

# Quantitative stochastic homogenization of nonlinearly elastic, random laminates

Stefan Neukamm, Mathias Schäffner, and Mario Varga

**Abstract.** In this paper we study quantitative stochastic homogenization of a nonlinearly elastic composite material with a laminate microstructure. We prove that for deformations close to the set of rotations, the homogenized stored energy function  $W_{\text{hom}}$  is  $C^3$  and that  $W_{\text{hom}}$ , the stress tensor  $DW_{\text{hom}}$ , and the tangent-moduli  $D^2W_{\text{hom}}$  can be represented with the help of stochastic correctors. Furthermore, we study the error of an approximation of these quantities via representative volume elements. More precisely, we consider periodic *representative volume elements (RVEs)* obtained by periodizing the distribution of the random material. For materials with a fast decay of correlations on scales larger than a unit scale, we establish error estimates on the random and systematic error of the RVE with optimal scaling in the size of the RVE and with a multiplicative random constant that has exponential moments.

## Contents

1. Introduction	1
2. Setting and main results	8
3. Deterministic intermediate results	15
4. Proofs of qualitative results: Theorem 2.4 and Corollary 2.6	28
5. Stochastic intermediate results	35
6. Proofs of quantitative results: Theorem 2.12 and Corollary 2.13	45
A. Linear corrector equation	56
References	57

## 1. Introduction

A standard model for a nonlinearly elastic, composite material is given by the nonconvex energy functional  $\mathcal{E}_\varepsilon(u) := \int_O W(\frac{x}{\varepsilon}, \nabla u) dx$ , where  $O \subseteq \mathbb{R}^d$  denotes the reference domain occupied by the elastic body,  $u: O \rightarrow \mathbb{R}^d$  its deformation, and  $W: \mathbb{R}^d \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R} \cup \{+\infty\}$  the stored energy function, which encodes the mechanical properties of the material. We are interested in the macroscopic behavior of composites that oscillate on a

---

*Mathematics Subject Classification 2020:* 74Q05 (primary); 74B20, 60H15 (secondary).

*Keywords:* nonlinear elasticity, stochastic homogenization, representative volume element method.

microscopic scale  $0 < \varepsilon \ll 1$ . Therefore, we consider the homogenization limit  $\varepsilon \rightarrow 0$  of  $\mathcal{E}_\varepsilon$  in the framework of  $\Gamma$ -convergence. The first results in this direction are due to Braides [10] and Müller [35]: for periodic composites (that is,  $x \mapsto W(x, F)$  is periodic for all  $F$ ) and assuming additional growth conditions for  $W$  (in particular, standard  $p$ -growth with  $1 < p < \infty$ ), they show that  $\mathcal{E}_\varepsilon$   $\Gamma$ -converges to an energy functional of the form  $u \mapsto \int_O W_{\text{hom}}(\nabla u) dx$  with a homogenized stored energy function given by the *multi-cell formula*

$$\begin{aligned} W_{\text{hom}}(F) &:= \lim_{L \rightarrow \infty} W_{\text{hom},L}(F), \\ W_{\text{hom},L}(F) &:= \inf_{\varphi \in W_{\text{per}}^{1,p}(\square_L; \mathbb{R}^d)} \int_{\square_L} W(x, F + \nabla \varphi) dx, \end{aligned} \tag{1.1}$$

where  $\square_L := [-\frac{L}{2}, \frac{L}{2}]^d$  and  $W_{\text{per}}^{1,p}(\square_L; \mathbb{R}^d)$  denotes the space of  $L$ -periodic Sobolev functions. A similar homogenization result and, in particular, formula (1.1) are also valid in the random case, that is, when  $x \mapsto W(x, F)$  is a stationary and ergodic random field; see [15, 16, 34].

The multi-cell formula for the homogenized stored energy function  $W_{\text{hom}}$  is the main subject of this paper. Its definition invokes a nonconvex minimization problem and an asymptotic limit  $L \rightarrow \infty$ , and, therefore, a priori not much is known about its analytic properties. The energy function  $W_{\text{hom}}$  describes the effective mechanical behavior of the composite. It is especially important to understand the *first Piola–Kirchhoff stress tensor* and the *tangent modulus*, i.e., the Jacobian  $DW_{\text{hom}}(F)$  and the Hessian  $D^2W_{\text{hom}}(F)$ . Therefore, it is natural and relevant to investigate the following questions:

(Q1) Is  $W_{\text{hom}}$  twice continuously differentiable?

(Q2) How can  $W_{\text{hom}}(F)$ ,  $DW_{\text{hom}}(F)$ , and  $D^2W_{\text{hom}}(F)$  be evaluated or approximated?

In this paper we study these questions in the regime of small but finite strains for random *laminates* composed of frame-indifferent materials with a stress-free, nondegenerate reference state (see Definition 2.1 below). Before we summarize our results we emphasize that the restriction to laminates (i.e.,  $W(x, F) = W(x_d, F)$ ) is a critical assumption for our method to work in the case of random materials and cannot be relaxed; see Remark 2.5 below. In a nutshell, the main results of this paper can be summarized as follows:

- (*Qualitative results and answer to Q1*) We show  $W_{\text{hom}} \in C^2(U)$  for an open neighborhood  $U$  of  $\text{SO}(d)$ ; see Theorem 2.4 (iii). This result is based on a representation of  $W_{\text{hom}}(F)$  for  $F \in U$  with the help of a sublinear corrector  $\varphi_F$  that solves the nonlinear corrector equation  $-\text{div } DW(x, F + \nabla \varphi_F) = 0$  in  $\mathbb{R}^d$ ; see Theorem 2.4 (ii). Since  $W$  is nonconvex, this corrector representation is highly nontrivial and extends recent results for periodic composites by the first two authors [39, 40].
- (*Quantitative results and answer to Q2*) We prove optimal error estimates for a periodic representative volume element approximation of  $W_{\text{hom}}(F)$ ,  $DW_{\text{hom}}(F)$ , and  $D^2W_{\text{hom}}(F)$  for  $F$  close to  $\text{SO}(d)$ ; see Theorem 2.12 and the discussion at the end of this introduction.

Before we present our results in detail, we discuss previous, related works.

**Convex case.** We first note that a complete understanding of (Q1) is currently only available for convex integrands with quadratic growth. Indeed, as shown by Müller [35], if  $W(x, F)$  is periodic in  $x$  and convex in  $F$ , the multi-cell formula reduces to a *one-cell formula* that can be represented with the help of a *corrector*, i.e.,  $W_{\text{hom}}(F) = W_{\text{hom},1}(F) = \int_{\square} W(y, F + \nabla \varphi_F) dy$ , where the corrector  $\varphi_F$  is a minimizer of the minimization problem in the definition of  $W_{\text{hom},1}(F)$ . Furthermore, in [24, Theorem 5.4] it is shown that, if  $W(x, F)$  is additionally  $C^2$  in  $F$  and satisfies a quadratic growth condition in  $F$ , then  $W_{\text{hom}}$  is  $C^2$ . The argument exploits the *corrector representation* of  $W_{\text{hom}}$  and only uses basic energy estimates for the corrector. It extends verbatim to the random, convex case.

To reach regularity beyond  $C^2$  is a nontrivial task and requires improved regularity of the corrector. For convex potentials with quadratic growth, the corrector  $\varphi_F$  can be characterized by a monotone, uniformly elliptic system with quadratic growth and thus, a stronger regularity theory is available; we refer to [39] for  $C^3$ -regularity of the homogenized potential in the periodic, vector-valued case, and [3, 12, 21] for related results for random, monotone equations.

**Nonconvex case.** The results mentioned so far do not extend to the case of nonlinear elasticity, since there the stored energy function is necessarily nonconvex. In fact, Müller [35, Theorem 4.3] provides a counterexample in the form of a laminate material that features a buckling instability under compressive loading; in particular, one has  $W_{\text{hom}}(F) < W_{\text{hom},L}(F)$  for some  $F \notin \text{SO}(d)$  and all  $L \in \mathbb{N}$ . This shows that we cannot even expect a one-cell formula or a corrector representation to hold for general deformations  $F$ . However, better behavior can be expected for deformations close to  $\text{SO}(d)$  for materials with a nondegenerate, stress-free reference state at the identity. Indeed, in [36] the first author and Müller show for periodic composites that  $W_{\text{hom}}$  admits a quadratic expansion at identity:

$$W_{\text{hom}}(\text{Id} + G) = Q_{\text{hom}}(G) + o(|G|^2), \quad (1.2)$$

where we have  $Q_{\text{hom}}(G) := \inf_{\varphi \in H_{\text{per}}^1(\square; \mathbb{R}^d)} \int_{\square} Q(x, G + \nabla \varphi) dx$ , and  $Q(x, G) := \frac{1}{2} D^2 W(x, \text{Id})[G, G]$  denotes the quadratic term in the expansion of  $W$  at identity. The argument uses soft properties of the corrector  $\varphi_F$  and appeals to the geometric rigidity estimate of [22]. An analogous statement holds in the random case; see [25]. Identity (1.2) says that the tangent modulus of  $W_{\text{hom}}$  at identity is given by the homogenization of the tangent modulus of  $W$ .

**Differentiability for small, but finite strains.** To establish an expansion similar to (1.2) for  $F \notin \text{SO}(d)$  is nontrivial – and, in fact, not always possible; cf. Müller’s counterexample mentioned above. Therefore, it is natural to focus on the regime of small strains, i.e., when  $F$  is close to  $\text{SO}(d)$ . The first differentiability result for  $W_{\text{hom}}$  away from  $\text{SO}(d)$  has recently been obtained by the first and second authors in [39, 40] for a periodic composite with a “regular” microstructure, e.g., a matrix material with smooth, possibly touching inclusions. It is shown that if  $W(x, \cdot)$  is  $C^2$  in a neighborhood of  $\text{SO}(d)$ , then

the homogenized stored energy function  $W_{\text{hom}}$  is  $C^2$  in a neighborhood of  $\text{SO}(d)$ , say in  $U_\rho := \{F \in \mathbb{R}^{d \times d} : \text{dist}(F, \text{SO}(d)) < \rho\}$  for some  $\rho > 0$ . The argument critically relies on the observation that for  $F \in U_\rho$  the multi-cell formula simplifies to a single-cell formula that features a representation with the help of a periodic corrector. More precisely, in [39, 40] we prove that for any  $F \in U_\rho$  there exists a periodic corrector  $\varphi_F \in H_{\text{per}}^1(\square; \mathbb{R}^d)$  such that

$$W_{\text{hom}}(F) = W_{\text{hom},1}(F) = \int_{\square} W(x, F + \nabla \varphi_F(x)) dx. \quad (1.3)$$

Moreover, we obtain a similar representation for the derivatives of  $W_{\text{hom}}$ ; in particular, we show that the tangent modulus admits the representation

$$D^2 W_{\text{hom}}(F) H \cdot G = \int_{\square} D^2 W(x, F + \nabla \varphi_F)(H + \nabla \varphi_{F,H}) \cdot G dx$$

for the periodic, linearized corrector  $\varphi_{F,H} \in H_{\text{per}}^1(\square; \mathbb{R}^d)$ . These corrector representations are the starting point for a quantitative analysis of  $W_{\text{hom}}$  and minimizers of  $\mathcal{E}_\varepsilon$ ; in particular, in [39, 40] we prove error estimates for the nonlinear two-scale expansion and we establish Lipschitz estimates for minimizers of  $\mathcal{E}_\varepsilon$  that are uniform in  $\varepsilon > 0$ . The general strategy of proof for (1.3) relies on a reduction to the convex case based on two major ingredients:

- (*A matching convex lower bound*) Inspired by [14] we construct in [39, Corollary 2.3] a strongly convex integrand  $V$  with quadratic growth such that

$$\begin{aligned} W(x, F) + \mu \det F &\geq V(x, F) \quad \text{for all } F \in \mathbb{R}^{d \times d}, \\ W(x, F) + \mu \det F &= V(x, F) \quad \text{for all } F \in U_\rho, \end{aligned} \quad (1.4)$$

where  $\mu > 0$  and  $\rho > 0$  only depend on  $W$ . With the help of  $V$ , convex homogenization theory and the fact that  $F \mapsto \det F$  is a null-Lagrangian, we obtain the lower bound  $W_{\text{hom}}(F) + \mu \det F \geq V_{\text{hom}}(F) = \int_{\square} V(x, F + \nabla \varphi_F)$  with a convex corrector  $\varphi_F$  given as the unique (up to a constant) sublinear solution to the monotone corrector problem,

$$-\nabla \cdot DV(x, F + \nabla \varphi_F) = 0 \quad \text{in } \mathbb{R}^d. \quad (1.5)$$

- (*Global Lipschitz regularity*) By the regularity theory for uniformly elliptic, monotone systems with piecewise constant, periodic coefficients, we establish the following global Lipschitz estimate for the convex corrector:

$$\|\text{dist}(F + \nabla \varphi_F, \text{SO}(d))\|_{L^\infty(\square)} \leq C \text{dist}(F, \text{SO}(d)),$$

where for  $d > 2$  we require the right-hand side to satisfy an additional smallness condition; see [40, Corollary 1]. In view of the matching property of  $V$  (cf. (1.4)), this allows us to deduce the sought-for corrector representation (1.3).

**Random composites and laminates.** The construction of the matching convex lower bound  $V$  verbatim extends to the random case; see Lemma 3.2 below. Nevertheless, the analysis in [39, 40] is restricted to periodic composites, since periodicity is critically used to obtain the required global Lipschitz estimate for the corrector: indeed, in [40] we obtain the Lipschitz estimate by combining a small-scale Lipschitz estimate (cf. [40, Theorem 4]) of the form  $\|\nabla\varphi_F\|_{L^\infty(\square_{1/2})} \leq C\|\nabla\varphi_F\|_{L^2(\square)}$  with the energy estimate  $\|\nabla\varphi_F\|_{L^2(\square)} \leq C \operatorname{dist}(F, \operatorname{SO}(d))$  for sublinear solutions to (1.5). While the latter is standard in the periodic case, in the random setting such an estimate generally only holds in the weaker form of a *large-scale*  $L^2$ -estimate: for all  $L$  larger than a random minimal radius we have  $L^{-\frac{d}{2}}\|\nabla\varphi_F\|_{L^2(\square_L)} \leq C \operatorname{dist}(F, \operatorname{SO}(d))$ ; we refer to [21] where such an estimate has been established for monotone systems. Thus, in the random setting we may only expect the bound

$$\|\operatorname{dist}(F + \nabla\varphi_F, \operatorname{SO}(d))\|_{L^\infty(B_1(x))} \leq \mathcal{C}(x) \operatorname{dist}(F, \operatorname{SO}(d)),$$

where  $\mathcal{C}(x)$  is a stationary random field with stretched exponential moments. We note that this Lipschitz estimate is not global, which prevents us from following the strategy of [40].

In contrast, for random laminates, as considered in this paper, the situation is better, since then the corrector problems for  $\varphi_F$  and the linearized corrector  $\varphi_{F,G}$  simplify to ordinary differential equations; see Theorem 2.4 (ii)(b). This allows us to retrieve global, deterministic Lipschitz estimates by appealing to ODE arguments; see Lemma 3.8 below.

**Quantitative periodic RVEs.** We finally comment on our quantitative analysis of the *representative volume element* (RVE) approximation for  $W_{\text{hom}}$ ,  $DW_{\text{hom}}$ , and  $D^2W_{\text{hom}}$ . As we explain in detail in the next section, in our paper we consider parametrized models to describe random laminates. More precisely, we denote by  $(\Omega, \mathbb{P})$  a probability space of parameter fields  $\omega: \mathbb{R} \rightarrow \mathbb{R}$ , where  $\mathbb{P}$  is stationary and ergodic with respect to the shifts  $\omega \mapsto \omega(\cdot + z)$ ,  $z \in \mathbb{R}$ . We then consider stored energy functions of the form  $(x, F) \mapsto W(\omega(x_d), F)$ , where  $x_d$  denotes the  $d$ th coordinate of  $x \in \mathbb{R}^d$ ; see Assumption 2.2. Our qualitative Theorem 2.4 then yields the corrector representation formula

$$W_{\text{hom}}(F) = \mathbb{E} \left[ \int_{\square} W(\omega(x_d), F + \nabla\varphi_F) dx \right], \quad (1.6)$$

for  $F$  sufficiently close to  $\operatorname{SO}(d)$ . A direct evaluation of (1.6) is not possible in practice, since the expression invokes two “infinities”: the domain of the corrector equation (1.5) is unbounded and the probability space has infinite degrees of freedom. Hence, a suitable approximation is required. The RVE method is a well-established procedure for this purpose. In this method, the corrector problem is considered (for a finite number of samples) on a domain of a *finite* size  $L$ , together with suitable boundary conditions. It is an ongoing discussion in the computational mechanics community how to choose the size of the RVE and the appropriate boundary conditions; see, e.g., [32, 42] and the references therein. In particular, we note that [32] also provides a numerical study of the convergence rate for the RVE approximation of the tangent modulus of a nonlinear material.

The first analytic convergence results with optimal scaling in  $L$  for the RVE approximation have been obtained by Gloria and Otto in [29, 30] for discrete, linear elliptic equations. Periodic RVEs were first investigated in [26] in the discrete setting and recently in [21] in the case of monotone systems. In periodic RVEs (in the form of [21, 26]) the original probability measure  $\mathbb{P}$  is approximated by a stationary, probability measure  $\mathbb{P}_L$  that is supported on  $L$ -periodic parametrizations. The effect of this approximation is that for  $\mathbb{P}_L$ -almost every realization, the corrector equation simplifies to an equation on a *finite* domain (of size  $L$ ) with *periodic boundary conditions*. These equations are well posed pathwise and thus for  $\mathbb{P}_L$ -a.e. parameter field  $\omega_L \in \Omega$  the proxy

$$W_{\text{hom},L}(\omega_L, F) := \int_{\square_L} W(\omega_L(x_d), F + \nabla \varphi_F(\omega_L, \cdot)) dx$$

is well defined (provided  $F \in U_\rho$ ). As already observed in [29] the RVE error naturally decomposes as

$$W_{\text{hom},L}(F) - W_{\text{hom}}(F) = \underbrace{W_{\text{hom},L}(F) - \mathbb{E}_L[W_{\text{hom},L}(F)]}_{=:\text{err}_{\text{rand}}(L)} + \underbrace{\mathbb{E}_L[W_{\text{hom},L}(F) - W_{\text{hom}}(F)]}_{=:\text{err}_{\text{sys}}(L)},$$

into a random error (due to random fluctuations with respect to  $\mathbb{P}_L$ ) and a systematic error due to differences of  $\mathbb{P}_L$  and  $\mathbb{P}$ . As already observed in [26], in order to quantify the error, the assumption of ergodicity needs to be quantified. Moreover, both errors are sensitive to the decay of correlations of the random material; see also [28] for a detailed analysis on the impact of the decay rate of correlations and the scaling of the sublinear corrector. In the present paper we study the periodic-RVE approximation under the strongest possible assumption on the decay of correlations, namely, materials that feature a rapid decay of correlations on scales larger than 1. In the existing literature two different approaches exist to quantify ergodicity: either by means of mixing conditions as in [5, 7] or in the form of functional inequalities; see [18, 19, 28–30, 37]. In the present paper we follow the second approach and work with a spectral gap estimate. In essence, we make the following assumptions:

- (*Existence of an  $L$ -periodic approximation*) There exists a shift-invariant probability measure  $\mathbb{P}_L$  that is supported on  $L$ -periodic fields in  $\Omega$  and that recovers the distribution of  $\mathbb{P}$  in  $\square_{L/2}$ ; see Definition 2.8 for details.
- (*Fast decorrelations*)  $(\Omega, \mathbb{P})$  and  $(\Omega, \mathbb{P}_L)$  feature *spectral gap estimates* that encode fast decorrelations on scales larger than 1; see Definition 2.9.

If  $W(\omega, F)$  is sufficiently regular in  $\omega$  and  $F$ , and for deformations  $F$  that are sufficiently close to  $\text{SO}(d)$ , our main result then yields estimates on the error of the RVE approximation of  $W_{\text{hom}}(F)$ , the stress tensor  $DW_{\text{hom}}$ , and the tangent modulus  $D^2W_{\text{hom}}$ . In particular, for  $W_{\text{hom},L}$  we get

$$|\text{err}_{\text{rand}}| \leq \mathcal{C} \text{dist}^2(F, \text{SO}(d)) L^{-\frac{1}{2}}, \quad |\text{err}_{\text{sys}}(L)| \leq C \text{dist}^2(F, \text{SO}(d)) \frac{\ln L}{L},$$

where  $\mathcal{C}$  denotes a random constant with  $\mathbb{E}_L[\exp(\frac{1}{C}\mathcal{C})] \leq 2$ , and  $C$  denotes a deterministic constant; see Theorem 2.12. These estimates are essentially optimal with respect to scaling in  $L$  and with respect to the integrability of  $\mathcal{C}$ . Moreover, they are the first analytical estimates for RVEs for geometrically nonlinear composites. Due to the different scaling of the random and systematic error, our results allow for optimizing the ratio between the number of Monte Carlo iterations and the size of the RVE; see Corollary 2.13. Next, we discuss the optimality of the scaling of  $|\text{err}_{\text{rand}}|$  and  $|\text{err}_{\text{sys}}|$  in  $L$ . The scaling of the random error  $\text{err}_{\text{rand}}$  coincides with the central limit theorem scaling in dimension  $d = 1$  and is thus optimal. For the systematic error  $|\text{err}_{\text{sys}}(L)|$ , we recover the previous findings of [26] and [21] for linear and nonlinear elliptic equations restricted to the case of laminates. Finally, we mention the recent paper [13], where for a linear elliptic equation with Gaussian coefficients, the systematic error is precisely identified and in particular the  $\ln L$ -factor is removed (in dimensions  $d > 2$ ).

**Structure of the paper.** We present the assumptions and main results of this paper in detail in Section 2. In Section 3 we collect and prove the deterministic auxiliary results needed for the proofs of the main results. In Section 4 the proof of the main qualitative result is presented. In Section 5 we collect and prove the stochastic auxiliary estimates that are needed for the proof of the quantitative error estimates. Section 6 is dedicated to the proof of the main quantitative results.

**Notation.**

- We write  $f_A \cdot$  for the averaged integral  $\frac{1}{|A|} \int_A \cdot$ ; we frequently write  $(f)_A$  to denote  $f_A f$ .
- For  $L > 0$  and  $d \in \mathbb{N}$ , we use the notation  $\square_L := (0, L)^d$ .
- We identify one-dimensional functions  $\mathbb{R} \ni x \mapsto u(x) \in \mathbb{R}$  with their extension to  $\mathbb{R}^d$  that depends only on the  $d$ th coordinate direction, i.e.,  $\mathbb{R}^d \ni x \mapsto u(x_d) \in \mathbb{R}$ . We usually write  $x_d$  for a one-dimensional variable in  $\mathbb{R}$ . Derivatives with respect to  $x_d$  are denoted by  $\partial_{x_d}$ . In particular, for  $\varphi \in H_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^m)$  we write  $\nabla \varphi$  to denote  $\partial_{x_d} \varphi \otimes e_d$  and not  $\partial_{x_d} \varphi$ . The same applies to divergence; in particular,  $\text{div}$  always denotes the  $d$ -dimensional divergence operator.
- For a matrix-valued mapping  $v: A \rightarrow \mathbb{R}^{m \times d}$ , we denote by  $v_d: A \rightarrow \mathbb{R}^m$  its  $d$ th column, i.e.,  $v_d(x) = [v_{1d}(x), \dots, v_{md}(x)]$ .

- We write

$$W_{\text{per},0}^{1,p}(\square_L) := \{\varphi \in W_{\text{per}}^{1,p}(\square_L) : \int_{\square_L} \varphi \, dx = 0\},$$

$$H_{\text{per},0}^1(\square_L) := W_{\text{per},0}^{1,2}(\square_L).$$

- We set

$$W_{\text{uloc}}^{1,p}(\mathbb{R}^d) := \{u \in W_{\text{loc}}^{1,p}(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \|u\|_{W^{1,p}(B_1(x))} < \infty\},$$

$$H_{\text{uloc}}^1(\mathbb{R}^d) = W_{\text{uloc}}^{1,2}(\mathbb{R}^d).$$

- For  $\delta > 0$ , we use the notation  $U_\delta := \{F \in \mathbb{R}^{d \times d} : \text{dist}(F, \text{SO}(d)) < \delta\}$ .

## 2. Setting and main results

In this section we first introduce the standard setting for the description of stationary random media. Consequently, we present the main results about nonconvex integral functionals in the vicinity of rotations: in Theorem 2.4 we argue that the RVE approximations are well defined and the so-called stochastic one-cell formula holds. Theorem 2.12 describes the precision of the RVE method in terms of the size of the representative elements.

We consider the parameter space  $\Omega = \{\omega : \mathbb{R} \rightarrow \mathbb{R} \text{ measurable}\}$  equipped with a  $\sigma$ -algebra  $\mathcal{S}$  and a probability measure  $\mathbb{P}$ . The mathematical expectation is denoted by  $[\cdot] := \int_{\Omega} \cdot d\mathbb{P}(\omega)$ . A sample  $\omega \in \Omega$  describes the spatially varying (layered) material properties and for this reason we frequently refer to it as a *configuration*. We assume the following standard assumptions for the random configuration space  $(\Omega, \mathcal{S}, \mathbb{P})$ :

- (P1) (*Stationarity*) For any  $z \in \mathbb{R}$ , the vectors  $(\omega(x_1), \dots, \omega(x_n))$  and  $(\omega(x_1 + z), \dots, \omega(x_n + z))$  have the same joint distribution for arbitrary  $x_1, \dots, x_n \in \mathbb{R}$  and  $n \in \mathbb{N}$ .
- (P2) (*Ergodicity*) For any random variable  $\mathcal{F} \in L^1(\Omega)$  we have

$$\lim_{R \rightarrow \infty} \int_0^R \mathcal{F}(\omega(\cdot + x)) dx = [\mathcal{F}].$$

We state the assumption on the considered energy density  $W$  as follows. We first introduce a class of frame-indifferent stored energy functions that are minimized, nondegenerate, and smooth at the identity, and satisfy the growth condition stated below.

**Definition 2.1.** For  $\alpha > 0$ ,  $p > 1$ , and a modulus of continuity  $\rho$ , we denote by  $\mathcal{W}_{\alpha, \rho}^p$  the class of Borel functions  $W : \mathbb{R}^{d \times d} \rightarrow [0, +\infty]$  which satisfies the following properties:

- (W1)  $W$  satisfies  $p$ -growth from below, i.e.,

$$\alpha |F|^p - \frac{1}{\alpha} \leq W(F) \quad \text{for all } F \in \mathbb{R}^{d \times d};$$

- (W2)  $W$  is frame indifferent, i.e.,

$$W(RF) = W(F) \quad \text{for all } R \in \text{SO}(d), F \in \mathbb{R}^{d \times d};$$

- (W3)  $F = \text{Id}$  is a natural state and  $W$  is nondegenerate, i.e.,  $W(\text{Id}) = 0$  and

$$W(F) \geq \alpha \text{dist}^2(F, \text{SO}(d)) \quad \text{for all } F \in \mathbb{R}^{d \times d};$$

- (W4)  $W$  is  $C^3$  in a neighborhood of  $\text{SO}(d)$ :

$$\begin{aligned} \|W\|_{C^3(\bar{U}_\alpha)} &< \frac{1}{\alpha}, \\ \forall F, F' \in \bar{U}_\alpha: |D^3 W(F) - D^3 W(F')| &\leq \rho(|F - F'|). \end{aligned} \tag{2.1}$$

**Assumption 2.2.** Fix  $\alpha > 0$ ,  $p \geq d$ , and a modulus of continuity  $\rho$ . We suppose that  $W : \mathbb{R} \times \mathbb{R}^{d \times d} \rightarrow [0, +\infty]$  is a Borel function and  $W(\omega, \cdot) \in \mathcal{W}_{\alpha, \rho}^p$  for almost every  $\omega \in \mathbb{R}$ .



**Remark 2.3.** Note that we use the same constant  $\alpha$  in the growth condition (W1), the nondegeneracy condition (W3), and the regularity assumption (W4). The only reason for this is that we want to reduce the number of parameters invoked in the assumption for  $W$ . Let us anticipate that the region, in which the multi-cell formula reduces to a single-cell expression, will depend (in a quite implicit way) on the constants in (W1), (W3), and (W4). Hence, working with the single parameter  $\alpha$  simplifies the presentation. Note that we have  $W_\alpha^p \subset W_{\alpha'}^p$  for  $0 < \alpha' < \alpha$ .

We associate with  $W$  the following approximation for the homogenized stored energy function:

$$W_{\text{hom},L}: \Omega \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R},$$

$$W_{\text{hom},L}(\omega, F) = \inf_{\varphi \in W_{\text{per},0}^{1,p}(\square_L; \mathbb{R}^d)} \int_{\square_L} W(\omega(x_d), F + \nabla \varphi) dx. \quad (2.2)$$

We state the first main result of this paper as follows. Based on a merely qualitative ergodicity assumption combined with the laminated structure of the material, we show that in the vicinity of rotations correctors exist and the homogenized stored energy function is characterized by a stochastic one-cell formula. We also discuss regularity properties of the homogenized stored energy function.

