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On graded division rings

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Abstract – Given an associative ring with identity R and a ring homomorphism $f: R \to D$ to a division ring D, the singular kernel of f is the set of square matrices of all sizes over R, which, on applying f, yield singular matrices over D. Paul M. Cohn characterized the sets of square matrices that can arise as singular kernels and showed that, up to isomorphism, the singular kernels characterize the different homomorphisms from R to division rings. In this work, we show that this characterization can be implemented in the context of graded rings. More precisely, given a ring R graded by a group Γ we adapt the theory of Cohn to determine the different homomorphisms of graded rings from R to Γ-graded division rings.

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Contents

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Introduction

Let R be a commutative ring. It is well known that the prime ideals of R classify the homomorphisms from R to division rings. Indeed, for any prime ideal P of R, we obtain a homomorphism from R to a division ring via the natural homomorphism $R \to Q(R/P)$, where $Q(R/P)$ denotes the field of fractions of R/P. Conversely, if $\varphi\colon R \to D$ is a homomorphism from R to a division ring D, then $P = \ker\varphi$ is a prime ideal of R, φ factors through $R \to Q(R/P)$ and therefore the division subring of D generated by the image of R is R-isomorphic to $Q(R/P)$. Moreover, let $P \subseteq P'$ be prime ideals of R. The localization of R/P at the prime ideal P'/P yields a local subring of $Q(R/P)$ with residue field isomorphic to $Q(R/P')$. This implies that any fraction $ab^{-1} \in Q(R)$ which is defined in $Q(R/P')$ is also defined in $Q(R/P)$. Also, looking at the determinants of matrices, one sees that any matrix with entries in R that becomes invertible in $Q(R/P')$ also becomes invertible in $Q(R/P)$.

If the ring R is not commutative, prime ideals no longer classify the homomorphisms to division rings. It may even be possible that R has infinitely many different "fields of fractions"; see for example [12, Section 9].

Let R be any ring. An epic R-division ring is a ring homomorphism $R \to K$, where K is a division ring generated by the image of R using sum, product and inversion of elements. Cohn [4] showed that the epic R-division rings are characterized up to R-isomorphism by the collection of square matrices over R which are carried to matrices singular over K. This set of matrices is called the singular kernel of $R \to K$. He also gave the precise conditions for a set of square matrices over R to be a singular kernel, calling such a collection a prime matrix ideal of R. The name comes from the fact that, if we endow the set of square matrices over R with a certain two operations of sum and product (the sum being a partial operation), those sets have similar behaviour to prime ideals. These operations are defined so that, when defined on square matrices over a commutative ring, the determinantal sum holds and the determinant of a product of matrices equals the product of the determinants. Also in [4], Cohn showed that if $\mathcal{P}$, $\mathcal{P}'$ are prime matrix ideals of R and $R \to K_{\mathcal{P}}$, $R \to K_{\mathcal{P}'}$ are the corresponding epic R-division rings, then $\mathcal{P} \subseteq \mathcal{P}'$ if and only if there exists a local subring of $K_{\mathcal{P}}$ containing the image of R with residue class division ring R-isomorphic to $K_{\mathcal{P}'}$.

We say that there exists a specialization from $K_{\mathcal{P}}$ to $K_{\mathcal{P}'}$. Furthermore, if a rational expression built up from elements of R makes sense in $K_{\mathcal{P}'}$, then it can also be evaluated in $K_{\mathcal{P}}$. Cohn also provided conditions on square matrices over R equivalent to the existence of (injective) homomorphisms from R to division rings and to the existence of a *best* epic R-division ring in the sense that a rational expression that makes sense in some epic R-division ring, makes sense in it.

The theory of group graded rings has played an important role in ring theory (see for example [8, 18]) and many results in classical ring theory have a mirrored version for group graded rings. Furthermore, if R is a filtered ring, it has proved fruitful to study the associated graded ring, which usually is a simpler object, in order to obtain information about the original ring.

The main aim of this article is to develop Cohn's theory on division rings in the context of group graded rings. More precisely, let Γ be a group and $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. A Γ-graded epic R-division ring is a homomorphism of Γ-graded rings $R \to K$, where K is a Γ-graded division ring generated by the image of R. Matrices over R represent homomorphisms between finitely generated free R-modules. Homomorphisms of Γ-graded modules between Γ-graded free R-modules are given by (what we call) *homogeneous matrices*. These are $m \times n$ matrices A for which there exist $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in \Gamma$ such that each (i, j) entry of A belongs to $R_{\alpha_i \beta_j^{-1}}$. We show that Γ-graded epic R-division rings $R \to K$ are characterized, up to R-isomorphism of Γ-graded rings, by the collection of homogeneous matrices which are carried to singular matrices over K. These sets are called the gr-singular kernel of $R \to K$. We give the precise conditions under which a collection of homogeneous matrices over R is a gr-singular kernel and thus define the concept of a gr-prime matrix ideal. If $\mathcal{P}$, $\mathcal{P}'$ are gr-prime matrix ideals of R and $R \to K_{\mathcal{P}}$, $R \to K_{\mathcal{P}'}$ are the corresponding Γ-graded epic R-division rings, then $\mathcal{P} \subseteq \mathcal{P}'$ if and only if there exists a Γ-graded local subring of $K_{\mathcal{P}}$ that contains the image of R with residue class Γ-graded division ring R-isomorphic to $K_{\mathcal{P}'}$ as Γ-graded rings. Furthermore, if a homogeneous rational expression obtained from elements of R makes sense in $K_{\mathcal{P}'}$ then it can also be evaluated in $K_{\mathcal{P}}$. We then provide conditions on the set of square homogeneous matrices over R that characterize when there exists an (injective) homomorphism of Γ-graded rings from R to a Γ-graded division ring and when there exists a *best* Γ-graded epic R-division ring.

In the study of division rings, one of the pioneering works carrying the information from the associated graded ring to the original filtered ring was [3]. Cohn showed that if a ring R endowed with a valuation with values in $\mathbb{Z}$ is such that its associated graded ring is a (graded) Ore domain, then R can be embedded in a division ring. Other proofs of this result can be found in [1, 13, 14]. More recently, a generalization of the result by Cohn has been given by Valitskas [20]. We believe that our work could be helpful in order to generalize the result by Cohn to a greater extent than has been done by Valitskas.

An elementary application of our theory is as follows. Suppose that R is a ring graded by a group Γ. As an immediate consequence of [18, Proposition 1.2.2], one obtains that if there exists an (injective) homomorphism from R to a division ring, then there exists an (injective) homomorphism of Γ-graded rings from R to a Γ-graded division ring. Thus, if one shows that there do not exist (injective) homomorphisms of Γ-graded rings from R to Γ-graded division rings, then there do not exist (injective) homomorphisms from R to division rings. See Section 8 for other similar results.

It is also interesting to remark that the existence of an (injective) homomorphism from a Γ-graded ring R to a division ring is not equivalent to the existence of a homomorphism of Γ-graded rings from R to a Γ-graded division ring; see Proposition 2.4 (4).

In Section 1 we introduce some of the notation that will be used throughout the paper and provide a short survey about the results on graded rings that will be used.

Let Γ be a group. A Γ-almost graded division ring is a (not necessarily graded) homomorphic image of a Γ-graded division ring. For example, let K be a field and consider the group ring $K[\Gamma]$. It is a Γ-graded division ring, and the augmentation map $K[\Gamma] \to K$, which is not a homomorphism of Γ-graded rings, endows K with a structure of Γ-almost graded division ring. In the nongraded context, this concept is not necessary because there are no nontrivial images of a division ring D other than D itself. In Section 2 we show that if R is a Γ-graded ring, $\varphi\colon R \to D$ is a homomorphism of Γ-graded rings with D a Γ-graded division ring and $\psi\colon D \to E$ is a ring homomorphism where E is a nonzero ring, then the homogeneous matrices over R that become invertible via φ and via $\psi\varphi$ are the same. Thus (a posteriori) $\psi(D)$ determines a Γ-graded epic R-division ring.

The main results in Section 3 are as follows. Let $\varphi\colon R \to D$ be a homomorphism of Γ-graded rings and let Σ be a set of square homogeneous matrices with entries in R. Suppose that the matrices of Σ become invertible in D via φ. Then, under certain natural conditions on Σ, the entries of the inverses of the matrices in Σ are the homogeneous elements of a Γ-graded subring of R. Moreover, if D is a Γ-graded division ring generated by the image of φ and Σ the set of homogeneous matrices that become invertible under φ, then any homogeneous element of D is an entry of the inverse of some matrix in Σ.

Section 4 begins showing that the universal localization R_Σ of the Γ-graded ring R at a set of homogeneous matrices is again a Γ-graded ring. Then it is shown that a homomorphism of Γ-graded rings $\varphi\colon R \to D$, where D is Γ-graded division ring, is an epimorphism in the category of Γ-graded rings if and only if D is generated by the image of φ. If this is the case, we say that (D,φ) is a Γ-graded epic R-division ring and we prove that if Σ is the set of square homogeneous matrices that become invertible in D via φ, then R_Σ is a Γ-graded local ring with Γ-graded residue division ring R-isomorphic to D. Then the concept of gr-specialization between Γ-graded epic R-division rings is defined. The section ends showing that the existence of a gr-specialization from (D,φ) to another Γ-graded epic R-division ring (D',φ') is equivalent to saying that all the homogeneous rational expressions (from elements of R) that make sense in (D',φ') make sense in (D,φ) too, and that it is also equivalent to the fact that any homogeneous matrix over R that becomes invertible in (D',φ') becomes invertible in (D,φ) too.

Section 5 is devoted to the proof of the graded version of the so-called Malcolmson criterion [16] and an important consequence (see also [7] for a generalization of [16]). This criterion determines the kernel of the natural homomorphism from R to the universal localization R_Σ of R at certain sets Σ of homogeneous matrices. As a corollary, one obtains a sufficient condition for the ring R_Σ not to be the zero ring. The long and technical proof of Malcolmson's criterion consists of an elementwise construction of the ring R_Σ. This construction will also be used in the next section.

The concept of gr-prime matrix ideal is given in Section 6 and it is shown that the different Γ-graded epic R-division rings are determined by the gr-prime matrix ideals up to R-isomorphism of Γ-graded rings.

In Section 7 the concepts of a gr-matrix ideal and of the radical of a gr-matrix ideal are defined and the gr-matrix ideal generated by a set of homogeneous square matrices is characterized. Then it is proved that gr-prime matrix ideals behave like prime ideals in a commutative ring. All these concepts are used to provide necessary and sufficient conditions for the existence of homomorphisms (embeddings) of Γ-graded rings to Γ-graded division rings.

In Section 8 we deal with a new situation that appears in the graded context. If Γ is a group and R is a Γ-graded ring, then the ring R can be considered as a Γ/Ω-graded ring for any normal subgroup Ω of Γ. Thus there are Γ-graded and Γ/Ω-graded versions of the concepts studied before. In this section we try to relate them. Note that when $\Omega=\Gamma$, a Γ/Ω-graded epic R-division ring is simply an R-division ring, and thus one can relate the theory of Γ-graded division rings and the theory of division rings as developed by Cohn.

We would like to finish this introduction by pointing out that most of the techniques used in this paper are adaptations of those from the works by Cohn and Malcolmson. We just take credit for realizing that they can be applied in the more general setting of group graded rings.

1. Basic definitions and notation

Rings are supposed to be associative with 1. We recall that a *domain* is a nonzero ring such that for elements x, y of the ring, the equality $xy = 0$ implies that either $x = 0$ or $y = 0$. A *division ring* is a nonzero ring such that every nonzero element is invertible. For a ring R, we define $\mathbb{M}(R)$ to be the set of all square matrices of any size. Also, for each i with $1 \leq i \leq n$, let e_i denote the column

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

in which the ith entry is 1 and the other entries are zero.

Let $A \in M_n(R)$. We say that A is *full* if whenever $A = PQ$, with $P \in M_{n\times r}(R)$ and $Q \in M_{r\times n}(R)$, then $r \geq n$. If we think of A as an endomorphism of the free (right) R-module R^n, it means that A does not factor through R^r with $r < n$. We say that A is *hollow* if it has an $r \times s$ block of zeros where $r + s > n$. It is well known that a hollow matrix is not full.

Let S be a ring and $f\colon R \to S$ be a ring homomorphism. For each matrix M with entries in R, we denote by M^f the matrix whose entries are the images of the entries of M by f, that is, if $a_{ij} \in R$ is the (i, j)-entry of M, then the (i, j)-entry of M^f is $f(a_{ij})$. Given a set of matrices Σ, we denote $\Sigma^f = \{M^f : M \in \Sigma\}$. We say that the ring homomorphism $f\colon R \to S$ is Σ-*inverting* if the matrix M^f is invertible over S for each $M \in \Sigma$.

We proceed to give some basics on group graded rings that can be found in [8, 18], for example.

If Γ is a group, the identity element of Γ will be denoted by e.

Let Γ be a group. A ring R is called a Γ-*graded ring* if $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$, where each R_γ is an additive subgroup of R and $R_\gamma R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The *support of* R is defined as the set $\operatorname{supp} R = \{\gamma \in \Gamma : R_\gamma \neq \{0\}\}$. The set $\mathrm{h}(R) = \bigcup_{\gamma\in\Gamma} R_\gamma$ is called the set of *homogeneous elements* of R. It is well known that the identity element $1 \in R$ belongs to R_e, that R_e is a subring of R and that if $x \in R_\gamma$ is invertible in R, then $x^{-1} \in R_{\gamma^{-1}}$. A (two-sided) ideal I of R is called a *graded ideal* if $I = \bigoplus_{\gamma\in\Gamma}(I \cap R_\gamma)$. Thus I is a graded ideal if and only if, for any $x \in I$, $x = \sum x_i$, where $x_i \in \mathrm{h}(R)$, implies that $x_i \in I$. Observe that if $X \subseteq \mathrm{h}(R)$, then the ideal of R generated by X is a graded ideal. If I is a graded ideal, then the quotient ring R/I is a Γ-graded ring with $R/I = \bigoplus_{\gamma\in\Gamma}(R/I)_\gamma$, where $(R/I)_\gamma = (R_\gamma + I)/I$.

A Γ-*graded domain* is a nonzero Γ-graded ring such that if $x, y \in \mathrm{h}(R)$, the equality $xy = 0$ implies that either $x = 0$ or $y = 0$. A Γ-*graded division ring* is a nonzero Γ-graded ring such that every nonzero homogeneous element is invertible. A commutative Γ-graded division ring is a Γ*-graded field*. Clearly, any Γ-graded division ring is a Γ-graded domain.

A nonzero Γ-graded ring R is called a Γ*-graded local ring* if the two-sided ideal $\mathfrak{m}$ generated by the noninvertible homogeneous elements is a proper ideal. In this case, the Γ-graded ring $R/\mathfrak{m}$ is a Γ-graded division ring and it will be called the *residue class* Γ*-graded division ring* of R.

For Γ-graded rings R and S, a *homomorphism of* Γ*-graded rings* $f\colon R\to S$ is a ring homomorphism such that $f(R_\gamma)\subseteq S_\gamma$ for all $\gamma\in\Gamma$. An *isomorphism of* Γ*-graded rings* is a homomorphism of Γ-graded rings which is bijective. Notice that the inverse is also an isomorphism of Γ-graded rings.

Let Ω be a normal subgroup of Γ. Consider the Γ-graded ring $R=\bigoplus_{\gamma\in\Gamma}R_\gamma$. It can be regarded as a Γ/Ω-graded ring as

$$R=\bigoplus_{\alpha\in\Gamma/\Omega}R_\alpha,\quad \text{where } R_\alpha=\bigoplus_{\gamma\in\alpha}R_\gamma.$$

Let R be a Γ-graded ring. A Γ*-graded (right) R-module* M is defined to be a right R-module with a direct sum decomposition $M=\bigoplus_{\gamma\in\Gamma}M_\gamma$, where each M_γ is an additive subgroup of M such that $M_\lambda R_\gamma\subseteq M_{\lambda\gamma}$ for all $\lambda,\gamma\in\Gamma$. A submodule N of M is called a graded submodule if $N=\bigoplus_{\gamma\in\Gamma}(N\cap M_\gamma)$. In this case, the factor module M/N forms a Γ-graded R-module with $M/N=\bigoplus_{\gamma\in\Gamma}(M/N)_\gamma$, where $(M/N)_\gamma=(M_\gamma+N)/N$.

For Γ-graded R-modules M and N, a *homomorphism of* Γ*-graded R-modules* $f\colon M\to N$ is a homomorphism of R-modules such that $f(M_\gamma)\subseteq N_\gamma$ for all $\gamma\in\Gamma$. In this case, $\ker f$ is a graded submodule of M and $\operatorname{Im} f$ is a graded submodule of N.

If Ω is a normal subgroup of Γ, then a Γ-graded R-module $M=\bigoplus_{\gamma\in\Gamma}M_\gamma$ can be regarded as a Γ/Ω-graded over the Γ/Ω-graded ring R as

$$M=\bigoplus_{\alpha\in\Gamma/\Omega}M_\alpha,\quad \text{where } M_\alpha=\bigoplus_{\gamma\in\alpha}M_\gamma.$$

Moreover, a homomorphism of Γ-graded R-modules is also a homomorphism of Γ/Ω-graded R-modules.

Let $\{M_i : i\in I\}$ be a set of Γ-graded R-modules. Then $\bigoplus_{i\in I}M_i$ has a natural structure of Γ-graded R-module given by $(\bigoplus_{i\in I}M_i)_\gamma=\bigoplus_{i\in I}(M_i)_\gamma$.

Let M be a Γ-graded R-module. For $\delta\in\Gamma$, we define the δ*-shifted* Γ-graded R-module $M(\delta)$ as

$$M(\delta)=\bigoplus_{\gamma\in\Gamma}M(\delta)_\gamma,\quad \text{where } M(\delta)_\gamma=M_{\delta\gamma}.$$

A Γ-graded R-module F is called a Γ-*graded free R-module* if F is a free R-module with a homogeneous basis. It is well known that the Γ-graded free R-modules are of the form

$$\bigoplus_{i\in I} R(\delta_i), \quad \text{where } I \text{ is an indexing set and } \delta_i \in \Gamma.$$

If $I = \{1, \dots, n\}$, then $\bigoplus_{i\in I} R(\delta_i) = R(\delta_1) \oplus \cdots \oplus R(\delta_n)$, will also be denoted by $R^n(\bar{\delta})$, where $\bar{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$.

Let Γ be a group and $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. Following [8], for $\bar{\alpha} = (\alpha_1, \dots, \alpha_m) \in \Gamma^m$ and $\bar{\beta} = (\beta_1, \dots, \beta_n) \in \Gamma^n$, set

$$M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}] = \begin{pmatrix} R_{\alpha_1\beta_1^{-1}} & R_{\alpha_1\beta_2^{-1}} & \cdots & R_{\alpha_1\beta_n^{-1}} \\ R_{\alpha_2\beta_1^{-1}} & R_{\alpha_2\beta_2^{-1}} & \cdots & R_{\alpha_2\beta_n^{-1}} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\alpha_m\beta_1^{-1}} & R_{\alpha_m\beta_2^{-1}} & \cdots & R_{\alpha_m\beta_n^{-1}} \end{pmatrix}.$$

That is, $M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}]$ consists of the matrices whose (i, j)-entry belongs to $R_{\alpha_i\beta_j^{-1}}$. Such a matrix $A \in M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}]$ gives a homomorphism of Γ-graded R-modules

$$R^n(\bar{\beta}) \to R^m(\bar{\alpha}), \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

and in this way $M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}]$ can be identified with the set of all homomorphisms of Γ-graded R-modules $R^n(\bar{\beta}) \to R^m(\bar{\alpha})$.

By $A \in \mathfrak{M}_{m\times n}(R)$, we mean that $A \in M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}]$ of some $\bar{\alpha} \in \Gamma^m$ and $\bar{\beta} \in \Gamma^n$. It is important to note that, for a matrix $A \in \mathfrak{M}_{m\times n}(R)$, it is possible that $A \in M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}] \cap M_{m\times n}(R)[\overline{\alpha'}][\overline{\beta'}]$ even if $\bar{\alpha} \neq \overline{\alpha'}$ or $\bar{\beta} \neq \overline{\beta'}$. The matrix A belongs to that intersection if whenever the (i, j)-entry of A is not zero, then $\alpha_i\beta_j^{-1} = \alpha_i'{\beta_j'}^{-1}$.

We set

$$\mathfrak{M}_\bullet(R) = \bigcup_{m,n} \mathfrak{M}_{m\times n}(R).$$

We remark that if $A \in M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}]$ and $B \in M_{n\times p}(R)[\bar{\beta}][\bar{\varepsilon}]$ then $AB \in M_{m\times p}(R)[\bar{\alpha}][\bar{\varepsilon}]$. We will say that A, B are *compatible*.

When $m = n$, we will write $M_n(R)[\bar{\alpha}][\bar{\beta}]$ and $\mathfrak{M}_n(R)$. The set of all such matrices will be denoted by $\mathfrak{M}(R)$, that is,

$$\mathfrak{M}(R) = \bigcup_n \mathfrak{M}_n(R).$$

If $A \in M_n(R)[\bar\alpha][\bar\beta]$ is an invertible matrix, then $A^{-1} \in M_n(R)[\bar\beta][\bar\alpha]$.

If $\Sigma \subseteq \mathfrak{M}(R)$, we will write $\Sigma_n[\bar\alpha][\bar\beta]$ to denote the set $\Sigma \cap M_n(R)[\bar\alpha][\bar\beta]$.

A matrix $A \in \mathfrak{M}_n(R)$ is *gr-full* if every time that $A = PQ$ for some matrices $P \in M_{n\times r}(R)[\bar\alpha][\bar\lambda]$, $Q \in M_{r\times n}(R)[\bar\lambda][\bar\beta]$, then $r \geq n$. If we think of A as a homomorphism of Γ-graded modules between two Γ-graded free R-modules, it means that for all $\bar\alpha, \bar\beta \in \Gamma^n$, such that A defines a graded homomorphism $R^n(\bar\beta) \to R^n(\bar\alpha)$, then it never factors by any graded homomorphism $R^n(\bar\beta) \to R^r(\bar\lambda)$ with $r < n$.

Suppose that $A \in M_n(R)[\bar\alpha][\bar\beta]$, $E \in M_n(R)$ is a permutation matrix obtained permuting the rows of I_n according to the permutation $\sigma \in S_n$. Then $E \in M_n(R)[\overline{\alpha'}][\bar\alpha]$, where $\overline{\alpha'} = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$, and $EA \in M_n(R)[\overline{\alpha'}][\bar\beta]$. Similarly, the matrix $E \in M_n(R)[\bar\beta][\overline{\beta'}]$, where $\overline{\beta'} = (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(n)})$, and $AE \in M_n(R)[\bar\alpha][\overline{\beta'}]$. Hence, for permutation matrices E, F of appropriate size, a matrix $A \in \mathfrak{M}(R)$ is gr-full if, and only if, EAF is gr-full.

A hollow matrix $A \in \mathfrak{M}(R)$ is not gr-full. Indeed, suppose that $A \in M_n(R)[\bar\alpha][\bar\beta]$ has an $r \times s$ block of zeros with $r + s > n$. There exist permutation matrices E, F such that $EAF = \left(\begin{smallmatrix} T & 0 \\ U & V\end{smallmatrix}\right)$, that is, the block of $r \times s$ zeros is in the north-east corner. Then

$$\begin{pmatrix} T & 0 \\ U & V\end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0 \\ U & V\end{pmatrix},$$

where $T \in M_{r\times(n-s)}(R)[\bar\alpha][\bar\beta]$, $U \in M_{(n-r)\times(n-s)}(R)[\bar\delta][\bar\beta]$, $V \in M_{(n-r)\times s}(R)[\bar\delta][\bar\varepsilon]$ for some sequences $\bar\alpha$, $\bar\beta$, $\bar\delta$, $\bar\varepsilon$ of elements of Γ. The result now follows because $\left(\begin{smallmatrix} T & 0 \\ 0 & I\end{smallmatrix}\right) \in M_{n\times(2n-r-s)}(R)[\bar\alpha * \bar\delta][\bar\beta * \bar\delta]$ and $\left(\begin{smallmatrix} I & 0 \\ U & V\end{smallmatrix}\right) \in M_{(2n-r-s)\times n}(R)[\bar\beta * \bar\delta][\bar\beta * \bar\varepsilon]$.

Let R be a Γ-graded local ring with maximal graded ideal $\mathfrak{m}$ and let $R \to R/\mathfrak{m}$, $a \mapsto \bar a$ be the natural projection. It is well known that $A = (a_{ij}) \in M_n(R)[\bar\alpha][\bar\beta]$ is invertible over R if and only if $\bar A = (\overline{a_{ij}}) \in M_n(R/\mathfrak{m})[\bar\alpha][\bar\beta]$ is invertible over the Γ-graded division ring $R/\mathfrak{m}$.

Let D be a Γ-graded division ring and M be a Γ-graded D-module. As in the ungraded case, the following assertions hold true:

(1) Any Γ-graded D-module is graded free.

(2) Any D-linearly independent subset of M consisting of homogeneous elements can be extended to a homogeneous basis of M.

(3) Any two homogeneous bases of M over D have the same cardinality.

(4) If N is a Γ-graded submodule of M, then $\dim_D(N) + \dim_D(M/N) = \dim_D(M)$.

We remark that, over a Γ-graded division ring, the concepts of gr-full matrix and of invertible matrix coincide.

2. Almost graded division rings

Throughout this section, let Γ *be a group.*

We say that a ring R is a Γ-*almost graded ring* if there is a family $\{R_\gamma : \gamma \in \Gamma\}$ of additive subgroups R_γ of R such that $1 \in R_e$, $R = \sum_{\gamma\in\Gamma} R_\gamma$ and $R_\gamma R_{\gamma'} \subseteq R_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. The name of almost graded rings was chosen to be compatible with the definition of almost strongly graded rings given in [18, p. 14]. We define $\operatorname{supp} R = \{\gamma \in \Gamma : R_\gamma \neq \{0\}\}$. Given two Γ-almost graded rings R and S, a ring homomorphism $f\colon R \to S$ is a *homomorphism of* Γ-*almost graded rings* if $f(R_\gamma) \subseteq S_\gamma$ for all $\gamma \in \Gamma$. Clearly, any Γ-graded ring $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ is a Γ-almost graded ring in the natural way. Given two Γ-graded rings R, S, a homomorphism of Γ-almost graded rings is in fact a homomorphism of Γ-graded rings.

The main examples of Γ-almost graded rings that we will consider are the following.

Example 2.1. Let R be a Γ-graded ring.

(1) Let S be ring and $f\colon R \to S$ be a ring homomorphism. Then S can be regarded as a Γ-almost graded ring with $S_\gamma = S$ for all $\gamma \in \Gamma$. Then $f\colon R \to S$ can also be regarded as homomorphism of Γ-almost graded rings.

(2) Again, let S be ring and $f\colon R \to S$ be a ring homomorphism. Then $\operatorname{Im} f$ can be regarded as a Γ-almost graded ring with $(\operatorname{Im} f)_\gamma = f(R_\gamma)$ for all $\gamma \in \Gamma$ and the restriction $f\colon R \to \operatorname{Im} f$ is a homomorphism of Γ-almost graded rings.

(3) Let Ω be a normal subgroup of Γ. If $S = \bigoplus_{\alpha\in\Gamma/\Omega} S_\alpha$ is a Γ/Ω-graded ring, then S can be endowed with a structure of Γ-almost graded ring defining $S_\gamma = S_\alpha$ for all $\gamma \in \alpha$, $\alpha \in \Gamma/\Omega$. The Γ-graded ring R can be considered as a Γ/Ω-graded ring defining $R_\alpha = \bigoplus_{\gamma\in\alpha} R_\gamma$ for each $\alpha \in \Gamma/\Omega$. If $f\colon R \to S$ is a homomorphism of Γ/Ω-graded rings, then it is a homomorphism of Γ-almost graded rings.

From a Γ-almost graded ring one can obtain a Γ-graded ring, as we proceed to describe; cf. [18, Proposition 1.2.2]. Let $S = \sum_{\gamma\in\Gamma} S_\gamma$ be a Γ-almost graded ring. The *lift* of S is the Γ-graded ring $\widetilde{S} = \bigoplus_{\gamma\in\Gamma} \widetilde{S}_\gamma$ defined as follows. Set $\widetilde{S}_\gamma$ to be a disjoint copy of S_γ. If $a \in S_\gamma$, denote by $\tilde{a} \in \widetilde{S}_\gamma$ the disjoint copy of $a \in S_\gamma$. Consider the Γ-graded additive group $\widetilde{S} = \bigoplus_{\gamma\in\Gamma} \widetilde{S}_\gamma$. Define $\widetilde{S}_\gamma \times \widetilde{S}_{\gamma'} \to \widetilde{S}_{\gamma\gamma'}$ by $(\tilde{a}, \tilde{b}) \mapsto \widetilde{ab}$, and extend it by distributivity to $\widetilde{S} \times \widetilde{S} \to \widetilde{S}$. This endows $\widetilde{S}$ with a structure of Γ-graded ring such that $\operatorname{supp} \widetilde{S} = \operatorname{supp} S$. The lift $\widetilde{S}$ of S has the following properties.

Proposition 2.2. *Let* $S = \sum_{\gamma\in\Gamma} S_\gamma$ *be a* Γ-*almost graded ring and* $\widetilde{S} = \bigoplus_{\gamma\in\Gamma} \widetilde{S}_\gamma$ *be the lift of* S. *Consider the map* $\pi\colon \widetilde{S} \to S$, $\sum_{\gamma\in\Gamma} \widetilde{a_\gamma} \mapsto \sum_{\gamma\in\Gamma} a_\gamma$. *The following statements hold true:*

(1) *π is a homomorphism of Γ-almost graded rings.*

(2) *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and $f\colon R \to S$ be a homomorphism of Γ-almost graded rings. Then there exists a unique homomorphism of Γ-graded rings $\tilde{f}\colon R \to \widetilde{S}$ such that $\pi\tilde{f} = f$. Such an $\tilde{f}$ is determined by $\tilde{f}(r_\gamma) = \widetilde{f(r_\gamma)} \in \widetilde{S}_\gamma$ for all $r_\gamma \in R_\gamma$, $\gamma \in \Gamma$.*

Proof. (1) follows easily from the definition.

(2) Since the restriction of π to $\widetilde{S}_\gamma$ is bijective for each $\gamma \in \Gamma$, we obtain the uniqueness of $\tilde{f}$. ∎

We say that the homomorphism of Γ-graded rings $\tilde{f}\colon R \to \widetilde{S}$ in Proposition 2.2 (2) is the *lift* of the homomorphism of Γ-almost graded rings $f\colon R \to S$.

We say that a nonzero ring E is a Γ-*almost graded division ring* if E is a Γ-almost graded ring such that every nonzero element $x \in E_\gamma$, $\gamma \in \Gamma$, is invertible with inverse $x^{-1} \in E_{\gamma^{-1}}$.

The following easy result tells us that Γ-almost graded division rings are graded division rings although not necessarily of type Γ.

Proposition 2.3. *Let E be a Γ-almost graded division ring. The following assertions hold true:*

(1) *If $0 \neq b \in E_\gamma$, then $bE_{\gamma'} = E_{\gamma\gamma'}$ and $E_{\gamma'}b = E_{\gamma'\gamma}$ for $\gamma \in \Gamma$.*

(2) *$E_\gamma \cdot E_{\gamma'} = E_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$.*

(3) supp *E is a subgroup of Γ.*

Proof. If $u \in E_{\gamma\gamma'}$, then $b \cdot b^{-1}u = u$, where $b^{-1}u \in E_{\gamma'}$. The other part is analogous. Thus (1) is proved.

(2) is a consequence of (1).

Since $1 \in E_e$, then (3) follows from (2). ∎

Now we give some easy relations between the existence of homomorphisms from a Γ-graded ring to division rings, to Γ-almost graded division rings and to Γ-graded division rings.

Proposition 2.4. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and $E = \sum_{\gamma\in\Gamma} E_\gamma$ be a Γ-almost graded division ring. The following assertions hold true:*

(1) *The lift $\widetilde{E} = \bigoplus_{\gamma\in\Gamma} E_\gamma$ of E is a Γ-graded division ring.*

(2) *There exists a homomorphism of Γ-almost graded rings from R to a Γ-almost graded division ring if and only if there exists a homomorphism of Γ-graded rings from R to a Γ-graded division ring.*

(3) *If there exists an (injective) homomorphism of rings from R to a division ring, then there exists an (injective) homomorphism of Γ-graded rings from R to a Γ-graded division ring.*

(4) *The converse of* (3) *is not true. That is, there exist groups Γ and Γ-graded rings for which there exist (injective) homomorphisms of Γ-graded rings to Γ-graded division rings, but for which there do not exist homomorphisms to division rings.*

Proof. (1) follows from the fact that if $a \in E_\gamma \setminus \{0\}$ for some $\gamma \in \Gamma$, then $a^{-1} \in E_{\gamma^{-1}}$. Thus $\tilde{a} \in \widetilde{E}_\gamma$ has inverse $\tilde{a}^{-1} = \widetilde{a^{-1}} \in \widetilde{E}_{\gamma^{-1}}$.

(2) Since every Γ-graded ring is a Γ-almost graded ring, we only have to prove one implication. Suppose that $f\colon R \to E$ is a homomorphism of Γ-almost graded rings. Then the lift $\tilde{f}\colon R \to \widetilde{E}$ is a homomorphism of Γ-graded rings with $\widetilde{E}$ a Γ-graded division ring by (1).

(3) Suppose $\varphi\colon R \to D$ is a homomorphism of rings where D is a division ring. Consider D as a Γ-almost graded rings as in Example 2.1 (1). Then φ can be regarded as a homomorphism of Γ-almost graded rings. Then the lift of φ, $\widetilde{\varphi}\colon \to \widetilde{D}$, gives a homomorphism of Γ-graded rings from R to the Γ-graded division ring $\widetilde{D}$.

(4) We produce an example of a graded ring for which there does not exist a homomorphism to a division ring but it is embeddable in a graded division ring. Let T be the ring obtained as a localization of $\mathbb{Z}$ at the prime ideal $3\mathbb{Z}$. Let R be the ring $T[i] \subseteq \mathbb{C}$. Let $C_2 = \langle x \rangle$ be the cyclic group of order two, and let $\sigma\colon C_2 \to \mathrm{Aut}(R)$ be the homomorphism of groups which sends x to the automorphism induced by complex conjugation. Now set $S = R[C_2; \sigma]$. That is, S is the skew group ring of G over R induced by σ. Hence S is a C_2-graded ring, $S = S_e + S_x$, where $S_e = R$ and $S_x = Rx$ and the product is determined by $xr = \bar{r}x$ for all $r \in R$. Clearly S is embeddable in the C_2 graded division ring $\mathbb{Q}(i)[C_2; \sigma]$. Suppose that there exists a homomorphism of rings from S to a division ring K. Let $\varphi\colon S \to K$ be such a homomorphism. Since $(1 - x)(1 + x) = 0$, then either $\varphi(1 + x) = 0$ or $\varphi(1 - x) = 0$. If $\varphi(1 + x) = 0$, then $0 = \varphi(1 + x) = 1 + \varphi(x)$. Thus $\varphi(x) = -1$. But then $(-1)\varphi(i) = \varphi(xi) = \varphi(-ix) = -\varphi(i)(-1) = \varphi(i)$. Since $\varphi(i) \neq 0$, then K has characteristic 2. This is a contradiction because φ induces a homomorphism from $R = S_e$ to K and 2 is invertible in R. In the same way, it can be shown that if $\varphi(1 - x) = 0$, then $\varphi(x) = 1$ and, again, it implies that the characteristic of K is 2, a contradiction. ■

Let R be a Γ-graded ring, S be a ring and $f\colon R \to S$ be a ring homomorphism. For each $\gamma \in \Gamma$, define

$$(S_0)_\gamma = f(R_\gamma).$$

If $n \geq 0$, and $(S_n)_\gamma$ has been defined for each $\gamma \in \Gamma$, define

$$\begin{aligned}(T_{n+1})_\gamma &= \{y^{-1} : y \in (S_n)_{\gamma^{-1}} \text{ and } y \text{ is invertible in } S\},\\ (S_{n+1})_\gamma &= \text{additive subgroup of } S \text{ generated by}\\ &\qquad \{x_1x_2\cdots x_r : r \in \mathbb{N},\ x_i \in (S_n)_{\gamma_i} \cup (T_{n+1})_{\gamma_i},\ \gamma_1\gamma_2\cdots\gamma_n = \gamma\}.\end{aligned}$$

Now set $(\mathrm{DC}(f))_\gamma$ = subgroup generated by $\bigcup_{n\geq 0}(S_n)_\gamma$. Then the subring of S defined by

$$\mathrm{DC}(f) = \text{additive subgroup generated by } \bigcup_{\gamma\in\Gamma}(\mathrm{DC}(f))_\gamma$$

is the *almost graded division closure of* $f\colon R \to S$. Note that $\mathrm{DC}(f)$ is a Γ-almost graded ring such that if $x \in (\mathrm{DC}(f))_\gamma$ and x is invertible in S, then $x^{-1} \in (\mathrm{DC}(f))_{\gamma^{-1}}$. It is the least subring of S that contains $\mathrm{Im}\, f$ and is closed under inversion of almost homogeneous elements.

