Rev. Mat. Iberoamericana **25** (2009), no. 3, 909–970

Asymptotic stability of solitons for the Benjamin-Ono equation

Carlos E. Kenig and Yvan Martel

Abstract

In this paper, we prove the asymptotic stability of the family of solitons of the Benjamin-Ono equation in the energy space. The proof is based on a Liouville property for solutions close to the solitons for this equation, in the spirit of [17], [19]. As a corollary of the proofs, we obtain the asymptotic stability of exact multi-solitons.

1. Introduction

We consider the Benjamin-Ono equation (BO)

$$u_t + \mathcal{H}u_{xx} + uu_x = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}u(x) = \frac{1}{\pi}\,\text{p.v.} \int_{-\infty}^{+\infty} \frac{u(y)}{y-x} dy = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \frac{u(y)}{y-x} dy. \tag{1.2}$$

Note that with this notation, $\int u_x \mathcal{H}u = \int |D^{\frac{1}{2}}u|^2 = \|u\|^2_{\dot{H}^{\frac{1}{2}}}$.

The Cauchy problem for (1.1) is globally well-posed in H^s, for any $s \geq 0$ (see Tao [27] for $s \geq 1$ and Ionescu and Kenig [11] for the case $s \geq 0$, see also Burq and Planchon [5] for the case $s > \frac{1}{4}$). Moreover, for solutions in the energy space $H^{\frac{1}{2}}$ the following quantities are invariant

$$\int u^2(t,x)dx = \int u^2(0,x)dx, \;\; E(t) = \int \Big(u_x \mathcal{H}u - \tfrac{1}{3}u^3\Big)(t,x)dx = E(0). \tag{1.3}$$

2000 Mathematics Subject Classification: 35Q53, 35Q51, 35B40.
Keywords: Benjamin-Ono equation, soliton, asymptotic stability, Liouville theorem.

Recall the scaling and translation invariances of equation (1.1)

$$\text{if } u(t,x) \text{ is solution then} \quad \forall c>0,\ x_0\in\mathbb{R},\ v(t,x)=c\,u(c^2t,c(x-x_0)) \text{ is also solution.} \tag{1.4}$$

We call soliton any travelling wave solution $u(t,x) = Q_c(x - x_0 - ct)$, where $c > 0$, $x_0 \in \mathbb{R}$, and $Q_c(x) = cQ(cx)$ solves:

$$-\mathcal{H}Q' + Q - \tfrac{1}{2}Q^2 = 0, \quad Q \in H^{\frac12}, \quad Q > 0. \tag{1.5}$$

It is known that there is a unique (up to translations) solution of (1.5), which is

$$Q(x) = \frac{4}{1+x^2}. \tag{1.6}$$

(see Benjamin [2] and Amick and Toland [1] for the uniqueness statement). This solution is stable (see Bennet et al. [3] and Weinstein [31]) in the following sense.

Stability of soliton in the energy space ([3], [31]). *There exist $C, \alpha_0 > 0$ such that if $u_0 \in H^{\frac12}$ satisfies $\|u_0 - Q\|_{H^{\frac12}} = \alpha \le \alpha_0$ then the solution $u(t)$ of* (1.1) *with $u(0) = u_0$ satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{y\in\mathbb{R}} \|u(t) - Q(.-y)\|_{H^{\frac12}} \le C\alpha.$$

See a sketch of proof of this result in Section 5.1.

The main result of this paper is the asymptotic stability of the family of solitons of (1.1). Then, we consider the multisoliton case (see Section 5).

Theorem 1. (Asymptotic stability of solitons in the energy space). *There exist $C, \alpha_0 > 0$ such that if $u_0 \in H^{\frac12}$ satisfies $\|u_0 - Q\|_{H^{\frac12}} = \alpha \le \alpha_0$, then there exists $c^+ > 0$ with $|c^+ - 1| \le C\alpha$ and a C^1 function $\rho(t)$ such that the solution $u(t)$ of* (1.1) *with $u(0) = u_0$ satisfies*

$$u(t,.+\rho(t)) \rightharpoonup Q_{c^+} \text{ in } H^{\frac12} \text{ weak, } \|u(t) - Q_{c^+}(.-\rho(t))\|_{L^2(x>\frac{t}{10})} \to 0, \tag{1.7}$$

$$\rho'(t) \to c^+ \quad \text{as } t\to+\infty. \tag{1.8}$$

The proof of Theorem 1 is based on the following rigidity result.

Theorem 2. (Nonlinear Liouville property). *There exist $C, \alpha_0 > 0$ such that if $u_0 \in H^{\frac12}$ satisfies $\|u_0 - Q\|_{H^{\frac12}} = \alpha \le \alpha_0$ and if the solution $u(t)$ of* (1.1) *with $u(0) = u_0$ satisfies for some function $\rho(t)$*

$$\forall \varepsilon > 0, \exists A_\varepsilon > 0, \text{ s.t. } \forall t \in \mathbb{R}, \quad \int_{|x|>A_\varepsilon} u^2(t, x+\rho(t))dx < \varepsilon, \tag{1.9}$$

then there exist $c_1 > 0$, $x_1 \in \mathbb{R}$, such that

$$u(t,x) = Q_{c_1}(x - x_1 - c_1 t), \quad |c_1 - 1| + |x_1| \le C\alpha. \tag{1.10}$$

Remark 1. In Theorem 1, the convergence of $u(t)$ to Q_{c^+} as $t \to +\infty$ is obtained strongly in L^2 in the region $x > \frac{t}{10}$. The value $\frac{1}{10}$ is somewhat arbitrary, the result holds for $x > \varepsilon t$, for any $\varepsilon > 0$, provided $\alpha_0 = \alpha_0(\varepsilon) > 0$ is small enough. Note that this result is optimal in L^2 since $u(t)$ could contain other small (and then slow) solitons and since in general $u(t)$ does not go to 0 in L^2 for $x < 0$. For example, if $\|u(t) - Q_{c^+}(. - \rho(t))\|_{H^{\frac{1}{2}}(\mathbb{R})} \to 0$ as $t \to +\infty$, then $E(u) = E(Q_{c^+})$ and $\int u^2 = \int Q_{c^+}^2$ and so by the variational characterization of $Q(x)$ (see [31]), $u(t) = Q_{c^+}(x - x_0 - c^+ t)$ is exactly a soliton.

Under the assumptions of Theorem 1, we expect strong convergence in $H^{\frac{1}{2}}$ to be true as well in the same local sense ($x > \varepsilon t$). This could require some more analysis.

By the methods of this paper, we are able to obtain the following weaker result (Section 4.3)

$$\lim_{t \to +\infty} \int_t^{t+1} \|u(s, . + \rho(s)) - Q_{c^+}\|^2_{H^{\frac{1}{2}}_{loc}} ds = 0. \tag{1.11}$$

Recall that the first result of asymptotic stability of solitons for generalized KdV equations was proved by Pego and Weinstein [24] in weighted spaces. The discussion p. 308 in [24] justifies the emergence of a pure dominant wave going to the right as time goes to $+\infty$, whereas small amplitude solitary waves and dispersion have small or negative speeds. The same discussion applies to the Benjamin-Ono equation. This is reflected in the monotonicity arguments we use to prove Theorem 1 (see Proposition 1). These arguments are in fact inspired by the proof of Kato's "local smoothing" effect for the generalized KdV equation, [12].

The proof of Theorem 1 follows the approach of [16], [17], concerning the case of the generalized KdV equations, where the asymptotic stability of the family of solitons is deduced from a Liouville type theorem such as Theorem 2. Moreover, similarly as in [17], the proof of Theorem 2 follows from a Liouville property on the linearized equation around Q, see Theorem 3 in Section 3.

With respect to the gKdV case, there are two main difficulties:

(1) L^2 monotonicity type results, which are similar to the ones for the gKdV equations ([17]), but whose proof are more subtle due to the nonlocal nature of the (BO) operator (see Section 2). For this part, we use a Kato type identity for (1.1) (see [9] and [25]).

(2) The proof of the linear Liouville theorem, which requires the analysis of some linear operators related to Q. Note that for this part, we use the fact that $Q(x)$ is explicit, and some known results about the linearized equation

around Q ([3], [31]). We point out that except for this part of the analysis, all the arguments are quite flexible and could be applied to generalized versions of the (BO) equation. In particular, we do not use the integrability property of the equation.

As a corollary of the proof of Theorem 1 and of Theorem 2, we obtain stability and asymptotic stability of multisoliton solutions. See Theorem 4 in Section 5 for a precise statement.

After the paper was finished and submitted, we learned that S. Gustafson, H. Takaoka, and T-P. Tsai [10] have obtained independently the stability part of Theorem 4. Note that the main result of the present paper, i.e. asymptotic stability of (single or multi-) solitons is not addressed in [10].

The rest of the paper is organized as follows. In Section 2, we prove L^2 monotonicity type results in the context of Theorem 1. In Section 3, we state and prove the linear Liouville Theorem, which is the main ingredient of the proof of Theorem 2. In Section 4, we prove Theorems 1 and 2 using Sections 2 and 3. Section 5 is devoted to the multisoliton case. In Section 6, we prove some weak convergence and well-posedness results used in the proofs. Finally, Appendix A contains the proof of some technical points.

2. Monotonicity arguments

2.1. Modulation

Lemma 1 (Choice of translation parameter)**.** *There exist $C, \alpha_0 > 0$ such that for any $0 < \alpha < \alpha_0$, if $u(t)$ is an $H^{\frac{1}{2}}$ solution of* (1.1) *such that*

$$\forall t \in \mathbb{R}, \quad \inf_{r\in\mathbb{R}} \|u(t) - Q(. - r)\|_{H^{\frac{1}{2}}} < \alpha, \tag{2.1}$$

then there exists $\rho(t) \in C^1(\mathbb{R})$ such that

$$\eta(t,x) = u(t, x + \rho(t)) - Q(x)$$

satisfies

$$\begin{aligned} &\forall t \in \mathbb{R}, \quad \int Q'(x)\eta(t,x)dx = 0, \quad \|\eta(t)\|_{H^{\frac{1}{2}}} \le C\alpha, \\ &|\rho'(t) - 1| \le C\left(\int \frac{\eta^2(t,x)}{1+x^2}dx\right)^{\frac{1}{2}} \le C\|\eta(t)\|_{L^2}. \end{aligned} \tag{2.2}$$

Proof of Lemma 1. This follows from standard arguments (see e.g. [4], Lemma 4.1, [15], Proposition 1 and Lemma 4).