**Theorem 2.4** (Corrector representation and regularity of  $W_{\text{hom}}$ ). *Let  $(\Omega, \mathcal{S}, \mathbb{P})$  satisfy (P1)–(P2) and let Assumption 2.2 hold. Then there exists  $\bar{\delta} = \bar{\delta}(\alpha, d, p) > 0$  and  $c = c(\alpha, d, p) \in [1, \infty)$  such that for all  $F \in U_{\bar{\delta}}$  the following statements hold true:*

- (i) (Homogenized energy density) *The following limit exists and it is deterministic:*

$$W_{\text{hom}}(F) := \lim_{L \rightarrow \infty} W_{\text{hom},L}(\omega, F) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

- (ii) (Corrector) *There exists a unique random field  $\varphi_F: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  with the following properties:*

- (a) *For  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  it holds that  $\varphi_F(\omega, \cdot) \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$ ,  $f_{\square} \varphi_F(\omega, \cdot) = 0$ , and*

$$-\text{div } DW(\omega(x_d), F + \nabla \varphi_F) = 0 \quad \text{in } \mathbb{R}^d, \quad (2.3)$$

*and the corrector  $\varphi_F$  is sublinear in the sense that*

$$\lim_{R \rightarrow \infty} \frac{1}{R} \|\varphi_F(\omega, \cdot)\|_{L^\infty(\square_R)} = 0. \quad (2.4)$$

- (b) *The corrector  $\varphi_F$  is one-dimensional in the sense of  $\varphi_F(\omega, x) = \varphi_F(\omega, x_d)$ .*  
 (c) *The gradient of the corrector  $\varphi_F$  is stationary and satisfies  $\mathbb{E}[\nabla \varphi] = 0$  and*

$$\|\nabla \varphi_F(\omega, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq c \text{dist}(F, \text{SO}(d)) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (2.5)$$

The homogenized energy density given in (i) satisfies the stochastic one-cell formula

$$W_{\text{hom}}(F) = \mathbb{E} \left[ \int_{\square} W(\omega(x_d), F + \nabla \varphi_F(\omega)) dx \right]. \quad (2.6)$$

(iii) (Regularity) The energy densities  $W_{\text{hom},L}(\omega, \cdot)$  and  $W_{\text{hom}}$  defined in (2.2) and in (i) are of class  $C^3(U_{\bar{\delta}})$  and it holds, for all  $G, H \in \mathbb{R}^{d \times d}$ , that

$$DW_{\text{hom}}(F) \cdot G = \mathbb{E} \left[ \int_{\square} DW(\omega(x_d), F + \nabla \varphi_F) \cdot G dx \right], \quad (2.7)$$

$$D^2 W_{\text{hom}}(F) H \cdot G = \mathbb{E} \left[ \int_{\square} D^2 W(\omega(x_d), F + \nabla \varphi_F)(H + \nabla \varphi_{F,H}) \cdot G dx \right], \quad (2.8)$$

where the linearized corrector  $\varphi_{F,H}: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is uniquely characterized by

$$\begin{aligned} & -\operatorname{div}(D^2 W(\omega(x_d), F + \nabla \varphi_F)(H + \nabla \varphi_{F,H})) \\ & = 0 \quad \text{in } \mathbb{R}^d, \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} & \int_{\square} \varphi_{F,H} dx = 0, \quad \nabla \varphi_{F,H} \text{ is stationary,} \\ & \mathbb{E}[\nabla \varphi_{F,H}] = 0, \quad \mathbb{E}[|\nabla \varphi_{F,H}|^2] < \infty. \end{aligned} \quad (2.10)$$

Moreover,  $\varphi_{F,H}$  is one-dimensional in the sense that  $\varphi_{F,H}(\omega, x) = \varphi_{F,H}(\omega, x_d)$  and, for all  $H \in \mathbb{R}^{d \times d}$ , it holds that  $\lim_{R \rightarrow \infty} \frac{1}{R} \|\varphi_{F,H}\|_{L^\infty(\square_R)} = 0$  and  $\|\nabla \varphi_{F,H}(\omega, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq c|H|$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ .

(iv) (Strong rank-one convexity)

$$D^2 W_{\text{hom}}(F)[a \otimes b, a \otimes b] \geq \frac{1}{c} |a \otimes b|^2 \quad \text{for all } a, b \in \mathbb{R}^d.$$

The proof is presented in Section 4.

For deterministic, periodic composites, the conclusion of Theorem 2.4 was proven in [39, 40] under various assumptions on the regularity of  $x \mapsto W(x, F)$ . However, Theorem 2.4 is the first corrector representation result for nonlinearly elastic materials with a random microstructure. Furthermore, the smallness assumption  $F \in U_{\bar{\delta}}$  in Theorem 2.4 cannot be relaxed, as can be seen by Müller's counterexample [35, Theorem 4.7] which features a periodic laminate material that undergoes buckling for sufficiently strong compressions.

**Remark 2.5** (The restriction to laminates in the random case). The restriction to laminates is a critical assumption for our method when applied to random materials: to our knowledge, the only current method for proving the existence of correctors for hyperelastic

composites (periodic and random) is the one introduced in [39] and described in the introduction. There, the corrector for the nonconvex problem is obtained by means of the corresponding convex lower bound (see Section 3.1 below). To relate the corrector for the convex problem to the corrector for the nonconvex problem, the key point is that correctors for the convex lower bound satisfy a deterministic bound on its gradient thanks to the laminate structure of the coefficients (see Lemma 3.6).

This is no longer true for general multidimensional random media: examples in [7], [27, Remark 5] show that for solutions of linear elliptic systems with random, rapidly decorrelating coefficients, global Lipschitz bounds can fail (even when the coefficients are smooth). The reason is that in suitable random media, with positive probability “bad” configurations of the coefficient field can be found in the ensemble. If we transfer this observation to nonlinear elasticity, this means that even for small macroscopic strains, we might locally observe concentration of large strains. On the other hand, even in the periodic case, the problem of quantitative homogenization in situations with large strains is completely open. Thus, we believe that substantially new ideas are needed to treat general random heterogeneous materials. Furthermore, it is unclear whether a corrector-type result in the sense of Theorem 2.4 can be expected for general random materials, since it is possible that a local concentration of strains leads to a buckling behavior.

Nevertheless, with the methods of the present paper, we could prove the following conditional result for multidimensional media: if there exists a random field  $\varphi_F: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfying (2.3), (2.4) and a uniform Lipschitz estimate in the form (2.5) with  $\text{dist}(F, \text{SO}(d)) \ll 1$ , then (2.6) follows. Furthermore, it seems possible that the method of the present manuscript could be extended to the almost-periodic, multidimensional case, since for linear elliptic systems with sufficiently smooth, almost-periodic coefficients, global Lipschitz estimates can be established; see, e.g., [4, 6].

We remark that Assumption 2.2 does not impose any condition on the growth of  $W$  from above;  $W(\omega, F)$  might even be equal to  $+\infty$  for  $F$  outside an open neighborhood of  $\text{SO}(d)$ . Therefore, it is unclear whether the energy functional  $u \mapsto \int_O W(\omega(\frac{x_d}{\varepsilon}), \nabla u) dx$  homogenizes (in the sense of  $\Gamma$ -convergence). However, if  $W$  additionally satisfies  $p$ -growth of the form

$$W(\omega, F) \lesssim 1 + |F|^p \quad \text{for all } F \in \mathbb{R}^{d \times d} \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega, \quad (2.11)$$

the corresponding  $\Gamma$ -limit takes the form  $u \mapsto \int_O \bar{W}_{\text{hom}}(\nabla u) dx$  where

$$\bar{W}_{\text{hom}}(F) = \lim_{L \rightarrow \infty} \inf_{\varphi \in W_0^{1,p}(\square_L; \mathbb{R}^d)} \int_{\square_L} W(\omega(x_d), F + \nabla \varphi(x)) dx.$$

Assuming  $p$ -growth of the form (2.11), standard  $\Gamma$ -convergence arguments yield  $\bar{W}_{\text{hom}} = W_{\text{hom}}$ ; see, e.g., [41, Lemma 3.2] in a general ergodic (and discrete) setting. The following corollary shows that under the assumptions of Theorem 2.4 (in particular without assuming (2.11)),  $W_{\text{hom}}$  and  $\bar{W}_{\text{hom}}$  coincide in the vicinity of rotations.

**Corollary 2.6.** *Consider the setting of Theorem 2.4. There exists  $\bar{\delta} = \bar{\delta}(\alpha, d, p) > 0$  such that*

$$W_{\text{hom}}(F) = \bar{W}_{\text{hom}}(F) \quad \text{for all } F \in U_{\bar{\delta}}.$$

See Section 4 for the proof.

**Remark 2.7.** Homogenization of  $u \mapsto \int_O W(\omega(\frac{x_d}{\varepsilon}), \nabla u) dx$  in the sense of  $\Gamma$ -convergence has been established under weaker conditions than the upper-growth condition (2.11); see, e.g., the classic textbooks [11, 31], and more recent works [2, 17]. However, to our knowledge all existing results require some control from above or some convexity condition on the set  $\{W < \infty\}$  that rules out the application to stored energy functions that satisfy physical growth conditions. In contrast, with the help of the corrector representation of  $W_{\text{hom}}$  it is possible to prove homogenization in a small-strain regime without imposing an additional growth condition for  $W$ . More precisely, by adapting the method of [39, Theorem 3] we get the following: Let  $O \subset \mathbb{R}^d$  be sufficiently smooth and let  $(u_\varepsilon)_\varepsilon = (u_\varepsilon(\omega, \cdot))_\varepsilon \subset g + W_0^{1,p}(O; \mathbb{R}^d)$  be such that

$$\mathcal{I}_\varepsilon(u_\varepsilon) = \inf_{g + W_0^{1,p}(O)} \mathcal{I}_\varepsilon + o(1), \quad \text{where } \mathcal{I}_\varepsilon(u) := \int_O W\left(\omega\left(\frac{x_d}{\varepsilon}\right), \nabla u\right) - f \cdot u dx.$$

Suppose that the assumptions of Theorem 2.4 are satisfied and assume the smallness condition  $\|f\|_{L^q(A)} + \|g - \text{id}\|_{W^{2,q}(A)} \ll 1$  for some  $q > d$ . Then it holds that

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(\omega, \cdot) - u_0\|_{L^p(O)} = 0 \quad \mathbb{P}\text{-a.e. } \omega \in \Omega, \quad (2.12)$$

where  $u_0$  denotes the unique minimizer of the homogenized energy functional

$$\mathcal{I}_{\text{hom}}(u) := \int_O W_{\text{hom}}(\nabla u) - f \cdot u dx \quad \text{subject to } u - g \in W_0^{1,p}(O).$$

Since in Theorem 2.4 we do not assume any growth conditions from above on  $F \mapsto W(\omega, F)$ , the convergence (2.12) does not follow from the known homogenization theory.

In the following we discuss an approximation of  $W_{\text{hom}}$  and its derivatives by periodic RVEs. In contrast to the approximation (2.2), which relies on the introduction of periodic boundary conditions, the periodic RVE that we discuss below is analogous to the one considered in [21] and is based on a periodic approximation of the probability space  $(\Omega, \mathcal{S}, \mathbb{P})$ .

**Definition 2.8** ( $L$ -periodic approximation of  $\mathbb{P}$ ). Let  $L \geq 1$ . A probability space  $(\Omega_L = \Omega, \mathcal{S}_L = \mathcal{S}, \mathbb{P}_L)$  is called an  $L$ -periodic approximation of  $(\Omega, \mathcal{S}, \mathbb{P})$  if the following conditions hold:

- (i)  $\mathbb{P}_L$  concentrates on  $L$ -periodic functions and it is stationary in the sense of (P1).
- (ii) If the random field  $\Omega \times \mathbb{R} \ni (\omega, x) \mapsto \omega(x)$  is distributed according to  $\mathbb{P}$  and  $\Omega_L \times \mathbb{R} \ni (\omega_L, x) \mapsto \omega_L(x)$  according to  $\mathbb{P}_L$ , then the random fields

$$\Omega \times B_{\frac{L}{4}}(0) \ni (\omega, x) \mapsto \omega(x) \quad \text{and} \quad \Omega_L \times B_{\frac{L}{4}}(0) \ni (\omega_L, x) \mapsto \omega_L(x)$$

have the same distribution.

In order to quantify the statement in Theorem 2.4 (i) we will make an additional quantitative ergodicity assumption on the probability space and its periodic approximation. This relies on a Malliavin-type functional calculus; see [18, 19] for a systematic discussion.

**Definition 2.9** (Spectral gaps). Let  $L \geq 1$ . Let  $(\Omega, \mathcal{S}, \mathbb{P})$  be a probability space and  $(\Omega_L, \mathcal{S}_L, \mathbb{P}_L)$  its  $L$ -periodic approximation.

- (i) We say that  $(\Omega, \mathcal{S}, \mathbb{P})$  satisfies a spectral gap estimate with constant  $\rho > 0$  if the following holds: for any random variable  $\mathcal{F}: \Omega \rightarrow \mathbb{R}$ , we have

$$\mathbb{E}[|\mathcal{F} - \mathbb{E}[\mathcal{F}]|^2] \leq \frac{1}{\rho^2} \mathbb{E} \left[ \int_{\mathbb{R}} \left( \int_{B_1(s)} \left| \frac{\partial \mathcal{F}}{\partial \omega} \right| \right)^2 ds \right],$$

where  $\int_{B_1(s)} \left| \frac{\partial \mathcal{F}}{\partial \omega} \right|$  denotes

$$\sup_{\delta \omega} \limsup_{t \rightarrow 0} \frac{|\mathcal{F}(\omega + t\delta \omega) - \mathcal{F}(\omega)|}{t},$$

where the supremum is taken over all configurations  $\delta \omega: \mathbb{R} \rightarrow \mathbb{R}$  supported in  $B_1(s)$  with  $\|\delta \omega\|_{L^\infty(\mathbb{R})} \leq 1$ .

- (ii) (*Periodic spectral gap*) We say that  $(\Omega_L, \mathcal{S}_L, \mathbb{P}_L)$  satisfies a periodic spectral gap estimate with constant  $\rho > 0$  if the following holds: for any random variable  $\mathcal{F}_L: \Omega_L \rightarrow \mathbb{R}$ , we have

$$\mathbb{E}_L[|\mathcal{F}_L - \mathbb{E}_L[\mathcal{F}_L]|^2] \leq \frac{1}{\rho^2} \mathbb{E}_L \left[ \int_0^L \left( \int_{B(s)} \left| \frac{\partial \mathcal{F}_L}{\partial \omega_L} \right| \right)^2 ds \right],$$

where  $\int_{B(s)} \left| \frac{\partial \mathcal{F}_L}{\partial \omega_L} \right|$  denotes

$$\sup_{\delta \omega_L} \limsup_{t \rightarrow 0} \frac{|\mathcal{F}_L(\omega_L + t\delta \omega_L) - \mathcal{F}_L(\omega_L)|}{t},$$

where the supremum is taken over all  $L$ -periodic configurations  $\delta \omega_L: \mathbb{R} \rightarrow \mathbb{R}$  supported in  $B_1(s) + LZ$  and with  $\|\delta \omega_L\|_{L^\infty(0,L)} \leq 1$ .

**Remark 2.10** (Example of Gaussian type). It is convenient to give examples of admissible distributions in the class of stationary Gaussian random fields, since the latter are uniquely determined by their covariance functions and allow us to apply Malliavin calculus to establish spectral gap inequalities. A specific example is the following: Suppose that  $\mathbb{P}$  describes a stationary, centered Gaussian random field  $\omega: \mathbb{R} \rightarrow \mathbb{R}$  with a bounded and compactly supported covariance function  $\mathcal{C}(s) := \text{Cov}(\omega(x+s), \omega(s))$  satisfying  $|\mathcal{C}(s)| = 0$  for  $|s| \geq \ell/2$  for some correlation length  $\ell > 0$ . Then  $(\Omega, \mathcal{S}, \mathbb{P})$  satisfies the spectral gap estimate of Definition 2.9 (i); see, e.g., [19]. For  $L \geq 4\ell$  we may define an  $L$ -periodic approximation by periodizing the covariance function: Set  $\mathcal{C}_L(x) := \mathcal{C}(\{x\}_L)$ , where  $\{x\}_L \in [-\frac{L}{2}, \frac{L}{2})$  is uniquely defined by  $x - \{x\}_L \in LZ$ . Let  $\mathbb{P}_L$  describe the stationary, centered Gaussian random field with covariance function  $\mathcal{C}_L$ . Then  $\mathbb{P}_L$  is indeed

an  $L$ -periodic approximation of  $\mathbb{P}$  in the sense of Definition 2.8 and additionally satisfies the spectral gap estimate of Definition 2.9 (ii) with a constant  $\rho$  that is independent of  $L$ .

**Remark 2.11.** There are several ways to *periodize* the coefficients. Here we follow [21, 26] and periodize the ensemble. A different approach would be to periodize the realizations, which is replacing  $\omega$  with the  $L$ -periodic extension of the restriction of  $\omega$  to  $(-L/2, L/2)$ . In general, different periodization procedures yield different systematic errors. The results of [13] for linear elliptic equations suggest that periodization of the ensemble yields a smaller systematic error than the “periodizations of the realizations” in dimensions  $d > 2$ . For laminates, which correspond to  $d = 1$ , it seems plausible that periodizing the realization yields a comparable systematic error to the periodization of the ensemble.

We present our main quantitative result:

**Theorem 2.12** (Quantitative RVE approximations). *Suppose Assumption 2.2 is satisfied and in addition that  $W$  satisfies*

$$\|D^4W(\omega, \cdot)\|_{C(\bar{U}_\alpha)} + \|\partial_\omega W(\omega, \cdot)\|_{C^3(\bar{U}_\alpha)} \leq \frac{1}{\alpha} \quad \text{for all } \omega \in \mathbb{R}.$$

We let  $(\Omega, \mathcal{S}, \mathbb{P})$  satisfy (P1) and the spectral gap estimate with constant  $\rho > 0$  of Definition 2.9 (i). Let  $L \geq 2$  and let  $(\Omega, \mathcal{S}, \mathbb{P}_L)$  be an  $L$ -periodic approximation of  $\mathbb{P}$  in the sense of Definition 2.8 which satisfies the periodic spectral gap estimate with constant  $\rho > 0$  of Definition 2.9 (ii). We denote by  $W_{\text{hom}, L}$  the corresponding representative volume element approximation defined in (2.2).

Then there exists  $c = c(\alpha, p, d, \rho) \in [1, \infty)$  such that for all  $F \in U_{\bar{\delta}}$  with  $\bar{\delta} > 0$  as in Theorem 2.4 the following statements hold true:

(i) (Estimate on random fluctuations)

$$|W_{\text{hom}, L}(\cdot, F) - \mathbb{E}_L[W_{\text{hom}, L}(F)]| \leq \mathcal{C}(\cdot) \text{dist}^2(F, \text{SO}(d))L^{-\frac{1}{2}}, \quad (2.13)$$

$$|DW_{\text{hom}, L}(\cdot, F) - \mathbb{E}_L[DW_{\text{hom}, L}(F)]| \leq \mathcal{C}(\cdot) \text{dist}(F, \text{SO}(d))L^{-\frac{1}{2}}, \quad (2.14)$$

$$|D^2W_{\text{hom}, L}(\cdot, F) - \mathbb{E}_L[D^2W_{\text{hom}, L}(F)]| \leq \mathcal{C}(\cdot)L^{-\frac{1}{2}}, \quad (2.15)$$

where  $\mathcal{C}$  denotes a random variable satisfying

$$\mathbb{E}_L \left[ \exp\left(\frac{1}{c}\mathcal{C}\right) \right] \leq 2.$$

(ii) (Systematic error)

$$|\mathbb{E}_L[W_{\text{hom}, L}(F)] - W_{\text{hom}}(F)| \leq c \text{dist}^2(F, \text{SO}(d)) \frac{\ln L}{L},$$

$$|\mathbb{E}_L[DW_{\text{hom}, L}(F)] - DW_{\text{hom}}(F)| \leq c \text{dist}(F, \text{SO}(d)) \frac{\ln L}{L},$$

$$|\mathbb{E}_L[D^2W_{\text{hom}, L}(F)] - D^2W_{\text{hom}}(F)| \leq c \frac{\ln L}{L}.$$

The proof is presented in Section 6.

The total error is dominated by the random component of order  $L^{-\frac{1}{2}}$ . However, the deviations from the mean may be reduced by empirical averaging: for  $N \in \mathbb{N}$ , we consider  $N$  independent copies of the coefficient field  $\{\omega_i\}_{i \in \{1, \dots, N\}}$  sampled according to  $\mathbb{P}_L$  and we define the Monte Carlo approximation by

$$W_{\text{hom}, L, N}(\omega, F) = \frac{1}{N} \sum_{i=1}^N W_{\text{hom}, L}(\omega_i, F).$$

With Theorem 2.12 at hand, we obtain the following:

**Corollary 2.13** (Monte Carlo approximation). *Let the assumptions of Theorem 2.12 be satisfied. There exists  $c = c(\alpha, d, p, \rho) > 0$  such that for all  $F \in U_{\bar{\delta}}$  with  $\bar{\delta} > 0$  as in Theorem 2.12 and for  $\ell \in \{0, 1, 2\}$ ,*

$$\mathbb{E}_L [|D^\ell W_{\text{hom}, L, N}(F) - D^\ell W_{\text{hom}}(F)|^2]^{\frac{1}{2}} \leq c \text{dist}^{2-\ell}(F, \text{SO}(d)) \left( \frac{1}{\sqrt{NL}} + \frac{\ln(L)}{L} \right).$$

The proof of Corollary 2.13 is standard and thus omitted here. Note that the total  $L^2$ -error is of order  $L^{-1} \ln L$  if we choose  $N \sim \frac{L}{\ln^2 L}$ . We note that the variance of the RVE can be reduced by discarding samples in the evaluation of the  $W_{\text{hom}, L, N}$  based on cheaply computable statistical quantifiers; see, e.g., [33] where this has been implemented for linear elliptic equations, and [20] for a rigorous error analysis.

### 3. Deterministic intermediate results

In the following we present the deterministic ingredients of our analysis. In particular, in the first part we recall a matching convex lower bound construction that allows us to relate the nonconvex corrector problem to a convex corrector problem. In the second part we collect some standard results about elliptic equations, and in the third part we establish deterministic Lipschitz estimates for correctors that rely on the laminate structure of the material.

#### 3.1. Reduction to a convex problem

The proof of Theorem 2.4 follows the strategy of [39]. The starting point is the observation that  $W \in \mathcal{W}_\alpha^p$  implies the existence of a “matching convex lower bound”. For the precise statement we introduce the following class of strongly convex functions.

**Definition 3.1** (Convex energy density). For  $\beta > 0$  we denote by  $\mathcal{V}_\beta$  the set of functions  $V \in C^2(\mathbb{R}^{d \times d})$  satisfying for all  $F, G \in \mathbb{R}^{d \times d}$ ,

$$\begin{aligned} \beta |F|^2 - \frac{1}{\beta} &\leq V(F) \leq \frac{1}{\beta} (|F|^2 + 1), \\ |DV(F)[G]| &\leq \frac{1}{\beta} (1 + |F|) |G|, \\ \beta |G|^2 &\leq D^2V(F)[G, G] \leq \frac{1}{\beta} |G|^2. \end{aligned}$$

The following lemma is proven in [39], extending a construction that appeared earlier in the context of discrete energies in [14, 23].

**Lemma 3.2** (Matching convex lower bound; see [39, Corollary 2.3]). *Let Assumption 2.2 be satisfied. Then there exist  $\delta, \mu, \beta > 0$  (depending on  $\alpha, d$ , and  $p$ ), and a Borel measurable function  $V: \mathbb{R} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$  satisfying for almost every  $\omega \in \mathbb{R}$ ,  $V(\omega, \cdot) \in \mathcal{V}_\beta$ ,  $V(\omega, \cdot) \in C^3(\mathbb{R}^{d \times d})$ , and*

$$W(\omega, F) + \mu \det F \geq V(\omega, F) \quad \text{for all } F \in \mathbb{R}^{d \times d}, \quad (3.1)$$

$$W(\omega, F) + \mu \det F = V(\omega, F) \quad \text{for all } F \in \mathbb{R}^{d \times d} \text{ with } \text{dist}(F, \text{SO}(d)) < \delta, \quad (3.2)$$

$$V(\omega, RF) = V(\omega, F) \quad \text{for all } F \in \mathbb{R}^{d \times d} \text{ and } R \in \text{SO}(d).$$

As a consequence of the matching property (3.2), together with the strong convexity of  $V$  and the fact that  $\det$  is a null-Lagrangian, we obtain the following corollary.

**Corollary 3.3.** *We consider the assumptions and setting of Lemma 3.2. Let  $\omega \in \Omega$  and  $\varphi \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$  satisfy*

$$\|\text{dist}(F + \nabla\varphi, \text{SO}(d))\|_{L^\infty(\mathbb{R}^d)} < \delta. \quad (3.3)$$

Then the following statements hold:

- (i) *For every  $u \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  and  $v, w \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$ , the following identities hold in the sense of distributions:*

$$\text{div } DW(\omega(x_d), F + \nabla\varphi) = \text{div } DV(\omega(x_d), F + \nabla\varphi), \quad (3.4)$$

$$\text{div}(D^2W(\omega(x_d), F + \nabla\varphi)\nabla u) = \text{div}(D^2V(\omega(x_d), F + \nabla\varphi)\nabla u), \quad (3.5)$$

$$\begin{aligned} \text{div}(D^3W(\omega(x_d), F + \nabla\varphi)\nabla v \cdot \nabla w) \\ = \text{div}(D^3V(\omega(x_d), F + \nabla\varphi)\nabla v \cdot \nabla w). \end{aligned} \quad (3.6)$$

- (ii) *There exists  $c = c(\alpha, d, p) > 0$  such that  $\mathbb{L}(x) := D^2W(\omega(x_d), F + \nabla\varphi(x))$  satisfies*

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla\eta|^2 dx &\leq c \int_{\mathbb{R}^d} \mathbb{L}(x)\nabla\eta \cdot \nabla\eta dx \quad \text{for all } \eta \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d), \\ |\mathbb{L}(\cdot)| &\leq c \quad \text{a.e.} \end{aligned}$$

The proof is presented in Section 3.4.

This corollary implies that the corrector equations corresponding to  $W$  and  $V$  have the same solutions for  $F$  in a sufficiently small neighborhood of  $\text{SO}(d)$ ; see, e.g., the proof of Lemma 3.8. In particular, this also implies that

$$W_{\text{hom},L}(F) + \mu \det(F) = V_{\text{hom},L}(F), \quad W_{\text{hom}}(F) + \mu \det(F) = V_{\text{hom}}(F),$$

where  $V_{\text{hom}}$  is the homogenized integrand corresponding to  $V$  and  $V_{\text{hom},L}$  its corresponding RVE approximation; see Section 4 for the details.



### 3.2. Standard estimates for elliptic equations

We formulate three simple deterministic existence, uniqueness, and regularity statements for uniformly elliptic equations.

