J. Spectr. Theory 13 (2023), 1145–1171
DOI 10.4171/JST/485

A uniqueness result for the Calderón problem for $U(N)$-connections coupled to spinors

Carlos Valero

Abstract. In this paper we define a Dirichlet-to-Neumann map for a twisted Dirac Laplacian acting on bundle-valued spinors over a spin manifold. We show that this map is a pseudodifferential operator of order 1 whose symbol determines the Taylor series of the metric and connection at the boundary. We go on to show that if two real-analytic connections couple to a spinor via the Yang–Mills–Dirac equations with appropriate boundary conditions, and have equal Dirichlet-to-Neumann maps, then the two connections are globally gauge equivalent in the smooth category. In the abelian case, the global gauge equivalence is in the real-analytic category.

1. Introduction

In this paper, we consider a Calderón inverse problem for unitary connections on Hermitian vector bundles over spin manifolds that couple to spinor fields via the Yang–Mills–Dirac system. In particular, we consider the Dirichlet-to-Neumann map for the twisted Dirac Laplacian acting on vector-valued spinors, and investigate how the introduction of the spin structure affects the recovery of the metric and connection from boundary data.

The Calderón problem has its origin in the physical question of whether one can determine the conductivity of a medium by making measurements on the boundary of potential functions and the induced currents. Geometrically, this corresponds to the question of whether one can determine the metric on a manifold with boundary, up to isometry, from knowledge of its Dirichlet-to-Neumann map, which sends a function on the boundary to the normal derivative of its harmonic extension. Much work has since been done on the Calderón problem for the scalar Laplacian; we refer the reader to [20], or the more recent [5, Section 1], for a survey of uniqueness results in the literature.

Mathematics Subject Classification 2020: 53C27 (primary); 35R01, 35R30, 53C07 (secondary).
Keywords: Dirichlet-to-Neumann map, Calderón problem, spinor bundle, Dirac Laplacian, Yang–Mills–Dirac system, gauge theory, inverse problems.

Many natural extensions of this problem arise when one considers other important second-order elliptic operators induced by some geometric structure. For example, one may fix a vector bundle E over a Riemannian manifold (M, g) with boundary, and consider a connection ∇ on this vector bundle. One may then ask to what extent the Dirichlet-to-Neumann map for the connection Laplacian $\nabla^* \nabla$ determines the connection, up to gauge equivalence. This problem has been studied in a few recent works, in which a number of uniqueness results are proved. In [11], a Riemannian manifold, Hermitian vector bundle, and connection are reconstructed from the hyperbolic Dirichlet-to-Neumann map associated to the wave equation of the connection Laplacian. In [3], the elliptic Dirichlet-to-Neumann map is considered, and it is shown using methods of complex geometrical optics that the Dirichlet-to-Neumann map for a connection Laplacian determines the connection up to gauge for a class of vector bundles over special Riemannian manifolds, namely conformally transversally anisotropic manifolds with injective ray transform. Using new methods of geometric analysis and Runge approximation, Cekić shows in [4] that a Hermitian vector bundle and Yang–Mills connection can be recovered, up to gauge transformations, from the Dirichlet-to-Neumann map of its connection Laplacian. Finally, in the recent preprint [7], the authors reconstruct a Euclidean vector bundle and connection from the connection Laplacian Dirichlet-to-Neumann map when all of the data is real-analytic and the dimension of M is at least 3.

On the other hand, inverse boundary problems for first-order Dirac operators have also been studied in the literature. One important uniqueness result is due to Kurylev and Lassas [10] who showed that a Riemannian manifold and super-vector bundle can be recovered from the spectrum and eigenfunctions of the corresponding Dirac operator on the boundary. There are also many works that consider the problem of recovering magnetic potentials from boundary data corresponding to first-order Dirac operators. Much in the spirit of the present paper, Salo and Tzou [16, 17] have shown, using the method of limiting Carleman weights, that a potential and magnetic field can be recovered from the Cauchy data of an associated Dirac equation over a compact domain in $\mathbb{R}^n$. In the terminology used in the present paper, the magnetic field corresponds to the curvature of an abelian connection.

In this paper, we consider the question of determining a $U(N)$-connection A up to gauge equivalence, from the Dirichlet-to-Neumann map of the twisted Dirac Laplacian $\not{D}_A^2$. More precisely, we consider a compact spin manifold M with boundary ∂M, and a Hermitian vector bundle E over M. Then the Dirac operator $\not{D}$ of M can be defined, acting on its bundle of complex spinors S. Now, given any connection A on E, we may endow the bundle $S \otimes E$ with the connection $\omega^{\mathrm{s}} \otimes A$, where ω^{s} is the spin connection on S, induced by the Levi-Civita connection. With this connection, we may define a twisted, or covariant, Dirac operator $\not{D}_A$, which like $\not{D}$ is a first-order, elliptic, and self-adjoint operator acting on sections of $S \otimes E$.

In order for the Dirichlet problem to be well defined, we consider the square of this operator, $\not{D}_A^2$. The Dirichlet-to-Neumann map associated to $\not{D}_A^2$ can be thus defined by sending any section χ of $S \otimes E|_{\partial M}$ to the covariant normal derivative of its harmonic extension with respect to $\not{D}_A^2$. It is straightforward to generalize this to include non-zero mass terms and zeroth-order potentials, provided that the Dirichlet-problem is well defined. We summarize with the following definition.

Definition 1.1. Let M be an n-dimensional compact spin manifold with boundary ∂M, and let g be a Riemannian metric on M. Let S be the spinor bundle associated to some fixed spin structure on M, and let E be a Hermitian bundle of rank N on M. Let A denote a $U(N)$-connection on E, and let Z be an endomorphism of $S \otimes E$. For any $m \in \mathbb{R}$ such that $m^2 \notin \operatorname{Spec}(\not{D}_A^2 + Z)$, we define the *Dirichlet-to-Neumann map*

$$\Lambda_{g,A,Z,m}: C^\infty(S \otimes E|_{\partial M}) \to C^\infty(S \otimes E|_{\partial M})$$

as follows. For $\chi \in C^\infty(S \otimes E|_{\partial M})$, we may solve the Dirichlet problem

$$\begin{cases} \not{D}_A^2 \varphi + Z\varphi - m^2\varphi = 0, \\ \varphi|_{\partial M} = \chi, \end{cases} \tag{1.1}$$

to obtain a unique solution $\varphi \in C^\infty(S \otimes E)$. We then define $\Lambda_{g,A,Z,m}(\chi) := \nabla_\nu^A \varphi|_{\partial M}$ where ν is the inward unit normal to ∂M.

We will often suppress subscripts on the Dirichlet-to-Neumann map that are understood to be fixed, and indicate only the relevant ones. For example, when we recover the connection with the background metric, mass parameter, and endomorphism fixed in Section 4, the Dirichlet-to-Neumann map is simply denoted Λ_A.

Remark 1.2. The Dirichlet-to-Neumann map can be extended to a map

$$\Lambda_{g,A,Z,m}: H^{\frac{1}{2}}(S \otimes E|_{\partial M}) \to H^{-\frac{1}{2}}(S \otimes E|_{\partial M}),$$

where for any vector bundle $\mathcal{E}$ and $s \in \mathbb{R}$, $H^s(\mathcal{E})$ denotes the Hilbert space of s-Sobolev sections of $\mathcal{E}$. That is, $H^s(\mathcal{E})$ denotes the space of distributional sections of $\mathcal{E}$ that are represented by a tuple of H^s functions in any smooth local trivialization. Indeed, there is a natural weak formulation of Definition 1.1 if we introduce a modified Dirichlet-to-Neumann map $\widehat{\Lambda}_{g,A,Z,m}$ as follows: for any $\chi \in H^{\frac{1}{2}}(S \otimes E|_{\partial M})$, we can again solve (1.1) to obtain a unique $\varphi \in H^1(S \otimes E)$. We then define $\widehat{\Lambda}_{g,A,Z,m}(\chi) \in H^{-\frac{1}{2}}(S \otimes E|_{\partial M})$ by the property that

$$\langle \widehat{\Lambda}_{g,A,Z,m}(\chi), \zeta \rangle = \int_M \langle \not{D}_A \varphi, \not{D}_A \psi \rangle - \int_M \langle (m^2 - Z)\varphi, \psi \rangle$$

holds for all $\zeta \in H^{\frac{1}{2}}(S \otimes E|_{\partial M})$, where $\psi \in H^1(S \otimes E)$ is any extension of ζ, which exists by the standard trace theorems for Sobolev spaces [2, Section 11]. For $\sigma_1, \sigma_2 \in \Gamma(S \otimes E)$, the Green's formula for the Dirac operator [12, equation 5.7] yields

$$\int_M \langle \not{D}_A \sigma_1, \sigma_2 \rangle = \int_M \langle \sigma_1, \not{D}_A \sigma_2 \rangle - \int_{\partial M} \langle \gamma(\nu) \cdot \sigma_1, \sigma_2 \rangle$$

where $\gamma(\nu)$ denotes Clifford multiplication by ν. This implies that, when restricted to smooth sections, $\widehat{\Lambda}_{g,A,Z,m}(\chi) = -\gamma(\nu)\, \not{D}_A\, \varphi|_{\partial M}$, and so differs from $\Lambda_{g,A,Z,m}(\chi)$ as given in Definition 1.1 by tangential derivatives. In particular, we may say that $\widehat{\Lambda}_{g,A,Z,m}$ and $\Lambda_{g,A,Z,m}$ contain the same information about the geometric data, such as the metric and connection. In this paper, we thus are free to restrict ourselves to considering $\Lambda_{g,A,Z,m}$.

We note that there are some natural gauge invariances that arise from Definition 1.1, which shall be explored in greater detail in Section 2. The first is the gauge-invariance of the connection A. Recall that two connections A and A' are called *gauge equivalent* if there exists a unitary automorphism G of E, otherwise called a *gauge transformation*, such that their covariant derivatives are related by

$$\nabla^{A'} = G^{-1} \circ \nabla^A \circ G. \tag{1.2}$$

We say that A and A' are locally gauge equivalent about a point $x \in M$ if there is an open neighbourhood U of x such that the restrictions of A and A' to U are gauge equivalent.