Time independent arguments. For $u \in H^{\frac{1}{2}}$ and $y \in \mathbb{R}$, set

$$I_y(u) = \int Q'(x)(u(x+y) - Q(x))dx \quad \text{so that} \quad \frac{\partial I_y}{\partial y}_{|y=0,u=Q} = \int (Q')^2 > 0.$$

Thus, by the implicit function theorem, there exists $\alpha_1 > 0$, V a neighborhood of 0 in $\mathbb{R}$ and a unique C^1 map:

$$\begin{aligned} &y : \{u \in H^{\frac{1}{2}},\ \|u - Q\|_{H^{\frac{1}{2}}} \leq \alpha_1\} \to V \\ &\text{such that } I_{y(u)}(u) = 0,\ |y(u)| \leq C\|u - Q\|_{H^{\frac{1}{2}}}\ . \end{aligned} \tag{2.3}$$

We uniquely extend the C^1 map $y(u)$ to $U_{\alpha_1} = \{u \in H^{\frac{1}{2}},\ \inf_r \|u(.+r) - Q\|_{H^{\frac{1}{2}}} \leq \alpha_1\}$ so that for all u and r, $y(u) = y(u(.+r)) + r$. Then, we set $\eta_u(x) = u(x + y(u)) - Q(x)$, so that

$$\int \eta_u Q' = 0 \quad \text{and} \quad \|\eta_u\|_{H^{\frac{1}{2}}} \leq C\|u - Q\|_{H^{\frac{1}{2}}}.$$

Estimates depending on t. For all t, we define $\rho(t) = y(u(t))$ and $\eta(t) = \eta_{u(t)}$. To conclude the proof of the lemma, we just have to prove the estimate on $\rho'(t) - 1$.

We perform formal computations which can be justified for $H^{\frac{1}{2}}$ solutions by density and continuous dependence arguments. The function $\eta(t,x)$ satisfies the following equation:

$$\eta_t = (\mathcal{L}\eta - \tfrac{1}{2}\eta^2)_x + (\rho' - 1)(Q + \eta)_x \quad \text{where } \mathcal{L}\eta = -\mathcal{H}\eta_x + \eta - Q\eta. \tag{2.4}$$

Thus, multiplying the equation of η by Q' and using $\int \eta Q' = 0$, we obtain

$$(\rho' - 1)\left[\int (Q')^2 - \int \eta Q''\right] = \int \eta\mathcal{L}(Q'') - \tfrac{1}{2}\int \eta^2 Q'', \tag{2.5}$$

which finishes the proof for α_0 small enough.

Remark 2. By the proof of Lemma 1, $\rho(t)$ depends continuously on $u(t)$ in $H^{\frac{1}{2}}$. In particular, let $u(t)$ satisfy the assumptions of Lemma 1 with $u(0) = u_0$. If $u_n(0) \to u_0$ in $H^{\frac{1}{2}}$ as $n \to +\infty$, then by continuous dependence (see [11]), we obtain for all $t \in \mathbb{R}$, $\rho_n(t) \to \rho(t)$ as $n \to +\infty$, where $\rho_n(t)$ is defined from $u_n(t)$ ($u_n(t)$ is the solution of (1.1) corresponding to $u_n(0) = u_{0n}$).

Note also that in the proof of Lemma 1, we can replace the space $H^{\frac{1}{2}}$ by L^2, so that in the same context if $u_n(0) \to u_0$ in L^2 as $n \to +\infty$ then for all $t \in \mathbb{R}$, $\rho_n(t) \to \rho(t)$ as $n \to +\infty$ (see continuous dependence in L^2 also in [11]).

Finally, for future reference, we justify that if $u_n \rightharpoonup u$ in $H^{\frac{1}{2}}$ weak, then $y(u_n) \to y(u)$, where $y(u)$ is defined in the proof of Lemma 1. Indeed, in this proof, by the decay of $Q'(x)$, we can also replace $H^{\frac{1}{2}}$ by the weighted space $L^2(\frac{1}{1+|x|}dx)$, so that if $u_n \to u$ in L^2_{loc} and $\|u_n\|_{L^2} + \|u\|_{L^2} \leq C$, then $y(u_n) \to y(u)$ as $n \to +\infty$.

In the rest of this section, we present monotonicity arguments on L^2 quantities for both $u(t)$ and $\eta(t)$, in the context of Lemma 1. These results are reminiscent of similar results for the gKdV equation in [17] and [20], but due to the nonlocal nature of the operator $\mathcal{H}$, the proofs are more involved.

2.2. Monotonicity results for $u(t)$

Let $A > 1$ to be chosen later and set

$$\varphi(x) = \varphi_A(x) = \frac{\pi}{2} + \arctan\left(\frac{x}{A}\right) \quad \text{so that} \quad \varphi'(x) = \frac{\frac{1}{A}}{1+(\frac{x}{A})^2} > 0. \tag{2.6}$$

Proposition 1. *Let $0 < \lambda < 1$. Under the assumptions of Lemma 1, for α_0 small enough and A large enough, there exists $C > 0$ such that for all $x_0 > 1$, $t_1 \leq t_2$,*

1. *Monotonicity on the right of the soliton:*

$$\begin{aligned}&\int u^2(t_2,x)\varphi(x-\rho(t_2)-x_0)dx \\ &\quad \leq \int u^2(t_1,x)\varphi(x-\rho(t_1)-\lambda(t_2-t_1)-x_0)dx + \frac{C}{x_0}.\end{aligned} \tag{2.7}$$

2. *Monotonicity on the left of the soliton:*

$$\begin{aligned}&\int u^2(t_2,x)\varphi(x-\rho(t_2)+\lambda(t_2-t_1)+x_0)dx \\ &\quad \leq \int u^2(t_1,x)\varphi(x-\rho(t_1)+x_0)dx + \frac{C}{x_0}.\end{aligned} \tag{2.8}$$

Proof of Proposition 1. First, we note that (2.8) is a consequence of (2.7) and the L^2 norm conservation. Indeed, let $v(t,x) = u(-t,-x)$. Then $v(t)$ is a solution of (1.1) satisfying the assumptions of Lemma 1 and $\rho_v(t) = -\rho(-t)$. Thus, from (2.7) applied on $v(t,x)$, we deduce

$$\begin{aligned}&\int u^2(-t_2,x)\varphi(-x+\rho(-t_2)-x_0)dx \\ &\quad \leq \int u^2(-t_1,x)\varphi(-x+\rho(-t_1)-\lambda(t_2-t_1)-x_0)dx + \frac{C}{x_0}.\end{aligned} \tag{2.9}$$

Since $\varphi(x) = \pi - \varphi(-x)$, from $\int u^2(-t_2) = \int u^2(-t_1)$, we obtain

$$\int u^2(-t_2,x)\varphi(x-\rho(-t_2)+x_0)dx + \frac{C}{x_0}$$
$$\geq \int u^2(-t_1,x)\varphi(x-\rho(-t_1)+\lambda(t_2-t_1)+x_0)dx, \tag{2.10}$$

which is exactly formula (2.8) for $t_2' = -t_1$, $t_1' = -t_2$.

We are reduced to prove (2.7). We perform calculations on regular solutions and then use density arguments and continuous dependence to obtain the result in the framework of Lemma 1.

First, we recall a Kato type identity for solutions of the BO equation. By direct computations, we have

$$\frac{1}{2}\frac{d}{dt}\int u^2(t,x)\varphi(x)dx = \int u_t u\varphi(x)dx = -\int (\mathcal{H}u_{xx}+uu_x)u\varphi(x)dx$$
$$= \int (\mathcal{H}u_x)(u\varphi'(x)+u_x\varphi(x))dx + \frac{1}{3}\int u^3\varphi'(x)dx. \tag{2.11}$$

For the first term in (2.11), we prove the following result.

Lemma 2. *For all $u \in H^1(\mathbb{R})$,*

$$\int (\mathcal{H}u_x)u\varphi'(x)dx \leq \frac{C}{A}\int u^2\varphi'(x)dx. \tag{2.12}$$

Proof of Lemma 2. For $f \in L^2(\mathbb{R})$, we define the harmonic extension of f on $\mathbb{R}\times\mathbb{R}_+ = \mathbb{R}^2_+$,

$$\forall x\in\mathbb{R},\quad F(x,0) = f(x)$$
$$\text{and}\quad F(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-x')^2+y^2}\,f(x')\,dx',\quad \text{if } y>0. \tag{2.13}$$

In particular, recall that $\mathcal{H}f'(x) = \partial_y F(x,0)$ (see Stein [26, Chapter III], and the Introduction of Toland [28]).

We denote by $\Phi(x,y)$ the harmonic extension of $\varphi'(x)$ and $U(x,y)$ the harmonic extension of $u(x)$ on $\mathbb{R}\times\mathbb{R}_+$. Note that $\Phi(x,y)$ is explicitly given by

$$\Phi(x,y) = \frac{1}{A}\frac{1+\frac{y}{A}}{(\frac{x}{A})^2+(1+\frac{y}{A})^2}. \tag{2.14}$$

Then, by the Green Formula on $\mathbb{R}^2_+$ (using decay properties of $\Phi(x,y)$ and $\Delta U^2 = 2|\nabla U|^2$), we obtain formally

$$\begin{aligned}\int (\mathcal{H}u_x)u\varphi' &= \int \partial_y U(t,x,0)U(t,x,0)\Phi(x,0)dx = \frac{1}{2}\int_{y=0} \partial_y(U^2)\Phi dx \\ &= -\frac{1}{2}\iint_{\mathbb{R}^2_+} (\Delta U^2)\Phi + \frac{1}{2}\iint_{\mathbb{R}^2_+} U^2\Delta\Phi + \frac{1}{2}\int_{y=0} U^2\partial_y\Phi \\ &= -\iint_{\mathbb{R}^2_+} |\nabla U|^2\Phi + \frac{1}{2}\int u^2(\mathcal{H}\varphi'')dx.\end{aligned} \tag{2.15}$$

See Appendix A.1 for a rigorous proof of (2.15). Since $\Phi \geq 0$ on $\mathbb{R}^2_+$, we obtain

$$\int (\mathcal{H}u_x)u\varphi' \leq \frac{1}{2}\int u^2(\mathcal{H}\varphi''). \tag{2.16}$$

By explicit computations, since $\mathcal{H}\left(\frac{1}{1+x^2}\right) = -\frac{x}{1+x^2}$, we have

$$\mathcal{H}\varphi' = -\frac{1}{A^2}\frac{x}{1+(\frac{x}{A})^2}, \quad \mathcal{H}\varphi'' = \frac{1}{A}\varphi' - 2(\varphi')^2 \quad \text{and} \quad \mathcal{H}\varphi'' \leq \frac{1}{A}\varphi'. \tag{2.17}$$

Lemma 2 follows.