**Definition 3.4.** Let  $d, m \in \mathbb{N}$ . For given  $0 < \lambda \leq \Lambda < \infty$  we denote by  $\mathcal{A}_\lambda^\Lambda$  the set of functions  $A: \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^{m \times d}$  satisfying for all  $F_1, F_2 \in \mathbb{R}^{m \times d}$ ,

$$\begin{aligned} (A(F_1) - A(F_2)) \cdot (F_1 - F_2) &\geq \lambda |F_1 - F_2|^2, \\ |A(F_1) - A(F_2)| &\leq \Lambda |F_1 - F_2|, \quad A(0) = 0. \end{aligned} \quad (3.7)$$

**Lemma 3.5.** For  $d, m \in \mathbb{N}$ , let  $A: \mathbb{R}^d \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^{m \times d}$  be Borel measurable, and there exists  $0 < \lambda \leq \Lambda < \infty$  such that  $A(x, \cdot) \in \mathcal{A}_{\lambda}^{\Lambda}$  for all  $x \in \mathbb{R}^d$ . Then the following statements hold:

- (i) For every  $g \in L^2(\mathbb{R}^d; \mathbb{R}^{m \times d})$ ,  $f \in L^2(\mathbb{R}^d; \mathbb{R}^m)$ , and  $T > 0$  there exists a unique weak solution  $\varphi \in H^1(\mathbb{R}^d; \mathbb{R}^m)$  of

$$\frac{1}{T} \varphi - \operatorname{div} A(x, \nabla \varphi) = \operatorname{div} g + f \quad \text{in } \mathbb{R}^d, \quad (3.8)$$

and it holds that

$$\int_{\mathbb{R}^d} \frac{1}{T} |\varphi|^2 + \lambda |\nabla \varphi|^2 dx \leq \int_{\mathbb{R}^d} \frac{1}{\lambda} |g|^2 + T |f|^2 dx. \quad (3.9)$$

- (ii) For every  $g \in L^\infty(\mathbb{R}^d; \mathbb{R}^{m \times d})$ ,  $f \in L^2(\mathbb{R}^d; \mathbb{R}^m)$ , and  $T > 0$ , there exists a unique  $\varphi \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  satisfying equation (3.8). Moreover, there exists  $c \in [1, \infty)$  such that for all  $x_0 \in \mathbb{R}^d$ ,  $R \geq \sqrt{T}$ ,

$$\int_{\mathbb{R}^d} \left( \frac{1}{T} |\varphi|^2 + \lambda |\nabla \varphi|^2 \right) \eta dx \leq c \int_{\mathbb{R}^d} \left( \frac{1}{\lambda} |g|^2 + T |f|^2 \right) \eta dx,$$

where  $\eta: \mathbb{R}^d \rightarrow \mathbb{R}$  is given by  $\eta(x) = \exp(-\frac{\gamma}{R} |x - x_0|)$  with  $\gamma = \gamma(\frac{\lambda}{\Lambda}) \in (0, 1]$ . In particular, we have

$$\begin{aligned} \int_{B_R(x_0)} \frac{1}{T} |\varphi|^2 + \lambda |\nabla \varphi|^2 dx \\ \leq c \left( \frac{1}{\lambda \gamma^d} \|g\|_{L^\infty(\mathbb{R}^d)}^2 + \frac{1}{R^d} \int_{\mathbb{R}^d} T \eta |f|^2 dx \right). \end{aligned} \quad (3.10)$$

Part (i) is completely standard and part (ii) follows for  $g \in (L^2 \cap L^\infty)(\mathbb{R}^d; \mathbb{R}^{m \times d})$  by testing (3.8) with  $\eta \varphi$  and a suitable choice of  $\gamma$  and the general case follows by approximation; see [21, Lemma 36] for details.

The following simple lemma is central to the analysis of the present paper. Based on the observation that in the case of laminates the PDE (3.8) reduces to an ODE, we establish the following crucial Lipschitz estimate.

**Lemma 3.6.** *Let  $d \in \mathbb{N}$ . Let  $A: \mathbb{R} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d}$  be a Borel measurable function and suppose that there exist  $\Lambda \geq \lambda > 0$  such that  $A(z, \cdot) \in \mathcal{A}_\lambda^\Lambda$  for all  $z \in \mathbb{R}$ . For  $T > 0$  and  $g \in L^\infty(\mathbb{R})^{d \times d}$ , let  $\varphi \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  be the unique solution to*

$$\frac{1}{T} \varphi(x) - \operatorname{div} A(x_d, \nabla \varphi(x)) = \operatorname{div}(g(x_d)) \quad \text{in } \mathbb{R}^d.$$

Then the following statements are true:

- (i)  $\varphi$  is one-dimensional in the sense that  $\varphi(x) = \varphi(x_d)$  and it solves

$$\frac{1}{T} \varphi(x_d) - \partial_{x_d} a(x_d, \partial_d \varphi(x_d)) = \partial_{x_d} g_d(x_d) \quad \text{in } \mathbb{R}, \quad (3.11)$$

where  $a: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is given by  $a(x_d, f) = A(x_d, f \otimes e_d) e_d$ , and  $g_d(x_d) = g(x_d) e_d$ .

If we replace  $g \in L^\infty(\mathbb{R}; \mathbb{R}^{d \times d})$  by  $g \in L^2(\mathbb{R}; \mathbb{R}^{d \times d})$ , an analogous claim holds for the weak solution of the equation.

- (ii) There exists  $c = c(\frac{\Lambda}{\lambda}) \in [1, \infty)$  such that

$$\|\nabla \varphi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{c}{\lambda} \|g\|_{L^\infty(\mathbb{R})}. \quad (3.12)$$

- (iii) Let  $L > 0$ . For given  $g \in L^\infty((0, L); \mathbb{R}^{d \times d})$ , there exists a unique weak solution  $\varphi \in H_{\text{per}}^1(\square_L; \mathbb{R}^d)$  to

$$-\operatorname{div} A(x_d, \nabla \varphi(x)) = \operatorname{div}(g(x_d)) \quad \text{in } \square_L$$

satisfying  $\int_{\square_L} \varphi \, dx = 0$ . Moreover,  $\varphi$  is one-dimensional in the sense that  $\varphi(x) = \varphi(x_d)$  and it weakly solves

$$-\partial_{x_d} a(x_d, \partial_d \varphi(x_d)) = \partial_{x_d} g_d(x_d) \quad \text{in } (0, L).$$

Finally, there exists  $c = c(\lambda, \Lambda) \in [1, \infty)$  such that (3.12) is valid.

The proof is presented in Section 3.4.

**Remark 3.7.** The properties (3.7) imply that the operator  $a: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  given in Lemma 3.6 satisfies for all  $x_d$  and all  $f_1, f_2 \in \mathbb{R}^d$ ,

$$\begin{aligned} (a(x_d, f_1) - a(x_d, f_2)) \cdot (f_1 - f_2) &\geq \lambda |f_1 - f_2|^2, \\ |a(x_d, f_1) - a(x_d, f_2)| &\leq \Lambda |f_1 - f_2|, \quad a(x_d, 0) = 0. \end{aligned}$$

If additionally  $A(x_d, \cdot)$  is linear and symmetric, then  $a$  is also linear and symmetric.

### 3.3. Lipschitz estimates and differentiability of localized correctors

For  $T > 0$ ,  $F \in \mathbb{R}^{d \times d}$ , and  $\omega \in \Omega$ , we consider the localized corrector equation

$$\frac{1}{T} \varphi_F^T - \operatorname{div} DW(\omega(x_d), F + \nabla \varphi_F^T) = 0 \quad \text{in } \mathbb{R}^d. \quad (3.13)$$

**Lemma 3.8.** *Let  $T > 0$  and  $\omega \in \Omega$ . Suppose Assumption 2.2 is satisfied. Then there exist  $\bar{\delta} = \bar{\delta}(\alpha, p, d) > 0$  and  $c = c(\alpha, p, d) > 0$  such that the following statements hold:*

- (i) *For all  $F \in U_{\bar{\delta}}$ , there exists a unique solution  $\varphi_F^T \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$  to (3.13) which satisfies*

$$\|F + \nabla \varphi_F^T\|_{L^\infty(\mathbb{R}^d)} < \delta,$$

where  $\delta = \delta(\alpha, d, p) > 0$  is as in Lemma 3.2. Moreover,  $\varphi_F^T$  is one-dimensional in the sense that  $\varphi_F^T(x) = \varphi_F^T(x_d)$  and satisfies

$$\|\nabla \varphi_F^T\|_{L^\infty(\mathbb{R}^d)} \leq c \operatorname{dist}(F, \operatorname{SO}(d)). \quad (3.14)$$

- (ii) *The function  $U_{\bar{\delta}} \ni F \mapsto \varphi_F^T \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$  is  $C^2$  in the sense of Fréchet. The first derivative  $\varphi_{F,G}^T := D_F \varphi_F^T(F)G \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$  at  $F \in U_{\bar{\delta}}$  in the direction  $G \in \mathbb{R}^{d \times d}$  is characterized by*

$$\frac{1}{T} \varphi_{F,G}^T - \operatorname{div}(D^2 W(\omega(x_d), F + \nabla \varphi_F^T)(G + \nabla \varphi_{F,G}^T)) = 0 \quad \text{in } \mathbb{R}^d. \quad (3.15)$$

Moreover, it is one-dimensional in the sense that  $\varphi_{F,G}^T(x) = \varphi_{F,G}^T(x_d)$  and satisfies

$$\|\nabla \varphi_{F,G}^T\|_{L^\infty(\mathbb{R}^d)} \leq c |G| \quad \text{for all } G \in \mathbb{R}^{d \times d}. \quad (3.16)$$

- (iii) *The second derivative  $\varphi_{F,G,H}^T := D_F^2 \varphi_F^T H G$  at  $F \in U_{\bar{\delta}}$  with  $G, H \in \mathbb{R}^{d \times d}$  is characterized by*

$$\begin{aligned} \frac{1}{T} \varphi_{F,G,H}^T - \operatorname{div}(D^2 W(\omega(x_d), F + \nabla \varphi_F^T) \nabla \varphi_{F,G,H}^T) \\ = \operatorname{div}(D^3 W(\omega(x_d), F + \nabla \varphi_F^T)(H + \nabla \varphi_{F,H}^T)(G + \nabla \varphi_{F,G}^T)) \quad \text{in } \mathbb{R}^d. \end{aligned} \quad (3.17)$$

Moreover, it is one-dimensional in the sense that  $\varphi_{F,G,H}^T(x) = \varphi_{F,G,H}^T(x_d)$  and satisfies

$$\|\nabla \varphi_{F,G,H}^T\|_{L^\infty(\mathbb{R}^d)} \leq c |H| |G| \quad \text{for all } G, H \in \mathbb{R}^{d \times d}. \quad (3.18)$$

The proof is presented in Section 3.4.

For the proof of the systematic error estimates of Theorem 2.12, we need the following result. In the following lemma, we denote by  $\varphi_F^T, \varphi_{F,G}^T, \varphi_{F,G,H}^T$  the correctors from Lemma 3.8 and by  $\hat{\varphi}_F^T, \hat{\varphi}_{F,G}^T, \hat{\varphi}_{F,G,H}^T$  the same objects if we replace  $\omega \in \Omega$  by another realization  $\hat{\omega} \in \Omega$ .

**Lemma 3.9.** *Suppose that Assumption 2.2 is satisfied and that it holds that  $W(\omega, \cdot) \in C^4(\bar{U}_\alpha)$  with*

$$\|D^4 W(\omega, \cdot)\|_{C(\bar{U}_\alpha)} \leq \frac{1}{\alpha} \quad \text{for all } \omega \in \mathbb{R}.$$

Let  $T, L > 0$ ,  $\omega, \widehat{\omega} \in \Omega$  with  $\omega(x_d) = \widehat{\omega}(x_d)$  for  $x_d \in [-\frac{L}{4}, \frac{L}{4}]$ . Then there exists  $c = c(\alpha, p, d) \in [4, \infty)$  such that for all  $F \in U_{\bar{\delta}}$  with  $\bar{\delta} = \bar{\delta}(\alpha, p, d) > 0$  as in Lemma 3.8, we have

$$\int_{B_{\sqrt{T}}} |\nabla \widehat{\varphi}_F^T - \nabla \varphi_F^T|^2 dx_d \leq c \operatorname{dist}^2(F, \operatorname{SO}(d)) \exp\left(-\frac{1}{c} \frac{L}{\sqrt{T}}\right), \quad (3.19)$$

$$\int_{B_{\sqrt{T}}} |\nabla \widehat{\varphi}_{F,G}^T - \nabla \varphi_{F,G}^T|^2 dx_d \leq c |G|^2 \exp\left(-\frac{1}{c} \frac{L}{\sqrt{T}}\right), \quad (3.20)$$

$$\int_{B_{\sqrt{T}}} |\nabla \widehat{\varphi}_{F,G,H}^T - \nabla \varphi_{F,G,H}^T|^2 dx_d \leq c |G|^2 |H|^2 \exp\left(-\frac{1}{c} \frac{L}{\sqrt{T}}\right). \quad (3.21)$$

The proof is presented in Section 3.4.

### 3.4. Proofs of deterministic intermediate results

*Proof of Corollary 3.3. Step 1. Proof of part (i).* Equation (3.4) is a direct consequence of (3.3), (3.2), and the fact that  $\operatorname{div} D \det(F + \nabla \varphi) = 0$  in the sense of distributions since  $\det$  is a null-Lagrangian.

Next we show (3.5). We suppose  $u \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ ; the general claim follows by approximation. For every  $\eta \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$  there exists  $h_0 > 0$  such that for all  $h \in [0, h_0]$  it holds that  $\|\operatorname{dist}(F + \nabla \varphi + h \nabla u, \operatorname{SO}(d))\|_{L^\infty(\operatorname{supp} \eta)} < \delta$  and thus

$$\begin{aligned} & \frac{1}{h} \int_{\mathbb{R}^d} (DW(\omega(x_d), F + \nabla \varphi + h \nabla u) - DW(\omega(x_d), F + \nabla \varphi)) \cdot \nabla \eta dx \\ &= \frac{1}{h} \int_{\mathbb{R}^d} (DV(\omega(x_d), F + \nabla \varphi + h \nabla u) - DV(\omega(x_d), F + \nabla \varphi)) \cdot \nabla \eta dx \quad \forall h \in (0, h_0]. \end{aligned}$$

Letting  $h \rightarrow 0$ , we obtain

$$\int_{\mathbb{R}^d} D^2 W(\omega(x_d), F + \nabla \varphi) \nabla u \cdot \nabla \eta dx = \int_{\mathbb{R}^d} D^2 V(\omega(x_d), F + \nabla \varphi) \nabla u \cdot \nabla \eta dx$$

and (3.5) follows from the arbitrariness of  $\eta \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ . The argument for (3.6) is analogous and left to the reader.

*Step 2.* By (3.3) and (3.2) we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \mathbb{L}(x) \nabla \eta \cdot \nabla \eta dx \\ &= \int_{\mathbb{R}^d} D^2 V(\omega(x_d), F + \nabla \varphi) \nabla \eta \cdot \nabla \eta - \mu D^2 \det(F + \nabla \varphi) \nabla \eta \cdot \nabla \eta dx \\ &\geq \beta \int_{\mathbb{R}^d} |\nabla \eta|^2 dx, \end{aligned}$$

where the inequality follows by strong convexity of  $V(\omega(x_d), \cdot)$  and  $\int_{\mathbb{R}^d} D^2 \det(F + \nabla \varphi) \nabla \eta \cdot \nabla \eta = 0$ , which follows from the fact that  $\det$  is a null-Lagrangian. Estimate  $|\mathbb{L}(\cdot)| \leq c$  is a consequence of  $W \in C^2(\bar{U}_\delta)$ . ■

*Proof of Lemma 3.6.* We only present the arguments for (i) and (ii), since (iii) is similar.

*Step 1. Proof of part (i).* For any  $h \in \mathbb{R}$  and  $i \in \{1, \dots, d-1\}$ , consider  $w := \varphi(\cdot + he_i) - \varphi$ . Obviously, we have  $w \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^m)$  and

$$\frac{1}{T} w(x) - \operatorname{div}(A(x_d, \nabla \varphi(x) + \nabla w(x)) - A(x_d, \nabla \varphi(x))) = 0 \quad \text{in } \mathbb{R}^d.$$

Hence, estimate (3.9) applied with  $\tilde{A}(x, F) := A(x_d, F + \nabla \varphi(x)) - A(x_d, \nabla \varphi(x))$  and  $g = 0$ ,  $f = 0$  yields  $w \equiv 0$  and thus  $\varphi$  depends only on  $x_d$  and satisfies (3.11).

*Step 2. Proof of part (ii).* Without loss of generality we suppose  $\lambda = 1$ ; the general case follows by replacing  $a$ ,  $T$ , and  $g$  by  $a/\lambda$ ,  $T\lambda$ , and  $g/\lambda$ . Assumption (3.7) and Remark 3.7 imply that for any ball  $B \subset \mathbb{R}$  we have

$$\begin{aligned} \int_B |\partial_{x_d} \varphi|^2 dx_d &\leq \int_B |a(x_d, \partial_{x_d} \varphi)|^2 dx_d \\ &\leq 2 \left( \int_B |a(x_d, \partial_{x_d} \varphi) + g_d|^2 dx_d + \int_B |g_d|^2 dx_d \right) \\ &\leq 2 \left( \int_B |\Psi(x_d)|^2 dx_d + \|g\|_{L^\infty(\mathbb{R})}^2 \right), \end{aligned} \quad (3.22)$$

where we use the shorthand  $\Psi(x_d) := a(x_d, \partial_{x_d} \varphi(x_d) \otimes e_d) + g_d(x_d)$ .

We prove the following: there exists  $c = c(\Lambda) \in [1, \infty)$  such that for every  $x_0 \in \mathbb{R}$  it holds that

$$\sup_{r \in (0,1)} \int_{r\tilde{B}} |\Psi(x_d)|^2 dx_d \leq c \|g\|_{L^\infty(\mathbb{R})}^2 \quad \text{where } r\tilde{B} := rB_{\sqrt{T}}(x_0) \subset \mathbb{R}. \quad (3.23)$$

Clearly, (3.23) together with (3.22) and the arbitrariness of  $x_0$  imply the claimed estimate (3.12).

In the following we use the notation  $(u)_B := \int_B u(x_d) dx_d$ . We have

$$\int_{r\tilde{B}} |\Psi(x_d)|^2 dx_d \leq 2 \int_{r\tilde{B}} |\Psi(x_d) - (\Psi)_{r\tilde{B}}|^2 dx_d + 2|(\Psi)_{r\tilde{B}}|^2.$$

With help of the (one-dimensional) Poincaré inequality, equations (3.11), and (3.10), we estimate the first term on the right-hand side:

$$\begin{aligned} \int_{r\tilde{B}} |\Psi(x_d) - (\Psi)_{r\tilde{B}}|^2 dx_d &\leq |r\tilde{B}|^2 \int_{r\tilde{B}} |\partial_d \Psi(x_d)|^2 dx_d \\ &\stackrel{(3.11)}{\leq} |r\tilde{B}| \int_{r\tilde{B}} \left| \frac{1}{T} \varphi \right|^2 dx_d \\ &\leq 4r \int_{B_{\sqrt{T}}(x_0)} \frac{1}{T} |\varphi|^2 dx_d \stackrel{(3.10)}{\leq} cr \|g\|_{L^\infty(\mathbb{R})}^2. \end{aligned} \quad (3.24)$$

To estimate  $|(\Psi)_{r\tilde{B}}|$ , we use (3.24) and a standard dyadic argument. First, we observe that for  $r \in [\frac{1}{4}, 1]$ , there exists  $c = c(\Lambda) \in [1, \infty)$  such that

$$\begin{aligned} |(\Psi)_{r\tilde{B}}| &\leq 2 \left( \int_{B_{\sqrt{T}}} |\Psi|^2 dx_d \right)^{\frac{1}{2}} \\ &\leq 2 \left( \Lambda \left( \int_{B_{\sqrt{T}}} |\partial_d \varphi|^2 dx_d \right)^{\frac{1}{2}} + \|g\|_{L^\infty(\mathbb{R})} \right) \stackrel{(3.10)}{\leq} c \|g\|_{L^\infty(\mathbb{R})}. \end{aligned} \quad (3.25)$$

For  $r \in (0, \frac{1}{4})$ , we choose  $q \in (\frac{1}{4}, \frac{1}{2})$  and  $k \in \mathbb{N}$  satisfying  $r = q^k$ . We have

$$\begin{aligned} |(\Psi)_{r\tilde{B}}| &\leq \left| \sum_{i=0}^{k-1} (\Psi)_{q^{i+1}\tilde{B}} - (\Psi)_{q^i\tilde{B}} \right| + |(\Psi)_{\tilde{B}}| \\ &\stackrel{(3.25)}{\leq} \sum_{i=0}^{k-1} |(\Psi)_{q^{i+1}\tilde{B}} - (\Psi)_{q^i\tilde{B}}| + c \left( \frac{\Lambda}{\lambda} \right) \|g\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Using

$$\begin{aligned} |(\Psi)_{q^{i+1}\tilde{B}} - (\Psi)_{q^i\tilde{B}}| &\leq \int_{q^{i+1}\tilde{B}} |\Psi - (\Psi)_{q^i\tilde{B}}| dx_d \\ &\leq \frac{1}{q} \left( \int_{q^i\tilde{B}} |\Psi - (\Psi)_{q^i\tilde{B}}|^2 dx_d \right)^{\frac{1}{2}} \stackrel{(3.24)}{\leq} c (\sqrt{q})^{i-2} \|g\|_{L^\infty(\mathbb{R})} \end{aligned}$$

we obtain

$$\sum_{i=0}^{k-1} |(\Psi)_{q^{i+1}\tilde{B}} - (\Psi)_{q^i\tilde{B}}| \leq \frac{c}{q} \|g\|_{L^\infty(\mathbb{R})} \sum_{i=0}^{k-1} (\sqrt{q})^i \leq \frac{c}{q(1-\sqrt{q})} \|g\|_{L^\infty(\mathbb{R})},$$

which concludes the proof.  $\blacksquare$

*Proof of Lemma 3.8.* Throughout the proof we write  $\lesssim$  if  $\leq$  holds up to a multiplicative constant depending only on  $\alpha$ ,  $d$ , and  $p$ .

*Step 1. Proof of part (i).* First, we consider the convex lower bound  $V$  for  $W$  from Lemma 3.2 and the equation

$$\frac{1}{T} \varphi_F^T - \operatorname{div} DV(\omega(x_d), F + \nabla \varphi_F^T) = 0 \quad \text{in } \mathbb{R}^d. \quad (3.26)$$

Since  $DW(\omega(x_d), R) = 0$  and thus  $DV(\omega(x_d), R) = \mu D \det(R)$  for any  $R \in \operatorname{SO}(d)$ , we may rewrite this equation as

$$\frac{1}{T} \varphi_F^T - \operatorname{div} A(x_d, \nabla \varphi_F^T) = \operatorname{div} g \quad \text{in } \mathbb{R}^d,$$

with

$$\begin{aligned} A(x_d, G) &:= DV(\omega(x_d), F + G) - DV(\omega(x_d), F), \\ g(x_d) &:= DV(\omega(x_d), F) - DV(\omega(x_d), R). \end{aligned}$$

Note that  $A$  satisfies the assumptions of Lemma 3.6 and it holds that  $\|g\|_{L^\infty(\mathbb{R})} \lesssim |F - R|$ . Hence, Lemmas 3.5 and 3.6 imply that there exists a unique  $\varphi_F^T \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  satisfying (3.26) and which is one-dimensional and satisfies  $\|\nabla \varphi_F^T\|_{L^\infty(\mathbb{R})} \lesssim |F - R|$ . Minimizing over  $R \in \text{SO}(d)$  we obtain  $\|\nabla \varphi_F^T\|_{L^\infty(\mathbb{R})} \lesssim \text{dist}(F, \text{SO}(d))$ . Hence, we find  $\bar{\delta} = \bar{\delta}(\alpha, d, p) > 0$  such that for all  $F \in U_{\bar{\delta}}$ , it holds that

$$\|\text{dist}(F + \nabla \varphi_F^T, \text{SO}(d))\|_{L^\infty(\mathbb{R}^d)} < \delta, \quad (3.27)$$

with  $\delta > 0$  as in Lemma 3.2. Combining (3.27) with Corollary 3.3 (i), we obtain that for all  $F \in U_{\bar{\delta}}$ ,  $\varphi_F^T$  solves (3.13). On the other hand, any solution of (3.13) satisfying (3.27) satisfies (3.26). However, the latter equation admits a unique solution. This implies the claimed uniqueness.

*Step 2. Proof of part (ii) except  $C^2$ -regularity of  $F \mapsto \varphi_F^T$ .*

*Substep 2.1. Lipschitz estimate on  $F \mapsto \varphi_F$ .* For  $F, F' \in \mathbb{R}^{d \times d}$  let  $\varphi_F^T$  and  $\varphi_{F'}^T$  be defined via (3.26). We claim

$$\|\nabla(\varphi_F^T - \varphi_{F'}^T)\|_{L^\infty(\mathbb{R}^d)} \lesssim |F - F'|. \quad (3.28)$$

Indeed,  $\Phi := \varphi_F^T - \varphi_{F'}^T$  solves

$$\begin{aligned} \frac{1}{T} \Phi - \text{div}(DV(\omega(x_d), F + \nabla \Phi + \nabla \varphi_{F'}^T) - DV(\omega(x_d), F + \nabla \varphi_{F'}^T)) \\ = \text{div}(DV(\omega(x_d), F + \nabla \varphi_{F'}^T) - DV(\omega(x_d), F' + \nabla \varphi_{F'}^T)) \quad \text{in } \mathbb{R}^d. \end{aligned}$$

Lemma 3.6 together with the Lipschitz continuity of  $DV(\omega(x_d), \cdot)$  implies (3.28).

*Substep 2.2. Differentiability of  $F \mapsto \varphi_F$ .* Lemma 3.6 and the one-dimensionality of  $\varphi_F^T$  imply that

$$\frac{1}{T} \varphi_{F,G}^T - \text{div}(D^2V(\omega(x_d), F + \nabla \varphi_F^T)(G + \nabla \varphi_{F,G}^T)) = 0 \quad \text{in } \mathbb{R}^d \quad (3.29)$$

admits a unique solution  $\varphi_{F,G}^T \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  which depends only on  $x_d$  and satisfies (3.16). Combining (3.26) and (3.29), we obtain that  $\Psi := \varphi_{F+G}^T - \varphi_F^T - \varphi_{F,G}^T$  solves

$$\frac{1}{T} \Psi - \text{div}(D^2V(\omega(x_d), F + \nabla \varphi_F^T) \nabla \Psi) = \text{div } g \quad \text{in } \mathbb{R}^d, \quad (3.30)$$

where

$$\begin{aligned} g &:= DV(\omega(x_d), F + G + \nabla \varphi_{F+G}^T) - DV(\omega(x_d), F + \nabla \varphi_F^T) \\ &\quad - D^2V(\omega(x_d), F + \nabla \varphi_F^T)(G + \nabla \varphi_{F+G}^T - \nabla \varphi_F^T). \end{aligned}$$

For  $|G|$  small enough, we have  $F + G \in U_{\bar{\delta}}$  and thus  $F + G + \nabla\varphi_{F+G}^T \in U_{\bar{\delta}}$  a.e. Since the smoothness of  $W$  and (3.2) imply  $\sup_{F \in U_{\bar{\delta}}} \|D^3V(\omega(x_d), F)\| \lesssim 1$  we obtain, by a Taylor expansion,

$$\|g\|_{L^\infty(\mathbb{R}^d)} \lesssim \|G + \nabla\varphi_{F+G}^T - \nabla\varphi_F^T\|_{L^\infty(\mathbb{R}^d)}^2 \stackrel{(3.28)}{\lesssim} |G|^2. \quad (3.31)$$

Hence, Lemmas 3.5 and 3.6 applied to (3.30) yield

$$\|\nabla\Psi\|_{L^\infty(\mathbb{R}^d)}^2 + \int_{B_{\sqrt{T}}(x_0)} \frac{1}{T} |\Psi|^2 + |\nabla\Psi|^2 dx \lesssim |G|^4 \quad \forall x_0 \in \mathbb{R}^d \quad (3.32)$$

and thus, by the Poincaré inequality,

$$\lim_{|G| \rightarrow 0} |G|^{-1} \sup_{x_0 \in \mathbb{R}^d} \|\varphi_{F+G}^T - \varphi_F^T - \varphi_{F,G}^T\|_{W^{1,\infty}(B_1(x_0))} = 0.$$

Hence,  $F \mapsto \varphi_F^T$  is differentiable with derivative  $\varphi_{F,G}^T = D_F\varphi_F^T G$ .

*Substep 2.3. Conclusion.* Let  $F \in U_{\bar{\delta}}$  and  $G \in \mathbb{R}^{d \times d}$  be given. Combining Corollary 3.3 and Step 1, we observe that equation (3.15) admits a unique solution in  $H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$ , which coincides with the solution of (3.29). Hence, claim (ii) (except  $C^2$ -regularity) is proven.