It is easy to see that if there exists a G as in (1.2) with $G|_{\partial M} = \mathrm{id}$, then one has $\Lambda_{g,A',Z',m} = \Lambda_{g,A,Z,m}$ where $Z' = (\mathrm{id}_S \otimes G)^{-1} Z (\mathrm{id}_S \otimes G)$. Indeed, if φ is the solution to (1.1) for A, then $(\mathrm{id}_S \otimes G)^{-1}\varphi$ is the solution to (1.1) with A replaced by A' and Z replaced with Z'. Therefore, we have

$$\begin{aligned} \Lambda_{g,A',Z',m}(\chi) &= \nabla_\nu^{A'}((\mathrm{id}_S \otimes G^{-1})\varphi)|_{\partial M} \\ &= (\mathrm{id}_S \otimes G^{-1}) \nabla_\nu^{A} \varphi|_{\partial M} = \Lambda_{g,A,Z,m}(\chi). \end{aligned}$$

Thus, given $\Lambda_{g,A,Z,m} = \Lambda_{g,A',Z',m}$, we can only every recover the connection up to a gauge transformation that is equal to the identity on the boundary.

Going further, we would like to say that the Definition 1.1, inasmuch as it depends on the geometry of the metric g, depends only on the isometry class of g. However, since spin structures, and hence spinor bundles, are defined with respect to a fixed metric, we must take care in relating the Dirichlet-to-Neumann maps of two different metrics. In Section 2, we explain how a diffeomorphism $\Phi: M \to M$ induces an isomorphism of associated spinor bundles $\widetilde{\Phi}: S_{\Phi^* g} \to S_g$. Then it is easy to prove that the Dirichlet-to-Neumann map is diffeomorphism-invariant.

Lemma 1.3. *Let* $\Phi: M \to M$ *be a diffeomorphism such that* $\Phi|_{\partial M} = \mathrm{id}$. *Then*

$$\Lambda_{\Phi^* g, A, Z', m} = (\widetilde{\Phi} \otimes \mathrm{id}_E)^{-1}|_{\partial M} \circ \Lambda_{g,A,Z,m} \circ (\widetilde{\Phi} \otimes \mathrm{id}_E)|_{\partial M},$$

where $Z' \in \mathrm{End}(S_{\Phi^* g} \otimes E)$ *is defined by* $Z' := (\widetilde{\Phi} \otimes \mathrm{id}_E)^{-1} Z (\widetilde{\Phi} \otimes \mathrm{id}_E)$.

Since we shall be primarily interested in the inverse problem for $U(N)$-connections on a Hermitian vector bundle, the diffeomorphism-invariance of the Dirichlet-to-Neumann map illustrated in Lemma 1.3 will not concern us.

In this paper, we want to consider a Calderón problem for a $U(N)$-connection A, which couples to an E-valued spinor ϕ through some natural equations arising from physics. To this end, recall that given a connection A, one may define its curvature F_A. If P denotes the principal $U(N)$-bundle of unitary frames of E, then F_A is an $\mathrm{ad}\, P$-valued 2-form. We can thus consider F_A as a 2-form with values in skew-Hermitian endomorphisms of E. In physics, the curvature of a connection corresponds to a force field; if the connection is abelian, then its curvature is the electromagnetic field. Now, let $m \in \mathbb{R}$ be such that m^2 is not in the Dirichlet spectrum of $D\!\!\!/_A^2$. We then assume that there exists an E-valued spinor ϕ such that (A, ϕ) satisfies the following second-order Yang–Mills–Dirac system:

$$\begin{cases} D\!\!\!/_A^2 \phi = m^2 \phi, \\ d_A^* F_A = J(\phi), \end{cases} \tag{1.3}$$

where the current $J(\phi)$ is an $\mathrm{ad}\, P$-valued 1-form depending quadratically on ϕ as defined in equation (4.3) below. Note that, in contrast to the case of a scalar field, the current for a spinor field does not depend on the connection A (see [1]). The system (1.3) arises when a gauge field interacts with fermionic matter, as represented by the spinor field ϕ. In this paper we prove the following result.

Theorem 1.4. *Let* (M, g) *be a compact* n*-dimensional real-analytic spin manifold with boundary, and let* E *be a real-analytic Hermitian vector bundle over* M. *Let* A *and* B *be real-analytic* $U(N)$*-connections on* E, *and let* ϕ *and* ψ *be smooth* E*-valued spinors on* M, *real-analytic in the interior, such that* (A, φ) *and* (B, ψ) *both satisfy the second-order Yang–Mills–Dirac system* (1.3), *and such that* $\varphi|_{\partial M} = \psi|_{\partial M}$. *If* $\Lambda_{g,A,0,m} = \Lambda_{g,B,0,m}$, *then near every point in* M, A *and* B *are locally gauge equivalent via a real-analytic gauge transformation. Moreover,* A *and* B *are globally gauge equivalent via a smooth gauge transformation. If* $N = 1$, *then* A *and* B *are globally gauge equivalent via a real-analytic gauge transformation.*

In other words, if a connection A couples to an auxiliary E-valued spinor ϕ via the second-order Yang–Mills–Dirac system (1.3), then by making boundary measurements $\Lambda_A(\chi)$ of other spinor fields, one can determine the curvature of the connection at any point up to conjugation. In the abelian case, one can determine the connection up to gauge.

2. Preliminaries

In this section, we recall some of the basic tools and constructions that we need to state the central problem and prove the main result. We review the construction of spinor bundles and the Dirac operator in Section 2.1, and extend this to bundle-valued spinors in Section 2.2. Lastly, we review a few key results from the theory of pseudodifferential operators in 2.3.

2.1. Spinors and the Dirac operator

In this section, we briefly recall some basic notions of spin geometry, such as Clifford algebras, Spin manifolds, the construction of the spinor bundle on a Spin manifold, and the definition and fundamental properties of Dirac operators. The important results for our purposes are the local formulae (2.1) for the spin connection, and (2.2) for the Dirac operator. For details on the results presented in this section, we refer the reader to [12].

Definition 2.1. The *Clifford algebra* $C\ell_n$ is the real associative algebra generated by vectors in $\mathbb{R}^n$, with a product $\cdot$ satisfying the relation

$$v \cdot w + w \cdot v = -2\langle v, w\rangle$$

for all $v, w \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Note that we have a natural inclusion $\mathbb{R}^n \subseteq C\ell_n$, which can be extended to a vector space isomorphism $\Lambda^*\mathbb{R}^n \to C\ell_n$ by fixing an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$ and sending

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto e_1 \cdots e_k.$$

In particular, we have $\dim C\ell_n = 2^n$.

Clifford algebras play an important role throughout geometry and physics. One reason for this is that $C\ell_n$ naturally contains the spin group $\mathrm{Spin}(n)$, which can be defined as a double cover of the rotation group $\mathrm{SO}(n)$. That is, there exists a surjective Lie group homomorphism $\lambda\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ such that $\ker \lambda = \{1, -1\}$. Thus, λ is a 2-sheeted covering map.

Remark 2.2. It follows from the preceding that $\mathrm{Spin}(n)$ is in fact the universal cover of $\mathrm{SO}(n)$ for $n \geq 3$, since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$.

Definition 2.3. Let (M, g) be an oriented n-dimensional Riemannian manifold *without* boundary. A *spin structure* P_{Spin} on M is a principal $\mathrm{Spin}(n)$-bundle, which covers the $\mathrm{SO}(n)$-bundle of orthonormal frames of M, in such a way that the covering map is compatible with the 2-fold covering map $\lambda\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$. When such a bundle P_{Spin} exists, we say M is *spin*.

Remark 2.4. Not every oriented Riemannian manifold is spin. In fact, the obstruction to being spin is entirely contained in the second Stiefel–Whitney class of M. Even when M is spin, the spin structure need not be unique. See [12, Chapter I, Sections 1–5] for details.

A spin manifold is endowed with a distinguished complex vector bundle S called the *spinor bundle*. It is defined as follows. First, fix a spin structure P_{Spin} on M. Let $\rho\colon \mathrm{C}\ell_n \to \mathrm{End}\,\mathbb{C}^k$ be an irreducible $\mathrm{C}\ell_n$-module. It follows from the structure of $\mathrm{C}\ell_n$ that $k = 2^{[\frac{n}{2}]}$. Then ρ restricts to a representation of $\mathrm{Spin}(n)$, called the *spinor representation*. The spinor bundle S is then the bundle associated to P_{Spin} and the spinor representation, $S := P_{\mathrm{Spin}} \times_\rho \mathbb{C}^k$. The sections of S are called *(complex) spinors*. One can also define real spinors but they shall not concern us here.

Remark 2.5. The spinor representations have the property that $-1 \in \mathrm{Spin}(n)$ acts non-trivially. Since $\lambda(-1) = 1$, where $\lambda\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the 2-fold covering map described above, it follows that the spinor representations do not descend to representations of $\mathrm{SO}(n)$. Therefore, the spinor representations are in some sense the simplest representations that do not correspond to representations of $\mathrm{SO}(n)$.

Since the typical fibre of S is a $\mathrm{C}\ell_n$-module, it follows that S is bundle of modules over $\mathrm{C}\ell(M)$, the Clifford bundle of M, which is defined by $\mathrm{C}\ell(M)_x := \mathrm{C}\ell(T_xM)$. In particular, since $TM \subseteq \mathrm{C}\ell(M)$, we have a map $\gamma\colon TM \to \mathrm{End}\,S$, called *Clifford multiplication*. It is possible to endow S with a Hermitian metric such that $\gamma(e)$ is skew-symmetric for all unit vectors $e \in TM$. We assume S has such a metric henceforward.

So, far we have presented these definitions for a spin manifold without boundary. A spin manifold with boundary M is defined to be any closed domain of a spin manifold N, whose spin structure is the restriction of the spin structure on N. All of the above definitions then extend to this setting.