For the second term in (2.11), we have the following.

Lemma 3. *For all $u \in H^1(\mathbb{R})$,*

$$\left|\int (\mathcal{H}u_x)u_x\varphi dx\right| \leq \frac{C}{A}\int u^2\varphi'(x)dx. \tag{2.18}$$

Proof of Lemma 3. We prove (2.18) for u smooth and compactly supported in $\mathbb{R}$, the general case will follow by a density argument.

Since the limit in (1.2) holds in L^2 (see Stein [26, Chapter II]), we have

$$\begin{aligned}\int (\mathcal{H}u_x)u_x\varphi dx &= \frac{1}{\pi}\int \text{p.v.}\left(\int \frac{u_x(y)}{y-x}dy\right)u_x(x)\varphi(x)dx \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{|y-x|>\varepsilon} u_x(y)u_x(x)\frac{\varphi(x)}{y-x}\,dy\,dx \\ &= \frac{1}{2\pi}\iint u_x(y)u_x(x)\,\frac{\varphi(x)-\varphi(y)}{y-x}dxdy = \frac{1}{2\pi}\iint u(y)u(x)K_\varphi(x,y)\,dx\,dy,\end{aligned} \tag{2.19}$$

by symmetry and then integration by parts, where

$$\begin{aligned}K_\varphi(x,y) &= -\frac{\partial^2}{\partial x\partial y}\left(\frac{\varphi(x)-\varphi(y)}{x-y}\right) \\ &= \frac{2(\varphi(x)-\varphi(y)) - (\varphi'(x)+\varphi'(y))(x-y)}{(x-y)^3}.\end{aligned} \tag{2.20}$$

Note that all the integrals in (2.19) make sense since $u(x)$ is compactly supported, $(\varphi(x)-\varphi(y))/(x-y)$ is bounded and moreover, by subtracting the following two Taylor formulas:

$$\varphi(x)=\varphi(y)+(x-y)\varphi'(y)+\frac{1}{2}(x-y)^2\varphi''(y)+\frac{1}{6}(x-y)^3\varphi'''(x_1),$$
$$\varphi(y)=\varphi(x)+(y-x)\varphi'(x)+\frac{1}{2}(y-x)^2\varphi''(x)+\frac{1}{6}(y-x)^3\varphi'''(x_2),$$

where $x_1,x_2\in(y,x)$, we find:

$$K_\varphi(x,y)=\frac{1}{2}\frac{\varphi''(y)-\varphi''(x)}{x-y}+\frac{1}{6}(\varphi'''(x_1)+\varphi'''(x_2)), \tag{2.21}$$

which is also bounded on $\mathbb{R}^2$. Note also that by explicit computations, we have

(2.22)

$$\varphi'''(x)=\frac{\varphi'(x)}{A^2}\left(\frac{-2}{1+\left(\frac{x}{A}\right)^2}+\frac{8\left(\frac{x}{A}\right)^2}{\left(1+\left(\frac{x}{A}\right)^2\right)^2}\right)=\frac{\varphi'(x)}{A}\left(-2\varphi'(x)+\frac{8}{A}x^2(\varphi')^2\right).$$

We are reduced to prove the following estimate

$$\left|\iint u(y)u(x)K_\varphi(x,y)dxdy\right|\leq\frac{C}{A}\int u^2\varphi'(x)dx. \tag{2.23}$$

We consider only the case $|y|<|x|$ (by symmetry), and we divide $\{(x,y)\,:\,|y|<|x|\}$ into the following regions:

• $\Sigma_1=\{(x,y)\,:\,x>A,\ 0<y<\frac{x}{2}\}$. For $(x,y)\in\Sigma_1$, by (2.20) and the fact that φ' is decreasing on $\mathbb{R}^+$, we have

$$|K_\varphi(x,y)|\leq\frac{4}{(x-y)^2}\sup_{[y,x]}\varphi'\leq\frac{16}{x^2}\varphi'(y)=\frac{16}{A^2}\frac{1}{(\frac{x}{A})^2}\varphi'(y)\leq\frac{32}{A}\varphi'(x)\varphi'(y).$$

Thus, by Cauchy-Schwarz inequality, since $\int\varphi'(x)=\pi$, we obtain

$$\left|\iint_{\Sigma_1}u(y)u(x)K_\varphi(x,y)dxdy\right|\leq\frac{C}{A}\int|u(x)|\varphi'(x)dx\int|u(y)|\varphi'(y)dy$$
$$\leq\frac{C\pi}{A}\int u^2(x)\varphi'(x)dx.$$

The case of the region $\Sigma_1^-=\{(x,y)\,:\,x<-A,\ \frac{x}{2}<y<0\}$ is similar.

• $\Sigma_2=\{(x,y)\,:\,x>A,\ -x<y<0\}$. For $(x,y)\in\Sigma_2$, we have by (2.20), $|x-y|=x-y>x>\frac{1}{2}(x+A)$, $\varphi'(y)>\varphi'(x)$ and so by (2.20) and φ bounded, we obtain

$$|K_\varphi(x,y)|\leq\frac{C}{(x+A)^3}+\frac{C\varphi'(y)}{x^2}.$$

For the term $\frac{C\varphi'(y)}{x^2}$, we argue as for Σ_1. For the other term, by Cauchy-Schwarz' inequality and the expression of φ', we have

$$\begin{aligned}&\iint_{\Sigma_2} |u(y)||u(x)|\frac{1}{(x+A)^3}\,dx\,dy\\ &\leq C\left(\iint_{\Sigma_2}\frac{u^2(x)}{(x+A)^3}\,dx\,dy\right)^{\frac12}\left(\iint_{\Sigma_2}\frac{u^2(y)}{(x+A)^3}\,dx\,dy\right)^{\frac12}\\ &\leq C\left(\frac12\int_{x>A}\frac{u^2(x)}{(x+A)^2}\,dx\right)^{\frac12}\left(\frac12\int_{y<0}\frac{u^2(y)}{(-y+A)^2}\,dy\right)^{\frac12}\leq \frac{C'}{A}\int u^2(x)\varphi'(x)dx.\end{aligned}$$

The case of $\Sigma_2^- = \{(x,y)\,:\,x<-A,\,0<y<-x\}$ is similar to Σ_2.

• $\Sigma_3 = \{(x,y)\,:\,|x|<A,\,|y|<|x|\}$. For $(x,y)\in\Sigma_3$, and $|s|<|x|$, we have $\frac{1}{2A}\leq\varphi'(s)\leq\frac{1}{A}$ and thus, from (2.21) and (2.22), we obtain

$$|K_\varphi(x,y)|\leq C\sup_{|s|<|x|}|\varphi'''(s)|\leq\frac{C}{A^3}\leq\frac{C}{A}\varphi'(x)\varphi'(y).$$

We finish as for Σ_1.

• $\Sigma_4 = \{(x,y)\,:\,x>A,\,\frac12 x<y<x\}$. For $(x,y)\in\Sigma_4$, and $y<s<x$, we have from (2.22):

$$|\varphi'''(s)|\leq\frac{10}{A}(\varphi'(s))^2\leq\frac{10}{A}\varphi'(y)\varphi'(s)\leq\frac{40}{A}\varphi'(y)\varphi'(x)$$

thus

$$|K_\varphi(x,y)|\leq\frac{C}{A}\varphi'(x)\varphi'(y),$$

and we conclude as for Σ_1. The case of $\Sigma_4^- = \{(x,y)\,:\,x<-A,\,x<y<\frac{x}{2}\}$ is similar

In conclusion, we have obtained (2.23) and Lemma 3 is proved.

From (2.11), Lemmas 2 and 3, there exists $C_0>0$ such that

$$\frac12\frac{d}{dt}\int u^2(t,x)\varphi(x)dx\leq\frac{C_0}{A}\int u^2(t,x)\varphi'(x)dx+\frac13\int|u^3(t,x)|\varphi'(x)dx. \tag{2.24}$$

Now, let $u(t)$ be a solution of (1.1) satisfying the assumptions of Lemma 1 on $\mathbb{R}$. Let $\eta(t)$, $\rho(t)$ be associated to the decomposition of $u(t)$ on I as in Lemma 1.