*Step 3. Proof of part (iii).* Fix  $F \in U_{\bar{\delta}}$ . For every  $G, H \in \mathbb{R}^{d \times d}$ , Lemma 3.6 implies that

$$\begin{aligned} & \frac{1}{T} \varphi_{F,G,H}^T - \operatorname{div}(D^2V(\omega(x_d), F + \nabla\varphi_F^T) \nabla\varphi_{F,G,H}^T) \\ &= \operatorname{div}(D^3V(\omega(x_d), F + \nabla\varphi_F^T)(H + \nabla\varphi_{F,H}^T)(G + \nabla\varphi_{F,G}^T)) \quad \text{in } \mathbb{R}^d \end{aligned} \quad (3.33)$$

admits a unique solution  $\varphi_{F,G,H}^T$  which depends only on  $x_d$  and satisfies (3.18). Set  $\bar{\Psi} := \varphi_{F+H,G}^T - \varphi_{F,G}^T - \varphi_{F,G,H}^T$ . In view of (3.33) and (3.29), we have

$$\frac{1}{T} \bar{\Psi} - \operatorname{div}(D^2V(\omega(x_d), F + \nabla\varphi_F^T) \nabla\bar{\Psi}) = \operatorname{div}(g_A + g_B + g_C) \quad \text{in } \mathbb{R}^d,$$

where

$$\begin{aligned} g_A &= (D^2V(\omega(x_d), F + H + \nabla\varphi_{F+H}^T) - D^2V(\omega(x_d), F + \nabla\varphi_F^T))(G + \nabla\varphi_{F+H,G}^T) \\ &\quad - D^3V(\omega(x_d), F + \nabla\varphi_F^T)(H + \nabla\varphi_{F+H}^T - \nabla\varphi_F^T)(G + \nabla\varphi_{F+H,G}^T), \\ g_B &= -D^3V(\omega(x_d), F + \nabla\varphi_F^T)(H + \nabla\varphi_{F,H}^T)(\nabla\varphi_{F,G}^T - \nabla\varphi_{F+H,G}^T), \\ g_C &= -D^3V(\omega(x_d), F + \nabla\varphi_F^T)(\nabla\varphi_{F,H}^T - \nabla\varphi_{F+H}^T + \nabla\varphi_F^T)(G + \nabla\varphi_{F+H,G}^T). \end{aligned}$$

Obviously, the functions  $g_A, g_B, g_C$  depend only on  $x_d$  and we claim that

$$\|g_A + g_B + g_C\|_{L^\infty(\mathbb{R}^d)} \lesssim |G| |H| \tilde{\rho}(|H|), \quad \text{where } \lim_{t \downarrow 0} \tilde{\rho}(t) = 0. \quad (3.34)$$



Before proving (3.34), we note that (3.34) implies the claim. Indeed, Lemmas 3.5, 3.6 and (3.34) yield

$$\|\nabla \bar{\Psi}\|_{L^\infty(\mathbb{R}^d)}^2 + \int_{B_{\sqrt{T}}(x_0)} \frac{1}{T} |\bar{\Psi}|^2 + |\nabla \bar{\Psi}|^2 dx \lesssim |G|^2 |H|^2 \tilde{\rho}(|H|)^2. \quad (3.35)$$

Taking the supremum over  $|G| \leq 1$  and letting  $H \rightarrow 0$ , the differentiability of  $F \mapsto D_F \varphi_F^T(\cdot)$  follows. Finally, in view of Corollary 3.3, the function  $\varphi_{F,G,H}^T$  is the unique solution of (3.17).

Hence it is left to prove (3.34). For  $|H|$  sufficiently small, the smoothness of  $W$  and a Taylor expansion imply

$$\begin{aligned} |g_A| &\lesssim \int_0^1 |(D^3 V(\omega(x_d), F + \nabla \varphi_F^T + s(H + \nabla \varphi_{F+H}^T - \nabla \varphi_F^T)) \\ &\quad - D^3 V(\omega(x_d), F + \nabla \varphi_F^T))| ds \\ &\quad \times |H + \nabla \varphi_{F+H}^T - \nabla \varphi_F^T| |G + \nabla \varphi_{F+H,G}^T| \\ &\stackrel{(2.1)}{\lesssim} \rho(|H + \nabla \varphi_{F+H}^T - \nabla \varphi_F^T|) |G| |H|^2. \end{aligned}$$

Clearly, estimate (3.28) implies that we find  $\tilde{\rho}$  satisfying  $\lim_{t \downarrow 0} \tilde{\rho}(t) = 0$  such that

$$\rho(|H + \nabla \varphi_{F+H}^T - \nabla \varphi_F^T|) \leq \tilde{\rho}(|H|).$$

Further, it holds that

$$\|g_B\|_{L^\infty(\mathbb{R}^d)} \lesssim |H| \|\nabla \varphi_{F,G}^T - \nabla \varphi_{F+H,G}^T\|_{L^\infty(\mathbb{R}^d)}.$$

We estimate the last expression by noting that  $\bar{\Phi} := \varphi_{F+H,G}^T - \varphi_{F,G}^T$  solves

$$\frac{1}{T} \bar{\Phi} - \operatorname{div}(D^2 V(\omega(x_d), F + \nabla \varphi_F^T) \nabla \bar{\Phi}) = \operatorname{div} \bar{g} \quad \text{in } \mathbb{R}^d,$$

where

$$\bar{g} := (D^2 V(\omega(x_d), F + H + \nabla \varphi_{F+H}^T) - D^2 V(\omega(x_d), F + \nabla \varphi_F^T))(G + \nabla \varphi_{F+H,G}^T)$$

satisfies  $\|\bar{g}\|_{L^\infty(\mathbb{R}^d)} \lesssim |G| |H|$ . Hence we obtain, with the help of Lemma 3.6,

$$\|g_B\|_{L^\infty(\mathbb{R}^d)} \lesssim |H| \|\nabla \bar{\Phi}\|_{L^\infty(\mathbb{R}^d)} \lesssim |G| |H|^2.$$

Finally, we estimate

$$\|g_C\|_{L^\infty(\mathbb{R}^d)} \lesssim |G| \|\nabla \varphi_{F,H}^T - \nabla \varphi_{F+H}^T + \nabla \varphi_F^T\|_{L^\infty(\mathbb{R}^d)} \lesssim |G| |H|^2,$$

where the last inequality follows by applying Lemma 3.6 to equation (3.30); cf. (3.31). Summarizing these bounds, we obtain the desired estimate (3.34).

*Step 4. Continuity of second derivatives.* Consider  $F \in U_{\delta}$ . We claim that  $F_n \rightarrow F$  implies  $\varphi_{F_n, G, H}^T \rightarrow \varphi_{F, G, H}^T$  in  $W^{1, \infty}(\mathbb{R}^d, \mathbb{R}^d)$  for every  $G, H \in \mathbb{R}^{d \times d}$ . Equation (3.33) implies that the difference  $\psi_n := \varphi_{F_n, G, H}^T - \varphi_{F, G, H}^T$  satisfies

$$\frac{1}{T} \psi_n - \operatorname{div} D^2 V(F_n + \nabla \varphi_{F_n}^T) \nabla \psi_n = \operatorname{div}(g_n^{(1)} + g_n^{(2)}),$$

where

$$\begin{aligned} g_n^{(1)} &:= D^3 V(F_n + \nabla \varphi_{F_n}^T)(H + \nabla \varphi_{F_n, H}^T)(G + \nabla \varphi_{F_n, G}^T) \\ &\quad - D^3 V(F + \nabla \varphi_F^T)(H + \nabla \varphi_{F, H}^T)(G + \nabla \varphi_{F, G}^T), \\ g_n^{(2)} &:= (D^2 V(F_n + \nabla \varphi_{F_n}^T) - D^2 V(F + \nabla \varphi_F^T)) \nabla \varphi_{F, G, H}^T. \end{aligned}$$

Combining  $\nabla \varphi_{F_n}^T \rightarrow \nabla \varphi_F^T$ ,  $\nabla \varphi_{F_n, H}^T \rightarrow \nabla \varphi_{F, H}^T$ ,  $\nabla \varphi_{F_n, G}^T \rightarrow \nabla \varphi_{F, G}^T$  in  $L^\infty$  as  $n \rightarrow \infty$  with (3.18) and the uniform continuity of  $D^3 V(x_d, \cdot)$  in the vicinity of  $\operatorname{SO}(d)$  (cf. (2.1)), we obtain that  $g_n^{(1)}, g_n^{(2)} \rightarrow 0$  in  $L^\infty(\mathbb{R}^d)$ . Hence the claim follows with the help of Lemmas 3.5 and 3.6.  $\blacksquare$

*Proof of Lemma 3.9.* Throughout the proof we suppose  $F \in U_{\delta}$  and we write  $\lesssim$  if  $\leq$  holds up to a multiplicative constant depending only on  $\alpha, d$ , and  $p$ .

*Step 1. Proof of (3.19).* We first note that by (3.26),  $v_F := \widehat{\varphi}_F^T - \varphi_F^T$  solves

$$\frac{1}{T} v_F - \operatorname{div}(A(x_d, \nabla v_F)) = \operatorname{div}(g_F(x_d)) \quad \text{in } \mathbb{R}^d, \quad (3.36)$$

where

$$A(\cdot, G) := DV(\widehat{\omega}, F + \nabla \varphi_F^T + G) - DV(\widehat{\omega}, F + \nabla \varphi_F^T)$$

and, for an arbitrary  $R \in \operatorname{SO}(d)$ ,

$$g_F := DV(\widehat{\omega}, F + \nabla \varphi_F^T) - DV(\widehat{\omega}, R) - (DV(\omega, F + \nabla \varphi_F^T) - DV(\omega, R)).$$

Lemma 3.6 applies to (3.36) and thus Lemma 3.5 yields

$$\int_{\mathbb{R}} \eta \frac{1}{T} |v_F|^2 + \eta |\nabla v_F|^2 dx_d \lesssim \int_{\mathbb{R}} \eta |g_F|^2 dx_d \quad (3.37)$$

with  $\eta(x_d) = \exp(-\gamma \frac{|x_d|}{\sqrt{T}})$  for some  $\gamma = \gamma(\alpha, p, d) \in (0, 1]$ . Since  $\omega = \widehat{\omega}$  in  $B_{\frac{L}{4}}(0)$ , we have  $g_F = 0$  in  $B_{\frac{L}{4}}(0)$  and (3.14) implies  $|g_F| \lesssim |F - R| + \|\nabla \varphi_F^T\|_{L^\infty}$ . In combination with (3.37), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \eta \frac{1}{T} |v_F|^2 + \eta |\nabla v_F|^2 dx_d &\lesssim \int_{\mathbb{R}} \eta |g_F|^2 dx_d \\ &\lesssim \operatorname{dist}^2(F, \operatorname{SO}(d)) \sqrt{T} \exp\left(-\frac{\gamma}{4} \frac{L}{\sqrt{T}}\right). \end{aligned} \quad (3.38)$$

Since  $\eta \geq \exp(-\gamma) > 0$  on  $B_{\sqrt{T}}$ , we obtain

$$\begin{aligned} \int_{\sqrt{T}} \frac{1}{T} |v_F|^2 + |\nabla v_F|^2 dx_d &\lesssim \frac{1}{\sqrt{T}} \int_{\mathbb{R}} \eta \frac{1}{T} |v_F|^2 + \eta |\nabla v_F|^2 dx_d \\ &\stackrel{(3.38)}{\lesssim} \text{dist}^2(F, \text{SO}(d)) \exp\left(-\frac{\gamma}{4} \frac{L}{\sqrt{T}}\right) \end{aligned}$$

and thus (3.19) (recall  $v_F := \widehat{\varphi}_F^T - \varphi_F^T$ ).

*Step 2. Proof of (3.20).* We first note that by (3.29),  $v_{F,G} := \widehat{\varphi}_{F,G}^T - \varphi_{F,G}^T$  solves

$$\frac{1}{T} v_{F,G} - \text{div}(D^2 V(\widehat{\omega}(x_d), F + \nabla \widehat{\varphi}_F^T) \nabla v_{F,G}) = \text{div} g_{F,G} \quad \text{in } \mathbb{R}^d,$$

where

$$g_{F,G} := (D^2 V(\widehat{\omega}, F + \nabla \widehat{\varphi}_F^T) - D^2 V(\omega, F + \nabla \varphi_F^T))(G + \nabla \varphi_{F,G}^T).$$

For  $\eta$  as in Step 1, we have

$$\int_{\mathbb{R}} \eta \frac{1}{T} |v_{F,G}|^2 + \eta |\nabla v_{F,G}|^2 dx_d \lesssim \int_{\mathbb{R}} \eta |g_{F,G}|^2 dx_d. \quad (3.39)$$

The Lipschitz estimate of Lemma 3.8 (ii) yields  $|G + \nabla \varphi_{F,G}^T| \lesssim |G|$  and we have

$$\begin{aligned} |g_{F,G}| &\lesssim |D^2 V(\widehat{\omega}(x_d), F + \nabla \widehat{\varphi}_F^T) - D^2 V(\omega(x_d), F + \nabla \varphi_F^T)| |G| \\ &\leq |D^2 V(\widehat{\omega}(x_d), F + \nabla \widehat{\varphi}_F^T) - D^2 V(\widehat{\omega}(x_d), F + \nabla \varphi_F^T)| |G| \\ &\quad + |D^2 V(\widehat{\omega}(x_d), F + \nabla \varphi_F^T) - D^2 V(\omega(x_d), F + \nabla \varphi_F^T)| |G| \\ &\lesssim |\nabla(\widehat{\varphi}_F^T - \varphi_F^T)| |G| + \mathbf{1}_{\mathbb{R} \setminus B_{\frac{L}{4}}} |G|. \end{aligned}$$

In combination with (3.39) and (3.38), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \eta |\nabla v_{F,G}|^2 dx_d &\lesssim |G|^2 \left( \int_{\mathbb{R}} \eta |\nabla(\widehat{\varphi}_F^T - \varphi_F^T)|^2 dx_d + \int_{\mathbb{R} \setminus B_{\frac{L}{4}}} \eta dx_d \right) \\ &\lesssim |G|^2 (\text{dist}^2(F, \text{SO}(d)) + 1) \sqrt{T} \exp\left(-\frac{\gamma}{4} \frac{L}{\sqrt{T}}\right) \end{aligned} \quad (3.40)$$

and (3.20) follows.

*Step 3. Proof of (3.21).* We first note that by (3.33),  $v_{F,G,H} := \widehat{\varphi}_{F,G,H}^T - \varphi_{F,G,H}^T$  solves

$$\frac{1}{T} v_{F,G,H} - \text{div}(D^2 V(\widehat{\omega}(x_d), F + \nabla \widehat{\varphi}_F^T) \nabla v_{F,G,H}) = \text{div}(g_{F,G,H} + f_{F,G,H}) \quad \text{in } \mathbb{R}^d,$$

where

$$\begin{aligned} g_{F,G,H} &= D^3 V(\widehat{\omega}(x_d), F + \nabla \widehat{\varphi}_F^T)(H + \nabla \widehat{\varphi}_{F,H}^T)(G + \widehat{\varphi}_{F,G}^T) \\ &\quad - D^3 V(\omega(x_d), F + \nabla \varphi_F^T)(H + \nabla \varphi_{F,H}^T)(G + \varphi_{F,G}^T), \\ f_{F,G,H} &= (D^2 V(\widehat{\omega}(x_d), F + \nabla \widehat{\varphi}_F^T) - D^2 V(\omega(x_d), F + \nabla \varphi_F^T)) \nabla \varphi_{F,G,H}^T. \end{aligned}$$

As before, we obtain

$$\int_{\mathbb{R}} \eta \frac{1}{T} |v_{F,G,H}|^2 + \eta |\nabla v_{F,G,H}|^2 dx_d \lesssim \int_{\mathbb{R}} \eta (|g_{F,G,H}|^2 + |f_{F,G,H}|^2) dx_d. \quad (3.41)$$

Furthermore, we compute, with the help of the estimates in Lemma 3.8 and a Taylor expansion,

$$\begin{aligned} |g_{F,G,H}| &\lesssim |H| |\nabla(\widehat{\varphi}_{F,G}^T - \varphi_{F,G}^T)| + |G| |\nabla(\widehat{\varphi}_{F,H}^T - \varphi_{F,H}^T)| \\ &\quad + (\mathbf{1}_{\mathbb{R} \setminus B_{\frac{L}{4}}} + |\nabla(\widehat{\varphi}_F^T - \varphi_F^T)|) |H| |G| \\ |f_{F,G,H}| &\lesssim (\mathbf{1}_{\mathbb{R} \setminus B_{\frac{L}{4}}} + |\nabla(\widehat{\varphi}_F^T - \varphi_F^T)|) |H| |G|. \end{aligned}$$

As a result of the previous two estimates and relying on (3.38) and (3.40), we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \eta (|g_{F,G,H}|^2 + |f_{F,G,H}|^2) dx_d \\ &\lesssim \int_{\mathbb{R}} \eta (|H|^2 |\nabla v_{F,G}|^2 + |G|^2 |\nabla v_{F,H}|^2 + (\mathbf{1}_{\mathbb{R} \setminus B_{\frac{L}{4}}} + |\nabla v_F|^2) |H|^2 |G|^2) dx_d \\ &\lesssim (\text{dist}^2(F, \text{SO}(d)) + 1) |G|^2 |H|^2 \sqrt{T} \exp\left(-\frac{\gamma}{4} \frac{L}{\sqrt{T}}\right) \end{aligned}$$

and in combination with (3.41) we have (3.21).  $\blacksquare$

#### 4. Proofs of qualitative results: Theorem 2.4 and Corollary 2.6

In this section, we denote by  $V$  the matching convex lower bound for  $W$  from Lemma 3.2. We also consider the corresponding RVE approximation

$$V_{\text{hom},L}(\omega, F) = \inf_{\varphi \in H_{\text{per},0}^1(\square_L)} \int_{\square_L} V(\omega(x_d), F + \nabla \varphi) dx. \quad (4.1)$$

In particular, the infimum on the right-hand side is attained by  $\varphi \in H_{\text{per},0}^1(\square_L; \mathbb{R}^d)$  characterized by the equation

$$-\text{div} DV(\omega(x_d), F + \nabla \varphi_F^L) = 0 \quad \text{in } \square_L. \quad (4.2)$$

Under the assumptions of Theorem 2.4, standard homogenization results (see for instance [9] in the convex quadratic case) imply

$$V_{\text{hom}}(F) := \mathbb{E} \left[ \int_{\square} V(\omega(x_d), F + \nabla \varphi_F) dx \right] = \lim_{L \rightarrow \infty} V_{\text{hom},L}(\omega, F) \quad \text{for } \mathbb{P}\text{-a.e. } \omega,$$

where  $\varphi_F: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is the unique solution to

$$-\text{div} DV(\omega(x_d), F + \nabla \varphi_F) = 0 \quad \text{in } \mathbb{R}^d, \text{ for } \mathbb{P}\text{-a.e. } \omega, \quad (4.3)$$

with the conditions

$$\begin{aligned} \int_{\square} \varphi_F dx &= 0, \quad \nabla \varphi_F \text{ is a stationary random field,} \\ \mathbb{E} \left[ \int_{\square} \nabla \varphi_F dx \right] &= 0, \quad \mathbb{E} \left[ \int_{\square} |\nabla \varphi_F|^2 dx \right] < \infty. \end{aligned} \quad (4.4)$$

The existence of  $\varphi_F$  can be obtained by considering the solution  $\varphi_F^T \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  of

$$\frac{1}{T} \varphi_F^T - \operatorname{div} DV(\omega(x_d), F + \nabla \varphi_F^T) = 0 \quad \text{in } \mathbb{R}^d,$$

and passing to the limit  $T \rightarrow \infty$ . In particular, it holds that

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[ \int_{\square} |\nabla \varphi_F^T - \nabla \varphi_F|^2 dx \right] = 0, \quad (4.5)$$

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[ \int_{\square} V(\omega(x_d), F + \nabla \varphi_F^T) dx \right] = V_{\text{hom}}(F). \quad (4.6)$$

*Proof of Theorem 2.4.* Throughout the proof we write  $\lesssim$  if  $\leq$  holds up to a multiplicative constant depending only on  $\alpha$ ,  $d$ , and  $p$ .

*Step 1. Lipschitz estimates for  $\varphi_F^L$  and  $\varphi_F$ .* Let  $\varphi_F^L = \varphi_F^L(\omega, \cdot)$  be given by (4.2). With Lemma 3.6 (iii) and arguing as in Lemma 3.8 (i), we obtain that  $\varphi_F^L$  is one-dimensional in the sense of  $\varphi_F^L(x) = \varphi_F^L(x_d)$  and satisfies

$$\|\nabla \varphi_F^L\|_{L^\infty(\mathbb{R}^d)} \lesssim \operatorname{dist}(F, \operatorname{SO}(d)) \quad \text{for all } L \geq 1. \quad (4.7)$$

In Step 1 of the proof of Lemma 3.8, we showed that  $\|\nabla \varphi_F^T\|_{L^\infty(\mathbb{R}^d)} \lesssim \operatorname{dist}(F, \operatorname{SO}(d))$  and that  $\varphi_F^T$  is one-dimensional. Hence, by (4.5) and weak\* lower semicontinuity of norms, we have

$$\begin{aligned} \|\nabla \varphi_F(\omega, \cdot)\|_{L^\infty(\mathbb{R}^d)} &\leq \liminf_{T \rightarrow \infty} \|\nabla \varphi_F^T(\omega, \cdot)\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim \operatorname{dist}(F, \operatorname{SO}(d)) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \end{aligned} \quad (4.8)$$

This means that we may choose  $\bar{\delta} = \bar{\delta}(\alpha, p, d) > 0$  such that for  $F \in U_{\bar{\delta}}$ , we have

$$\max_{\varphi \in \{\varphi_F^L, \varphi_F^T\}} \|\operatorname{dist}(F + \nabla \varphi, \operatorname{SO}(d))\|_{L^\infty(\mathbb{R}^d)} < \delta \quad \forall L, T \geq 1, \quad (4.9)$$

where  $\delta$  is given by Lemma 3.2. Moreover, for  $\varphi(\omega, \cdot) = \varphi_F(\omega, \cdot)$  the estimate in (4.9) holds  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , and (4.8) improves  $\mathbb{E}[|\nabla \varphi_F|^2] < \infty$  to

$$\mathbb{P}[|\nabla \varphi_F| \leq c \operatorname{dist}(F, \operatorname{SO}(d))] = 1, \quad (4.10)$$

where  $c = c(\alpha, p, d) < \infty$  and  $F \in U_{\bar{\delta}}$ .

*Step 2.* We show that for  $F \in U_{\bar{g}}$ , we have

$$W_{\text{hom},L}(\omega, F) = V_{\text{hom},L}(\omega, F) - \mu \det F. \quad (4.11)$$

This implies (i), in particular,

$$W_{\text{hom}}(F) = \lim_{L \rightarrow \infty} W_{\text{hom},L}(\omega, F) = V_{\text{hom}}(F) - \mu \det F \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (4.12)$$

We first prove

$$W_{\text{hom},L}(\omega, F) \geq V_{\text{hom},L}(\omega, F) - \mu \det(F) \quad \text{for all } F \in \mathbb{R}^{d \times d}. \quad (4.13)$$

For every  $\varphi \in W_{\text{per}}^{1,p}(\square_L; \mathbb{R}^d)$ , inequality (3.1) implies

$$\begin{aligned} \int_{\square_L} W(\omega(x_d), F + \nabla \varphi) dx &\geq \int_{\square_L} V(\omega(x_d), F + \nabla \varphi) - \mu \det(F + \nabla \varphi) dx \\ &\geq V_{\text{hom},L}(\omega, F) - \mu \det(F), \end{aligned}$$

where we use that  $\det$  is a null-Lagrangian in the form  $\int_{\square_L} \det(F + \nabla \varphi) dx = \det(F)$  for all  $\varphi \in W_{\text{per}}^{1,p}(\square_L; \mathbb{R}^d)$  with  $p \geq d$ . Taking the infimum over  $\varphi$ , we obtain (4.13).

Finally, we prove (4.11). Fix  $F \in U_{\bar{g}}$  and let  $\varphi_F^L \in W_{\text{per}}^{1,p}(\square_L; \mathbb{R}^d)$  be as in Step 1. Then, it holds that

$$\begin{aligned} W_{\text{hom},L}(\omega, F) &\leq \int_{\square_L} W(\omega(x_d), F + \nabla \varphi_F^L) dx \\ &\stackrel{(4.9),(3.2)}{=} \int_{\square_L} V(\omega(x_d), F + \nabla \varphi_F^L) - \mu \det(F + \nabla \varphi_F^L) dx \\ &= V_{\text{hom},L}(\omega, F) - \mu \det F, \end{aligned}$$

and in combination with (4.13) we have (4.11).

*Step 3. Proof of (ii).* For  $F \in U_{\bar{g}}$ , let  $\varphi_F: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be characterized by (4.3) and (4.4). A combination of (4.9), (3.2), and Corollary 3.3 (i) implies that  $\varphi_F(\omega, \cdot)$  is a solution to (2.3) for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Moreover, (4.4) and (4.10) imply sublinearity of  $\varphi_F$  in the sense of (2.4). Hence, we have existence of a random field satisfying properties (a)–(c) of Theorem 2.4 (ii). Uniqueness follows as in the proof of Lemma 3.8 (i) (based on uniqueness for solutions to (4.3) and (4.4)).

Finally, combining  $\mathbb{E}[\nabla \varphi_F] = 0$  and the one-dimensionality of  $\varphi_F$  in the form  $\nabla \varphi_F = (\partial_{x_d} \varphi_F) \otimes e_d$ , together with the identity  $\det(F + u \otimes v) = \det(F)(1 + v^T F^{-1}u)$ , we obtain  $\mathbb{E}[\det(F + \nabla \varphi_F)] = \det(F)$  and thus

$$\begin{aligned} W_{\text{hom}}(F) &\stackrel{(4.12)}{=} V_{\text{hom}}(F) - \mu \det(F) = \mathbb{E} \left[ \int_{\square} V(\omega(x_d), F + \nabla \varphi_F) - \mu \det(F + \nabla \varphi_F) \right] \\ &\stackrel{(4.9)}{=} \mathbb{E} \left[ \int_{\square} W(\omega(x_d), F + \nabla \varphi_F) dx \right]. \end{aligned}$$

This completes the proof of (ii).

*Step 4.*  $W_{\text{hom},L}(\omega, \cdot) \in C^3(U_{\bar{\delta}})$ . The mapping  $F \mapsto \varphi_F^L$  belongs to  $C^2(U_{\bar{\delta}}; W_{\text{per}}^{1,\infty}(\square, \mathbb{R}^d))$  and the derivatives  $\varphi_{F,G}^L := D\varphi_F^L G$  and  $\varphi_{F,G,H}^L := D^2\varphi_F^L HG$  are characterized as the unique (up to a constant) solutions to

$$-\operatorname{div}(D^2W(\omega(x_d), F + \nabla\varphi_F^L)(G + \nabla\varphi_{F,G}^L)) = 0 \quad \text{in } \square_L \quad (4.14)$$

and

$$\begin{aligned} & -\operatorname{div}(D^2W(\omega(x_d), F + \nabla\varphi_F^L)\nabla\varphi_{F,G,H}^L) \\ & = \operatorname{div}(D^3W(\omega(x_d), F + \nabla\varphi_F^L)(H + \nabla\varphi_{F,H}^L)(G + \nabla\varphi_{F,G}^L)) \quad \text{in } \square_L. \end{aligned} \quad (4.15)$$

Moreover, they satisfy the Lipschitz bounds

$$\|\nabla\varphi_{F,G}^L\|_{L^\infty(\mathbb{R}^d)} \lesssim |G|, \quad \|\nabla\varphi_{F,G,H}^L\|_{L^\infty(\mathbb{R}^d)} \lesssim |G||H|. \quad (4.16)$$

All of these statements follow analogously to the proof of Lemma 3.8 (with the help of Lemma 3.6 (iii)) and we omit their proofs. As a result of these facts, the claim follows with

$$\begin{aligned} DW_{\text{hom},L}(F)G &= \int_{\square_L} DW(\omega(x_d), F + \nabla\varphi_F^L) \cdot G \, dx, \\ D^2W_{\text{hom},L}(F)HG &= \int_{\square_L} D^2W(\omega(x_d), F + \nabla\varphi_F^L)(H + \nabla\varphi_{F,H}^L) \cdot G \\ D^3W_{\text{hom},L}(F)IHG &= \int_{\square_L} D^3W(\omega(x_d), F + \nabla\varphi_F^L)(I + \nabla\varphi_{F,I}^L)(H + \nabla\varphi_{F,H}^L) \\ & \quad \cdot (G + \nabla\varphi_{F,G}^L) \, dx, \end{aligned}$$

where in the last identity we used (4.14) and (4.15) in the form

$$\begin{aligned} & \int_{\square_L} D^2W(\omega(x_d), F + \nabla\varphi_F^L)\nabla\varphi_{F,H,I}^L \cdot G \, dx \\ & = \int_{\square_L} D^3W(\omega(x_d), F + \nabla\varphi_F^L)(I + \nabla\varphi_{F,I}^L)(H + \nabla\varphi_{F,H}^L) \cdot \nabla\varphi_{F,G}^L \, dx. \end{aligned}$$

*Step 5.* We show that  $W_{\text{hom}} \in C^3(U_{\bar{\delta}})$ . Consider  $F \in U_{\bar{\delta}}$ , where  $\bar{\delta}$  is chosen as in Step 1. We note that in view of (4.9) and (3.2), the unique function  $\varphi_F: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfying (4.3) and (4.4) also satisfies (2.3).

*Substep 5.1. Differentiability of  $U_{\bar{\delta}} \ni F \mapsto \varphi_F$ .* We claim that the mapping  $U_{\bar{\delta}} \ni F \mapsto \varphi_F \in L^2(\Omega, W^{1,\infty}(\square; \mathbb{R}^d))$  is differentiable with derivative  $D\varphi_F G := \varphi_{F,G}$  for every  $G \in \mathbb{R}^{d \times d}$ , where  $\varphi_{F,G}: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is uniquely characterized via (2.9) and (2.10) (with  $H$  replaced by  $G$ ).

