x)\psi(t))dxdt \\ &= \int_{\mathbb{R}^3} \varphi(x)\psi(t)dt = \int_{\mathbb{R}^2} \varphi(x)dx.\end{aligned}$$

Therefore, $(-\Delta + V)u = 1$ is a weak solution on $\mathbb{R}^2$. Inequality (3.7) follows from (3.4). As shown before, we have $\tilde{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $|\nabla u(x)| = |\nabla \tilde{u}(x,t)|$ as $u(x,t)$ is independent of t. Hence, $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$.

The case $d = 1$ follows similarly to the case $d = 2$.

Finally, continuity follows from [27, Corollary 1.5]. ∎

We point out that the weak solution u constructed above does in general not belong to the form domain of H, and we will therefore often have to work with the Lax–Milgram solution u_L instead of u. If V is a polynomial, the maximal function $m(\cdot, V)$ is equivalent to the function introduced in [46, 54]

$$M(x,V) = \sum_{\alpha\in\mathbb{N}_0^n} |\partial^\alpha V(x)|^{1/(|\alpha|+2)}, \tag{3.10}$$

see (5.4) below. The sum is of course finite for a polynomial. We now consider $V(x) = |x|^2$ on $\mathbb{R}^d$. Then $M(x,V)$ is comparable to $1 + |x|$ and hence, by (3.7) and (5.4), $u(x)$ is comparable to $(1 + |x|)^{-2}$ which is not square integrable for $d > 2$.

In Lemma 5.3 we show that the landscape function, for polynomial potentials, belongs to the form domain if and only if the landscape function is integrable.

The equivalence of the landscape function and the Fefferman–Phong–Shen maximal function exhibited in Proposition 3.3 allows one to prove a Harnack inequality for the landscape function, see also [33, Corollary 1.38] for the case $d \geq 3$.

Corollary 3.4. *Let V be as in Theorem 2.1. Then there exists a constant $C_{\mathrm{H}} \geq 1$, depending only on $C_{\mathrm{K}}, C_{\mathrm{D}}, \delta$, and d, such that for almost every $x \in \mathbb{R}^d$ and almost every $y \in Q(x, 2\sqrt{u(x)})$ we have*

$$C_{\mathrm{H}}^{-1} u(x) \leq u(y) \leq C_{\mathrm{H}} u(x). \tag{3.11}$$

Proof. This follows immediately from (3.2) and (3.7). ∎

4. Proof of Theorem 2.1

In this section we show that we can estimate the rank $\mathcal{N}^V(\mu)$ of the spectral projection of H in terms of the measure of the sublevel set $\mathcal{V}(\mu)$ of the effective potential $\frac{1}{u}$, both defined in Theorem 2.1.

For this, we introduce two types of coarse-grained volumes. A box of sidelength ℓ is a set of the form $\times_{i=1}^{d}[a_i, b_i]$ where $b_i - a_i = \ell$. For any $\ell > 0$, we consider a collection $\mathcal{Q}_\ell$ of boxes of sidelength ℓ such that $\bigcup_{Q \in \mathcal{Q}_\ell} Q = \mathbb{R}^d$ and $\mathring{Q} \cap \mathring{Q}' = \emptyset$ whenever $Q \neq Q'$. We define for any $\mu > 0$

$$N(\mu) = \left| \left\{ Q \in \mathcal{Q}_{\mu^{-1/2}} : \inf_Q \frac{1}{u} \leq \mu \right\} \right|$$

and

$$n(\mu) = \left| \left\{ Q \in \mathcal{Q}_{\mu^{-1/2}} : \sup_Q \frac{1}{u} \leq \mu \right\} \right|,$$

where $\inf_Q$, $\sup_Q$ denote the essential infimum, respectively the essential supremum.

For the class of potentials considered here, namely those satisfying the Kato-type and doubling conditions, both coarse-grained volumes are directly related to the measure $\mathcal{V}(\mu)$ of the sublevel set. We now pick the cubes in $\mathcal{Q}_\ell$ as having their corners on $\ell\mathbb{Z}^d$.

Lemma 4.1. *Let V satisfy the conditions of Theorem 2.1. Then*

$$n(\mu) \leq \mu^{d/2} \mathcal{V}(\mu) \leq N(\mu) \leq n(2^{\lceil \log_2 C_{\mathrm{H}} \rceil} \mu)$$

for all $\mu \in \mathbb{R}$.

Proof. The first two inequalities are immediate as, up to null sets, $n(\mu)/\mu^{d/2}$ is the measure of all boxes that are strictly contained in the sublevel set $\{1/u \leq \mu\}$ and $N(\mu)/\mu^{d/2}$ is the measure of all the boxes that intersect the sublevel set.

With our specific choice of cubes, the smaller ones are completely included in exactly one larger one and hence, $N(\mu) \leq n(2^{\lceil \log_2 C_{\mathrm{H}} \rceil}\mu)$. ■

We now turn to the proof of the main theorem, namely the bounds (2.3). Our arguments are variational and adapted from the proofs of [10], which are themselves inspired by Fefferman and Phong [12]. We start with the upper bound.

Lemma 4.2. *Let V satisfy the conditions of Theorem 2.1. Then*

$$\mathcal{N}^V(\mu) \leq N(C\mu)$$

for all $C > \max\{2, \frac{2d}{\pi^2}\}$ and all $\mu \in \mathbb{R}$.

Proof. In order to have that $\mathcal{N}^V(\mu) \leq N$ it suffices, by the Min–Max principle (see [35, Theorem XIII.2]), to exhibit a subspace $\mathcal{H}_N \subseteq \mathrm{dom}(H^{1/2})$ with codimension at most N such that

$$\int_{\mathbb{R}^d} (|\nabla v|^2 + V|v|^2) > \mu \int_{\mathbb{R}^d} |v|^2$$

for all $v \in \mathcal{H}_N$. Let $\mathcal{F}$ be the collection of boxes such that

$$\mathcal{F} = \Big\{ Q \in \mathcal{Q}_{(C\mu)^{-1/2}} : \inf_Q \frac{1}{u} \leq C\mu \Big\},$$

where $C > 0$ will be chosen later, and let

$$\mathcal{H}_N = \Big\{ v \in \mathrm{dom}(H^{1/2}) : \int_Q v = 0 \text{ for all } Q \in \mathcal{F} \Big\}.$$

Since the cubes are disjoint, the codimension of $\mathcal{H}_N$ is equal to $|\mathcal{F}| = N(C\mu)$.