Let $0<\lambda<1$, $t_0\in[t_1,t_2]$ and $x_0\geq 1$. For any $t\in[t_1,t_0]$, $x\in\mathbb{R}$, we set

$$\widetilde{x}=x-x_0-\rho(t)-\lambda(t_0-t),\quad M_\varphi(t)=\frac12\int u^2(t,x)\varphi(\widetilde{x})dx. \tag{2.25}$$

Then, by (2.24), we find

$$M'_\varphi(t) \leq -\frac{1}{2}\left(\rho'(t) - \lambda - \frac{2C_0}{A}\right)\int u^2(t)\varphi'(\widetilde{x}) + \frac{1}{3}\int |u(t)|^3\varphi'(\widetilde{x}). \tag{2.26}$$

Fix now $A > 0$ large enough so that $\frac{2C_0}{A} \leq \frac{1}{4}(1-\lambda)$. Then, by (2.2), we choose $\alpha_0 > 0$ small enough so that $\forall t \in I$, $\rho'(t) - \lambda > \frac{1}{2}(1-\lambda)$. Therefore, we obtain

$$M'_\varphi(t) \leq -\frac{1}{8}(1-\lambda)\int u^2(t)\varphi'(\widetilde{x}) + \frac{1}{3}\int |u(t)|^3\varphi'(\widetilde{x}). \tag{2.27}$$

Finally, we estimate the nonlinear term $\int |u(t)|^3\varphi'(\widetilde{x})$. We first observe:

$$\int |u(t)|^3\varphi'(\widetilde{x}) \leq C\int Q^3(x-\rho(t))\varphi'(\widetilde{x})dx + C\int |\eta(t,x)|^3\varphi'(\widetilde{x})dx. \tag{2.28}$$

For the first term, we distinguish two regions in x:

• $\Omega_1 = \{x : x < \rho(t) + \frac{1}{2}x_0 + \frac{1}{2}\lambda(t_0 - t)\}$. For $x \in \Omega_1$, we have $\widetilde{x} < -\frac{1}{2}x_0 - \frac{1}{2}\lambda(t_0 - t)$, and thus

$$\varphi'(\widetilde{x}) \leq \frac{C}{(x_0 + \lambda(t_0 - t))^2}.$$

This implies

$$\int_{\Omega_1} Q^3(x-\rho(t))\varphi'(\widetilde{x}) \leq \frac{C}{(x_0 + \lambda(t_0 - t))^2}\int Q^3 \leq \frac{C}{(x_0 + \lambda(t_0 - t))^2}. \tag{2.29}$$

• $\Omega_2 = \{x > \rho(t) + \frac{1}{2}x_0 + \frac{1}{2}\lambda(t_0 - t)\}$. For $x \in \Omega_2$, we have $x - \rho(t) > \frac{1}{2}x_0 + \frac{1}{2}\lambda(t_0 - t)$ and thus $Q^3(x-\rho(t)) \leq \frac{C}{(x_0+\lambda(t_0-t))^6}$, and

$$\int_{\Omega_2} Q^3(x-\rho(t))\varphi'(\widetilde{x})dx \leq \frac{C}{(x_0 + \lambda(t_0 - t))^6}.$$

Now, we claim

$$\int |\eta(t, x-\rho(t))|^3\varphi'(\widetilde{x})dx \leq C\alpha_0\int \eta^2(t, x-\rho(t))\varphi'(\widetilde{x})dx, \tag{2.30}$$

where C is independent of A. See proof of (2.30) in Appendix A.2. Moreover, as before, we find

$$\begin{aligned}\int \eta^2(t, x-\rho(t))\varphi'(\widetilde{x})dx &\leq C\int (u^2(t,x) + Q^2(x-\rho(t)))\varphi'(\widetilde{x})dx \\ &\leq C\int u^2(t,x)\varphi'(\widetilde{x})dx + \frac{C}{(x_0 + \lambda(t_0 - t))^2}.\end{aligned} \tag{2.31}$$

Thus, it follows from (2.27)–(2.30) that for $\alpha_0 > 0$ small enough, $\forall t \in [t_1, t_0]$,

$$M'_\varphi(t) \le -\frac{1}{8}(1-\lambda)\int u^2(t)\varphi'(\widetilde{x}) + C\alpha_0 \int u^2(t)\varphi'(\widetilde{x}) + \frac{C}{(x_0+\lambda(t_0-t))^2}$$

$$\le -\frac{1}{16}(1-\lambda)\int u^2(t)\varphi'(\widetilde{x}) + \frac{C}{(x_0+\lambda(t_0-t))^2}. \tag{2.32}$$

Let $t \in [t_1, t_0]$. By integration of (2.32) on $[t, t_0]$, since

$$\int_t^{t_0} \frac{dt'}{(x_0+\lambda(t_0-t'))^2} = \frac{1}{\lambda x_0}\int_0^{\frac{\lambda(t_0-t)}{x_0}} \frac{dt''}{(1+t'')^2} \le \frac{C}{x_0}, \quad \left(t'' = \frac{\lambda}{x_0}(t_0-t')\right)$$

we find:

$$\int u^2(t_0,x)\varphi(x-x_0-\rho(t_0))dx + \frac{1}{C}\int_t^{t_0}\int u^2(t',x)\varphi'(x-x_0-\rho(t')-\lambda(t-t'))dx\,dt'$$

$$\le \int u^2(t,x)\varphi(x-x_0-\rho(t)-\lambda(t_0-t))dx + \frac{C}{x_0}. \tag{2.33}$$

By density and continuous dependence ([11]) estimate (2.33) also holds for $H^{\frac{1}{2}}$ solutions.

2.3. Monotonicity results for $\eta(t)$

Here, we present similar monotonicity arguments for $\eta(t)$. See [20] for similar results in the case of the gKdV equations.

Proposition 2. *Let $0 < \lambda < 1$. Under the assumptions of Lemma 1, for α_0 small enough and A large enough, there exists $C > 0$ such that for all $x_0 > 1$, $t_1 \le t_2$,*

$$\int \eta^2(t_2,x)(\varphi(x-x_0)-\varphi(-x_0))\,dx$$
$$\le \int \eta^2(t_1,x)(\varphi(x-\lambda(t_2-t_1)-x_0)-\varphi(-x_0-\lambda(t_2-t_1)))dx$$
$$+ C\int_{t_1}^{t_2} \frac{\|\eta(t)\|_{L^2}^2}{(x_0+\lambda(t_2-t))^2}dt.$$

Remark 3. With respect to Proposition 1, we need to modify slighty the function in the integral ($\varphi(x-x_0)-\varphi(-x_0)$ instead of $\varphi(x-x_0)$) to remove some terms in the second member, see comments in the proof. This estimate is clearly improving Proposition 1 since the remainder term can now be controlled by $\frac{C}{x_0}\sup_t \|\eta(t)\|_{L^2}^2$.

As for $u(t)$ in the proof of Proposition 1, we have by direct computations using (2.4),

$$\frac{1}{2}\frac{d}{dt}\int \eta^2(t,x)\varphi(x)dx = \int \eta_t\eta\varphi(x)dx$$
$$= -\int (\mathcal{L}\eta)(\eta\varphi' + \eta_x\varphi) + \frac{1}{3}\int \eta^3\varphi' + (\rho'-1)\left(\int Q'\eta\varphi - \tfrac{1}{2}\int \eta^2\varphi'\right)$$
$$= \int(\mathcal{H}\eta_x)\eta\varphi' + \int(\mathcal{H}\eta_x)\eta_x\varphi - \frac{1}{2}\int\eta^2\varphi' + \frac{1}{2}\int\eta^2(Q\varphi' - Q'\varphi) + \frac{1}{3}\int\eta^3\varphi'$$
$$+ (\rho'-1)\left(\int Q'\eta\varphi - \tfrac{1}{2}\int \eta^2\varphi'\right). \tag{2.34}$$

Let $0<\lambda<1$ and $\overline{x} = x - x_0 - \lambda(t_0 - t)$. Then, by Lemmas 2 and 3, we get

$$\frac{d}{dt}\int \eta^2\varphi(\overline{x}) \leq -\left(\rho'(t) - \lambda - \frac{2C_0}{A}\right)\int \eta^2\varphi'(\overline{x}) + \int \eta^2(Q\varphi'(\overline{x}) - Q'\varphi(\overline{x}))$$
$$+ \frac{2}{3}\int |\eta|^3\varphi'(\overline{x}) + 2(\rho'-1)\int Q'\eta\varphi(\overline{x}).$$

Now, as in the proof of Proposition 1, we fix $A>1$ such that $\frac{2C_0}{A} \leq \frac{1}{4}(1-\lambda)$ and α_0 small enough so that $\rho' - \lambda > \frac{1}{2}(1-\lambda)$ by (2.2). Then, by (2.30) and (2.2), we can choose $\alpha_0 > 0$ small enough so that

$$\frac{2}{3}\int |\eta|^3\varphi'(\overline{x}) \leq \frac{1}{8}(1-\lambda)\int \eta^2\varphi'(\overline{x}).$$

Thus, we obtain

$$\frac{d}{dt}\int\eta^2\varphi(\overline{x}) \leq \frac{-1}{8}(1-\lambda)\int\eta^2\varphi'(\overline{x}) + \int\eta^2(Q\varphi'(\overline{x}) - Q'\varphi(\overline{x})) + 2(\rho'-1)\int Q'\eta\varphi(\overline{x}).$$

At this point, note that the term $\int \eta^2 Q'\varphi(\overline{x})$ has no sign, and since $\varphi(y) \sim \frac{C}{|y|}$ as $y \to -\infty$, this term can only be controlled by $\frac{C}{(x_0+\lambda(t_0-t))}\int \eta^2$, which is not sufficient for our purposes. We modify slightly the functional to cancel the main order of this term.

Indeed, since $\int \eta Q' = 0$, using (2.4), we have

$$\frac{d}{dt}\int\eta^2 = 2\int Q\eta\eta_x = -\int Q'\eta^2.$$

Therefore, using also $\int Q'\eta = 0$, we get

$$\frac{d}{dt}\int \eta^2\left(\varphi(\overline{x}) - \varphi(-x_0 - \lambda(t_0 - t))\right) \leq -\frac{1}{8}(1-\lambda)\int \eta^2\varphi'(\overline{x})$$
$$+ \int \eta^2\left(Q\varphi'(\overline{x}) - Q'(\varphi(\overline{x}) - \varphi(-x_0 - \lambda(t_0 - t)))\right)$$
$$+ 2(\rho' - 1)\int \eta Q'(\varphi(\overline{x}) - \varphi(-x_0 - \lambda(t_0 - t))) - \lambda\varphi'(-(x_0 + \lambda(t_0 - t)))\int \eta^2.$$

Now, we claim the following estimate $\forall x \in \mathbb{R}$,

$$Q(x)\varphi'(\overline{x}) + \left|Q(x)\left(\varphi(\overline{x}) - \varphi(-(x_0 + \lambda(t_0 - t)))\right)\right| \leq \frac{C}{(x_0 + \lambda(t_0 - t))^2}. \tag{2.35}$$

Since

$$Q(x)\varphi'(\overline{x}) \leq \frac{C}{(1 + x^2)(1 + (x - x_0 - \lambda(t_0 - t))^2)}$$

(recall that the value of A has been fixed) estimate (2.35) is clear for $Q(x)\varphi'(\overline{x})$ by considering the two regions $|x| > \frac{1}{2}(x_0 + \lambda(t_0 - t))$ and $|x| < \frac{1}{2}(x_0 + \lambda(t_0 - t))$.