For  $G \in \mathbb{R}^{d \times d}$  and  $\omega \in \Omega$ , we consider  $\varphi_{F,G}^T(\omega) \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  as in Lemma 3.8 (ii). Appealing to Corollary 3.3 and Proposition A.1, we have

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[ \int_{\square} |\varphi_{F,G}^T - \varphi_{F,G}|^2 \, dx \right] = 0.$$

Since  $\varphi_{F,G}^T$  satisfies  $\varphi_{F,G}^T(\omega, x) = \varphi_{F,G}^T(\omega, x_d)$  and  $\|\nabla\varphi_{F,G}^T\|_{L^\infty(\mathbb{R}^d)} \lesssim |G|$  (see Lemma 3.8 (ii)), we obtain  $\varphi_{F,G}(\omega, x) = \varphi_{F,G}(\omega, x_d)$  and  $\|\nabla\varphi_{F,G}\|_{L^\infty(\mathbb{R}^d)} \lesssim |G|$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Furthermore, the proof of Lemma 3.8 (ii) implies for  $|G|$  small enough,

$$\begin{aligned} \mathbb{E}\left[\int_{\square} |\nabla(\varphi_{F+G} - \varphi_F - \varphi_{F,G})|^2\right] &= \lim_{T \rightarrow \infty} \mathbb{E}\left[\int_{\square} |\nabla(\varphi_{F+G}^T - \varphi_F^T - \varphi_{F,G}^T)|^2\right] \\ &\leq \limsup_{T \rightarrow \infty} \mathbb{E}[\|\nabla(\varphi_{F+G}^T - \varphi_F^T - \varphi_{F,G}^T)\|_{L^\infty(\square)}^2] \\ &\stackrel{(3.32)}{\lesssim} |G|^4. \end{aligned} \quad (4.17)$$

This means that the mapping  $U_{\bar{\delta}} \in F \mapsto \varphi_F \in L^2(\Omega, H^1(\square; \mathbb{R}^d))$  is differentiable with derivative  $D\varphi_F G := \varphi_{F,G}$  (recall  $\int_{\square} \varphi_F dx = 0$ ). From (4.17) and the weak\* lower semi-continuity, we deduce

$$\mathbb{E}[\|\nabla(\varphi_{F+G} - \varphi_F - \varphi_{F,G})\|_{L^\infty(\square)}^2]^{\frac{1}{2}} \lesssim |G|^2,$$

and thus the claimed differentiability in  $L^2(\Omega, W^{1,\infty}(\square; \mathbb{R}^d))$ .

*Substep 5.2. Differentiability of  $U_{\bar{\delta}} \ni F \mapsto \varphi_{F,G}$ .* We claim that for every  $G \in \mathbb{R}^{d \times d}$  the mapping  $U_{\bar{\delta}} \in F \mapsto \varphi_{F,G} \in L^2(\Omega, W^{1,\infty}(\square; \mathbb{R}^d))$  is differentiable.

For  $G, H \in \mathbb{R}^{d \times d}$  and  $\omega \in \Omega$ , we consider  $\varphi_{F,G,H}^T(\omega) \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  as in Lemma 3.8 (iii). Appealing to Corollary 3.3 and Proposition A.1, we have

$$\lim_{T \rightarrow \infty} \mathbb{E}\left[\int_{\square} |\varphi_{F,G,H}^T - \varphi_{F,G,H}|^2 dx\right] = 0,$$

where  $\varphi_{F,G,H}: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is uniquely characterized by

$$\begin{aligned} &-\operatorname{div}(D^2 W(\omega(x_d), F + \nabla\varphi_F) \nabla\varphi_{F,G,H}) \\ &= \operatorname{div}(D^3 W(\omega(x_d), F + \nabla\varphi_F)(H + \nabla\varphi_{F,H})(G + \nabla\varphi_{F,G})) \quad \text{in } \mathbb{R}^d \end{aligned} \quad (4.18)$$

for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , and

$$\begin{aligned} \int_{\square} \varphi_{F,G,H} dx &= 0, \quad \nabla\varphi_{F,G,H} \text{ is stationary,} \\ \mathbb{E}[\nabla\varphi_{F,G,H}] &= 0, \quad \mathbb{E}[\|\nabla\varphi_{F,G,H}\|^2] < \infty. \end{aligned} \quad (4.19)$$

As in the previous substep, we deduce from Lemma 3.8 (iii) that  $\varphi_{F,G,H}$  is one-dimensional and satisfies  $\|\nabla\varphi_{F,G,H}\|_{L^\infty(\mathbb{R}^d)} \lesssim |G| |H|$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ . Furthermore, we obtain for  $|H| > 0$  small enough,

$$\begin{aligned} \mathbb{E}\left[\int_{\square} |\nabla(\varphi_{F+H,G} - \varphi_{F,G} - \varphi_{F,G,H})|^2\right] \\ \leq \limsup_{T \rightarrow \infty} \mathbb{E}[\|\nabla(\varphi_{F+H,G}^T - \varphi_{F,G}^T - \varphi_{F,G,H}^T)\|_{L^\infty(\square)}^2] \\ \stackrel{(3.35)}{\lesssim} |G|^2 |H|^4. \end{aligned}$$

From the above estimate, we conclude, as in the previous substep, that  $U_{\bar{\delta}} \in F \mapsto \varphi_{F,G} \in L^2(\Omega, W^{1,\infty}(\square; \mathbb{R}^d))$  is differentiable with derivative  $D\varphi_{F,G} H := \varphi_{F,G,H}$ .



*Substep 5.3. Differentiability of  $W_{\text{hom}}$ .* Since  $W_{\text{hom},L}(\omega; \cdot)$  is bounded in  $C^3(U_{\bar{\delta}})$  uniformly in  $L$  and  $\omega$  (see Step 4), we directly conclude from (4.12) that  $W_{\text{hom}} \in C^{2,\gamma}(U_{\bar{\delta}})$  for every  $\gamma \in (0, 1)$ .

The  $L^2(\Omega; W^{1,\infty}(\square; \mathbb{R}^d))$  differentiability of  $U_{\bar{\delta}} \ni F \mapsto \varphi_F$  combined with (4.9) implies

$$DW_{\text{hom}}(F)G = \mathbb{E} \left[ \int_{\square} DW(\omega(x_d), F + \nabla \varphi_F)(G + \nabla \varphi_{F,G}) dx \right],$$

and with the help of

$$\mathbb{E} \left[ \int_{\square} DW(\omega(x_d), F + \nabla \varphi_F) \nabla \varphi_{F,G} dx \right] = 0 \quad (4.20)$$

we obtain the claimed formula (2.7) for  $DW_{\text{hom}}$ . The identity (4.20) is standard and relies on the corrector equation (2.3), together with the stationarity of  $\nabla \varphi_F$  and  $\nabla \varphi_{F,G}$ : recall that for any stationary field  $\mathcal{F}$  and  $\eta \in C_c^\infty(\mathbb{R})$  it holds that

$$\mathbb{E} \left[ \int_{\mathbb{R}} \eta \mathcal{F} \right] = \mathbb{E}[\mathcal{F}] \int_{\mathbb{R}} \eta dx_d \quad \text{and} \quad \mathbb{E} \left[ \int_{\mathbb{R}} \partial_{x_d} \eta \mathcal{F} \right] = \mathbb{E}[\mathcal{F}] \int_{\mathbb{R}} \partial_{x_d} \eta dx_d = 0. \quad (4.21)$$

Hence, by stationarity and one-dimensionality of  $\nabla \varphi_F$  and  $\nabla \varphi_{F,G}$ , we have for any  $\eta \in C_c^\infty(\mathbb{R})$  satisfying  $\int_{\mathbb{R}} \eta dx = 1$ ,

$$\begin{aligned} & \mathbb{E} \left[ \int_{\square} DW(\omega(x_d), F + \nabla \varphi_F) \nabla \varphi_{F,G} dx \right] \\ & \stackrel{(4.21)}{=} \mathbb{E} \left[ \int_{\mathbb{R}} \eta DW(\omega(x_d), F + \nabla \varphi_F) \nabla \varphi_{F,G} dx_d \right] \\ & = \left[ \int_{\mathbb{R}} DW(\omega(x_d), F + \nabla \varphi_F) \nabla(\eta \varphi_{F,G}) \right. \\ & \quad \left. - (\partial_{x_d} \eta) DW(\omega(x_d), F + \nabla \varphi_F) \varphi_{F,G} \otimes e_d dx_d \right] \\ & \stackrel{(2.3)}{=} -\mathbb{E} \left[ \int_{\mathbb{R}} (\partial_{x_d} \eta) DW(\omega(x_d), F + \nabla \varphi_F) \varphi_{F,G} \otimes e_d dx_d \right]. \end{aligned} \quad (4.22)$$

Next, we choose  $\eta \in C_c^\infty(\mathbb{R})$  satisfying  $\int_{\mathbb{R}} \eta dx = 1$  with  $\text{supp } \eta \subset (-\frac{1}{2}, \frac{1}{2})$ , and set  $\eta_R := R^{-1} \eta(\cdot/R)$ . Combining (4.22) with (2.1), (2.5), and the sublinearity of  $\varphi_{F,G}$ , we obtain

$$\begin{aligned} & \left| \mathbb{E} \left[ \int_{\square} DW(\omega(x_d), F + \nabla \varphi_F) \nabla \varphi_{F,G} dx \right] \right| \\ & \stackrel{(4.22)}{=} \left| \mathbb{E} \left[ \int_{\mathbb{R}} (\partial_{x_d} \eta) DW(\omega(x_d), F + \nabla \varphi_F) \varphi_{F,G} \otimes e_d dx_d \right] \right| \\ & \lesssim \mathbb{E} \left[ \frac{1}{R^2} \int_{-\frac{R}{2}}^{\frac{R}{2}} |\varphi_{F,G}| dx_d \right] \lesssim \frac{1}{R} \|\varphi_{F,G}\|_{L^\infty(\square_R)} \xrightarrow{R \rightarrow \infty} 0. \end{aligned}$$

Next, we come to the second and third derivatives of  $W_{\text{hom}}$ . The  $L^2(\Omega; W^{1,\infty}(\square; \mathbb{R}^d))$  differentiability of  $U_{\bar{g}} \ni F \mapsto \varphi_F$ ,  $U_{\bar{g}} \ni F \mapsto \varphi_{F,G}$ , together with (2.7) and (4.9), implies  $W_{\text{hom}} \in C^3(U_{\bar{g}})$  and

$$\begin{aligned} D^2 W_{\text{hom}}(F)HG &= \mathbb{E} \left[ \int_{\square} D^2 W(\omega(x_d), F + \nabla \varphi_F)(H + \nabla \varphi_{F,H}) \cdot G \, dx \right], \\ D^3 W_{\text{hom}}(F)GHI &= \mathbb{E} \left[ \int_{\square} D^3 W(\omega(x_d), F + \nabla \varphi_F)(I + \nabla \varphi_{F,I})(H + \nabla \varphi_{F,H}) \cdot G \, dx \right] \\ &\quad + \mathbb{E} \left[ \int_{\square} D^2 W(\omega(x_d), F + \nabla \varphi_F) \nabla \varphi_{F,H,I} \cdot G \, dx \right]. \end{aligned}$$

Hence, we obtain (2.8) and with the help of

$$\begin{aligned} &\mathbb{E} \left[ \int_{\square} D^2 W(\omega(x_d), F + \nabla \varphi_F) \nabla \varphi_{F,H,I} \cdot G \, dx \right] \\ &= \mathbb{E} \left[ \int_{\square} D^3 W(\omega(x_d), F + \nabla \varphi_F)(I + \nabla \varphi_{F,I})(H + \nabla \varphi_{F,H}) \nabla \varphi_{F,G} \, dx \right] \end{aligned}$$

(which follows from (2.9), (2.10), (4.18), (4.19), and a similar calculation to (4.22)), we arrive at

$$\begin{aligned} D^3 W_{\text{hom}}(F)IHG &= \mathbb{E} \left[ \int_{\square} D^3 W(\omega(x_d), F + \nabla \varphi_F)(I + \nabla \varphi_{F,I})(H + \nabla \varphi_{F,H}) \cdot (G + \nabla \varphi_{F,G}) \, dx \right]. \end{aligned}$$

*Step 6. Strong rank-one convexity.* This follows from the strong convexity of  $V$  and thus  $V_{\text{hom},L}$  and  $V_{\text{hom}}$ , combined with the fact that  $F \mapsto \det(F)$  is rank-one affine (see [39, proof of Theorem 2, Step 6] for details).  $\blacksquare$

*Proof of Corollary 2.6.* We only show  $W_{\text{hom}}(F) \geq \bar{W}_{\text{hom}}(F)$  since the reverse inequality is trivial. For given  $L \in \mathbb{N}$  and  $\varepsilon \in (0, 1)$ , we consider a cut-off function  $\eta \in C_0^\infty(\square_L)$  satisfying  $\eta = 1$  in  $\square_{(1-\varepsilon)L}$  and  $|\nabla \eta| \lesssim \frac{1}{\varepsilon L}$ , where here and for the rest of the proof  $\lesssim$  means  $\leq$  up to a multiplicative constant that depends on  $\alpha$ ,  $d$ , and  $p$ . For  $\varphi_F$  as in Theorem 2.4, we have

$$\begin{aligned} &\inf_{\varphi \in W_0^{1,p}(\square_L; \mathbb{R}^d)} \int_{\square_L} W(\omega(x_d), F + \nabla \varphi(x)) \, dx \\ &\leq \int_{\square_L} W(\omega(x_d), F + \nabla(\eta \varphi_F)) \, dx \\ &\leq \int_{\square_L} W(\omega(x_d), F + \nabla \varphi_F) \, dx \\ &\quad + \frac{1}{|\square_L|} \int_{\square_L \setminus \square_{(1-\varepsilon)L}} W(\omega(x_d), F + \nabla(\eta \varphi_F)) \, dx, \end{aligned}$$

where we used  $W \geq 0$  in the last inequality. By the subadditive ergodic theorem (cf. [1]), the left-hand side converges to  $\bar{W}_{\text{hom}}(F)$  as  $L \rightarrow \infty$  and thus by (2.6) we obtain

$$\begin{aligned} \bar{W}_{\text{hom}}(F) &\leq W_{\text{hom}}(F) \\ &+ \limsup_{L \rightarrow \infty} \frac{1}{|\square_L|} \int_{\square_L \setminus \square_{(1-\varepsilon)L}} W(\omega(x_d), F + \nabla(\eta\varphi_F)) dx \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \end{aligned} \quad (4.23)$$

In order to bound the second term on the right-hand side in (4.23), we observe that

$$\begin{aligned} &\|\text{dist}(F + \nabla(\eta\varphi_F), \text{SO}(d))\|_{L^\infty(\square_L)} \\ &\leq \text{dist}(F, \text{SO}(d)) + \|\nabla\varphi_F\|_{L^\infty(\mathbb{R}^d)} + \|\varphi_F \otimes \nabla\eta\|_{L^\infty(\square_L)} \\ &\stackrel{(2.5)}{\lesssim} \text{dist}(F, \text{SO}(d)) + \frac{1}{\varepsilon L} \|\varphi_F\|_{L^\infty(\square_L)}, \end{aligned}$$

where we use  $|\nabla\eta| \leq (\varepsilon L)^{-1}$  in the second estimate. In view of the sublinearity of the corrector (2.4), we have for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,

$$\|\text{dist}(F + \nabla(\eta\varphi_F), \text{SO}(d))\|_{L^\infty(\square_L)} \lesssim \text{dist}(F, \text{SO}(d))$$

provided  $L$  is sufficiently large (depending on  $\alpha, d, \varepsilon, p$ , and  $\omega$ ). Hence, for  $\bar{\delta} = \bar{\delta}(\alpha, d, p) > 0$  sufficiently small, we have  $\|\text{dist}(F + \nabla(\eta\varphi_F), \text{SO}(d))\|_{L^\infty(\square_L)} \leq \alpha$  for all  $F \in U_{\bar{\delta}}$  and thus

$$\limsup_{L \rightarrow \infty} \frac{1}{|\square_L|} \int_{\square_L \setminus \square_{(1-\varepsilon)L}} W(\omega(x_d), F + \nabla(\eta\varphi_F)) dx \stackrel{(2.1)}{\lesssim} \limsup_{L \rightarrow \infty} \frac{|\square_L \setminus \square_{(1-\varepsilon)L}|}{|\square_L|} \lesssim \varepsilon$$

and the desired inequality  $\bar{W}_{\text{hom}}(F) \leq W_{\text{hom}}(F)$  follows from the arbitrariness of  $\varepsilon > 0$ . ■

## 5. Stochastic intermediate results

In this section, we provide optimal estimates on the growth of the correctors  $\varphi_F^T$ ,  $\varphi_{F,G}^T$ , and  $\varphi_{F,G,H}^T$  which are constructed in Lemma 3.8. For this, we recall in Section 5.1 some results from the literature. In Section 5.2 we state the estimates, and the corresponding proofs are in Section 5.3.

### 5.1. Multiscale decomposition and the spectral gap

The spectral gap inequalities of Definition 2.9, imply  $L^q$ -versions of the spectral gap inequality:

**Lemma 5.1** ([18, Proposition 1.10]). *Suppose  $(\Omega, \mathcal{S}, \mathbb{P})$  satisfies a spectral gap estimate of Definition 2.9 (i) and  $\mathbb{P}$  satisfies (P1). Then there exists  $C = C(\rho) < \infty$  such that for any  $q \geq 1$ ,*

$$\mathbb{E}[|\mathcal{F}(\omega) - \mathbb{E}[\mathcal{F}(\omega)]|^{2q}]^{\frac{1}{2q}} \leq Cq\mathbb{E}\left[\int_{\mathbb{R}} \left| \left( \int_{B_1(s)} \left| \frac{\partial \mathcal{F}}{\partial \omega} \right| \right)^2 ds \right|^q\right]^{\frac{1}{2q}}. \quad (5.1)$$

Moreover, suppose that the  $L$ -periodic approximation (in the sense of Definition 2.8)  $(\Omega_L, \mathcal{S}_L, \mathbb{P}_L)$  of  $(\Omega, \mathcal{S}, \mathbb{P})$  satisfies a (periodic) spectral gap estimate of Definition 2.9 (ii). Then there exists  $C = C(\rho) < \infty$  such that for any  $q \geq 1$ ,

$$\mathbb{E}[|\mathcal{F}(\omega_L) - \mathbb{E}[\mathcal{F}(\omega_L)]|^{2q}]^{\frac{1}{2q}} \leq Cq\mathbb{E}\left[\int_0^L \left|\left(\int_{B_1(s)} \frac{\partial \mathcal{F}}{\partial \omega}\right)^2 ds\right|^q\right]^{\frac{1}{2q}}.$$

**Remark 5.2.** From Lemma 5.1, we will frequently deduce estimates of the form  $\mathbb{E}[\mathcal{F}(\omega)^k]^{\frac{1}{k}} \leq ck$  for all  $k \in \mathbb{N}$  and a random variable  $\mathcal{F} \geq 0$ . By the elementary inequality

$$\frac{(2ce)^{-k}\mathbb{E}[\mathcal{F}(\omega)^k]}{k!} \leq \frac{1}{2^k} \frac{k^k}{e^k k!} \leq \frac{1}{2^k},$$

this implies exponential moments of  $\mathcal{F}$  in the form  $\mathbb{E}[\exp(\frac{1}{C}\mathcal{F})] \leq 2$  with  $C = 2ce$ .

The following lemma is a special case of [8, Lemma 4.8] and can be interpreted as an improved Poincaré inequality in which spatial oscillations of the gradient are taken into account (see also [5, Proposition 6.1] for a related deterministic estimate).

**Lemma 5.3** ([8, Lemma 4.8]). *Let  $m \geq 2$  and  $K \geq 0$ . Let  $u: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function such that*

$$\mathbb{E}\left[\left(\int_{B_1(x_0)} |\nabla u|^2 dx\right)^{\frac{m}{2}}\right]^{\frac{1}{m}} \leq K \quad \text{and} \quad \mathbb{E}\left[\left(\int_{\mathbb{R}} \nabla u \cdot g\right)^m\right]^{\frac{1}{m}} \leq Kr^{-\frac{1}{2}}$$

for all  $r \geq 2$ , all  $x_0 \in \mathbb{R}$ , and all measurable  $g: \mathbb{R} \rightarrow \mathbb{R}$  supported in  $B_r(x_0)$  satisfying

$$\left(\int_{B_r(x_0)} |g|^3\right)^{\frac{1}{3}} \leq r^{-1}.$$

Then it holds that

$$\mathbb{E}\left[\left(\int_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^2 dx\right)^{\frac{m}{2}}\right]^{\frac{1}{m}} \leq cKr^{\frac{1}{2}}.$$

## 5.2. Decay estimates for localized correctors

**Lemma 5.4.** *Let the assumptions of Theorem 2.12 hold and let  $\bar{\delta} = \bar{\delta}(\alpha, p, d) > 0$  be as in Lemma 3.8. Then for any  $F \in U_{\bar{\delta}}$ , we have the following estimates:*

(i) *For all  $r \geq 2$ ,  $x_0 \in \mathbb{R}^d$ ,  $T \geq 2$ , and  $R \geq \sqrt{T}$ , it holds that*

$$\left(\int_{B_r(x_0)} |\varphi_F^T - (\varphi_F^T)_{B_r(x_0)}|^2 dx\right)^{\frac{1}{2}} \leq \mathcal{C}r^{\frac{1}{2}} \text{dist}(F, \text{SO}(d)), \quad (5.2)$$

$$\frac{1}{\sqrt{R}} \left(\int_{B_R(x_0)} |\varphi_F^T|^2 dx\right)^{\frac{1}{2}} + \sqrt{\frac{R}{T}} \left|\int_{B_R(x_0)} \varphi_F^T dx\right| \leq \mathcal{C} \text{dist}(F, \text{SO}(d)). \quad (5.3)$$

(ii) For all  $x_0 \in \mathbb{R}^d$ ,  $T \geq 2$ , and  $R \geq \sqrt{T}$ , it holds that

$$\begin{aligned} & \left( \int_{B_R(x_0)} \frac{1}{T} |\varphi_F^{2T} - \varphi_F^T|^2 + |\nabla \varphi_F^{2T} - \nabla \varphi_F^T|^2 dx \right)^{\frac{1}{2}} \\ & \leq \mathfrak{C} \left( \frac{R}{T} \right)^{\frac{1}{2}} \text{dist}(F, \text{SO}(d)). \end{aligned} \quad (5.4)$$

In (5.2)–(5.4),  $\mathfrak{C} = \mathfrak{C}(x_0, \omega)$  denotes a random field that, in particular depends on  $F$ ,  $T$ , and  $r$  but satisfies the uniform moment bound  $\mathbb{E}[\exp(\frac{1}{c}\mathfrak{C})] \leq 2$ , where  $c = c(\alpha, p, d, \rho) > 0$ .

The proof is presented in Section 5.3.

**Lemma 5.5.** Consider the situation of Lemma 5.4. For any  $F \in U_{\delta}$ , it holds that

(i) For all  $r \geq 2$ ,  $x_0 \in \mathbb{R}^d$ ,  $T \geq 2$ ,  $R \geq \sqrt{T}$ , and  $G \in \mathbb{R}^{d \times d}$ , we have

$$\left( \int_{B_r(x_0)} |\varphi_{F,G}^T - (\varphi_{F,G}^T)_{B_r(x_0)}|^2 dx \right)^{\frac{1}{2}} \leq \mathfrak{C} r^{\frac{1}{2}} |G|, \quad (5.5)$$

$$\frac{1}{\sqrt{R}} \left( \int_{B_R(x_0)} |\varphi_{F,G}^T|^2 dx \right)^{\frac{1}{2}} + \sqrt{\frac{R}{T}} \left| \int_{B_R(x_0)} \varphi_{F,G}^T dx \right| \leq \mathfrak{C} |G|. \quad (5.6)$$

(ii) For all  $x_0 \in \mathbb{R}^d$ ,  $T \geq 2$ ,  $R \geq \sqrt{T}$ , and  $G \in \mathbb{R}^{d \times d}$ , it holds that

$$\left( \int_{B_R(x_0)} \frac{1}{T} |\varphi_{F,G}^{2T} - \varphi_{F,G}^T|^2 + |\nabla \varphi_{F,G}^{2T} - \nabla \varphi_{F,G}^T|^2 dx \right)^{\frac{1}{2}} \leq \mathfrak{C} \left( \frac{R}{T} \right)^{\frac{1}{2}} |G|. \quad (5.7)$$

In (5.5)–(5.7),  $\mathfrak{C} = \mathfrak{C}(x_0, \omega)$  denotes a random field that, in particular, depends on  $F$ ,  $G$ ,  $T$ , and  $r$ , but satisfies the uniform moment bound  $\mathbb{E}[\exp(\frac{1}{c}\mathfrak{C})] \leq 2$ , where  $c = c(\alpha, p, d, \rho) > 0$ .

The proof is presented in Section 5.3.

**Lemma 5.6.** Consider the situation of Lemma 5.4. For any  $F \in U_{\delta}$ , it holds that

(i) For all  $r \geq 2$ ,  $x_0 \in \mathbb{R}^d$ ,  $T \geq 2$ ,  $R \geq \sqrt{T}$ , and  $G, H \in \mathbb{R}^{d \times d}$ , we have

$$\left( \int_{B_r(x_0)} |\varphi_{F,G,H}^T - (\varphi_{F,G,H}^T)_{B_r(x_0)}|^2 dx \right)^{\frac{1}{2}} \leq \mathfrak{C} |G| |H| r^{\frac{1}{2}}, \quad (5.8)$$

$$\frac{1}{\sqrt{R}} \left( \int_{B_R(x_0)} |\varphi_{F,G,H}^T|^2 dx \right)^{\frac{1}{2}} + \sqrt{\frac{R}{T}} \left| \int_{B_R(x_0)} \varphi_{F,G,H}^T dx \right| \leq \mathfrak{C} |H| |G|. \quad (5.9)$$

(ii) For all  $x_0 \in \mathbb{R}^d$ ,  $T \geq 2$ ,  $R \geq \sqrt{T}$ , and  $G, H \in \mathbb{R}^{d \times d}$ , it holds that

$$\begin{aligned} & \left( \int_{B_R(x_0)} \frac{1}{T} |\varphi_{F,G,H}^{2T} - \varphi_{F,G,H}^T|^2 + |\nabla \varphi_{F,G,H}^{2T} - \nabla \varphi_{F,G,H}^T|^2 dx \right)^{\frac{1}{2}} \\ & \leq \mathfrak{C} |G| |H| \left( \frac{R}{T} \right)^{\frac{1}{2}}. \end{aligned} \quad (5.10)$$

In (5.8)–(5.10),  $\mathcal{C} = \mathcal{C}(x_0, \omega)$  denotes a random field that, in particular, depends on  $F, G, H, T$ , and  $r$ , but satisfies the uniform moment bound  $\mathbb{E}[\exp(\frac{1}{c}\mathcal{C})] \leq 2$ , where  $c = c(\alpha, p, d, \rho) > 0$ .

The proof is presented in Section 5.3.

### 5.3. Proofs of stochastic intermediate results

*Proof of Lemma 5.4.* Throughout the proof we write  $\lesssim$  if  $\leq$  holds up to a multiplicative constant depending only on  $\alpha, d, p$ , and  $\rho$ . Suppose  $F \in U_{\bar{\delta}}$  and recall that, according to the proof of Lemma 3.8,  $\varphi_F^T$  is the unique solution to

$$\frac{1}{T}\varphi_F^T - \operatorname{div} DV(\omega(x_d), F + \nabla\varphi_F^T) = 0 \quad \text{in } \mathbb{R}^d, \quad (5.11)$$

where  $V$  corresponds to the matching convex lower bound for  $W$  from Lemma 3.2.