First, we want to show that

$$\langle (-\Delta + V)\varphi, \varphi \rangle_{L^2(\mathbb{R}^d)} \geq \Big\langle \frac{1}{u}\varphi, \varphi \Big\rangle_{L^2(\mathbb{R}^d)} \tag{4.1}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. We start by considering $d \geq 3$. Denote by u_L the Lax–Milgram solution of (3.8). By (3.3) and (3.9) we know that $1/u_L \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d) \cap H^1_{\mathrm{loc}}(\mathbb{R}^d)$, using the chain rule for Sobolev functions [25, Theorem 6.16]. This readily implies that

$|\varphi|^2/u_L$ is in the form domain of H for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. As u_L is a Lax–Milgram solution of (3.8), we get

$$\int_{\mathbb{R}^d} \Big(\nabla u_L \cdot \nabla\Big(\frac{|\varphi|^2}{u_L}\Big) + V u_L \frac{|\varphi|^2}{u_L}\Big) = \int_{\mathbb{R}^d} \frac{\mathbb{1}_{B(0,L)}}{u_L}|\varphi|^2.$$

Furthermore, using the product rule [25, Lemma 7.4] yields

$$\nabla u_L \cdot \nabla(|\varphi|^2/u_L) = |\nabla\varphi|^2 - u_L^2|\nabla(\varphi/u_L)|^2.$$

Combining the last two equalities and taking $L \to \infty$ implies (4.1) for $d \geq 3$.

For $d \leq 2$, we set $\widetilde{V}(x,t) = V(x)$ for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^{3-d}$ and denote by $\tilde{u}$ the landscape function of $\widetilde{V}$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\psi \in C_c^\infty(\mathbb{R}^{3-d})$ with $\int_{\mathbb{R}^{3-d}} \psi(t)dt = 1$ and $(\varphi \otimes \psi)(x,t) = \varphi(x)\psi(t)$ for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^{3-d}$. Then we have by the previous computations for $d = 3$

$$\begin{aligned}\langle(-\Delta + V)\varphi, \varphi\rangle_{L^2(\mathbb{R}^d)} &= \langle(-\Delta + \widetilde{V})(\varphi\otimes\psi), \varphi\otimes\psi\rangle_{L^2(\mathbb{R}^3)} \\ &\geq \Big\langle\frac{1}{\tilde{u}}(\varphi\otimes\psi), \varphi\otimes\psi\Big\rangle_{L^2(\mathbb{R}^3)} = \Big\langle\frac{1}{u}\varphi, \varphi\Big\rangle_{L^2(\mathbb{R}^d)}.\end{aligned}$$

The bound (4.1) extends, for all $d \geq 1$, by density of $C_c^\infty(\mathbb{R}^d)$ in the form domain of H (see [11, Theorem 8.2.1]) to all $v \in \mathrm{dom}(H^{1/2})$. This implies that

$$2\int_{\mathbb{R}^d} (|\nabla v|^2 + V|v|^2) \geq \int_{\mathbb{R}^d} \Big(|\nabla v|^2 + \frac{1}{u}|v|^2\Big)$$

for all $v \in \mathrm{dom}(H^{1/2})$. With this, the statement of the lemma follows from the claim that if $v \in \mathcal{H}_N \setminus \{0\}$, then

$$\int_{\mathbb{R}^d} \Big(|\nabla v|^2 + \frac{1}{u}|v|^2\Big) > 2\mu \int_{\mathbb{R}^d} |v|^2.$$

We check this inequality using the partition into boxes. In any box $Q \notin \mathcal{F}$, we simply use the bound $\min_Q 1/u > C\mu$. If $Q \in \mathcal{F}$, we recall that the integral of v vanishes and use the Poincaré inequality with optimal constant $\frac{\pi^2}{d}(C\mu)$ since the boxes have sidelength $(C\mu)^{-1/2}$, see [32]. Hence, the claimed lower bound holds for all $C > \max\{2, \frac{2d}{\pi^2}\}$ indeed. ∎

Next we turn to the lower bound in (2.3).

Lemma 4.3. *Let V satisfy the conditions of Theorem 2.1. Then*

$$n(\mu) \leq \mathcal{N}^V((1 + (4C_{\mathrm{H}})^2)\mu)$$

for all $\mu \in \mathbb{R}$.

Proof. For a lower bound $N \le \mathcal{N}^V(C\mu)$, it suffices, again by the Min–Max principle, to find a subspace $\mathcal{H}_N \subseteq \operatorname{dom}(H^{1/2})$ of dimension at least N such that

$$\int_{\mathbb{R}^d} (|\nabla v|^2 + V|v|^2) \le C\mu \int_{\mathbb{R}^d} |v|^2.$$

We define

$$\mathcal{F} = \Big\{ Q \in \mathcal{Q}_{\mu^{-1/2}} : \sup_Q \frac{1}{u} \le \mu \Big\}.$$

Furthermore, for a box Q we pick $\chi_Q \in H^1(\mathbb{R}^d)$ with $0 \le \chi_Q \le 1$, $\|\nabla \chi_Q\|_{L^\infty(\mathbb{R}^d)} \le 4\mu^{1/2}$, $\chi_Q \equiv 1$ on $Q/2$ and $\chi_Q \equiv 0$ on $\mathbb{R}^d \setminus Q$ (a possible choice for χ_Q is to interpolate linearly from $\partial(Q/2)$ to ∂Q). Since the functions $\chi_Q u$ are non-zero and orthogonal to each other, the space

$$\mathcal{H}_N = \operatorname{span}\{\chi_Q u : Q \in \mathcal{F}\}$$

is of dimension $|\mathcal{F}| = n(\mu)$.