Now, the Levi-Civita connection ω on the oriented orthonormal frame bundle of M lifts to a connection ω^{s} on any given spin structure, which we call the *spin connection*. This connection induces a covariant derivative ∇^{s} on sections of S, which can be explicitly described with respect to a local trivialization as follows.

Let $(e_i)_i$ be a local orthonormal frame for M, and let ${\omega^i}_j$ be the Levi-Civita connection 1-form with respect to this frame, defined by the property that if ∇ is the Levi-Civita connection on TM, then for all vector fields X in the domain of $(e_i)_i$, we have

$$\nabla_X e_j = \omega^i{}_j(X) e_i,$$

where here ∇ is the covariant derivative on TM induced by the Levi-Civita connection. Note that here, and in the rest of this paper, we use the summation convention, so that a sum is implied over any index that occurs once in the superscript and once in the subscript. For clarity, however, we shall write the sum explicitly when the Clifford multiplication map γ is involved.

Now, this frame $(e_i)_i$ can be lifted to a local section of the spin structure P_{Spin}, which we can regard as a local frame $(\sigma_\alpha)_\alpha$ for S (although there is another lifted frame, namely $(-\sigma_\alpha)_\alpha$, the choice of lift is immaterial here). Then, with respect to this frame of spinors, the spin connection takes the form

$$\nabla^{\text{s}} \sigma_\alpha = -\frac{1}{2} \sum_{i<j} \omega^i{}_j \otimes \gamma(e_i)\gamma(e_j)\sigma_\alpha. \tag{2.1}$$

The structure of a Clifford module allows us to define the Dirac operator $\not{D}$ on sections of S as follows. For $\varphi \in \Gamma(S)$, let $(e_i)_i$ be any orthonormal frame in an open set U. Then, in U,

$$\not{D}\varphi := \sum_{i=1}^{n} \gamma(e_i) \nabla^s_{e_i} \varphi. \tag{2.2}$$

This definition does not depend on the choice of orthonormal frame. The Dirac operator plays a crucial role in physics, where it occurs in the equations of motion for spinor fields, which represent fermionic matter. The square of the Dirac operator satisfies the famous Lichnerowicz formula,

$$\not{D}^2 \varphi = (\nabla^{\text{s}})^* \nabla^{\text{s}} \varphi + \frac{1}{4} R\varphi,$$

where $(\nabla^s)^*$ denotes the formal adjoint of ∇^s with respect to the L^2-inner product, and R is the scalar curvature of g. Thus, one can transfer questions about the Dirac Laplacian $\not{D}^2$ to questions about the spin connection Laplacian at the expense of a curvature term.

2.2. $U(N)$-connections and bundle-valued spinors

We now want to introduce an auxiliary Hermitian vector bundle (E, h) equipped with a connection A, whose curvature F_A corresponds to some physical force. Moreover, we want to introduce a mechanism to couple this connection to spinor fields, corresponding to the physical interaction between the force field F_A and fermions. If the connection is abelian, then its curvature is the familiar electromagnetic field.

Let (E, h) be a Hermitian vector bundle of rank N, which is associated to its bundle of complex orthonormal frames P, which is a principal $U(N)$-bundle. Consider the bundle $S \otimes E$, the bundle of E-valued spinors, which is associated to the spliced principal bundle $P_{\mathrm{Spin}} \times_M P$, defined to be $\Delta_M^*(P_{\mathrm{Spin}} \times P)$ where $\Delta_M \colon M \to M \times M$ is the diagonal embedding. It has structure group $\mathrm{Spin}(n) \times U(N)$. Given a $U(N)$-connection A on P, we can equip $S \otimes E$ with the connection $\omega^{\mathrm{s}} \otimes A$. The corresponding covariant derivative on sections of $S \otimes E$ is denoted by ∇^A. Thus, with respect to a local trivialization of E, an E-valued spinor ψ is given by a tuple of N complex spinors, and

$$(\nabla_X^A \psi)^a = \nabla_X^{\mathrm{s}} \psi^a + A^a{}_b(X)\psi^b \quad a \in \{1, \ldots, N\},$$

where $A^a{}_b$ are the components of the $\mathfrak{u}(N)$-valued 1-form representing the connection A in this local trivialization. We can also define a twisted Dirac operator acting on $\psi \in \Gamma(S \otimes E)$ by

$$\not{D}_A \psi := \sum_{i=1}^{n} \gamma(e_i)\nabla_{e_i}^A \psi$$

where $(e_i)_i$ is any local orthonormal frame on M. This definition does not depend on the orthonormal frame $(e_i)_i$. Note that Clifford multiplication on E-valued spinors acts on the S factor; that is, we have identified $\gamma(e_i)$ with $\gamma(e_i) \otimes \mathrm{id}$. With respect to a local trivialization of E, we can view $\psi \in \Gamma(S \otimes E)$ as N complex spinors, and the twisted Dirac operator takes the form

$$(\not{D}_A \psi)^a = \sum_{i=1}^{n} \gamma(e_i)\nabla_{e_i}^s \psi^a + A^a{}_b(e_i)\gamma(e_i)\psi^b, \quad a \in \{1, \ldots, N\}.$$

The twisted Dirac operator satisfies a twisted Lichnerowicz formula [12, Chapter II, Theorem 8.17],

$$\not{D}_A^2 \psi = (\nabla^A)^*\nabla^A \psi + \frac{1}{4}R\psi + \mathfrak{F}_A \cdot \psi, \tag{2.3}$$

where the curvature operator $\mathfrak{F}_A \colon S \otimes E \to S \otimes E$ is defined by

$$\mathfrak{F}_A(\sigma \otimes \eta) := \frac{1}{2}\sum_{j,k=1}^{n} (\gamma(e_j)\gamma(e_k)\sigma) \otimes (F_A(e_j, e_k)\eta).$$

A *gauge transformation* is a section of $U(E)$, the bundle of unitary automorphisms of E. These sections form a group called the *gauge group,* which we denote $\mathcal{G}(E)$. A gauge transformation $G \in \mathcal{G}(E)$ acts on a connection A by taking it to the connection A' defined by

$$\nabla^{A'} = G^{-1} \circ \nabla^A \circ G.$$

Two connections A and A' are considered to be gauge equivalent if they lie in the same $\mathcal{G}(E)$-orbit. The notion of local gauge equivalence over an open set is similarly defined using the restricted bundles and connections. If A' is related to A by a local gauge transformation G, then with respect to a local trivialization, the connection 1-forms representing A and A' are related by

$$A' = G^{-1}AG + G^{-1}dG.$$

Note that $G^{-1}dG$ is indeed a $\mathfrak{u}(n)$-valued 1-form.

2.3. Pseudodifferential calculus

The proof of Theorem 1.4 requires a few key results from the theory of pseudodifferential operators, which we recall briefly here. For details and proofs, we refer the reader to [19], for example.

Let $W \subseteq \mathbb{R}^n$ be open. Then the symbol class $S^m(W)$ is defined to be the space of all functions $p \in C^\infty(W \times \mathbb{R}^n)$ satisfying for all $\alpha, \beta \in \mathbb{N}^n$,

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|},$$

where $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ is the usual regularization of $|\xi|$. The symbol class $S^m(W, \mathbb{C}^{k\times k})$ is then the space of all matrix-valued functions whose entries are in $S^m(W)$. Each $p \in S^m(W, \mathbb{C}^{k\times k})$ yields a map $P: C_c^\infty(W, \mathbb{C}^k) \to C^\infty(W, \mathbb{C}^k)$ given by

$$(Pw)(x) := \int e^{ix\cdot\xi} p(x,\xi)\widehat{w}(\xi)\,d\xi, \tag{2.4}$$

where $\widehat{w}$ denotes the Fourier transform of w. We say that $P \in \Psi^m(W, \mathbb{C}^k)$ if P has the form (2.4) for a symbol $p \in S^m(W, \mathbb{C}^{k\times k})$. A pseudodifferential operator $P \in \Psi^m(W, \mathbb{C}^k)$ is called *classical* if its symbol $p(x,\xi)$ is given by an asymptotic series in the sense that for any large positive M, there is an integer J such that

$$p(x,\xi) - \sum_{j=1}^{J} p_{m_j}(x,\xi) \in S^{-M}(W, \mathbb{C}^{k\times k}),$$

where each term p_{m_j} is positive-homogeneous of degree m_j in ξ, and the sequence of real numbers m_j is decreasing to $-\infty$. In this case, we write

$$p(x,\xi) \sim \sum_{j\geq 1} p_{m_j}(x,\xi).$$

Now, let E be a vector bundle of rank k over a smooth manifold M, and let $\mathcal{D}'(E)$ be the space of E-valued distributions on M. A map $P: C_c^\infty(M, E) \to C^\infty(M, E)$ is called a *pseudodifferential operator of order* M if for every chart W of M and every local trivialization of E over W, the induced map is in $\Psi^m(W, \mathbb{C}^k)$. The space of pseudodifferential operators on E of order m is denoted $\Psi^m(M, E)$. If $P \in \Psi^m(M, E)$ for all $m \in \mathbb{R}$, then we call P a *smoothing operator.* The space of smoothing operators is denoted $\Psi^{-\infty}(M, E)$. Moreover, if $P \in \Psi^m(M, E)$, then it extends to a map $\mathcal{E}'(E) \to \mathcal{D}'(E)$, and if M is compact, then P extends to a map $H^s(E) \to H^{s-m}(E)$.

We often work with pseudodifferential operators modulo $\Psi^{-\infty}(M, E)$, since then composition of pseudodifferential operators becomes well defined. Thus, we will treat two pseudodifferential operators as equivalent if their difference is a smoothing operator. Each equivalence class then corresponds to a symbol modulo $S^{-\infty}$, the intersection of all symbol classes S^m. Note that in particular two classical pseudodifferential operators differ by a smoothing operator if and only if their formal symbols are equal modulo $S^{-\infty}$.