If $\mathrm{DC}(f) = S$ and $\mathrm{DC}(f)$ is a Γ-almost graded division ring, we say that S is the Γ*-almost graded division ring generated by* $\mathrm{Im}\, f$.

Notice also that if S is a division ring, then $\mathrm{DC}(f)$ is a Γ-almost graded division ring.

Note that if S is a Γ-graded ring, and $f\colon R \to S$ is a homomorphism of Γ-graded rings, then $(S_n)_\gamma \subseteq S_\gamma$ for each $n \geq 0$. Therefore $(\mathrm{DC}(f))_\gamma \subseteq S_\gamma$ and $\mathrm{DC}(f)$ is a Γ-graded subring of S. It is the least subring of S that contains $\mathrm{Im}\, f$ and is closed under inversion of homogeneous elements. Moreover, if S is a Γ-graded division ring, then $\mathrm{DC}(f)$ is a Γ-graded division subring of S. In this case, if $S = \mathrm{DC}(f)$ we say that S is the Γ*-graded division ring generated by* $\mathrm{Im}\, f$.

Proposition 2.5. *Let Γ be a group, $D = \bigoplus_{\gamma\in\Gamma} D_\gamma$ be a Γ-graded division ring, and let $f\colon D \to S$ be a ring homomorphism with S a nonzero ring. The following assertions hold true:*

(1) $\mathrm{DC}(f)$ *is a Γ-almost graded division ring with*

$$\mathrm{DC}(f)_\gamma = (\mathrm{Im}\, f)_\gamma = \{f(x) : x \in D_\gamma\}$$

and $D = \widetilde{\mathrm{DC}(f)}$.

(2) *The sets*

$$\begin{aligned}\Upsilon &= \{A \in \mathfrak{M}(D) : A \text{ is invertible over } D\},\\ \Sigma &= \{A \in \mathfrak{M}(D) : A^f \text{ is invertible over } S\}\end{aligned}$$

coincide.

(3) *If R is a Γ-graded ring and $\varphi\colon R \to D$ is a homomorphism of Γ-graded rings then the sets*

$$\Upsilon_\varphi = \{A \in \mathfrak{M}(R) : A^\varphi \textit{ is invertible over } D\},$$
$$\Sigma_\varphi = \{A \in \mathfrak{M}(R) : A^{(f\varphi)} \textit{ is invertible over } S\}$$

coincide.

Proof. (1) has already been proved.

(2) Clearly, if $A \in \Upsilon$, then $A \in \Sigma$. Suppose now that $A \in M_n(R)[\bar{\alpha}][\bar{\beta}]$ such that $A \notin \Upsilon$. Then there exists a nonzero homogeneous column

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M_{n\times 1}(R)[\bar{\beta}][\delta]$$

of degree δ as an element of $R^n(\bar{\beta})$ such that

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

Note that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^f \neq 0,$$

because D is a graded division ring and S is not the zero ring. Thus

$$A^f \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^f = 0,$$

which implies that $A \notin \Sigma$.

(3) follows from (2) because $\Sigma_\varphi = \{A \in \mathfrak{M}(R) : A^\varphi \in \Sigma\}$ and $\Upsilon_\varphi = \{A \in \mathfrak{M}(R) : A^\varphi \in \Upsilon\}$. ■

3. Graded rational closure

Throughout this section, let Γ be a group.

We begin this section introducing some important notation that will be used throughout.

Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \Gamma^n$, $\overline{\alpha'} = (\alpha'_1, \dots, \alpha'_m) \in \Gamma^m$ and $\delta \in \Gamma$; then we define

$$\bar{\alpha} * \overline{\alpha'} := (\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_m) \in \Gamma^{n+m},$$
$$\bar{\alpha} \cdot \delta := (\alpha_1 \delta, \dots, \alpha_n \delta) \in \Gamma^n.$$

Let R be a Γ-graded ring and S be a ring.

For each $A \in \mathfrak{M}_n(R)$, the last column will be called A_∞ and the matrix consisting of the remaining $n-1$ columns will be called $A_\bullet$. We will write $A = (A_\bullet \ A_\infty)$.

For each sequence $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \Gamma^n$, the last element α_n will be denoted α_∞, and $(\alpha_1, \dots, \alpha_{n-1})$ will be denoted by $\alpha_\bullet$. Thus $\bar{\alpha} = \alpha_\bullet * \alpha_\infty$.

For $u \in M_{n\times 1}(S)$, the last entry of u will be denoted by u_∞ and the $(n-1) \times 1$ column consisting of the remaining entries will be denoted by $u_\bullet$. Hence $u = \left(\begin{smallmatrix} u_\bullet \\ u_\infty \end{smallmatrix}\right)$. We remark that if $n = 1$, then $A_\bullet$, $\alpha_\bullet$, $u_\bullet$ are empty and thus $A = A_\infty$, $\bar{\alpha} = \alpha_\infty$ and $u = u_\infty$.

If $A \in \mathfrak{M}_{n\times(n+1)}(R)$, we will denote by A_0 its first column, by A_∞ its last column and by $A_\bullet$ the matrix consisting of the other $n-1$ columns; that is, we will write $A = (A_0 \ A_\bullet \ A_\infty)$. We will call the matrix $(A_0 \ A_\bullet)$ the *numerator* of A and the matrix $(A_\bullet \ A_\infty)$ the *denominator* of A. If $A \in M_{n\times(n+1)}(R)[\bar{\alpha}][\bar{\beta}]$, we suppose $\bar{\beta}$ is divided as $\beta_0 * \beta_\bullet * \beta_\infty$. If $u \in M_{(n+1)\times 1}(S)$, we will write $u = \left(\begin{smallmatrix} u_0 \\ u_\bullet \\ u_\infty \end{smallmatrix}\right)$. Again, we remark that if $n = 1$, then $A_\bullet$, $\beta_\bullet$, $u_\bullet$ are empty and thus $A = (A_0 \ A_\infty)$, $\bar{\beta} = (\beta_0, \beta_\infty)$ and $u = \left(\begin{smallmatrix} u_0 \\ u_\infty \end{smallmatrix}\right)$.

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and $\Sigma \subseteq \mathfrak{M}(R)$.

We say that the subset Σ of $\mathfrak{M}(R)$ is *gr-lower semimultiplicative* if it satisfies the following two conditions:

(i) $(1) \in \Sigma$, i.e. the identity matrix of size 1×1 belongs to Σ.

(ii) If $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$ and $B \in \Sigma_m[\overline{\alpha'}][\overline{\beta'}]$, then the matrix $\left(\begin{smallmatrix} A & 0 \\ C & B \end{smallmatrix}\right) \in \Sigma$ for any $C \in M_{m\times n}(R)[\overline{\alpha'}][\bar{\beta}]$. Notice that the matrix $\left(\begin{smallmatrix} A & 0 \\ C & B \end{smallmatrix}\right) \in M_{(n+m)}(R)[\bar{\alpha} * \overline{\alpha'}][\bar{\beta} * \overline{\beta'}]$.

A gr-upper semimultiplicative subset of $\mathfrak{M}(R)$ is defined analogously.

A subset Σ of $\mathfrak{M}(R)$ is *gr-multiplicative* if it satisfies the following two conditions:

(i) Σ is gr-lower semimultiplicative.

(ii) If $A \in \Sigma$, then $EAF \in \Sigma$ for any permutation matrices E, F of appropriate size.

REMARK 3.1. We remark that if Σ is gr-multiplicative then it is also an upper gr-semimultiplicative subset of $\mathfrak{M}(R)$. Indeed, suppose that $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$, $B \in \Sigma_m[\overline{\alpha'}][\overline{\beta'}]$ and $C \in M_{n\times m}(R)[\bar{\alpha}][\overline{\beta'}]$. Then, since Σ is lower gr-semimultiplicative, $\left(\begin{smallmatrix} B & 0 \\ C & A \end{smallmatrix}\right) \in \Sigma$. But now $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = E^{-1}\left(\begin{smallmatrix} B & 0 \\ C & A \end{smallmatrix}\right)E \in \Sigma$ for some permutation matrix E, as desired.

Proposition 3.2. *Let R be a Γ-graded ring, S be a ring and $f\colon R\to S$ be a ring homomorphism. Then the set*

$$\Sigma = \left\{M \in \mathfrak{M}(R) : M^f \text{ is invertible over } S\right\}$$

is gr-multiplicative.

Proof. Clearly the 1×1 matrix $(1)\in\Sigma$.

Let $A\in\Sigma_n[\bar\alpha][\bar\beta]$, $B\in\Sigma_m[\overline{\alpha'}][\overline{\beta'}]$ and $C\in M_{m\times n}[\overline{\alpha'}][\bar\beta]$. Then the matrix $\left(\begin{smallmatrix} A & 0\\ C & B\end{smallmatrix}\right)^f$ belongs to Σ because it is invertible with inverse

$$\begin{pmatrix} (A^f)^{-1} & 0\\ -(B^f)^{-1}(C^f)(A^f)^{-1} & (B^f)^{-1}\end{pmatrix}.$$

Notice that if E, F are permutation matrices, then E^f, F^f are also permutation matrices. Hence, if $A\in\Sigma$, then the matrix $(EAF)^f$ is invertible with inverse $(F^f)^{-1}(A^f)^{-1}(E^f)^{-1}$. ■

Note that if S is a Γ-graded ring, $f\colon R\to S$ is a graded homomorphism and $A\in M_n(R)[\bar\alpha][\bar\beta]$, then $A^f\in M_n(S)[\bar\alpha][\bar\beta]$. Moreover, if A^f is invertible, then $(A^f)^{-1}\in M_n(S)[\bar\beta][\bar\alpha]$, and the (j,i)-entry of $(A^f)^{-1}$ belongs to $R_{\beta_j\alpha_i^{-1}}$. With this in mind, we make the following definition.

Let $R=\bigoplus_{\gamma\in\Gamma}R_\gamma$ be a Γ-graded ring and $\Sigma\subseteq\mathfrak{M}(R)$. Let S be a ring (not necessarily graded) and $f\colon R\to S$ be a Σ-inverting ring homomorphism. For $\gamma\in\Gamma$, we define the *homogeneous rational closure of degree* γ as the set $(Q_f(\Sigma))_\gamma$ consisting of all $x\in S$ such that there exist $\bar\alpha,\bar\beta\in\Gamma^n$ and $A\in\Sigma_n[\bar\alpha][\bar\beta]$ such that $\gamma=(\alpha_i\beta_j^{-1})^{-1}=\beta_j\alpha_i^{-1}$ and x is the (j,i)-entry of $(A^f)^{-1}$ (for some positive integer n and $i,j\in\{1,\dots,n\}$). The *homogeneous rational closure* is the set

$$Q_f(\Sigma)=\bigcup_{\gamma\in\Gamma}(Q_f(\Sigma))_\gamma.$$

The *graded rational closure*, denoted by $R_f(\Sigma)$, is the additive subgroup of S generated by $Q_f(\Sigma)$.

When the set Σ is gr-lower semimultiplicative, the graded rational closure $R_f(\Sigma)$ is a subring of S as the following results show.

Lemma 3.3. *Let $R=\bigoplus_{\gamma\in\Gamma}R_\gamma$ be a Γ-graded ring and Σ be a gr-lower semimultiplicative subset of $\mathfrak{M}(R)$. Let S be a ring and $f\colon R\to S$ be a Σ-inverting ring homomorphism. Fix $\gamma\in\Gamma$. For $x\in S$, the following conditions are equivalent:*

(1) $x\in(Q_f(\Sigma))_\gamma$.

(2) *There exist* $\bar{\alpha}, \bar{\beta} \in \Gamma^n$ *and* $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$ *such that* $\alpha_i = e$, $\beta_j = \gamma$ *and* x *is the* (j, i)*-entry of* $(A^f)^{-1}$.

(3) *There exist* $\bar{\alpha}, \bar{\beta} \in \Gamma^n$, $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$ *and* $u \in M_{n\times 1}(S)$ *such that* $\alpha_i = e$, $\beta_j = \gamma$, $u_j = x$ *and* $A^f u = e_i$.

(4) *There exist* $\bar{\alpha}, \bar{\beta} \in \Gamma^n$, $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$, $a \in M_{n\times 1}(R)[\bar{\alpha}][e]$ *and* $u \in M_{n\times 1}(S)$ *such that* $\beta_j = \gamma$, $u_j = x$ *and* $A^f u = a^f$.

(5) *There exist* $\bar{\alpha}, \bar{\beta} \in \Gamma^n$, $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$, $a \in M_{n\times 1}(R)[\bar{\alpha}][e]$ *and* $u \in M_{n\times 1}(S)$ *such that* $\beta_\infty = \gamma$, $u_\infty = x$ *and* $A^f u = a^f$.

(6) *There exist* $\bar{\alpha}, \bar{\beta} \in \Gamma^n$, $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$, $b \in M_{1\times n}(R)[\gamma][\bar{\beta}]$ *and* $c \in M_{n\times 1}(R)[\bar{\alpha}][e]$ *such that* $x = b^f (A^f)^{-1} c^f$.

(7) *There exist* $\bar{\alpha} \in \Gamma^n$, $\bar{\beta} \in \Gamma^{n+1}$, $A \in M_{n\times(n+1)}(R)[\bar{\alpha}][\bar{\beta}]$ *and* $u \in M_{(n+1)\times 1}(S)$ *such that* $\beta_0 = e$, $\beta_\infty = \gamma$, $u_0 = 1$, $u_\infty = x$, $(A_\bullet\ A_\infty) \in \Sigma$ *and* $A^f u = 0$.

Proof. (1) ⇒ (2) Let $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$ such that x is the (j, i)-entry of $(A^f)^{-1}$ and $\gamma = (\alpha_i \beta_j^{-1})^{-1} = \beta_j \alpha_i^{-1}$ for some i, j. Then A can be regarded as a matrix in $A \in \Sigma_n[\bar{\alpha} \cdot \alpha_i^{-1}][\bar{\beta} \cdot \alpha_i^{-1}]$ and thus (2) follows.

(2) ⇒ (3) Suppose that (2) holds. Let u be the ith column of $(A^f)^{-1}$. Then $A^f u = e_i$, as desired.

(3) ⇒ (4) This is clear because $e_i \in M_{n\times 1}(R)[\bar{\alpha}][e]$ and $e_i^f = e_i$.

(4) ⇒ (5) Let $A \in \Sigma$, i, j, a and u be as in (4). Suppose that $A^f u = a^f$ with $u_j = x$. The matrix $\left(\begin{smallmatrix} A & 0 \\ -e_j^t & 1 \end{smallmatrix}\right) \in \Sigma_{n+1}[\bar{\alpha} * \beta_j][\bar{\beta} * \beta_j]$. Notice that it belongs to Σ because Σ is gr-lower semimultiplicative. The matrix $\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) \in M_{(n+1)\times 1}(R)[\bar{\alpha} * \beta_j][e]$. Now (5) follows from the equality

$$\begin{pmatrix} A^f & 0 \\ -e_j^t & 1 \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} a^f \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}^f .$$

(5) ⇒ (6) From (5) we obtain that $u = (A^f)^{-1} a^f$. Hence

$$x = (e_n^t)^f u = (e_n^t)^f (A^f)^{-1} a^f .$$

Now (6) follows because $e_n^t \in M_{1\times n}(R)[\gamma][\bar{\beta}]$.

(6) ⇒ (1) Let A, b and c as in (6). Then

$$\begin{pmatrix} 1 & 0 & 0 \\ c & A & 0 \\ 0 & b & 1 \end{pmatrix} \in \Sigma_{(n+2)\times(n+2)}[e * \bar{\alpha} * \gamma][e * \bar{\beta} * \gamma].$$

Moreover,

$$\begin{pmatrix} 1 & 0 & 0 \\ c^f & A^f & 0 \\ 0 & b^f & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -(A^f)^{-1}c^f & (A^f)^{-1} & 0 \\ b^f(A^f)^{-1}c^f & -b^f(A^f)^{-1} & 1 \end{pmatrix}.$$

Thus $x = b^f(A^f)^{-1}c^f$ belongs to $(Q_f(\Sigma))_\gamma$.

(5) $\Leftrightarrow$ (7) Suppose $A \in M_n(R)[\bar{\alpha}][\bar{\beta}]$ with $\beta_\infty = \gamma$, $a \in M_{n\times 1}(R)[\bar{\alpha}][e]$ and $u \in M_{n\times 1}(S)$ with $u_\infty = x$. Then the equality $A^f u = a^f$ is equivalent to the equality

$$\begin{pmatrix} -a^f & A^f \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} = 0.$$

Notice that $(-a \;\; A) \in M_{n\times(n+1)}(R)[\bar{\alpha}][e * \bar{\beta}]$. ■

Theorem 3.4. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and Σ be a gr-lower semimultiplicative subset of $\mathfrak{M}(R)$. Let S be a ring and $f\colon R \to S$ a Σ-inverting ring homomorphism. Then*

(1) *for each $\gamma \in \Gamma$, $f(R_\gamma) \subseteq (Q_f(\Sigma))_\gamma$;*

(2) *if $\gamma \in \Gamma$ and $x, y \in (Q_f(\Sigma))_\gamma$, then $x + y \in (Q_f(\Sigma))_\gamma$;*

(3) *if $\gamma, \delta \in \Gamma$ and $x \in (Q_f(\Sigma))_\gamma$, $y \in (Q_f(\Sigma))_\delta$, then $xy \in (Q_f(\Sigma))_{\gamma\delta}$.*

Hence $R_f(\Sigma)$ is a Γ-almost graded ring (which is a subring of S) that contains $\mathrm{Im}(f)$. *Furthermore,*

(4) *the restriction $f\colon R \to R_f(\Sigma)$ is a ring epimorphism;*

(5) *if S is a Γ-graded ring and $f\colon R \to S$ is a homomorphism of Γ-graded rings, then $(Q_f(\Sigma))_\gamma \subseteq S_\gamma$ for each $\gamma \in \Gamma$ and $R_f(\Sigma) = \bigoplus_{\gamma\in\Gamma}(Q_f(\Sigma))_\gamma$ is a Γ-graded subring of S such that* $\mathrm{h}(R_f(\Sigma)) = Q_f(\Sigma)$.

Proof. (1) Let $r \in R_\gamma$. Then $f(1)f(r) = f(r)$, where $1 \in M_1(R)[\gamma][\gamma]$ and $r \in M_1(R)[\gamma][e]$. Then Lemma 3.3 (5) implies that $f(r) \in (Q_f(\Sigma))_\gamma$.

(2) Let $x, y \in (Q_f(\Sigma))_\gamma$. By Lemma 3.3 (5), there exist $\bar{\alpha}, \bar{\beta} \in \Gamma^n$, $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$, $a \in M_{n\times 1}(R)[\bar{\alpha}][e]$ and $u \in M_{n\times 1}(S)$ such that $\beta_\infty = \gamma$, $u_\infty = x$ and

$$A^f u = \begin{pmatrix} A^f_\bullet & A^f_\infty \end{pmatrix} \begin{pmatrix} u_\bullet \\ x \end{pmatrix} = a^f.$$

There also exist $B \in \Sigma_{n'}[\overline{\alpha'}][\overline{\beta'}]$, $b \in M_{n'\times 1}(R)[\overline{\alpha'}][e]$ and $v \in M_{n'\times 1}(S)$ such that $\beta'_\infty = \gamma$, $v_\infty = y$ and

$$B^f v = \begin{pmatrix} B^f_\bullet & B^f_\infty \end{pmatrix} \begin{pmatrix} v_\bullet \\ y \end{pmatrix} = b^f.$$

Then the matrix

$$\left(\begin{array}{cc|c} A_\bullet & A_\infty & 0 \\ \hline 0 & -B_\infty & B \end{array}\right) \in \Sigma_{n+n'}[\bar{\alpha} * \overline{\alpha'}][\bar{\beta} * \overline{\beta'}],$$

the column $\binom{a}{b} \in M_{(n+n')\times 1}[\bar{\alpha} * \overline{\alpha'}][e]$ and we have the equality

$$\left(\begin{array}{cc|c} A^f_\bullet & A^f_\infty & 0 \\ \hline 0 & -B^f_\infty & B^f \end{array}\right)\left(\begin{array}{c} u_\bullet \\ x \\ \hline v_\bullet \\ x+y \end{array}\right) = \begin{pmatrix} a^f \\ b^f \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}^f.$$

Hence $x + y \in (Q_f(\Sigma))_\gamma$.

(3) Let $x \in (Q_f(\Sigma))_\gamma$ and $y \in (Q_f(\Sigma))_\delta$. There exist $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$, $a \in M_{n\times 1}(R)[\bar{\alpha}][e]$ and $u \in M_{n\times 1}(S)$ such that $\beta_\infty = \gamma$, $u_\infty = x$ and $A^f u = a^f$. There also exist $B \in \Sigma_{n'}[\overline{\alpha'}][\overline{\beta'}]$, $b \in M_{n'\times 1}(R)[\overline{\alpha'}][e]$ and $v \in M_{n'\times 1}(S)$ such that $\beta'_\infty = \delta$, $v_\infty = y$ and $B^f v = b^f$. Now

$$\left(\begin{array}{cc|c} B_\bullet & B_\infty & 0 \\ \hline 0 & -a & A \end{array}\right) \in \Sigma_{n'+n}[\overline{\alpha'} * \bar{\alpha}\beta'_\infty][\overline{\beta'} * \bar{\beta}\beta'_\infty],$$

with $(\overline{\beta'} * \bar{\beta}\beta'_\infty)_\infty = \gamma\delta$, $\binom{b}{0} \in M_{(n'+n)\times 1}(R)[\overline{\alpha'} * \bar{\alpha}\beta'_\infty][e]$ and we have the equality

$$\left(\begin{array}{cc|c} B^f_\bullet & B^f_\infty & 0 \\ \hline 0 & -a^f & A^f \end{array}\right)\left(\begin{array}{c} v_\bullet \\ y \\ \hline u_\bullet y \\ xy \end{array}\right) = \begin{pmatrix} b^f \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}^f.$$

Hence $xy \in (Q_f(\Sigma))_{\gamma\delta}$.

From (1)–(3), it is easy to show that $R_f(\Sigma)$ is a Γ-almost graded ring and a subring of S.

(4) Let $g, h \colon R_f(\Sigma) \to T$ be ring homomorphisms such that $gf = hf$. If $x \in (Q_f(\Sigma))_\gamma$, then x is an entry of a square matrix B which is the inverse of A^f for some $A \in \Sigma$. From $A^f B = BA^f = I$, it follows that $A^{gf} B^g = B^g A^{gf} = I$ and $A^{hf} B^h = B^h A^{hf} = I$. Thus $B^g = B^h$, and $g(x) = h(x)$. Since $R_f(\Sigma)$ is generated by $(Q_f(\Sigma))_\gamma$, $\gamma \in \Gamma$, then $f \colon R \to R_f(\Sigma)$ is a ring epimorphism.

(5) Now suppose that S is a Γ-graded ring and $f \colon R \to S$ is a homomorphism of Γ-graded rings. Let $x \in (Q_f(\Sigma))_\gamma$. There exist $A \in \Sigma_n[\bar{\alpha}][\bar{\beta}]$, $a \in M_{n\times 1}(R)[\bar{\alpha}][e]$ and $u \in M_{n\times 1}(S)$ such that $\beta_\infty = \gamma$, $u_\infty = x$ and $A^f u = a^f$. Notice that $A^f \in M_n(S)[\bar{\alpha}][\bar{\beta}]$ is an invertible matrix and that $a^f \in M_{n\times 1}(S)[\bar{\alpha}][e]$. The matrix

$(A^f)^{-1} \in M_n(S)[\bar{\beta}][\bar{\alpha}]$. Now $(A^f)^{-1}$ and a^f are compatible and $u = (A^f)^{-1}a^f$. Then $x = u_\infty \in S_{\beta_\infty}$, that is, $x \in S_\gamma$.

By (1)–(3), it is easy to prove that $R_f(\Sigma)$ is a graded subring of S whose set of homogeneous elements equals $Q_f(\Sigma)$. ■

Lemma 3.5 (Cramer's rule). *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and Σ be a subset of $\mathfrak{M}(R)$. Let S be a ring and $f\colon R \to S$ be a Σ-inverting ring homomorphism.*

Let $\gamma \in \Gamma$ and $x \in (Q_f(\Sigma))_\gamma$. We suppose that $\bar{\alpha} \in \Gamma^n$, $\bar{\beta} \in \Gamma^{n+1}$, $A \in M_{n\times(n+1)}(R)[\bar{\alpha}][\bar{\beta}]$ and $u \in M_{(n+1)\times 1}(S)$ such that $\beta_0 = e$, $\beta_\infty = \gamma$, $u_0 = 1$, $u_\infty = x$, $(A_\bullet\ A_\infty) \in \Sigma$ and $A^f u = 0$. Then the following assertions hold true:

(1) *x is invertible in S if, and only if, the matrix $(A_0\ A_\bullet)^f$ is invertible in $M_n(S)$.*

(2) *x is a regular element of S if, and only if, the matrix $(A_0\ A_\bullet)^f$ is a regular element of $M_n(S)$.*

(3) *If $x = 0$, then the matrix $(A_0\ A_\bullet)^f$ is not full over S. Furthermore, if S is a Γ-graded ring and $f\colon R \to S$ is a homomorphism of graded rings, then the matrix $(A_0\ A_\bullet)^f \in M_n(S)[\bar{\alpha}][\beta_0 * \beta_\bullet]$ is not gr-full over S.*

Proof. First note the equality

$$\begin{pmatrix} A_\bullet^f & -A_0^f \end{pmatrix} = \begin{pmatrix} A_\bullet^f & A_\infty^f \end{pmatrix} \begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix}. \tag{3.1}$$

Also notice that the homogeneous matrix $(A_\bullet^f\ A_\infty^f)$ is invertible in $M_n(S)$ because f is Σ-inverting.

(1) Suppose that x is invertible in S. Then

$$\begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix}$$

is invertible in $M_n(S)$. Hence $(A_\bullet^f\ -A_0^f)$ is invertible, and therefore $(A_0^f\ A_\bullet^f)$ is invertible in $M_n(S)$.

Conversely, suppose that $(A_0^f\ A_\bullet^f)$ is invertible in $M_n(S)$. Hence the fact that $(A_\bullet^f\ -A_0^f)$ is invertible and (3.1) imply that

$$\begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix}$$

is invertible in $M_n(S)$. Thus there exists $\left(\begin{smallmatrix} v & w \\ y & z \end{smallmatrix}\right) \in M_n(S)$ such that

$$\begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix}\begin{pmatrix} v & w \\ y & z \end{pmatrix} = I, \quad \begin{pmatrix} v & w \\ y & z \end{pmatrix}\begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix} = I.$$

Thus $xz = 1$, $yI = 0$ and $yu_\bullet + zx = 1$. Therefore x is invertible in S.

(2) follows easily from (3.1).

(3) Suppose that $x = 0$. Then (3.1) can be expressed as

$$\begin{aligned} \begin{pmatrix} A_\bullet^f & -A_0^f \end{pmatrix} &= \begin{pmatrix} A_\bullet^f & A_\infty^f \end{pmatrix}\begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix} \\ &= \begin{pmatrix} A_\bullet^f & A_\infty^f \end{pmatrix}\begin{pmatrix} I & u_\bullet \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A_\bullet^f & A_\infty^f \end{pmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix}\begin{pmatrix} I & u_\bullet \end{pmatrix}, \end{aligned}$$

which implies that $(A_\bullet^f \;\; -A_0^f)$ is not full and therefore $(A_0^f \;\; A_\bullet^f)$ is not full.

If, moreover, $f\colon R \to S$ is a homomorphism of Γ-graded rings, then we have that $(A_\bullet \;\; -A_0)^f \in M_n(S)[\bar{\alpha}][\beta_\bullet * e]$, $(A_\bullet \;\; A_\infty)^f \in M_n(S)[\bar{\alpha}][\beta_\bullet * \beta_\infty]$, $\left(\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right) \in M_{n\times(n-1)}(S)[\beta_\bullet * \beta_\infty][\beta_\bullet]$ and $(I \;\; u_\bullet) \in M_{(n-1)\times n}(S)[\beta_\bullet][\beta_\bullet * e]$. It implies that the matrix $(A_\bullet \;\; -A_0)^f$ is not gr-full, which in turn implies that $(A_0 \;\; A_\bullet)^f$ is not gr-full, as desired. ■

Given A and x as in Lemma 3.5, we say that $(A_0 \;\; A_\bullet)$ is the *numerator of* x and $(A_\bullet \;\; A_\infty)$ is the *denominator of* x. Thus x is invertible in S if and only if its numerator is invertible in $M_n(S)$.

Theorem 3.6. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. Let S be a ring and $f\colon R \to S$ be a ring homomorphism. Set*

$$\Sigma = \left\{A \in \mathfrak{M}(R) : A^f \text{ is invertible over } S\right\}.$$

If $x \in (Q_f(\Sigma))_\gamma$ is invertible in S, then $x^{-1} \in (Q_f(\Sigma))_{\gamma^{-1}}$.

Moreover, if S is a Γ-almost graded division ring and $f\colon R \to S$ is a homomorphism of Γ-almost graded rings, then $R_f(\Sigma)$ is a Γ-almost graded division subring of S that equals DC(f).

Proof. Let $x \in (Q_f(\Sigma))_\gamma$. By Lemma 3.3 (7), there exist $A \in M_{n\times(n+1)}(R)[\bar{\alpha}][\bar{\beta}]$ and $u \in M_{(n+1)\times 1}(S)$ such that $\beta_0 = e$, $\beta_\infty = \gamma$, $u_0 = 1$, $u_\infty = x$, $(A_\bullet \;\; A_\infty) \in \Sigma$ and

$A^f u = (A_0^f \ A_\bullet^f \ A_\infty^f)\left(\begin{smallmatrix}1\\ u_\bullet \\ x\end{smallmatrix}\right) = 0$. Equivalently, $A_0^f + A_\bullet^f u_\bullet + A_\infty^f u_\infty = 0$. Hence $A_0^f x^{-1} + A_\bullet^f u_\bullet x^{-1} + A_\infty^f = 0$, or equivalently

$$\begin{pmatrix} A_\infty^f & A_\bullet^f & A_0^f \end{pmatrix} \begin{pmatrix} 1 \\ u_\bullet x^{-1} \\ x^{-1} \end{pmatrix} = 0.$$

Since x is invertible, Cramer's rule implies that the matrix $(A_0^f \ A_\bullet^f)$ is invertible over S. Thus $(A_\bullet^f \ A_0^f)$ is invertible over S, and therefore $(A_\bullet \ A_0) \in \Sigma$. Moreover, notice that $(A_\infty \ A_\bullet \ A_0) \in M_{n\times(n+1)}(R)[\bar{\alpha}][\beta_\infty * \beta_\bullet * \beta_0]$. This can also be expressed as $A \in M_{n\times(n+1)}(R)[\bar{\alpha}\beta_\infty^{-1}][\beta_\infty\beta_\infty^{-1} * \beta_\bullet\beta_\infty^{-1} * \beta_0\beta_\infty^{-1}]$. By Lemma 3.3 (7), and observing the equality $\beta_\infty\beta_\infty^{-1} * \beta_\bullet\beta_\infty^{-1} * \beta_0\beta_\infty^{-1} = e * \beta_\bullet\beta_\infty^{-1} * \gamma^{-1}$ in Γ^{n+1}, we get that $x^{-1} \in (Q_f(\Sigma))_{\gamma^{-1}}$.

As noted in Propositions 2.4 and 2.2, the lift $\widetilde{S}$ of S is a Γ-graded division ring and the lift $\tilde{f} \colon R \to \widetilde{S}$ of f is a homomorphism of Γ-graded rings such that $f = \pi\tilde{f}$, where $\pi \colon \widetilde{S} \to S$ is the natural homomorphism. Note that the sets Σ and $\{A \in \mathfrak{M}(R) : A^{\tilde{f}}$ is invertible over $\widetilde{S}\}$ coincide by Proposition 2.5 (3). By the foregoing and Theorem 3.4 (5), $R_{\tilde{f}}(\Sigma)$ is a Γ-graded division ring which equals $\mathrm{DC}(\tilde{f})$. Now observe that $\varphi(R_{\tilde{f}}(\Sigma)) = R_f(\Sigma)$ and $\varphi(\mathrm{DC}(\tilde{f})) = \mathrm{DC}(f)$. ∎

Corollary 3.7. *Let R be a Γ-graded ring, K be a Γ-graded division ring and $f \colon R \to K$ be a homomorphism of Γ-graded rings. If*

$$\Sigma = \left\{A \in \mathfrak{M}(R) : A^f \text{ is invertible over } K\right\}$$

and K is generated as a Γ-graded division ring by the image of f, then $K = R_f(\Sigma)$. ∎

We end this section with an interesting result, but one that will not be used in later sections. We show that two elements (and by induction any finite number of elements) can be brought to a common denominator.

Lemma 3.8. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and Σ be a gr-lower semi-multiplicative subset of $\mathfrak{M}(R)$. Let S be a ring and $f \colon R \to S$ a Σ-inverting ring homomorphism.*

If $x \in (Q_f(\Sigma))_\gamma$ and $y \in (Q_f(\Sigma))_\delta$ for some $\gamma, \delta \in \Gamma$, then they can be brought to a common denominator.

Proof. Let $x \in (Q_f(\Sigma))_\gamma$ and $y \in (Q_f(\Sigma))_\delta$. There exist $A \in M_{n\times(n+1)}(R)[\bar{\alpha}][\bar{\beta}]$ and $u \in M_{(n+1)\times 1}(S)$ such that $\beta_0 = e$, $\beta_\infty = \gamma$, $u_0 = 1$, $u_\infty = x$, $(A_\bullet \ A_\infty) \in \Sigma$

and $A^f u = (A_0^f \ A_\bullet^f \ A_\infty^f)\left(\begin{smallmatrix}1\\ u_\bullet\\ x\end{smallmatrix}\right) = 0$. There also exist $B \in M_{n'\times(n'+1)}(R)[\overline{\alpha'}][\overline{\beta'}]$ and $v \in M_{(n'+1)\times 1}(S)$ such that $\beta'_0 = e$, $\beta'_\infty = \delta$, $v_0 = 1$, $v_\infty = y$, $(B_\bullet \ B_\infty) \in \Sigma$ and $B^f v = (B_0^f \ B_\bullet^f \ B_\infty^f)\left(\begin{smallmatrix}1\\ v_\bullet\\ y\end{smallmatrix}\right) = 0$. Then

$$\left(\begin{array}{c|cc|cc} A_0 & A_\bullet & A_\infty & 0 & 0 \\ \hline 0 & 0 & -B_\infty & B_\bullet & B_\infty \end{array}\right)^f \begin{pmatrix} 1 \\ \hline u_\bullet \\ x \\ \hline 0 \\ x \end{pmatrix} = 0,$$

$$\left(\begin{array}{c|cc|cc} 0 & A_\bullet & A_\infty & 0 & 0 \\ \hline B_0 & 0 & -B_\infty & B_\bullet & B_\infty \end{array}\right)^f \begin{pmatrix} 1 \\ \hline 0 \\ 0 \\ \hline v_\bullet \\ y \end{pmatrix} = 0.$$

Now,

$$\left(\begin{array}{c|cc|cc} A_0 & A_\bullet & A_\infty & 0 & 0 \\ \hline 0 & 0 & -B_\infty & B_\bullet & B_\infty \end{array}\right) \in M_{(n+n')\times(n+n'+1)}(R)[\bar{\alpha} * \overline{\alpha'}{\beta'_\infty}^{-1}\beta_\infty][\bar{\beta} * \bar{\varepsilon}],$$

where $\bar{\varepsilon} = \beta'_\bullet {\beta'_\infty}^{-1}\beta_\infty * \beta_\infty$ and

$$\left(\begin{array}{c|cc|cc} 0 & A_\bullet & A_\infty & 0 & 0 \\ \hline B_0 & 0 & -B_\infty & B_\bullet & B_\infty \end{array}\right) \in M_{(n+n')\times(n+n'+1)}(R)[\bar{\alpha}\beta_\infty^{-1}\beta'_\infty * \overline{\alpha'}][\bar{\nu}],$$

where $\bar{\nu} = \beta'_0 * \beta_\bullet\beta_\infty^{-1}\beta'_\infty * \beta'_\infty * \beta'_\bullet * \beta'_\infty$. ∎

4. The category of graded R-division rings and gr-specializations

This section is an adaptation of [5, Section 7.2] to the graded situation.