By Proposition 3.3, we have $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and thus $\chi_Q u$ is in the form domain of H. Using the product rule for Sobolev function [25, Lemma 7.4] and the fact that u solves the landscape equation, we get for all $\varphi, \psi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
&\langle \nabla(\psi u), \nabla\varphi \rangle_{L^2(\mathbb{R}^d)} + \langle \psi u, V\varphi \rangle_{L^2(\mathbb{R}^d)} \\
&\quad = \langle \nabla u, \nabla(\psi\varphi) \rangle_{L^2(\mathbb{R}^d)} + \langle u, V\psi\varphi \rangle_{L^2(\mathbb{R}^d)} - \langle \nabla u, (\nabla\psi)\varphi \rangle_{L^2(\mathbb{R}^d)} \\
&\qquad + \langle (\nabla\psi)u, \nabla\varphi \rangle_{L^2(\mathbb{R}^d)} \\
&\quad = \langle \psi, \varphi \rangle_{L^2(\mathbb{R}^d)} - \langle \nabla u, (\nabla\psi)\varphi \rangle_{L^2(\mathbb{R}^d)} + \langle (\nabla\psi)u, \nabla\varphi \rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Now, we pick a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ such that

$$\operatorname{supp}(\varphi_n) \subseteq 2Q, \quad \sup_n \|\varphi_n\|_{L^\infty(\mathbb{R}^d)} < \infty$$

and $\varphi_n \to \chi_Q u$ in $H^1(\mathbb{R}^d)$ and a similar approximation $\psi_n \to \chi_Q u$ and we get

$$\int_{\mathbb{R}^d} (|\nabla(\chi_Q u)|^2 + V\chi_Q^2 u^2) = \int_{\mathbb{R}^d} (\chi_Q^2 u + |\nabla\chi_Q|^2 u^2)$$

and in turn

$$\begin{aligned}
\int_{\mathbb{R}^d} (|\nabla(\chi_Q u)|^2 + V\chi_Q^2 u^2) &\le \Big(\sup_Q \frac{1}{u}\Big) \int_Q \chi_Q^2 u^2 + 4^2\mu \int_Q u^2 \\
&\le \mu\Big(\int_Q \chi_Q^2 u^2 + 4^2 \int_Q u^2 \Big).
\end{aligned}$$

Now, (3.11) implies that

$$\int_Q u^2 \leq |Q| \sup_Q u^2 \leq |Q| \, C_{\mathrm{H}}^2 \inf_{Q/2} \operatorname{ess} u^2 \leq C_{\mathrm{H}}^2 \int_{Q/2} u^2 \leq C_{\mathrm{H}}^2 \int_{\mathbb{R}^d} \chi_Q^2 u^2,$$

where the last inequality follows from the properties of χ_Q. This yields the claim we had set to prove. ∎

Together, Lemmas 4.2 and 4.3 yield the claim of Theorem 2.1. Finally, we prove Corollary 2.2.

Proof of Corollary 2.2. If u vanishes at infinity, i.e., $\limsup_{R\to\infty} \sup_{\mathbb{R}^d \setminus B(0,R)} u = 0$, then each sublevel set of $1/u$ is bounded up to a null set and thus H has discrete spectrum by (2.3). Assume on the other hand that u does not vanish at infinity. There is $\mu > 0$ and a sequence of points $(x_n)_{n\geq 1}$ such that $\lim_{n\to\infty} |x_n| = \infty$ and $\liminf_{\varepsilon\to 0^+} \inf_{B(x_n,\varepsilon)} u \geq \frac{C_{\mathrm{H}}}{\mu}$ for all n. Then by (3.11) we have

$$\bigcup_{n\geq 1} Q(x_n, 2\sqrt{C_{\mathrm{H}}/\mu}) \subseteq \{x \in \mathbb{R}^d : 1/u(x) \leq \mu\}$$

and hence, by (2.3), the spectrum of H is not discrete. ∎

5. The case of polynomial potentials

When the potential V is a polynomial, as in the original setting of Fefferman and Phong, one can obtain more precise information of the landscape function. We start by giving the proof for Corollary 2.3.

Proof of Corollary 2.3. Since the addition of a constant does not change the structure of the spectrum, we assume that the polynomial satisfies $V \geq 1$. We check first that these polynomials satisfy (2.1) and (2.2). Condition (2.1) holds with $\delta = 2$ due to the inequality

$$c \sup_{B(x,r)} V \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} V(y)dy \leq \sup_{B(x,r)} V,$$

where c can be chosen to depend only on d and the total degree of V, but neither x nor r. The upper bound is immediate. It is enough to show the lower bound for $r = 1$ and $x = 0$ by scaling and translation. In that case, the claim follows from the fact that the space of all polynomials in d variables and total degree at most D is a finite-dimensional vector space and thus all norms are equivalent.

For the same reason, and since polynomials are analytic functions, there exists a constant $C > 0$ depending only on d and D such that

$$\int_{B(0,2)} V(y)dy \le C \int_{B(0,1)} V(y)dy,$$

which implies doubling after rescaling and translation. In particular, C_{D} can be chosen to only depend on d and D.

Now, Corollary 2.2 and (3.7) imply that the spectrum of H is discrete if and only if $\lim_{|x|\to\infty} m(x, V) = \infty$. For polynomials the Fefferman–Phong–Shen maximal function $m(x, V)$ is in fact equivalent to $M(x, V)$ introduced in (3.10), in the sense that

$$cM(x, V) \le m(x, V) \le CM(x, V). \tag{5.1}$$

The equivalence was already noted in [40] and we provide a proof below for completeness, see Lemma 5.4.

With these preliminaries, we can now turn to the central claim of the corollary. If one of the directional derivative vanishes, then $-\Delta + V$ is unitarily equivalent (via a suitable rotation) to $-\Delta + W$ where $\partial_1 W \equiv 0$. In this case, $M((t, 0, \ldots, 0), W) = M(0, W)$, which implies by the remarks above that the spectrum of $-\Delta + W$ is not discrete and hence also the spectrum of $-\Delta + V$ is not discrete.