If $P_i \in \Psi^{m_i}(M, E)$ for $i = 1, 2$, then their composition $Q := P_1 \circ P_2$ is well defined modulo smoothing operators, and its symbol q modulo $S^{-\infty}$ is given in local coordinates by

$$q(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha p_1(x, \xi) D_x^\alpha p_2(x, \xi). \tag{2.5}$$

3. Boundary determination

In this section, we prove that the Dirichlet-to-Neumann map $\Lambda_{g,A,Z,m}$ is a pseudodifferential operator of order 1 whose symbol with respect to a local trivialization determines the Taylor series of g, Z, and A at the boundary, when A is in an appropriate gauge. We shall see upon applying the recipe of Lee and Uhlmann that unlike the analogous proofs for the scalar Laplacian [13] or the connection Laplacian [4], we need not place any restrictions on the metric. In particular, we shall see that because the connection coefficients of $\omega^s \otimes A$ involve the Levi-Civita connection, and thus the derivatives of the metric, the Taylor series of the metric can be recovered without fixing a representative in its conformal class as is done in [4].

Theorem 3.1. *The Dirichlet-to-Neumann map in Definition 1.1 is an elliptic pseudodifferential operator of order* 1. *Moreover, in any local trivialization where the normal component* A_n *of the connection* 1*-form* A *vanishes, the total symbol of* $\Lambda_{g,A,Z,m}$ *determines the Taylor series of* g, A, *and* Z *at the boundary.*

We would like to say, therefore, that if $\Lambda_{g,A,Z,m} = \Lambda_{\tilde{g},\tilde{A},\tilde{Z},m}$, then the Taylor series of the data (g, A, Z) and $(\tilde{g}, \widetilde{A}, \widetilde{Z})$ agree at the boundary. This equality of Dirichlet-to-Neumann maps, however, does not make sense, since they act on different bundles. We can nevertheless compare these two maps by proceeding as follows.

Suppose we have a Riemannian metric g on M inducing a spinor bundle S for a fixed choice of spin structure. Then for a connection A on E, and an endomorphism Z of $S \otimes E$, we can construct the corresponding Dirichlet-to-Neumann map $\Lambda_{g,A,Z,m}$ on $(S \otimes E)|_{\partial M}$. Suppose we have also another Riemannian metric $\tilde{g}$ on M, which induces another spinor bundle $\widetilde{S}$ corresponding to a choice of spin structure. Then for any connection $\widetilde{A}$ on E, and endomorphism $\widetilde{Z}$ of $\widetilde{S} \otimes E$, we can construct the corresponding Dirichlet-to-Neumann map $\Lambda_{\tilde{g},\widetilde{A},\widetilde{Z},m}$ on $(\widetilde{S} \otimes E)|_{\partial M}$. To say that these two Dirichlet-to-Neumann maps are equal is of course to mean that they are equal up to isomorphism, but we must take care to specify what kind of isomorphism.

Definition 3.2. We say that the Dirichlet-to-Neumann maps $\Lambda_{g,A,Z,m}$ and $\Lambda_{\tilde{g},\widetilde{A},\widetilde{Z},m}$ corresponding to data (g, A, Z) and $(\tilde{g}, \widetilde{A}, \widetilde{Z})$ respectively are *isomorphic* if there exists an isomorphism $\Phi\colon S|_{\partial M} \to \widetilde{S}|_{\partial M}$ induced by an isomorphism of the restriction of the SO(n)-structures of g and $\tilde{g}$ to ∂M, such that

$$\Lambda_{\tilde{g},\widetilde{A},\widetilde{Z},m} = (\Phi \otimes \mathrm{id}_E) \circ \Lambda_{g,A,Z,m} \circ (\Phi^{-1} \otimes \mathrm{id}_E).$$

This is equivalent to saying that the local matrix representations of $\Lambda_{g,A,Z,m}$ and $\Lambda_{\tilde{g},\widetilde{A},\widetilde{Z},m}$ are equal when we choose local trivializations of $S|_{\partial M}$ and $\widetilde{S}|_{\partial M}$ induced by orthonormal frames for g and $\tilde{g}$ respectively.

Remark 3.3. An isomorphism between the SO(n)-structures of two metrics g and $\tilde{g}$ is equivalent to an isometry between g and $\tilde{g}$.

The precise meaning of Theorem 3.1 is then that if two sets of data (g, A, Z) and $(\tilde{g}, \widetilde{A}, \widetilde{Z})$ lead to equivalent Dirichlet-to-Neumann maps, then the Taylor series of g and $\tilde{g}$ are equal at the boundary, as are the Taylor series of Z and $\widetilde{Z}$, and moreover, after possibly making a gauge transformation of $\widetilde{A}$, the Taylor series of A and $\widetilde{A}$ are equal at the boundary in any local trivialization where $A_n = 0$ and $\widetilde{A}_n = 0$ near the boundary.

The proof of Theorem 3.1 follows the recipe of Lee and Uhlmann, which is by now standard in the literature. The idea is to factor the second-order differential operator $\not{D}_A^2 + Z - m^2$ into a product of first-order pseudodifferential operators modulo smoothing:

$$\not{D}_A^2 + Z - m^2 \sim (D_n + i(E - \theta_n) - iB)(D_n - i\theta_n + iB)$$

where $D_n = -i\partial_n$, $E = -\frac{1}{2}g^{\alpha\beta}\partial_n g_{\alpha\beta}$, $\theta_n = \mathrm{id}_S \otimes A_n + \omega_n^{\mathrm{s}} \otimes \mathrm{id}_E$, and B is some pseudodifferential operator to be determined. We can inductively solve for the total symbol of B to show that such a pseudodifferential operator indeed exists. Then, using this factorization and the theory of generalized heat equations, one can show that $B|_{\partial M} \sim \Lambda_{g,A,Z,m}$. Finally, we note that the inductive procedure used to determine the symbol of B in terms of the known data (g, A, Z) can be inverted to inductively solve for the normal derivatives of this data at the boundary.

Proof of Theorem 3.1. For this proof, we let x denote the coordinates in a fixed boundary chart for g, and write $x = (x', x^n)$, where x^n is the normal coordinate, and $x' = (x^1, \ldots, x^{n-1})$ are the tangential coordinates. Greek letters run over $\{1, \ldots, n-1\}$ while Latin indices run over $\{1, \ldots, n\}$. So, in particular, $(x^\alpha, 0)$ form coordinates on the boundary. We let $D_k := -i\partial_k$. Finally, let θ be the matrix valued 1-form representing the connection $\omega^s \otimes A$ in a local trivialization to be determined, where ω^s is the spin connection. We now use the twisted Lichnerowicz formula (2.3),

$$\not{D}_A^2 = (\nabla^A)^* \nabla^A + \frac{1}{4} R + \frac{1}{2} \mathfrak{F}_A$$

and the following formula for the connection Laplacian in a coordinate chart [15, equation 2.7.31],

$$(\nabla^A)^* \nabla^A = -g^{ij} \nabla_i^A \nabla_j^A + g^{ij} \Gamma_{ij}^k \nabla_k^A,$$

where Γ_{ij}^k are the Christoffel symbols of g with respect to this chart, given by

$$\nabla_i \partial_j = \Gamma_{ij}^k \partial_k.$$

It is a straightforward exercise to verify that the Christoffel symbols are also given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

Now, using the expression for the connection Laplacian and the twisted Lichnerowicz formula, we can write out the Dirac Laplacian in this boundary chart, and in this local trivialization. Separating out the normal derivatives from the tangential derivatives, we have

$$\not{D}_A^2 + Z - m^2 = D_n^2 + i(E - 2\theta_n) D_n + Q_2 + Q_1 + Q_0 \tag{3.1}$$

where $E = -\frac{1}{2} g^{\alpha\beta} \partial_n g_{\alpha\beta}$, and where

$$\begin{aligned}
Q_2 &:= -g^{\alpha\beta} \partial_a \partial_\beta, \\
Q_1 &:= -2g^{\alpha\beta} \theta_\alpha \partial_\beta + g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma \partial_\gamma, \\
Q_0 &:= -g^{ij} (\partial_i \theta_j) - g^{ij} \theta_i \theta_j + g^{ij} \Gamma_{ij}^k \theta_k + \frac{1}{4} R + \frac{1}{2} \mathfrak{F}_A + Z - m^2.
\end{aligned}$$

Here, ${\Gamma^i}_{jk}$ are the Christoffel symbols of g corresponding to the boundary chart. Note that Q_i is an i-th order differential operator involving only tangential derivatives. We want to show that there exists a pseudodifferential operator $B(x, D')$ of order 1 such that

$$\not{D}_A^2 + Z - m^2 = (D_n + i(E - \theta_n) - iB)(D_n - i\theta_n + iB) + \mathcal{S} \tag{3.2}$$

where $\mathcal{S}$ is a smoothing operator. In writing $B(x, D')$, we mean that B is a map $[0, \varepsilon) \to \Psi(\partial M)$, which for each value x_n yields a pseudodifferential operator $B(x_n, x', D')$ on the corresponding tangential leaf. In particular, we have a well-defined notion of restricting B to the boundary, $B(0, x', D')$. The motivation for this factorization comes from the following Lemma, a proof of which is given following the current proof.

Lemma 3.4. *If $B(x, D')$ exists, then $B|_{\partial M} = \Lambda_{g,A,Z,m} + \mathcal{R}$ where $\mathcal{R}$ is a smoothing operator.*

An important implication of the preceding lemma is that $B(0, x', D')$ has the same total symbol as the Dirichlet-to-Neumann map, and so in doing symbol computations, it suffices to work with the operator B appearing in the factorization.