Throughout this section, let Γ *be a group.*

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring.

A Γ*-graded* R*-ring* is a pair (K, φ), where K is a Γ-graded ring and $\varphi\colon R \to K$ is a homomorphism of graded rings. A *graded* R*-subring* of (K, φ) is a graded subring L of K such that $\varphi(R) \subseteq L$.

A Γ*-graded* R*-division ring* is a Γ-graded R-ring (K, φ) such that K is a Γ-graded division ring. If $K = \mathrm{DC}(\varphi)$, that is, K is the Γ-graded division ring generated by the image of φ, we say that (K, φ) is a Γ*-graded epic* R*-field*.

A *homomorphism of* Γ*-graded* R*-rings* between Γ-graded R-rings (K, φ) and (K', φ') is a homomorphism of graded rings $f\colon K \to K'$ such that $\varphi' = f \circ \varphi$. If, moreover, $f\colon K \to K'$ is an isomorphism of Γ-graded rings, we say that f is an *isomorphism of* Γ*-graded* R*-rings*.

Now let $\Sigma \subseteq \mathfrak{M}(R)$. The *universal localization of* R *at* Σ is a pair (R_Σ, λ), where R_Σ is a ring and $\lambda\colon R \to R_\Sigma$ is a Σ-inverting homomorphism such that for any other Σ-inverting ring homomorphism $f\colon R \to S$ there exists a unique ring homomorphism $F\colon R_\Sigma \to S$ with $f = F\lambda$.

Now we give some important properties of R_Σ.

Proposition 4.1. *Let* $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ *be a* Γ*-graded ring and let* $\Sigma \subseteq \mathfrak{M}(R)$. *Then the following statements hold true:*

(1) *There exists the universal localization* (R_Σ, λ) *of* R *at* Σ.

(2) $\lambda\colon R \to R_\Sigma$ *is a ring epimorphism.*

(3) *The ring* R_Σ *is a* Γ*-graded ring,* $\lambda\colon R \to R_\Sigma$ *is a homomorphism of* Γ*-graded rings, and* (R_Σ, λ) *is unique up to isomorphism of* Γ*-graded* R*-rings.*

(4) *Suppose that* $S = \bigoplus_{\gamma\in\Gamma} S_\gamma$ *is a* Γ*-graded ring,* $f\colon R \to S$ *is a* Σ*-inverting homomorphism of* Γ*-graded rings and* $F\colon R_\Sigma \to S$ *is the unique homomorphism of rings such that* $f = F\lambda$. *Then* $F\colon R_\Sigma \to S$ *is a homomorphism of* Γ*-graded* R*-rings. Moreover, if* Σ *is gr-lower semimultiplicative, then* $\operatorname{Im} F = R_f(\Sigma)$.

(5) *Suppose that* $S = \sum_{\gamma\in\Gamma} S_\gamma$ *is a* Γ*-almost graded division ring,* $f\colon R \to S$ *is a* Σ*-inverting homomorphism of* Γ*-almost graded rings and* $F\colon R_\Sigma \to S$ *is the unique homomorphism of rings such that* $f = F\lambda$. *Then* $F\colon R_\Sigma \to S$ *is a homomorphism of* Γ*-graded* R*-rings.*

Proof. First we construct a free ring $\mathbb{Z}\langle X\rangle$, where X is constructed as follows. For each $\gamma \in \Gamma$ and $r \in R_\gamma$, consider a symbol x_r^γ. For each matrix $A = (a_{ij}) \in \Sigma$, fix $(\bar\alpha, \bar\beta)$ such that $A \in M_n(R)[\bar\alpha][\bar\beta]$ and consider a matrix A^* whose entries are symbols $A^* = (a_{ij}^*)$. Then let X be the disjoint union

$$X = \left\{x_r^\gamma : r \in R_\gamma, \gamma \in \Gamma\right\} \cup \left\{a_{ij}^* : a_{ij} \text{ is the } (i,j)\text{-entry of } A \in \Sigma\right\}.$$

Now we turn $\mathbb{Z}\langle X\rangle$ into a Γ-graded ring by giving degrees to the elements of X. If $r \in R_\gamma$, we set x_r^γ to be of degree γ. If $A = (a_{ij}) \in \Sigma$ with fixed $(\bar\alpha, \bar\beta)$, then $a_{ij} \in R_{\alpha_i\beta_j^{-1}}$, thus we set a_{ij}^* to be of degree $\beta_i\alpha_j^{-1}$. Notice that $A^* \in M_n(\mathbb{Z}\langle X\rangle)[\bar\beta][\bar\alpha]$.

Let I be the ideal of $\mathbb{Z}\langle X\rangle$ generated by the homogeneous elements of any of the forms

- $x_{r+s}^\gamma - x_r^\gamma - x_s^\gamma$ for $r, s \in R_\gamma$;

- $x_{rs}^{\gamma\delta} - x_r^{\gamma} x_s^{\delta}$ for $r \in R_\gamma$ and $s \in R_\delta$;
- $x_1^e - 1$;
- $\sum_k x_{a_{ik}}^{\alpha_i \beta_k^{-1}} a_{kj}^* - \delta_{i,j}$ for $A \in \Sigma$;
- $\sum_k a_{ik}^* x_{a_{kj}}^{\alpha_k \beta_j^{-1}} - \delta_{i,j}$ for $A \in \Sigma$.

Set $R_\Sigma = \mathbb{Z}\langle X\rangle/I$ and $\lambda\colon R \to R_\Sigma$ be the homomorphism of Γ-graded rings determined by $\lambda(r) = \overline{x_r^\gamma}$ for each $r \in R_\gamma$, $\gamma \in \Gamma$. Since I is a graded ideal of $\mathbb{Z}\langle X\rangle$, then R_Σ is a Γ-graded ring and λ is a homomorphism of graded rings.

Suppose that $f\colon R \to S$ is a Σ-inverting ring homomorphism. For each $A = (a_{ij}) \in \Sigma[\bar\alpha][\bar\beta]$, suppose that $(A^f)^{-1} = (b_{ij})$. Then there exists a unique homomorphism of rings $F'\colon \mathbb{Z}\langle X\rangle \to S$ such that $F'(x_r^\gamma) = f(r)$ for each $r \in R_\gamma$, $\gamma \in \Gamma$, and $F'(a_{ij}^*) = b_{ij}$. Note that $I \subseteq \ker F$, and let $F\colon R_\Sigma \to S$ be the induced homomorphism. Hence $F\lambda = f$, as desired. To prove the uniqueness and the fact that $\lambda\colon R \to R_\Sigma$ is a ring epimorphism, notice that from $F\lambda = f$, we obtain that $F(\overline{x_r^\gamma}) = f(r)$, and now the same argument as Theorem 3.4 (4) shows that $F(\overline{a_{ij}^*}) = b_{ij}$.

Now we proceed to show (4). Suppose that S is a Γ-graded ring and $f\colon R \to S$ is a Σ-inverting homomorphism of graded rings. Notice that $A^f \in M_n(S)[\bar\alpha][\bar\beta]$ and $(A^f)^{-1} \in M_n(S)[\bar\beta][\bar\alpha]$. Then $f(r) \in S_\gamma$ for each $r \in R_\gamma$, $\gamma \in \Gamma$, and $b_{ji} \in S_{\beta_j \alpha_i^{-1}}$ for each $A = (a_{ij}) \in \Sigma_n[\bar\alpha][\bar\beta]$. Hence F' and F are homomorphisms of Γ-graded rings. Now, if Σ is gr-lower semimultiplicative, then $R_f(\Sigma)$ is a subring of S generated by $\operatorname{Im} f$ and the entries of the inverses of the matrices in Σ^f, and that is exactly the image of F.

(5) As noted in Propositions 2.4 and 2.2, the lift $\tilde S$ of S is a Γ-graded division ring and there exists a homomorphism of Γ-graded rings $\tilde f\colon R \to \tilde S$ such that $f = \pi \tilde f$, where $\pi\colon \tilde S \to S$ is the natural homomorphism of Γ-almost graded rings. Note that the sets Σ and $\{A \in \mathfrak{M}(R) : A^{\tilde f}$ is invertible over $\tilde S\}$ coincide by Proposition 2.5 (3). Thus $\Sigma^{\tilde f}$ consists of invertible matrices over $\tilde S$ and, by (4), there exists a unique homomorphism of Γ-graded rings $\tilde F\colon R_\Sigma \to \tilde S$ such that $\tilde f = \tilde F\lambda$. Observe that $F = \pi \tilde F$ because, if $A \in \Sigma$, then $A^{(\pi \tilde F)} = (A^{\tilde F})^{\varphi}$ is invertible. Now the result follows because $\tilde F$ and π are homomorphisms of Γ-almost graded rings. ■

Now our aim is to show that if (K, φ) is a *Γ-graded epic R-field*, then $\varphi\colon R \to K$ is in fact an epimorphism of (Γ-graded) rings. For the sake of completion, we preferred to give the proof of the following lemma, but this could be shown as a direct consequence of [5, Proposition 7.2.1] and the fact that if $f\colon R \to S$ is a homomorphism of Γ-graded rings that is an epimorphism in the category of Γ-graded rings, then it is an epimorphism in the category of rings. The proof of this fact is as follows: If $g_1, g_2 \colon S \to T$ are homomorphisms of rings such that $g_1 f = g_2 f$, there exist homomorphisms of Γ-graded rings $\widetilde{g_1} \colon S \to \widetilde{\operatorname{Im} g_1 f}$, $\widetilde{g_2} \colon S \to \widetilde{\operatorname{Im} g_2 f}$ and a homomorphism of rings $\pi \colon \widetilde{\operatorname{Im} g_1 f} \to T$ such that $\widetilde{g_1} f = \widetilde{g_2} f$ and $g_1 = \pi \widetilde{g_1}$, $g_2 = \pi \widetilde{g_2}$. Since f is an epimorphism of Γ-graded rings, then $\widetilde{g_1} = \widetilde{g_2}$. Thus $g_1 = g_2$.

Lemma 4.2. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$, $S = \bigoplus_{\gamma\in\Gamma} S_\gamma$ be Γ-graded rings and $f \colon R \to S$ be a homomorphism of Γ-graded rings. The following statements are equivalent:*

(1) *f is an epimorphism of Γ-graded rings.*

(2) *In the Γ-graded S-bimodule $S \otimes_R S$, $x \otimes 1 = 1 \otimes x$ for all $x \in S$.*

(3) *The natural map $p \colon S \otimes_R S \to S$ determined by $p(x \otimes y) = xy$ is an isomorphism of graded S-bimodules.*

Proof. (1) $\Rightarrow$ (2) Consider the Γ-graded additive group $M = S \oplus (S \otimes_R S)$. It can be endowed with a structure of Γ-graded ring via the multiplication $(x, u)(y, v) = (xy, xv + uy)$. Notice that if $(x, u) \in M_\gamma$ and $(y, v) \in M_\delta$, then x, u have degree γ and y, v have degree δ. Hence xy and $xv + uy$ have degree $\gamma\delta$.

Consider the homomorphisms of Γ-graded rings $g, h \colon S \to M$ defined by $g(x) = (x, 0)$ and $h(x) = (x, x \otimes 1 - 1 \otimes x)$. Since $gf = hf$ and f is an epimorphism of graded rings, then $x \otimes 1 = 1 \otimes x$.

(2) $\Rightarrow$ (1) Let $g, h \colon S \to T$ be homomorphisms of Γ-graded rings such that $gf = hf$. Then there exists a well-defined map $F \colon S \otimes_R S \to T$, $x \otimes y \mapsto g(x)h(y)$. For each $x \in S$, since $x \otimes 1 = 1 \otimes x$, we obtain that $g(x) = F(x \otimes 1) = F(1 \otimes x) = h(x)$. Thus $g = h$, as desired.

(2) $\Rightarrow$ (3) First note that p is a homomorphism of Γ-graded S-bimodules. Clearly p is surjective. Now, since $p(\sum_i x_i \otimes y_i) = \sum_i x_i y_i$, injectivity follows from the fact that $\sum_i x_i \otimes y_i = \sum_i x_i (1 \otimes y_i) = \sum_i x_i (y_i \otimes 1) = \sum_i x_i y_i \otimes 1 = (\sum_i x_i y_i) \otimes 1$.

(3) $\Rightarrow$ (2) Since for each $x \in S$, $p(x \otimes 1) = x = p(1 \otimes x)$ and p is an isomorphism, the result follows. ∎

Proposition 4.3. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring, $K = \bigoplus_{\gamma\in\Gamma} K_\gamma$ be a Γ-graded division ring and $f \colon R \to K$ be a homomorphism of Γ-graded rings. Then f is an epimorphism of graded rings if, and only if, $K = \mathrm{DC}(f)$.*

Proof. Suppose that $f \colon R \to K$ is an epimorphism of Γ-graded rings. Consider the graded division subring $\mathrm{DC}(f)$ of K. Let $\mathcal{B}$ be a set of homogeneous elements of K that is a basis of K as a right $\mathrm{DC}(f)$-module. Then we have the following isomorphisms of graded right K-modules:

$$\begin{aligned} K \cong K \otimes_{\mathrm{DC}(f)} K &\cong \Big(\bigoplus_{b\in\mathcal{B}} b\,\mathrm{DC}(f)\Big) \otimes_{\mathrm{DC}(f)} K \\ &\cong \bigoplus_{b\in\mathcal{B}} (b\,\mathrm{DC}(f) \otimes_{\mathrm{DC}(f)} K) \\ &\cong \bigoplus_{b\in\mathcal{B}} b \otimes_{\mathrm{DC}(f)} K \cong \bigoplus_{b\in\mathcal{B}} K(\gamma_b), \end{aligned}$$

for some $\gamma_b \in \Gamma$. Hence $\mathcal{B}$ must consist of just one element.

Conversely, suppose that $\mathrm{DC}(f) = K$. Let

$$\Sigma = \big\{A \in \mathfrak{M}(R) : A^f \text{ is invertible over } K\big\}.$$

By Corollary 3.7, $K = \mathrm{DC}(f) = R_f(\Sigma)$. By Theorem 3.4 (4), $f\colon R \to K$ is a ring epimorphism, and therefore an epimorphism of Γ-graded rings. ■

Theorem 4.4. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring.*

(1) *If $\Sigma \subseteq \mathfrak{M}(R)$ is such that the universal localization R_Σ is a Γ-graded local ring with maximal graded ideal $\mathfrak{m}$, then $R_\Sigma/\mathfrak{m}$ is a Γ-graded epic R-division ring.*

(2) *Let $K = \sum_{\gamma\in\Gamma} K_\gamma$ be a Γ-almost graded division ring and $f\colon R \to K$ be a homomorphism of Γ-almost graded rings such that $\mathrm{DC}(f) = K$. Let*

$$\Sigma = \big\{A \in \mathfrak{M}(R) : A^f \textit{ is invertible over } K\big\}.$$

The following assertions hold true:

(a) *R_Σ is a Γ-graded local ring.*

(b) *If $\mathfrak{m}$ is the maximal graded ideal of R_Σ, then $R_\Sigma/\mathfrak{m}$ is a Γ-graded epic R-division ring satisfying the following statements:*

(i) *There exists a surjective homomorphism of Γ-almost graded rings $\widetilde{F}\colon R_\Sigma/\mathfrak{m} \to K$ such that the following diagram is commutative:*

$$\begin{array}{ccccc} R & \xrightarrow{\lambda} & R_\Sigma & \xrightarrow{\pi} & R_\Sigma/\mathfrak{m}. \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle F} & \swarrow{\scriptstyle \widetilde{F}} & \\ & & K & & \end{array}$$

(ii) *If K is a Γ-graded division ring, then $\widetilde{F}\colon R_\Sigma/\mathfrak{m} \to K$ is an isomorphism of Γ-graded epic R-division rings.*

Proof. (1) The homomorphism $\lambda\colon R \to R_\Sigma$ is a ring epimorphism by Proposition 4.1 (2). The natural homomorphism $\pi\colon R_\Sigma \to R_\Sigma/\mathfrak{m}$ is surjective. Therefore $\pi\lambda\colon R \to R_\Sigma/\mathfrak{m}$ is a ring epimorphism, thus a Γ-graded epic R-division ring.

(2) Let $\lambda\colon R \to R_\Sigma$ be the canonical homomorphism. Hence, by Proposition 4.1 (5), there exists a unique homomorphism of Γ-almost graded R-rings $F\colon R_\Sigma \to K$ such that $F\lambda = f$. Set $\mathfrak{m} = (\ker F)_g$, in other words, $\mathfrak{m} = \bigoplus_{\gamma\in\Gamma}(\ker F \cap (R_\Sigma)_\gamma) \subseteq \ker F$. Let $x \in (R_\Sigma)_\gamma \setminus \mathfrak{m}$. Then $F(x) \neq 0$ and then $F(x) \in K_\gamma$ is invertible in K. By Proposition 4.1 (4), $R_\Sigma = R_\lambda(\Sigma)$. Thus there exist $\bar{\alpha} \in \Gamma^n$, $\bar{\beta} \in \Gamma^{n+1}$, $A \in M_{n\times(n+1)}(R)[\bar{\alpha}][\bar{\beta}]$ and $u \in M_{(n+1)\times 1}(S)$ such that $\beta_0 = e$, $\beta_\infty = \gamma$, $u_0 = 1$, $u_\infty = x$, $(A_\bullet \;\; A_\infty) \in \Sigma$ and

$$\begin{pmatrix} A_0^\lambda & A_\bullet^\lambda & A_\infty^\lambda \end{pmatrix}\begin{pmatrix} 1 \\ u_\bullet \\ x \end{pmatrix} = 0.$$

Applying F to the entries of the matrices involved we obtain

$$\begin{pmatrix} A_0^f & A_\bullet^f & A_\infty^f \end{pmatrix}\begin{pmatrix} 1 \\ u_\bullet^F \\ F(x) \end{pmatrix} = 0.$$

Since $F(x)$ is invertible, by Cramer's rule (Lemma 3.5), $(A_0^f \;\; A_\bullet^f)$ is invertible in K. Therefore $(A_0 \;\; A_\bullet) \in \Sigma$ and $(A_0^\lambda \;\; A_\bullet^\lambda)$ is invertible over R_Σ. Again by Cramer's rule, x is invertible in R_Σ. Hence R_Σ is a Γ-graded local ring, where $\mathfrak{m}$ is the ideal generated by the noninvertible homogeneous elements of R_Σ, and (a) is proved. The ring $R_\Sigma/\mathfrak{m}$ is a Γ-graded division ring and, by (1), (b) follows.

(i) and (ii) follow because, respectively, $\mathfrak{m} \subseteq \ker F$ and $\mathfrak{m} = \ker F$ if K is a Γ-graded division ring. ∎

We proceed to give a result that characterizes when a universal localization R_Σ is a graded local ring. Its proof follows that given in the ungraded result in [5, Proposition 7.2.6]. But before that, we need the following result, which is well known and can be found, for example, in [8, Proposition 1.1.31].

Lemma 4.5. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. Then R is a Γ-graded local ring if and only if R_e is a local ring.* ∎

Proposition 4.6. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring, Σ be a gr-multiplicative subset of $\mathfrak{M}(R)$ and $\lambda\colon R \to R_\Sigma$ be the natural homomorphism of Γ-graded rings. Then R_Σ is a Γ-graded local ring if and only if it satisfies the following two conditions:*

(1) $R_\Sigma \neq \{0\}$.

(2) *For a matrix* $A \in \Sigma_n[\overline{\alpha'} * e][\overline{\beta'} * e]$*, if* B*, the* (n,n)*-minor of* A*, is such that* B^λ *is not invertible over* R_Σ*, then* $(A - e_{nn})^\lambda$ *is invertible over* R_Σ*, where* e_{nn} *denotes the matrix with* 1 *in the* (n,n)*-entry and zeros everywhere else.*

Proof. Consider the canonical homomorphism of Γ-graded local rings $\lambda\colon R \to R_\Sigma$.

Suppose that R_Σ is a Γ-graded local ring with maximal graded ideal $\mathfrak{m}$ and canonical homomorphism $\pi\colon R_\Sigma \to R_\Sigma/\mathfrak{m}$. Since R_Σ is graded local, by definition, $R_\Sigma \neq \{0\}$. Recall that any matrix $C \in \mathfrak{M}(R_\Sigma)$ is invertible if and only if C^π is invertible over $R_\Sigma/\mathfrak{m}$. Let $A \in \Sigma_n[\overline{\alpha'} * e][\overline{\beta'} * e]$ such that its (n,n)-minor B is not invertible over R_Σ. It is enough to show that $(A - e_{nn})^\pi$ is invertible. Some nontrivial left linear combination (over the graded division ring $R/\mathfrak{m}$) with homogeneous coefficients of the rows of B^π is zero. If we take the corresponding left linear combination of the first $n-1$ rows of A^π, we obtain $(0, 0, \dots, 0, c)$, where c is homogeneous and $c \neq 0$, because A^π is invertible. From the last row of A we now subtract c^{-1} times this combination of the other rows and obtain the matrix $A - e_{nn}$, which is therefore invertible in $R_\Sigma/\mathfrak{m}$ because it is the product of the matrix corresponding to those elementary operations on A^π times A^π.

Conversely, suppose now that conditions (1) and (2) are satisfied. By Lemma 4.5, it is enough to prove that $(R_\Sigma)_e$ is a local ring. Let $x \in (R_\Sigma)_e$. By Lemma 3.3 (3), there exist $\bar\alpha, \bar\beta \in \Gamma^n$, $A \in \Sigma_n[\bar\alpha][\bar\beta]$ and $u \in M_{n\times 1}(R_\Sigma)$ such that $\alpha_i = e$, $\beta_j = e$, $u_j = x$ and $A^\lambda u = e_i$. Since Σ is gr-multiplicative, we may suppose that $A \in \Sigma_n[\overline{\alpha'} * e][\overline{\beta'} * e]$, $u_n = x$ and $A^\lambda u = e_n$. Suppose x is not invertible in R_Σ. Equivalently, by Lemma 3.5, the matrix $(A^\lambda_\bullet\ e^\lambda_n)$ is not invertible in R_Σ. This implies that the (n,n)-minor of $(A^\lambda_\bullet\ e^\lambda_n)$, which is the (n,n)-minor of A, is not invertible in R_Σ. Hence $(A - e_{nn})^\lambda$ is invertible over R_Σ. Then the matrix $(A^\lambda)^{-1}(A - e_{nn})^\lambda = I - (A^\lambda)^{-1} e_{nn}$ is invertible in R_Σ. Since this matrix is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \cdots & \ddots & 1 & * \\ 0 & \cdots & \cdots & 0 & 1-x \end{pmatrix},$$

we obtain that $1 - x$ is invertible in R_Σ, as desired. ■

Now we proceed to define the category of graded epic R-division rings andgr-specializations.

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring.

Suppose that (K,φ), (L,ψ) are Γ-graded epic R-division rings and set

$$\Sigma = \{A \in \mathfrak{M}(R) : A^{\psi} \text{ is invertible over } L\}.$$

If there exists a homomorphism of Γ-graded R-rings $\Phi\colon R_\Sigma \to K$, we define the *core of L in K* as $\mathfrak{C}_L(K) = \Phi(R_\Sigma)$. We remark that, if it exists, it is unique and observe that, by Proposition 4.1 (4), $\mathfrak{C}_L(K) = R_\varphi(\Sigma)$. By Theorem 4.4 (2)(a), R_Σ is a Γ-graded local ring. Therefore $\mathfrak{C}_L(K)$ is a Γ-graded local subring of K that contains R. Moreover, the natural homomorphism of Γ-graded R-rings $\Psi\colon R_\Sigma \to L$ factors through $\mathfrak{C}_L(K)$ in a unique way, because $L \cong R_\Sigma/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal graded ideal of R_Σ.

A *gr-subhomomorphism* is a homomorphism of Γ-graded R-rings $f\colon K_f \to L$ where K_f is a graded R-subring of K such that $x^{-1} \in K_f$ for each $x \in \mathrm{h}(K_f) \setminus \ker f$. Note that K_f is a graded local subring of K because any homogeneous element not in the graded ideal $\ker f$ is invertible. Hence $K_f/\ker f$ is a Γ-graded R-division ring contained in L. This implies that f is a surjective homomorphism of Γ-graded R-rings and that $K_f/\ker f \cong L$ is a Γ-graded epic R-division ring. For each $A \in \Sigma$, consider A^{φ} which belongs to $\mathfrak{M}(K)$. Since K_f is a Γ-graded local R-ring whose residue graded division ring is L, we get that A^{φ} is invertible over K_f. Thus there exists a unique homomorphism of graded R-rings $\Phi\colon R_\Sigma \to K_f \subseteq K$ and a commutative diagram of homomorphisms of Γ-graded R-rings

$$\begin{array}{ccc} & \overset{\Phi}{\nearrow} & K_f \\ R_\Sigma & & \downarrow f \\ & \underset{\Psi}{\searrow} & L. \end{array} \tag{4.1}$$

Thus $\mathfrak{C}_L(K)$ is contained in the domain of any subhomomorphism from K to L, it is a Γ-graded local R-subring of K, the restriction of any subhomomorphism to $\mathfrak{C}_L(K)$ is a subhomomorphism and all such restrictions coincide in $\mathfrak{C}_L(K)$, because of the commutativity of (4.1).

Now we give another description of $\mathfrak{C}_L(K)$. Let $f\colon K_f \to L$ be a gr-subhomomorphism between the Γ-graded epic R-fields (K,φ), (L,ψ). For each $\gamma \in \Gamma$ define $(c(f)_0)_\gamma = \varphi(R_\gamma)$, and if $n \geq 0$, set

$$\begin{aligned}(c(f)_{n+1})_\gamma = {} & \text{additive subgroup of } K \text{ generated by} \\ & \{x_1\cdots x_r : r \geq 1, x_i \in (c(f)_n)_{\gamma_i} \text{ or } x_i = y_i^{-1} \\ & \qquad \text{where } y_i \in (c(f)_n)_{\gamma_i^{-1}} \setminus \ker f, \gamma_1\cdots\gamma_r = \gamma\}.\end{aligned}$$

Then define $c(f)_\gamma = \bigcup_{n\geq 0}(c(f)_n)_\gamma$, and $C_L(K) = \bigoplus_{\gamma\in\Gamma} c(f)_\gamma$. Note that $C_L(K)$ is a Γ-graded local R-subring of K_f with maximal graded ideal $C_L(K) \cap \ker f$

and such that the restriction $f\colon C_L(K) \to L$ is a gr-subhomomorphism. If we take $K_f = \mathfrak{C}_L(K)$, then we obtain that $C_L(K) \subseteq \mathfrak{C}_L(K)$, but since $\mathfrak{C}_L(K)$ is contained in the domain of any gr-subhomomorphism, we get that $C_L(K) = \mathfrak{C}_L(K)$. Roughly speaking, this equality means that any rational homogeneous expression obtained from the elements of (the image of) R in L makes sense in K and the elements obtained with those rational expressions from the elements of (the image of) R in K form $\mathfrak{C}_L(K)$.

Since, if there exist gr-subhomomorphisms between the Γ-graded epic R-division rings (K, φ) and (L, ψ), they all coincide in the core, we make the following definition. A *gr-specialization* is the unique homomorphism of Γ-graded R-rings $f\colon \mathfrak{C}_L(K) \to L$.

Suppose that (K, φ), (L, ψ) and (M, ϕ) are Γ-graded epic R-division rings. If $f\colon K_f \to L$ and $g\colon L_g \to M$ are gr-subhomomorphisms, then the restriction $gf\colon P = f^{-1}(L_g) \to M$ is a gr-subhomomorphism which will be called the *composition gr-subhomomorphism* of f and g. Indeed, suppose that $z \in \mathrm{h}(P) \setminus \ker(gf)$. Since $g(f(z)) \neq 0$, then $f(z)^{-1} \in L_g$. As $f(z) \neq 0$, and thus $z^{-1} \in K_f$, then $z^{-1} \in P$. We define the composition of the corresponding gr-specializations, as the gr-specialization corresponding to the composition gr-subhomomorphism of f and g. In other words, it is the unique homomorphism of Γ-graded R-rings $\mathfrak{C}_M(K) \to M$. It follows that the composition of gr-specializations is associative.

Note that the only subhomomorphism from the Γ-graded epic R-division ring (K, φ) to (K, φ) is the identity map on K. Therefore $\mathfrak{C}_K(K) = K$ and the corresponding specialization is the identity map.

We define the category $\mathcal{E}_R$ as the category whose objects are the Γ-graded epic R-division rings and whose morphisms are the gr-specializations. We remark that there is at most one morphism between two objects in this category and that isomorphisms correspond to isomorphisms of Γ-graded R-rings. Indeed, if the composition of two gr-specializations f and g is the identity gr-specialization, then they have to be isomorphisms of Γ-graded R-rings.

An initial object (K, φ) in the category $\mathcal{E}_R$ is a *universal Γ-graded epic R-division ring*. In other words, there exists a gr-specialization from (K, φ) to any other Γ-graded epic R-division ring (L, ψ). If, moreover, $\varphi\colon R \to K$ is injective, we say that this initial object is a *universal Γ-graded epic R-division ring of fractions of R*.

Now we give the following important result.

THEOREM 4.7. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and let (K_1, φ_1), (K_2, φ_2) be Γ-graded epic R-division rings. Set*

$$\Sigma_i = \left\{A \in \mathfrak{M}(R) : A^{\varphi_i} \text{ is invertible over } K_i\right\}, \quad i = 1, 2.$$

The following statements are equivalent:

(1) *There exists a gr-specialization from* (K_1, φ_1) *to* (K_2, φ_2).

(2) $\Sigma_2 \subseteq \Sigma_1$.

(3) *There exists a homomorphism* $R_{\Sigma_2} \to R_{\Sigma_1}$ *of* Γ*-graded* R*-rings.*

Furthermore, if there exists a gr-specialization from (K_1, φ_1) *to* (K_2, φ_2) *and another gr-specialization from* (K_2, φ_2) *to* (K_1, φ_1)*, then* K_1 *and* K_2 *are isomorphic* Γ*-graded* R*-rings.*

Proof. (1) $\Rightarrow$ (2) By definition, there exists a homomorphism of Γ-graded R rings $\mathfrak{C}_{K_2}(K_1) \to K_2$. By definition of $\mathfrak{C}_{K_2}(K_1)$, any matrix in Σ_2 is invertible over $\mathfrak{C}_{K_2}(K_1) \subseteq K_1$. Thus $\Sigma_2 \subseteq \Sigma_1$.

(2) $\Rightarrow$ (3) If $\Sigma_2 \subseteq \Sigma_1$, the universal property of R_{Σ_2} implies the existence of a homomorphism of Γ-graded R-rings $R_{\Sigma_2} \to R_{\Sigma_1}$.

(3) $\Rightarrow$ (1) Consider the unique homomorphisms of Γ-graded R-rings $\Phi_i \colon R_{\Sigma_i} \to K_i$, $i = 1, 2$. Let $h \colon R_{\Sigma_2} \to R_{\Sigma_1}$ be a homomorphism of Γ-graded R-rings. Then there exists the homomorphism of graded R-rings $\Phi_1 h \colon R_{\Sigma_2} \to K_1$. Then, by what has been explained above, Φ_2 factors through $\mathfrak{C}_{K_2}(K_1)$, and gives the desired specialization.

Now suppose that there exist gr-specializations $f \colon \mathfrak{C}_{K_2}(K_1) \to K_2$ and $g \colon \mathfrak{C}_{K_1}(K_2) \to K_1$. Then the composition gf gives a gr-specialization from K_1 in itself. Thus it has to be the identity. Similarly, the composition fg gives a gr-specialization from K_2 in itself. Hence f is an isomorphism in the category $\mathcal{E}_R$ of Γ-graded epic R-division rings. Therefore, f is an isomorphism of graded R-rings. ■

Corollary 4.8. *Let* $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ *be a* Γ*-graded ring. Suppose that there exists* $\Omega \subseteq \mathfrak{M}(R)$ *such that* (R_Ω, λ)*, where* $\lambda \colon R \to R_\Omega$ *is the canonical homomorphism, is a* Γ*-graded (epic)* R*-division ring. Then the only gr-specializations to* (R_Ω, λ) *are isomorphisms of* Γ*-graded* R*-rings.*

Proof. Suppose there exists a gr-specialization from the Γ-graded epic R-division ring (K, φ) to (R_Ω, λ). By Theorem 4.7 (3), then there exists a (unique) homomorphism of Γ-graded R-rings $R_\Omega \to R_\Sigma \to K$, where

$$\Sigma = \{A \in \mathfrak{M}(R) : A^\varphi \text{ is invertible over } K\}.$$

Now, since R_Ω and K are Γ-graded epic R-division rings, the image of R_Ω must be K and therefore they are isomorphic as Γ-graded R-rings. ■

Corollary 4.9. *Let* $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ *be a* Γ*-graded ring with a universal* Γ*-graded epic* R*-division ring* (U, ρ)*. Let* $\Sigma \subseteq \mathfrak{M}(R)$ *and consider the canonical homomorphism* $\lambda \colon R \to R_\Sigma$*. Suppose that there exists a homomorphism of* Γ*-graded rings* $R_\Sigma \to L$

for some Γ*-graded division ring* L*. Then there exists a unique homomorphism of* Γ*-graded rings* $\tilde{\rho}\colon R_\Sigma \to U$ *such that* $\tilde{\rho}\lambda = \rho$ *and* $(U, \tilde{\rho})$ *is a universal* Γ*-graded epic* R_Σ*-division ring.*

Proof. Let $f\colon R_\Sigma \to L$ be a homomorphism of Γ-graded rings with L a Γ-graded division ring. Then $(\mathrm{DC}(f\lambda), f\lambda)$ is a Γ-graded epic R-division ring such that the matrices in Σ become invertible. Hence, by Theorem 4.7, Σ^ρ consists of invertible matrices in U. Thus there exists a unique homomorphism of Γ-graded rings $\tilde{\rho}\colon R_\Sigma \to U$ and $(U, \tilde{\rho})$ is a Γ-graded epic R_Σ-division ring.

Consider a Γ-graded epic R_Σ-division ring (K, φ). The composition $\varphi\lambda\colon R \to K$ is an epimorphism of Γ-graded rings, because λ and φ are. Hence $(K, \varphi\lambda)$ is a Γ-graded epic R-division ring and therefore there exists a specialization from (U, ρ) to $(K, \varphi\lambda)$ that can be regarded as specialization from $(U, \tilde{\rho})$ to (K, φ). ∎

Adapting [5, p. 426] to the graded context, we give some examples to illustrate the concepts of universal graded division ring and graded division rings that are universal localizations.

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a commutative Γ-graded domain. Then the localization of R at the set $\mathrm{h}(R) \setminus \{0\}$ of nonzero homogeneous elements yields a Γ-graded epic R-field (F, φ). We point out that $F = \bigoplus_{\gamma\in\Gamma} F_\gamma$ is a Γ-graded field with

$$F_\gamma = \left\{ab^{-1} \mid a \in R_\delta,\ b \in R_\varepsilon,\ \delta\varepsilon^{-1} = \gamma\right\}$$

for each $\gamma \in \Gamma$. Furthermore, if (K, ψ) is a Γ-graded epic R-division ring, then $\ker\psi$ is a graded prime ideal of R. That is, $\ker\varphi \neq R$ and if $x, y \in \mathrm{h}(R)$ with $xy \in \ker\psi$, then $x \in \ker\psi$ or $y \in \ker\psi$. Hence $\mathrm{h}(R) \setminus \ker\psi$ is a multiplicative subset of R. Then the localization of R at $\mathrm{h}(R) \setminus \ker\psi$ is a Γ-graded local subring of F with Γ-graded residue division ring R-isomorphic to K. Therefore (F, φ) is a Γ-graded universal R-division ring of fractions that is a universal localization.