Next, we are going to show that if $-\Delta + V$ does not have discrete spectrum, then some directional derivative of V vanishes identically. As $-\Delta + V$ does not have discrete spectrum, we must have that

$$\liminf_{|x|\to\infty} M(x, V) =: M_0 < \infty. \tag{5.2}$$

We consider the semi-algebraic set

$$A = \{x \in \mathbb{R}^d : (\partial^\alpha V(x))^2 < 2M_0^{2(2+|\alpha|)} \text{ for all } \alpha \in \mathbb{N}^d\}$$

and the polynomial function

$$F : \mathbb{R}^d \to \mathbb{R}^{(D+1)^d}, \quad x \mapsto (\partial^\alpha V(x)^2)_{\alpha \in [0,D]^d \cap \mathbb{Z}^d}.$$

Now, (5.2) implies that A is an unbounded set and we can therefore pick a sequence $(x^{(n)})_{n\in\mathbb{N}} \subseteq A$ such that $|x^{(n)}| \to \infty$ and

$$\lim_{n\to\infty} F(x^{(n)}) =: y = (y_\alpha)_{\alpha \in [0,D]^d \cap \mathbb{Z}^d}$$

with $|y_\alpha| \le M_0^{2(2+|\alpha|)}$.

Next, we would like to pass from a mere sequence to an analytic curve. This is done by the following curve selection lemma at infinity.

Lemma 5.1 ([17, Lemma 2.17]). *Let $A \subset \mathbb{R}^d$ be a semi-algebraic set, and let the function $F: \mathbb{R}^d \to \mathbb{R}^N$ be a semi-algebraic map. Assume that there exists a sequence $(x^{(n)})_{n \in \mathbb{N}} \subset A$ such that $\lim_{n\to\infty} |x^{(n)}| = \infty$ and $\lim_{n\to\infty} F(x^{(n)}) = y \in (\mathbb{R} \cup \{\pm\infty\})^N$. Then there exists an analytic curve $\gamma: (0,\delta) \to A$ of the form*

$$\gamma(t) = \sum_{j=-m}^{\infty} a^{(j)} t^j \tag{5.3}$$

such that $a^{(-m)} \in \mathbb{R}^N \setminus \{0\}$, $m \in \mathbb{Z}_{>0}$ and $\lim_{t\to 0^+} F(\gamma(t)) = y$.

Let γ be a curve as given by the previous lemma. We would like to say that V remains constant along γ and thus get a direction in which the gradient of V vanishes identically. However, analytic functions can remain bounded on an unbounded set without being constant. Thus, we truncate the series (5.3) at $j = 0$, thereby obtaining a polynomial approximation of the curve γ, and F will still remain bounded along the truncation.

Lemma 5.2. *For every $\varepsilon > 0$, there exists $C > 0$ such that for all $v \in \mathbb{R}^d$ with $|v| < \varepsilon$ we have for all $t \in (0, \delta/2)$*

$$0 \le V(\gamma(t) + v) \le C.$$

Proof. By Taylor's theorem, $V(\gamma(t) + v) = \sum_{\alpha \in \mathbb{N}^d} \frac{v^\alpha}{\alpha!} (\partial^\alpha V)(\gamma(t))$. The claim follows from the fact that $|(\partial^\alpha V)(\gamma(t))|$ are all uniformly bounded for $t \in (0, \delta/2)$. ∎

With this, we define polynomial function

$$G(s) = \sum_{j=0}^{m} a^{(-j)} s^j.$$

Note that every component of G is single variable polynomial. For every $\varepsilon > 0$ there exists $0 < \delta_\varepsilon < \delta$ such that

$$|\gamma(t) - G(t^{-1})| < \frac{\varepsilon}{2}$$

for all $t \in (0, \delta_\varepsilon)$, and hence by the previous lemma

$$0 \le V(G(t^{-1})) \le C.$$

Let now

$$P(s, x_1, \ldots, x_d) = V(G(s) + x).$$

For $s > \frac{1}{\delta_\varepsilon}$ and $|x| < \frac{\varepsilon}{2}$, we get

$$0 \leq P(s, x_1, \ldots, x_d) \leq C$$

again by Lemma 5.2. Now, for any $|x| < \frac{\varepsilon}{2}$, the function $s \mapsto P(s,x)$ is a polynomial that is bounded on an unbounded interval and thus constant. Therefore, $\partial_s P(s,x) = 0$ on $\mathbb{R} \times B_{\varepsilon/2}(0)$. By the identity theorem, we get that $\partial_s P(s,x) = 0$ on $\mathbb{R}^{d+1}$. But

$$0 = \partial_s P(s,x) = (\nabla V)(G(s) + x) \cdot G'(s).$$

As G is not constant, there is $s_0 \in \mathbb{R}$ such that $G'(s_0) \neq 0$ and so the derivative of V in direction $G'(s_0)$ vanishes identically. ∎

As mentioned before, the landscape function will not belong to the form domain of H. For polynomial potentials, there is an easy criterion to check whether $u \in \operatorname{dom}(H^{1/2})$.

Lemma 5.3. *Let $V \geq 0$ be a non-zero polynomial. Then $u \in L^1(\mathbb{R}^d)$ if and only if $u \in \operatorname{dom}(H^{1/2})$.*

Proof. As V is smooth, we get by standard elliptic regularity theory that the landscape function is a classical solution of the landscape equation. Multiplying the landscape equation by $\varphi \in C_c^\infty(\mathbb{R}^d)$ and integrating yields after integration by parts

$$\int_{\mathbb{R}^d} \varphi |\nabla u|^2 + \int_{\mathbb{R}^d} V u^2 \varphi = \int_{\mathbb{R}^d} u\varphi + \int_{\mathbb{R}^d} u^2 (\Delta\varphi).$$

We saw in the proof of Corollary 2.3 that either $\lim_{|x|\to\infty} u(x) = 0$ or that there is one spatial coordinate along which u is constant. Hence, if $u \in L^1(\mathbb{R}^d)$, then u vanishes at infinity and automatically $u \in L^2(\mathbb{R}^d)$. However, then we can choose a sequence of $\varphi_n \in C_c^\infty(\mathbb{R}^d)$ converging to 1 and obtain by monotone convergence

$$\int_{\mathbb{R}^d} (|\nabla u|^2 + V u^2) \leq \int_{\mathbb{R}^d} u + C \int_{\mathbb{R}^d} u^2 < \infty$$

and therefore $u \in \operatorname{dom}(H^{1/2})$.