We now proceed with the proof of Theorem 3.1. In order to show that such a $B(x, D')$ exists, we will use the factorization (3.2) to determine a formal symbol whose corresponding pseudodifferential operator satisfies (3.2) in each degree. So, by equations (3.1) and (3.2), we have

$$\begin{aligned} s(D_n + i(E - \theta_n) - iB)(D_n - i\theta_n + iB) + \mathcal{S} \\ = D_n^2 + i(E - 2\theta_n)D_n + Q_2 + Q_1 + Q_0, \end{aligned} \tag{3.3}$$

where $\mathcal{S}$ is a smoothing operator. Expanding the left-hand side of (3.3), we see that it equals

$$D_n^2 + i(E - 2\theta_n)D_n + i[D_n, B] - \partial_n\theta_n + (E - \theta_n)\theta_n - [B, \theta_n] - EB + B^2. \tag{3.4}$$

Replacing the left-hand side of equation (3.3) with (3.4) and rearranging, we get

$$i[D_n, B] - [B, \theta_n] - EB + B^2 = Q_2 + Q_1 + Q_0' \tag{3.5}$$

where

$$Q_0' := Q_0 + \partial_n\theta_n - (E - \theta_n)\theta_n.$$

We want to consider the total symbols of the left and right-hand sides of (3.5). Let $b(x, \xi')$ be the symbol of $B(x, D')$, and similarly for Q_2, Q_1 and Q_0'. Then equation (3.5) implies

$$\begin{aligned} \partial_n b + [\theta_n, b] - \sum_{\nu \neq 0} \frac{1}{\nu!} \partial_{\xi'}^{\nu} b \cdot D_{x'}^{\nu} \theta_n - Eb + \sum_{\nu} \frac{1}{\nu!} (\partial_{\xi'}^{\nu} b)(D_{x'}^{\nu} b) \\ = q_2 + q_1 + q_0', \end{aligned} \tag{3.6}$$

where ν runs over multi-indices. Let us assume that $b(x, \xi')$ has a formal symbol given by

$$b(x, \xi') = \sum_{k \leq 1} b_k(x, \xi') \tag{3.7}$$

where $b_k(x,\xi')$ is homogeneous in ξ' of degree k away from 0. Plugging (3.7) into (3.6) and taking the degree m part of the equation, we get

$$b_1^2 = q_2, \tag{3.8}$$

$$2b_1 b_0 + \partial_n b_1 - E b_1 + \sum (\partial_{\xi'} b_1)(D_{x'} b_1) = q_1, \tag{3.9}$$

$$\begin{aligned} 2b_1 b_{-1} + \partial_n b_0 + [\theta_n, b_0] - \sum_\alpha (\partial_{\xi_\alpha} b_1)(D_{x^\alpha}\theta_n) - E b_0 \\ + \sum_{\substack{j+k=|\nu| \\ 0\le j,k\le 1}} \frac{1}{\nu!}(\partial_{\xi'}^\nu b_j)(D_{x'}^\nu b_k) = q_0', \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} 2b_1 b_{m-1} = -\partial_n b_m - [\theta_n, b_m] - \sum_{0<|\nu|\le 1-m} \frac{1}{\nu!}\partial_{\xi'}^\nu b_{m+|\nu|}\cdot D_{x'}^\nu \theta_n + E b_m \\ - \sum_{\substack{j+k-|\nu|=m \\ m\le j,k\le 1}} \frac{1}{\nu!}(\partial_{\xi'}^\nu b_j)(D_{x'}^\nu b_k) \end{aligned} \tag{3.11}$$

for $m \le -1$. Note that since $q_2(x,\xi') = g^{\alpha\beta}\xi_\alpha\xi_\beta$, equation (3.8) yields the solution

$$b_1(x,\xi') := -|\xi'|_g.$$

Note that we choose the principal symbol b_1 to be negative as opposed to positive, as was needed in the proof of Lemma 3.4 above. It is clear from equations (3.9)–(3.11) that for $m < 1$, we can inductively determine b_{m-1} from the first $|m|+1$ terms in (3.7). Therefore, since we have a formal symbol that satisfies (3.6), the corresponding pseudodifferential operator satisfies (3.3). This completes the proof that the pseudodifferential operator B exists, and so along with Lemma 3.4, whose proof is given at the end of this section, it follows that the Dirichlet-to-Neumann map is an elliptic pseudodifferential operator of order 1.

In the other direction, suppose we know the Dirichlet-to-Neumann map $\Lambda_{g,A,Z,m}$ associated to data (g, A, Z, m). Then we know its total symbol in any boundary chart and local trivialization of E over the boundary. By using equations (3.8)–(3.11), we can work backwards and inductively extract the boundary data from the homogeneous terms in the expansion of the symbol. To do this, we fix a boundary chart for ∂M, and an orthonormal frame e_α on ∂M, which we extend into M via parallel transport along e_n, the inward pointing normal. This orthonormal frame e_i now induces a local trivialization of the spinor bundle near ∂M in which the spin connection takes the form (2.1). We also pick a local trivialization of E.

To start with, the restriction of the metric at the boundary $g|_{\partial M}$ is encoded in the function q_2 and can therefore be recovered from the principal symbol b_1 as per equation (3.8). Upon simplifying and rearranging, equation (3.9) yields

$$b_0 = i g^{\alpha\beta}\theta_\alpha \frac{\xi_\beta}{|\xi|} - \frac{1}{2}(E g^{\alpha\beta} + \frac{1}{2}\partial_n g^{\alpha\beta})\frac{\xi_\alpha\xi_\beta}{|\xi|^2} + F_0(g_{\alpha\beta}|_{\partial M}) \tag{3.12}$$

for all $\xi \in T^*\partial M \setminus \{0\}$, where F_0 is some function of its arguments involving only tangential derivatives along ∂M. In particular, F_0 is known once $g|_{\partial M}$ is known. Thus, from b_0 we can determine the first two terms on the right-hand side of (3.12). Moreover, since these first two terms have opposite parity with respect to ξ, we can determine each term separately by substituting different values for ξ. Therefore, knowing $g^{\alpha\beta}|_{\partial M}$, we can recover $\theta_\alpha|_{\partial M}$, which by (2.1) takes the form

$$\theta_\alpha|_{\partial M} = \mathrm{id}_S \otimes A_\alpha|_{\partial M} - \frac{1}{2}\sum_{i<j} \omega^i{}_j(\partial_\alpha)|_{\partial M}\gamma(e_i)\gamma(e_j) \otimes \mathrm{id}_E \tag{3.13}$$

where $\omega^i{}_j(\partial_\alpha)$ denotes the connection 1-form for the Levi-Civita connection with respect to the orthonormal frame e_i, evaluated on the coordinate vector ∂_α. By the Clifford relations, the matrices $\gamma(e_i)\gamma(e_j)$ occurring in the second term of (3.13) are traceless for $i \neq j$. We can therefore recover A_α as a partial trace over the spinor bundle:

$$A_\alpha(0, x') = \frac{1}{\mathrm{rank}\, S}\mathrm{Tr}_S\, \theta_\alpha|_{\partial M}.$$

Since $A_n = 0$ in this gauge by assumption, we have recovered $A|_{\partial M}$. Knowing $A|_{\partial M}$, from equation (3.13) we can recover the matrix

$$-\frac{1}{2}\sum_{i<j} \omega^i{}_j(\partial_\alpha)|_{\partial M}\gamma(e_i)\gamma(e_j).$$

Using the fact that $\gamma(e_i)\gamma(e_j)$ provide an orthonormal set of matrices with respect to the trace inner product, we can extract $\omega^i{}_j(\partial_\alpha)|_{\partial M}$. The final obstacle is to obtain the Christoffel symbols of g with respect to the boundary normal chart. For this, let h be the matrix defined by $\partial_\alpha = h^\beta{}_\alpha e_\beta$. Then, if $\Gamma^i{}_{jk}$ denotes the connection coefficients in the boundary normal chart, we have

$$\Gamma^i{}_{\alpha j} = (h^{-1})^i{}_k(\omega^k{}_\ell(\partial_\alpha))h^\ell{}_j + (h^{-1})^i{}_k\partial_\alpha h^k{}_j. \tag{3.14}$$

Since h is known on the boundary, so is $\omega^i{}_j(\partial_\alpha)|_{\partial M}$, and we can recover $\Gamma^i_{\alpha j}$. In particular, we recover the normal derivative of the metric at the boundary, since

$$\Gamma^n_{\alpha\beta} = -\frac{1}{2}\partial_n g_{\alpha\beta}.$$

Thus, we have recovered $A_\alpha|_{\partial M}$ and $\partial_n g|_{\partial M}$. The next equation (3.10), upon rearranging, leads to an equation of the form

$$b_{-1} = \frac{i}{2} g^{\alpha\beta} \partial_n \theta_\alpha \frac{\xi_\beta}{|\xi|^2} + (T_{-1}^{\alpha\beta}(g|_{\partial M}, \partial_n g|_{\partial M}, \partial_n^2 g|_{\partial M}, \partial_n A|_{\partial M}) + Z|_{\partial M} g^{\alpha\beta}) \frac{\xi_\alpha \xi_\beta}{|\xi|^3} + F_{-1}(g|_{\partial M}, \partial_n g|_{\partial M}, A|_{\partial M}), \tag{3.15}$$

where F_{-1} is a function of its arguments involving only tangential derivatives along ∂M, and the $T_{-1}^{\alpha\beta}$ term includes quantities which are not yet known, including the curvature terms. Now, as before, we can recover $\partial_n \theta_\alpha|_{\partial M}$. Moreover, since the local trivialization of S is induced by an orthonormal frame, the matrices $\gamma(e_i)$ are constant, and thus $\partial_n \theta_\alpha$ takes the form

$$\partial_n \theta_\alpha = -\frac{1}{4} \partial_n(\omega^i{}_j(\partial_\alpha)) \gamma(e_i)\gamma(e_j) \otimes \mathrm{id}_E + \mathrm{id}_S \otimes \partial_n A_\alpha.$$

Taking a partial trace over S as before, we can determine $\partial_n A_\alpha|_{\partial M}$, and therefore also $\partial_n(\omega^i{}_j(\partial_\alpha))$. Using the fact that e_β is parallel along ∂_n, and the fact that we have determined $\Gamma^i{}_{jk}|_{\partial M}$, it follows that $\partial_n h|_{\partial M}$ is known. Therefore, taking the normal derivative of the transformation law (3.14) for the connection coefficients, we get

$$\partial_n \Gamma^i{}_{\alpha j} = (h^{-1})^i{}_k \partial_n(\omega^k_\ell(\partial_\alpha)) h^\ell{}_j + S(h|_{\partial M}, \partial_n h|_{\partial M}, g|_{\partial M}, \partial_n g|_{\partial M}),$$

where S is a function of known quantities. We can therefore recover the normal derivatives of the Christoffel symbols of g in the boundary normal chart, and in particular, we can recover

$$\partial_n \Gamma^n{}_{\alpha\beta}|_{\partial M} = -\frac{1}{2} \partial_n^2 g_{\alpha\beta}|_{\partial M}.$$

Thus, having recovered the second derivatives of the metric at ∂M, and the first derivatives of the connection at ∂M, we also recover the curvature terms $R|_{\partial M}$ and $\mathfrak{F}_A|_{\partial M}$. The only unknown remaining is the endomorphism $Z|_{\partial M}$ in the even term of (3.15), which is now easily recovered.