For the other term, we first note that since $|Q(x)| \leq \frac{C}{1+x^2}$ and φ is bounded, the estimate is clear for $|x| > \frac{1}{2}(x_0 + \lambda(t_0 - t))$. For $|x| < \frac{1}{2}(x_0 + \lambda(t_0 - t))$, we have

$$|\varphi(\overline{x}) - \varphi(-x_0 - \lambda(t_0 - t))| \leq |x| \sup_{[\frac{1}{2}(x_0+\lambda(t_0-t), \frac{3}{2}(x_0+\lambda(t_0-t)]} \varphi' \leq \frac{C|x|}{(x_0 + \lambda(t_0 - t))^2};$$

thus, for such x, we obtain the following estimate which finishes the proof of (2.35):

$$\left|Q(x)\left(\varphi(\overline{x}) - \varphi(-x_0 - \lambda(t_0 - t))\right)\right| \leq \frac{C}{(x_0 + \lambda(t_0 - t))^2}.$$

By (2.2) and (2.35), and since $|Q'(x)| \leq \frac{C}{1+|x|}Q(x)$, we obtain

$$\left|\int \eta^2\Big(Q\varphi'(\overline{x}) - Q'\left(\varphi(\overline{x}) - \varphi(-x_0 - \lambda(t_0 - t))\right)\Big)\right| \leq \frac{C\|\eta(t)\|_{L^2}^2}{(x_0 + \lambda(t_0 - t))^2}, \tag{2.36}$$

$$\left|(\rho' - 1)\int Q'\eta(\varphi(\overline{x}) - \varphi(-x_0 - \lambda(t_0 - t)))\right| \leq \frac{C\|\eta(t)\|_{L^2}}{(x_0 + \lambda(t_0 - t))^2}\int \frac{|\eta|}{1 + |x|}$$
$$\leq \frac{C\|\eta(t)\|_{L^2}^2}{(x_0 + \lambda(t_0 - t))^2}. \tag{2.37}$$

The conclusion is thus:

$$\frac{d}{dt}\int \eta^2\Big(\varphi(\overline{x}) - \varphi(-(x_0 + \lambda(t_0 - t)))\Big) \leq -\frac{1}{8}(1-\lambda)\int \eta^2\varphi'(\overline{x}) + \frac{C\|\eta(t)\|_{L^2}^2}{(x_0 + \lambda(t_0 - t))^2}.$$

By integration on $[t, t_0]$, we get

$$\int \eta^2(t_0,x)\Big(\varphi(x-x_0)-\varphi(-x_0)\Big)dx \\ +\frac{1}{C}\int_t^{t_0}\int \eta^2(t',x)\varphi'(x-x_0-\lambda(t_0-t'))dxdt' \\ \leq \int \eta^2(t,x)\left(\varphi(x-x_0-\lambda(t_0-t))-\varphi(-x_0-\lambda(t_0-t))\right)dx \\ +C\int_t^{t_0}\frac{\|\eta(t')\|_{L^2}^2dt'}{(x_0+\lambda(t_0-t'))^2}.$$

3. Linear Liouville property

In this section, we prove the following result.

Theorem 3. *Let $w \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^\infty(\mathbb{R}, L^2(\mathbb{R}))$ be a solution of*

(3.1) $$w_t = (\mathcal{L}w)_x + \beta(t)Q', \quad (t,x)\in\mathbb{R}^2, \quad \textit{where } \beta \textit{ is continuous,}$$

satisfying

(3.2) $$\forall t\in\mathbb{R}, \quad \int w(t,x)Q(x)dx = \int w(t,x)Q'(x)dx = 0,$$

(3.3) $$\forall t\in\mathbb{R},\ \forall x_0>1, \quad \int_{|x|>x_0} w^2(t,x)dx \leq \frac{C}{x_0}.$$

Then

(3.4) $$w\equiv 0 \quad \textit{on } \mathbb{R}^2.$$

This result is similar to Theorem 3 in [16]. For the proof, we follow the strategy of [14], [19], introducing a dual problem whose operator has better spectral properties. Since $w(t)$ is only L^2 and has a weak decay at infinity in space, we will need to regularize and localize the dual solution.

For the sake of clarity, we now present the formal argument. The complete justification will be presented in Sections 3.1 and 3.2.

Multiplying the equation of $w(t)$ by $xw(t)$, we get

$$\frac{d}{dt}\int xw^2 = -2\int(\mathcal{H}w)w_x - \int w^2 + \int w^2(Q-xQ') + 2\beta(t)\int xQ'w,$$

where $(\int (Q')^2)\beta(t) = \int w\mathcal{L}(Q'')$ (multiply the equation of w by Q' and use $\int wQ' = 0$).

But it is not clear how to study the spectral properties of the operator

$$2\int(\mathcal{H}w)w_x+\int w^2-\int w^2(Q-xQ')+\frac{2}{\int(Q')^2}\left(\int w\mathcal{L}Q''\right)\left(\int xQ'w\right).$$

Moreover, the decay estimate (3.3) is not quite enough to control $\int xw^2$.

Therefore, we instead rely on the dual problem, setting $v=\mathcal{L}w$. Since $\mathcal{L}Q'=0$ (direct calculation), we obtain the following equation for $v(t)$: $v_t=\mathcal{L}(v_x)$. Multiplying the equation by xv, we obtain

$$-\frac{d}{dt}\int xv^2=2\int(\mathcal{H}v)v_x+\int v^2-\int v^2(Q+xQ').$$

Note that the operator in v is much easier to study since now the potential xQ' has a positive contribution ($xQ'\leq 0$), moreover, there is no scalar product. In fact, we will obtain (see Proposition 4) the positivity of this operator under the orthogonality condition $\int v(xQ)'=0$. Observe that $\int v(xQ)'=\int(\mathcal{L}w)(xQ)'=-\int wQ=0$ since $\mathcal{L}((xQ)')=-Q$ (see (3.35)).

Provided that $\int|x|v^2(t)\leq C$, we would obtain from the above identity

$$\int_{-\infty}^{+\infty}\|v(t)\|^2_{H^{\frac12}}dt\leq C,$$

which says that for a subsequence $t_n\to+\infty$, $v(t_n)\to 0$, $w(t_n)\to 0$. Combined with energy conservation ($(\mathcal{L}w(t),w(t))=C$) and Lemma 15 below, this gives $w\equiv 0$. But (3.3) is not enough to obtain the estimate $\int|x|v^2(t)\leq C$ In fact, since $w(t)$ is only in L^2, we both need to localize and regularize the dual problem.

3.1. Proof of Theorem 3 assuming the positivity of a quadratic form

Lemma 4. (Regularized dual problem). *There exists $\gamma_0>0$ such that for any $0<\gamma<\gamma_0$, the following is true. Let $v=(1-\gamma\partial_x^2)^{-1}(\mathcal{L}w)$. Then, $v\in C(\mathbb{R},H^1(\mathbb{R}))\cap L^\infty(\mathbb{R},H^1(\mathbb{R}))$ and*

1. *Equation of v.*

$$v_t=\mathcal{L}(v_x)-\gamma(1-\gamma\partial_x^2)^{-1}(2v_{xx}Q'+v_xQ''). \tag{3.5}$$

2. *Decay of v.*

$$\forall t\in\mathbb{R},x_0>1,\quad \int_{|x|>x_0}(v_x^2(t,x)+v^2(t,x))dx\leq\frac{C_\gamma}{x_0^{\frac34}}. \tag{3.6}$$

3. *Virial type estimate.*

$$\int_{-\infty}^{+\infty}\frac{1}{(1+t^2)^{\frac25}}\|v(t)\|^2_{H^1}dt<C. \tag{3.7}$$

Proof of Lemma 4. First, since $\sup_t \|w(t)\|_{L^2} \leq C$, we obtain $\sup_t \|v(t)\|_{H^1} \leq C_\gamma$ (see Claim 1 below).

1. *Equation of* v. Let $\widetilde{v} = \mathcal{L}w$ so that $w_t = \widetilde{v}_x + \beta Q'$. Since $\mathcal{L}Q' = 0$, the function $\widetilde{v}$ satisfies $\widetilde{v}_t = \mathcal{L}w_t = \mathcal{L}(\widetilde{v}_x)$. Now, we introduce a regularization of the function $\widetilde{v}$. For $0 < \gamma < \frac{1}{2}$ to be chosen later small enough, we set:

$$v(t,x) = (1-\gamma\partial_x^2)^{-1}\widetilde{v}(t,x) \quad \text{or equivalently} \quad v - \gamma v_{xx} = \widetilde{v} = \mathcal{L}w. \tag{3.8}$$

Then, $v(t,x)$ satisfies the following equation

$$v_t = (1-\gamma\partial_x^2)^{-1}\widetilde{v}_t = (1-\gamma\partial_x^2)^{-1}\mathcal{L}(\widetilde{v}_x) = \mathcal{L}(v_x) - (1-\gamma\partial_x^2)^{-1}(\widetilde{v}_x Q) + v_x Q.$$

But $-(1-\gamma\partial_x^2)^{-1}(\widetilde{v}_x Q) + v_x Q = (1-\gamma\partial_x^2)^{-1}(-2\gamma v_{xx}Q' - \gamma v_x Q'')$, and so

$$v_t = \mathcal{L}(v_x) - \gamma(1-\gamma\partial_x^2)^{-1}(2v_{xx}Q' + v_x Q''). \tag{3.9}$$

2. *Decay estimate on* v. By using the decay on $w(t)$, we claim

$$\forall x_0 > 1, \forall t, \quad \int_{|x|\geq x_0} (v_x^2(t,x) + v^2(t,x))dx \leq \frac{C_\gamma}{x_0^{\frac{3}{4}}}. \tag{3.10}$$

Indeed, let $(x_0 > 1)$

$$h(x) = h_{x_0}(x) = \varphi^2_{\sqrt{x_0}}(x - x_0) = \left(\frac{\pi}{2} + \arctan\left(\frac{x - x_0}{\sqrt{x_0}}\right)\right)^2.$$

Note that $0 \leq |h'| + |h''| \leq Ch$. Since $v - \gamma v_{xx} = \mathcal{L}w$, multiplying by vh, we have

$$\int v^2 h + \gamma \int v_x^2 h - \frac{1}{2}\gamma \int v^2 h'' = \int w\mathcal{L}(vh) = \int wD(vh) + \int wvh - \int Qwvh. \tag{3.11}$$