On the other hand, if $u \in \operatorname{dom}(H^{1/2})$, then $u \in H^1(\mathbb{R}^d)$ and in particular $u \in L^2(\mathbb{R}^d)$. Thus, we can take again a suitable sequence of test functions to obtain by dominated convergence

$$\int_{\mathbb{R}^d} u = \int_{\mathbb{R}^d} |\nabla u|^2 + \int_{\mathbb{R}^d} V u^2 - \int_{\mathbb{R}^d} u^2 < \infty,$$

namely $u \in L^1(\mathbb{R}^d)$. ∎

Let us now return to the concrete example of Simon's potential $V(x,y) = x^2y^2$. First of all, we can now prove that the corresponding landscape function is in the form domain of $-\Delta + V$. Combining Proposition 3.3 and Lemma 5.4, it is enough to check that $M(\cdot, x^2y^2)^{-2} \in L^1(\mathbb{R}^2)$. An explicit calculation yields $M((x,y), x^2y^2) \geq |xy| + \sqrt{|x|} + \sqrt{|y|} + 1$, and thus its inverse is indeed square integrable in $\mathbb{R}^2$. Similarly, we obtain the following two-sided estimate on the effective potential:

$$c(x^2y^2 + |x| + |y| + 1) \leq \frac{1}{u(x,y)} \leq C(x^2y^2 + |x| + |y| + 1).$$

This yields for μ sufficiently large

$$\frac{\mu}{3C}\log\Big(\frac{\mu}{3C}\Big) = \Big|\Big\{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq \frac{\mu}{3C}, 0 \leq y \leq \frac{\mu}{3Cx}\Big\}\Big| \leq \mathcal{V}(\mu).$$

we have

$$\begin{aligned}\mathcal{V}(\mu) &\leq 1 + 4\Big|\Big\{(x,y) \in \Big[1, \frac{\mu}{c}\Big] \times \mathbb{R}_{\geq 0} : x^2y^2 + |x| + |y| + 1 \leq \frac{\mu}{c}\Big\}\Big| \\ &\leq 1 + 4\Big|\Big\{(x,y) \in \Big[1, \frac{\mu}{c}\Big] \times \mathbb{R}_{\geq 0} : |xy| \leq \frac{\mu}{c}\Big\}\Big| \\ &= 1 + \frac{4\mu}{c}\log\Big(\frac{\mu}{c}\Big).\end{aligned}$$

The combination of those two estimates with Theorem 2.1 recovers Simon's asymptotics [44, Theorem 1.4]

$$\mathcal{N}^{x^2y^2}(\mu) = \frac{1}{\pi}\mu^{3/2}\log(\mu) + o(\mu^{3/2}\log(\mu)).$$

up to multiplicative constants.

We conclude this section with a proof of the equivalence of the functions $m(\cdot, V)$ and $M(\cdot, V)$ in the case of polynomials. We point out that the arguments in this section show that in the case of polynomials, the constants appearing in Theorem 2.1, and *a fortiori* the Harnack constant, depend only on the spatial dimension and the degree of the polynomial.

Lemma 5.4. *Let $V \geq 0$ a polynomial on $\mathbb{R}^d$ of total degree $D \geq 0$. Then there exist constants $C, c > 0$ depending only on d, D such that*

$$cM(x,V) \leq m(x,V) \leq CM(x,V), \tag{5.4}$$

where $m(\cdot, V)$, $M(\cdot, V)$ were defined in (3.1) and (3.10).

Proof. By translating the potential, we can pick $x = 0$. Furthermore, for all $\lambda > 0$ we have $m(x, V_\lambda) = \lambda m(\lambda x, V)$ and $M(x, V_\lambda) = \lambda M(\lambda x, V)$, where $V_\lambda(x) = \lambda^2 V(\lambda x)$.

Hence, we can assume that $M(0, V) = 1$ and then need to show that there exists $c > 0$ depending only on d, D such that $m(0, V) \geq c$.

Since $V \mapsto \partial^\alpha V$ is a linear map on a finite-dimensional space,

$$|\partial^\alpha V(0)| \leq \sup_{x \in B(0,1)} |\partial^\alpha V(x)| \leq c \sup_{x \in B(0,1)} |V(x)|,$$

and so

$$\begin{aligned} 1 = M(0, V) &\leq \sum_{\alpha \in \mathbb{N}_0^n} (c \sup_{x \in B(0,1)} V(x))^{1/(|\alpha|+2)} \\ &\leq c((\sup_{x \in B(0,1)} V(x))^{1/2} + (\sup_{x \in B(0,1)} V(x))^{1/(D+2)}). \end{aligned}$$

Hence, we have $\sup_{x \in B(0,1)} V(x) \geq c > 0$. For $r \geq 1$, we get

$$cr^2 \leq r^2 \sup_{x \in B(0,r)} V(x) \leq \frac{c}{r^{n-2}} \int_{B(0,r)} V(y) dy.$$

Recall that C_{D} can be chosen to only depend on d, D, therefore the right-hand side is greater than C_{D} for r large enough, which yields an upper bound on $\frac{1}{m(0,V)}$. Hence,

$$m(0, V) \geq c = cM(0, V).$$

We turn to the lower bound. First of all, a simple Taylor expansion yields (see [46, Lemma 2.5])

$$|\partial^\alpha V(y)| \leq CM(x, V)^{|\alpha|+2}(1 + |x - y|M(x, V))^D,$$

so that

$$M(y, V) \leq CM(x, V)(1 + |x - y|M(x, V))^{D/2}, \tag{5.5}$$

for all $x, y \in \mathbb{R}^d$. Thus, if $m(0, V) = 1$ then

$$\begin{aligned} C_{\mathrm{D}}^{-(2+D)} m(0, V)^{2+D} &= \int_{B(0,1)} V(y) dy \leq |B(0, 1)| \sup_{y \in B(0,1)} V(y) \\ &\leq |B(0, 1)| \sup_{y \in B(0,1)} M(y, V)^2 \\ &\leq C^2 M(0, V)^{2+D}, \end{aligned}$$

by (5.5), which again yields the desired estimate by translating and rescaling. ■

6. The potential well

In this section we explicitly compute the landscape function for potential wells $\varepsilon \mathbb{1}_{B(0,\delta)^c}$, where $B(0,\delta)^c = \mathbb{R}^d \setminus B(0,\delta)$ and $\varepsilon, \delta > 0$. We shall observe first of all that the minimum of the effective potential properly reflects the value of the bottom of the spectrum in the sense that both are of order ε as $\varepsilon \to 0$. Secondly, we will see that the estimates of the main theorem are not tight enough to distinguish the difference between $d = 1, 2$, where an eigenvalue is present for all $\varepsilon > 0$, and $d \geq 3$ where this is not the case.