Let $S = E \times F$ be the direct product of two Γ-graded fields $E = \bigoplus_{\gamma\in\Gamma} E_\gamma$ and $F = \bigoplus_{\gamma\in\Gamma} F_\gamma$. Then $S = \bigoplus_{\gamma\in\Gamma} S_\gamma$ is a Γ-graded ring with $S_\gamma = E_\gamma \times F_\gamma$. Suppose (D, ρ) is a Γ-graded epic S-division ring. Since $(1, 1) = (1, 0) + (0, 1)$ and $(1, 0)(0, 1) = (0, 0)$, then either $\rho(1, 0) = 0$ or $\rho(0, 1) = 0$. If $\rho(1, 0) = 0$, then $\rho(E \times \{0\}) = 0$ and if $\rho(0, 1) = 0$, then $\rho(\{0\} \times F) = 0$. Hence S has only two epic S-division rings, which are E and F. Note that neither of them is a universal Γ-graded epic S-division ring. On the other hand, both are universal localizations. For example, E is the universal localization of S at $\{(1, 1)\} \cup \{(a, 0) \mid a \in \mathrm{h}(E) \setminus \{0\}\}$.

Now let $E = \bigoplus_{\gamma\in\Gamma} E_\gamma$ be a Γ-graded field. Then the polynomial ring $E[x] = \bigoplus_{\gamma\in\Gamma} E[x]_\gamma$ is a Γ-graded ring with

$$E[x]_\gamma = E_\gamma[x] = \left\{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in E_\gamma,\ n \in \mathbb{N}\right\}.$$

The ideal (x^2) is a graded ideal of $E[x]$. Hence $T = E[x]/(x^2)$ is a Γ-graded local ring with maximal graded ideal $(x)/(x^2)$. Then E is the unique Γ-graded epic T-division ring, and thus E is a universal Γ-graded epic T-division ring. Notice that E is not a universal localization at matrices in $\mathfrak{M}(R)$ because the matrices which become invertible in E are already invertible in T since E is the Γ-graded residue division ring of T.

The ring $U = T \times F$, with T as before and F a Γ-graded field, has E and F as Γ-graded epic U-division rings, but only F is a universal localization.

5. Malcolmson's construction of the universal localization

Throughout this section, let Γ be a group.

This section is devoted to showing that the natural extension of the construction of the ring R_Σ given by Malcolmson [16] works in the context of graded rings. Although technical, this construction will be important for us in the next section. In Section 5.5 we give the graded version of the main results in [16], the so-called Malcolmson criterion and a sufficient condition for the universal localization R_Σ not to be the zero ring.

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and $\lambda\colon R \to R_\Sigma$ the universal localization at a gr-lower semimultiplicative subset of $\mathfrak{M}(R)$. By Lemma 3.3 (6) and Proposition 4.1 (4), every homogeneous element of $(R_\Sigma)_\gamma$ is of the form $F^\lambda(A^\lambda)^{-1}X^\lambda$, where $A \in \Sigma_n[\bar\alpha][\bar\beta]$, $F \in M_{1\times n}(R)[\gamma][\bar\beta]$, $X \in M_{n\times 1}[\bar\alpha][e]$. For each $\gamma \in \Gamma$, $(R_\Sigma)_\gamma$ is constructed as a set of equivalent classes of 5-tuples (F, A, X, α, β). The equivalence class $[(F, A, X, \alpha, \beta)]$ of (F, A, X, α, β) is interpreted as the element $F^\lambda(A^\lambda)^{-1}X^\lambda$ of R_Σ and addition and product are defined according to this interpretation. Thus, for $[F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta] \in (R_\Sigma)_\gamma$,

$$[F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta] = \left[\begin{pmatrix} F' & F \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} X' \\ X \end{pmatrix}, \alpha' * \alpha, \beta' * \beta\right],$$

and for $(F', A', X', \alpha', \beta') \in (R_\Sigma)_{\gamma'}$ and $(F, A, X, \alpha, \beta) \in (R_\Sigma)_\gamma$,

$$[F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta] = \left[\begin{pmatrix} 0 & F' \end{pmatrix}, \begin{pmatrix} A & 0 \\ -X'F & A' \end{pmatrix}, \begin{pmatrix} X \\ 0 \end{pmatrix}, \alpha * \alpha'\gamma, \beta * \beta'\gamma\right].$$

Also $-[(F, A, X, \alpha, \beta)] = [(-F, A, X, \alpha, \beta)]$ and $\lambda(r) = [(r, 1, 1, e, e)]$ for $r \in R_\gamma$.

In this section, for ease of exposition, we use the following notation. By "A is a homogeneous matrix", we mean $A \in \mathfrak{M}_\bullet(R)$. We will also use the terms *homogeneous row*, *homogeneous column* to emphasize that the matrix in question is a row or a column, respectively. If $A \in M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}]$, but we do not want to make reference to the size of A, we will say "A is a homogeneous matrix of distribution (α, β)". Also, the sequence $\bar{\alpha}\gamma$ will be denoted by $\alpha\gamma$ for each $\bar{\alpha} \in \Gamma^n$ and $\gamma \in \Gamma$.

5.1 – *Equivalence relation*

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and Σ be a gr-lower semimultiplicative subset of $\mathfrak{M}(R)$.

For $\gamma \in \Gamma$, let $(T_\Sigma)_\gamma$ be the set of 5-tuples (F, A, X, α, β), where $A \in \Sigma$ is of distribution (α, β), F is a homogeneous row of distribution (γ, β), and X is a homogeneous column of distribution (α, e).

Let $(F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon) \in (T_\Sigma)_\gamma$. We say that

$$(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$$

if and only if there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V such that

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \\ \hline F & -G & 0 & J & 0 \end{array}\right) = \left(\begin{array}{c} P \\ \hline U \end{array}\right) \left(\begin{array}{c|c} Q & V \end{array}\right), \tag{5.1}$$

where P, U, Q, V have distributions (π, ω), (γ, ω), (ω, θ), (ω, e), respectively, and, if we think of

$$\pi = \pi_1 * \pi_2 * \pi_3 * \pi_4 \quad \text{and} \quad \theta = \theta_1 * \theta_2 * \theta_3 * \theta_4$$

then $\pi_1 = \alpha$, $\pi_2 = \delta$, $\theta_1 = \beta$, $\theta_2 = \varepsilon$.

The right-hand side of (5.1) will also be denoted by

$$\left(\begin{array}{cccc} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \\ \hline U_1 & U_2 & U_3 & U_4 \end{array}\right) \left(\begin{array}{cccc|c} Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4 \end{array}\right).$$

Lemma 5.1. *Let $(F,A,X,\alpha,\beta),(G,B,Y,\delta,\varepsilon)\in(T_\Sigma)_\gamma$ such that there is a factorization as a product of homogeneous matrices of any of these forms with L, M, P, $Q\in\Sigma$ and with the corresponding distributions*

$$(1)\quad \left(\begin{array}{cc|c} A & 0 & X \\ 0 & B & Y \\ \hline F & -G & 0 \end{array}\right) = \left(\begin{array}{c} P \\ \hline U \end{array}\right)\left(\begin{array}{c|c} Q & V \end{array}\right),$$

$$(2)\quad \left(\begin{array}{ccc|c} A & 0 & 0 & X \\ 0 & B & 0 & Y \\ 0 & 0 & M & W \\ \hline F & -G & 0 & 0 \end{array}\right) = \left(\begin{array}{c} P \\ \hline U \end{array}\right)\left(\begin{array}{c|c} Q & V \end{array}\right),$$

$$(3)\quad \left(\begin{array}{ccc|c} A & 0 & 0 & X \\ 0 & B & 0 & Y \\ 0 & 0 & L & 0 \\ \hline F & -G & J & 0 \end{array}\right) = \left(\begin{array}{c} P \\ \hline U \end{array}\right)\left(\begin{array}{c|c} Q & V \end{array}\right),$$

then $(F,A,X,\alpha,\beta)\sim(G,B,Y,\delta,\varepsilon)$.

Proof. (1) Suppose P, U, Q, V have distributions (π,ω), (γ,ω), (ω,θ), (ω,e), where $\pi_1=\alpha$, $\pi_2=\delta$, $\theta_1=\beta$ and $\theta_2=\varepsilon$, and that we have the factorization

$$\left(\begin{array}{cc|c} A & 0 & X \\ 0 & B & Y \\ \hline F & -G & 0 \end{array}\right) = \left(\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \\ \hline U_1 & U_2 \end{array}\right)\left(\begin{array}{cc|c} Q_{11} & Q_{12} & V_1 \\ Q_{21} & Q_{22} & V_2 \end{array}\right).$$

Thus we have the factorization

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline F & -G & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{cccc} P_{11} & P_{12} & 0 & 0 \\ P_{21} & P_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline U_1 & U_2 & 0 & 1 \end{array}\right)\left(\begin{array}{cccc|c} Q_{11} & Q_{12} & 0 & 0 & V_1 \\ Q_{21} & Q_{22} & 0 & 0 & V_2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right),$$

where the factors on the right-hand side have distributions

$$(\alpha*\delta*e*\gamma*\gamma,\omega_1*\omega_2*e*\gamma),$$
$$(\omega_1*\omega_2*e*\gamma,\beta*\varepsilon*e*\gamma*e)$$

(2) Suppose P, U, Q, V have distributions (π,ω), (γ,ω), (ω,θ), (ω,e), where $\pi_1=\alpha$, $\pi_2=\delta$, $\theta_1=\beta$ and $\theta_2=\varepsilon$ and we have the factorization

$$\left(\begin{array}{ccc|c} A & 0 & 0 & X \\ 0 & B & 0 & Y \\ 0 & 0 & L & W \\ \hline F & -G & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \\ \hline U_1 & U_2 & U_3 \end{array}\right) \left(\begin{array}{ccc|c} Q_{11} & Q_{12} & Q_{13} & V_1 \\ Q_{21} & Q_{22} & Q_{23} & V_2 \\ Q_{31} & Q_{32} & Q_{33} & V_3 \end{array}\right).$$

Thus we have the equality

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & 1 & 0 \\ \hline F & -G & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{cccc} P_{11} & P_{12} & P_{13} & 0 \\ P_{21} & P_{22} & P_{23} & 0 \\ P_{31} & P_{32} & P_{33} & 0 \\ 0 & 0 & 0 & 1 \\ \hline U_1 & U_2 & U_3 & 1 \end{array}\right) \left(\begin{array}{cccc|c} Q_{11} & Q_{12} & Q_{13} & 0 & V_1 \\ Q_{21} & Q_{22} & Q_{23} & 0 & V_2 \\ Q_{31} & Q_{32} & Q_{33} & 0 & V_3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right),$$

where the factors on the right-hand side have distributions

$$(\alpha*\delta*\pi_3*\gamma*\gamma,\omega_1*\omega_2*\omega_3*\gamma),$$
$$(\omega_1*\omega_2*\omega_3*\gamma,\beta*\varepsilon*\theta_3*\gamma*e).$$

(3) Suppose P, U, Q, V have distributions (π,ω), (γ,ω), (ω,θ), (ω,e), where $\pi_1=\alpha$, $\pi_2=\delta$, $\theta_1=\beta$ and $\theta_2=\varepsilon$ and that we have the factorization

$$\left(\begin{array}{ccc|c} A & 0 & 0 & X \\ 0 & B & 0 & Y \\ 0 & 0 & M & 0 \\ \hline F & -G & J & 0 \end{array}\right) = \left(\begin{array}{ccc} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \\ \hline U_1 & U_2 & U_3 \end{array}\right) \left(\begin{array}{ccc|c} Q_{11} & Q_{12} & Q_{13} & V_1 \\ Q_{21} & Q_{22} & Q_{23} & V_2 \\ Q_{31} & Q_{32} & Q_{33} & V_3 \end{array}\right).$$

Thus we have the factorization

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & M & 0 \\ \hline F & -G & 0 & J & 0 \end{array}\right) = \left(\begin{array}{cccc} P_{11} & P_{12} & P_{13} & 0 \\ P_{21} & P_{22} & P_{23} & 0 \\ P_{31} & P_{32} & P_{33} & 0 \\ P_{31} & P_{32} & P_{33} & M \\ \hline U_1 & U_2 & U_3 & J \end{array}\right) \left(\begin{array}{cccc|c} Q_{11} & Q_{12} & Q_{13} & 0 & V_1 \\ Q_{21} & Q_{22} & Q_{23} & 0 & V_2 \\ Q_{31} & Q_{32} & Q_{33} & 0 & V_3 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right),$$

where the factors on the right-hand side have distributions

$$(\alpha*\delta*\pi_3*\pi_3*\gamma,\omega_1*\omega_2*\omega_3*\theta_3),\quad (\omega_1*\omega_2*\omega_3*\theta_3,\beta*\varepsilon*\theta_3*\theta_3*e),$$

respectively. ∎

Lemma 5.2. *For each $\gamma \in \Gamma$, the relation $\sim$ defined in $(T_\Sigma)_\gamma$ is an equivalence relation.*

Proof. Let $(F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon), (H, C, Z, \zeta, \eta) \in (T_\Sigma)_\gamma$.

The relation $\sim$ is reflexive. Indeed, we have the factorization

$$\left(\begin{array}{cc|c} A & 0 & X \\ 0 & A & X \\ \hline F & -F & 0 \end{array}\right) = \left(\begin{array}{cc} I & 0 \\ I & A \\ \hline 0 & -F \end{array}\right) \left(\begin{array}{cc|c} A & 0 & X \\ -I & I & 0 \end{array}\right),$$

where the factors are homogeneous matrices that have distributions

$$(\alpha * \alpha * \gamma, \alpha * \beta), \quad (\alpha * \beta, \beta * \beta * e),$$

respectively. This shows that $(F, A, X, \alpha, \beta) \sim (F, A, X, \alpha, \beta)$.

The relation $\sim$ is symmetric. Indeed, suppose that $(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$. There exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V, such that

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \\ \hline F & -G & 0 & J & 0 \end{array}\right) = \left(\begin{array}{cccc} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \\ \hline U_1 & U_2 & U_3 & U_4 \end{array}\right) \left(\begin{array}{cccc|c} Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4 \end{array}\right), \tag{5.2}$$

where P, U, Q, V have distributions (π, ω), (γ, ω), (ω, θ), (ω, e), respectively, and $\pi_1 = \alpha$, $\pi_2 = \delta$, $\theta_1 = \beta$, $\theta_2 = \varepsilon$. Then we have the factorization

$$\left(\begin{array}{cccc|c} B & 0 & 0 & 0 & Y \\ 0 & A & 0 & 0 & X \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \\ \hline G & -F & 0 & -J & 0 \end{array}\right) = \left(\begin{array}{cccc} P_{21} & P_{22} & P_{23} & P_{24} \\ P_{11} & P_{12} & P_{13} & P_{14} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \\ \hline -U_1 & -U_2 & -U_3 & -U_4 \end{array}\right) \left(\begin{array}{cccc|c} Q_{12} & Q_{11} & Q_{13} & Q_{14} & V_1 \\ Q_{22} & Q_{21} & Q_{23} & Q_{24} & V_2 \\ Q_{32} & Q_{31} & Q_{33} & Q_{34} & V_3 \\ Q_{42} & Q_{41} & Q_{43} & Q_{44} & V_4 \end{array}\right),$$

where the factors have distributions

$$(\pi_2 * \pi_1 * \pi_3 * \pi * 4 * \gamma, \omega), \quad (\omega, \theta_2 * \theta_1 * \theta_3 * \theta_4 * e),$$

respectively. Hence $(G, B, Y, \delta, \varepsilon) \sim (F, A, X, \alpha, \beta)$, and the symmetric property of the relation $\sim$ is proved.

Now we proceed to prove that $\sim$ satisfies the transitive property. Suppose that $(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$ and $(G, B, Y, \delta, \varepsilon) \sim (H, C, Z, \zeta, \eta)$. Hence there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V, as in (5.2), and there exist $L', M', P', Q' \in \Sigma$, homogeneous rows J', U' and homogeneous columns W', V' such that

$$\left(\begin{array}{cccc|c} B & 0 & 0 & 0 & Y \\ 0 & C & 0 & 0 & Z \\ 0 & 0 & L' & 0 & W' \\ 0 & 0 & 0 & M' & 0 \\ \hline G & -H & 0 & J' & 0 \end{array}\right) = \left(\begin{array}{cccc} P'_{11} & P'_{12} & P'_{13} & P'_{14} \\ P'_{21} & P'_{22} & P'_{23} & P'_{24} \\ P'_{31} & P'_{32} & P'_{33} & P'_{34} \\ P'_{41} & P'_{42} & P'_{43} & P'_{44} \\ \hline U'_1 & U'_2 & U'_3 & U'_4 \end{array}\right) \left(\begin{array}{cccc|c} Q'_{11} & Q'_{12} & Q'_{13} & Q'_{14} & V'_1 \\ Q'_{21} & Q'_{22} & Q'_{23} & Q'_{24} & V'_2 \\ Q'_{31} & Q'_{32} & Q'_{33} & Q'_{34} & V'_3 \\ Q'_{41} & Q'_{42} & Q'_{43} & Q'_{44} & V'_4 \end{array}\right),$$

where P', U', Q', V' have distributions (π', ω'), (γ, ω'), (ω', θ'), (ω', e), respectively, and $\pi'_1 = \delta$, $\pi'_2 = \zeta$, $\theta'_1 = \varepsilon$, $\theta'_2 = \eta$. Then we have the factorization of the matrix

$$\left(\begin{array}{ccccccccccc|c} C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z \\ 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\ 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W \\ 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L' & 0 & 0 & 0 & 0 & 0 & W' \\ 0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 \\ \hline H & -F & 0 & 0 & 0 & 0 & -G & H & 0 & -J' & J & 0 \end{array}\right)$$

as a product of the matrices

$$\left(\begin{array}{ccccccccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{11} & P_{12} & P_{13} & P_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{21} & P_{22} & P_{23} & P_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{31} & P_{32} & P_{33} & P_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{41} & P_{42} & P_{43} & P_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & -P_{21} & -P_{22} & -P_{23} & -P_{24} & 0 & P'_{11} & P'_{12} & P'_{13} & P'_{14} & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & P'_{21} & P'_{22} & P'_{23} & P'_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & P'_{31} & P'_{32} & P'_{33} & P'_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P'_{41} & P'_{42} & P'_{43} & P'_{44} & 0 \\ 0 & -P_{41} & -P_{42} & -P_{43} & -P_{44} & 0 & 0 & 0 & 0 & 0 & M \\ \hline 0 & -U_1 & -U_2 & -U_3 & -U_4 & 0 & -U'_1 & -U'_2 & -U'_3 & -U'_4 & J \end{array}\right) \left(\begin{array}{ccccccccccc|c} C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z \\ 0 & Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 & 0 & 0 & 0 & 0 & V_1 \\ 0 & Q_{21} & Q_{22} & Q_{23} & Q_{24} & 0 & 0 & 0 & 0 & 0 & 0 & V_2 \\ 0 & Q_{31} & Q_{32} & Q_{33} & Q_{34} & 0 & 0 & 0 & 0 & 0 & 0 & V_3 \\ 0 & Q_{41} & Q_{42} & Q_{43} & Q_{44} & 0 & 0 & 0 & 0 & 0 & 0 & V_4 \\ 0 & 0 & 0 & 0 & 0 & L' & 0 & 0 & 0 & 0 & 0 & W' \\ Q'_{12} & 0 & Q'_{11} & 0 & 0 & Q'_{13} & Q'_{11} & Q'_{12} & Q'_{13} & Q'_{14} & 0 & V'_1 \\ Q'_{22} & 0 & Q'_{21} & 0 & 0 & Q'_{23} & Q'_{21} & Q'_{22} & Q'_{23} & Q'_{24} & 0 & V'_2 \\ Q'_{32} & 0 & Q'_{31} & 0 & 0 & Q'_{33} & Q'_{31} & Q'_{32} & Q'_{33} & Q'_{34} & 0 & V'_3 \\ Q'_{42} & 0 & Q'_{41} & 0 & 0 & Q'_{43} & Q'_{41} & Q'_{42} & Q'_{43} & Q'_{44} & 0 & V'_4 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & I & 0 \end{array}\right),$$

where the factors have distributions

$$(\zeta * \alpha * \delta * \pi_3 * \pi_4 * \pi'_3 * \delta * \zeta * \pi'_3 * \pi'_4 * \pi_4 * \gamma,$$
$$\zeta * \omega_1 * \omega_2 * \omega_3 * \omega_4 * \pi'_3 * \omega'_1 * \omega'_2 * \omega'_3 * \omega'_4 * \theta_4),$$

$$(\zeta * \omega_1 * \omega_2 * \omega_3 * \omega_4 * \pi'_3 * \omega'_1 * \omega'_2 * \omega'_3 * \omega'_4 * \theta_4,$$
$$\eta * \beta * \varepsilon * \theta_3 * \theta_4 * \theta'_3 * \varepsilon * \eta * \theta'_3 * \theta'_4 * \theta_4 * e),$$

respectively. This factorization implies that $(F, A, X, \alpha, \beta) \sim (H, C, Z, \zeta, \eta)$, as desired. ■

5.2 – *Operations*

Let $\gamma \in \Gamma$.

If $(F', A', X', \alpha', \beta'), (F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$, then we define

$$\begin{aligned}(F', A', X', \alpha', \beta') &+ (F, A, X, \alpha, \beta) \\ &= \left(\begin{pmatrix} F' & F \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} X' \\ X \end{pmatrix}, \alpha' * \alpha, \beta' * \beta\right).\end{aligned}$$

Note that it belongs to $(T_\Sigma)_\gamma$.

If $(F', A', X', \alpha', \beta') \in (T_\Sigma)_{\gamma'}$ and $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$, then we define

$$\begin{aligned}(F', A', X', \alpha', \beta') &\cdot (F, A, X, \alpha, \beta) \\ &= \left(\begin{pmatrix} 0 & F' \end{pmatrix}, \begin{pmatrix} A & 0 \\ -X'F & A' \end{pmatrix}, \begin{pmatrix} X \\ 0 \end{pmatrix}, \alpha * \alpha'\gamma, \beta * \beta'\gamma\right).\end{aligned}$$

Note that this element belongs to $(T_\Sigma)_{\gamma'\gamma}$ because the homogeneous matrix $(0 \;\; F')$ has distribution $(\gamma'\gamma, \beta * \beta'\gamma)$ and the homogeneous matrix $\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right)$ has distribution $(\alpha * \alpha'\gamma, e)$.

If $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$, we define

$$-(F, A, X, \alpha, \beta) = (-F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma.$$

Finally, if $r \in R_\gamma$, we define

$$\mu(r) = (r, 1, 1, e, e) \in (T_\Sigma)_\gamma.$$

Now we prove a series of lemmas that show the compatibility of the operations just defined and the equivalence relation $\sim$.

Lemma 5.3. *The following assertions hold true:*

(1) *If* $(F', A, X, \alpha, \beta), (F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$, *then*

$$(F', A, X, \alpha, \beta) + (F, A, X, \alpha, \beta) \sim (F' + F, A, X, \alpha, \beta).$$

(2) *If* $(F, A, X', \alpha, \beta), (F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$, *then*

$$(F, A, X', \alpha, \beta) + (F, A, X, \alpha, \beta) \sim (F, A, X' + X, \alpha, \beta).$$

(3) *If* $r \in R_{\gamma'}$ *and* $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$, *then*

$$\mu(r) \cdot (F, A, X, \alpha, \beta) \sim (rF, A, X, \alpha, \beta) \in (T_\Sigma)_{\gamma'\gamma}.$$

(4) *If* $(F', A', X', \alpha', \beta') \in (T_\Sigma)_{\gamma'}$ *and* $r \in R_\gamma$, *then*

$$(F', A', X', \alpha', \beta') \cdot \mu(r) \sim (F', A', X'r, \alpha'\gamma, \beta'\gamma) \in (T_\Sigma)_{\gamma'\gamma}.$$

Proof. (1) This follows from the factorization

$$\left(\begin{array}{ccc|c} A & 0 & 0 & X \\ 0 & A & 0 & X \\ 0 & 0 & A & X \\ \hline F' & F & -F'-F & 0 \end{array}\right) = \left(\begin{array}{ccc} I & 0 & 0 \\ I & A & 0 \\ I & 0 & A \\ \hline 0 & F & -F'-F \end{array}\right) \left(\begin{array}{ccc|c} A & 0 & 0 & X \\ -I & I & 0 & 0 \\ -I & 0 & I & 0 \end{array}\right),$$

where the factors on the right-hand side have distributions

$$(\alpha * \alpha * \alpha * \gamma, \alpha * \beta * \beta), \quad (\alpha * \beta * \beta, \beta * \beta * \beta * e),$$

respectively.

(2) This follows from the equality

$$\left(\begin{array}{ccc|c} A & 0 & 0 & X' \\ 0 & A & 0 & X \\ 0 & 0 & A & X'+X \\ \hline F & F & -F & 0 \end{array}\right) = \left(\begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ I & I & A \\ \hline 0 & 0 & -F \end{array}\right) \left(\begin{array}{ccc|c} A & 0 & 0 & X' \\ 0 & A & 0 & X \\ -I & -I & I & 0 \end{array}\right),$$

where the factors on the right-hand side have distributions

$$(\alpha * \alpha * \alpha * \gamma, \alpha * \alpha * \beta), \quad (\alpha * \alpha * \beta, \beta * \beta * \beta * e),$$

respectively.

(3) This follows from the factorization

$$\left(\begin{array}{ccc|c} A & 0 & 0 & X \\ -F & 1 & 0 & 0 \\ 0 & 0 & A & X \\ \hline 0 & r & -rF & 0 \end{array}\right) = \left(\begin{array}{ccc} I & 0 & 0 \\ 0 & 1 & 0 \\ I & 0 & A \\ \hline 0 & r & -rF \end{array}\right) \left(\begin{array}{ccc|c} A & 0 & 0 & X \\ -F & 1 & 0 & 0 \\ -I & 0 & I & 0 \end{array}\right),$$

where the factors on the right-hand side have distributions

$$(\alpha * \gamma * \alpha * \gamma'\gamma, \alpha * \gamma * \beta), \quad (\alpha * \gamma * \beta, \beta * \gamma * \beta * e),$$

respectively.

(4) This follows from the factorization

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -X'r & A' & 0 & 0 \\ 0 & 0 & A' & X'r \\ \hline 0 & F' & -F' & 0 \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & I & 0 \\ X'r & I & A' \\ \hline 0 & 0 & -F' \end{array}\right) \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -X'r & A' & 0 & 0 \\ 0 & -I & I & 0 \end{array}\right),$$

where the factors on the right-hand side have distributions

$$(e * \alpha'\gamma * \alpha'\gamma * \gamma'\gamma, e * \alpha'\gamma * \beta'\gamma), \quad (e * \alpha'\gamma * \beta'\gamma, e * \beta'\gamma * \beta'\gamma * e),$$

respectively. ∎

Lemma 5.4. *The relation $\sim$ is compatible with the operations defined on the $(T_\Sigma)_\gamma$. More precisely, the following assertions hold true:*

(1) *For $x', x \in (T_\Sigma)_\gamma$, then $x + x' \sim x' + x$.*

(2) *For $x', x, y \in (T_\Sigma)_\gamma$ such that $x \sim y$, then $x' + x \sim x' + y$ and $x + x' \sim y + x'$.*

(3) *For $x, y \in (T_\Sigma)_\gamma$ and $x' \in (T_\Sigma)_{\gamma'}$ such that $x \sim y$, then $x'x \sim x'y$ and $xx' \sim yx'$.*

(4) *For $x, y \in (T_\Sigma)_\gamma$ such that $x \sim y$, then $-x \sim -y$.*

Proof. (1) Let $(F', A', X', \alpha', \beta'), (F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$. The equality

$$\left(\begin{array}{cccc|c} A' & 0 & 0 & 0 & X' \\ 0 & A & 0 & 0 & X \\ 0 & 0 & A & 0 & X \\ 0 & 0 & 0 & A' & X' \\ \hline F' & F & -F & -F' & 0 \end{array}\right) = \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & I & A & 0 \\ I & 0 & 0 & A' \\ \hline 0 & 0 & -F & -F' \end{array}\right) \left(\begin{array}{cccc|c} A' & 0 & 0 & 0 & X' \\ 0 & A & 0 & 0 & X \\ 0 & -I & I & 0 & 0 \\ -I & 0 & 0 & I & 0 \end{array}\right),$$

where the factors on the right-hand side have distributions

$$(\alpha' * \alpha * \alpha * \alpha' * \gamma, \alpha' * \alpha * \beta * \beta') \quad (\alpha' * \alpha * \beta * \beta', \beta' * \beta * \beta * \beta' * e),$$

respectively, shows (1).

(2) First note that, by (1), it is enough to prove that $x' + x \sim x' + y$. Now let $\gamma \in \Gamma$, let $(F', A', X', \alpha', \beta') \in (T_\Sigma)_\gamma$ and let $(F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon) \in (T_\Sigma)_\gamma$ be such that $(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$. Thus there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U, and homogeneous columns W, V, as in (5.1). The result follows because

the matrix

$$\left(\begin{array}{ccccccccc|c}
A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X' \\
0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
0 & 0 & A' & 0 & 0 & 0 & 0 & 0 & 0 & X' \\
0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & Y \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & W \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 \\
\hline
F' & F & -F' & -G & 0 & F & -G & 0 & J & 0
\end{array}\right)$$

can be expressed as a product of the homogeneous matrices

$$\left(\begin{array}{ccccccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & A' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & -I & 0 & 0 & 0 & P_{11} & P_{12} & P_{13} & P_{14} \\
0 & 0 & 0 & -I & 0 & P_{21} & P_{22} & P_{23} & P_{24} \\
0 & 0 & 0 & 0 & -I & P_{31} & P_{32} & P_{33} & P_{34} \\
0 & 0 & 0 & 0 & 0 & P_{41} & P_{42} & P_{43} & P_{44} \\
\hline
0 & 0 & -F' & 0 & 0 & U_1 & U_2 & U_3 & U_4
\end{array}\right)
\left(\begin{array}{ccccccccc|c}
A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X' \\
0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
-I & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & Y \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & W \\
0 & Q_{11} & 0 & Q_{12} & Q_{13} & Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\
0 & Q_{21} & 0 & Q_{22} & Q_{23} & Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\
0 & Q_{31} & 0 & Q_{32} & Q_{33} & Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\
0 & Q_{41} & 0 & Q_{42} & Q_{43} & Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4
\end{array}\right)$$

that have distributions

$$\begin{aligned}
&(\alpha' * \alpha * \alpha' * \delta * \pi_3 * \alpha * \delta * \pi_3 * \pi_4 * \gamma, \\
&\qquad \alpha' * \alpha * \beta' * \delta * \pi_3 * \omega_1 * \omega_2 * \omega_3 * \omega_4), \\
&(\alpha' * \alpha * \beta' * \delta * \pi_3 * \omega_1 * \omega_2 * \omega_3 * \omega_4, \\
&\qquad \beta' * \beta * \beta' * \varepsilon * \theta_3 * \beta * \varepsilon * \theta_3 * \theta_4 * e),
\end{aligned}$$

respectively.

(3) Let $\gamma, \gamma' \in \Gamma$, let $(F', A', X', \alpha', \beta') \in (T_\Sigma)_{\gamma'}$ and let (F, A, X, α, β), $(G, B, Y, \delta, \varepsilon) \in (T_\Sigma)_\gamma$ be such that $(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$. Thus there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U, and homogeneous columns W, V, as in (5.1).

We prove first that

$$(F', A', X', \alpha', \beta') \cdot (F, A, X, \alpha, \beta) \sim (F', A', X', \alpha', \beta') \cdot (G, B, Y, \delta, \varepsilon).$$

This follows because the homogeneous matrix

$$\left(\begin{array}{cccccccccc|c}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
-X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y \\
0 & 0 & -X'G & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & W \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -X'F & X'G & 0 & -XJ & A' & 0 \\
\hline
0 & F' & 0 & -F' & 0 & 0 & 0 & 0 & 0 & F' & 0
\end{array}\right)$$

has the factorization, as a product of homogeneous matrices,

$$\left(\begin{array}{cccccccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
-I & 0 & 0 & 0 & 0 & P_{11} & P_{12} & P_{13} & P_{14} & 0 \\
0 & 0 & -I & 0 & 0 & P_{21} & P_{22} & P_{23} & P_{24} & 0 \\
0 & 0 & 0 & 0 & -I & P_{31} & P_{32} & P_{33} & P_{34} & 0 \\
0 & 0 & 0 & 0 & 0 & P_{41} & P_{42} & P_{43} & P_{44} & 0 \\
0 & -I & 0 & I & 0 & -X'U_1 & -X'U_2 & -X'U_3 & -X'U_4 & A' \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F'
\end{array}\right)
\left(\begin{array}{cccccccccc|c}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
-X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y \\
0 & 0 & -X'G & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & W \\
Q_{11} & 0 & Q_{12} & 0 & Q_{13} & Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & V_1 \\
Q_{21} & 0 & Q_{22} & 0 & Q_{23} & Q_{21} & Q_{22} & Q_{23} & Q_{24} & 0 & V_2 \\
Q_{31} & 0 & Q_{32} & 0 & Q_{33} & Q_{31} & Q_{32} & Q_{33} & Q_{34} & 0 & V_3 \\
Q_{41} & 0 & Q_{42} & 0 & Q_{43} & Q_{41} & Q_{42} & Q_{43} & Q_{44} & 0 & V_4 \\
0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & I & 0
\end{array}\right),$$

where the factors have distributions

$$\begin{aligned}
&(\alpha * \alpha'\gamma * \delta * \alpha'\gamma * \pi_3 * \alpha * \delta * \pi_3 * \pi_4 * \alpha'\gamma * \gamma'\gamma, \\
&\qquad \alpha * \alpha'\gamma * \delta * \alpha'\gamma * \pi_3 * \omega_1 * \omega_2 * \omega_3 * \omega_4 * \beta'\gamma), \\
&(\alpha * \alpha'\gamma * \delta * \alpha'\gamma * \pi_3 * \omega_1 * \omega_2 * \omega_3 * \omega_4 * \beta'\gamma, \\
&\qquad \beta * \beta'\gamma * \varepsilon * \beta'\gamma * \theta_3 * \beta * \varepsilon * \theta_3 * \theta_4 * \beta'\gamma * e),
\end{aligned}$$

respectively.

Now let $\gamma, \gamma' \in \Gamma$, let $(F, A, X, \alpha, \beta), \in (T_\Sigma)_\gamma$ and let $(F', A', X', \alpha', \beta')$, $(G', B', Y', \delta', \varepsilon') \in (T_\Sigma)_{\gamma'}$ be such that $(F', A', X', \alpha', \beta') \sim (G', B', Y', \delta', \varepsilon')$. Thus there exist $L', M', P, Q \in \Sigma$, homogeneous rows J', U, and homogeneous columns W', V such that

$$\left(\begin{array}{cccc|c}
A' & 0 & 0 & 0 & X' \\
0 & B' & 0 & 0 & Y' \\
0 & 0 & L' & 0 & W' \\
0 & 0 & 0 & M' & 0 \\
\hline
F' & -G' & 0 & J' & 0
\end{array}\right)
=
\left(\begin{array}{cccc}
P_{11} & P_{12} & P_{13} & P_{14} \\
P_{21} & P_{22} & P_{23} & P_{24} \\
P_{31} & P_{32} & P_{33} & P_{34} \\
P_{41} & P_{42} & P_{43} & P_{44} \\
\hline
U_1 & U_2 & U_3 & U_4
\end{array}\right)
\left(\begin{array}{cccc|c}
Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\
Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4
\end{array}\right),$$

where P, U, Q, V have distributions (π', ω'), (γ', ω'), (ω', θ'), (ω', e), respectively, and $\pi'_1 = \alpha'$, $\pi'_2 = \delta'$, $\theta'_1 = \beta'$, $\theta'_2 = \varepsilon'$. We show that

$$(F', A', X', \alpha', \beta') \cdot (F, A, X, \alpha, \beta) \sim (G', B', Y', \delta', \varepsilon') \cdot (F, A, X.\alpha, \delta).$$

This follows because the homogeneous matrix

$$\left(\begin{array}{ccccccccccc|c}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
-X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
0 & 0 & -Y'F & B' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & X \\
0 & 0 & 0 & 0 & -W'F & L' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -X'F & A' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L' & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M' & 0 \\
\hline
0 & F' & 0 & -G' & 0 & 0 & 0 & F' & -G' & 0 & J' & 0
\end{array}\right)$$

can be expressed as the product of homogeneous matrices

$$\left(\begin{array}{ccccccccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & -I & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 \\
0 & -I & 0 & 0 & 0 & 0 & -X'F & P_{11} & P_{12} & P_{13} & P_{14} \\
0 & 0 & 0 & -I & 0 & 0 & 0 & P_{21} & P_{22} & P_{23} & P_{24} \\
0 & 0 & 0 & 0 & -W'F & -I & 0 & P_{31} & P_{32} & P_{33} & P_{34} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{41} & P_{42} & P_{43} & P_{44} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & U_1 & U_2 & 0U_3 & U_4
\end{array}\right)
\left(\begin{array}{ccccccccccc|c}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
-X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
0 & 0 & -Y'F & B' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -I & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -W'F & L' & 0 & 0 & 0 & 0 & 0 & 0 \\
-I & 0 & -I & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & Q_{11} & -V_1F & Q_{12} & 0 & Q_{13} & 0 & Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 \\
0 & Q_{21} & -V_2F & Q_{22} & 0 & Q_{23} & 0 & Q_{21} & Q_{22} & Q_{23} & Q_{24} & 0 \\
0 & Q_{31} & -V_3F & Q_{32} & 0 & Q_{33} & 0 & Q_{31} & Q_{32} & Q_{33} & Q_{34} & 0 \\
0 & Q_{41} & -V_4F & Q_{34} & 0 & Q_{43} & 0 & Q_{41} & Q_{42} & Q_{43} & Q_{44} & 0
\end{array}\right),$$

where the factors have distributions

$$(\alpha * \alpha'\gamma * \alpha * \delta'\gamma * \alpha * \pi'_3\gamma * \alpha * \alpha'\gamma * \delta'\gamma * \pi'_3\gamma * \pi'_4\gamma * \gamma'\gamma,$$
$$\alpha * \alpha'\gamma * \alpha * \delta'\gamma * \beta * \pi'_3\gamma * \beta * \omega'_1\gamma * \omega'_2\gamma * \omega'_3\gamma * \omega'_4\gamma),$$

$$(\alpha * \alpha'\gamma * \alpha * \delta'\gamma * \beta * \pi'_3\gamma * \beta * \omega'_1\gamma * \omega'_2\gamma * \omega'_3\gamma * \omega'_4\gamma,$$
$$\beta * \beta'\gamma * \beta * \varepsilon'\gamma * \beta * \theta'_3\gamma * \beta * \beta'\gamma * \varepsilon'\gamma * \theta'_3\gamma * \theta'_4\gamma * e),$$

respectively.