*Step 1. Estimate (5.2).* We consider the random variable  $\mathcal{F}(\omega) = \int_{\mathbb{R}} \partial_{x_d} \varphi_F^T(\omega) \cdot g \, dx_d$ , where  $g: \mathbb{R} \rightarrow \mathbb{R}^d$  is supported in  $B_r(x_0)$  and it satisfies

$$\left( \int_{B_r(x_0)} |g|^3 \, dx_d \right)^{\frac{1}{3}} \leq r^{-1}. \quad (5.12)$$

We compute the derivative of equation (5.11) with respect to a bounded perturbation  $\delta\omega$  supported in  $B_1(s)$ : the change  $\delta\varphi_F^T$  of the corrector under such a perturbation, which is the weak limit in  $H^1(\mathbb{R}^d, \mathbb{R}^d)$  of

$$\delta_t \varphi_F^T(\omega) := \frac{\varphi_F^T(\omega + t\delta\omega) - \varphi_F^T(\omega)}{t},$$

satisfies the linear equation

$$\begin{aligned} \frac{1}{T}\delta\varphi_F^T - \operatorname{div}(D^2V(\omega(x_d), F + \nabla\varphi_F^T)\nabla\delta\varphi_F^T) \\ = \operatorname{div}(\partial_\omega DV(\omega(x_d), F + \nabla\varphi_F^T)\delta\omega) \quad \text{in } \mathbb{R}^d. \end{aligned} \quad (5.13)$$

Let us briefly sketch the argument for this claim. Observe that  $\delta_t \varphi_F^T(\omega)$  satisfies the linear equation

$$\frac{1}{T}\delta_t \varphi_F^T - \operatorname{div} A_t \nabla \delta_t \varphi_F^T = \operatorname{div} g_t,$$

where

$$\begin{aligned} A_t(\omega, x_d) &:= \int_0^1 D^2V(\omega, F + \nabla\varphi_F^T(\omega, x_d) + st\nabla\delta_t\varphi_F^T(\omega)) \, ds, \\ g_t(\omega, x_d) &:= \int_0^1 \partial_\omega DV(\omega + st\delta\omega, F + \nabla\varphi_F^T(\omega, x_d) + t\nabla\delta_t\varphi_F^T(\omega, x_d)) \, ds \, \delta\omega. \end{aligned}$$

The deterministic estimate (3.14) on  $\nabla\varphi_F^T$  (and thus on  $t\nabla\delta_t\varphi_F^T$ ) and the fact that  $\delta\omega$  is bounded and supported in  $B_1(s)$  yield that  $g_t$  is bounded (independent of  $t$  as  $t \downarrow 0$ ) with compact support. In combination with Lemmas 3.5 and 3.6 this implies that  $\|\nabla\delta_t\varphi_F^T\|_{L^\infty(\mathbb{R}^d)} + \|\delta_t\varphi_F^T\|_{H^1(\mathbb{R}^d)}$  is equibounded as  $t \downarrow 0$  and thus

$$A_t(\omega, \cdot) \rightarrow D^2V(\omega, F + \nabla\varphi_F^T(\omega, \cdot)), \quad g_t(\omega, \cdot) \rightarrow \partial_\omega DV(\omega, F + \nabla\varphi_F^T(\omega, \cdot))\delta\omega \quad \text{a.e.}$$

Altogether, we conclude that from every sequence  $(t_j)_j$  with  $t_j \downarrow 0$  we can extract a subsequence (not relabeled) such that  $\delta_{t_j}\varphi_F^T$  converges weakly in  $H^1(\mathbb{R}^d, \mathbb{R}^d)$  to the unique solution  $\delta\varphi_F^T$  of (5.13) and uniqueness of the limit yields weak convergence of the whole sequence.

According to Lemma 3.6,  $\delta\varphi_F^T$  depends only on  $x_d$  and since

$$x_d \mapsto v_d(x_d) := \partial_\omega D_d V(\omega(x_d), F + \nabla\varphi_F^T(x_d))\delta\omega(x_d) \quad (5.14)$$

is bounded and compactly supported,  $(x_d \mapsto \delta\varphi_F^T(x_d)) \in H^1(\mathbb{R}; \mathbb{R}^d)$  is the unique weak solution to

$$\frac{1}{T}\delta\varphi_F^T - \partial_{x_d}(a(x_d)\partial_{x_d}\delta\varphi_F^T) = \partial_{x_d}v_d \quad \text{in } \mathbb{R}, \quad (5.15)$$

where  $a(x_d): \mathbb{R}^d \rightarrow \mathbb{R}^d$  is given by

$$a(x_d)f := (D^2V(\omega(x_d), F + \nabla\varphi_F^T)(f \otimes e_d))e_d. \quad (5.16)$$

We denote by  $h \in H^1(\mathbb{R}; \mathbb{R}^d)$  the unique weak solution to

$$\frac{1}{T}h - \partial_{x_d}(a(x_d)\partial_{x_d}h) = \partial_{x_d}g. \quad (5.17)$$

We compute  $\delta\mathcal{F} := \lim_{t \rightarrow 0} \frac{\mathcal{F}(\omega+t\delta\omega) - \mathcal{F}(\omega)}{t}$  as follows:

$$\begin{aligned} \delta\mathcal{F} &= \int_{\mathbb{R}} \partial_{x_d}\delta\varphi_F^T \cdot g \, dx_d \stackrel{(5.17)}{=} \int_{\mathbb{R}} \frac{1}{T}h \cdot \delta\varphi_F^T + a\partial_{x_d}h \cdot \partial_{x_d}\varphi_F^T \, dx_d \\ &\stackrel{(5.15)}{=} \int_{\mathbb{R}} v_d \cdot \partial_{x_d}h \, dx_d. \end{aligned}$$

Note that  $v_d$  is supported in  $B_1(s)$  (for  $s \in \mathbb{R}$ ). Also, we have the estimate

$$|\partial_\omega D_d V(\omega, F + \nabla\varphi_F^T)| = |\partial_\omega D_d V(\omega, F + \nabla\varphi_F^T) - \partial_\omega D_d V(\omega, R)| \lesssim |F - R + \nabla\varphi_F^T|$$

for an arbitrary  $R \in \text{SO}(d)$ , where we use  $DV(\omega, R) = D \det(R)$  and thus  $\partial_\omega DV(\omega, R) = 0$ . In particular, with the help of (3.14), this implies  $|v_d| \lesssim \text{dist}(F, \text{SO}(d))|\delta\omega|$ . Thus we have

$$\int_{B_1(s)} \left| \frac{\partial\mathcal{F}}{\partial\omega} \right| \lesssim \text{dist}(F, \text{SO}(d)) \int_{B_1(s)} |\partial_{x_d}h| \, dx_d.$$

The  $q$ th version of the spectral gap inequality (see (5.1)) implies

$$\begin{aligned}
 \mathbb{E}[|\mathcal{F} - \mathbb{E}[\mathcal{F}]|^{2q}]^{\frac{1}{2q}} &\lesssim q \operatorname{dist}(F, \operatorname{SO}(d)) \mathbb{E} \left[ \left| \int_{\mathbb{R}} \left( \int_{B_1(s)} |\partial_{x_d} h| dx_d \right)^2 ds \right|^q \right]^{\frac{1}{2q}} \\
 &\lesssim q \operatorname{dist}(F, \operatorname{SO}(d)) \mathbb{E} \left[ \left| \int_{\mathbb{R}} |\partial_{x_d} h|^2 dx_d \right|^q \right]^{\frac{1}{2q}} \\
 &\stackrel{(3.9), (5.17)}{\lesssim} q \operatorname{dist}(F, \operatorname{SO}(d)) \left( \int_{B_r(x_0)} |g|^2 dx_d \right)^{\frac{1}{2}} \\
 &\stackrel{(5.12)}{\lesssim} q \operatorname{dist}(F, \operatorname{SO}(d)) r^{-\frac{1}{2}}. \tag{5.18}
 \end{aligned}$$

Since  $\varphi_F^T$  is a stationary field, we have that  $\mathbb{E}[\varphi_F^T]$  is constant and thus  $\mathbb{E}[\mathcal{F}] = \int_{\mathbb{R}} g \cdot \partial_{x_d} \mathbb{E}[\varphi_F^T] dx_d = 0$ . Hence, (3.14), (5.18), and Lemma 5.3 imply

$$\mathbb{E} \left[ \left( \int_{B_r(x_0)} |\varphi_F^T - (\varphi_F^T)_{B_r(x_0)}|^2 dx \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \lesssim q \operatorname{dist}(F, \operatorname{SO}(d)) r^{\frac{1}{2}},$$

for every  $q \geq 2$  and we obtain (5.2) (see Remark 5.2).

*Step 2. Estimates (5.3).* In view of (5.2) and the triangle inequality it suffices to show

$$\left| \int_{B_R(x_0)} \varphi_F^T dx \right| \leq \mathcal{C} \sqrt{\frac{T}{R}} \operatorname{dist}(F, \operatorname{SO}(d)). \tag{5.19}$$

We define random variables, for  $i \in \{1, \dots, d\}$ ,

$$\mathcal{F}_i(\omega) := \int_{B_R} \varphi_F^T(\omega) \cdot e_i dx_d = \int_{\mathbb{R}} \varphi_F^T(\omega) \cdot f_i dx_d,$$

where  $f_i := \frac{1}{|B_R|} \mathbf{1}_{B_R} e_i$ . Analogously to Step 1, we obtain

$$\delta \mathcal{F}_i = \int_{\mathbb{R}} \delta \varphi_F^T \cdot f_i dx_d = - \int_{\mathbb{R}} v_d \cdot \partial_{x_d} h_i dx_d,$$

where  $v_d$  is given in (5.14) and  $h_i$  solves

$$\frac{1}{T} h_i - \partial_{x_d} (a(x_d) \partial_{x_d} h_i) = f_i \tag{5.20}$$

(see (5.16) for the definition of  $a$ ). Using  $\mathbb{E}[\mathcal{F}_i] = 0$  and (5.1), we obtain

$$\begin{aligned}
 \mathbb{E}[|\mathcal{F}_i|^{2q}]^{\frac{1}{2q}} &\lesssim q \operatorname{dist}(F, \operatorname{SO}(d)) \mathbb{E} \left[ \left| \int_{\mathbb{R}} |\partial_{x_d} h_i|^2 ds \right|^q \right]^{\frac{1}{2q}} \\
 &\stackrel{(3.9), (5.20)}{\lesssim} q \operatorname{dist}(F, \operatorname{SO}(d)) \sqrt{T} \left( \int_{\mathbb{R}} |f_i|^2 ds \right)^{\frac{1}{2}} \\
 &\lesssim q \operatorname{dist}(F, \operatorname{SO}(d)) \sqrt{\frac{T}{R}}. \tag{5.21}
 \end{aligned}$$

From (5.21) we deduce (5.19), which together with (5.2) yields (5.3).



*Step 3. Estimate (5.4).* Using (5.11), we obtain that  $\Psi := \varphi_F^{2T} - \varphi_F^T$  solves

$$\frac{1}{2T}\Psi - \operatorname{div} B(x_d, \nabla\Psi) = \frac{1}{2T}\varphi_F^T \quad \text{in } \mathbb{R}^d,$$

where

$$B(x_d, G) = DV(\omega(x_d), F + \nabla\varphi_F^T + G) - DV(\omega(x_d), F + \nabla\varphi_F^T).$$

Lemma 3.5 (ii) in combination with the fact that  $\varphi_F^T$  depends only on  $x_d$  yields

$$\int_{B_R} \frac{1}{T} |\Psi|^2 + |\nabla\Psi|^2 dx_d \lesssim \frac{1}{RT} \int_{\mathbb{R}} \eta(x_d) |\varphi_F^T|^2 dx_d,$$

where  $\eta(z) := \exp(-\gamma \frac{|z|}{R})$  for some  $\gamma = \gamma(\alpha, p, d) \in (0, 1]$ . Finally, a dyadic decomposition of the integral on the right-hand side, together with (5.3) and the exponential decay of  $\eta$ , yields

$$\begin{aligned} \frac{1}{\sqrt{RT}} \left( \int_{\mathbb{R}} \eta(x_d) |\varphi_F^T|^2 dx_d \right)^{\frac{1}{2}} &\leq \frac{1}{\sqrt{T}} \sum_{i=1}^{\infty} \exp(-\gamma 2^{i-1}) 2^{\frac{i}{2}} \left( \int_{B_{2^i R}} |\varphi_F^T|^2 dx_d \right)^{\frac{1}{2}} \\ &\stackrel{(5.3)}{\leq} \sqrt{\frac{R}{T}} \mathcal{C} \operatorname{dist}(F, \operatorname{SO}(d)) \sum_{i=1}^{\infty} \exp(-\gamma 2^{i-1}) 2^i \end{aligned}$$

and thus (5.4) follows (since  $\sum_{i=1}^{\infty} \exp(-\gamma 2^{i-1}) 2^i \lesssim 1$ ).  $\blacksquare$

*Proof of Lemma 5.5.* Throughout the proof we write  $\lesssim$  if  $\leq$  holds up to a multiplicative constant depending only on  $\alpha, d, p$ , and  $\rho$ . As in the proof of Lemma 5.4, we recall that for  $F \in U_{\delta}^g$  the function  $\varphi_{F,G}^T$  is the unique one-dimensional solution to

$$\frac{1}{T} \varphi_{F,G}^T - \operatorname{div}(D^2V(\omega(x_d), F + \nabla\varphi_F^T)(G + \nabla\varphi_{F,G}^T)) = 0 \quad \text{in } \mathbb{R}^d, \quad (5.22)$$

where  $V$  corresponds to the matching convex lower bound for  $W$  from Lemma 3.2.

*Step 1. Estimate (5.5).* We consider the random variable

$$\mathcal{F}(\omega) = \int_{\mathbb{R}} \partial_{x_d} \varphi_{F,G}^T(\omega) \cdot g dx_d,$$

where  $g: \mathbb{R} \rightarrow \mathbb{R}^d$  is supported in  $B_r(x_0)$  and satisfies (5.12). We compute the derivative of equation (5.22) with respect to a bounded perturbation  $\delta\omega$  supported in  $B_1(s)$ : First, we note that  $\delta\varphi_{F,G}^T \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  is one-dimensional in the sense that  $\delta\varphi_{F,G}^T = \delta\varphi_{F,G}^T(x_d)$  and it is the unique weak solution to the ODE (cf. Step 1 of proof of Lemma 5.4)

$$\begin{aligned} &\frac{1}{T} \delta\varphi_{F,G}^T - \partial_{x_d}(a(x_d) \partial_{x_d} \delta\varphi_{F,G}^T) \\ &= \partial_{x_d} \partial_{\omega} D^2V(\omega(x_d), F + \nabla\varphi_F^T)(G + \nabla\varphi_{F,G}^T \otimes e_d) e_d \delta\omega \\ &\quad + \partial_{x_d} D^3V(\omega(x_d), F + \nabla\varphi_F^T)(\nabla\delta\varphi_F^T)(G + \nabla\varphi_{F,G}^T) e_d \quad \text{in } \mathbb{R}, \end{aligned} \quad (5.23)$$

where  $a(x_d)$  is given in (5.16). We denote by  $h_1$  the unique weak solution in  $H^1(\mathbb{R}; \mathbb{R}^d)$  to

$$\frac{1}{T} h_1 - \partial_{x_d}(a(x_d) \partial_{x_d} h_1) = \partial_{x_d} g \quad \text{in } \mathbb{R} \quad (5.24)$$

and by  $h_2 \in H^1(\mathbb{R}; \mathbb{R}^d)$  the unique weak solution to

$$\begin{aligned} & \frac{1}{T} h_2 - \partial_{x_d}(a(x_d) \partial_{x_d} h_2) \\ &= \partial_{x_d}(D^3 V(x_d, F + \nabla \varphi_F^T)(G + \nabla \varphi_{F,G}^T)(\partial_{x_d} h_1 \otimes e_d)) \quad \text{in } \mathbb{R}. \end{aligned} \quad (5.25)$$

Similarly to Step 1 of the proof of Lemma 5.4, by testing (5.24) with  $\delta \varphi_{F,G}^T$ , (5.23) with  $h_1$ , (5.25) with  $\delta \varphi_F^T$ , and finally (5.15) with  $h_2$ , we obtain

$$\begin{aligned} \delta F &= \int_{\mathbb{R}} \partial_{x_d} \delta \varphi_{F,G}^T \cdot g \, dx_d \\ &= \int_{\mathbb{R}} \partial_{\omega} a(x_d) ((G + \nabla \varphi_{F,G}^T) e_d) \delta \omega \cdot \partial_{x_d} h_1 \, dx_d \\ &\quad + \int_{\mathbb{R}} D^3 V(\omega(x_d), F + \nabla \varphi_F^T) (\nabla \delta \varphi_F^T) (G + \nabla \varphi_{F,G}^T) e_d \cdot \partial_{x_d} h_1 \, dx_d \\ &= \int_{\mathbb{R}} \partial_{\omega} a(x_d) ((G + \nabla \varphi_{F,G}^T) e_d) \delta \omega \cdot \partial_{x_d} h_1 \, dx_d \\ &\quad + \int_{\mathbb{R}} \partial_{\omega} D V(\omega(x_d), F + \nabla \varphi_F^T) \delta \omega e_d \cdot \partial_{x_d} h_2 \, dx_d, \end{aligned}$$

and thus for all  $s \in \mathbb{R}$  (using (3.14) and (3.16)),

$$\int_{B_1(s)} \left| \frac{\partial \mathcal{F}}{\partial \omega} \right| \lesssim \int_{B_1(s)} |G| |\partial_{x_d} h_1| + \text{dist}(F, \text{SO}(d)) |\partial_{x_d} h_2| \, dx_d.$$

Using

$$\int_{\mathbb{R}} |\partial_{x_d} h_2|^2 \lesssim |G|^2 \int_{\mathbb{R}} |\partial_{x_d} h_1|^2 \lesssim |G|^2 \int_{B_r(x_0)} |g|^2,$$

$|F| \lesssim 1$ , and  $\mathbb{E}[\mathcal{F}] = 0$ , we obtain

$$\begin{aligned} \mathbb{E}[|\mathcal{F}|^{2q}]^{\frac{1}{2q}} &\lesssim q |G| \mathbb{E} \left[ \left| \int_{\mathbb{R}} |\partial_{x_d} h_1|^2 ds \right|^q \right]^{\frac{1}{2q}} \\ &\lesssim q |G| \left( \int_{B_r(x_0)} |g|^2 ds \right)^{\frac{1}{2}} \\ &\lesssim q |G| r^{-\frac{1}{2}}, \end{aligned}$$

and thus applying Lemma 5.3 we deduce (5.5).

*Step 2. Estimate (5.6).* This follows analogously to Step 2 of the proof of Lemma 5.4 and using similar arguments to Step 1 of this proof.

*Step 3. Estimate (5.7).* Using equation (5.22), we obtain that  $\Psi := \varphi_{F,G}^{2T} - \varphi_{F,G}^T$  solves

$$\frac{1}{2T}\Psi - \operatorname{div}(D^2V(\omega(x_d), F + \nabla\varphi_F^T)\nabla\Psi) = \frac{1}{2T}\varphi_{F,G}^T + \operatorname{div}\tilde{g} \quad \text{in } \mathbb{R}^d,$$

where

$$\tilde{g} = (D^2V(\omega(x_d), F + \nabla\varphi_F^{2T}) - D^2V(\omega(x_d), F + \nabla\varphi_F^T))(G + \nabla\varphi_{F,G}^{2T}).$$

Note that  $\Psi$  is one-dimensional and solves an ODE corresponding to the above equation. Thus Lemma 3.5 and (3.16) yield

$$\begin{aligned} \int_{B_R} \frac{1}{T}|\Psi|^2 + |\nabla\Psi|^2 dx_d &\lesssim \frac{1}{R} \int_{\mathbb{R}} \eta|\tilde{g}|^2 + \frac{1}{T}\eta|\varphi_{F,G}^T|^2 dx_d \\ &\lesssim \frac{1}{R} \int_{\mathbb{R}} |G|^2\eta|\nabla\varphi_F^{2T} - \nabla\varphi_F^T|^2 + \frac{1}{T}\eta|\varphi_{F,G}^T|^2 dx_d, \end{aligned}$$

where  $\eta(z) = \exp(-\gamma\frac{|z|}{R})$ ,  $\gamma = \gamma(\alpha, p, d) > 0$ . Combining estimates (5.6) and (5.4) with a dyadic decomposition of  $\mathbb{R}$  and the exponential decay of  $\eta$ , we obtain (as in Step 3 of Lemma 5.4)

$$\left( \frac{1}{R} \int_{\mathbb{R}} \eta \left( \frac{1}{T} |\varphi_{F,G}^T|^2 + |G|^2 |\nabla(\varphi_F^{2T} - \varphi_F^T)|^2 \right) dx_d \right)^{\frac{1}{2}} \leq \mathcal{C} |G| \left( \frac{R}{T} \right)^{\frac{1}{2}}.$$

This completes the proof.  $\blacksquare$

*Proof of Lemma 5.6.* Throughout the proof we write  $\lesssim$  if  $\leq$  holds up to a multiplicative constant depending only on  $\alpha, d, p$ , and  $\rho$ . As before, we note that  $\varphi_{F,G,H}^T$  is the unique one-dimensional solution to

$$\begin{aligned} \frac{1}{T}\varphi_{F,G,H}^T - \operatorname{div}(D^2V(\omega(x_d), F + \nabla\varphi_F^T)\nabla\varphi_{F,G,H}^T) \\ = \operatorname{div}(D^3V(\omega(x_d), F + \nabla\varphi_F^T)(H + \nabla\varphi_{F,H}^T)(G + \nabla\varphi_{F,G}^T)) \quad \text{in } \mathbb{R}^d. \end{aligned} \quad (5.26)$$

*Step 1. Estimate (5.8).* We consider the random variable

$$\mathcal{F}(\omega) = \int_{\mathbb{R}} \partial_{x_d}\varphi_{F,G,H}^T(\omega) \cdot g dx_d,$$

where  $g: \mathbb{R} \rightarrow \mathbb{R}^d$  is supported in  $B_r(x_0)$  and satisfies (5.12). We compute the derivative of equation (5.26) with respect to a bounded perturbation  $\delta\omega$  supported in  $B_1(s)$ : First, we note that  $\delta\varphi_{F,G,H}^T \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  is one-dimensional in the sense that  $\delta\varphi_{F,G,H}^T = \delta\varphi_{F,G,H}^T(x_d)$  and it is the unique weak solution to the ODE (cf. Step 1 of proof of Lemma 5.4)

$$\frac{1}{T}\delta\varphi_{F,G,H}^T - \partial_{x_d}a(x_d)\partial_{x_d}\delta\varphi_{F,G,H}^T = \partial_{x_d}v(x_d)e_d \quad \text{in } \mathbb{R},$$

where  $a(x_d)$  is given as in (5.16) and

$$\begin{aligned}
v &= \partial_\omega D^2 V(\omega(x_d), F + \nabla \varphi_F^T) \delta \omega \nabla \varphi_{F,G,H}^T \\
&\quad + D^3 V(\omega(x_d), F + \nabla \varphi_F^T) (\nabla \delta \varphi_F^T) (\nabla \varphi_{F,G,H}^T) \\
&\quad + \partial_\omega D^3 V(\omega(x_d), F + \nabla \varphi_F^T) (H + \nabla \varphi_{F,H}^T) (G + \nabla \varphi_{F,G}^T) \delta \omega \\
&\quad + D^4 V(\omega(x_d), F + \nabla \varphi_F^T) (\nabla \delta \varphi_F^T) (H + \nabla \varphi_{F,H}^T) (G + \nabla \varphi_{F,G}^T) \\
&\quad + D^3 V(\omega(x_d), F + \nabla \varphi_F^T) (\nabla \delta \varphi_{F,H}^T) (G + \nabla \varphi_{F,G}^T) \\
&\quad + D^3 V(\omega(x_d), F + \nabla \varphi_F^T) (H + \nabla \varphi_{F,H}^T) (\nabla \delta \varphi_{F,G}^T).
\end{aligned}$$

As in Step 1 of the proof of Lemma 5.5, we compute  $\delta \mathcal{F} := \lim_{t \rightarrow 0} \frac{\mathcal{F}(\omega+t\delta\omega) - \mathcal{F}(\omega)}{t}$  as follows:

$$\delta \mathcal{F} = \int_{\mathbb{R}} v_d \cdot \partial_{x_d} h_1,$$

where  $h_1 \in H^1(\mathbb{R}; \mathbb{R}^d)$  denotes the unique weak solution of (5.17). In a very similar manner to Step 1 of the proof of Lemma 5.5, thus omitting the details here, we obtain the following estimate, for  $q \geq 1$ :

$$\mathbb{E}[|\mathcal{F}|^{2q}]^{\frac{1}{2q}} \lesssim q |G| |H| r^{-\frac{1}{2}}.$$

This yields the claim.

*Step 2. Estimate (5.9).* These estimates follow analogously to Step 1 of this proof and Step 2 of the proof of Lemma 5.4.

*Step 3. Estimate (5.10).* We note that using equation (5.26),  $\Psi := \varphi_{F,G,H}^{2T} - \varphi_{F,G,H}^T$  solves

$$\frac{1}{2T} \Psi - \operatorname{div}(D^2 V(\omega(x_d), F + \nabla \varphi_F^T) \nabla \Psi) = \frac{1}{2T} \varphi_{F,G,H}^T + \operatorname{div} \tilde{g} \quad \text{in } \mathbb{R}^d, \quad (5.27)$$

where

$$\begin{aligned}
\tilde{g} &= (D^2 V(\omega(x_d), F + \nabla \varphi_F^{2T}) - D^2 V(\omega(x_d), F + \nabla \varphi_F^T)) \nabla \varphi_{F,G,H}^{2T} \\
&\quad + D^3 V(\omega(x_d), F + \nabla \varphi_F^{2T}) (H + \nabla \varphi_{F,H}^{2T}) (G + \nabla \varphi_{F,G}^{2T}) \\
&\quad - D^3 V(\omega(x_d), F + \nabla \varphi_F^T) (H + \nabla \varphi_{F,H}^T) (G + \nabla \varphi_{F,G}^T) \\
&= (D^2 V(\omega(x_d), F + \nabla \varphi_F^{2T}) - D^2 V(\omega(x_d), F + \nabla \varphi_F^T)) \nabla \varphi_{F,G,H}^{2T} \\
&\quad + D^3 V(\omega(x_d), F + \nabla \varphi_F^{2T}) (H + \nabla \varphi_{F,H}^{2T}) (\nabla \varphi_{F,G}^{2T} - \nabla \varphi_{F,G}^T) \\
&\quad + D^3 V(\omega(x_d), F + \nabla \varphi_F^{2T}) (\nabla \varphi_{F,H}^{2T} - \nabla \varphi_{F,H}^T) (G + \nabla \varphi_{F,G}^T) \\
&\quad + (D^3 V(\omega(x_d), F + \nabla \varphi_F^{2T}) - D^3 V(\omega(x_d), F + \nabla \varphi_F^T)) (H + \nabla \varphi_{F,H}^T) (G + \nabla \varphi_{F,G}^T).
\end{aligned}$$

The Lipschitz continuity of  $D^2V(\omega, \cdot)$  and  $D^3V(\omega, \cdot)$ , combined with estimates (3.16) and (3.18), yields

$$|\tilde{g}| \lesssim |H| |G| |\nabla \varphi_F^{2T} - \nabla \varphi_F^T| + |H| |\nabla \varphi_{F,G}^{2T} - \nabla \varphi_{F,G}^T| + |G| |\nabla \varphi_{F,H}^{2T} - \nabla \varphi_{F,H}^T|.$$

We remark that  $\Psi$  is one-dimensional and thus (5.27) boils down to an ODE, hence Lemma 3.5 yields

$$\begin{aligned} & \int_{B_R(x_0)} \frac{1}{T} |\Psi|^2 + |\nabla \Psi|^2 dx_d \\ & \lesssim \frac{1}{R} \left( \int_{\mathbb{R}} \eta |\tilde{g}|^2 dx_d + \frac{1}{T} \int_{\mathbb{R}} \eta |\varphi_{F,G,H}^T|^2 dx_d \right) \\ & \lesssim \frac{1}{R} \int_{\mathbb{R}} \eta (|H|^2 |G|^2 |\nabla \varphi_F^{2T} - \nabla \varphi_F^T|^2) dx_d \\ & \quad + \frac{1}{R} \int_{\mathbb{R}} \eta (|H|^2 |\nabla \varphi_{F,G}^{2T} - \nabla \varphi_{F,G}^T|^2 + |G|^2 |\nabla \varphi_{F,H}^{2T} - \nabla \varphi_{F,H}^T|^2) dx_d \\ & \quad + \frac{1}{RT} \int_{\mathbb{R}} \eta |\varphi_{F,G,H}^T|^2 dx_d, \end{aligned}$$

where  $\eta(z) = \exp(-\gamma \frac{|z|}{R})$  with  $\gamma = \gamma(\alpha, p, d) > 0$ . Using a dyadic decomposition of  $\mathbb{R}$  combined with (5.4) and (5.7), we conclude (5.10).  $\blacksquare$

## 6. Proofs of quantitative results: Theorem 2.12 and Corollary 2.13

### 6.1. Random fluctuations, proof of Theorem 2.12 (i)

*Proof of Theorem 2.12 (i).* Throughout the proof we write  $\lesssim$  if  $\leq$  holds up to a multiplicative constant depending only on  $\alpha, d, p$ , and  $\rho$ .