We start by observing that the landscape function corresponding to the spherical well are radially symmetric. Indeed, all the Lax–Milgram solutions (3.8) are invariant under rotation of the first d variables and thus the landscape function, given as a pointwise limit of those solutions, shares the same symmetry. Passing to spherical coordinates, we see that the radial part $f(|x|) = u(x)$ solves the ODE

$$-f''(r) - \frac{d-1}{r} f'(r) + \varepsilon \mathbb{1}_{[\delta,\infty)}(r) f(r) = 1$$

on $(0,\infty)$. The general solution of this ODE on $(0,\delta)$ is given, for $d \neq 2$, by

$$f(r) = -\frac{r^2}{2d} + a_1 + \frac{a_2}{r^{d-2}},$$

respectively by the same expression with $r^{-(d-2)}$ replaced by $\log(r)$ for $d = 2$. As $\lim_{r\to 0+} f(r) = \lim_{r\to 0+} u(re_1) = u(0)$, we conclude in the case $d \geq 2$ that $a_2 = 0$. The same follows for $d = 1$ as u is even and $C^1(\mathbb{R})$.

On the other hand, on (δ,∞), the general solution is given by

$$f(r) = \frac{1}{\varepsilon} + b_1 r^{1-\frac{d}{2}} K_{-1+d/2}(\sqrt{\varepsilon} r) + b_2 r^{-1+\frac{d}{2}} I_{-1+d/2}(\sqrt{\varepsilon} r),$$

where I_m, K_m denote the modified Bessel function of the first, respectively the second kind. We have $\lim_{r\to\infty} I_m(r) = \infty$ and $\lim_{r\to\infty} K_m(r) = 0$ for $m \geq -1/2$ (use [1, (9.6.10) and 9.6.23] and $K_{-1/2}(x) = \sqrt{\pi} e^{-x}/\sqrt{2x}$, $I_{-1/2}(x) = \sqrt{2}\cosh(x)/\sqrt{\pi x}$). As 0 is not in the spectrum of $-\Delta + \varepsilon \mathbb{1}_{B(0,\delta)^c}$, we get that u is bounded and hence $b_2 = 0$. This yields

$$f(r) = \begin{cases} -\dfrac{r^2}{2d} + a_1, & r \in (0,\delta), \\ \dfrac{1}{\varepsilon} + b_1 r^{1-\frac{d}{2}} K_{-1+d/2}(\sqrt{\varepsilon} r), & r \in (\delta,\infty). \end{cases}$$

Finally, the coefficients can be determined by the fact that $f \in C^1(\mathbb{R}_{>0})$. In dimensions $d = 1, 3$ the Bessel functions can be expressed in elementary functions and the solutions are given by

$$u(x) = \begin{cases} -\frac{|x|^2}{2} + \frac{1}{\varepsilon} + \frac{\delta}{\sqrt{\varepsilon}} + \frac{\delta^2}{2}, & |x| \le \delta, \\ \frac{1}{\varepsilon} + \frac{\delta}{\sqrt{\varepsilon}} e^{-\sqrt{\varepsilon}(|x|-\delta)}, & |x| > \delta \end{cases}$$

for $d = 1$ and by

$$u(x) = \begin{cases} -\frac{|x|^2}{6} + \frac{1}{\varepsilon} + \delta^2\Big(\frac{1}{6} + \frac{1}{1+\sqrt{\varepsilon}\delta}\Big), & |x| \le \delta, \\ \frac{1}{\varepsilon} + \frac{\delta^3}{1+\sqrt{\varepsilon}\delta} \frac{e^{-\sqrt{\varepsilon}(|x|-\delta)}}{|x|}, & |x| > \delta \end{cases} \tag{6.1}$$

for $d = 3$.

In all dimensions, we have that u is radially symmetric and its radial part is monotone decreasing (even exponentially). Furthermore, we have $\lim_{|x|\to\infty} u(x) = \frac{1}{\varepsilon}$. This implies that the sublevel set $\mathcal{V}(\mu)$ of the effective potential $1/u$ is monotone increasing, remains finite for $\mu < \varepsilon$ and $\lim_{\mu\to\varepsilon^-} \mathcal{V}(\mu) = \infty$. This is consistent with the fact that the bottom of the essential spectrum is ϵ and $c < 1, C > 1$ in Theorem 2.1.

For $d = 1$, the smallest eigenvalue μ_0 of $-\Delta + \varepsilon \mathbb{1}_{B(0,\delta)^c}$, for $0 < \varepsilon$ sufficiently small, is the smallest positive solution of

$$\sqrt{\varepsilon - \mu_0} = \sqrt{\mu_0} \tan(\sqrt{\mu_0}\delta).$$

Thus, for $\delta > 0$ fixed, we obtain $\mu_0 = \varepsilon(1 - O(\sqrt{\varepsilon}))$ as $\varepsilon \to 0^+$. As discussed at the beginning of the section, this is the same asymptotic behaviour as that of the minimum of $1/u$, see (6.1). The same holds for $d = 3$ where however the bottom of the spectrum is the bottom of the essential spectrum, namely $\mathcal{N}(\mu) = 0$ for $\mu < \varepsilon$ and $\mathcal{N}(\mu) = \infty$ for $\mu \ge \varepsilon$. Here, $\mathcal{V}(\mu)$ is arbitrarily large for $\mu \to \varepsilon^-$, showing that $c < 1$ in (2.3).