Continuing in this fashion, at step m we get an equation of the form

$$b_{1-m} = \frac{i}{2} g^{\alpha\beta} \partial_n^{m-1} \theta_\alpha \frac{\xi_\beta}{|\xi|^m} + (T_{1-m}^{\alpha\beta} + \partial_n^{m-2} Z|_{\partial M}) \frac{\xi_\alpha \xi_\beta}{|\xi|^{m+1}} + F_{1-m}(g_{\alpha\beta}|_{\partial M}, \dots, \partial_n^{m-1} g_{\alpha\beta}|_{\partial M}, A_\alpha|_{\partial M}, \dots, \partial_n^{m-2} A_\alpha|_{\partial M}, Z|_{\partial M}, \dots, \partial_n^{m-3} Z|_{\partial M}),$$

where $T_{1-m}^{\alpha\beta}$ is a known quantity depending on derivatives of g up to order m, derivatives of A up to order $m-1$, and derivatives of Z up to order $m-3$. Therefore, as before we can recover $\partial_n^{m-1}\theta_\alpha$, and by taking traces, $\partial_n^{m-1} A_\alpha$ and $\partial_n^{m-1} \omega^i{}_j(\partial_\alpha)$. Then, taking derivatives of the transformation law, we can recover $\partial_n^{m-1} \Gamma^n{}_{\alpha\beta}$ as above. In this fashion we recover the normal derivatives of A and g at the boundary. ∎

We now give a proof of Lemma 3.4, which is essentially identical to the proof given in [13]; we nonetheless include it here in the interest of being self-contained.

Proof of Lemma 3.4. Let us suppose that a first-order pseudodifferential operator $B(x, D')$ exists that satisfies equation (3.2). Let $\chi \in H^{\frac{1}{2}}(S|_{\partial M})$, and let $\varphi \in \mathcal{D}'(S)$ be the solution to the Dirichlet problem (1.1). Since $(\not{D}_A^2 + Z - m^2)\varphi = 0$, the factorization (3.2) yields the following system of equations:

$$\psi := (D_n - i\theta_n + iB)\varphi, \quad \varphi|_{x^n=0} = \chi, \tag{3.16}$$

$$h = (D_n + i(E - \theta_n) - iB)\psi, \tag{3.17}$$

where h is some smooth spinor field near ∂M given by $h := -\mathcal{S}\psi$. Note that, although the regularity of ψ is not a priori known, we know that h is smooth since $\mathcal{S}$ is a smoothing operator. Writing $t := T - x^n$, we can write equation (3.17) as

$$\partial_t\psi - (B - E + \theta_n)\psi = -ih. \tag{3.18}$$

Equation (3.18) is a generalized backwards heat equation. Now, by elliptic regularity, we know that φ is smooth in the interior, and therefore so is ψ by equation (3.16). In particular, $\psi|_{x^n=T}$ is smooth. As was shown in the proof of Theorem 3.1, we can choose the principal symbol of B to be a negative scalar, which implies that the backwards heat equation (3.18) with initial condition $\psi|_{x_n=T}$ is well posed. Thus, the solution operator is a smoothing operator, and since $\psi|_{x_n=T}$ is smooth, so is ψ. In particular, $\psi|_{x^n=0}$ is smooth. So, let us define an operator $\mathcal{R}$ by

$$\mathcal{R}\chi := \psi|_{\partial M}.$$

Then we have that $\mathcal{R}$ is smoothing by construction, and moreover

$$\begin{aligned}\mathcal{R}\chi := \psi|_{\partial M} &= ((D_n - i\theta_n + iB(x, D'))\varphi)|_{\partial M}\\ &= -i(\nabla_n^A\varphi)|_{\partial M} + iB(0, x', D')\chi\\ &= -i\Lambda_{g,A,Z,m}\chi + iB(0, x', D')\chi.\end{aligned}$$

So, indeed, we have $B|_{\partial M} = \Lambda_{g,A,Z,m}$ modulo smoothing. ∎

Remark 3.5. The proof of Theorem 3.1, unlike in the case of the scalar or connection Laplacian, holds for $\dim M = 2$. Moreover, it is unnecessary to normalize the metric in order to obtain the Taylor series of the endomorphism Z at the boundary, as is done for the connection Laplacian in [4]. These observations can be attributed to the fact that conformal covariance of the Dirac operator does not extend to even powers thereof; see [6] for details.

4. Recovering real-analytic Yang–Mills–Dirac connections

From here on, we fix a background metric g on a compact spin manifold M with boundary ∂M, and let S be the spinor bundle of M. Let E be a Hermitian vector bundle of rank N over M, associated to its principal $U(N)$-bundle P of unitary frames, and let A be a $U(N)$-connection on P, whose spinorial Dirichlet-to-Neumann map Λ_A is given. We want to introduce an auxiliary E-valued spinor ϕ to which A couples in a physically interesting way, and whose boundary value is known, and investigate the extent to which Λ_A determines the connection A modulo the action of the gauge group. To this end, we introduce the Yang–Mills–Dirac equations:

$$\not{D}_A \phi = m\phi, \tag{4.1}$$

$$d_A^* F_A = J(\phi), \tag{4.2}$$

where $m \in \mathbb{R}$, d_A^* is the L^2-adjoint of the covariant derivative d_A acting on sections of $\Lambda^2(M, \operatorname{ad} P)$, and $J \colon \Gamma(S \otimes E) \to \Lambda^1(M, \operatorname{ad} P)$ associates to each E-valued spinor φ its *current*, and is defined as follows: for $\varphi \in \Gamma(S \otimes E)$, there is a unique $J(\varphi) \in \Lambda^1(M, \operatorname{ad} P)$ such that

$$\langle \varphi, \xi \cdot \varphi \rangle = \langle J(\varphi), \xi \rangle \tag{4.3}$$

for all $\xi \in \Lambda^1(M, \operatorname{ad} P)$, where the action of $\Lambda^1(M, \operatorname{ad} P)$ on $S \otimes E$ is the tensor product of Clifford multiplication and the action of ad P on E. The current map J clearly defines a pointwise map between $S \otimes E$ and the bundle of ad P-valued 1-forms. Note that for any gauge transformation G, we have $J(G\varphi) = GJ(\varphi)G^{-1}$. The physical significance of equations (4.1)–(4.2) is that they represent a gauge field A interacting with some matter field ϕ of mass m, by means of its charge current $J(\phi)$. For details on the Yang–Mills–Dirac system, we refer the reader to [1].

Because we are concerned primarily with the Dirichlet-to-Neumann map of the second-order operator $\not{D}_A^2$, we will assume that the connection A and the auxiliary spinor field ϕ satisfy the more general second-order system:

$$\not{D}_A^2 \phi = m^2\phi, \tag{4.4}$$

$$d_A^* F_A = J(\phi). \tag{4.5}$$

Note that every solution of (4.1)–(4.2) is a solution of (4.4)–(4.5), but not vice versa.

In this section, we prove the following result.

Theorem 4.1. *Let A and B be real-analytic $U(N)$-connections as above and let ϕ and ψ be smooth E-valued spinors on M, real-analytic in the interior, such that (A, ϕ) and (B, ψ) both satisfy the second-order Yang–Mills–Dirac system (4.4)–(4.5), and such that $\phi|_{\partial M} = \psi|_{\partial M}$. If $\Lambda_{g,A,0,m} = \Lambda_{g,B,0,m}$, then A and B are locally gauge equivalent in a neighbourhood of any point in M via a real-analytic gauge transformation.*

Theorem 1.4 follows from Theorem 4.1, as well as Corollaries 4.5 and 4.7 below.

Proof. We work in a real-analytic trivialization over an open set U intersecting the boundary ∂M. We can find a smooth global gauge transformation F satisfying

$$\begin{cases} \partial_n F = -A_n F \text{ in } U, \\ F|_{\partial M} = \mathrm{id}, \end{cases}$$

which is moreover real-analytic in U, since A_n is real-analytic there. By applying this gauge transformation F, we get a pair (A', ϕ') that is real-analytic in U, such that $A'_n = 0$ in the local trivialization over U. Moreover, since $F|_{\partial M} = \mathrm{id}$, it follows that $\Lambda_{A'} = \Lambda_A$. Therefore, doing the same for B, we may assume without loss of generality that A and B are smooth connections, satisfying $A_n = B_n = 0$ in the local trivialization over U, and real-analytic there.

Since $\Lambda_A = \Lambda_B$, Theorem 3.1, implies that in this local trivialization we have

$$\partial_n^k (A - B)|_{\partial M} = 0, \quad k \geq 0.$$

That is, A and B have the same Taylor series at the boundary in this local trivialization.