*Step 1. Preparation.* For given  $\omega_L \in \Omega_L$  and  $F \in \mathbb{R}^{d \times d}$ , we denote by  $\varphi_F^L$  the unique corrector for the matching convex lower bound, i.e., the unique minimizer of (4.1) (and solution to (4.2)). We recall that  $\varphi_F^L \in W^{1,\infty}(\square_L; \mathbb{R}^d)$  is one-dimensional in the sense that  $\varphi_F^L(x) = \varphi_F^L(x_d)$  and satisfies estimate (4.7). Moreover, for all  $F \in U_{\bar{\delta}}$  with  $\bar{\delta}(\alpha, d, p) > 0$  as in Theorem 2.4, we have  $\|\text{dist}(F + \nabla \varphi_F^L, \text{SO}(d))\|_{L^\infty(\square_L)} < \delta$  with  $\delta > 0$  as in Lemma 3.2 and thus

$$\begin{aligned} W_{\text{hom},L}(\omega_L, F) &= \int_{\square_L} W(\omega_L(x_d), F + \nabla \varphi_F^L) dx \\ &= \int_0^L V(\omega_L(x_d), F + \nabla \varphi_F^L(x_d)) dx_d - \mu \det(F). \end{aligned} \quad (6.1)$$

*Step 2. Fluctuations of  $W_{\text{hom},L}$ , proof of (2.13).* We define a random variable  $\mathcal{F}: \Omega_L \rightarrow [0, \infty)$  by

$$\mathcal{F}(\omega_L) := W_{\text{hom},L}(\omega_L, F) \stackrel{(6.1)}{=} \int_{(0,L)} V(\omega_L(x_d), F + \nabla \varphi_F^L) dx_d - \mu \det(F).$$

We compute its derivative with respect to a bounded periodic perturbation  $\delta\omega_L$  supported in  $B_1(s) + LZ$ :

$$\begin{aligned} \delta\mathcal{F}(\omega_L, \delta\omega_L) &:= \lim_{t \rightarrow 0} \frac{\mathcal{F}(\omega_L + t\delta\omega_L) - \mathcal{F}(\omega_L)}{t} \\ &= \int_{(0,L)} \partial_\omega V(\omega_L(x_d), F + \nabla\varphi_F^L)\delta\omega_L \\ &\quad + DV(\omega_L(x_d), F + \nabla\varphi_F^L) \cdot \nabla\delta\varphi_F^L dx_d, \end{aligned} \quad (6.2)$$

where  $\delta\varphi_F^L$  is an  $L$ -periodic function with zero mean (solution to (6.3)). Hence, the second term on the right-hand side in (6.2) vanishes. In order to estimate the first term on the right-hand side above we make use of  $W(\cdot, R) = 0$ ,  $DW(\cdot, R) = 0$  for all  $R \in \text{SO}(d)$  (hence  $\partial_\omega V(\cdot, R) = 0$ ,  $\partial_\omega DV(\cdot, R) = 0$ ), and thus we obtain with  $R_F \in \text{SO}(d)$  such that  $|R_F - F| = \text{dist}(F, \text{SO}(d))$  that

$$\begin{aligned} &\partial_\omega V(\omega_L(x_d), F + \nabla\varphi_F^L) \\ &= \int_0^1 (1-t) D^2 \partial_\omega V(\omega_L(x_d), R_F + t(F - R_F + \nabla\varphi_F^L)) \\ &\quad \times [F - R_F + \nabla\varphi_F^L, F - R_F + \nabla\varphi_F^L] dt \\ &\leq \frac{1}{2} \|\partial_\omega D^2 V\|_{C^0(\bar{U}_\omega)} |F - R_F + \nabla\varphi_F^L|^2. \end{aligned}$$

Therefore, (4.7) and  $|R_F - F| = \text{dist}(F, \text{SO}(d))$  yield

$$\int_{B_1(s)} \left| \frac{\partial\mathcal{F}}{\partial\omega_L} \right| = \sup_{\delta\omega_L} |\delta\mathcal{F}(\omega_L, \delta\omega_L)| \lesssim \frac{1}{L} \text{dist}^2(F, \text{SO}(d)).$$

Finally, the  $q$ th version of the spectral gap inequality in the form of (5.1) yields

$$\mathbb{E}_L[|\mathcal{F} - \mathbb{E}_L[\mathcal{F}]|^{2q}]^{\frac{1}{2q}} \lesssim qL^{-1} \text{dist}^2(F, \text{SO}(d)).$$

This implies the claim (2.13) (see Remark 5.2).

*Step 3. Fluctuations of  $DW_{\text{hom},L}$ , proof of (2.14).* As in the previous step it suffices to show the corresponding claim for  $DV$  instead of  $DW$ . For  $F \in U_{\bar{\delta}}$  and  $G \in \mathbb{R}^{d \times d}$ , we consider the random variable

$$\mathcal{F}(\omega_L) = \int_{(0,L)} DV(\omega_L(x_d), F + \nabla\varphi_F^L) \cdot G dx_d = DV_{\text{hom},L}(F)[G].$$

We compute its derivative with respect to a bounded periodic perturbation  $\delta\omega_L$  supported in  $B_1(s) + LZ$ :

$$\delta F(\omega_L) = \int_{(0,L)} \partial_\omega DV(\omega_L(x_d), F + \nabla\varphi_F^L)\delta\omega_L \cdot G + \widehat{\mathbb{L}}_L(\omega_L(x_d))\nabla\delta\varphi_F^L \cdot G dx_d.$$

where

$$\widehat{\mathbb{L}}_L(\omega_L(x_d)) := D^2V(\omega_L(x_d), F + \nabla\varphi_F^L(x_d)),$$

and  $\delta\varphi_F^L \in W_{\text{per},0}^{1,2}(\square_L)$  solves

$$-\operatorname{div} \widehat{\mathbb{L}}_F^L \nabla \delta\varphi_F^L = \operatorname{div}(\partial_\omega DV(\omega_L(x_d), F + \nabla\varphi_F^L)\delta\omega_L) \quad \text{in } \square_L. \quad (6.3)$$

As in [21, proof of Theorem 9a] we introduce the auxiliary function  $h \in W_{\text{per},0}^{1,2}(\square_L)$  satisfying

$$-\operatorname{div} \widehat{\mathbb{L}}_L \nabla h = \operatorname{div} \widehat{\mathbb{L}}_L G \quad \text{in } \square_L \quad (6.4)$$

and obtain (by testing (6.4) with  $\delta\varphi_F^L$  and (6.3) with  $h$ )

$$\delta F(\omega_L) = \int_{\square_L} \partial_\omega DV(\omega_L(x_d), F + \nabla\varphi_F^L)\delta\omega_L \cdot (G + \nabla h) dx.$$

Appealing to  $\partial_\omega DV(\omega, R) = 0$  for all  $R \in \operatorname{SO}(d)$ , we have

$$\begin{aligned} & \|\partial_\omega DV(\omega_L, F + \nabla\varphi_F^L)\|_{L^\infty(\square_L)} \\ &= \|\partial_\omega DV(\omega_L, F + \nabla\varphi_F^L)\|_{L^\infty(0,L)} \\ &\lesssim \operatorname{dist}(F, \operatorname{SO}(d)), \end{aligned}$$

and, by Lemma 3.6 (iii), it holds that  $h(x) = h(x_d)$  with

$$\|\nabla h\|_{L^\infty(\square_L)} \lesssim \|\nabla h\|_{L^\infty(0,L)} \lesssim |G|.$$

Combining the previous three estimates, we obtain

$$\sup_s \int_{B_1(s)} \left| \frac{\partial \mathcal{F}}{\partial \omega_L} \right| \lesssim \operatorname{dist}(F, \operatorname{SO}(d)) |G| L^{-1},$$

and thus, appealing to the  $q$ th version of the spectral gap inequality in the form of (5.1),

$$\mathbb{E}_L[|\mathcal{F} - \mathbb{E}_L[\mathcal{F}]|^{2q}]^{\frac{1}{2q}} \lesssim qL^{-1} \operatorname{dist}(F, \operatorname{SO}(d)) |G|.$$

Hence, (2.14) follows.

*Step 4. Fluctuations of  $D^2W_{\text{hom},L}$ , proof of (2.15).* For  $G, H \in \mathbb{R}^{d \times d}$ , we define the random variable

$$\begin{aligned} \mathcal{F}(\omega_L) &:= D^2V_{\text{hom},L}(\omega_L, F)H \cdot G \\ &= \int_{[0,L]} D^2V(\omega_L(x_d), F + \nabla\varphi_F^L)(H + \nabla\varphi_{F,H}^L) \cdot G dx_d. \end{aligned}$$

We compute its derivative with respect to a bounded periodic perturbation  $\delta\omega_L$  supported in  $B_1(s) + LZ$ :

$$\begin{aligned}\delta\mathcal{F} &= \int_{(0,L)} \partial_\omega \widehat{\mathbb{L}}_F^L \delta\omega_L (H + \nabla\varphi_{F,H}^L) \cdot G \, dx_d \\ &\quad + \int_{(0,L)} D^3V(\omega_L(x_d), F + \nabla\varphi_F^L) \nabla\delta\varphi_F^L (H + \nabla\varphi_{F,H}^L) \cdot G \, dx_d \\ &\quad + \int_{(0,L)} \widehat{\mathbb{L}}_F^L \nabla\delta\varphi_{F,H}^L \cdot G \, dx_d \\ &=: I_1 + I_2 + I_3,\end{aligned}\tag{6.5}$$

where  $\delta\varphi_F^L$  solves (6.3) and  $\delta\varphi_{F,H}^L \in W_{\text{per},0}^{1,2}(\square_L; \mathbb{R}^d)$  solves

$$-\operatorname{div}(\widehat{\mathbb{L}}_F^L \nabla\delta\varphi_{F,H}^L) = \operatorname{div} f \quad \text{in } \square_L,\tag{6.6}$$

where

$$f := \partial_\omega \widehat{\mathbb{L}}_F^L (H + \nabla\varphi_{F,H}^L) \delta\omega_L + D^3V(\omega_L(x_d), F + \nabla\varphi_F^L) \nabla\delta\varphi_F^L (H + \nabla\varphi_{F,H}^L).$$

We estimate the three terms on the right-hand side of (6.5) separately: For the first term, we use

$$|I_1| \lesssim |G| \|H + \nabla\varphi_{F,H}^L\|_{L^\infty(\square_L)} \int_0^L |\delta\omega_L| \, dx_1 \stackrel{(4.16)}{\lesssim} |G| |H| \int_0^L |\delta\omega_L| \, dx_1.$$

To estimate  $I_2$ , we introduce  $h_1 \in W_{\text{per},0}^{1,2}(\square_L; \mathbb{R}^d)$  satisfying

$$-\operatorname{div} \widehat{\mathbb{L}}_L \nabla h_1 = \operatorname{div} D^3V(\omega_L(x_d), F + \nabla\varphi_F^L) (H + \nabla\varphi_{F,H}^L) G \quad \text{in } \square_L,\tag{6.7}$$

and obtain (by testing (6.7) with  $\delta\varphi_F^L$  and (6.3) with  $h_1$ )

$$\begin{aligned}|I_2| &= \left| \int_{\square_L} \partial_\omega DV(\omega_L(x_d), F + \nabla\varphi_F^L) \delta\omega_L \cdot \nabla h_1 \, dx \right| \\ &\lesssim \operatorname{dist}(F, \operatorname{SO}(d)) |H| |G| \int_0^L |\delta\omega_L|.\end{aligned}$$

It remains to estimate  $|I_3|$ . Considering  $h$ , the solution to (6.4), we obtain (testing (6.4) with  $\delta\varphi_{F,H}^L$  and (6.6) with  $h$ )

$$\begin{aligned}I_3 &= \int_{(0,L)} \partial_\omega \widehat{\mathbb{L}}_F^L \delta\omega_L (H + \nabla\varphi_{F,H}^L) \cdot \nabla h \, dx_d \\ &\quad + \int_{(0,L)} D^3V(\omega_L(x_d), F + \nabla\varphi_F^L) \nabla\delta\varphi_F^L (H + \nabla\varphi_{F,H}^L) \cdot \nabla h \, dx_d \\ &=: I_4 + I_5.\end{aligned}$$



As before, we have  $|I_4| \lesssim |H| |G| \int_{(0,L)} |\delta\omega_L| dx_d$ . The same argument as for the estimate of  $I_2$  but with  $G$  replaced by  $\nabla h$  yields

$$|I_5| \lesssim \text{dist}(F, \text{SO}(d)) |G| |H| \int_{(0,L)} |\delta\omega_L| dx_d.$$

Collecting all these estimates, we obtain

$$\sup_s \int_{B_1(s)} \left| \frac{\partial \mathcal{F}}{\partial \omega_L} \right| \lesssim (1 + \text{dist}(F, \text{SO}(d))) |G| |H| L^{-1}$$

and the claim follows via a further application of the spectral gap inequality.  $\blacksquare$

## 6.2. Systematic error, proof of Theorem 2.12 (ii)

In order to treat the systematic error, we introduce a localized version of the RVE approximation: for any configuration  $\omega: \mathbb{R} \rightarrow \mathbb{R}$ , and for a compactly supported cut-off function  $\eta$  satisfying

$$\eta \in C_c^\infty(\mathbb{R}), \quad \eta \geq 0, \quad \int_{\mathbb{R}} \eta(x_d) dx_d = 1, \quad (6.8)$$

we set

$$W_{\eta,T}(F) = \int_{\mathbb{R}} \eta(x_d) W(\omega(x_d), F + \nabla \varphi_F^T(x_d)) dx_d \quad \text{for } F \in U_{\bar{\delta}}, \quad (6.9)$$

where the localized corrector  $\varphi_F^T$  and  $\bar{\delta}$  are as in Lemma 3.8. Furthermore, we define a cut-off for a realization  $\omega \in \Omega$ :

$$\pi_L \omega(x_d) = \begin{cases} \omega(x_d) & \text{if } x_d \in [-\frac{L}{4}, \frac{L}{4}], \\ 0 & \text{otherwise.} \end{cases} \quad (6.10)$$

We define  $W_{\eta,T}(\pi_L \omega, F)$  by replacing  $\omega$  by  $\pi_L \omega$  in (6.9). Note that  $\omega$  and  $\omega_L$  admit the same distribution on  $B_{\frac{L}{4}}(0)$  according to Definition 2.8. Thus we have  $\mathbb{E}_L[\mathcal{F}(\pi_L \omega_L)] = \mathbb{E}[\mathcal{F}(\pi_L \omega)]$  for any random variable  $\mathcal{F}$ . This motivates the following decomposition of the systematic error of  $W_{\text{hom}}$ :

$$\begin{aligned} \mathbb{E}_L[W_{\text{hom},L}(F)] - W_{\text{hom}}(F) &= \mathbb{E}_L[W_{\text{hom},L}(F)] - \mathbb{E}_L[W_{\eta,T}(F)] \\ &\quad + \mathbb{E}_L[W_{\eta,T}(F)] - \mathbb{E}_L[W_{\eta,T}(\pi_L \omega, F)] \\ &\quad + \mathbb{E}[W_{\eta,T}(\pi_L \omega, F)] - \mathbb{E}[W_{\eta,T}(F)] \\ &\quad + \mathbb{E}[W_{\eta,T}(F)] - W_{\text{hom}}(F). \end{aligned} \quad (6.11)$$

The first and last terms on the right-hand side in (6.11) measure the localization error and the middle terms correspond to the error made by the ensemble periodization. In the following we treat these two sources of error separately; see Lemmas 6.2 and 6.3 below.

In order to estimate the systematic error related to  $DW_{\text{hom}}$  and  $D^2W_{\text{hom}}$  we use an analogous decomposition to (6.11), where  $DW_{\eta,T}$  and  $D^2W_{\eta,T}$  are given via

$$\begin{aligned} & DW_{\eta,T}(F) \cdot G \\ &= \int_{\mathbb{R}} \eta(x_d) DW(\omega(x_d), F + \nabla\varphi_F^T(x_d)) \cdot (G + \nabla\varphi_{F,G}^T(x_d)) dx_d, \end{aligned} \quad (6.12)$$

$$\begin{aligned} & D^2W_{\eta,T}(F)H \cdot G \\ &= \int_{\mathbb{R}} \eta(x_d) D^2W(\omega(x_d), F + \nabla\varphi_F^T(x_d))(H + \nabla\varphi_{F,H}^T(x_d)) \cdot (G + \nabla\varphi_{F,G}^T(x_d)) \\ & \quad + \eta(x_d) DW(\omega(x_d), F + \nabla\varphi_F^T(x_d)) \cdot \nabla\varphi_{F,G,H}^T(x_d) dx_d. \end{aligned} \quad (6.13)$$

**Remark 6.1.** Assumption (6.8) and the stationarity of the random field  $(x_d, \omega) \mapsto \mathcal{F}(x_d, \omega) := W(\omega(x_d), F + \nabla\varphi_F^T(x_d))$  imply

$$\mathbb{E}[W_{\eta,T}(F)] = \mathbb{E}\left[\int_{\mathbb{R}} \eta(x_d) \mathcal{F}(x_d, \omega) dx_d\right] = \mathbb{E}[\mathcal{F}] \int_{\mathbb{R}} \eta dx_d = \mathbb{E}[\mathcal{F}].$$

Hence,  $\mathbb{E}[W_{\eta,T}]$  coincides for any choice of  $\eta$  satisfying (6.8). Clearly, analogous statements hold for  $DW_{\eta,T}$  and  $D^2W_{\eta,T}$ .

**Lemma 6.2** (Cost of localization). *Suppose the assumptions of Theorem 2.12 are satisfied. Let  $L \geq 3$  and let  $\mathbb{P}_L$  be an  $L$ -periodic approximation of  $\mathbb{P}$  in the sense of Definition 2.8, and denote by  $W_{\text{hom},L}$  the corresponding representative volume approximation. There exists  $c = c(\alpha, p, d, \rho) \in [1, \infty)$  such that for all  $F \in U_{\bar{\delta}}$  with  $\bar{\delta} > 0$  as in Lemma 3.8 the following estimates are valid:*

(i) For all  $T \geq \sqrt{2}$ , we have for  $\ell \in \{0, 1, 2\}$  and all  $\eta$  satisfying (6.8),

$$|\mathbb{E}[D^\ell W_{\eta,T}(F)] - D^\ell W_{\text{hom}}(F)| \leq c \text{dist}^{2-\ell}(F, \text{SO}(d)) \frac{1}{\sqrt{T}}. \quad (6.14)$$

(ii) For all  $T \in [\sqrt{2}, L^2]$ , we have for  $\ell \in \{0, 1, 2\}$  and all  $\eta$  satisfying (6.8) with  $\text{supp } \eta \subset (-\frac{L}{2}, \frac{L}{2})$ ,

$$|\mathbb{E}_L[D^\ell W_{\eta,T}(F)] - \mathbb{E}_L[D^\ell W_{\text{hom},L}(F)]| \leq c \text{dist}^{2-\ell}(F, \text{SO}(d)) \frac{1}{\sqrt{T}}.$$

**Lemma 6.3** (Cost of ensemble periodization). *Suppose the assumptions of Theorem 2.12 are satisfied. Let  $L \geq 3$  and let  $\mathbb{P}_L$  be an  $L$ -periodic approximation of  $\mathbb{P}$  in the sense of Definition 2.8, and denote by  $W_{\text{hom},L}$  the corresponding representative volume approximation. There exists  $c_1 = c_1(\alpha, p, d) \in [1, \infty)$  such that for all  $F \in U_{\bar{\delta}}$  with  $\bar{\delta} > 0$  as in Lemma 3.8 the following estimates are valid:*

$$|\mathbb{E}[D^\ell W_{\eta_L,T}(\pi_L(\cdot), F) - D^\ell W_{\eta_L,T}(\cdot, F)]| \leq c \text{dist}^{2-\ell}(F, \text{SO}(d)) \frac{1}{L}, \quad (6.15)$$

where  $T := (\frac{1}{2c} \frac{L}{\ln(L)})^2$  with  $c = c(\alpha, p, d) \in [4, \infty)$  as in Lemma 3.9 and  $\eta_L$  denotes a nonnegative weight supported in  $B_{\sqrt{T}}(0)$  with  $|\eta_L| \leq \frac{2}{\sqrt{T}}$ . Moreover, the same estimates are valid with  $\mathbb{E}$  replaced by  $\mathbb{E}_L$ .

*Proof of Theorem 2.12 (ii).* Follows directly from Lemmas 6.2 and 6.3 and the decomposition of the systematic error (6.11). ■

*Proof of Lemma 6.2.* We provide the argument only for part (i); the argument for part (ii) follows the same pattern. Throughout the proof we write  $\lesssim$  if  $\leq$  holds up to a multiplicative constant depending only on  $\alpha$ ,  $d$ ,  $p$ , and  $\rho$ . Moreover, we suppose that  $\bar{\delta}$  is as in Lemma 3.8.

*Step 1. Estimate (6.14) with  $\ell = 0$ .* Since  $\lim_{T \rightarrow \infty} \mathbb{E}[W_{\eta,T}(F)] = W_{\text{hom}}(F)$ , which follows from  $\mathbb{E}[f_{\square} \det(F + \nabla \varphi_F^T)] = \det(F)$ , Remark 6.1, (4.6), and (3.2), we have

$$W_{\text{hom}}(F) - \mathbb{E}[W_{\eta,T}(F)] = \sum_{i=0}^{\infty} (\mathbb{E}[W_{\eta,2^{i+1}T}(F)] - \mathbb{E}[W_{\eta,2^i T}(F)]),$$

and thus it is sufficient to prove

$$|\mathbb{E}[W_{\eta,2T}(F)] - \mathbb{E}[W_{\eta,T}(F)]| \lesssim \text{dist}^2(F, \text{SO}(d)) \frac{1}{\sqrt{T}} \quad \text{for all } T \geq 1.$$

We compute

$$\begin{aligned} & W_{\eta,2T}(F) - W_{\eta,T}(F) \\ &= \int_{\mathbb{R}} \eta (W(\omega(x_d), F + \nabla \varphi_F^{2T}) - W(\omega(x_d), F + \nabla \varphi_F^T)) \\ &\quad - \eta DW(\omega(x_d), F + \nabla \varphi_F^T) \cdot (\nabla \varphi_F^{2T} - \nabla \varphi_F^T) \\ &\quad + \eta DW(\omega(x_d), F + \nabla \varphi_F^T) \cdot (\nabla \varphi_F^{2T} - \nabla \varphi_F^T) dx_d \\ &= \int_{\mathbb{R}} \eta (W(\omega(x_d), F + \nabla \varphi_F^{2T}) - W(\omega(x_d), F + \nabla \varphi_F^T)) \\ &\quad - \eta DW(\omega(x_d), F + \nabla \varphi_F^T) \cdot (\nabla \varphi_F^{2T} - \nabla \varphi_F^T) \\ &\quad - \frac{1}{T} \eta \varphi_F^T \cdot (\varphi_F^{2T} - \varphi_F^T) - DW(\omega(x_d), F + \nabla \varphi_F^T) e_d \cdot \partial_{x_d} \eta (\varphi_F^{2T} - \varphi_F^T) dx_d. \end{aligned}$$

Taking the expectation of the estimate above and using  $\mathbb{E}[\int_{\mathbb{R}} \partial_{x_d} \eta \mathcal{F}] = \mathbb{E}[\mathcal{F}] \int_{\mathbb{R}} \partial_{x_d} \eta dx_d = 0$  for any stationary field  $\mathcal{F}$ , we obtain

$$\begin{aligned} & |\mathbb{E}[W_{\eta,2T}(F)] - \mathbb{E}[W_{\eta,T}(F)]| \\ &\lesssim \mathbb{E} \left[ \int_{\mathbb{R}} \eta |\nabla \varphi_F^{2T} - \nabla \varphi_F^T|^2 dx_d \right] + \mathbb{E} \left[ \int_{\mathbb{R}} \frac{1}{T} \eta |\varphi_F^T| |\varphi_F^{2T} - \varphi_F^T| dx_d \right] \\ &\leq \mathbb{E} \left[ \int_{\mathbb{R}} \eta |\nabla \varphi_F^{2T} - \nabla \varphi_F^T|^2 dx_d \right] \\ &\quad + \mathbb{E} \left[ \int_{\mathbb{R}} \frac{1}{T} \eta |\varphi_F^{2T} - \varphi_F^T|^2 dx_d \right]^{\frac{1}{2}} \mathbb{E} \left[ \int_{\mathbb{R}} \frac{1}{T} \eta |\varphi_F^T|^2 dx_d \right]^{\frac{1}{2}}. \end{aligned}$$

According to Remark 6.1, we may choose  $\eta$  supported in  $B_{\sqrt{T}}(0)$  with  $|\eta| \lesssim \frac{1}{\sqrt{T}}$ . In this way, (5.3) and (5.4) yield the claim.

Step 2. Estimate (6.14) with  $\ell = 1$ . As in Step 1, it is sufficient to show that

$$|\mathbb{E}[DW_{\eta,2T}(F) \cdot G] - \mathbb{E}[DW_{\eta,T}(F) \cdot G]| \lesssim \text{dist}(F, \text{SO}(d)) |G| \frac{1}{\sqrt{T}}.$$

We compute

$$\begin{aligned} & DW_{\eta,2T}(F) \cdot G - DW_{\eta,T}(F) \cdot G \\ &= \int_{\mathbb{R}} \eta DW(\omega(x_d), F + \nabla\varphi_F^{2T}) \cdot (G + \nabla\varphi_{F,G}^{2T}) \\ &\quad - \eta DW(\omega(x_d), F + \nabla\varphi_F^T) \cdot (G + \nabla\varphi_{F,G}^T) dx_d \\ &= \int_{\mathbb{R}} \eta (DW(\omega(x_d), F + \nabla\varphi_F^{2T}) - DW(\omega(x_d), F + \nabla\varphi_F^T)) \cdot (G + \nabla\varphi_{F,G}^T) dx_d \\ &\quad + \int_{\mathbb{R}} \eta DW(\omega(x_d), F + \nabla\varphi_F^{2T}) \cdot (\nabla\varphi_{F,G}^{2T} - \nabla\varphi_{F,G}^T) dx_d \\ &= \int_{\mathbb{R}} \eta (DW(\omega(x_d), F + \nabla\varphi_F^{2T}) - DW(\omega(x_d), F + \nabla\varphi_F^T)) \cdot (G + \nabla\varphi_{F,G}^T) dx_d \\ &\quad - \int_{\mathbb{R}} \eta D^2W(\omega(x_d), F + \nabla\varphi_F^T) (\nabla\varphi_F^{2T} - \nabla\varphi_F^T) \cdot (G + \nabla\varphi_{F,G}^T) dx_d \\ &\quad + \int_{\mathbb{R}} \eta DW(\omega(x_d), F + \nabla\varphi_F^{2T}) \cdot (\nabla\varphi_{F,G}^{2T} - \nabla\varphi_{F,G}^T) dx_d \\ &\quad + \int_{\mathbb{R}} \eta D^2W(\omega(x_d), F + \nabla\varphi_F^T) (\nabla\varphi_F^{2T} - \nabla\varphi_F^T) \cdot (G + \nabla\varphi_{F,G}^T) dx_d \\ &=: I_0 + I_1 + I_2 + I_3. \end{aligned}$$

We treat the four integrals on the right-hand side using the following arguments. In particular, for this purpose we rely on the cancellations coming from (5.4). Specifically, a Taylor expansion, combined with (3.16), (5.4), and a suitable choice for  $\eta$  (that is,  $\text{supp } \eta \subset B_{\sqrt{T}}(0)$  with  $|\eta| \lesssim \frac{1}{\sqrt{T}}$ ) yields

$$\mathbb{E}[I_0 + I_1] \lesssim |G| \mathbb{E} \left[ \int_{B_{\sqrt{T}}(0)} |\nabla\varphi_F^{2T} - \nabla\varphi_F^T|^2 dx_d \right] \lesssim |G| \text{dist}^2(F, \text{SO}(d)) \frac{1}{\sqrt{T}}.$$

Further,

$$\begin{aligned} I_2 &\stackrel{(3.13)}{=} - \int_{\mathbb{R}} \frac{1}{T} \eta \varphi_F^{2T} (\varphi_{F,G}^{2T} - \varphi_{F,G}^T) \\ &\quad + \partial_{x_d} \eta DW(\omega(x_d), F + \nabla\varphi_F^{2T}) e_d \cdot (\varphi_{F,G}^{2T} - \varphi_{F,G}^T) dx_d. \end{aligned}$$

The second term on the right-hand side has zero expectation and thus (5.3) and (5.7) yield

$$\mathbb{E}[I_2] \lesssim \text{dist}(F, \text{SO}(d)) |G| \frac{1}{\sqrt{T}}.$$

Finally, we have

$$\begin{aligned}
 I_3 &= \int_{\mathbb{R}} D^2 W(\omega(x_d), F + \nabla \varphi_F^T)(G + \nabla \varphi_{F,G}^T) \cdot \eta(\nabla \varphi_F^{2T} - \nabla \varphi_F^T) dx_d \\
 &\stackrel{(3.15)}{=} - \int_{\mathbb{R}} \frac{1}{T} \eta \varphi_{F,G}^T (\varphi_F^{2T} - \varphi_F^T) \\
 &\quad + \partial_{x_d} \eta(D^2 W(\omega(x_d), F + \nabla \varphi_F^T)(G + \nabla \varphi_{F,G}^T) e_d) \cdot (\varphi_F^{2T} - \varphi_F^T) dx_d.
 \end{aligned}$$

Taking the expectation, the second expression on the right-hand side vanishes. Therefore, using (5.4) and (5.6) we conclude  $\mathbb{E}[I_3] \lesssim \text{dist}(F, \text{SO}(d)) |G| \frac{1}{\sqrt{T}}$ . Collecting all the estimates, the claim follows.