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, No. 55, U. S. Government Printing Office, Washington, DC, 1964
Zbl 0171.38503 MR 167642

[2] D. Arnold, M. Filoche, S. Mayboroda, W. Wang, and S. Zhang, The landscape law for tight binding Hamiltonians. *Comm. Math. Phys.* **396** (2022), no. 3, 1339–1391
Zbl 1511.35109 MR 4507924

[3] D. N. Arnold, G. David, M. Filoche, D. Jerison, and S. Mayboroda, Localization of eigenfunctions via an effective potential. *Comm. Partial Differential Equations* **44** (2019), no. 11, 1186–1216 Zbl 1432.35061 MR 3995095

[4] S. Balasubramanian, Y. Liao, and V. Galitski, Many-body localization landscape. *Phy. Rev. B* 101 (2020), article no. 014201

[5] F. Calogero, Upper and lower limits for the number of bound states in a given central potential. *Comm. Math. Phys.* **1** (1965), 80–88 Zbl 0128.21907 MR 181250

[6] I. Chenn, W. Wang, and S. Zhang, Approximating the ground state eigenvalue via the effective potential. *Nonlinearity* **35** (2022), no. 6, 3004–3035 Zbl 1491.82011 MR 4443926

[7] M. Christ, On the $\overline{\partial}$ equation in weighted L^2 norms in $\mathbf{C}^1$. *J. Geom. Anal.* **1** (1991), no. 3, 193–230 Zbl 0737.35011 MR 1120680

[8] M. Cwikel, Weak type estimates for singular values and the number of bound states of Schrödinger operators. *Ann. of Math. (2)* **106** (1977), no. 1, 93–100 Zbl 0362.47006 MR 473576

[9] B. Davey, J. Hill, and S. Mayboroda, Fundamental matrices and Green matrices for non-homogeneous elliptic systems. *Publ. Mat.* **62** (2018), no. 2, 537–614 Zbl 1400.35098 MR 3815288

[10] G. David, M. Filoche, and S. Mayboroda, The landscape law for the integrated density of states. *Adv. Math.* **390** (2021), article no. 107946 Zbl 1479.35267 MR 4298594

[11] E. B. Davies, *Spectral theory and differential operators*. Cambridge Stud. Adv. Math 42, Cambridge University Press, Cambridge, 1995 Zbl 0893.47004 MR 1349825

[12] C. L. Fefferman, The uncertainty principle. *Bull. Amer. Math. Soc. (N.S.)* **9** (1983), no. 2, 129–206 Zbl 0526.35080 MR 707957

[13] M. Filoche, S. Mayboroda, and T. Tao, The effective potential of an M-matrix. *J. Math. Phys.* **62** (2021), no. 4, article no. 041902, 15 Zbl 1462.81088 MR 4246780

[14] R. L. Frank, Cwikel's theorem and the CLR inequality. *J. Spectr. Theory* **4** (2014), no. 1, 1–21 Zbl 1295.35347 MR 3181383

[15] R. L. Frank, A. Laptev, and T. Weidl, *Schrödinger operators: eigenvalues and Lieb-Thirring inequalities*. Cambridge Stud. Adv. Math 200, Cambridge University Press, Cambridge, 2023 Zbl 07595814 MR 4496335

[16] A. Grigor'yan and N. Nadirashvili, Negative eigenvalues of two-dimensional Schrödinger operators. *Arch. Ration. Mech. Anal.* **217** (2015), no. 3, 975–1028 Zbl 1319.35128 MR 3356993

[17] H. V. Ha and T. T. Nguyen, Łojasiewicz gradient inequalities for polynomial functions and some applications. *J. Math. Anal. Appl.* **509** (2022), no. 1, article no. 125950 Zbl 1516.26010 MR 4358608

[18] V. Hoang, D. Hundertmark, J. Richter, and S. Vugalter, Quantitative bounds versus existence of weakly coupled bound states for Schrödinger type operators. *Ann. Henri Poincaré* **24** (2023), no. 3, 783–842 Zbl 1511.35247 MR 4562137

[19] D. Hundertmark, P. Kunstmann, T. Ried, and S. Vugalter, Cwikel's bound reloaded. *Invent. Math.* **231** (2023), no. 1, 111–167 Zbl 1510.35193 MR 4526822

[20] V. Ivrii, 100 years of Weyl's law. *Bull. Math. Sci.* **6** (2016), no. 3, 379–452 Zbl 1358.35075 MR 3556544

[21] A. Laptev, The negative spectrum of the class of two-dimensional Schrödinger operators with potentials that depend on the radius. *Funktsional. Anal. i Prilozhen.* **34** (2000), no. 4, 85–87; English transl., *Funct. Anal. Appl.* **34** (2000), no. 4, 305–307 Zbl 0972.35079 MR 1819649

[22] A. Laptev and Y. Netrusov, On the negative eigenvalues of a class of Schrödinger operators. In *Differential operators and spectral theory*, pp. 173–186, Amer. Math. Soc. Transl. Ser. 2 189, American Mathematical Society, Providence, RI, 1999 Zbl 0941.35055 MR 1730512

[23] A. Laptev, L. Read, and L. Schimmer, Calogero type bounds in two dimensions. *Arch. Ration. Mech. Anal.* **245** (2022), no. 3, 1491–1505 Zbl 1495.35135 MR 4467323

[24] E. Lieb, Bounds on the eigenvalues of the Laplace and Schroedinger operators. *Bull. Amer. Math. Soc.* **82** (1976), no. 5, 751–753 Zbl 0329.35018 MR 407909

[25] E. H. Lieb, and M. Loss, *Analysis*. Second edn., Grad. Stud. Math. 14, American Mathematical Society, Providence, RI, 2001 Zbl 0966.26002 MR 1817225

[26] E. H. Lieb and W. E. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. In *Essays Honor Valentine Bargmann,* pp. 269-303, Stud. Math. Phys., Princeton University Press, Princeton, NJ, 1976 Zbl 0342.35044