First, from

$$\left|\int wvh\right| + \left|\int Qwvh\right| \leq C\|w\sqrt{h}\|_{L^2}\|v\sqrt{h}\|_{L^2}$$

and

$$\int w^2 h \leq \int_{x<\frac{x_0}{2}} w^2 h + \int_{x>\frac{x_0}{2}} w^2 \leq C\frac{1}{x_0}$$

(using the definition of h and (3.3)) it follows that

$$\left|\int wvh\right| + \left|\int Qwvh\right| \leq \frac{C}{x_0^{\frac{1}{2}}}\|v\sqrt{h}\|_{L^2}.$$

Second, by Lemma 14, we have

$$\left|\int wD(vh)-\int D(v\sqrt{h})\sqrt{h}w\right|\le\|w\|_{L^2}\|v\sqrt{h}\|_{L^4}\|D(\sqrt{h})\|_{L^4}\le\frac{C}{x_0^{\frac{3}{8}}}\|v\sqrt{h}\|_{H^1}.$$

Since

$$\left|\int D(v\sqrt{h})\sqrt{h}w\right|\le\|w\sqrt{h}\|_{L^2}\|v\sqrt{h}\|_{H^1}\quad\text{and}\quad\|v\sqrt{h}\|_{H^1}^2=\int(v_x^2+v^2)h+O(\frac{1}{x_0}),$$

we obtain from (3.11)

$$\int_{x>x_0}(v_x^2+v^2)\le\int(v_x^2+v^2)h\le\frac{C_\gamma}{x_0^{\frac{3}{4}}}.$$

3. *Virial type estimate on* $v(t)$. Let $\frac{1}{3}<\theta<\frac{1}{2}$, $B>1$ to be chosen later and set

$$I(t)=\frac{1}{2}\int g\Big(\frac{x}{(B+t^2)^\theta}\Big)v^2(t,x)dx,\quad z=v\sqrt{g'\Big(\frac{x}{(B+t^2)^\theta}\Big)}$$

where $g(x)=\arctan(x)$.

$$(3.12)\qquad (\widetilde{\mathcal{L}}z,z)=-2(\mathcal{L}(z_x),xz)=2\int|D^{\frac{1}{2}}z|^2+\int z^2-\int(xQ'+Q)z^2.$$

For any $0<\sigma_0<1$, we claim

(3.13)
$$\left|2I'(t)+\frac{1}{(B+t^2)^\theta}(\widetilde{\mathcal{L}}z,z)\right|\le\frac{\sigma_0}{(B+t^2)^\theta}\|z\|_{L^2}^2+\frac{C}{\sigma_0(B+t^2)^{1-\theta}}\|v\|_{L^2}^2$$
$$+\frac{C}{(B+t^2)^{\frac{7}{4}\theta}}\|z\|_{H^{\frac{1}{2}}}\|v\|_{L^2}+\frac{C}{(B+t^2)^\theta}\gamma^{\frac{1}{4}}\|z\|_{H^{\frac{1}{2}}}\|v\|_{L^2}+\frac{C}{(B+t^2)^{2\theta}}\|z\|_{L^2}^2.$$

Proof of (3.13). We compute $I'(t)$:

$$I'(t)=-\frac{\theta t}{(B+t^2)^{\theta+1}}\int xg'\Big(\frac{x}{(B+t^2)^\theta}\Big)v^2+\int g\Big(\frac{x}{(B+t^2)^\theta}\Big)vv_t.$$

First, note that by Cauchy-Schwarz' inequality, for any $\sigma_0>0$,

$$\left|\frac{\theta t}{(B+t^2)^{\theta+1}}\int xg'\Big(\frac{x}{(B+t^2)^\theta}\Big)v^2\right|\le\frac{\sigma_0}{(B+t^2)^\theta}\int g'\Big(\frac{x}{(B+t^2)^\theta}\Big)v^2$$
$$+\frac{\theta^2t^2}{4\sigma_0(B+t^2)^{2-\theta}}\int\Big(\frac{x}{(B+t^2)^\theta}\Big)^2g'\Big(\frac{x}{(B+t^2)^\theta}\Big)v^2.$$

Since $s^2g'(s) \leq 1$, we obtain

$$\left|\frac{\theta t}{(B+t^2)^{\theta+1}}\int xg'\Big(\frac{x}{(B+t^2)^{\theta}}\Big)v^2\right| \leq \frac{\sigma_0}{(B+t^2)^{\theta}}\int z^2 + \frac{C\theta^2}{\sigma_0(B+t^2)^{1-\theta}}\int v^2.$$

Second, we use the equation of v to compute the term $\int g\big(\frac{x}{(B+t^2)^{\theta}}\big)vv_t$.

$$\begin{aligned}
&\int g\Big(\frac{x}{(B+t^2)^{\theta}}\Big)vv_t\\
&= \int g\Big(\frac{x}{(B+t^2)^{\theta}}\Big)v\mathcal{L}v_x - \gamma\int g\Big(\frac{x}{(B+t^2)^{\theta}}\Big)v(1-\gamma\partial_x^2)^{-1}(2v_{xx}Q'+v_xQ'')\\
&= \mathbf{A}+\mathbf{B}.
\end{aligned}$$

Estimate on $\mathbf{A}$.

$$\begin{aligned}
\mathbf{A} &= \int g\Big(\frac{x}{(B+t^2)^{\theta}}\Big)v(-\mathcal{H}v_{xx}+v_x) - \int g\Big(\frac{x}{(B+t^2)^{\theta}}\Big)Qvv_x\\
&= -\int\Big(\frac{1}{(B+t^2)^{\theta}}g'\Big(\frac{x}{(B+t^2)^{\theta}}\Big)v + g\Big(\frac{x}{(B+t^2)^{\theta}}\Big)v_x\Big)(-\mathcal{H}v_x+v)\\
&\quad + \frac{1}{2}\int\Big(\frac{1}{(B+t^2)^{\theta}}g'\Big(\frac{x}{(B+t^2)^{\theta}}\Big)Q + g\Big(\frac{x}{(B+t^2)^{\theta}}\Big)Q'\Big)v^2.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{A} =& \frac{-1}{(B+t^2)^{\theta}}\int |D^{\frac{1}{2}}z|^2 + v\Big(D\big(vg'\big(\tfrac{x}{(B+t^2)^{\theta}}\big)\big) - D\Big(v\sqrt{g'\big(\tfrac{x}{(B+t^2)^{\theta}}\big)}\Big)\sqrt{g'\big(\tfrac{x}{(B+t^2)^{\theta}}\big)}\Big)\\
&+ \int(\mathcal{H}v_x)v_xg\Big(\frac{x}{(B+t^2)^{\theta}}\Big) - \frac{1}{2}\frac{1}{(B+t^2)^{\theta}}\int z^2\\
&+ \frac{1}{2}\frac{1}{(B+t^2)^{\theta}}\int(xQ'+Q)z^2\\
&+ \frac{1}{2}\int\Big(g\Big(\frac{x}{(B+t^2)^{\theta}}\Big) - \frac{x}{(B+t^2)^{\theta}}g'\Big(\frac{x}{(B+t^2)^{\theta}}\Big)\Big)Q'v^2\\
=& -\frac{1}{2}\frac{1}{(B+t^2)^{\theta}}(\widetilde{\mathcal{L}}z,z) + \mathbf{A_1}+\mathbf{A_2}+\mathbf{A_3},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{A_1} &= -\frac{1}{(B+t^2)^{\theta}}\int v\Big(D\Big(z\sqrt{g'\big(\tfrac{x}{(B+t^2)^{\theta}}\big)}\Big) - (Dz)\sqrt{g'\big(\tfrac{x}{(B+t^2)^{\theta}}\big)}\Big),\\
\mathbf{A_2} &= \int(\mathcal{H}v_x)v_xg\Big(\frac{x}{(B+t^2)^{\theta}}\Big),\\
\mathbf{A_3} &= \frac{1}{2}\int\Big(g\Big(\frac{x}{(B+t^2)^{\theta}}\Big) - \frac{x}{(B+t^2)^{\theta}}g'\Big(\frac{x}{(B+t^2)^{\theta}}\Big)\Big)Q'v^2.
\end{aligned}$$

Estimate on $\mathbf{A_1}$. By Lemma 14, we have

$$|\mathbf{A}_1| \leq \frac{C}{(B+t^2)^\theta}\|v\|_{L^2}\|z\|_{L^4}\Big\|D\sqrt{g'\big(\tfrac{x}{(B+t^2)^\theta}\big)}\Big\|_{L^4} \leq \frac{C}{(B+t^2)^{\frac{7\theta}{4}}}\|v\|_{L^2}\|z\|_{H^{\frac12}}.$$

Estimate on $\mathbf{A_2}$. Since $\int(\mathcal{H}v_x)v_x = 0$, Lemma 3, applied to $A=(B+t^2)^\theta$ gives

$$|\mathbf{A}_2| \leq \frac{C}{(B+t^2)^{2\theta}}\|z\|_{L^2}^2.$$

Estimate on $\mathbf{A_3}$. Since for all $y\in\mathbb{R}$, $|\arctan y - \frac{y}{1+y^2}| \leq Cy^2$, we have, for all $x\in\mathbb{R}$,

$$\left|\left(g\left(\frac{x}{(B+t^2)^\theta}\right) - \frac{x}{(B+t^2)^\theta}\, g'\left(\frac{x}{(B+t^2)^\theta}\right)\right)Q'(x)\right| \leq \frac{x^2|Q'(x)|}{(B+t^2)^{2\theta}} \leq \frac{C}{(B+t^2)^{2\theta}}\frac{1}{1+|x|}. \tag{3.14}$$

Thus,

$$|\mathbf{A}_3| \leq \frac{C}{(B+t^2)^{2\theta}}\|v\|_{L^2}\|z\|_{L^2}.$$

Estimate on $\mathbf{B}$. First, we claim the following.

Claim 1. (i) $x(1-\gamma\partial_x^2)^{-1}f = (1-\gamma\partial_x^2)^{-1}(xf) - 2\gamma(1-\gamma\partial_x^2)^{-2}(f')$.