We want to extend this observation to ϕ and ψ. Note that by assumption $\phi|_{\partial M} = \psi|_{\partial M}$. Using this fact, as well as the fact that $A_n = B_n = 0$ in U, the equality $\Lambda_A = \Lambda_B$ yields

$$\partial_n (\phi - \psi)|_{\partial M} = (\nabla_n^A \phi - \nabla_n^B \psi)|_{\partial M} = \Lambda_A(\phi|_{\partial M}) - \Lambda_B(\psi|_{\partial M}) = 0.$$

So, $\partial_n(\phi - \psi)|_{\partial M} = 0$ in this local trivialization. Moreover, (4.4) yields

$$\partial_n^2 (\phi - \psi)|_{\partial M} = (\mathbb{D}_B^2 \psi - \mathbb{D}_A^2 \phi)|_{\partial M} = m^2(\psi - \phi)|_{\partial M} = 0,$$

where we again have used $\partial_n(\phi - \psi)|_{\partial M} = 0$, $(\phi - \psi)|_{\partial M} = 0$, and $\partial_n^k (A - B)|_{\partial M} = 0$ for $k \leq 1$. By taking derivatives of (4.4), similar arguments yield $\partial_n^k (\phi - \psi)|_{\partial M} = 0$.

Now, we want to make a local gauge transformation of A in U so that the new connection A' satisfies $d^* A' = 0$. The key ingredient in this step is the Cauchy–Kovalevskaya theorem; this is one place where the real-analyticity hypothesis plays a crucial role. To this end, we recall the following well-known result; see [8, Theorem 5.4], for example.

Lemma 4.2. *Let $\mathfrak{g}$ be a matrix Lie algebra. Then for $S : \mathbb{R}^n \to \mathfrak{g}$, we have*

$$d(e^S) = e^S \Big(\frac{1 - e^{-\operatorname{ad} S}}{\operatorname{ad} S} \Big)(dS) \tag{4.6}$$

where $\operatorname{ad} S \in \operatorname{End} \mathfrak{g}$ *is the endomorphism* $X \mapsto [S, X]$.

For $S \in \mathfrak{u}(N)$, let us denote the endomorphism in brackets acting on dS in (4.6) by $\Theta(S)$. Note that $\Theta(S)$ is defined by a power series in ad S and is therefore real-analytic in S. Moreover, note that $\Theta(0) = \mathrm{id}$, and therefore $\Theta(S)$ is invertible for small values of S.

Now, let us consider the following Cauchy problem for $S\colon U \to \mathfrak{u}(N)$,

$$\begin{cases} d^*(e^{-S}Ae^S + e^{-S}d(e^S)) = 0, \\ S|_{U\cap\partial M} = 0, \\ \partial_n S|_{U\cap\partial M} = 0. \end{cases} \tag{4.7}$$

Using Lemma 4.2, equation (4.7) can be written in the form

$$\begin{cases} -g^{ij}\nabla_i\nabla_j S = \Theta(S)^{-1}F(x, S, dS; A) \\ S|_{U\cap\partial M} = 0, \\ \partial_n S|_{U\cap\partial M} = 0, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} F(x, S, dS; A) = g^{ij}e^{-S}\Theta(-S)(\partial_i S)A_j e^S + g^{ij}e^{-S}(\nabla_i A_j)e^S \\ + g^{ij}e^{-S}A_j e^S\Theta(S)(\partial_i S) + g^{ij}(D\Theta)(S)(\partial_i S, \partial_j S). \end{aligned} \tag{4.9}$$

Note that since $S \mapsto \Theta(S)$ is real-analytic near $S = 0$, as are the connection A and metric g, the function $F(x, S, dS; A)$ is real-analytic in a neighbourhood of $(x, 0, 0; A)$ for $x \in U \cap \partial M$. Therefore, since $\Theta(S)^{-1}$ is well defined and real analytic in a neighbourhood of $S = 0$, the Cauchy–Kovalevskaya theorem (see [18, Section 16.4] for example) yields, after possibly shrinking U, the existence of a real-analytic solution $S\colon U \to \mathfrak{u}(N)$ to (4.8).

Therefore, e^S yields a local gauge transformation over U. Applying this gauge transformation to A, we get a new connection $A' := e^{-S}Ae^S + e^{-S}d(e^S)$ over U, which satisfies $d^*A' = 0$. Moreover, since the Yang–Mills–Dirac equations (4.4)–(4.5) are gauge-invariant, the pair (A', ϕ'), where $\phi' := e^{-S}\phi$, continues to satisfy the Yang–Mills–Dirac system. In particular, the pair (A', ϕ') satisfies the following nonlinear elliptic system over U,

$$\begin{cases} \not{D}_{A'}^2\phi' = m^2\phi', \\ d_{A'}^*F_{A'} = J(\phi'), \\ dd^*A' = 0. \end{cases} \tag{4.10}$$

In the same manner, we may conclude, after possibly shrinking U again, that there exists an analytic function $T\colon U \to \mathfrak{u}(N)$ such that the connection B' defined by

$B' := e^{-T} B e^{T} + e^{-T} d(e^{T})$, and the spinor $\psi' := e^{-T}\psi$, also satisfy the nonlinear elliptic system (4.10) over U.

We now want to show that S and T have the same Taylor series at the boundary. To this end, note that since $\Theta(0) = \mathrm{id}$, equation (4.9) yields $\Theta^{-1}(0)F(x,0,0;A) = d^*A(x)$ for any $x \in \partial M$. Furthermore, since $S|_{\partial M} = T|_{\partial M} = 0$ and $\partial_n S|_{\partial M} = \partial_n T|_{\partial M} = 0$, it follows that $dS|_{\partial M} = dT|_{\partial M} = 0$. These observations, along with (4.8) give us

$$\begin{aligned}\partial_n^2(S-T)|_{\partial M} = (\Delta S - \Delta T)|_{\partial M} &= (F(\cdot,0,0;A) - F(\cdot,0,0;B))|_{\partial M}\\ &= (d^*A - d^*B)|_{\partial M} = 0,\end{aligned}$$

the last equality holding since A and B have the same Taylor series at the boundary. By taking derivatives of (4.8) and using $\partial_n^k(A-B)|_{\partial M} = 0$ for all k, we get $\partial_n^k(T-S) = 0$ for all k. Therefore, it follows that $\partial_n^k(A'-B')|_{\partial M} = 0$ and $\partial_n^k(\phi'-\psi')|_{\partial M} = 0$ for all $k \geq 0$. Since (A',ϕ') and (B',ψ') both satisfy (4.10) in U, and have equal Taylor series at the boundary, it follows from the unique continuation principle for elliptic systems with scalar principal part (see [9, Theorem 3.5.2] for example) that $(A',\phi') = (B',\psi')$ in U. In particular, this shows that (A,ϕ) is gauge equivalent to (B,ψ) over U, by a gauge transformation that is equal to the identity on ∂M.

We have so far shown that the original pairs (A,ϕ) and (B,ψ) are locally gauge equivalent near the boundary ∂M by a real-analytic gauge transformation. We now want to extend this to the interior of M. For this final step, we use the real-analyticity of A and B in the interior as follows.

First, we have the following lemma from [4], which is stated there for the case $\mathcal{L} = d_A^* d_A$ on some Hermitian vector bundle E. The proof easily extends to the case $\mathcal{L} = \not{D}_A^2 - m^2$ presently under consideration, provided of course that $m^2 \notin \mathrm{Spec}\, \not{D}_A^2$.

Lemma 4.3. *Let $\beta \subseteq M^{\mathrm{int}}$ be an embedding of $[0,1]$ into M. Then there exists smooth sections $\varphi_1, \dots, \varphi_\ell$, harmonic with respect to $\mathcal{L}$, and having* $\mathrm{supp}\,(\varphi_i|_{\partial M}) \subseteq \Gamma$ *for some given non-empty open subset $\Gamma \subseteq \partial M$, that form a frame for $E \otimes S$ at every point along β.*

Remark 4.4. The proof involves solving the Dirichlet problem for $\mathcal{L}$, and then extending the resulting sections to global ones. See [4, Lemma 6.1] for details.

Armed with Lemma 4.3, we fix a point $x \in M$, and let β be a path from x to a point y near ∂M such that A and B are locally gauge equivalent near y. We consider a tubular neighbourhood W of β, over which $S \otimes E$ is trivial. Since A is real-analytic, the harmonic sections φ_i given by Lemma 4.3 are real-analytic. Therefore, they provide a real-analytic trivialization of $S \otimes E$ over U.

Now, let ψ_i be the solution to

$$\begin{cases} \not{D}_B^2 \psi_i = m^2 \psi_i, \\ \psi_i|_{\partial M} = \varphi_i|_{\partial M}. \end{cases}$$

Since B is real-analytic, so are the ψ_i. Let us construct the endomorphism

$$H_0 = \sum_j \psi_j \otimes \widehat{\varphi}_j \tag{4.11}$$

where $\hat{\varphi}_j$ indicates the dual frame of $(S \otimes E)^*$ defined by

$$\widehat{\varphi}_j(\varphi_k) = \delta_{jk}.$$

Then H_0 is real-analytic in W. Moreover, near y, we know that there exists a real-analytic gauge transformation G taking A to B, satisfying $G|_{\partial M} = \mathrm{id}$. Let $G_0 := \mathrm{id} \otimes G$. Then we have that $G_0^{-1}\varphi_i = \psi_i$ near ∂M. Indeed, both $G_0^{-1}\varphi_i$ and ψ_i satisfy the same elliptic equation,

$$\not{D}_B^2(G_0^{-1}\varphi_i) = G_0^{-1} \not{D}_A^2 \varphi_i = m^2 G_0^{-1}\varphi_i$$

and boundary condition $G_0^{-1}\varphi_i|_{\partial M} = \psi_i|_{\partial M}$. Also, by equality of the Dirichlet-to-Neumann maps,

$$\nabla_n^B(G_0^{-1}\varphi_i)|_{\partial M} = G_0^{-1}\nabla_n^A \varphi_i|_{\partial M} = \nabla_n^B \psi_i|_{\partial M}.$$

So, again, by unique continuation, we have that $G_0^{-1}\varphi_i = \psi_i$ near ∂M. This is precisely the property of H_0 defined in (4.11). Therefore, $H_0 = G_0$ near ∂M. In particular, taking partial traces, we find

$$G = \frac{1}{\operatorname{rank} S} \operatorname{Tr}_S H_0. \tag{4.12}$$

Since G is unitary near ∂M, it follows from real-analyticity that the right-hand of (4.12) is unitary in W. Denoting this right-hand side by H, it follows that H is a unitary automorphism of E over W. Moreover, since $B = H^{-1}AH + H^{-1}dH$ near ∂M, it follows again from real-analyticity that this holds over W. This proves that A and B are gauge equivalent about any point in M, and are thus locally gauge equivalent. ∎

The last part of the proof of Theorem 4.1 above constructs a local real-analytic gauge transformation between A and B along a curve. By modifying the argument given there, we can easily adapt the method of [4, Section 6] to show that A and B in fact have equal holonomies along embedded loops at a point on the boundary, from which it follows that they are globally gauge equivalent in the smooth category. We therefore have the following result.