[27] V. Liskevich and I. I. Skrypnik, Harnack inequality and continuity of solutions to elliptic equations with nonstandard growth conditions and lower order terms. *Ann. Mat. Pura Appl. (4)* **189** (2010), no. 2, 335–356 Zbl 1187.35073 MR 2602154

[28] A. Mohamed, P. Lévy-Bruhl, and J. Nourrigat, Étude spectrale d'opérateurs liés à des représentations de groupes nilpotents. *J. Funct. Anal.* **113** (1993), no. 1, 65–93 Zbl 0777.35047 MR 1214898

[29] A. Mohamed and J. Nourrigat, Encadrement du $N(\lambda)$ pour un opérateur de Schrödinger avec un champ magnétique et un potentiel électrique. *J. Math. Pures Appl. (9)* **70** (1991), no. 1, 87–99 Zbl 0725.35068 MR 1091921

[30] M. Otelbaev, Bounds for eigenvalues of singular differential operators. *Mat. Zametki* **20** (1976), no. 6, 859–867; English transl., *Math. Notes* **20** (1976), 1038–1042 Zbl 0354.34022 MR 435503

[31] M. Otelbaev, Imbedding theorems for spaces with a weight and their application to the study of the spectrum of a Schrödinger operator. *Trudy Mat. Inst. Steklov.* **150** (1979), 265–305, 323–324 Zbl 0416.46019 MR 544014

[32] L. E. Payne and H. F. Weinberger, An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.* **5** (1960), 286–292 (1960) Zbl 0099.08402 MR 117419

[33] B. Poggi, Applications of the landscape function for Schrödinger operators with singular potentials and irregular magnetic fields. 2021, arXiv:2107.14103

[34] B. Poggi and S. Mayboroda, Exponential decay of fundamental solutions to Schrödinger operators and the landscape function. Spring Western Virtual Sectional Meeting, American Mathematical Society, 2021

[35] M. Reed and B. Simon, *Methods of modern mathematical physics.* IV. Analysis of operators. Academic Press, New York etc., 1978 Zbl 0401.47001 MR 493421

[36] D. Robert, Comportement asymptotique des valeurs propres d'opérateurs du type Schrödinger à potentiel "dégénéré". *J. Math. Pures Appl. (9)* **61** (1982), no. 3, 275–300 (1983) Zbl 0511.35069 MR 690397

[37] G. V. Rozenblyum, Estimates of the spectrum of the Schrödinger operator. In *Problems of mathematical analysis, No. 5: Linear and nonlinear differential equations, differential operators,* pp. 152–166 Leningrad University, Leningrad, 1975; English translation, *J. Sov. Math.* **10** (1978), no. 6, 934-944. Zbl 0432.35062 MR 0410068

[38] G. V. Rozenblyum, Distribution of the discrete spectrum of singular differential operators. *Izv. Vysš. Učebn. Zaved. Matematika* (1976), no. 1, 75–86 (in Russian) MR 0430557

[39] E. Shargorodsky, On negative eigenvalues of two-dimensional Schrödinger operators. *Proc. Lond. Math. Soc. (3)* **108** (2014), no. 2, 441–483 Zbl 1327.35274 MR 3166359

[40] Z. Shen, Eigenvalue asymptotics and exponential decay of eigenfunctions for Schrödinger operators with magnetic fields. *Trans. Amer. Math. Soc.* **348** (1996), no. 11, 4465–4488 Zbl 0866.35088 MR 1370650

[41] Z. Shen, On bounds of $N(\lambda)$ for a magnetic Schrödinger operator. *Duke Math. J.* **94** (1998), no. 3, 479–507 Zbl 0948.35088 MR 1639527

[42] Z. Shen, On fundamental solutions of generalized Schrödinger operators. *J. Funct. Anal.* **167** (1999), no. 2, 521–564 Zbl 0936.35051 MR 1716207

[43] Z. W. Shen, L^p estimates for Schrödinger operators with certain potentials. *Ann. Inst. Fourier (Grenoble)* **45** (1995), no. 2, 513–546 Zbl 0818.35021 MR 1343560

[44] B. Simon, Nonclassical eigenvalue asymptotics. *J. Funct. Anal.* **53** (1983), no. 1, 84–98 Zbl 0529.35064 MR 715548

[45] B. Simon, Some quantum operators with discrete spectrum but classically continuous spectrum. *Ann. Physics* **146** (1983), no. 1, 209–220 Zbl 0547.35039

[46] H. F. Smith, Parametrix construction for a class of subelliptic differential operators. *Duke Math. J.* **63** (1991), no. 2, 343–354 Zbl 0777.35002 MR 1115111

[47] S. Steinerberger, Localization of quantum states and landscape functions. *Proc. Amer. Math. Soc.* **145** (2017), no. 7, 2895–2907 Zbl 1365.35098 MR 3637939

[48] S. Steinerberger, Regularized potentials of Schrödinger operators and a local landscape function. *Comm. Partial Differential Equations* **46** (2021), no. 7, 1262–1279 Zbl 1487.35200 MR 4279965

[49] W. Wang and S. Zhang, The exponential decay of eigenfunctions for tight-binding Hamiltonians via landscape and dual landscape functions. *Ann. Henri Poincaré* **22** (2021), no. 5, 1429–1457 Zbl 1462.81092 MR 4250739

[50] H. Weyl, Über die asymptotische Verteilung der Eigenwerte. *Gött. Nachr* (1911), 110–117 JFM 43.0435.04

[51] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Math. Ann.* **71** (1912), no. 4, 441–479 JFM 43.0436.01 MR 1511670

[52] H. Weyl, Über die Abhängigkeit der Eigenschwingungen einer Membran und deren Begrenzung. *J. Reine Angew. Math.* **141** (1912), 1–11 JFM 43.0948.03 MR 1580843

[53] H. Weyl, Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetze. *J. Reine Angew. Math.* **143** (1913), 177–202 JFM 44.1053.02 MR 1580880

[54] J. Zhong, *Harmonic analysis for some Schroedinger type operators*. Ph.D. thesis, Princeton, NJ, 1993 MR 2689454

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