*Step 3. Estimate (6.14) with  $\ell = 2$ .* As in the first step, it suffices to prove

$$\left| \mathbb{E}[D^2 W_{\eta,2T}(F)H \cdot G] - \mathbb{E}[D^2 W_{\eta,T}(F)H \cdot G] \right| \lesssim |H| |G| \frac{1}{\sqrt{T}}. \quad (6.16)$$

We compute (dropping  $\omega$  from the notation)

$$\begin{aligned}
 &D^2 W_{\eta,2T}(F)H \cdot G - D^2 W_{\eta,T}(F)H \cdot G \\
 &= \int_{\mathbb{R}} \eta(D^2 W(F + \nabla \varphi_F^{2T})(H + \nabla \varphi_{F,H}^{2T}) \cdot (G + \nabla \varphi_{F,G}^{2T}) \\
 &\quad - D^2 W(F + \nabla \varphi_F^T)(H + \nabla \varphi_{F,H}^T) \cdot (G + \nabla \varphi_{F,G}^T)) dx_d \\
 &\quad + \int_{\mathbb{R}} \eta(DW(F + \nabla \varphi_F^{2T}) \cdot \nabla \varphi_{F,G,H}^{2T} - DW(F + \nabla \varphi_F^T) \cdot \nabla \varphi_{F,G,H}^T) dx_d \\
 &= \int_{\mathbb{R}} \eta(D^2 W(F + \nabla \varphi_F^{2T}) - D^2 W(F + \nabla \varphi_F^T))(H + \nabla \varphi_{F,H}^T) \cdot (G + \nabla \varphi_{F,G}^T) dx_d \\
 &\quad - \int_{\mathbb{R}} \eta D^3 W(F + \nabla \varphi_F^T)(\nabla \varphi_F^{2T} - \nabla \varphi_F^T)(H + \nabla \varphi_{F,H}^T) \cdot (G + \nabla \varphi_{F,G}^T) dx_d \\
 &\quad + \int_{\mathbb{R}} \eta D^3 W(F + \nabla \varphi_F^T)(\nabla \varphi_F^{2T} - \nabla \varphi_F^T)(H + \nabla \varphi_{F,H}^T) \cdot (G + \nabla \varphi_{F,G}^T) dx_d \\
 &\quad + \int_{\mathbb{R}} \eta D^2 W(F + \nabla \varphi_F^{2T})(\nabla \varphi_{F,H}^{2T} - \nabla \varphi_{F,H}^T) \cdot (G + \nabla \varphi_{F,G}^T) dx_d \\
 &\quad + \int_{\mathbb{R}} \eta D^2 W(F + \nabla \varphi_F^{2T})(H + \nabla \varphi_{F,H}^{2T}) \cdot (\nabla \varphi_{F,G}^{2T} - \nabla \varphi_{F,G}^T) dx_d \\
 &\quad + \int_{\mathbb{R}} \eta(DW(F + \nabla \varphi_F^{2T}) - DW(F + \nabla \varphi_F^T) \\
 &\quad \quad - D^2 W(F + \nabla \varphi_F^T)(\nabla \varphi_F^{2T} - \nabla \varphi_F^T)) \cdot \nabla \varphi_{F,G,H}^T dx_d \\
 &\quad + \int_{\mathbb{R}} \eta DW(F + \nabla \varphi_F^{2T}) \cdot (\nabla \varphi_{F,G,H}^{2T} - \nabla \varphi_{F,G,H}^T) dx_d \\
 &\quad + \int_{\mathbb{R}} \eta D^2 W(F + \nabla \varphi_F^T)(\nabla \varphi_F^{2T} - \nabla \varphi_F^T) \cdot \nabla \varphi_{F,G,H}^T dx_d \\
 &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
 \end{aligned}$$

A Taylor expansion combined with (5.4) and (3.16) yields

$$|\mathbb{E}[I_1 + I_2]| \lesssim |G| |H| \mathbb{E} \left[ \int_{B_{\sqrt{T}}} |\nabla \varphi_F^{2T} - \nabla \varphi_F^T|^2 \right] \lesssim |G| |H| |\text{dist}^2(F, \text{SO}(d))| \frac{1}{\sqrt{T}}.$$

Testing equation (3.17) with  $\eta(\varphi_F^{2T} - \varphi_F^T)$ , we obtain

$$I_3 + I_8 = - \int_{\mathbb{R}} \frac{1}{T} \eta \varphi_{F,G,H}^T (\varphi_F^{2T} - \varphi_F^T) dx_d - \int_{\mathbb{R}} \nabla \eta \otimes (\varphi_F^{2T} - \varphi_F^T) \cdot S dx_3,$$

where

$$S := D^2 W(F + \nabla \varphi_F^T) \nabla \varphi_{F,G,H}^T + D^3 W(F + \nabla \varphi_F^T) (H + \nabla \varphi_{F,H}^T) (G + \nabla \varphi_{F,G}^T)$$

is a stationary random field. Hence, the expectation of the second term above vanishes and thus (5.4) and (5.9) yield  $|\mathbb{E}[I_3 + I_8]| \lesssim |G| |H| \frac{1}{\sqrt{T}}$ . Analogously to the treatment of  $I_1 + I_2$ , we obtain with the help of a Taylor expansion, (3.18), and (5.4),

$$|\mathbb{E}[I_6]| \lesssim \text{dist}^2(F, \text{SO}(d)) |G| |H| \frac{1}{\sqrt{T}}.$$

Similar computations to the estimate for  $I_3 + I_8$  yield (using (3.15) and (3.13))

$$|\mathbb{E}[I_4 + I_5]| + |\mathbb{E}[I_7]| \lesssim |G| |H| \frac{1}{\sqrt{T}}.$$

Collecting all these bounds we conclude (6.16).  $\blacksquare$

*Proof of Lemma 6.3.* Throughout this proof, we set  $\hat{\omega} := \pi_L \omega$  (see (6.10)) and denote by  $\hat{\varphi}_F^T$ ,  $\hat{\varphi}_{F,G}^T$ , and  $\hat{\varphi}_{F,G,H}^T$  the solutions to (3.13), (3.15), and (3.17) with  $\omega$  replaced by  $\hat{\omega}$ . The relevant estimates for  $\hat{\varphi}_F^T$ ,  $\hat{\varphi}_{F,G}^T$ ,  $\hat{\varphi}_{F,G,H}^T$  are contained in Lemma 3.9.

*Step 1. Proof of (6.15) with  $\ell = 0$ .* Note that  $\sqrt{T} < \frac{L}{4}$  (recall  $\sqrt{T} = \frac{L}{2c \ln(L)}$  with  $c \geq 4$ ) and thus  $\omega = \hat{\omega}$  on the support of  $\eta_L$ . Hence, a Taylor expansion implies

$$\begin{aligned} & W_{\eta_L, T}(\hat{\omega}, F) - W_{\eta_L, T}(\omega, F) \\ &= \int_{\mathbb{R}} \eta_L(W(\omega(x_d), F + \nabla \hat{\varphi}_F^T) - W(\omega(x_d), F + \nabla \varphi_F^T)) dx_d \\ &= \int_{\mathbb{R}} \eta_L \int_0^1 DW(\omega(x_d), F + \nabla \varphi_F^T + t \nabla(\varphi_F^T - \hat{\varphi}_F^T)) (\nabla(\varphi_F^T - \hat{\varphi}_F^T)) dx_d. \end{aligned}$$

Using  $DW(\cdot, R) = 0$  for all  $R \in \text{SO}(d)$  and the deterministic estimates  $\|\nabla \varphi_F^T\|_{L^\infty} + \|\nabla \hat{\varphi}_F^T\|_{L^\infty} \lesssim \text{dist}(F, \text{SO}(d))$  (see (3.14)), we obtain

$$\begin{aligned} \mathbb{E}[W_{\eta_L, T}(\pi_L \cdot, F) - W_{\eta_L, T}(F)] &\lesssim \text{dist}(F, \text{SO}(d)) \mathbb{E} \left[ \int_{B_{\sqrt{T}}} |\nabla \varphi_F^T - \nabla \hat{\varphi}_F^T|^2 dx_3 \right]^{\frac{1}{2}} \\ &\stackrel{(3.19)}{\lesssim} \text{dist}^2(F; \text{SO}(d)) \exp\left(-\frac{1}{2c} \frac{L}{\sqrt{T}}\right), \end{aligned}$$

and in combination with the choice of  $T$  we obtain (6.15).

Step 2. Proof of (6.15) with  $\ell = 1$ . As in the previous step, we use  $\omega = \hat{\omega}$  on  $\text{supp } \eta_L$ , and Taylor expansion to obtain

$$\begin{aligned}
 & DW_{\eta_L, T}(F) \cdot G - DW_{\eta_L, T}(\pi_L \omega, F) \cdot G \\
 & \stackrel{(6.12)}{=} \int_{\mathbb{R}} \eta_L DW(\omega(x_d), F + \nabla \varphi_F^T) \cdot (G + \nabla \varphi_{F, G}^T) \\
 & \quad - \eta_L DW(\omega(x_d), F + \nabla \hat{\varphi}_F^T) \cdot (G + \nabla \hat{\varphi}_{F, G}^T) dx_d \\
 & = \int_{\mathbb{R}} \eta_L ((DW(\omega(x_d), F + \nabla \varphi_F^T) - DW(\omega(x_d), F + \nabla \hat{\varphi}_F^T)) \cdot (G + \nabla \varphi_{F, G}^T)) dx_d \\
 & \quad + \int_{\mathbb{R}} \eta_L DW(\omega(x_d), F + \nabla \hat{\varphi}_F^T) \cdot \nabla (\varphi_{F, G}^T - \hat{\varphi}_{F, G}^T) dx_d.
 \end{aligned}$$

Hence, the deterministic estimates (3.14) and (3.16) imply

$$\begin{aligned}
 & (DW_{\eta_L, T}(F) - DW_{\eta_L, T}(\pi_L(\cdot), F)) \cdot G \\
 & \lesssim |G| \left( \int_{B_{\sqrt{T}}} |\nabla (\varphi_F^T - \hat{\varphi}_F^T)|^2 dx_d \right)^{\frac{1}{2}} \\
 & \quad + \text{dist}(F, \text{SO}(d)) \left( \int_{B_{\sqrt{T}}} |\nabla (\varphi_{F, G}^T - \hat{\varphi}_{F, G}^T)|^2 dx_d \right)^{\frac{1}{2}}.
 \end{aligned}$$

Finally, (3.19) and (3.20) in combination with the choice of  $T$  yield the claim.

Step 3. Proof of (6.15) with  $\ell = 2$ . Using  $\hat{\omega} = \omega$  on  $\text{supp } \eta_L$ , we compute

$$\begin{aligned}
 & D^2 W_{\eta_L, T}(\pi_L \omega, F) H \cdot G - D^2 W_{\eta_L, T}(F) H \cdot G \\
 & \stackrel{(6.13)}{=} \int_{\mathbb{R}} \eta_L [D^2 W(\hat{\omega}, F + \nabla \hat{\varphi}_F^T)(H + \nabla \hat{\varphi}_{F, H}^T) \cdot (G + \nabla \hat{\varphi}_{F, G}^T) dx_d \\
 & \quad + DW(\hat{\omega}, F + \nabla \hat{\varphi}_F^T) \cdot \nabla \hat{\varphi}_{F, G, H}^T] \\
 & \quad - \int_{\mathbb{R}} \eta_L [D^2 W(\omega, F + \nabla \varphi_F^T)(H + \nabla \varphi_{F, H}^T) \cdot (G + \nabla \varphi_{F, G}^T) \\
 & \quad + DW(\omega, F + \nabla \varphi_F^T) \cdot \nabla \varphi_{F, G, H}^T] dx_d \\
 & = \int_{\mathbb{R}} \eta_L (D^2 W(\omega, F + \nabla \hat{\varphi}_F^T) - D^2 W(\omega, F + \nabla \varphi_F^T))(H + \nabla \varphi_{F, H}^T) \\
 & \quad \times (G + \nabla \varphi_{F, G}^T) dx_d \\
 & \quad + \int_{\mathbb{R}} \eta_L D^2 W(\omega, F + \nabla \hat{\varphi}_F^T)(\nabla \hat{\varphi}_{F, H}^T - \nabla \varphi_{F, H}^T)(G + \nabla \varphi_{F, G}^T) dx_d \\
 & \quad + \int_{\mathbb{R}} \eta_L D^2 W(\omega, F + \nabla \hat{\varphi}_F^T)(H + \nabla \hat{\varphi}_{F, H}^T)(\nabla \hat{\varphi}_{F, G}^T - \nabla \varphi_{F, G}^T) dx_d \\
 & \quad + \int_{\mathbb{R}} \eta_L (DW(\omega, F + \nabla \hat{\varphi}_F^T) - DW(\omega, F + \nabla \varphi_F^T))(\nabla \varphi_{F, G, H}^T) dx_d \\
 & \quad + \int_{\mathbb{R}} \eta_L DW(\omega, F + \nabla \hat{\varphi}_F^T)(\nabla \hat{\varphi}_{F, G, H}^T - \nabla \varphi_{F, G, H}^T) dx_d
 \end{aligned}$$

$$\begin{aligned}
&\lesssim |H| |G| \left( \int_{B_{\sqrt{T}}} |\nabla \widehat{\varphi}_F^T - \nabla \varphi_F^T|^2 dx_d \right)^{\frac{1}{2}} + |G| \left( \int_{B_{\sqrt{T}}} |\nabla \widehat{\varphi}_{F,H}^T - \nabla \varphi_{F,H}^T|^2 dx_d \right)^{\frac{1}{2}} \\
&\quad + |H| \left( \int_{B_{\sqrt{T}}} |\nabla \widehat{\varphi}_{F,G}^T - \nabla \varphi_{F,G}^T|^2 dx_d \right)^{\frac{1}{2}} + \text{dist}(F, \text{SO}(d)) |G| |H| \\
&\quad + \text{dist}(F, \text{SO}(d)) \left( \int_{B_{\sqrt{T}}} |\nabla \widehat{\varphi}_{F,G,H}^T - \nabla \varphi_{F,G,H}^T|^2 dx_d \right)^{\frac{1}{2}},
\end{aligned}$$

where the last estimate follows from the properties of  $W$  and (3.14), (3.16), and (3.18). Finally, estimates (3.19)–(3.21) and the choice of  $T$  yield the claim.  $\blacksquare$

## A. Linear corrector equation

We recall the following standard result. We state it in terms of our specific probability space; however, it also holds in a more general stationary and ergodic setting.

**Proposition A.1.** *Let  $(\Omega, \mathcal{S}, \mathbb{P})$  satisfy (P1)–(P2) and for  $T > 0$  we consider a measurable function  $\mathbb{L}_T: \Omega \times \mathbb{R}^d \rightarrow L(\mathbb{R}^{d \times d}, \mathbb{R}^{d \times d})$ . We assume that there exists  $c > 0$  such that for all  $T > 0$  and for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , it holds that*

$$\begin{aligned}
\frac{1}{c} \int_{\mathbb{R}^d} |\nabla \eta|^2 dx &\leq \int_{\mathbb{R}^d} \mathbb{L}_T(\omega, x) \nabla \eta \cdot \nabla \eta dx \quad \text{for all } \eta \in C_c^\infty(\mathbb{R}^d), \\
|\mathbb{L}_T(\omega(x_d))| &\leq c.
\end{aligned} \tag{A.1}$$

- (i) *Then, for any  $g_T \in L^\infty(\Omega \times \mathbb{R}^d; \mathbb{R}^{d \times d})$ , there exists  $\varphi^T \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$  a unique solution to*

$$\frac{1}{T} \varphi^T - \text{div}(\mathbb{L}_T(\omega, x) \nabla \varphi^T) = \text{div}(g_T(\omega, x)) \quad \text{in } \mathbb{R}^d.$$

- (ii) *We additionally assume the following:*

- (a) *There exists  $\mathbb{L}: \Omega \times \mathbb{R}^d \rightarrow L(\mathbb{R}^{d \times d}, \mathbb{R}^{d \times d})$  satisfying an analogous assumption to (A.1). There exists  $g \in L^\infty(\Omega \times \mathbb{R}^d; \mathbb{R}^{d \times d})$ .*
- (b) *The mapping  $(\omega, x) \mapsto (\mathbb{L}_T(\omega, x), \mathbb{L}(\omega, x), g_T(\omega, x), g(\omega, x))$  is a stationary random field.*
- (c) *It holds that*

$$\begin{aligned}
\limsup_{T \rightarrow \infty} \|g_T\|_{L^\infty(\Omega \times \mathbb{R}^d)} &< \infty, \\
\lim_{T \rightarrow \infty} (\mathbb{L}_T(\omega, x), g_T(\omega, x)) &= (\mathbb{L}(\omega, x), g(\omega, x)) \quad \text{a.e.}
\end{aligned}$$

*Then there exists  $\varphi \in H_{\text{uloc}}^1(\mathbb{R}^d; \mathbb{R}^d)$ , a unique solution to*

$$-\text{div}(\mathbb{L}(\omega, x) \nabla \varphi) = \text{div}(g(\omega, x)) \quad \text{in } \mathbb{R}^d,$$



with the following properties:

$$\begin{aligned} \int_{\square} \varphi \, dx &= 0, \quad \nabla \varphi \text{ is a stationary random field,} \\ \mathbb{E} \left[ \int_{\square} \nabla \varphi \, dx \right] &= 0, \quad \mathbb{E} \left[ \int_{\square} |\nabla \varphi|^2 \, dx \right] < \infty. \end{aligned}$$

Moreover, it holds that

$$\begin{aligned} \limsup_{R \rightarrow \infty} \frac{1}{R^2} \int_{\square_R} |\varphi|^2 \, dx &= 0, \\ \lim_{T \rightarrow \infty} \mathbb{E} \left[ \int_{\square} |\nabla \varphi_T - \nabla \varphi|^2 \, dx \right] &= 0. \end{aligned}$$

The proof of this proposition is a simple variation of [38] and for this reason we omit it.

**Funding.** SN and MV acknowledge funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – project number 405009441.

## References

- [1] M. A. Akcoglu and U. Krengel, [Ergodic theorems for superadditive processes](#). *J. Reine Angew. Math.* **323** (1981), 53–67 Zbl [0453.60039](#) MR [0611442](#)
- [2] O. Anza Hafsa and J.-P. Mandallena, [Homogenization of nonconvex integrals with convex growth](#). *J. Math. Pures Appl. (9)* **96** (2011), no. 2, 167–189 Zbl [1242.49026](#) MR [2818711](#)
- [3] S. Armstrong, S. J. Ferguson, and T. Kuusi, [Higher-order linearization and regularity in non-linear homogenization](#). *Arch. Ration. Mech. Anal.* **237** (2020), no. 2, 631–741 Zbl [1442.35131](#) MR [4097329](#)
- [4] S. Armstrong, A. Gloria, and T. Kuusi, [Bounded correctors in almost periodic homogenization](#). *Arch. Ration. Mech. Anal.* **222** (2016), no. 1, 393–426 Zbl [1355.35011](#) MR [3519974](#)
- [5] S. Armstrong, T. Kuusi, and J.-C. Mourrat, [Mesoscopic higher regularity and subadditivity in elliptic homogenization](#). *Comm. Math. Phys.* **347** (2016), no. 2, 315–361 Zbl [1357.35025](#) MR [3545509](#)
- [6] S. N. Armstrong and Z. Shen, [Lipschitz estimates in almost-periodic homogenization](#). *Comm. Pure Appl. Math.* **69** (2016), no. 10, 1882–1923 Zbl [1367.35075](#) MR [3541853](#)
- [7] S. N. Armstrong and C. K. Smart, [Quantitative stochastic homogenization of convex integral functionals](#). *Ann. Sci. Éc. Norm. Supér. (4)* **49** (2016), no. 2, 423–481 Zbl [1344.49014](#) MR [3481355](#)
- [8] P. Bella, B. Fehrman, J. Fischer, and F. Otto, [Stochastic homogenization of linear elliptic equations: Higher-order error estimates in weak norms via second-order correctors](#). *SIAM J. Math. Anal.* **49** (2017), no. 6, 4658–4703 Zbl [1380.35011](#) MR [3725278](#)
- [9] A. Bourgeat and A. Piatnitski, [Approximations of effective coefficients in stochastic homogenization](#). *Ann. Inst. H. Poincaré Probab. Statist.* **40** (2004), no. 2, 153–165 Zbl [1058.35023](#) MR [2044813](#)
- [10] A. Braides, [Homogenization of some almost periodic coercive functional](#). *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)* **9** (1985), no. 1, 313–321 Zbl [0582.49014](#) MR [0899255](#)

- [11] A. Braides and A. Defranceschi, *Homogenization of multiple integrals*. Oxford Lecture Ser. Math. Appl. 12, The Clarendon Press, Oxford University Press, New York, 1998  
Zbl [0911.49010](#) MR [1684713](#)
- [12] N. Clozeau and A. Gloria, [Quantitative nonlinear homogenization: Control of oscillations](#). *Arch. Ration. Mech. Anal.* **247** (2023), no. 4, article no. 67 Zbl [1522.35037](#) MR [4619005](#)
- [13] N. Clozeau, M. Josien, F. Otto, and Q. Xu, [Bias in the representative volume element method: Periodize the ensemble instead of its realizations](#). *Found. Comput. Math.* (2023)  
DOI <https://doi.org/10.1007/s10208-023-09613-y>
- [14] S. Conti, G. Dolzmann, B. Kirchheim, and S. Müller, [Sufficient conditions for the validity of the Cauchy–Born rule close to  \$SO\(n\)\$](#) . *J. Eur. Math. Soc. (JEMS)* **8** (2006), no. 3, 515–530  
Zbl [1146.74006](#) MR [2250170](#)
- [15] G. Dal Maso and L. Modica, [Nonlinear stochastic homogenization](#). *Ann. Mat. Pura Appl. (4)* **144** (1986), 347–389 Zbl [0607.49010](#) MR [0870884](#)
- [16] G. Dal Maso and L. Modica, [Nonlinear stochastic homogenization and ergodic theory](#). *J. Reine Angew. Math.* **368** (1986), 28–42 Zbl [0582.60034](#) MR [0850613](#)
- [17] M. Duerinckx and A. Gloria, [Stochastic homogenization of nonconvex unbounded integral functionals with convex growth](#). *Arch. Ration. Mech. Anal.* **221** (2016), no. 3, 1511–1584  
Zbl [1347.49018](#) MR [3509007](#)
- [18] M. Duerinckx and A. Gloria, [Multiscale functional inequalities in probability: Concentration properties](#). *ALEA Lat. Am. J. Probab. Math. Stat.* **17** (2020), no. 1, 133–157 Zbl [1456.60053](#)  
MR [4057186](#)
- [19] M. Duerinckx and A. Gloria, [Multiscale functional inequalities in probability: Constructive approach](#). *Ann. H. Lebesgue* **3** (2020), 825–872 Zbl [1472.60086](#) MR [4149827](#)
- [20] J. Fischer, [The choice of representative volumes in the approximation of effective properties of random materials](#). *Arch. Ration. Mech. Anal.* **234** (2019), no. 2, 635–726 Zbl [1431.65201](#)  
MR [3995049](#)
- [21] J. Fischer and S. Neukamm, [Optimal homogenization rates in stochastic homogenization of nonlinear uniformly elliptic equations and systems](#). *Arch. Ration. Mech. Anal.* **242** (2021), no. 1, 343–452 Zbl [1471.35024](#) MR [4302762](#)
- [22] G. Friesecke, R. D. James, and S. Müller, [A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity](#). *Comm. Pure Appl. Math.* **55** (2002), no. 11, 1461–1506 Zbl [1021.74024](#) MR [1916989](#)
- [23] G. Friesecke and F. Theil, [Validity and failure of the Cauchy–Born hypothesis in a two-dimensional mass-spring lattice](#). *J. Nonlinear Sci.* **12** (2002), no. 5, 445–478 Zbl [1084.74501](#)  
MR [1923388](#)
- [24] G. Geymonat, S. Müller, and N. Triantafyllidis, [Homogenization of non-linearly elastic materials, microscopic bifurcation and macroscopic loss of rank-one convexity](#). *Arch. Rational Mech. Anal.* **122** (1993), no. 3, 231–290 Zbl [0801.73008](#) MR [1219609](#)
- [25] A. Gloria and S. Neukamm, [Commutability of homogenization and linearization at identity in finite elasticity and applications](#). *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **28** (2011), no. 6, 941–964 Zbl [1256.35004](#) MR [2859933](#)
- [26] A. Gloria, S. Neukamm, and F. Otto, [Quantification of ergodicity in stochastic homogenization: Optimal bounds via spectral gap on Glauber dynamics](#). *Invent. Math.* **199** (2015), no. 2, 455–515 Zbl [1314.39020](#) MR [3302119](#)
- [27] A. Gloria, S. Neukamm, and F. Otto, [A regularity theory for random elliptic operators](#). *Milan J. Math.* **88** (2020), no. 1, 99–170 Zbl [1440.35064](#) MR [4103433](#)

- [28] A. Gloria, S. Neukamm, and F. Otto, [Quantitative estimates in stochastic homogenization for correlated coefficient fields](#). *Anal. PDE* **14** (2021), no. 8, 2497–2537 Zbl 1485.35156 MR 4377865
- [29] A. Gloria and F. Otto, [An optimal variance estimate in stochastic homogenization of discrete elliptic equations](#). *Ann. Probab.* **39** (2011), no. 3, 779–856 Zbl 1215.35025 MR 2789576
- [30] A. Gloria and F. Otto, [An optimal error estimate in stochastic homogenization of discrete elliptic equations](#). *Ann. Appl. Probab.* **22** (2012), no. 1, 1–28 Zbl 1387.35031 MR 2932541
- [31] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*. Springer, Berlin, 1994 Zbl 0801.35001 MR 1329546
- [32] T. Kanit, S. Forest, I. Galliet, V. Mounoury, and D. Jeulin, [Determination of the size of the representative volume element for random composites: Statistical and numerical approach](#). *Int. J. Solids Struct.* **40** (2003), no. 13-14, 3647–3679 Zbl 1038.74605
- [33] C. Le Bris, F. Legoll, and W. Minvielle, [Special quasirandom structures: A selection approach for stochastic homogenization](#). *Monte Carlo Methods Appl.* **22** (2016), no. 1, 25–54 Zbl 1334.35450 MR 3466605
- [34] K. Messaoudi and G. Michaille, [Stochastic homogenization of nonconvex integral functionals](#). *RAIRO Modél. Math. Anal. Numér.* **28** (1994), no. 3, 329–356 Zbl 0818.60029 MR 1275348
- [35] S. Müller, [Homogenization of nonconvex integral functionals and cellular elastic materials](#). *Arch. Rational Mech. Anal.* **99** (1987), no. 3, 189–212 Zbl 0629.73009 MR 0888450
- [36] S. Müller and S. Neukamm, [On the commutability of homogenization and linearization in finite elasticity](#). *Arch. Ration. Mech. Anal.* **201** (2011), no. 2, 465–500 Zbl 1262.74029 MR 2820355
- [37] A. Naddaf and T. Spencer, [Estimates on the variance of some homogenization problems](#). 1998, unpublished preprint
- [38] S. Neukamm, [An introduction to the qualitative and quantitative theory of homogenization](#). *Interdiscip. Inform. Sci.* **24** (2018), no. 1, 1–48 Zbl 1481.35010 MR 3833281
- [39] S. Neukamm and M. Schäffner, [Quantitative homogenization in nonlinear elasticity for small loads](#). *Arch. Ration. Mech. Anal.* **230** (2018), no. 1, 343–396 Zbl 1401.74044 MR 3840917
- [40] S. Neukamm and M. Schäffner, [Lipschitz estimates and existence of correctors for nonlinearly elastic, periodic composites subject to small strains](#). *Calc. Var. Partial Differential Equations* **58** (2019), no. 2, article no. 46 Zbl 1461.35028 MR 3910888
- [41] S. Neukamm, M. Schäffner, and A. Schlömerkemper, [Stochastic homogenization of non-convex discrete energies with degenerate growth](#). *SIAM J. Math. Anal.* **49** (2017), no. 3, 1761–1809 Zbl 1362.74028 MR 3650427
- [42] M. Schneider, M. Josien, and F. Otto, [Representative volume elements for matrix-inclusion composites—A computational study on the effects of an improper treatment of particles intersecting the boundary and the benefits of periodizing the ensemble](#). *J. Mech. Phys. Solids* **158** (2022), article no. 104652 MR 4332066

Received 27 March 2023; revised 2 November 2023; accepted 16 November 2023.

### Stefan Neukamm

Fakultät Mathematik, Technische Universität Dresden, Zellescher Weg 25, 01217 Dresden, Germany; [stefan.neukamm@tu-dresden.de](mailto:stefan.neukamm@tu-dresden.de)

**Mathias Schäffner**

Institut für Mathematik, MLU Halle, Theodor-Lieser-Straße 5, 06120 Halle (Saale), Germany;  
[mathias.schaeffner@mathematik.uni-halle.de](mailto:mathias.schaeffner@mathematik.uni-halle.de)

**Mario Varga**

Fakultät Mathematik, Technische Universität Dresden, Zellescher Weg 25, 01217 Dresden,  
Germany; [mario.varga@tu-dresden.de](mailto:mario.varga@tu-dresden.de)