(ii) $\|(1-\gamma\partial_x^2)^{-1}f\|_{L^2}+\gamma^{\frac12}\|(1-\gamma\partial_x^2)^{-1}(f')\|_{L^2}+\gamma\|(1-\gamma\partial_x^2)^{-1}(f'')\|_{L^2}\leq C\|f\|_{L^2}$,
$\|(1-\gamma\partial_x^2)^{-1}(f'')\|_{L^2} \leq C\gamma^{-\frac34}\|f\|_{\dot H^{\frac12}}$.

Proof of Claim 1. (i) Let $h=(1-\gamma\partial_x^2)^{-1}f$. Then, $xh-\gamma(xh)'' = xf-2\gamma h'$ and so $xh = (1-\gamma\partial_x^2)^{-1}(xf-2\gamma(1-\gamma\partial_x^2)^{-1}f')$.

(ii) $\int|f|^2 = \int|h-\gamma h''|^2 = \int h^2 + 2\gamma\int(h')^2+\gamma^2\int(h'')^2$, which proves the first estimate.

Next,

$$\|(1-\gamma\partial_x^2)^{-1}f''\|_{L^2} \leq C\|(\frac{\xi^2}{1+\gamma\xi^2})\hat f\|_{L^2} \leq C\gamma^{-\frac34}\||\xi|^{\frac12}\hat f\|_{L^2}\,,$$

since $\forall\xi\in\mathbb{R}$, $\forall\gamma>0$,

$$\frac{\xi^2}{1+\gamma\xi^2} \leq \gamma^{-\frac34}|\xi|^{\frac12}\,.$$

The claim is proved.

Using (i) of Claim 1, we obtain

$$\mathbf{B} = -\gamma \int \frac{1}{x} g\Big(\frac{x}{(B+t^2)^\theta}\Big) v\,(1-\gamma\partial_x^2)^{-1} H,$$

where

$$H = 2xv_{xx}Q' + xv_xQ'' - 2\gamma(1-\gamma\partial_x^2)^{-1}(2v_{xx}Q' + v_xQ'')_x.$$

Since $|g(y)| \le C|y|$, for all y, we have

$$|\mathbf{B}| \le \frac{C\gamma}{(B+t^2)^\theta}\|v\|_{L^2}\|(1-\gamma\partial_x^2)^{-1}H\|_{L^2}.$$

Now, we use Claim 1 (ii) to estimate $\|(1-\gamma\partial_x^2)^{-1}H\|_{L^2}$. We can rewrite H under the form:

$$H = (2vxQ')'' + (vF_1)' + vF_2 - 2\gamma(1-\gamma\partial_x^2)^{-1}((2vQ')'' + (vF_3)' + vF_4)_x,$$

where for $j = 1,\dots,4$, $|F_j(x)| \le C\frac{1}{1+x^2}$. Thus,

$$\begin{aligned}\|(1-\gamma\partial^2)^{-1}H\|_{L^2} &\le C\gamma^{-\frac34}\|vxQ'\|_{\dot H^{\frac12}} + C\gamma^{-\frac12}\|v\tfrac{1}{1+x^2}\|_{L^2}\\ &\quad + \gamma^{\frac12}\|(1-\gamma\partial_x^2)^{-1}((2vQ')'' + (vF_3)' + vF_4)\|_{L^2}\\ &\le C\gamma^{-\frac34}\|vxQ'\|_{\dot H^{\frac12}} + C\gamma^{-\frac12}\|v\tfrac{1}{1+x^2}\|_{L^2}.\end{aligned}$$

Now, we claim

$$\|v\tfrac{1}{1+x^2}\|_{L^2} \le C\|z\|_{L^2}, \quad \|vxQ'\|_{\dot H^{\frac12}} \le C\|z\|_{H^{\frac12}}. \tag{3.15}$$

The first estimate is clear since $\frac{1}{1+x^2} \le C\sqrt{g'}$. Let

$$f(x) = \frac{xQ'(x)}{\sqrt{g'(\frac{x}{(B+t^2)^\theta})}}.$$

Then, by Lemma 14,

$$\|D^{\frac12}(vxQ')\|_{L^2} = \|D^{\frac12}(zf)\|_{L^2} \le \|(D^{\frac12}z)f\|_{L^2} + C\|z\|_{L^4}\|D^{\frac12}f\|_{L^4} \le C\|z\|_{H^{\frac12}},$$

since $\|f\|_{L^\infty} + \|D^{\frac12}f\|_{L^4} \le \|f\|_{H^1} \le C$.

Thus, $\|(1-\gamma\partial^2)^{-1}H\|_{L^2} \le C\gamma^{-\frac34}\|z\|_{H^{\frac12}}$ and in conclusion for the term $\mathbf{B}$:

$$|\mathbf{B}| \le \frac{C\gamma^{\frac14}}{(B+t^2)^\theta}\|z\|_{H^{\frac12}}\|v\|_{L^2}.$$

Putting together the above estimates, we obtain (3.13).

We now claim the following (see proof in Section 3.2):

Proposition 3. *There exist $\lambda > 0$, $\gamma_0 > 0$ and $B_0 > 1$ such that, for $0 < \gamma < \gamma_0$, $B \geq B_0$,*

$$\forall t, \quad (\widetilde{\mathcal{L}} z(t), z(t)) \geq \lambda \|z(t)\|^2_{H^{\frac{1}{2}}}, \quad \text{where } z \text{ is as above.}$$

Remark 4. The operator $\widetilde{\mathcal{L}}$ does not depend on γ and B, but the orthogonality conditions on w imply almost orthogonality conditions on z that depend on γ, B, see proof of Proposition 3.

Choose $\theta = \frac{2}{5}$ and fix $\sigma_0 = \frac{\lambda}{4}$. Then,

$$-2I'(t) \geq \frac{\lambda}{2(B+t^2)^\theta} \|z(t)\|^2_{H^{\frac{1}{2}}} - \frac{C}{(B+t^2)^\theta}\left(\frac{1}{(B+t^2)^{\frac{1}{5}}} + \gamma^{\frac{1}{2}}\right) \|v\|^2_{L^2}.$$

By the decay property (3.6),

$$\int v^2(t) \leq \int_{|x| \leq \frac{1}{2}(B+t^2)^\theta} v^2(t) + \frac{C_\gamma}{(B+t^2)^{\frac{3}{4}\theta}} \leq C \int z^2(t) + \frac{C_\gamma}{(B+t^2)^{\frac{3}{10}}}.$$

For $\gamma > 0$ small enough and B large enough, and by $\|v\|_{H^{\frac{1}{2}}} \leq C$, we get

$$-2I'(t) \geq \frac{\lambda}{4(B+t^2)^\theta} \|z(t)\|^2_{H^{\frac{1}{2}}} - \frac{C_\gamma}{(B+t^2)^{\frac{3}{5}}}.$$

Since $I(t)$ is bounded, we obtain by integration

$$\int_{-\infty}^{+\infty} \frac{1}{(B+t^2)^\theta} \|z(t)\|^2_{H^{\frac{1}{2}}} dt < C_\gamma. \tag{3.16}$$

We claim that (3.16) and (3.6) imply

$$\int_{-\infty}^{+\infty} \frac{1}{(B+t^2)^\theta} \|v(t)\|^2_{H^{\frac{1}{2}}} dt < C. \tag{3.17}$$

Indeed, by (3.6) and the expression of g', and considering the two regions $x > \frac{1}{(B+t^2)^{\frac{\theta}{2}}}$, $x < \frac{1}{(B+t^2)^{\frac{\theta}{2}}}$, we have

$$\|v - z\|^2_{H^1} = \|v(1 - \sqrt{g'})\|^2_{H^1} \leq \frac{C}{(B+t^2)^{\frac{3}{8}\theta}} = \frac{C}{(B+t^2)^{\frac{3}{20}}}. \tag{3.18}$$

Thus, by $\|v\|_{H^{\frac{1}{2}}} \leq \|z\|_{H^{\frac{1}{2}}} + \|v-z\|_{H^{\frac{1}{2}}}$, and (3.16)

$$\int_{-\infty}^{+\infty} \frac{1}{(B+t^2)^\theta} \|v(t)\|^2_{H^{\frac{1}{2}}} dt \leq 2C_\gamma + \int_{-\infty}^{+\infty} \frac{1}{(B+t^2)^{\frac{11}{20}}} dt \leq C.$$

Using another virial argument, we claim

$$\int_{-\infty}^{+\infty} \frac{1}{(B+t^2)^\theta} \|z(t)\|_{H^{\frac{3}{2}}}^2 dt < C. \tag{3.19}$$

Proof of (3.19). We set

$$J(t) = \frac{1}{2}\int g\left(\frac{x}{(B+t^2)^\theta}\right) v_x^2(t).$$

Proceeding as in the proof of (3.13) (the equation for v_x is very similar to the one for v), we obtain

$$\left| J'(t) + \frac{1}{(B+t^2)^\theta}\int (D^{\frac{3}{2}} z)^2 \right| \leq \frac{C}{(B+t^2)^\theta} \|v\|_{H^1}(\|v\|_{H^1} + \|z\|_{H^{\frac{3}{2}}}).$$

Using $\|v\|_{H^1} \leq \|z\|_{H^1} + \|v-z\|_{H^1}$, (3.18) and the following estimate

$$\|z\|_{H^1} \leq \varepsilon \|D^{\frac{3}{2}} z\|_{L^2} + C_\varepsilon \|z\|_{L^2},$$

we obtain, for $\varepsilon > 0$ small enough,

$$-J'(t) \geq \frac{1}{2}\frac{1}{(B+t^2)^\theta}\|D^{\frac{3}{2}} z\|_{L^2}^2 + C\frac{1}{(B+t^2)^\theta}\|z\|_{L^2}^2.$$

Since $J(t)$ is bounded and using (3.16), we obtain (3.19).

Finally, by (3.16), (3.18) and (3.19), we get (3.7). Lemma 4 is proved.