(4) Let $(F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon) \in (T_\Sigma)_\gamma$ be such that $(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$. Thus there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V, as in (5.1). The result follows because we have the factorization

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \\ \hline -F & G & 0 & -J & 0 \end{array}\right) = \left(\begin{array}{cccc} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \\ \hline -U_1 & -U_2 & -U_3 & -U_4 \end{array}\right) \left(\begin{array}{cccc|c} Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4 \end{array}\right),$$

where the factors have distributions

$$(\alpha * \delta * \pi_3 * \pi_4 * \gamma, \omega_1 * \omega_2 * \omega_3 * \omega_4),$$

$$(\omega_1 * \omega_2 * \omega_3 * \omega_4, \beta * \varepsilon * \theta_3 * \theta_4 * e),$$

respectively. ■

5.3 – *Graded ring structure*

We define $(\mathcal{R}_\Sigma)_\gamma$ as the set of equivalence classes in $(T_\Sigma)_\gamma$ under the equivalence relation $\sim$. The equivalent class of $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$ will be denoted by $[F, A, X, \alpha, \beta]$.

In Section 5.2 we proved that the operation + is well defined in $(\mathcal{R}_\Sigma)_\gamma$ for each $\gamma \in \Gamma$.

Lemma 5.5. *Let $\gamma \in \Gamma$. Then $(\mathcal{R}_\Sigma)_\gamma$ is an abelian group with sum defined by*

$$[F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta] = \left[\begin{pmatrix} F' & F \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} X' \\ X \end{pmatrix}, \alpha' * \alpha, \beta' * \beta\right]$$

Proof. The operation is well defined and commutative by Lemma 5.4 (2) and (1). Now we show that the operation is associative. Let $[F'', A'', X'', \alpha'', \beta'']$, $[F', A', X', \alpha', \beta']$, $[F, A, X, \alpha, \beta] \in (T_\Sigma)_\gamma$. Then

$$[F'', A'', X'', \alpha'', \beta''] + ([F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta])$$
$$= [F'', A'', X'', \alpha'', \beta''] + \left[\begin{pmatrix} F' & F \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} X' \\ X \end{pmatrix}, \alpha' * \alpha, \beta' * \beta\right]$$

$$= \left[\begin{pmatrix} F'' & F' & F \end{pmatrix}, \begin{pmatrix} A'' & 0 & 0 \\ 0 & A' & 0 \\ 0 & 0 & A \end{pmatrix}, \begin{pmatrix} X'' \\ X' \\ X \end{pmatrix}, \alpha'' * \alpha' * \alpha, \beta'' * \beta' * \beta \right]$$

$$= \left[\begin{pmatrix} F'' & F' \end{pmatrix}, \begin{pmatrix} A'' & 0 \\ 0 & A' \end{pmatrix}, \begin{pmatrix} X'' \\ X' \end{pmatrix}, \alpha'' * \alpha', \beta'' * \beta' \right] + [F, A, X, \alpha, \beta]$$

$$= ([F'', A'', X'', \alpha'', \beta''] + [F', A', X', \alpha', \beta']) + [F, A, X, \alpha, \beta],$$

as desired.

The element $\mu(0) = [0, 1, 1, e, e]$ is the zero element. Indeed, let $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$. Then we have the factorization

$$\left(\begin{array}{ccc|c} A & 0 & 0 & X \\ 0 & 1 & 0 & 1 \\ 0 & 0 & A & X \\ \hline F & 0 & -F & 0 \end{array}\right) = \left(\begin{array}{ccc} I & 0 & 0 \\ 0 & 1 & 0 \\ I & 0 & A \\ \hline 0 & 0 & -F \end{array}\right) \left(\begin{array}{ccc|c} A & 0 & 0 & X \\ 0 & 1 & 0 & 1 \\ -I & 0 & I & 0 \end{array}\right)$$

by Lemma 5.3, where the factors have distributions

$$(\alpha * e * \alpha * \gamma, \alpha * e * \beta), \quad (\alpha * e * \beta, \beta * e * \beta * e),$$

respectively. Thus $[F, A, X, \alpha, \beta] + [0, 1, 1, e, e] = [F, A, X, \alpha, \beta]$.

Given $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$, the element $[-F, A, X, \alpha, \beta]$ is well defined by Lemma 5.4 (4). We claim that it is the additive inverse of $[F, A, X, \alpha, \beta]$ in $\mathcal{R}_\gamma$. Thus consider the factorization

$$\left(\begin{array}{ccc|c} A & 0 & 0 & X \\ 0 & A & 0 & X \\ 0 & 0 & 1 & 1 \\ \hline F & -F & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc} I & 0 & 0 \\ I & A & 0 \\ 0 & 0 & 1 \\ \hline 0 & -F & 0 \end{array}\right) \left(\begin{array}{ccc|c} A & 0 & 0 & X \\ -I & I & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right),$$

where the factors have distributions

$$(\alpha * \alpha * e * \gamma, \alpha * \beta * e), \quad (\alpha * \beta * e, \beta * \beta * e * e),$$

respectively. It shows that $[F, A, X, \alpha, \beta] + [-F, A, X, \alpha, \beta] = [0, 1, 1, e, e]$, as claimed. ∎

In Section 5.2 we showed that the product functions $(\mathcal{R}_\Sigma)_{\gamma'} \times (\mathcal{R}_\Sigma)_\gamma \to (\mathcal{R}_\Sigma)_{\gamma'\gamma}$ are well defined. Now we define $\mathcal{R}_\Sigma = \bigoplus_{\gamma \in \Gamma} (\mathcal{R}_\Sigma)_\gamma$. By the foregoing lemma, it is an additive group. We now prove that it is a Γ-graded ring with the induced product.

Lemma 5.6. *$\mathcal{R}_\Sigma$ is a Γ-graded ring with the product determined by the rule*

$$[F',A',X',\alpha',\beta']\cdot[F,A,X,\alpha,\beta]=\left[\begin{pmatrix}0 & F'\end{pmatrix},\begin{pmatrix}A & 0\\ -X'F & A'\end{pmatrix},\begin{pmatrix}X\\0\end{pmatrix},\alpha*\alpha'\gamma,\beta*\beta'\gamma\right],$$

for any $(F',A',X',\alpha',\beta')\in(T_\Sigma)_{\gamma'}$ and $(F,A,X,\alpha,\beta)\in(T_\Sigma)_\gamma$.

Proof. By Lemma 5.4 (3), the product is well defined.

By Lemma 5.3 (3) and (4), the identity element is $[1,1,1,e,e]$.

Now we proceed to show that the product is associative. Let $(F'',A'',X'',\alpha'',\beta'')\in(T_\Sigma)_{\gamma''}$, $(F',A',X',\alpha',\beta')\in(T_\Sigma)_{\gamma'}$ and $(F,A,X,\alpha,\beta)\in(T_\Sigma)_\gamma$. Then

$$\begin{aligned}
&\big([F'',A'',X'',\alpha'',\beta'']\cdot[F',A',X',\alpha',\beta']\big)\cdot[F,A,X,\alpha,\beta]\\
&\quad=\left[\begin{pmatrix}0 & F''\end{pmatrix},\begin{pmatrix}A' & 0\\ -X''F' & A''\end{pmatrix},\begin{pmatrix}X'\\0\end{pmatrix},\alpha'*\alpha''\gamma',\beta'*\beta''\gamma'\right]\cdot[F,A,X,\alpha,\beta]\\
&\quad=\left[\begin{pmatrix}0 & 0 & F''\end{pmatrix},\begin{pmatrix}A & 0 & 0\\ -X'F & A' & 0\\ 0 & -X''F' & A''\end{pmatrix},\begin{pmatrix}X\\0\\0\end{pmatrix},\alpha*\alpha'\gamma*\alpha''\gamma'\gamma,\beta*\beta'\gamma*\beta''\gamma'\gamma\right]\\
&\quad=[F'',A'',X'',\alpha'',\beta'']\cdot\left[\begin{pmatrix}0 & F'\end{pmatrix},\begin{pmatrix}A & 0\\ -X'F & A'\end{pmatrix},\begin{pmatrix}X\\0\end{pmatrix},\alpha*\alpha'\gamma,\beta*\beta'\gamma\right]\\
&\quad=[F'',A'',X'',\alpha'',\beta'']\cdot([F',A',X',\alpha',\beta']\cdot[F,A,X,\alpha,\beta]),
\end{aligned}$$

which shows that the product is associative.

It remains to show that the distributive laws are satisfied. Let $(F',A',X',\alpha',\beta')$, $(G',B',Y',\delta',\varepsilon')\in(T_\Sigma)_{\gamma'}$ and $(F,A,X,\alpha,\beta)\in(T_\Sigma)_\gamma$. First note that

$$\begin{aligned}
&\big([F',A',X',\alpha',\beta']+[G',B',Y',\delta',\varepsilon']\big)\cdot[F,A,X,\alpha,\beta]\\
&\quad=\left[\begin{pmatrix}0 & F' & G'\end{pmatrix},\begin{pmatrix}A & 0 & 0\\ -X'F & A' & 0\\ -Y'F & 0 & B'\end{pmatrix},\begin{pmatrix}X\\0\\0\end{pmatrix},\alpha*\alpha'\gamma*\delta'\gamma,\beta*\beta'\gamma*\varepsilon'\gamma\right].
\end{aligned}\tag{5.3}$$

Second, observe that

$$\begin{aligned}
&[F',A',X',\alpha',\beta']\cdot[F,A,X,\alpha,\beta]+[G',B',Y',\delta',\varepsilon']\cdot[F',A',X',\alpha',\beta']\\
&\quad=\left[\begin{pmatrix}0 & F' & 0 & G'\end{pmatrix},\begin{pmatrix}A & 0 & 0 & 0\\ -X'F & A' & 0 & 0\\ 0 & 0 & A & 0\\ 0 & 0 & -Y'F & B'\end{pmatrix},\begin{pmatrix}X\\0\\X\\0\end{pmatrix},\alpha*\alpha'\gamma*\alpha*\delta'\gamma,\beta*\beta'\gamma*\beta*\varepsilon'\gamma\right].
\end{aligned}\tag{5.4}$$

The fact that (5.3) equals (5.4) follows because the homogeneous matrix

$$\left(\begin{array}{ccccccc|c} A & 0 & 0 & 0 & 0 & 0 & 0 & X \\ -X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 \\ -Y'F & 0 & B' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & 0 & 0 & X \\ 0 & 0 & 0 & -X'F & A' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & 0 & X \\ 0 & 0 & 0 & 0 & 0 & -Y'F & B' & 0 \\ \hline 0 & F' & G' & 0 & -F' & 0 & -G' & 0 \end{array}\right)$$

factorizes as the product of homogeneous matrices

$$\left(\begin{array}{ccccccc} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & A & 0 & 0 & 0 \\ 0 & I & 0 & -X'F & A' & 0 & 0 \\ I & 0 & 0 & 0 & 0 & A & 0 \\ 0 & 0 & I & 0 & 0 & -Y'F & B' \\ \hline 0 & 0 & 0 & 0 & -F' & 0 & -G' \end{array}\right)\left(\begin{array}{ccccccc|c} A & 0 & 0 & 0 & 0 & 0 & 0 & X \\ -X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 \\ -Y'F & 0 & B' & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & I & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & I & 0 \end{array}\right),$$

where the factors have distributions

$$(\alpha * \alpha'\gamma * \delta'\gamma * \alpha * \alpha'\gamma * \alpha * \delta'\gamma * \gamma'\gamma, \alpha * \alpha'\gamma * \delta'\gamma * \beta * \beta'\gamma * \beta * \varepsilon'\gamma),$$
$$(\alpha * \alpha'\gamma * \delta'\gamma * \beta * \beta'\gamma * \beta * \varepsilon'\gamma, \beta * \beta'\gamma * \varepsilon'\gamma * \beta * \beta'\gamma * \beta * \varepsilon'\gamma * e),$$

respectively.

Now let $(F', A', X', \alpha', \beta') \in (T_\Sigma)_{\gamma'}$ and (F, A, X, α, β),$(G, B, Y, \delta, \varepsilon) \in (T_\Sigma)_\gamma$. First note that

$$\begin{aligned} &[F', A', X', \alpha', \beta'] \cdot ([F, A, X, \alpha, \beta] + [G, B, Y, \delta, \varepsilon]) \\ &= \left[\begin{pmatrix} 0 & 0 & F' \end{pmatrix}, \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ -X'F & -X'G & A' \end{pmatrix}, \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix}, \alpha * \delta * \alpha'\gamma, \beta * \varepsilon * \beta'\gamma \right]. \end{aligned} \tag{5.5}$$

Second, observe that

$$\begin{aligned} &[F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta] + [F', A', X', \alpha', \beta'] \cdot [G, B, Y, \delta, \varepsilon] \\ &= \left[\begin{pmatrix} 0 & F' & 0 & F' \end{pmatrix}, \begin{pmatrix} A & 0 & 0 & 0 \\ -X'F & A' & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & -X'G & A' \end{pmatrix}, \begin{pmatrix} X \\ 0 \\ Y \\ 0 \end{pmatrix}, \alpha * \alpha'\gamma * \delta * \alpha'\gamma, \beta * \beta'\gamma * \varepsilon * \beta'\gamma \right]. \end{aligned} \tag{5.6}$$

The fact that (5.5) equals (5.6) follows because the homogeneous matrix

$$\left(\begin{array}{ccccccc|c} A & 0 & 0 & 0 & 0 & 0 & 0 & X \\ -X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & 0 & 0 & Y \\ 0 & 0 & -X'G & A' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & X \\ 0 & 0 & 0 & 0 & 0 & B & 0 & Y \\ 0 & 0 & 0 & 0 & -X'F & -X'G & A' & 0 \\ \hline 0 & F' & 0 & F' & 0 & 0 & -F' & 0 \end{array}\right)$$

factorizes as the product of homogeneous matrices

$$\left(\begin{array}{ccccccc} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & A & 0 & 0 \\ 0 & 0 & I & 0 & 0 & B & 0 \\ 0 & I & 0 & I & -X'F & -X'G & A' \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -F' \end{array}\right)\left(\begin{array}{ccccccc|c} A & 0 & 0 & 0 & 0 & 0 & 0 & X \\ -X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & 0 & 0 & Y \\ 0 & 0 & -X'G & A' & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & I & 0 & 0 \\ 0 & -I & 0 & -I & 0 & 0 & I & 0 \end{array}\right),$$

where the factors have distributions

$$(\alpha*\alpha'\gamma*\delta*\alpha'\gamma*\alpha*\delta*\alpha'\gamma*\gamma'\gamma, \alpha*\alpha'\gamma*\delta*\alpha'\gamma*\beta*\varepsilon*\beta'\gamma),$$
$$(\alpha*\alpha'\gamma*\delta*\alpha'\gamma*\beta*\varepsilon*\beta'\gamma, \beta*\beta'\gamma*\varepsilon*\beta'\gamma*\beta*\varepsilon*\beta'\gamma*e),$$

respectively. ■

5.4 – *Universal localization property*

Proposition 5.7. *Consider the map $\mu\colon R\to\mathcal{R}_\Sigma$ determined by $\mu(r)=[r,1,1,e,e]$ for all $r\in R_\gamma$, $\gamma\in\Gamma$. Then the pair $(\mathcal{R}_\Sigma,\mu)$ is the universal localization of R at Σ.*

Proof. By Lemma 5.3 (1) and (3), μ is a homomorphism of Γ-graded rings.

By E_i we will denote the column matrix consisting of 1 as its i-entry and all the other entries are zero, and by E_i^{T} its transpose, the row matrix consisting of 1 as its i-entry and all other entries are zero.

Let $A=(a_{ij})\in\Sigma$ be an $n\times n$ homogeneous matrix of distribution (α,β).

We claim that the $n\times n$ matrix $B=([E_i^{\mathsf{T}},A,E_j,\alpha\alpha_j^{-1},\beta\alpha_j^{-1}])_{ij}$ is the inverse of A^μ.

First observe that $[E_i^\top, A, E_j, \alpha\alpha_j^{-1}, \beta\alpha_j^{-1}] \in \mathcal{R}_{\beta_i\alpha_j^{-1}}$ because $E_i^\top$ has distribution $(\beta_i\alpha_j^{-1}, \beta\alpha_j^{-1})$, A has distribution $(\alpha\alpha_j^{-1}, \beta\alpha_j^{-1})$ and E_j has distribution $(\alpha\alpha_j^{-1}, e)$. Thus B is homogeneous of distribution (β, α).

Second, using Lemma 5.3 (3) and (1), we obtain that the product of the ith line of A^μ with the jth column of B equals

$$\begin{aligned}\sum_k \mu(a_{i,k})[E_k^\top, A, E_j, \alpha\alpha_j^{-1}, \beta\alpha_j^{-1}] &= \sum_k [a_{i,k}E_k^\top, A, E_j, \alpha\alpha_j^{-1}, \beta\alpha_j^{-1}] \\ &= \left[\sum_k a_{i,k}E_k^\top, A, E_j, \alpha\alpha_j^{-1}, \beta\alpha_j^{-1}\right] \\ &= [E_i^\top A, A, E_j, \alpha\alpha_j^{-1}, \beta\alpha_j^{-1}] \in \mathcal{R}_{\alpha_i\alpha_j^{-1}}.\end{aligned}$$

Third, we show that

$$[E_i^\top A, A, E_j, \alpha\alpha_j^{-1}, \beta\alpha_j^{-1}] = \mu(\delta_{ij}) = [\delta_{ij}, 1, 1, e, e] = \begin{cases}[1, 1, 1, e, e] & \text{if } i = j, \\ [0, 1, 1, e, e] & \text{if } i \neq j.\end{cases}$$

This follows from the factorization

$$\left(\begin{array}{cc|c} A & 0 & E_j \\ 0 & 1 & 1 \\ \hline E_i^\top A & -\delta_{ij} & 0 \end{array}\right) = \left(\begin{array}{cc} I & 0 \\ 0 & 1 \\ \hline E_i^\top & -\delta_{ij} \end{array}\right)\left(\begin{array}{cc|c} A & 0 & E_j \\ 0 & 1 & 1 \end{array}\right),$$

where the factors have distributions

$$(\alpha\alpha_j^{-1} * e * \alpha_i\alpha_j^{-1}, \alpha\alpha_j^{-1} * e), \quad (\alpha\alpha_j^{-1} * e, \beta\alpha_j^{-1} * e * e),$$

respectively. Therefore B is the right inverse of A^μ.

Now we proceed to prove that B is the left inverse of A^μ. Using Lemma 5.3 (4) and (2) and considering that $a_{kj} \in R_{\alpha_k\beta_j^{-1}}$, we obtain that the product of the ith line of B with the jth column of A^μ equals

$$\begin{aligned}&\sum_k [E_i^\top, A, E_k, \alpha\alpha_k^{-1}, \beta\alpha_k^{-1}]\mu(a_{k,j}) \\ &\quad = \sum_k [E_i^\top, A, E_k a_{k,j}, \alpha\alpha_k^{-1}\cdot\alpha_k\beta_j^{-1}, \beta\alpha_k^{-1}\cdot\alpha_k\beta_j^{-1}] \\ &\quad = \left[E_i^\top, A, \sum_k E_k a_{k,j}, \alpha\beta_j^{-1}, \beta\beta_j^{-1}\right] \\ &\quad = [E_i^\top, A, AE_j, \alpha\beta_j^{-1}, \beta\beta_j^{-1}] \in \mathcal{R}_{\beta_i\beta_j^{-1}}.\end{aligned}$$

As before, we show that $[E_i^{\mathsf{T}}, A, AE_j, \alpha\beta_j^{-1}, \beta\beta_j^{-1}] = \mu(\delta_{ij})$. This follows from

$$\left(\begin{array}{cc|c} A & 0 & AE_j \\ 0 & 1 & 1 \\ \hline E_i^{\mathsf{T}} & \delta_{ij} & 0 \end{array}\right) = \left(\begin{array}{cc} A & 0 \\ 0 & 1 \\ \hline E_i^{\mathsf{T}} & -\delta_{ij} \end{array}\right) \left(\begin{array}{cc|c} I & 0 & E_j \\ 0 & 1 & 1 \end{array}\right),$$

where the factors have distributions

$$(\alpha\beta_j^{-1} * e * \beta_i\beta_j^{-1}, \beta\beta_j^{-1} * e), \quad (\beta\beta_j^{-1} * e, \beta\beta_j^{-1} * e * e),$$

respectively.

Therefore, the claim is proved.

It remains to prove that $\mu\colon R \to \mathcal{R}_\Sigma$ is universal.

Note that if $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$, and we suppose that $F = (f_1, \ldots, f_n)$ and $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then

$$\begin{aligned} F^\mu (A^\mu)^{-1} X^\mu &= \sum_{i,j} \mu(f_i)[E_i^{\mathsf{T}}, A, E_j, \alpha\alpha_j^{-1}, \beta\alpha_j^{-1}]\mu(x_j) \\ &= \sum_{i,j} [f_i E_i^{\mathsf{T}}, A, E_j x_j, \alpha\alpha_j^{-1} \cdot \alpha_j, \beta\alpha_j^{-1} \cdot \alpha_j] \\ &= \left[\sum_i f_i E_i^{\mathsf{T}}, A, \sum_j E_j x_j, \alpha, \beta\right] \\ &= [F, A, X, \alpha, \beta]. \end{aligned} \tag{5.7}$$

Now let S be a Γ-graded ring and $\varphi\colon R \to S$ be a Σ-inverting homomorphism of graded rings. We define $\Phi\colon \mathcal{R}_\Sigma \to S$ as follows. Let $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$; then

$$\Phi([F, A, X, \alpha, \beta]) = F^\varphi (A^\varphi)^{-1} X^\varphi \in S_\gamma.$$

Now we show that Φ is well defined. Let $(F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon) \in (T_\Sigma)_\gamma$ be such that $(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$. Then there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V, such that

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \\ \hline F & -G & 0 & J & 0 \end{array}\right) = \left(\frac{P}{U}\right) \left(\begin{array}{c|c} Q & V \end{array}\right),$$

where P, U, Q, V have distributions (π,ω), (γ,ω), (ω,θ), (ω,e), respectively, and $\pi_1=\alpha, \pi_2=\delta, \theta_1=\beta, \theta_2=\varepsilon$. Then

$$\begin{aligned}
0 &= U^\varphi V^\varphi \\
&= U^\varphi Q^\varphi (Q^\varphi)^{-1}(P^\varphi)^{-1}P^\varphi V^\varphi \\
&= (UQ)^\varphi((PQ)^\varphi)^{-1}(PV)^\varphi \\
&= \begin{pmatrix} F^\varphi & -G^\varphi & 0 & J^\varphi \end{pmatrix} \begin{pmatrix} A^\varphi & 0 & 0 & 0 \\ 0 & B^\varphi & 0 & 0 \\ 0 & 0 & L^\varphi & 0 \\ 0 & 0 & 0 & M^\varphi \end{pmatrix}^{-1} \begin{pmatrix} X^\varphi \\ Y^\varphi \\ W^\varphi \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} F^\varphi & -G^\varphi & 0 & J^\varphi \end{pmatrix} \begin{pmatrix} (A^\varphi)^{-1} & 0 & 0 & 0 \\ 0 & (B^\varphi)^{-1} & 0 & 0 \\ 0 & 0 & (L^\varphi)^{-1} & 0 \\ 0 & 0 & 0 & (M^\varphi)^{-1} \end{pmatrix} \begin{pmatrix} X^\varphi \\ Y^\varphi \\ W^\varphi \\ 0 \end{pmatrix} \\
&= F^\varphi(A^\varphi)^{-1}X^\varphi - G^\varphi(B^\varphi)^{-1}Y^\varphi + 0(L^\varphi)^{-1}W^\varphi + J^\varphi(M^\varphi)^{-1}0 \\
&= F^\varphi(A^\varphi)^{-1}X^\varphi - G^\varphi(B^\varphi)^{-1}Y^\varphi,
\end{aligned}$$

which shows that Φ is well defined.

Now let $(F', A', X', \alpha', \beta')$, $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$. Then

$$\begin{aligned}
\Phi([F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta]) &= \begin{pmatrix} F' & F \end{pmatrix}^\varphi \left(\begin{pmatrix} A' & 0 \\ 0 & A \end{pmatrix}^\varphi \right)^{-1} \begin{pmatrix} X' \\ X \end{pmatrix}^\varphi \\
&= \begin{pmatrix} F'^\varphi & F \end{pmatrix} \begin{pmatrix} (A'^\varphi)^{-1} & 0 \\ 0 & (A^\varphi)^{-1} \end{pmatrix} \begin{pmatrix} X'^\varphi \\ X^\varphi \end{pmatrix} \\
&= F'^\varphi(A'^\varphi)^{-1}X'^\varphi + F^\varphi(A^\varphi)^{-1}X^\varphi \\
&= \Phi([F', A', X', \alpha', \beta']) \\
&\quad + \Phi([F, A, X, \alpha, \beta]).
\end{aligned}$$

Thus Φ is an additive map.

Let $(F', A', X', \alpha', \beta') \in (T_\Sigma)_{\gamma'}$ and $(F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma$. Then

$$\begin{aligned}
&\Phi([F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta]) \\
&\quad = \begin{pmatrix} 0 & F' \end{pmatrix}^\varphi \left(\begin{pmatrix} A & 0 \\ -X'F & A' \end{pmatrix}^\varphi \right)^{-1} \begin{pmatrix} X \\ 0 \end{pmatrix}^\varphi
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} 0 & F'^{\varphi} \end{pmatrix} \begin{pmatrix} A^{\varphi} & 0 \\ -X'^{\varphi} F^{\varphi} & A'^{\varphi} \end{pmatrix}^{-1} \begin{pmatrix} X^{\varphi} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & F'^{\varphi} \end{pmatrix} \begin{pmatrix} (A^{\varphi})^{-1} & 0 \\ -(A'^{\varphi})^{-1} X'^{\varphi} F^{\varphi} (A^{\varphi})^{-1} & (A'^{\varphi})^{-1} \end{pmatrix} \begin{pmatrix} X^{\varphi} \\ 0 \end{pmatrix} \\
&= F'^{\varphi} (A'^{\varphi})^{-1} X'^{\varphi} F^{\varphi} (A^{\varphi})^{-1} X^{\varphi} \\
&= \Phi([F', A', X', \alpha', \beta']) \cdot \Phi([F, A, X, \alpha, \beta]).
\end{aligned}
$$

Hence Φ is a homomorphism of graded rings. Clearly $\Phi\mu = \varphi$. The uniqueness of Φ now follows from (5.7). ■

5.5 – *Malcolmson's criterion*

Now we proceed to prove two results that determine the kernel of the natural homomorphism $R \to R_\Sigma$ and an important case in which the ring R_Σ is not zero. The following theorem is known as Malcolmson's criterion.

Theorem 5.8. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and Σ be a gr-lower semi-multiplicative subset of $\mathfrak{M}(R)$. Consider the canonical homomorphism of Γ-graded rings $\lambda\colon R \to R_\Sigma$. For $\gamma \in \Gamma$, a homogeneous element $r \in R_\gamma$ belongs to* $\ker\lambda$ *if and only if there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V, such that*

$$
\left(\begin{array}{cc|c} L & 0 & W \\ 0 & M & 0 \\ \hline 0 & J & r \end{array}\right) = \left(\begin{array}{c} P \\ \hline U \end{array}\right) \left(\begin{array}{c|c} Q & V \end{array}\right),
$$

where P, U, Q, V have distributions (π, ω), (γ, ω), (ω, θ), (ω, e), respectively.

Proof. By Proposition 5.7, $\mu\colon R \to \mathcal{R}_\Sigma$ is the universal localization of R at Σ. Thus $\lambda(r) = 0$ if and only if $\mu(r) = 0$.

Hence suppose that $r \in R_\gamma$ is such that $\mu(r) = 0$. It means that $[r, 1, 1, e, e] \sim [0, 1, 1, e, e]$. Thus there exist $L, M, P, Q \in \Sigma$, homogeneous lines J, U and homogeneous columns W, V, such that

$$
\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \\ \hline r & 0 & 0 & J & 0 \end{array}\right) = \left(\begin{array}{cccc} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \\ \hline U_1 & U_2 & U_3 & U_4 \end{array}\right) \left(\begin{array}{cccc|c} Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4 \end{array}\right),
$$

where P has distribution (π, ω), U has distribution (γ, ω), Q has distribution (ω, θ) and V has distribution (ω, e). Now the equality

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & L & 0 & 0 & W \\ 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -J & r & r \end{array}\right) = \left(\begin{array}{ccccc} P_{11} & P_{12} & P_{13} & P_{14} & 0 \\ P_{21} & P_{22} & P_{23} & P_{24} & 0 \\ P_{31} & P_{32} & P_{33} & P_{34} & 0 \\ P_{41} & P_{42} & P_{43} & P_{44} & 0 \\ -P_{11} & -P_{12} & -P_{13} & -P_{14} & 1 \\ \hline -U_1 & -U_2 & -U_3 & -U_4 & r \end{array}\right) \left(\begin{array}{ccccc|c} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & V_1 \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & 0 & V_2 \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & 0 & V_3 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & 0 & V_4 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right),$$

where the homogeneous matrices of the right-hand side have distributions

$$(e * e * \pi_3 * \pi_4 * e\gamma, \omega_1 * \omega_2 * \omega_3 * \omega_4 * e),$$
$$(\omega_1 * \omega_2 * \omega_3 * \omega_4 * e, e * e * \theta_3 * \theta_4 * e * e),$$

respectively, shows the result.

Conversely, suppose there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V, such that

$$\left(\begin{array}{cc|c} L & 0 & W \\ 0 & M & 0 \\ \hline 0 & J & r \end{array}\right) = \left(\frac{P}{U}\right) \left(Q \mid V \right),$$

where P, U, Q, V have distributions (π, ω), (γ, ω), (ω, θ), (ω, e), respectively. It follows that $[0, 1, 1, e, e] \sim [r, 1, 1, e, e]$ because

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \\ \hline 0 & -r & 0 & J & 0 \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & P \\ \hline 0 & -r & U \end{array}\right) \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & Q & V \end{array}\right)$$

where the factors on the right-hand side have distributions

$$(e * e * \pi * \gamma, e * e * \omega),$$
$$(e * e * \omega, e * e * \theta * e),$$

respectively. ∎

Corollary 5.9. *Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring and Σ be a gr-multiplicative subset of $\mathfrak{M}(R)$ consisting of gr-full matrices. Then R_Σ is a nonzero Γ-graded ring.*

Proof. It is enough to prove that $1 \in R_e$ is not in the kernel of the canonical homomorphism of graded rings $\lambda\colon R \to R_\Sigma$. Suppose that $1 \in \ker \lambda$. Then, by Theorem 5.8, there exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V, such that

$$\left(\begin{array}{cc|c} L & 0 & W \\ 0 & M & 0 \\ \hline 0 & J & 1 \end{array}\right) = \left(\frac{P}{U}\right)\left(\, Q \mid V \,\right),$$

where P, U, Q, V have distributions (π, ω), (e, ω), (ω, θ), (ω, e), respectively. Making elementary column operations, we obtain

$$\left(\begin{array}{cc|c} L & -WJ & W \\ 0 & M & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\frac{P}{U}\right)\left(\, Q' \mid V \,\right),$$

where P, U, Q', V have distributions (π, ω), (e, ω), (ω, θ), (ω, e), respectively. Since Σ is gr-multiplicative, it is also upper gr-semimultiplicative by Remark 3.1. Thus the matrix

$$\begin{pmatrix} L & -WJ & W \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Sigma,$$

but it is not gr-full, a contradiction. Therefore, $1 \notin \ker \lambda$. ∎

6. A gr-prime matrix ideal yields a graded division ring, and vice versa

The first part of this section is the adaptation to the graded context of the first part of [5, Section 7.3]. The proof of the main result Theorem 6.3 is the graded version of [15] using the construction of Section 5. It could also have been proved without using the results in Section 5 via a graded version of [15] that can be found in [11].

Throughout this section, let Γ *be a group.*

Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring. If (K, φ) is a Γ-graded epic R-division ring, the set

$$\left\{ A \in \mathfrak{M}(R) : A^\varphi \text{ is not invertible over } K \right\}$$

will be called the *singular kernel of* (K, φ). Now we show that gr-singular kernels are gr-prime matrix ideals. The aim of this section is to show that gr-singular kernels determine graded epic R-division rings similarly to the way that commutative R-fields are determined by prime ideals of R.

Given an $n \times n$ matrix A with entries in R, if we write $A = (A_1 \;\; A_2 \;\; \dots \;\; A_n)$ we understand that $A_1, \dots, A_n$ are the columns of A, and if we write

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}$$

we understand that $A_1, \dots, A_n$ are the rows of A.

Given two matrices $A, B \in \mathfrak{M}(R)$, we define the *diagonal sum* of A and B as

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Notice that if $A \in M_m(R)[\bar{\alpha}][\bar{\beta}]$ and $B \in M_n(R)[\overline{\alpha'}][\overline{\beta'}]$, then $A \oplus B \in M_{m+n}(R)[\bar{\alpha} * \overline{\alpha'}][\bar{\beta} * \overline{\beta'}]$.

Let $A, B \in M_n(R)[\bar{\alpha}][\bar{\beta}]$. If they differ at most in the ith column, then we define the *determinantal sum* of A and B with respect to the ith column as

$$A \nabla B = (A_1 \; \dots \; A_i + B_i \; \dots \; A_n).$$

Similarly, if they differ at most in the ith row we define the determinantal sum of A and B with respect to the ith row as

$$A \nabla B = \begin{pmatrix} A_1 \\ \vdots \\ A_i + B_i \\ \vdots \\ A_n \end{pmatrix}.$$

The matrix $A \nabla B$, when defined, has the same distribution as A and B.

Note that the operation $\oplus$ is associative. On the other hand, the operation ∇ is not always defined, and as a consequence it is not associative.

Notice that distributive laws are satisfied. More precisely, if C is another homogeneous matrix, then $C \oplus (A \nabla B) = (C \oplus A) \nabla (C \oplus B)$ and $(A \nabla B) \oplus C = (A \oplus C) \nabla (B \oplus C)$ whenever $A \nabla B$ is defined.