Corollary 4.5. *Let A and B be as above. Then A and B are globally gauge equivalent via a smooth gauge transformation.*

Proof. First suppose that $\dim M > 2$. Let $p \in \partial M$, and let U be a neighbourhood of p intersecting ∂M over which A and B are gauge equivalent by a real-analytic gauge transformation G satisfying $G|_{\partial M} = \mathrm{id}$. Let $\gamma\colon [0,1] \to M$ be a smooth, embedded loop based at p. We can approximate γ in the C^1 norm by curves of the form $\beta := \beta_1 \cdot \beta_2$, where β_1 is a curve from p to some point $q \in U$ lying entirely in U, β_2 is a curve from q to p, and $\cdot$ denotes concatenation. We want to show that A and B have the same holonomy along β. Let W be a tubular neighbourhood of β_2. By constructing H_0 as in (4.11), we obtain a real-analytic endomorphism on $S \otimes E$ over W. The argument above shows that (4.12) holds in $U \cap W$. Therefore, gluing G and the right-hand side of (4.12) yields a well defined real-analytic unitary automorphism K of $\operatorname{End} E$ over $U \cup W$ satisfying $K|_{\partial M} = \mathrm{id}$. We therefore have that K is a gauge equivalence between A and B in $U \cup W$ satisfying $K(p) = \mathrm{id}$, from which we see that A and B have the same holonomy along β. It follows therefore that there exists a smooth global gauge transformation between A and B [14, Theorem 4.4].

For $\dim M = 2$, we note that γ may have self-intersections. To deal with this case, we first quote the following easy lemma [4, Lemma 6.3].

Lemma 4.6. *Let Σ be a smooth compact surface with boundary, and E a vector bundle over Σ. Let A and B be two connections on E, and denote their corresponding parallel transport maps by $\mathcal{P}^A$ and $\mathcal{P}^B$. If $\mathcal{P}^A = \mathcal{P}^B$ for all embedded simple closed curves at a point $p \in \Sigma$, then A and B have the same holonomy at p.*

Since the above argument for $\dim M > 2$ holds in the case where $\dim M = 2$ and γ has no self-intersections, the result follows. ∎

For abelian connections, the global gauge transformation is real-analytic.

Corollary 4.7. *Let A and B be as above. If $\operatorname{rank} E = 1$, then A and B are globally gauge equivalent via a real-analytic gauge transformation G satisfying $G|_{\partial M} = 1$.*

Proof. In the case $N = 1$, global gauge transformations are elements of $C^\infty(M, S^1)$. The same arguments use in preceding proof show that (A, ϕ) is locally gauge equivalent to (B, ψ) everywhere in M. That is, about every point in M there exists a locally defined smooth S^1-valued function satisfying $f\phi = \psi$. Since ϕ and ψ are solutions to elliptic equations, the set on which they do not vanish is dense. It thus follows that one can patch these local S^1-valued functions to obtain a global well defined S^1-valued function, which is by definition a gauge transformation from A to B, and equal to 1 on ∂M. ∎

Remark 4.8. The ease with which we obtain the preceding corollary can be attributed to the fact that Theorem 1.4 yields not only a gauge transformation between the connections, but also between the spinors, on which the gauge action takes a particularly simple form when $N = 1$.

Remark 4.9. In the proof of [4, Theorem 1.3], a unitary connection on a Hermitian vector bundle is recovered up to gauge from the Dirichlet-to-Neumann map of its connection Laplacian in the smooth category, by showing that the Dirichlet-to-Neumann map determines the holonomy along any loop γ starting and ending at a point on the boundary. This is done by solving the Dirichlet-problem to get a smooth frame along the curve γ, and then constructing a local gauge transformation over γ out of this frame. While applying this method to the Dirichlet-to-Neumann map of the Dirac Laplacian seems to produce an candidate endomorphism of $S \otimes E$ over γ, in the smooth case, this local gauge transformation is not a priori of the form $\mathrm{id}_S \otimes G$, as desired to conclude that A and B are locally gauge equivalent along γ. Nevertheless, the methods used in [4] provide very useful tools for studying inverse problems of connections on vector bundles, and so it is reasonable to suspect that they could also be used to improve upon Theorem 1.4.

Here, as well as in the proof of Theorem 1.4, we have seen how the introduction of a spinor field coupled to a unitary connection affects the recovery of the connection up to gauge from the Dirichlet-to-Neumann map of its twisted Dirac Laplacian. This motivates the following question: in what ways does the introduction of a spin structure affect the recovery of other geometric structures from boundary data associated to a Dirac Laplacian? For instance, regarding the Calderón problem for the metric, it would be worthwhile to study if the spin structure can be exploited to facilitate the recovery of the metric up to isometry, even in dimension 2, from the Dirichlet-to-Neumann map of the Dirac Laplacian.

Acknowledgements. The author thanks Professor Niky Kamran for his helpful comments, as well as the anonymous reviewer for their detailed report and suggestions.

Funding. This work was partially supported by the Natural Sciences and Engineering Research Council of Canada via a Canada Graduate Scholarship for Doctoral studies.

References

[1] D. Bleecker, *Gauge theory and variational principles*. Global Analysis Pure and Applied: Series A 1, Addison-Wesley, Reading, MA, 1981 Zbl 0481.58002 MR 643361

[2] B. Booß-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators*. Math. Theor. Appl., Birkhäuser , Boston, MA, 1993 Zbl 0797.58004 MR 1233386

[3] M. Cekić, Calderón problem for connections. *Comm. Partial Differential Equations* **42** (2017), no. 11, 1781–1836 Zbl 1382.35333 MR 3764927

[4] M. Cekić, Calderón problem for Yang-Mills connections. *J. Spectr. Theory* **10** (2020), no. 2, 463–513 Zbl 1445.35326 MR 4107522

[5] T. Daudé, N. Kamran, and F. Nicoleau, A survey of non-uniqueness results for the anisotropic Calderón problem with disjoint data. In *Nonlinear analysis in geometry and applied mathematics*. Part 2, pp. 77–101, Harv. Univ. Cent. Math. Sci. Appl. Ser. Math. 2, International Press, Somerville, MA, 2018 Zbl 1405.35123 MR 3823882

[6] M. Fischmann, On conformal powers of the Dirac operator on spin manifolds. *Arch. Math. (Brno)* **50** (2014), no. 4, 237–254 Zbl 1349.53016 MR 3291852

[7] R. Gabdurakhmanov and G. Kokarev, On Calderon's problem for the connection Laplacian. 2021, arXiv:2112.13419

[8] B. C. Hall, *Lie groups, Lie algebras, and representations*. Grad. Texts in Math. 222, Springer, New York, 2003 Zbl 1026.22001 MR 1997306

[9] V. Isakov, *Inverse problems for partial differential equations*. Third edn., Applied Mathematical Sciences 127, Springer, Cham, 2017 Zbl 1366.65087 MR 3616276

[10] Y. Kurylev and M. Lassas, Inverse problems and index formulae for Dirac operators. *Adv. Math.* **221** (2009), no. 1, 170–216 Zbl 1163.35041 MR 2509324

[11] Y. Kurylev, L. Oksanen, and G. P. Paternain, Inverse problems for the connection Laplacian. *J. Differential Geom.* **110** (2018), no. 3, 457–494 Zbl 1415.53024 MR 3880231

[12] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*. Princeton Math. Series 38, Princeton University Press, Princeton, NJ, 1989 Zbl 0688.57001 MR 1031992

[13] J. M. Lee and G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements. *Comm. Pure Appl. Math.* **42** (1989), no. 8, 1097–1112 Zbl 0702.35036 MR 1029119

[14] J. Lewandowski, Group of loops, holonomy maps, path bundle and path connection. *Classical Quantum Gravity* **10** (1993), no. 5, 879–904

[15] G. Rudolph and M. Schmidt, *Differential geometry and mathematical physics*. Part II. Fibre bundles, topology and gauge fields. Theoretical and Mathematical Physics, Springer, Dordrecht, 2017 Zbl 1364.53001 MR 3616138

[16] M. Salo and L. Tzou, Carleman estimates and inverse problems for Dirac operators. *Math. Ann.* **344** (2009), no. 1, 161–184 Zbl 1169.35063 MR 2481057

[17] M. Salo and L. Tzou, Inverse problems with partial data for a Dirac system: a Carleman estimate approach. *Adv. Math.* **225** (2010), no. 1, 487–513 Zbl 1197.35329 MR 2669360

[18] M. E. Taylor, *Partial differential equations*. III. Nonlinear equations. Second edn., Appl. Math. Sci. 117, Springer, New York, 2011 Zbl 1206.35004 MR 2744149

[19] F. Trèves, *Introduction to pseudodifferential and Fourier integral operators*. Vol. 1. Pseudodifferential operators. University Series in Mathematics, Plenum Press, New York and London, 1980 Zbl 0453.47027 MR 597144

[20] G. Uhlmann, Electrical impedance tomography and Calderón's problem. *Inverse Problems* **25** (2009), no. 12, article no. 123011 Zbl 1181.35339 MR 3460047

Received 13 April 2022; revised 1 November 2023.

Carlos Valero
Department of Mathematics and Statistics, McGill University, 845 Sherbrooke St W, Montréal, H3A 0G4, Canada; charles.valero@mail.mcgill.ca