Also, if $B, C \in M_n(R)[\bar{\alpha}][\bar{\beta}]$ differ in at most one column (row) and $A \in M_n(R)[\overline{\alpha'}][\bar{\alpha}]$, $D \in M_n(R)[\bar{\beta}][\overline{\beta'}]$, then

$$A(B \nabla C) = AB \nabla AC, \quad (B \nabla C)D = BD \nabla CD.$$

On the other hand, if $B, C \in M_n(R)[\bar{\alpha}][\bar{\beta}]$ differ in at most one row (column), then it may happen that

$$A(B\nabla C) \neq AB\nabla AC, \quad (B\nabla C)D \neq BD\nabla CD,$$

because, for example, AB and AC (BD, BC) may differ in more than one row (column) and thus the right-hand side does not make sense. But in some cases one can apply the distributive law. Let $X \in \mathfrak{M}(R)$ and suppose that either X is a diagonal matrix, or X is a permutation matrix; then

$$X(B\nabla C) = XB\nabla XC, \quad (B\nabla C)X = BX\nabla CX.$$

Moreover, there exist $\alpha', \beta' \in \Gamma^n$ such that X can be considered as an element of $M_n(R)[\overline{\alpha'}][\bar{\alpha}] \cap M_n(R)[\bar{\beta}][\overline{\beta'}]$. Thus $X(B\nabla C) \in M_n(R)[\overline{\alpha'}][\bar{\beta}]$ and $(B\nabla C)X \in M_n(R)[\bar{\alpha}][\overline{\beta'}]$.

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. A subset $\mathcal{P}$ of $\mathfrak{M}(R)$ is a gr-prime matrix ideal if the following conditions are satisfied:

(PM1) $\mathcal{P}$ contains all the homogeneous matrices that are not gr-full.

(PM2) If $A, B \in \mathcal{P}$ and their determinantal sum (with respect to a row or column) exists, then $A\nabla B \in \mathcal{P}$.

(PM3) If $A \in \mathcal{P}$, then $A \oplus B \in \mathcal{P}$ for all $B \in \mathfrak{M}(R)$.

(PM4) For $A, B \in \mathfrak{M}(R)$, $A \oplus B \in \mathcal{P}$ implies that $A \in \mathcal{P}$ or $B \in \mathcal{P}$.

(PM5) $1 \notin \mathcal{P}$.

(PM6) If $A \in \mathcal{P}$ and E, F are permutation matrices of appropriate size, then $EAF \in \mathcal{P}$.

We remark that when $\Gamma = \{1\}$, that is, the ungraded case, (PM6) is a consequence of (PM1)–(PM5) as shown in [5, (g) on p. 431]. We have not been able to obtain (PM6) from the others in the general graded case.

Proposition 6.1. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. Let $K = \sum_{\gamma\in\Gamma} K_\gamma$ be a Γ-almost graded division ring and $\varphi\colon R \to K$ be a homomorphism of Γ-almost graded rings. Then*

$$\mathcal{P} = \{A \in \mathfrak{M}(R) : A^\varphi \textit{ is not invertible}\}$$

is a gr-prime matrix ideal. Therefore, the following assertions hold true:

(1) *If (K, φ) is a Γ-graded epic R-division ring, then the gr-singular kernel of (K, φ) is a Γ-gr-prime matrix ideal.*

(2) *Let N be a normal subgroup of Γ and consider R as a Γ/N-graded ring. Let (K,φ) be a Γ/N-graded epic R-division ring. Then*

$$\mathcal{P} = \left\{A \in \mathfrak{M}_\Gamma(R) : A^\varphi \textit{ is not invertible}\right\}$$

is a Γ-gr-prime matrix ideal.

Proof. Let K be a Γ-almost graded division ring and $\varphi\colon R \to K$ be a homomorphism of Γ-almost graded rings. As noted in Propositions 2.4 and 2.2, the lift $\widetilde{K}$ of K is a Γ-graded division ring and the lift $\widetilde{\varphi}\colon R \to \widetilde{S}$ of f is a homomorphism of Γ-graded rings such that $\varphi = \pi\widetilde{\varphi}$, where $\pi\colon \widetilde{K} \to K$ is the natural homomorphism of Γ-almost graded rings. Note that the sets $\mathcal{P}$ and $\{A \in \mathfrak{M}(R) : A^{\widetilde{\varphi}}$ is not invertible over $\widetilde{K}\}$ coincide by Proposition 2.5 (3). Thus we may suppose that K is a Γ-graded epic R-division ring and $\varphi\colon R \to K$ is a homomorphism of Γ-graded rings. Let $\mathcal{P} = \{A \in \mathfrak{M}(R) : A^\varphi$ is not invertible over $K\}$.

If $A \in \mathfrak{M}(R)$ is not gr-full, then A^φ is not gr-full. Since K is a Γ-graded division ring, A^φ is not invertible over K. Thus (PM1) is satisfied.

Now let $A, B \in \mathcal{P}_n[\bar{\alpha}][\bar{\beta}]$ such that $A\nabla B$ is defined. We may suppose that A, B differ in the first column. Hence $A = (A_1\ C_2\ \dots\ C_n)$ and $B = (B_1\ C_2\ \dots\ C_n)$. Since A^φ and B^φ are not invertible over K, the columns of A^φ and B^φ are right linearly dependent over K. If the columns $C_2^\varphi, \dots, C_n^\varphi$ are right linearly dependent over K, then the columns of $(A\nabla B)^\varphi$ are right linearly dependent over K and thus $A\nabla B \in \mathcal{P}$. Hence we can suppose that there exist homogeneous elements $a_1, \dots, a_n, b_1, \dots, b_n \in K$, with $a_1, b_1 \neq 0$, such that

$$A_1^\varphi a_1 + C_2^\varphi a_2 + \cdots + C_n^\varphi a_n = 0, \quad B_1^\varphi b_1 + C_2^\varphi b_2 + \cdots + C_n^\varphi b_n = 0.$$

But then

$$A_1^\varphi + B_1^\varphi + C_2^\varphi(a_2a_1^{-1} + b_2b_1^{-1}) + \cdots + C_n^\varphi(a_na_1^{-1} + b_nb_1^{-1}) = 0,$$

which shows that $A\nabla B \in \mathcal{P}$. Thus (PM2) is proved.

Let $A \in \mathcal{P}$ and $B \in \mathfrak{M}(R)$; then A^φ is not invertible over K, but then $A^\varphi \oplus B^\varphi = (A \oplus B)^\varphi$ is not invertible over K. This implies (PM3).

Now suppose that $A, B \in \mathfrak{M}(R)$ are such that $A \oplus B \in \mathcal{P}$. This means that the homogeneous matrix $A^\varphi \oplus B^\varphi$ is not invertible over K. This implies that either A^φ or B^φ is not invertible. That is, $A \in \mathcal{P}$ or $B \in \mathcal{P}$ and (PM4) follows.

Clearly, (PM5) is satisfied.

Let $A \in \mathcal{P}$ and E, F be permutation matrices with entries in R. Notice that E^φ, F^φ are permutation matrices with entries in K. Thus, if $(EAF)^\varphi = E^\varphi A^\varphi F^\varphi$ were invertible over K, then $A^\varphi = (E^\varphi)^{-1}(EAF)^\varphi(F^\varphi)^{-1}$ would be invertible over K, a contradiction. Thus $EAF \in \mathcal{P}$ and (PM6) is shown. ∎

Lemma 6.2. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and $\mathcal{P}$ be a gr-prime matrix ideal. Let $A, B \in \mathfrak{M}(R)$. The following assertions hold true:*

(1) *If A and B are such that $C = A\nabla B$ exists and B is not gr-full, then $A \in \mathcal{P}$ if and only if $C \in \mathcal{P}$.*

(2) *Let $A \in \mathcal{P}$. The result of adding a suitable right multiple of one column of A to another column again lies in $\mathcal{P}$. More precisely, if $A \in M_n(R)[\bar{\alpha}][\bar{\beta}]$ and $a \in R_{\beta_i\beta_j^{-1}}$, then $(A_1 \ \dots \ A_{j-1} \ A_j{+}A_ia \ A_{j+1} \ \dots \ A_n)$ belongs to $\mathcal{P}$.*

(3) *If $A \oplus B \in \mathcal{P}$, then $B \oplus A \in \mathcal{P}$.*

(4) *Suppose that $A \in M_m(R)[\bar{\alpha}][\bar{\beta}]$ and $B \in M_n(R)[\bar{\delta}][\bar{\varepsilon}]$. For $C \in M_{n\times m}(R)[\bar{\delta}][\bar{\beta}]$,*

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \in \mathcal{P} \quad \textit{if and only if} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{P}.$$

Similarly, for $C \in M_{m\times n}(R)[\bar{\beta}][\bar{\varepsilon}]$,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{P} \quad \textit{if and only if} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{P}.$$

(5) *The set $\mathfrak{M}(R) \setminus \mathcal{P}$ is gr-multiplicative.*

(6) *No identity matrix belongs to $\mathcal{P}$.*

(7) *Suppose that $A \in M_n(R)[\bar{\alpha}][\bar{\beta}]$ and $B \in M_n(R)[\bar{\beta}][\bar{\delta}]$. Then $AB \in \mathcal{P}$ if, and only if, $A \oplus B \in \mathcal{P}$.*

(8) *No invertible matrix in $\mathfrak{M}(R)$ belongs to $\mathcal{P}$.*

(9) *Suppose that A and B are such that $C = A\nabla B$ exists and $B \in \mathcal{P}$. Then $A \in \mathcal{P}$ if, and only if, $C \in \mathcal{P}$.*

Proof. (1) By (PM1) and (PM2), if $A \in \mathcal{P}$, then $C \in \mathcal{P}$. Conversely, suppose that $C \in \mathcal{P}$. Clearly $A = C\nabla B'$, where B' is obtained from B by changing the sign of a row or column. Now $A \in \mathcal{P}$ because B' is not gr-full.

(2) Suppose that $\bar{\beta} = \beta_1 * \overline{\beta'}$ and $c \in R_{\beta_2\beta_1^{-1}}$. If $A = (A_1 \ A_2 \ \dots \ A_n)$, then

$$\begin{aligned}(A_1{+}A_2c \ \ A_2 \ \dots \ A_n) &= (A_1 \ A_2 \ \dots \ A_n)\nabla(A_2c \ \ A_2 \ \dots \ A_n) \\ &= A\nabla(A_2 \ A_3 \ \dots \ A_n)\begin{pmatrix} c & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.\end{aligned}$$

Thus the right-hand side is a determinantal sum of A and $(A_2c\ \ A_2\ \ \dots\ \ A_n)$, which is not gr-full. Indeed, it is the product of $(A_2\ \ A_3\ \ \dots\ \ A_n) \in M_{n\times(n-1)}(R)[\bar\alpha][\overline{\beta'}]$ and

$$\begin{pmatrix} c & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \in M_{(n-1)\times n}(R)[\overline{\beta'}][\bar\beta],$$

respectively.

(3) This follows from (PM6).

(4) We show the first statement; the other can be proved analogously. If we write $A = (A_1\ \ A')$ and $C = (C_1\ \ C')$, then

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \begin{pmatrix} A_1 & A' & 0 \\ 0 & C' & B \end{pmatrix} \nabla \begin{pmatrix} 0 & A' & 0 \\ C_1 & C' & B \end{pmatrix}.$$

The second matrix on the right-hand side is a matrix with a submatrix that is a block of zeros of size $m \times (n+1)$. Since $m+n+1 > m+n$, that matrix is hollow and therefore not gr-full. By (1),

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \in \mathcal{P} \quad \text{if and only if} \quad \begin{pmatrix} A_1 & A' & 0 \\ 0 & C' & B \end{pmatrix} \in \mathcal{P}.$$

Similarly, one can repeat the argument applied to columns of A' and C' and so on, to obtain the desired result.

(5) Let $\Sigma = \mathfrak{M}(R) \setminus \mathcal{P}$. By (PM5), $1 \in \Sigma$. By (PM4), $A \oplus B \in \Sigma$ if $A, B \in \Sigma$. Now (4) implies that Σ is lower gr-semimultiplicative. Finally, (PM6) shows that Σ is gr-multiplicative.

(6) This follows from (PM4) and (PM5).

(7) First notice that, by (6) and (PM4), a matrix $C \in \mathfrak{M}(R)$ belongs to $\mathcal{P}$ if and only if $C \oplus I \in \mathcal{P}$ for the identity matrix I of the same size as C.

We claim that $C \in \mathcal{P}$ if and only if $-C \in \mathcal{P}$. Indeed,

$$\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \in \mathcal{P} \overset{(2)}{\Longleftrightarrow} \begin{pmatrix} C & -C \\ 0 & I \end{pmatrix} \in \mathcal{P} \overset{(2)}{\Longleftrightarrow} \begin{pmatrix} 0 & -C \\ I & I \end{pmatrix} \in \mathcal{P}$$

$$\overset{\text{(PM6)}}{\Longleftrightarrow} \begin{pmatrix} -C & 0 \\ I & I \end{pmatrix} \in \mathcal{P} \overset{(4)}{\Longleftrightarrow} \begin{pmatrix} -C & 0 \\ 0 & I \end{pmatrix} \in \mathcal{P},$$

and the claim is proved. Then

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{P} \overset{(4)}{\Longleftrightarrow} \begin{pmatrix} A & 0 \\ I & B \end{pmatrix} \in \mathcal{P} \qquad \overset{(2)}{\Longleftrightarrow} \begin{pmatrix} A & -AB \\ I & 0 \end{pmatrix} \in \mathcal{P}$$

$$\overset{\text{(PM6)}}{\Longleftrightarrow} \begin{pmatrix} -AB & A \\ 0 & I \end{pmatrix} \in \mathcal{P} \overset{(4)}{\Longleftrightarrow} \begin{pmatrix} -AB & 0 \\ 0 & I \end{pmatrix},$$

and, by the claim, the result follows.

(8) If $A \in M_n(R)[\bar{\alpha}][\bar{\beta}]$ is invertible, then $A^{-1} \in M_n(R)[\bar{\beta}][\bar{\alpha}]$. Since $AA^{-1} = I \notin \mathcal{P}$, (7) implies that $A \oplus A^{-1} \notin \mathcal{P}$. Now (PM3) shows that $A \notin \mathcal{P}$.

(9) By (PM2), if $A \in \mathcal{P}$, then $C \in \mathcal{P}$. Conversely, suppose that $C \in \mathcal{P}$. Clearly $A = C \nabla B'$, where B' is obtained from B by changing the sign of a row or column. More precisely, B' is the product of B by a diagonal matrix D whose diagonal elements are 1 or -1. Now $B \oplus D \in \mathcal{P}$ because $B \in \mathcal{P}$. Thus $B' \in \mathcal{P}$ by (7). Therefore $A \in \mathcal{P}$ by (PM2). ■

The proof of Lemma 6.2 is very similar to that for the ungraded case; see for example [5, pp. 430–431]. The main difference is that we were not able to show [5, (d) p. 430], because not every multiple of a column can be added to another column so that the matrix remains homogeneous. As a consequence, the proof of Lemma 6.2 (7) is also different.

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a graded ring and let $\mathcal{P}$ be a gr-prime matrix ideal. The universal localization of R at the set $\Sigma = \mathfrak{M}(R) \setminus \mathcal{P}$ will be denoted by $R_{\mathcal{P}}$ (instead of R_Σ).

Theorem 6.3. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. The following assertions hold true:*

(1) *If $\mathcal{P}$ is any gr-prime matrix ideal of R, then the localization $R_{\mathcal{P}}$ is a Γ-graded local ring. Moreover, its residue class Γ-graded division ring is a Γ-graded epic R-division ring such that its gr-singular kernel equals $\mathcal{P}$.*

(2) *If (K, φ) is a Γ-graded epic R-division ring, with gr-singular kernel $\mathcal{P}$, then $\mathcal{P}$ is a gr-prime matrix ideal and the Γ-graded local ring $R_{\mathcal{P}}$ has residue class graded division ring R-isomorphic to K.*

Proof. (2) By Proposition 6.1, $\mathcal{P}$ is a gr-prime matrix ideal.

By (1), $R_{\mathcal{P}}$ is a Γ-graded local ring and its residue class graded division ring is a graded epic R-division ring with singular kernel $\mathcal{P}$. Then, by Theorem 4.4 (b)(ii), K and the residue class graded division ring of $R_{\mathcal{P}}$ are isomorphic Γ-graded R-rings.

(1) Let $\Sigma = \mathfrak{M}(R) \setminus \mathcal{P}$. We will use the identification of $R_{\mathcal{P}}$ with the ring $\mathcal{R}_\Sigma = \bigoplus_{\gamma\in\Gamma}(\mathcal{R}_\Sigma)_\gamma$ given in Sections 5.3 and 5.4. The elements of $(\mathcal{R}_\Sigma)_\gamma$ are equivalence classes $[F, A, X, \alpha, \beta]$ under the equivalence relation $\sim$ defined in Section 5.1 of 5-tuples (F, A, X, α, β), where $A \in \Sigma$ is of distribution (α, β), F is a homogeneous row of distribution (γ, β) and X is a homogeneous column of distribution (α, e).

For each $\gamma \in \Gamma$, let $\mathfrak{P}_\gamma$ be the subset of $(\mathcal{R}_\Sigma)_\gamma$ consisting of the elements $[F, A, X, \alpha, \beta] \in (\mathcal{R}_\Sigma)_\gamma$ such that $\left(\begin{smallmatrix} A & X \\ F & 0 \end{smallmatrix}\right) \in \mathcal{P}$. Notice that this matrix is homogeneous of distribution $(\alpha * \gamma, \beta * e)$.

Step 1: For each $\gamma \in \Gamma$, $\mathfrak{P}_\gamma$ is well defined, that is, if $(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$ and $\left(\begin{smallmatrix} A & X \\ F & 0 \end{smallmatrix}\right) \in \mathcal{P}$, then $\left(\begin{smallmatrix} B & Y \\ G & 0 \end{smallmatrix}\right) \in \mathcal{P}$.

Suppose $(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)$. There exist $L, M, P, Q \in \Sigma$, homogeneous rows J, U and homogeneous columns W, V, such that

$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \\ \hline F & -G & 0 & J & 0 \end{array}\right) \in \mathfrak{M}(R)$$

is not gr-full, by (5.1), and thus belongs to $\mathcal{P}$ by (PM1). Applying permutations of rows and columns we obtain that

$$\begin{pmatrix} A & 0 & 0 & 0 & X \\ 0 & B & 0 & 0 & Y \\ F & -G & 0 & J & 0 \\ 0 & 0 & L & 0 & W \\ 0 & 0 & 0 & M & 0 \end{pmatrix} \in \mathcal{P}, \quad \begin{pmatrix} A & 0 & 0 & X & 0 \\ 0 & B & 0 & Y & 0 \\ F & -G & 0 & 0 & J \\ 0 & 0 & L & W & 0 \\ 0 & 0 & 0 & 0 & M \end{pmatrix} \in \mathcal{P}.$$

Since $M \notin \mathcal{P}$, then

$$\begin{pmatrix} A & 0 & 0 & X \\ 0 & B & 0 & Y \\ F & -G & 0 & 0 \\ 0 & 0 & L & W \end{pmatrix} \in \mathcal{P}$$

by Lemma 6.2 (4) and (PM4). Again, applying column permutations, Lemma 6.2 (4), (PM4) and the fact that $L \notin \mathcal{P}$ we obtain

$$\begin{pmatrix} A & 0 & X \\ 0 & B & Y \\ F & -G & 0 \end{pmatrix} \in \mathcal{P}.$$

After permuting some rows and columns, we obtain

$$\begin{pmatrix} A & X & 0 \\ 0 & Y & B \\ F & 0 & -G \end{pmatrix} \in \mathcal{P}, \qquad \begin{pmatrix} A & X & 0 \\ F & 0 & -G \\ 0 & Y & B \end{pmatrix} \in \mathcal{P}.$$

Now observe that

$$\begin{pmatrix} A & X & 0 \\ F & 0 & -G \\ 0 & 0 & B \end{pmatrix} \in \mathcal{P},$$

because $[F, A, X, \alpha, \beta] \in \mathfrak{P}_\gamma$ and because of Lemma 6.2 (4) and (PM3). Hence the equality

$$\begin{pmatrix} A & 0 & 0 \\ F & 0 & -G \\ 0 & Y & B \end{pmatrix} \nabla \begin{pmatrix} A & X & 0 \\ F & 0 & -G \\ 0 & 0 & B \end{pmatrix} = \begin{pmatrix} A & X & 0 \\ F & 0 & -G \\ 0 & Y & B \end{pmatrix}$$

implies that

$$\begin{pmatrix} A & 0 & 0 \\ F & 0 & -G \\ 0 & Y & B \end{pmatrix} \in \mathcal{P}$$

by Lemma 6.2 (9). Then, since $A \notin \mathcal{P}$, $\left(\begin{smallmatrix} 0 & -G \\ Y & B \end{smallmatrix}\right) \in \mathcal{P}$ by Lemma 6.2 (4) and (PM4). After permuting some rows and columns, $\left(\begin{smallmatrix} B & Y \\ -G & 0 \end{smallmatrix}\right) \in \mathcal{P}$. Now

$$\begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} B & Y \\ -G & 0 \end{pmatrix} = \begin{pmatrix} B & Y \\ G & 0 \end{pmatrix} \in \mathcal{P} \tag{6.1}$$

by (PM3) and Lemma 6.2 (7).

Step 2: For each $\gamma \in \Gamma$, $\mathfrak{P}_\gamma$ is an additive subgroup of $(\mathcal{R}_\Sigma)_\gamma$.

The 2×2 matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{P}$ is hollow and therefore not gr-full. Thus it belongs to $\mathcal{P}$, and $[0, 1, 1, e, e] \in \mathfrak{P}_\gamma$. Thus the zero element of $(\mathcal{R}_\Sigma)_\gamma$ belongs to $\mathfrak{P}_\gamma$.

If $[F, A, X, \alpha, \beta] \in \mathfrak{P}_\gamma$, then $-[F, A, X, \alpha, \beta] = [-F, A, X, \alpha, \beta] \in \mathfrak{P}_\gamma$ because if $\left(\begin{smallmatrix} A & X \\ F & 0 \end{smallmatrix}\right) \in \mathcal{P}$, then $\left(\begin{smallmatrix} A & X \\ -F & 0 \end{smallmatrix}\right) \in \mathcal{P}$ by the same argument as (6.1).

So now let $[F', A', X', \alpha', \beta'], [F, A, X, \alpha, \beta] \in \mathfrak{P}_\gamma$. Then

$$[F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta] = \left[\begin{pmatrix} F' & F \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} X' \\ X \end{pmatrix}, \alpha' * \alpha, \beta' * \beta \right].$$

To show that the previous sum belongs to $\mathfrak{P}_\gamma$, since

$$\begin{pmatrix} A' & 0 & 0 \\ 0 & A & X \\ F' & F & 0 \end{pmatrix} \nabla \begin{pmatrix} A' & 0 & X' \\ 0 & A & 0 \\ F' & F & 0 \end{pmatrix} = \begin{pmatrix} A' & 0 & X' \\ 0 & A & X \\ F' & F & 0 \end{pmatrix},$$

it is enough to show that both summands belong to $\mathcal{P}$ by (PM2). The homogeneous matrix

$$\begin{pmatrix} A' & 0 & 0 \\ 0 & A & X \\ F' & F & 0 \end{pmatrix} \in \mathcal{P}$$

by (PM3) and Lemma 6.2 (4), because $\left(\begin{smallmatrix} A & X \\ F & 0 \end{smallmatrix}\right) \in \mathcal{P}$. By a similar argument,

$$\begin{pmatrix} A' & X' & 0 \\ F' & 0 & F \\ 0 & 0 & A \end{pmatrix} \in \mathcal{P}.$$

Permuting rows and columns, we obtain that

$$\begin{pmatrix} A' & 0 & X' \\ 0 & A & 0 \\ F' & F & 0 \end{pmatrix} \in \mathcal{P}.$$

Step 3: If $[F', A', X', \alpha', \beta'] \in (\mathcal{R}_\Sigma)_{\gamma'}$, and $[F, A, X, \alpha, \beta] \in \mathfrak{P}_\gamma$, then

$$[F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta] \in \mathfrak{P}_{\gamma'\gamma}.$$

Similarly, if $[F', A', X', \alpha', \beta'] \in \mathfrak{P}_{\gamma'}$, and $[F, A, X, \alpha, \beta] \in (\mathcal{R}_\Sigma)_\gamma$, then

$$[F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta] \in \mathfrak{P}_{\gamma'\gamma}.$$

We prove both cases at the same time. Observe that

$$[F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta] = \left[\begin{pmatrix} 0 & F' \end{pmatrix}, \begin{pmatrix} A & 0 \\ -X'F & A' \end{pmatrix}, \begin{pmatrix} X \\ 0 \end{pmatrix}, \alpha * \alpha'\gamma, \beta * \beta'\gamma \right].$$

First note that the matrix

$$\begin{pmatrix} A & X & 0 & 0 \\ F & 0 & 0 & 1 \\ 0 & 0 & A' & X' \\ 0 & 0 & F' & 0 \end{pmatrix} \in \mathcal{P}$$

by Lemma 6.2 (4) and (PM3), and that it is of distribution $(\alpha * \gamma * \alpha'\gamma * \gamma'\gamma, \beta * e * \beta'\gamma * \gamma)$. Now the matrix

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -X' & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is invertible, homogeneous and of distribution $(\alpha * \gamma * \alpha'\gamma * \gamma^{-1}\gamma'^{-1}, \alpha * \gamma * \alpha'\gamma * \gamma'\gamma)$. By Lemma 6.2 (7) and (PM3), we obtain

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -X' & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A & X & 0 & 0 \\ F & 0 & 0 & 1 \\ 0 & 0 & A' & X' \\ 0 & 0 & F' & 0 \end{pmatrix} = \begin{pmatrix} A & X & 0 & 0 \\ F & 0 & 0 & 1 \\ -X'F & 0 & A' & 0 \\ 0 & 0 & F' & 0 \end{pmatrix} \in \mathcal{P}.$$

Permuting rows and columns, we get

$$\begin{pmatrix} A & 0 & X & 0 \\ -X'F & A' & 0 & 0 \\ 0 & F' & 0 & 0 \\ F & 0 & 0 & 1 \end{pmatrix} \in \mathcal{P}.$$

Now Lemma 6.2 (4), (PM5) and (PM4) imply that

$$\begin{pmatrix} A & 0 & X \\ -X'F & A' & 0 \\ 0 & F' & 0 \end{pmatrix} \in \mathcal{P},$$

as desired.

Step 4: Define $\mathfrak{P} = \bigoplus_{\gamma \in \Gamma} \mathfrak{P}_\gamma$. Then $\mathfrak{P}$ is a graded ideal of $\mathcal{R}_\Sigma$ by Steps 1–3. Moreover, $\mathfrak{P} \neq \mathcal{R}_\Sigma$ because $[1, 1, 1, e, e]$, the identity element of $\mathcal{R}_\Sigma$, does not belong to $\mathfrak{P}$ by Lemma 6.2 (8), since the 2×2 matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is invertible.

Step 5: $\mathcal{R}_\Sigma$ is a Γ-graded local ring with graded maximal ideal $\mathfrak{P}$.

Let $\varphi\colon R \to R_{\mathcal{P}}$ be the universal localization at Σ. By (the proof of) Proposition 5.7, the isomorphism $\Phi\colon \mathcal{R}_\Sigma \to R_{\mathcal{P}}$ sends $[F, A, X, \alpha, \beta] \in (\mathcal{R}_\Sigma)_\gamma$ to $F^\varphi(A^\varphi)^{-1}X^\varphi \in (R_{\mathfrak{P}})_\gamma$.

Let $[F, A, X, \alpha, \beta] \in (\mathcal{R}_\Sigma)_\gamma \setminus \mathfrak{P}_\gamma$. Thus $\left(\begin{smallmatrix} A & X \\ F & 0 \end{smallmatrix}\right) \notin \mathcal{P}$ and $\left(\begin{smallmatrix} A & X \\ F & 0 \end{smallmatrix}\right)^\varphi$ is invertible in $R_{\mathcal{P}}$. Also, the matrices $\left(\begin{smallmatrix} A^\varphi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} I & 0 \\ -F^\varphi(A^\varphi)^{-1} & 1 \end{smallmatrix}\right)$ are invertible in $R_{\mathcal{P}}$. Hence

$$\begin{pmatrix} A^\varphi & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ -F^\varphi(A^\varphi)^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & X \\ F & 0 \end{pmatrix}^\varphi = \begin{pmatrix} I & (A^\varphi)^{-1}X \\ 0 & -F^\varphi(A^\varphi)^{-1}X^\varphi \end{pmatrix}$$

is invertible in $R_{\mathcal{P}}$. Thus the element $F^{\varphi}(A^{\varphi})^{-1}X^{\varphi}$ is invertible in $R_{\mathcal{P}}$, and therefore $[F, A, X, \alpha, \beta]$ is invertible in $\mathcal{R}_{\Sigma}$.

Step 6: Set $\mathcal{K} = \mathcal{R}_{\Sigma}/\mathfrak{P}$ and let $\Psi\colon R \to \mathcal{K}$ be the composition of $\mu\colon R \to \mathcal{R}_{\Sigma}$ with the natural projection $\mathcal{R}_{\Sigma} \to \mathcal{K}$, $[F, A, X, \alpha, \beta] \mapsto \overline{[F, A, X, \alpha, \beta]}$. Then $(\mathcal{K}, \Psi)$ is a Γ-graded epic R-division ring by Propositions 4.1 (2) and 4.3.

Step 7: For $x \in \mathrm{h}(R)$, $\Psi(x) = 0$ if, and only if, the 1×1 matrix $x \in \mathcal{P}$.

Indeed,

$$\Psi(x) = \overline{[x, 1, 1, e, e]} = 0 \Leftrightarrow [x, 1, 1, e, e] \in \mathfrak{P} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ x & 0 \end{pmatrix} \in \mathcal{P} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & x \end{pmatrix} \in \mathcal{P}.$$

By Lemma 6.2 (4) and (PM4) the last condition is equivalent to $x \in \mathcal{P}$.

Step 8: For $r \in R_{\gamma}$ and $[F, A, X, \alpha, \beta] \in (\mathcal{R}_{\Sigma})_{\gamma}$, $\Psi(r) = [F, A, X, \alpha, \beta]$ if and only if $\left(\begin{smallmatrix} A & X \\ F & r \end{smallmatrix}\right) \in \mathcal{P}$.

First notice that $\Psi(r) = \overline{[F, A, X, \alpha, \beta]}$ if and only if $\overline{[r, 1, 1, e, e]} = \overline{[F, A, X, \alpha, \beta]}$. Equivalently,

$$[F, A, X, \alpha, \beta] - [r, 1, 1, e, e] = \left[\begin{pmatrix} F & -r \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} X \\ 1 \end{pmatrix}, \alpha * e, \beta * e \right] \in \mathfrak{P}_{\gamma},$$

which means that

$$\begin{pmatrix} A & 0 & X \\ 0 & 1 & 1 \\ F & -r & 0 \end{pmatrix} \in \mathcal{P}.$$

This matrix belongs to $\mathcal{P}$ if and only if

$$\begin{pmatrix} A & 0 & X \\ 0 & 1 & 0 \\ F & -r & r \end{pmatrix} \in \mathcal{P} \tag{6.2}$$

by Lemma 6.2 (2), since the last matrix is obtained after subtracting the second-last column from the last one. After permuting rows and columns, we get that the matrix (6.2) belongs to $\mathcal{P}$ if and only if

$$\begin{pmatrix} A & X & 0 \\ F & r & -r \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{P}.$$

By Lemma 6.2 (4), (PM3), (PM4) and (PM5) this is equivalent to $\left(\begin{smallmatrix} A & X \\ F & r \end{smallmatrix}\right) \in \mathcal{P}$.

Step 9: The singular kernel of $(\mathcal{K}, \Psi)$ is $\mathcal{P}$.

Since $\mathcal{R}_\Sigma$ is R-isomorphic to $R_{\mathcal{P}}$, we get that any matrix in $\mathfrak{M}(R) \setminus \mathcal{P}$ is inverted in $\mathcal{R}_\Sigma$ via μ and therefore in $\mathcal{K}$ via Ψ.

It remains to show that the matrices in $\mathcal{P}$ do not become invertible via Ψ. Let $A \in \mathcal{P}$ be of size $n \times n$. If all square submatrices (obtained by eliminating an equal number of rows and columns) of A belong to $\mathcal{P}$, then, in particular, all entries of A belong to $\mathcal{P}$. By Step 7, A^Ψ is the zero matrix and therefore not invertible. Hence suppose that A_1, of size $m \times m$ with $m > 1$, is a submatrix of A of largest size that does not belong to $\mathcal{P}$. We will show that A^Ψ is singular in $\mathcal{K}$ by expressing one column of A^Ψ as a homogeneous linear combination of the others.

Since rearrangement of rows and columns does not affect the singularity of A, we may suppose that

$$A = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \end{pmatrix}$$

where $\left(\begin{smallmatrix} A_2 \\ A_5 \end{smallmatrix}\right)$ is a column of A and $A \in M_n(R)[\overline{\alpha_1} * \overline{\alpha_2}][\overline{\beta_1} * e * \overline{\beta_2}]$. First observe that for every $j \in \{1, \dots, n\}$,

$$\begin{pmatrix} A_1 & A_2 \\ E_j^\top \begin{pmatrix} A_1 \\ A_4 \end{pmatrix} & E_j^\top \begin{pmatrix} A_2 \\ A_5 \end{pmatrix} \end{pmatrix} \in \mathcal{P},$$

because if $j \le m$, then it has a repeated row and if $j > m$ then it is a submatrix of A of greater size than A. By Step 8, this means that

$$\left(E_j^\top \begin{pmatrix} A_2 \\ A_5 \end{pmatrix} \right)^\Psi = \overline{\left[E_j^\top \begin{pmatrix} A_1 \\ A_4 \end{pmatrix}, A_1, A_2, \alpha_1, \beta_1 \right]} \quad \text{for } j = 1, \dots, n.$$

Hence, if we suppose that $\begin{pmatrix} A_1 \\ A_4 \end{pmatrix} = (b_{kl})$,

$$\begin{pmatrix} A_2 \\ A_5 \end{pmatrix}^\Psi = \begin{pmatrix} E_1^\top \begin{pmatrix} A_2 \\ A_5 \end{pmatrix} \\ E_2^\top \begin{pmatrix} A_2 \\ A_5 \end{pmatrix} \\ \vdots \\ E_n^\top \begin{pmatrix} A_2 \\ A_5 \end{pmatrix} \end{pmatrix}^\Psi = \begin{pmatrix} \overline{\left[E_1^\top \begin{pmatrix} A_1 \\ A_4 \end{pmatrix}, A_1, A_2, \alpha_1, \beta_1 \right]} \\ \overline{\left[E_2^\top \begin{pmatrix} A_1 \\ A_4 \end{pmatrix}, A_1, A_2, \alpha_1, \beta_1 \right]} \\ \vdots \\ \overline{\left[E_n^\top \begin{pmatrix} A_1 \\ A_4 \end{pmatrix}, A_1, A_2, \alpha_1, \beta_1 \right]} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{[b_{11},1,1,e,e]} & \cdots & \overline{[b_{1m},1,1,e,e]} \\ \vdots & \ddots & \vdots \\ \overline{[b_{n1},1,1,e,e]} & \cdots & \overline{[b_{nm},1,1,e,e]} \end{pmatrix} \cdot \begin{pmatrix} \overline{[E_1^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \\ \overline{[E_2^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \\ \vdots \\ \overline{[E_m^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \end{pmatrix}$$

$$= \begin{pmatrix} A_1 \\ A_4 \end{pmatrix}^{\Psi} \cdot \begin{pmatrix} \overline{[E_1^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \\ \overline{[E_2^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \\ \vdots \\ \overline{[E_m^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \end{pmatrix},$$

where we have used Lemma 5.3 (3) in the second equality. The result follows noting that $\binom{A_1}{A_4} \in M_{n\times m}(R)[\overline{\alpha_1} * \overline{\alpha_2}][\overline{\beta_1}]$ and

$$\begin{pmatrix} \overline{[E_1^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \\ \overline{[E_2^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \\ \vdots \\ \overline{[E_m^{\mathsf{T}},A_1,A_2,\alpha_1,\beta_1]} \end{pmatrix} \in M_{m\times 1}(\mathcal{K})[\overline{\beta_1}][e]. \qquad \blacksquare$$

The following is Theorem 4.7, but expressed in terms of gr-prime matrix ideals.

Corollary 6.4. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring, and let (K_i, φ_i), $i = 1,2$, be Γ-graded epic R-division rings with singular kernels $\mathcal{P}_i$, respectively. The following statements are equivalent:*

(1) *There exists a gr-specialization from K_1 to K_2.*

(2) $\mathcal{P}_1 \subseteq \mathcal{P}_2$.

(3) *There exists a homomorphism $R_{\mathcal{P}_2} \to R_{\mathcal{P}_1}$ of Γ-graded R-rings.*

Furthermore, if there exists a gr-specialization from K_1 to K_2 and another gr-specialization from K_2 to K_1, then K_1 and K_2 are isomorphic graded R-rings. ■

The following corollaries are the graded versions of the results in [5, p. 442].

Corollary 6.5. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and $(K = \bigoplus_{\gamma\in\Gamma} K_\gamma, \varphi)$ be a graded epic R-division ring with singular kernel $\mathcal{P}$. Suppose that $\gamma \in \Gamma$. Consider the universal localization $\lambda\colon R \to R_{\mathcal{P}}$ and let $\Phi\colon R_{\mathcal{P}} \to K$ be the homomorphism of Γ-graded rings such that $\varphi = \Phi\lambda$.*

(1) *Let* $x \in K_\gamma$*. Then* $x = 0$ *if and only if its numerator belongs to* $\mathcal{P}$*.*

(2) *Let* $x \in (R_{\mathcal{P}})_\gamma$*. Then* $x \in \ker \Phi$ *if and only if its numerator belongs to* $\mathcal{P}$*.*

Proof. Suppose that $(A_0 \ \ A_\bullet)$ is the numerator of x.

(1) By Lemma 3.5 (1), x is invertible if and only if $(A_0 \ \ A_\bullet)^\varphi$ is invertible over K, that is, if and only if $(A_0 \ \ A_\bullet)$ belongs to $\mathcal{P}$.

(2) By Lemma 3.5 (1), x is invertible if and only if $(A_0 \ \ A_\bullet)^\lambda$ is invertible over $R_{\mathcal{P}}$. Since $R_{\mathcal{P}}$ is a local ring with residue class graded division ring R-isomorphic to K, x is invertible if and only if $(A_0 \ \ A_\bullet)^{\Phi\lambda}$ is invertible over K. That is, $x \in \ker \Phi$ if and only if $(A_0 \ \ A_\bullet)$ belongs to $\mathcal{P}$. ∎

Corollary 6.6. *Let* $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ *and* $R' = \bigoplus_{\gamma\in\Gamma} R'_\gamma$ *be* Γ*-graded rings with gr-prime matrix ideals* $\mathcal{P}$ *and* $\mathcal{P}'$*, respectively, with corresponding graded epic* R*-division rings* (K,φ) *and* (K',φ') *respectively. Let* $f\colon R \to R'$ *be a homomorphism of* Γ*-graded rings. The following assertions hold true:*

(1) f *extends to a gr-specialization if, and only if,* $\mathcal{P}^f \subseteq \mathcal{P}'$*.*

(2) f *extends to a homomorphism* $K \to K'$ *if, and only if,* $\mathcal{P}^f \subseteq \mathcal{P}'$ *and* $\Sigma^f \subseteq \Sigma'$*, where* $\Sigma = \mathfrak{M}(R) \setminus \mathcal{P}$ *and* $\Sigma' = \mathfrak{M}(R') \setminus \mathcal{P}'$*.*

Proof. (1) First note that the set $\mathcal{P}'' = \{A \in \mathfrak{M}(R) : A^f \in \mathcal{P}'\}$ is a gr-prime matrix ideal whose corresponding graded epic R-division ring is $\varphi' f\colon R \to \mathrm{DC}(\varphi' f)$.
By Corollary 6.4, there exists a specialization from (K,φ) to $(\mathrm{DC}(\varphi' f))$ if, and only if, $\mathcal{P} \subseteq \mathcal{P}''$.

(2) If $\mathcal{P}^f \subseteq \mathcal{P}'$ and $\Sigma^f \subseteq \Sigma'$, then $\mathcal{P} = \mathcal{P}''$, and therefore the gr-specialization of (1) is in fact an isomorphism by Corollary 6.4. ∎

7. gr-matrix ideals

In this section, the concepts, arguments and proofs are an adaptation of those in [5, Section 7.3] to the graded context.

Throughout this section, let Γ *be a group.*

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. A subset $\mathcal{I}$ of $\mathfrak{M}(R)$ is a *gr-matrix pre-ideal* if the following conditions are satisfied.

(I1) $\mathcal{I}$ contains all the homogeneous matrices that are not gr-full.

(I2) If $A, B \in \mathcal{I}$ and their determinantal sum (with respect to a row or column) exists, then $A \nabla B \in \mathcal{I}$.

(I3) If $A \in \mathcal{I}$, then $A \oplus B \in \mathcal{I}$ for all $B \in \mathfrak{M}(R)$.

(I4) If $A \in \mathcal{I}$ and E, F are permutation matrices of appropriate size, then $EAF \in \mathcal{I}$.

If, moreover, we have

(I5) for $A \in \mathfrak{M}(R)$, if $A \oplus 1 \in \mathcal{I}$, then $A \in \mathcal{I}$,

we call $\mathcal{I}$ a *gr-matrix ideal*.

Clearly, $\mathfrak{M}(R)$ is a gr-matrix ideal. A *proper* gr-matrix ideal is a gr-matrix ideal different from $\mathfrak{M}(R)$.

Lemma 7.1. *Let R be a Γ-graded ring and $\mathcal{I}$ be a gr-matrix pre-ideal. Let $A, B \in \mathfrak{M}(R)$. The following assertions hold true:*

(1) *If A and B are such that $C = A\nabla B$ exists and B is not gr-full, then $A \in \mathcal{I}$ if and only if $C \in \mathcal{I}$.*

(2) *Let $A \in \mathcal{I}$. The result of adding a suitable right multiple of one column of A to another column again lies in $\mathcal{I}$. More precisely, if $A \in M_n(R)[\bar{\alpha}][\bar{\beta}]$ and $a \in R_{\beta_i \beta_j^{-1}}$, then $(A_1 \ \dots \ A_{j-1} \ \ A_j + A_i a \ \ A_{j+1} \ \dots \ A_n)$ belongs to $\mathcal{I}$.*

(3) *If $A \oplus B \in \mathcal{I}$, then $B \oplus A \in \mathcal{I}$.*

(4) *Suppose that $A \in M_m(R)[\bar{\alpha}][\bar{\beta}]$ and $B \in M_n(R)[\bar{\delta}][\bar{\varepsilon}]$. For $C \in M_{n\times m}(R)[\bar{\delta}][\bar{\beta}]$,*

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \in \mathcal{I} \quad \text{if and only if} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{I}.$$

Similarly, for $C \in M_{m\times n}(R)[\bar{\beta}][\bar{\varepsilon}]$,

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{I} \quad \text{if and only if} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{I}.$$

If, moreover, $\mathcal{I}$ is a gr-matrix ideal, then the following assertions hold true:

(5) *Suppose that $A \in M_n(R)[\bar{\alpha}][\bar{\beta}]$, $B \in M_n(R)[\bar{\beta}][\bar{\delta}]$. Then $AB \in \mathcal{I}$ if and only if $A \oplus B \in \mathcal{I}$.*

(6) *If A and B are such that $C = A\nabla B$ exists and $B \in \mathcal{I}$, then $A \in \mathcal{I}$ if and only if $C \in \mathcal{I}$.*

(7) *If an identity matrix I_n, $n \geq 1$, belongs to $\mathcal{I}$, then $\mathcal{I} = \mathfrak{M}(R)$*

Proof. Note that (I1), (I2), (I3), (I4) are the same as (PM1), (PM2), (PM3), (PM6). Hence (1)–(6) follow in exactly the same way as in Lemma 6.2.

To prove (7), note that if $I_n \in \mathcal{I}$, for some $n \geq 1$, an application of (I5) shows that the 1×1 matrix $1 \in \mathcal{I}$. By (I3), any identity matrix I_m, $m \geq 1$, belongs to $\mathcal{I}$. Again using (I3), $I_m \oplus A \in \mathcal{I}$ for any positive integer m and matrix $A \in \mathfrak{M}(R)$. By (5), any $A \in \mathfrak{M}(R)$ belongs to $\mathcal{I}$, as desired. ∎

One could think of defining a gr-prime matrix ideal as a gr-matrix ideal $\mathcal{I}$ such that the following two conditions are satisfied:

(I6) $\mathcal{I}$ is a proper gr-matrix ideal.

(I7) $\mathcal{I}$ satisfies (PM4).

We proceed to show that both definitions are equivalent. Let $\mathcal{P}$ be a gr-prime matrix ideal, i.e. (PM1)–(PM6) preceding Proposition 6.1 are satisfied. Clearly, $\mathcal{P}$ satisfies (I1)–(I4) and (I7). By (PM5), $1 \notin \mathcal{P}$. Therefore, by (PM4), if $A \oplus 1 \in \mathcal{P}$, then $A \in \mathcal{P}$ for any $A \in \mathfrak{M}(R)$. Hence (I5) is satisfied. Again by (PM5), $\mathcal{P}$ is a proper gr-matrix ideal. Conversely, suppose that $\mathcal{I}$ satisfies (I1)–(I7). Clearly (PM1)–(PM4), (PM6) are satisfied. By Lemma 7.1 (7) and (I6), (PM5) is satisfied, as desired.

It is easy to prove that any intersection of gr-matrix (pre-)ideals is again a gr-matrix (pre-)ideal. Thus, given a subset $\mathcal{S} \subseteq \mathfrak{M}(R)$, we define the *gr-matrix (pre-)ideal generated by* $\mathcal{S}$ as the intersection of gr-matrix (pre-)ideals $\mathcal{I}$ that contain $\mathcal{S}$. That is, $\bigcap_{\mathcal{S}\subseteq\mathcal{I}} \mathcal{I}$. Note that this gr-matrix (pre)-ideal is contained in any gr-matrix (pre-)ideal that contains $\mathcal{S}$.

Now we fix some notation that will be used in what follows.

Let $\mathcal{W} \subseteq \mathfrak{M}(R)$. We say that a matrix $C \in \mathfrak{M}(R)$ is a determinantal sum of elements of $\mathcal{W}$ if there exist $A_1, \dots, A_m \in \mathcal{W}$, $m \geq 1$, such that $A_1 \nabla A_2 \nabla \cdots \nabla A_m$ exists for some choice of parenthesis and equals C.

We will write $\mathcal{N}$ to denote the subset of $\mathfrak{M}(R)$ consisting of the matrices which are not gr-full.

We will denote the set of all identity matrices by $\mathfrak{I}$.

If $\mathcal{X} \subseteq \mathfrak{M}(R)$, we denote by $\mathcal{D}(\mathcal{X})$ the set of all matrices in $\mathfrak{M}(R)$ which are of the form $E(X \oplus A)F$, where $X \in \mathcal{X}$, $A \in \mathfrak{M}(R)$ and E, F are permutation matrices of appropriate sizes. We remark that we allow A to be the empty matrix $\mathbb{O}$.

Lemma 7.2. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and $\mathcal{A}$ be a gr-matrix pre-ideal. Suppose that $\Sigma \subset \mathfrak{M}(R)$ satisfies the following two conditions:*

(i) $1 \in \Sigma$.

(ii) *If $P, Q \in \Sigma$, then $P \oplus Q \in \Sigma$.*

Then the following assertions hold true:

(1) *The set $\mathcal{A}/\Sigma := \{A \in \mathfrak{M}(R) : A \oplus P \in \mathcal{A}$ for some $P \in \Sigma\}$ is a gr-matrix ideal containing $\mathcal{A}$.*

(2) *The gr-matrix ideal $\mathcal{A}/\Sigma$ is proper if and only if $\mathcal{A} \cap \Sigma = \emptyset$.*

(3) *The gr-matrix ideal $\mathcal{A}/\mathfrak{I}$ is the gr-matrix ideal generated by $\mathcal{A}$.*

Proof. (1) Let $A \in \mathcal{A}$. By (I3), $A \oplus 1 \in \mathcal{A}$. Since $1 \in \Sigma$, $A \in \mathcal{A}/\Sigma$. Hence $\mathcal{A} \subseteq \mathcal{A}/\Sigma$ and, by (I1), all non-gr-full matrices belong to $\mathcal{A}$. Therefore $\mathcal{A}/\Sigma$ satisfies (I1).

Let $A, B \in \mathcal{A}/\Sigma$ be such that $A\nabla B$ is well defined. There exist $P, Q \in \Sigma$ such that $A \oplus P, B \oplus Q \in \mathcal{A}$. By (I3), $A \oplus P \oplus Q$ and $B \oplus Q \oplus P$ belong to $\mathcal{A}$. By (I4), $B \oplus P \oplus Q \in \mathcal{A}$. Now $(A\nabla B) \oplus P \oplus Q = (A \oplus P \oplus Q)\nabla(B \oplus P \oplus Q) \in \mathcal{A}$ by (I2). Hence $A\nabla B \in \mathcal{A}/\Sigma$ and $\mathcal{A}/\Sigma$ satisfies (I2).

Let $A \in \mathcal{A}/\Sigma$ and $B \in \mathfrak{M}(R)$. There exists $P \in \Sigma$ such that $A \oplus P \in \mathcal{A}$. By (I3), $A \oplus P \oplus B \in \mathcal{A}$. Now (I4) implies that $A \oplus B \oplus P \in \mathcal{A}$. Hence $A \oplus B \in \mathcal{A}/\Sigma$, and $\mathcal{A}/\Sigma$ satisfies (I3).

Let $A \in \mathcal{A}/\Sigma$ and E, F be permutation matrices of the same size as A. There exists $P \in \Sigma$ such that $A \oplus P \in \mathcal{A}$. Since $E \oplus I$ and $F \oplus I$ are also permutation matrices, (I4) implies that $(E \oplus I)(A \oplus P)(F \oplus I) = EAF \oplus P \in \mathcal{A}$. Hence $EAF \in \mathcal{A}/\Sigma$ and (I4) is satisfied.

Now let $A \in \mathfrak{M}(R)$ be such that $A \oplus 1 \in \mathcal{A}/\Sigma$. Thus there exists $P \in \Sigma$ such that $A \oplus 1 \oplus P \in \mathcal{A}$. Since $1 \oplus P \in \Sigma$, then $A \in \mathcal{A}/\Sigma$ and A/Σ satisfies (I5).

(2) Suppose that $\mathcal{A} \cap \Sigma \neq \emptyset$. Let $P \in \mathcal{A} \cap \Sigma$ and $M \in \mathfrak{M}(R)$. Then $P \oplus M \in \mathcal{A}$ by (I3). By (I4), $M \oplus P \in \mathcal{A}$. Hence $M \in \mathcal{A}/\Sigma$. Therefore, $\mathcal{A}/\Sigma = \mathfrak{M}(R)$.

Conversely, suppose that $\mathcal{A}/\Sigma = \mathfrak{M}(R)$. Thus $1 \in \mathcal{A}/\Sigma$ and there exists $P \in \Sigma$ such that $1 \oplus P \in \mathcal{A}$. Notice that $1 \oplus P \in \Sigma$, by (i) and (ii). Therefore $\mathcal{A} \cap \Sigma \neq \emptyset$.

(3) Clearly $\mathfrak{I}$ satisfies conditions (i) and (ii). Thus $\mathcal{A}/\mathfrak{I}$ is a gr-matrix ideal that contains $\mathcal{A}$ by (1). Now let $\mathcal{B}$ be a gr-matrix ideal such that $\mathcal{A} \subseteq \mathcal{B}$. If $A \in \mathcal{A}/\mathfrak{I}$, then there exists $n \geq 1$ such that $A \oplus I_n \in \mathcal{A} \subseteq \mathcal{B}$. By applying (I5) repeatedly, we obtain that $A \in \mathcal{B}$, as desired. ■

Lemma 7.3. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and let $\mathcal{X} \subseteq \mathfrak{M}(R)$. Let $\mathcal{A}(\mathcal{X})$ be the subset of $\mathfrak{M}(R)$ consisting of all the matrices that can be expressed as a determinantal sum of elements of $\mathcal{N} \cup \mathcal{D}(\mathcal{X})$. The following assertions hold true:*

(1) *$\mathcal{A}(\mathcal{X})$ is the gr-matrix pre-ideal generated by $\mathcal{X}$.*

(2) *$\mathcal{A}(\mathcal{X})/\mathfrak{I}$ is the gr-matrix ideal generated by $\mathcal{X}$.*

(3) *The gr-matrix ideal generated by $\mathcal{X}$ is proper if and only if $\mathcal{A}(\mathcal{X}) \cap \mathfrak{I} = \emptyset$.*

Proof. (1) $\mathcal{X} \subseteq \mathcal{A}(\mathcal{X})$ because $X = I(X \oplus \mathbb{O})I$ for all $X \in \mathcal{X}$. By definition of $\mathcal{A}(\mathcal{X})$, every homogeneous matrix that is not gr-full belongs to $\mathcal{A}(\mathcal{X})$. By the same reason, if $A, B \in \mathcal{A}(\mathcal{X})$ and $A\nabla B$ is defined, then $A\nabla B \in \mathcal{A}(\mathcal{X})$.

Let $A \in \mathcal{A}(\mathcal{X})$ and $B \in \mathfrak{M}(R)$. That $A \oplus B \in \mathcal{A}(\mathcal{X})$ follows from the following three facts: First, for any $U, V \in \mathfrak{M}(R)$, when defined, $(U \nabla V) \oplus M = (U \oplus M)\nabla(V \oplus M)$. Second, for $X \in \mathcal{X}$ and $U, M \in \mathfrak{M}(R)$ and permutation matrices E, F of suitable size, $E(X \oplus U)F \oplus M = (E \oplus I)(X \oplus U \oplus M)(F \oplus I)$. Third, if U is not gr-full, then, for all $M \in \mathfrak{M}(R)$, $U \oplus M$ is not full for all $M \in \mathfrak{M}(R)$. Indeed, if $U = U_1U_2$, then $U \oplus M = (U_1 \oplus M)(U_2 \oplus I)$.

If $A \in \mathcal{A}(\mathcal{X})$ and E, F are permutation matrices of appropriate size, then $EAF \in \mathcal{A}(\mathcal{X})$. This follows from the following facts: First, if $A, B \in \mathfrak{M}(R)$ and E, F are permutation matrices such that $E(A\nabla B)F$ is defined, then $E(A\nabla B)F = EAF\nabla EBF$. Second, for $X \in \mathcal{X}$, $U \in \mathfrak{M}(R)$ and permutation matrices E, F, P, Q of appropriate sizes then $P(E(X \oplus U)F)Q = (PE)(X \oplus U)(FQ)$. Third, if $U \in \mathfrak{M}(R)$ is not gr-full, and E, F are permutation matrices of appropriate size, then EUF is not gr-full. Indeed, if $U = U_1U_2$, then $EUF = (EU_1)(U_2F)$.

Therefore, $\mathcal{A}(\mathcal{X})$ is a gr-matrix pre-ideal that contains $\mathcal{X}$.

Now let $\mathcal{B}$ be a gr-matrix pre-ideal such that $\mathcal{X} \subseteq \mathcal{B}$. By (I1), $\mathcal{N} \subseteq \mathcal{B}$. By (I3) and (I4), $E(X \oplus A)F \in \mathcal{B}$ for all $X \in \mathcal{X}$, $A \in \mathfrak{M}(R)$ and permutation matrices E, F of appropriate size. By (I2), $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{B}$.

(2) Any gr-matrix ideal containing $\mathcal{X}$, must contain $\mathcal{A}(\mathcal{X})$. By Lemma 7.2 (3), the result follows.

(3) By (2), the gr-matrix ideal generated by $\mathcal{X}$ equals $\mathcal{A}(\mathcal{X})/\mathfrak{J}$. By Lemma 7.2 (2), $\mathcal{A}(\mathcal{X})/\mathfrak{J}$ is proper if and only if $\mathcal{A}(\mathcal{X}) \cap \mathfrak{J} = \emptyset$. ∎

Corollary 7.4. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ graded ring. The set $\mathcal{A}(\mathcal{N})/\mathfrak{J}$ is the least gr-matrix ideal. Hence R has proper gr-matrix ideals if and only if no matrix of $\mathfrak{J}$ can be expressed as a determinantal sum of matrices of $\mathcal{N}$.*

Proof. The set $\mathcal{N}$ is contained in each gr-matrix ideal. By Lemma 7.3 (2), $\mathcal{A}(\mathcal{N})/\mathfrak{J}$ is the gr-matrix ideal generated by $\mathcal{N}$. Thus all gr-matrix ideals contain the gr-matrix ideal $\mathcal{A}(\mathcal{N})/\mathfrak{J}$.

Since any matrix in $\mathfrak{M}(R)$ of the form $E(X \oplus A)F$, where $X \in \mathcal{N}$, $A \in \mathfrak{M}(R)$ and E, F are permutation matrices of appropriate sizes, again belongs to $\mathcal{N}$, then $\mathcal{D}(\mathcal{N}) = \mathcal{N}$. Thus $\mathcal{A}(\mathcal{N})$ consists of the matrices in $\mathfrak{M}(R)$ that can be expressed as a determinantal sum of matrices from $\mathcal{N}$.

Now R has proper gr-matrix ideals if and only if $\mathcal{A}(\mathcal{N})/\mathfrak{J}$ is proper. By Lemma 7.2 (3), this is equivalent to $\mathcal{A}(\mathcal{N}) \cap \mathfrak{J} = \emptyset$. In other words, no matrix of $\mathfrak{J}$ can be expressed as a determinantal sum of matrices of $\mathcal{N}$. ∎

Lemma 7.5. *Let* $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ *be a* Γ*-graded ring,* $\mathcal{I}$ *be a gr-matrix ideal and* $\mathcal{Z} \subseteq \mathfrak{M}(R)$*. Then the set* $\mathcal{I}_{\mathcal{Z}} = \{A \in \mathfrak{M}(R) : A \oplus Z \in \mathcal{I} \text{ for all } Z \in \mathcal{Z}\}$ *is a gr-matrix ideal.*

Proof. Let $A \in \mathfrak{M}(R)$ and suppose it is not gr-full. If $A = BC$, then $A \oplus Z = (B \oplus Z)(C \oplus I)$ for all $Z \in \mathcal{Z}$. Thus $A \in \mathcal{I}_{\mathcal{Z}}$ and (I1) is satisfied.

Let $A, B \in \mathcal{I}_{\mathcal{Z}}$ and suppose that $A\nabla B$ exists. Then $(A\nabla B) \oplus Z = (A \oplus Z)\nabla(B \oplus Z)$ for all $Z \in \mathcal{Z}$. Since $A \oplus Z, B \oplus Z \in \mathcal{I}$, then $(A\nabla B) \oplus Z \in \mathcal{I}$ for all $Z \in \mathcal{Z}$. Hence $A\nabla B \in \mathcal{I}_{\mathcal{Z}}$, and (I2) is satisfied.

Let $A \in \mathcal{I}_{\mathcal{Z}}$ and $B \in \mathfrak{M}(R)$. Since $A \oplus Z \in \mathcal{I}$ for all $Z \in \mathcal{Z}$ and $\mathcal{I}$ is a gr-matrix ideal, then $A \oplus Z \oplus B \in \mathcal{I}$ for all $Z \in \mathcal{Z}$. By (I4), $A \oplus B \oplus Z \in \mathcal{I}$ for all $Z \in \mathcal{Z}$. Therefore $A \oplus B \in \mathcal{I}_{\mathcal{Z}}$ and (I3) is satisfied.

If $A \in \mathcal{I}_{\mathcal{Z}}$, $Z \in \mathcal{Z}$ and E, F are permutation matrices of appropriate size, then $EAF \oplus Z = (E \oplus I)(A \oplus Z)(F \oplus I)$. This shows that $EAF \in \mathcal{I}_{\mathcal{Z}}$ and (I4) is satisfied.

Suppose now that $A \in \mathfrak{M}(R)$ and that $A \oplus 1 \in \mathcal{I}_{\mathcal{Z}}$. Hence $A \oplus 1 \oplus Z \in \mathcal{I}$ for all $Z \in \mathcal{Z}$. By (I4), $A \oplus Z \oplus 1 \in \mathcal{I}$ for all $Z \in \mathcal{Z}$. Now, by (I5), $A \oplus Z \in \mathcal{I}$ for all $Z \in \mathcal{Z}$, which shows that $A \in \mathcal{I}_{\mathcal{Z}}$. Therefore (I5) is satisfied. ■

Let $\mathcal{A}_1, \mathcal{A}_2$ be two gr-matrix ideals of a Γ-graded ring $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$. The *product of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$, denoted by $\mathcal{A}_1\mathcal{A}_2$, is the gr-matrix ideal generated by the set

$$\{A_1 \oplus A_2 : A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2\}.$$

A helpful description of $\mathcal{A}_1\mathcal{A}_2$ is given in the following lemma.

Lemma 7.6. *Let* $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ *be* Γ*-graded ring and* $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathfrak{M}(R)$*. Set*

$$\mathcal{X} = \{X_1 \oplus X_2 : X_1 \in \mathcal{X}_1,\ X_2 \in \mathcal{X}_2\}.$$

Let $\mathcal{A}_1$ *be the gr-matrix ideal generated by* $\mathcal{X}_1$*,* $\mathcal{A}_2$ *be the gr-matrix ideal generated by* $\mathcal{X}_2$ *and* $\mathcal{A}$ *be the gr-matrix ideal generated by* $\mathcal{X}$*. Then* $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$*.*

As a consequence, for any $A, B \in \mathfrak{M}(R)$*,* $\langle A\rangle\langle B\rangle = \langle A \oplus B\rangle$*, where* $\langle A\rangle$ *denotes the gr-matrix ideal generated by* $\{A\}$*.*

Proof. First, $\mathcal{A} \subseteq \mathcal{A}_1\mathcal{A}_2$ because $X_1 \oplus X_2 \in \mathcal{A}_1\mathcal{A}_2$ for all $X_1 \in \mathcal{X}_1$, $X_2 \in \mathcal{X}_2$.

Now observe that $X_1 \oplus X_2 \in \mathcal{X} \subseteq \mathcal{A}$ for all $X_1 \in \mathcal{X}_1$, $X_2 \in \mathcal{X}_2$. By (I4), $X_2 \oplus X_1 \in \mathcal{X} \subseteq \mathcal{A}$ for all $X_1 \in \mathcal{X}_1$, $X_2 \in \mathcal{X}_2$. Hence $\mathcal{X}_2$ is contained in the gr-matrix ideal $\mathcal{A}_{\mathcal{X}_1}$. Thus $\mathcal{A}_2 \subseteq \mathcal{A}_{\mathcal{X}_1}$. It implies that $A_2 \oplus X_1 \in \mathcal{A}$ for all $A_2 \in \mathcal{A}_2$ and $X_1 \in \mathcal{X}_1$. Again by (I4), $X_1 \oplus A_2 \in \mathcal{A}$ for all $A_2 \in \mathcal{A}_2$ and $X_1 \in \mathcal{X}_1$. Therefore $\mathcal{X}_1$ is contained in the gr-matrix ideal $\mathcal{A}_{\mathcal{A}_2}$. Thus $\mathcal{A}_1 \subseteq \mathcal{A}_{\mathcal{A}_2}$. This means that $A_1 \oplus A_2 \in \mathcal{A}$ for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Therefore $\mathcal{A}_1\mathcal{A}_2 \subseteq \mathcal{A}$. ■

Now we show that gr-prime matrix ideals behave like graded prime ideals of graded rings.

Proposition 7.7. *Let* $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ *be a* Γ*-graded ring. For a proper gr-matrix ideal* $\mathcal{P}$*, the following are equivalent:*

(1) $\mathcal{P}$ *is a gr-prime matrix ideal.*

(2) *For gr-matrix ideals* $\mathcal{A}_1$, $\mathcal{A}_2$*, if* $\mathcal{A}_1\mathcal{A}_2 \subseteq \mathcal{P}$*, then* $\mathcal{A}_1 \subseteq \mathcal{P}$ *or* $\mathcal{A}_2 \subseteq \mathcal{P}$*.*

(3) *For gr-matrix ideals* $\mathcal{A}_1$, $\mathcal{A}_2$ *that contain* $\mathcal{P}$*, if* $\mathcal{A}_1\mathcal{A}_2 \subseteq \mathcal{P}$*, then* $\mathcal{A}_1 = \mathcal{P}$ *or* $\mathcal{A}_2 = \mathcal{P}$*.*

Proof. Suppose (1) holds true. Let $\mathcal{A}_1, \mathcal{A}_2$ be gr-matrix ideals such that $\mathcal{A}_1 \not\subseteq \mathcal{P}$ and $\mathcal{A}_2 \not\subseteq \mathcal{P}$. Hence there exist $A_1 \in \mathcal{A}_1 \setminus \mathcal{P}$ and $A_2 \in \mathcal{A}_2 \setminus \mathcal{P}$. Hence $A_1 \oplus A_2 \notin \mathcal{P}$. This implies that $\mathcal{A}_1\mathcal{A}_2 \not\subseteq \mathcal{P}$. Therefore (2) holds true.

Clearly (2) implies (3).

Suppose (3) holds true and let $A_1, A_2 \in \mathfrak{M}(R)$ be such that $A_1 \oplus A_2 \in \mathcal{P}$. Let $\mathcal{A}_1, \mathcal{A}_2$ be the gr-matrix ideals generated by $\mathcal{P} \cup \{A_1\}$ and $\mathcal{P} \cup \{A_2\}$, respectively. Notice that $X_1 \oplus X_2 \in \mathcal{P}$ for $X_1 \in \mathcal{A}_1$, $X_2 \in \mathcal{A}_2$. Hence $\mathcal{A}_1\mathcal{A}_2 \subseteq \mathcal{P}$. By (3), either $\mathcal{A}_1 = \mathcal{P}$ or $\mathcal{A}_2 = \mathcal{P}$. Hence $A_1 \in \mathcal{P}$ or $A_2 \in \mathcal{P}$, and (1) is satisfied. ∎

Let $\mathcal{A}$ be a gr-matrix ideal. The *radical of* $\mathcal{A}$ is defined as the set

$$\sqrt{\mathcal{A}} = \left\{A \in \mathfrak{M}(R) : \oplus^r A \in \mathcal{A} \text{ for some positive integer } r\right\}.$$

We say that a proper gr-matrix ideal $\mathcal{A}$ is *gr-semiprime* if $\sqrt{\mathcal{A}} = \mathcal{A}$.

Lemma 7.8. *Let* $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ *be a* Γ*-graded ring and let* $\mathcal{A}$ *be a gr-matrix ideal. The following assertions hold true:*

(1) $\sqrt{\mathcal{A}}$ *is a gr-matrix ideal that contains* $\mathcal{A}$*.*

(2) $\sqrt{\sqrt{\mathcal{A}}} = \sqrt{\mathcal{A}}$*.*

(3) *If* $\mathcal{A}$ *is a gr-prime matrix ideal, then* $\sqrt{\mathcal{A}} = \mathcal{A}$*.*

Proof. (1) If $A \in \mathcal{A}$, then, for $r = 1$, we obtain that $A = \oplus^1 A \in \mathcal{A}$. Hence $\mathcal{A} \subseteq \sqrt{\mathcal{A}}$. In particular, all homogeneous matrices which are not gr-full belong to $\sqrt{\mathcal{A}}$. Thus $\sqrt{\mathcal{A}}$ satisfies (I1).

Let $A, B \in \sqrt{\mathcal{A}}$ be such that $A\nabla B$ exists. There exist $r, s \geq 1$ such that $\oplus^r A$, $\oplus^s B \in \mathcal{A}$. Set $n = r + s + 1$. To prove that $\sqrt{\mathcal{A}}$ satisfies (I2), it is enough to show that $\oplus^n (A\nabla B) \in \mathcal{A}$. For that aim, using $(A\nabla B) \oplus P = (A \oplus P)\nabla(B \oplus P)$, one can prove by induction on n that $\oplus^n (A\nabla B)$ is a determinantal sum of elements of the form

$$C_1 \oplus C_2 \oplus \cdots \oplus C_n, \tag{7.1}$$

where each C_i equals A or B. By the choice of n, there are at least r C_i's equal to A or at least s C_i's equal to B. In either case, there exist permutation matrices E, F of appropriate size such that

$$C_1 \oplus C_2 \oplus \cdots \oplus C_n = \begin{cases} E((\oplus^r A) \oplus C'_{r+1} \oplus \cdots \oplus C'_n)F, \\ E((\oplus^s B) \oplus C'_{s+1} \oplus \cdots \oplus C'_n)F. \end{cases}$$

This implies that the elements in (7.1) belong to $\mathcal{A}$ by (I3). Now (I2) implies that $\oplus^n(A \nabla B) \in \mathcal{A}$, as desired.

Now let $A \in \sqrt{\mathcal{A}}$ and $B \in \mathfrak{M}(R)$. There exists $r \geq 1$ such that $\oplus^r A \in \mathcal{A}$. The equality $\oplus^r(A \oplus B) = E((\oplus^r A) \oplus (\oplus^r B))F$ holds for some permutation matrices E, F. Hence $\oplus^r(A \oplus B) \in \mathcal{A}$. Thus $A \oplus B \in \sqrt{\mathcal{A}}$ and $\sqrt{\mathcal{A}}$ satisfies (I3).

Let $A \in \sqrt{\mathcal{A}}$ be such that $\oplus^r A \in \mathcal{A}$. For permutation matrices E, F of appropriate size,

$$\oplus^r(EAF) = (\oplus^r E)(\oplus^r A)(\oplus^r F) \in \mathcal{A}.$$

Therefore $EAF \in \sqrt{\mathcal{A}}$ and $\sqrt{\mathcal{A}}$ satisfies (I4).

If $X \in \mathfrak{M}(R)$ is such that $X \oplus 1 \in \sqrt{\mathcal{A}}$, then there exists $t \geq 1$ such that $\oplus^t(X \oplus 1) \in \mathcal{A}$. But now $(\oplus^t X) \oplus I_t = E(\oplus^t(X \oplus 1))F \in \mathcal{A}$. Applying (I5), we get that $\oplus^t X \in \mathcal{A}$, and therefore $X \in \sqrt{\mathcal{A}}$. Hence $\sqrt{\mathcal{A}}$ satisfies (I5).

(2) By (1), $\sqrt{\mathcal{A}} \subseteq \sqrt{\sqrt{\mathcal{A}}}$. Now let $A \in \sqrt{\sqrt{\mathcal{A}}}$. This means that $\oplus^r A \in \sqrt{\mathcal{A}}$ for some positive integer r. Hence there exists a positive integer s such that $\oplus^s(\oplus^r A) \in \mathcal{A}$. Thus $\oplus^{rs} A = \oplus^s(\oplus^r A) \in \mathcal{A}$. Therefore $A \in \sqrt{\mathcal{A}}$, as desired.

(3) Suppose $\mathcal{A}$ is a gr-prime matrix ideal and let $A \in \sqrt{\mathcal{A}}$. Hence $\oplus^r A \in \mathcal{A}$. By (PM4), $A \in \mathcal{A}$, as desired. ∎

Proposition 7.9. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. Suppose that the nonempty subset Σ of $\mathfrak{M}(R)$ and the gr-matrix ideal $\mathcal{A}$ satisfy the following two conditions:*

(i) *$A \oplus B \in \Sigma$ for all $A, B \in \Sigma$.*

(ii) *$\mathcal{A} \cap \Sigma = \emptyset$.*

Then the set W of gr-matrix ideals $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \cap \Sigma = \emptyset$ has maximal elements and each such maximal element is a gr-prime matrix ideal.

Proof. Let $(\mathcal{C}_i)_{i\in I}$ be a nonempty chain in W. Set $\mathcal{C} = \bigcup_{i\in I} \mathcal{C}_i$. It is not difficult to show that $\mathcal{C}$ is a gr-matrix ideal. Then clearly $\mathcal{A} \subseteq \mathcal{C}_i \subseteq \mathcal{C}$ and $\mathcal{C} \cap \Sigma = (\bigcup_{i\in I} \mathcal{C}_i) \cap \Sigma = \bigcup_{i\in I}(\mathcal{C}_i \cap \Sigma) = \emptyset$. By Zorn's lemma, W has maximal elements. Suppose that $\mathcal{P}$ is a maximal element of W. Since $\mathcal{P} \cap \Sigma = \emptyset$, $\mathcal{P}$ is a proper gr-matrix ideal. Let $\mathcal{A}_1, \mathcal{A}_2$ be gr-matrix ideals such that $\mathcal{P} \subsetneq \mathcal{A}_1$, $\mathcal{P} \subsetneq \mathcal{A}_2$. Since $\mathcal{P}$ is maximal in W, there exist $A_1 \in \mathcal{A}_1 \cap \Sigma$, $A_2 \in \mathcal{A}_2 \cap \Sigma$. Then $A_1 \oplus A_2 \in \Sigma$ and $A_1 \oplus A_2 \notin \mathcal{P}$. Therefore $\mathcal{A}_1\mathcal{A}_2 \neq \mathcal{P}$. ■

Corollary 7.10. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. Let $\mathcal{A}$ be a proper gr-matrix ideal. Then there exist maximal gr-matrix ideals $\mathcal{P}$ with $\mathcal{A} \subseteq \mathcal{P}$, and such maximal gr-matrix ideals are gr-prime matrix ideals. In particular, if there are proper gr-matrix ideals, then gr-prime matrix ideals exist.*

Proof. By Lemma 7.1 (7), no identity matrix belongs to $\mathcal{A}$. Now apply Proposition 7.9 to $\mathcal{A}$ and $\Sigma = \mathfrak{I}$. ■

Proposition 7.11. *Let R be a Γ-graded ring. For each proper gr-matrix ideal $\mathcal{A}$, the radical $\sqrt{\mathcal{A}}$ is the intersection of all gr-prime matrix ideals that contain $\mathcal{A}$.*

Proof. Let $\mathcal{P}$ be a prime matrix ideal such that $\mathcal{A} \subseteq \mathcal{P}$. If $A \in \sqrt{\mathcal{A}}$, then $\oplus^r A \in \mathcal{A} \subseteq \mathcal{P}$ for some positive integer r. By (PM4), $A \in \mathcal{P}$. Thus $\sqrt{\mathcal{A}} \subseteq \mathcal{P}$.

Now let $A \in \mathfrak{M}(R) \setminus \sqrt{\mathcal{A}}$. Notice that such an A exists because $\sqrt{\mathcal{A}} \subseteq \mathcal{P}$. If we apply Proposition 7.9 to $\mathcal{A}$ and $\Sigma = \{\oplus^r A : r \text{ positive integer}\}$, we obtain a gr-prime matrix ideal $\mathcal{P}$ such that $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{P} \cap \Sigma = \emptyset$. Therefore A does not belong to the intersection of the gr-prime matrix ideals that contain $\mathcal{A}$. ■

Corollary 7.12. *Let R be a Γ-graded ring. A proper gr-matrix ideal is gr-semiprime if and only if it is the intersection of gr-prime matrix ideals.* ■

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. By Corollary 7.4, $\mathcal{A}(\mathcal{N})/\mathfrak{I}$ is the least gr-matrix ideal. We define the *gr-matrix nilradical* of R as the gr-matrix ideal $\mathfrak{N} = \sqrt{\mathcal{A}(\mathcal{N})/\mathfrak{I}}$.

Theorem 7.13. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. The following assertions are equivalent:*

(1) *There exists a Γ-graded epic R-division ring $(K = \bigoplus_{\gamma\in\Gamma} K_\gamma, \varphi)$.*

(2) *There exists a homomorphism of Γ-almost graded rings from R to a Γ-almost graded division ring.*

(3) *The gr-matrix nilradical is a proper gr-matrix ideal.*

(4) *No identity matrix can be expressed as a determinantal sum of elements of $\mathcal{N}$.*

Proof. (1) is equivalent to (2) by Theorem 4.4 (2)(b). One could also argue as follows. By Proposition 6.1, (2) implies the existence of gr-prime matrix ideals, and therefore of Γ-graded epic R-division rings by Theorem 6.3.

If (1) holds, the gr-singular kernel of φ is a gr-prime matrix ideal by Theorem 6.3. Thus (3) holds.

If (3) holds, then $\mathcal{A}(\mathcal{N})/\mathfrak{I}$ is a proper gr-matrix ideal. By Corollary 7.4, (4) holds.

Suppose that (4) holds true. Again by Corollary 7.4, there exist proper gr-matrix ideals. By Corollary 7.10, a gr-prime matrix ideal exists. Now Theorem 6.3 implies (1). ∎

Theorem 7.14. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. There exists a universal Γ-graded epic R-division ring if and only if the gr-matrix nilradical is a gr-prime matrix ideal.*

Proof. By Corollary 6.4, the existence of a universal Γ-graded epic R-division ring is equivalent to the existence of a least gr-prime matrix ideal $\mathcal{P}$. Hence the least gr-matrix ideal $\mathcal{A}(\mathcal{N})/\mathfrak{I} \subseteq \mathcal{P}$ is proper. By Proposition 7.11, $\mathfrak{N}$ is the intersection of all gr-prime matrix ideals. Hence $\mathfrak{N} = \mathcal{P}$.

Conversely, if $\mathfrak{N}$ is a gr-prime matrix ideal, then $\mathcal{A}(\mathcal{N})/\mathfrak{I}$ is proper and, by Proposition 7.11, $\mathfrak{N}$ is the intersection of all gr-prime matrix ideals. Therefore $\mathfrak{N}$ is the least gr-prime matrix ideal. ∎

Proposition 7.15. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and let $P, Q \in \mathfrak{M}(R)$. There exists a homomorphism of Γ-graded rings $\varphi\colon R \to K$ to a Γ-graded division ring $K = \bigoplus_{\gamma\in\Gamma} K_\gamma$ such that P^φ is invertible over K and Q^φ is not invertible over K if and only if no matrix of the form $I \oplus (\oplus^r P)$ can be expressed as a determinantal sum of matrices of $\mathcal{N} \cup \mathcal{D}(\{Q\})$.*

Proof. The existence of such a (K,φ) is equivalent to the existence of gr-prime matrix ideals $\mathcal{P}$ such that $Q \in \mathcal{P}$ and $P \notin \mathcal{P}$. The existence of such gr-prime matrix ideals is equivalent to the condition $P \notin \sqrt{\langle Q\rangle}$, where $\langle Q\rangle$ denotes the gr-matrix ideal generated by Q. Hence it is equivalent to the condition that no matrix of the form $\oplus^r P \in \langle Q\rangle$. By Lemma 7.3 (2), $\langle Q\rangle$ is of the form $\mathcal{A}(\{Q\})/\mathfrak{I}$. Therefore, by Lemmas 7.2 and 7.3, everything is equivalent to the condition that no matrix of the form $I \oplus (\oplus^r P)$ can be expressed as a determinantal sum of matrices of $\mathcal{N} \cup \mathcal{D}(\{Q\})$, as desired. ∎

Corollary 7.16. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring and let $P, Q \in \mathfrak{M}(R)$. The following assertions hold true:*

(1) *There exists a Γ-graded epic R-division ring (K,φ) such that P^φ is invertible over K if and only if no matrix of the form $I \oplus (\oplus^r P)$ can be expressed as a determinantal sum of matrices of $\mathcal{N}$.*

(2) *There exists a Γ-graded epic R-division ring (K,φ) such that Q^φ is not invertible over K if and only if no identity matrix can be expressed as a determinantal sum of matrices of $\mathcal{N} \cup \mathcal{D}(\{Q\})$.*

Proof. (1) In Proposition 7.15, let $Q = 0$.

(2) In Proposition 7.15, let $P = 1$. ■

Let I be a nonempty set. A *filter* on I is a set $\mathfrak{F}$ of subsets of I which has the following properties:

(F1) Every subset of I that contains a set of $\mathfrak{F}$ belongs to $\mathfrak{F}$.

(F2) Every finite intersection of sets of $\mathfrak{F}$ belongs to $\mathfrak{F}$.

(F3) The empty set is not in $\mathfrak{F}$.

For details on filters and ultrafilters we refer the reader to [2]. The set of filters on I is partially ordered by inclusion. An *ultrafilter* on I is a maximal filter. By [2, Theorem 1, p. 60], each filter is contained in an ultrafilter. An ultrafilter $\mathfrak{U}$ on I has the following property: if J, K are subsets of I such that $J \cup K = I$, then either $J \in \mathfrak{U}$ or $K \in \mathfrak{U}$.

Let I be a set and $\mathfrak{U}$ be an ultrafilter on I. For each $i \in I$, let $R_i = \bigoplus_{i\in I} R_{i\gamma}$ be a Γ-graded ring. Following [10], we define the graded ultraproduct of the family $\{R_i\}_{i\in I}$ as follows. Consider the ring $P = \prod_{i\in I} R_i$ and consider the following subset S of P:

$$S = \bigoplus_{\gamma\in\Gamma}\Big(\prod_{i\in I} R_{i\gamma}\Big).$$

Note that S is a subring of P which is Γ-graded with $S_\gamma = \prod_{i\in I} R_{i\gamma}$. For each $\gamma \in \Gamma$, if $x = (x_{i\gamma})_{i\in I} \in S_\gamma$, let $z(x) = \{i \in I : x_{i\gamma} = 0\}$. The set $Z_\gamma = \{x \in S_\gamma : z(x) \in \mathfrak{U}\}$ is an additive subgroup of S_γ. Moreover, if $y \in S_\delta$ and $x \in Z_\gamma$, then $yx \in Z_{\delta\gamma}$ and $xy \in Z_{\gamma\delta}$. Therefore $Z = \bigoplus_{\gamma\in\Gamma} Z_\gamma$ is a graded ideal of S. Then the Γ-graded ring $U = S/Z$ is called the *graded ultraproduct* of the family of Γ-graded rings $\{R_i\}_{i\in I}$.

A homogeneous element $x \in U_\gamma$ is the class of an element $(x_i)_{i\in I} \in S_\gamma$, where each $x_i \in R_{i\gamma}$. We will write $x = [(x_i)_{i\in I}]_\mathfrak{U}$. Observe that if $x = [(x_i)_{i\in I}]_\mathfrak{U}$ and $y = [(y_i)_{i\in I}]_\mathfrak{U}$, then $x = y$ if and only if the set $\{i \in I : x_i = y_i\} \in \mathfrak{U}$.

Let R be a Γ-graded ring. Suppose that (R_i, φ_i) is a Γ-graded R-ring for each $i \in I$. Hence $\varphi_i : R \to R_i$ is a homomorphism of Γ-graded rings. Then there exists a unique homomorphism of rings $\varphi' : R \to \prod_{i\in I} R_i$ such that $\pi_i\varphi' = \varphi_i$ for each $i \in I$. Observe that $\operatorname{Im}\varphi' \subseteq S$. Composing with the natural homomorphism $S \to S/Z = U$, we obtain a homomorphism of Γ-graded rings $\varphi : R \to U$. Hence U is a Γ-graded R-ring in a natural way.

Lemma 7.17. *Let I be a nonempty set and $\mathfrak{U}$ be an ultrafilter on I. If R_i is a Γ-graded division ring for each $i \in I$, then the ultraproduct U of the family $\{R_i\}_{i\in I}$ is a Γ-graded division ring.*

Proof. Let $x \in U_\gamma$. Then $x = [(x_i)_{i\in I}]_{\mathfrak{U}}$ for some $x_i \in R_{i\gamma}$. If x is nonzero, then $J = \{i \in I : x_i \neq 0\} \in \mathfrak{U}$. For each $i \in I$, define

$$x_i{}' = \begin{cases} x_i^{-1} & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

Notice that $x_i{}' \in R_{i\gamma^{-1}}$ for each $i \in I$. Then $x' = [(x'{}_i)_{i\in I}]_{\mathfrak{U}} \in U_{\gamma^{-1}}$ and $xx' = x'x = 1$, as desired. ■

Lemma 7.18. *Let R be a Γ-graded domain. Suppose that, for each $a \in \mathrm{h}(R) \setminus \{0\}$, there exists a homomorphism of Γ-graded rings $\varphi_a : R \to K_a$, where K_a is a Γ-graded division ring such that $\varphi_a(a) \neq 0$. Then there exists a Γ-graded epic R-division ring of fractions.*

Proof. Let $I = \mathrm{h}(R) \setminus \{0\}$. For each $a \in I$, let $I_a = \{\lambda \in I : \varphi_\lambda(a) \neq 0\}$. Let $E = \{a_1, \dots, a_n\}$ be a finite subset of I. Then $\bigcap_{i=1}^n I_{a_i} \neq \emptyset$, because $\varphi_{a_1\cdots a_n}(a_i) \neq 0$ for each $i = 1, \dots, n$. Hence the set $\mathfrak{B} = \{I_a : a \in I\}$ is a set of subsets of I such that no finite subset of $\mathfrak{B}$ has empty intersection. By [2, Proposition 1, p. 58], there exists a filter on I containing $\mathfrak{B}$. By [2, Theorem 1, p. 60], there exists an ultrafilter $\mathfrak{U}$ on I containing $\mathfrak{B}$. By Lemma 7.17, the ultraproduct U of the family $\{K_a\}_{a\in I}$ is a Γ-graded division ring and there exists a homomorphism of Γ-graded rings $\varphi : R \to U$, defined by $\varphi(x) = [(\varphi_a(x))_{a\in I}]_{\mathfrak{U}}$. Since the set $I_x \in \mathfrak{U}$, then $\varphi(x) \neq 0$ for each $x \in \mathrm{h}(R) \setminus \{0\}$. Therefore φ is injective. ■

Theorem 7.19. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. The following assertions are equivalent:*

(1) *There exists a Γ-graded epic R-division ring of fractions (K, φ).*

(2) *There exists a homomorphism of Γ-almost graded rings $\varphi : R \to K$ with K a Γ-almost graded division ring such that $\varphi(x) \neq 0$ for each $x \in \mathrm{h}(R) \setminus \{0\}$.*

(3) *R is a Γ-graded domain and no matrix of the form aI with $a \in \mathrm{h}(R) \setminus \{0\}$ can be expressed as a determinantal sum of matrices of $\mathcal{N}$.*

(4) *No diagonal matrix with nonzero homogeneous elements on the main diagonal can be expressed as a determinantal sum of matrices of $\mathcal{N}$.*

Proof. (1) and (2) are equivalent by Theorem 4.4 (b).

Suppose that (1) holds true. Then, for each diagonal matrix A as in (4), A^{φ} is invertible. Thus $A \notin \mathcal{P}$, the gr-prime matrix ideal given as the gr-singular kernel of φ. In particular, A cannot be expressed as the determinantal sum of matrices in $\mathcal{N}$. Thus (4) holds.

Suppose (4) holds. Clearly no matrix of the form aI with $a \in \mathrm{h}(R) \setminus \{0\}$ can be expressed as a determinantal sum of matrices of $\mathcal{N}$. Thus, to prove (3), it remains to show that R is a Γ-graded domain. Thus let $a, b \in \mathrm{h}(R)$ of degrees $\gamma, \delta \in \Gamma$, respectively. If $ab = 0$, then $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in M_2(R)[(\gamma, e)][(e, \delta^{-1})]$. Then we can express

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \nabla \begin{pmatrix} 0 & 0 \\ -1 & b \end{pmatrix}$$

as a determinantal sum of matrices in $M_2(R)[(\gamma, e)][(e, \delta^{-1})]$. Note that $\left(\begin{smallmatrix} 0 & 0 \\ -1 & b \end{smallmatrix}\right)$ is hollow, and hence it is not gr-full. Furthermore,

$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & b \end{pmatrix},$$

where the factors belong to $M_{2\times 1}(R)[(\gamma, e)][e]$ and $M_{1\times 2}(R)[e][(e, \delta^{-1})]$, respectively. Hence $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ can be expressed as a determinantal sum of matrices from $\mathcal{N}$. By (4), either $a = 0$ or $b = 0$. Hence R is a Γ-graded domain and (3) holds.

Suppose now that (3) holds. If there does not exist a Γ-graded epic R-division ring of fractions, then, by Lemma 7.18, there exists a nonzero $a \in \mathrm{h}(R)$ such that a^{φ} is not invertible for every homomorphism of Γ-graded rings $\varphi \colon R \to K$ with K a Γ-graded division ring. Hence the 1×1 homogeneous matrix (a) belongs to the intersection of all gr-prime matrix ideals, i.e. $(a) \in \mathfrak{N}$. Hence $\oplus^r(a) \in \mathcal{A}(\mathcal{N})/\mathfrak{I}$. Thus $I_s \oplus (\oplus^r(a)) = I_s \oplus aI_r$ can be written as a determinantal sum of matrices of $\mathcal{N}$. Then, since $aI_s \oplus I_r \in \mathfrak{M}(R)$ and it is diagonal, $aI_{r+s} = (aI_s \oplus I_r)(I_s \oplus aI_r)$ is a determinantal sum of matrices of $\mathcal{N}$, a contradiction. Therefore (1) holds. ■

8. gr-prime spectrum

Throughout this section, let Γ be a group and $\Omega \subseteq \Omega'$ be normal subgroups of Γ.

Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring. It can be considered as a Γ/Ω-graded ring too. Now we introduce some notation in order to clarify which structure of a graded object is being considered. We will denote by $\mathfrak{M}^{\Gamma}(R)$ and by $\mathfrak{M}^{\Gamma/\Omega}(R)$ the corresponding sets of homogeneous matrices. Notice that $\mathfrak{M}^{\Gamma}(R) \subseteq \mathfrak{M}^{\Gamma/\Omega}(R)$. We denote by $\mathrm{Spec}_\Gamma(R)$ the set of all Γ-gr-prime matrix ideals and by $\mathrm{Spec}_{\Gamma/\Omega}(R)$ the set of all Γ/Ω-gr-prime matrix ideals. If $\Omega = \Gamma$, we will write $\mathrm{Spec}(R)$ instead of $\mathrm{Spec}_{\Gamma/\Gamma}(R)$. Note that $\mathrm{Spec}(R)$ is the usual set of prime matrix ideals.

It follows directly from the definition that if $\mathcal{P}$ is a Γ/Ω-gr-prime matrix ideal, then $\mathcal{P} \cap \mathfrak{M}^\Gamma(R)$ is a Γ-gr-prime matrix ideal. Hence there exists a map

$$\mathrm{Spec}_{\Gamma/\Omega}(R) \to \mathrm{Spec}_\Gamma(R), \quad \mathcal{P} \mapsto \mathcal{P} \cap \mathfrak{M}^\Gamma(R).$$

Considering R as a Γ/Ω-graded ring and Ω'/Ω as a normal subgroup of Γ/Ω, we obtain maps $\mathrm{Spec}_{\Gamma/\Omega'}(R) \to \mathrm{Spec}_{\Gamma/\Omega}(R)$. Hence if $\mathrm{Spec}_{\Gamma/\Omega}(R)$ is empty, then $\mathrm{Spec}_{\Gamma/\Omega'}(R)$ is also empty. In other words, if there does not exist a Γ/Ω-graded epic R-division ring, then there does not exist a Γ/Ω'-graded epic R-division ring. In particular, for $\Omega' = \Gamma$ we obtain maps $\mathrm{Spec}(R) \to \mathrm{Spec}_{\Gamma/\Omega}(R)$, $\mathcal{Q} \mapsto \mathcal{Q} \cap \mathfrak{M}^{\Gamma/\Omega}(R)$, for each normal subgroup Ω of Γ. Therefore, if there exists a normal subgroup Ω of Γ such that there does not exist a Γ/Ω-graded epic R-division ring, then there does not exist an epic R-division ring.

Let $\mathcal{Q}' \in \mathrm{Spec}_{\Gamma/\Omega'}(R)$ and let $\mathcal{Q} = \mathcal{Q}' \cap \mathfrak{M}^{\Gamma/\Omega}(R)$ be the corresponding element in $\mathrm{Spec}_{\Gamma/\Omega}(R)$. Let $(K_{\mathcal{Q}'}, \varphi_{\mathcal{Q}'})$ be the Γ/Ω'-graded epic R-division ring determined by $\mathcal{Q}'$, and let $(K_{\mathcal{Q}}, \varphi_{\mathcal{Q}})$ be the Γ/Ω-graded epic R-division ring determined by $\mathcal{Q}$. Let x be a homogeneous element of R considered as a Γ/Ω-graded ring. Notice it is also a homogeneous element of R considered as a Γ/Ω'-graded ring. If $x \notin \ker \varphi_{\mathcal{Q}'}$, then $x \in \mathfrak{M}^{\Gamma/\Omega'}(R) \setminus \mathcal{Q}'$. Thus $x \in \mathfrak{M}^{\Gamma/\Omega}(R) \setminus \mathcal{Q}$, and therefore $x \notin \ker \varphi_{\mathcal{Q}}$. Hence if $\varphi_{\mathcal{Q}'}$ is injective, then $\varphi_{\mathcal{Q}}$ is also injective. In other words, if $(K_{\mathcal{Q}'}, \varphi_{Q'})$ is a Γ/Ω'-graded epic R-division ring of fractions, then $(K_{\mathcal{Q}}, \varphi_Q)$ is also a Γ/Ω-graded epic R-division ring of fractions. Therefore, if there exists a normal subgroup Ω of Γ such that there does not exist a Γ/Ω-graded epic R-division ring of fractions, then there does not exist an epic R-division ring of fractions.

Let $\mathcal{P}' \in \mathrm{Spec}_{\Gamma/\Omega'}(R)$ and set $\mathcal{P} = \mathcal{P}' \cap \mathfrak{M}^{\Gamma/\Omega}(R) \in \mathrm{Spec}_{\Gamma/\Omega}(R)$. If $\mathcal{P}' \subseteq \mathcal{Q}'$, then $\mathcal{P} \subseteq \mathcal{Q}$. Hence a specialization from $(K_{\mathcal{P}'}, \varphi_{P'})$ to $(K_{\mathcal{Q}'}, \varphi_{Q'})$ implies the existence of a specialization from $(K_{\mathcal{P}}, \varphi_{\mathcal{P}})$ to $(K_{\mathcal{Q}}, \varphi_{\mathcal{Q}})$ by Corollary 6.4. Notice that it could happen that $\mathcal{Q} = \mathcal{P}$.

Also, if the map $\mathrm{Spec}_{\Gamma/\Omega'}(R) \to \mathrm{Spec}_{\Gamma/\Omega}(R)$ is surjective and R has a universal Γ/Ω'-graded epic R-division ring (of fractions), then R has a universal Γ/Ω-graded epic R-division ring (of fractions).

Suppose that for each Γ-graded epic R-division ring D there exist ring homomorphisms to division rings. Then $\mathrm{Spec}_{\Gamma/\Omega}(R) \to \mathrm{Spec}_\Gamma(R)$ is surjective for each $\Omega \lhd \Gamma$. Let (D, φ) be a Γ-graded epic R-division ring with Γ-singular kernel $\mathcal{P}$. Let $\phi\colon D \to E$ be a ring homomorphism with E a division ring. Consider the composition $\phi \circ \varphi\colon R \to E$. It is a homomorphism of Γ/Ω-almost graded rings with E

a Γ/Ω-almost graded division ring. By Theorem 4.4 (2)(b), there exist $\psi\colon R \to D'$ a Γ/Ω-graded epic R-division ring, and a homomorphism $\rho\colon D' \to E$ such that $\phi\varphi = \rho\psi$. By Proposition 2.5,

$$\begin{aligned}&\{A \in \mathfrak{M}^{\Gamma}(R) : A^{(\phi\varphi)} \text{ is invertible over } E\}\\ &\qquad = \{A \in \mathfrak{M}^{\Gamma}(R) : A^{\varphi} \text{ is invertible over } D\},\\ &\{A \in \mathfrak{M}^{\Gamma/\Omega}(R) : A \text{ is invertible over } D'\}\\ &\qquad = \{A \in \mathfrak{M}^{\Gamma/\Omega}(R) : A^{(\rho\psi)} \text{ is invertible over } E\}.\end{aligned}$$

Now, since $\mathfrak{M}^{\Gamma}(R) \subseteq \mathfrak{M}^{\Gamma/\Omega}(R)$, we get that

$$\begin{aligned}&\{A \in \mathfrak{M}^{\Gamma}(R) : A^{\varphi} \text{ inverts over } D\}\\ &\qquad = \{A \in \mathfrak{M}^{\Gamma/\Omega}(R) : A \text{ inverts over } D'\} \cap \mathfrak{M}^{\Gamma}(R).\end{aligned}$$

Hence, if $\mathcal{P}'$ is the Γ/Ω-singular kernel of (D', ψ), then $\mathcal{P} = \mathcal{P}' \cap \mathfrak{M}^{\Gamma}(R)$.

We gather together what we have just proved in the following result.

THEOREM 8.1. *Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. The following assertions hold true:*

(1) *If there does not exist a Γ/Ω-graded epic R-division ring (of fractions), then there does not exist a Γ/Ω'-graded epic R-division ring (of fractions). Therefore, if there exists a normal subgroup Ω of Γ such that there does not exist a Γ/Ω-graded epic R-division ring (of fractions), then there does not exist an epic R-division ring (of fractions).*

(2) *Let $(K_{\mathcal{P}'}, \varphi_{\mathcal{P}'})$, $(K_{\mathcal{Q}'}, \varphi_{\mathcal{Q}'})$ be Γ/Ω'-epic R-division rings, such that there exists a specialization from $(K_{\mathcal{P}'}, \varphi_{\mathcal{P}'})$ to $(K_{\mathcal{Q}'}, \varphi_{\mathcal{Q}'})$. Then there exists a gr-specialization between the corresponding Γ/Ω-graded epic R-division rings.*

(3) *If the map $\mathrm{Spec}_{\Gamma/\Omega'}(R) \to \mathrm{Spec}_{\Gamma/\Omega}(R)$, $\mathcal{Q}' \mapsto \mathcal{Q}' \cap \mathfrak{M}^{\Gamma/\Omega}(R)$, is surjective, then the existence of a universal Γ/Ω'-graded epic R-division ring implies the existence of a universal Γ/Ω-graded epic R-division ring. Therefore, if $\mathrm{Spec}(R) \to \mathrm{Spec}_{\Gamma/\Omega}(R)$, $\mathcal{Q}' \mapsto \mathcal{Q}' \cap \mathfrak{M}^{\Gamma/\Omega}(R)$, is surjective, the existence of a universal R-division ring implies the existence of a universal Γ/Ω-graded epic R-division ring.*

(4) *If for each Γ-graded epic R-division ring there exist ring homomorphisms to division rings, then $\mathrm{Spec}_{\Gamma/\Omega}(R) \to \mathrm{Spec}_{\Gamma}(R)$, $\mathcal{Q} \mapsto \mathcal{Q} \cap \mathfrak{M}^{\Gamma}(R)$ is surjective.*

■

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. In the foregoing, we gave a correspondence from the set of Γ/Ω-graded epic R-division rings to the set of Γ-graded epic R-division rings. We proceed to give a more down-to-earth description of such correspondence. Recall that R can be regarded as a Γ/Ω-graded ring making $R = \bigoplus_{\alpha\in\Gamma/\Omega} R_\alpha$, where $R_\alpha = \bigoplus_{\gamma\in\alpha} R_\gamma$ for each $\alpha \in \Gamma/\Omega$.

Let $E = \bigoplus_{\alpha\in\Gamma/\Omega} E_\alpha$ be a Γ/Ω-graded division ring. Consider the group ring $E[\Gamma] = \bigoplus_{\gamma\in\Gamma} E\gamma$. We construct a Γ-graded division ring $D = \bigoplus_{\gamma\in\Gamma} D_\gamma$ which is a Γ-graded subring of $E[\Gamma]$ in the same way as in [18, Proposition 1.2.2]. For each $\gamma \in \Gamma$, there exists a unique $\alpha \in \Gamma/\Omega$ such that $\gamma \in \alpha$. Set $D_\gamma = E_\alpha\gamma \subseteq E\gamma$. Note that

$$D_\gamma D_{\gamma'} = E_\alpha \gamma E_{\alpha'}\gamma = E_\alpha E_{\alpha'}\gamma\gamma' \subseteq E_{\alpha\alpha'}\gamma\gamma' = D_{\gamma\gamma'}.$$

Hence D is a Γ-graded ring. Since E is a Γ/Ω-graded division ring, any nonzero homogeneous element of D is invertible. Thus D is a Γ-graded division ring.

Suppose that (E,φ) is a Γ/Ω-graded epic R-division ring. Let $\gamma\in\Gamma$ and $\alpha\in\Gamma/\Omega$ be such that $\gamma\in\alpha$. For each $a_\gamma \in R_\gamma$, $\varphi(a_\gamma)\in E_\alpha$. Then define $\psi(a_\gamma) = \varphi(a_\gamma)\gamma \in D_\gamma$. In this way, we obtain a homomorphism of Γ-graded rings $\psi\colon R\to D$.

Let $A = (a_{ij}) \in M_n(R)[\bar\delta][\bar\varepsilon]$. We claim that A^φ is invertible in E if and only if A^ψ is invertible in D. Indeed, let $\alpha_i, \beta_j \in \Gamma/\Omega$ be such that $\delta_i \in \alpha_i$, $\varepsilon_j \in \beta_j$. Then $A^\varphi = (b_{ij})$ with $b_{ij} \in E_{\alpha_i\beta_j^{-1}}$ and $(A^\varphi)^{-1} = (c_{ij})$ with $c_{ij}\in E_{\beta_i\alpha_j^{-1}}$. Then $A^\psi = (b_{ij}\alpha_i\beta_j^{-1})$ is invertible in D with inverse $(A^\psi)^{-1} = (c_{ij}\beta_i\alpha_j^{-1})$. Conversely, if A^ψ is invertible with inverse $(A^\psi)^{-1} = (d_{ij}\beta_i\alpha_j^{-1})$, where $d_{ij} \in E_{\beta_i\alpha_j^{-1}}$, then $(A^\varphi)^{-1} = (d_{ij})$. Hence let $\mathcal{P} \in \operatorname{Spec}_{\Gamma/\Omega}(R)$. If (E,φ) is the Γ/Ω-graded epic R-division ring associated to $\mathcal{P}$, then the Γ-graded epic R-division ring associated to $\mathcal{P}\cap\mathfrak{M}^\Gamma(R)$ is determined by the Γ-graded division ring $\psi\colon R\to D$, that is, the Γ-graded epic R-division ring $\psi\colon R\to D'$, where D' is the graded division ring generated by $\operatorname{Im}\psi$.

Now we proceed to give an important family of examples of Theorem 8.1 (4).

Let $(\Gamma,<)$ be an ordered group. Let $D = \bigoplus_{\gamma\in\Gamma} D_\gamma$ be a Γ-graded division ring. Given a map $f\colon\Gamma\to D$, let $\operatorname{supp} f = \{\gamma\in\Gamma : f(\gamma)\neq 0\}$. We will write f as a series. Thus $f = \sum_{\gamma\in\Gamma} a_\gamma$ means that $f(\gamma) = a_\gamma \in D$ for each $\gamma\in\Gamma$. Consider the set

$$D((\Gamma;<)) = \left\{ f = \textstyle\sum_{\gamma\in\Gamma} a_\gamma : a_\gamma \in D_\gamma \text{ for all } \gamma\in\Gamma, \operatorname{supp} f \text{ is well ordered}\right\},$$

where $D((\Gamma;<))$ is an abelian group under the natural sum. That is, for $f = \sum_{\gamma\in\Gamma} a_\gamma$, $f' = \sum_{\gamma\in\Gamma} a'_\gamma$, then

$$f + f' = \sum_{\gamma\in\Gamma}(a_\gamma + a'_\gamma).$$

One can then define the product in $D((\Gamma;<))$ as

$$ff' = \sum_{\gamma\in\Gamma}\Bigg(\sum_{\delta\varepsilon=\gamma} a_\delta a'_\varepsilon\Bigg).$$

These operations endow $D((\Gamma;<))$ with a ring structure. We regard D as a subring of $D((\Gamma;<))$ identifying D with the series of $D((\Gamma;<))$ of finite support. Mal'cev and Neumann independently showed that $D((\Gamma;<))$ is in fact a division ring [17, 19]. Hence we have just seen that for every Γ-graded division ring there exists a homomorphism of rings to a division ring.

Now we proceed to show that every $D((\Gamma;<))$ contains a Γ/Ω-graded division ring and that it corresponds to D via $\mathrm{Spec}_{\Gamma/\Omega}(R)\to\mathrm{Spec}_\Gamma(R)$. Let Ω be a normal subgroup of Γ. Consider D as a Γ/Ω-graded ring. For each $\alpha\in\Gamma/\Omega$, define the subset of $D((\Gamma;<))$,

$$E_\alpha=\left\{f=\textstyle\sum_{\gamma\in\Gamma}a_\gamma\in D((\Gamma;<)) : \operatorname{supp} f\subseteq\alpha\right\}.$$

Note that E_α is an additive subgroup of $D((\Gamma;<))$. Let $\alpha,\beta\in\Gamma/\Omega$. Suppose that $f=\sum_{\gamma\in\Gamma}a_\gamma\in E_\alpha$ and $f'=\sum_{\gamma\in\Gamma}a'_\gamma\in E_\beta$. Then

$$ff'=\sum_{\gamma\in\Gamma}\left(\sum_{\delta\varepsilon=\gamma}a_\delta a'_\varepsilon\right)\in E_{\alpha\beta}.$$

Hence $E_\alpha E_\beta\subseteq E_{\alpha\beta}$. Moreover, if $\alpha\in\Gamma/\Omega$,

$$E_\alpha\cap\left(\sum_{\substack{\beta\in\Gamma/\Omega\\ \beta\neq\alpha}}E_\beta\right)=\{0\},$$

because Γ is the disjoint union $\Gamma=\bigcup_{\beta\in\Gamma/\Omega}\beta$. Hence $E(\Omega)=\bigoplus_{\alpha\in\Gamma/\Omega}E_\alpha$ is a Γ/Ω-graded ring. Furthermore, let $f=\sum_{\gamma\in\Gamma}a_\gamma\in E_\alpha$, $f\neq 0$. Then f is invertible in $D((\Gamma;<))$ with inverse

$$f^{-1}=\left(\sum_{n\geq 0}(-1)^n g^n\right)a_{\gamma_0}^{-1},$$

where $\gamma_0=\min\operatorname{supp} f$ and $g=\sum_{\gamma\in\Gamma}a_{\gamma_0}^{-1}a_\gamma$. Since $\operatorname{supp}\gamma\subseteq\alpha$, $\gamma_0\in\alpha$ and $\gamma_0^{-1}\in\alpha^{-1}$, then $\operatorname{supp} g\subseteq E_e$, where e denotes the identity element in Γ/Ω. Hence $\operatorname{supp} g^n\subseteq E_e$ for each integer $n\geq 0$ and $\operatorname{supp}(\sum_{n\geq 0}(-1)^n g^n)\subseteq E_e$. Thus $\operatorname{supp} f^{-1}\subseteq\alpha^{-1}$. Therefore $E(\Omega)$ is a Γ/Ω-graded division ring and the embedding $\phi_\Omega\colon D\hookrightarrow E(\Omega)$ is a homomorphism of Γ/Ω-graded rings. Let $D(\Omega)$ be the Γ/Ω graded division subring of $E(\Omega)$ generated by D. Then $(D(\Omega),\phi_\Omega\colon D\hookrightarrow D(\Omega))$ is a Γ/Ω-graded epic D-division ring.

Let $R=\bigoplus_{\gamma\in\Gamma}R_\gamma$ be a Γ-graded ring, where $(\Gamma,<)$ is an ordered group. Let $\mathscr{P}\in\mathrm{Spec}_\Gamma(R)$ with corresponding epic R-division ring (K,φ). Consider $K((\Gamma,<))$. Then, for each $\Omega\lhd\Gamma$, we get that $\mathrm{Spec}_{\Gamma/\Omega}(R)\to\mathrm{Spec}_\Gamma(R)$ is surjective. Indeed, if

$\mathcal{Q} \in \mathrm{Spec}_{\Gamma/\Omega}(R)$ is the corresponding Γ/Ω-graded prime matrix ideal to the Γ/Ω-graded epic R-division ring $(K(\Omega), \phi_\Omega\varphi)$, then $\mathcal{Q} \mapsto \mathcal{P}$ by Proposition 2.5.

We would like to remark that $D(\Gamma)$, the division subring of $D((\Gamma; <))$ generated by D, does not depend on the order $<$ of Γ by [9] or [6]. Hence, since $D(\Omega)$ is just $\mathrm{DC}(\phi_\Omega)$, then $D(\Omega)$ does not depend on the order $<$ of Γ.

We end this section with a concrete application of the results in this section. Let K be a field, X be a nonempty set and $K\langle X\rangle$ be the free K-algebra on X. It is well known that $K\langle X\rangle$ has a universal division ring of fractions [5, Section 7.5]. Now let Γ be a group and $X \to \Gamma$, $x \mapsto \hat{x}$, be a map. Then $K\langle X\rangle = \bigoplus_{\gamma\in\Gamma} K\langle X\rangle_\gamma$ is a Γ-graded ring, where $K\langle X\rangle_\gamma$ is the K-vector space spanned by the monomials $x_1x_2\ldots x_r$ such that $\hat{x}_1\hat{x}_2\cdots\hat{x}_r = \gamma$. If $(\Gamma, <)$ is an ordered group, then $K\langle X\rangle$ has a Γ-graded universal division ring of fractions by the foregoing example and Theorem 8.1 (3),(4).

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