Lemma 5 (Decay estimate on $w(t)$)**.** *The following hold*

$$\int_{-\infty}^{+\infty} \frac{1}{(1+t^2)^{\frac{2}{5}}} \|w(t)\|_{L^2}^2 dt < C, \tag{3.20}$$

$$\sup_{t\in\mathbb{R}} \int |x| w^2(t,x) dx \leq C. \tag{3.21}$$

Proof of Lemma 5. Estimate (3.20) is a consequence of Lemma 4 by comparing v and w. Let $\gamma > 0$ small. We have by the definition of v: $(1-\gamma\partial_x^2)v = \mathcal{L}w$. Let $\widetilde{w} = (1-\gamma\partial_x^2)^{-\frac{1}{4}} w$. Then,

$$\int w(1-\gamma\partial_x^2)^{\frac{1}{2}} v = \int \widetilde{w}(1-\gamma\partial_x^2)^{-\frac{1}{4}}(\mathcal{L}w).$$

On the one hand, we have

$$\left|\int w(1-\gamma\partial_x^2)^{\frac{1}{2}} v\right| \leq C\|w\|_{L^2}\|v\|_{H^1}.$$

On the other hand, as in the proof of Claim 1

$$\|(1-\gamma\partial_x^2)^{-\frac14}(\mathcal{L}w) - \mathcal{L}\widetilde{w}\|_{L^2} \\ \leq \|(1-\gamma\partial_x^2)^{-\frac14}(Qw) - Qw\|_{L^2} + \|Q(w-\widetilde{w})\|_{L^2} \leq \gamma^{\frac14}\|w\|_{L^2}.$$

Thus,

$$\left|\int \widetilde{w}(1-\gamma\partial_x^2)^{-\frac14}(\mathcal{L}w) - (\mathcal{L}\widetilde{w},\widetilde{w})\right| \leq C\gamma^{\frac14}\|w\|_{L^2}^2$$

and since $(\mathcal{L}\widetilde{w},\widetilde{w}) \geq \frac12\lambda\|\widetilde{w}\|_{H^{\frac12}}$ for $\gamma > 0$ small enough (this is a consequence of Lemma 15 and the orthogonality conditions on w –see Section 3.2, in particular the proof of Proposition 3), we obtain

$$\int \widetilde{w}(1-\gamma\partial_x^2)^{-\frac14}(\mathcal{L}w) \geq \frac{\lambda}{2}\|\widetilde{w}\|_{H^{\frac12}}^2 - C\gamma^{\frac14}\|w\|_{L^2}^2 \geq \lambda_1\|w\|_{L^2}^2.$$

In conclusion, we have obtained

$$\|w\|_{L^2} \leq C\|v\|_{H^1},$$

and Lemma 4 then implies (3.20).

Now, we prove (3.21). Indeed, the integrability property (3.20) allows us to obtain the decay on $w(t,x)$ by monotonicity properties.

By the proof of Proposition 2, we have, for any $\lambda \in (0,1)$, for any t_0, $t \in (-\infty, t_0]$, $x_0 > 1$,

$$\begin{aligned} &\int w^2(t_0,x)\big(\varphi(x-x_0) - \varphi(-x_0)\big)dx \\ &\leq \int w^2(t,x)\left(\varphi(x-x_0-\lambda(t_0-t)) - \varphi(-x_0-\lambda(t_0-t))\right)dx \\ &\quad + C\int_t^{t_0} \frac{\|w(t')\|_{L^2}^2 dt'}{(x_0+\lambda(t_0-t'))^2}. \end{aligned} \tag{3.22}$$

The last term in (3.22) is treated as follows ($x_0 > 1$)

$$\int_t^{t_0} \frac{\|w(t')\|_{L^2}^2 dt'}{(x_0+\lambda(t_0-t'))^2} \leq Cx_0^{-\frac65}\int_{-\infty}^{+\infty} \frac{\|w(t')\|_{L^2}^2 dt'}{(1+(t_0-t'))^{\frac45}}$$

Thus, by (3.20) (applied to $w(t+t_0)$) and (3.3), letting $t \to -\infty$ in (3.22), we obtain

$$\int w^2(t_0)\left(\varphi(x-x_0) - \varphi(-x_0)\right)dx \leq Cx_0^{-\frac65}.$$

By the change of variable $x \to -x$, $t \to -t$, which leaves the equation invariant, we get:

$$\int w^2(t_0)\,(\varphi(x_0) - \varphi(x + x_0))\,dx \leq C x_0^{-\frac{6}{5}},$$

and thus, summing up the two estimates,

$$\int w^2(t_0)\,(\varphi(x - x_0) - \varphi(x + x_0) + \varphi(x_0) - \varphi(-x_0))\,dx \leq C x_0^{-\frac{6}{5}}.$$

We verify easily that for all $|x| > x_0 \geq 1$,

$$\begin{aligned}\varphi(x - x_0) &- \varphi(x + x_0) + \varphi(x_0) - \varphi(-x_0) \\ &\geq \varphi(0) - \varphi(2x_0) + \varphi(x_0) - \varphi(-x_0) \geq \tfrac{\pi}{2} - \arctan(2) > 0.\end{aligned} \tag{3.23}$$

Thus, for all $x_0 > 1$,

$$\int_{|x| \geq x_0} w^2(t_0) \leq C x_0^{-\frac{6}{5}}. \tag{3.24}$$

By integrating in x_0, we obtain the following estimate

$$\forall t \in \mathbb{R}, \quad \int |x| w^2(t) \leq C. \tag{3.25}$$

Thus Lemma 5 is proved.

Now, we claim that estimate (3.21) implies a gain of regularity on $w(t)$.

Lemma 6 (Gain of regularity on $w(t)$)**.** *Let $w \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^\infty(\mathbb{R}, L^2(\mathbb{R}))$ be a solution of* (3.1) *satisfying* (3.21). *Then, $w(t) \in C(\mathbb{R}, H^{\frac{1}{2}}(\mathbb{R}))$ and the following identity holds*

$$\begin{aligned}\int x w^2(t_2) - \int x w^2(t_1) = &- \int_{t_1}^{t_2} \int \left(2|D^{\frac{1}{2}} w|^2 + w^2 + w^2(xQ' - Q)\right) \\ &+ 2 \int_{t_1}^{t_2} \beta(t) \int x Q' w.\end{aligned} \tag{3.26}$$

End of the proof of Theorem 3 assuming Lemma 6. Note first that multiplying the equation of $w(t)$ by Q' and using $\int wQ' = 0$, we find $\left(\int (Q')^2\right) \beta(t) = \int w \mathcal{L}(Q'')$, so that

$$|\beta(t)| \leq C \|w\|_{L^2}. \tag{3.27}$$

Multiplying the equation of $w(t)$ by $\mathcal{L}w$ and using $\mathcal{L}Q' = 0$, we also have

$$\forall t \in \mathbb{R}, \quad (\mathcal{L}w(t), w(t)) = (\mathcal{L}w(0), w(0)).$$

By (3.26), the estimates on $\int |x|w^2(t)$ and on $\beta(t)$, and Lemma 5, we have

$$\int_{-\infty}^{+\infty} \frac{1}{(1+t^2)^{\frac{2}{5}}} \|w(t)\|^2_{H^{\frac{1}{2}}} dt < C.$$

This implies that for a sequence $t_n \to +\infty$, we have $\|w(t_n)\|_{H^{\frac{1}{2}}} \to 0$ as $n \to +\infty$.

Since $(\mathcal{L}w(t), w(t)) = \lim_{t_n \to \infty} (\mathcal{L}w(t_n), w(t_n))$, we obtain $(\mathcal{L}w(t), w(t)) = 0$ and so by the orthogonality conditions on $w(t)$ and Lemma 15, we finally obtain $\forall t$, $w(t) = 0$.

Proof of Lemma 6. Formally, identity (3.26) follows from multiplying equation (3.1) by xw, integration by parts and properties of the Hilbert transform. To justify (3.26), we use a regularization of $w(t)$.

We set $w_n = (1 - \frac{1}{n}\partial_x^2)^{-1} w$, so that for all t, $w_n(t) \to w(t)$ in $L^2(\mathbb{R})$ as $n \to +\infty$. Then, w_n satisfies the following equation

$$w_{nt} = (\mathcal{L}w_n)_x - \tfrac{1}{n}(1 - \tfrac{1}{n}\partial_x^2)^{-1}(2Q'w_{nx} + w_n Q'')_x + \beta(1 - \tfrac{1}{n}\partial_x^2)^{-1}Q'. \tag{3.28}$$

Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth nondecreasing function such that $h(x) = x$ if $x > 1$ and $h(x) = 0$ if $x < 0$. Then,

$$\begin{aligned}
\int h(x) w^2 &= \int h(x)(w_n - \tfrac{1}{n} w_{nxx})^2 \\
&= \int h(x) w_n^2 - \tfrac{2}{n} \int w_{nxx} w_n h(x) + \frac{1}{n^2} \int w_{nxx}^2 h(x) \\
&= \int h(x) w_n^2 + \tfrac{2}{n} \int w_{nx}^2 h(x) - \tfrac{1}{n} \int w_n^2 h''(x) + \frac{1}{n^2} \int w_{nxx}^2 h(x)
\end{aligned}$$

implies that

$$\int_{x>0} x w_n^2 \leq C \quad \text{and} \quad \int_{x>0} x(w - w_n)^2 \to 0 \quad \text{as } n \to +\infty. \tag{3.29}$$

The same holds true in the region $x < 0$.

For the functions w_n, we have the following identity, for any $t_1 < t_2$:

$$\begin{aligned}
\int x w_n^2(t_2) - \int x w_n^2(t_1) = &- \int_{t_1}^{t_2} \int \Big(2|D^{\frac{1}{2}} w_n|^2 + w_n^2 + w_n^2(xQ' - Q)\Big) dx dt \\
&+ \int_{t_1}^{t_2} \int \big(- \tfrac{2}{n} x (1 - \tfrac{1}{n}\partial_x^2)^{-1} (2Q' w_{nx} + w_n Q'')_x w_n \big) \\
&+ 2 \int_{t_1}^{t_2} \beta \int x((1 - \tfrac{1}{n}\partial_x^2)^{-1} Q') w_n dx dt.
\end{aligned} \tag{3.30}$$

Indeed, multiplying the equation of w_n by $Ag(\frac{x}{A})w_n$ where $g(x) = \arctan(x)$, we find

$$\begin{aligned}\int Ag(\tfrac{x}{A})w_n^2(t_2) - \int Ag(\tfrac{x}{A})w_n^2(t_1)\\ = -\int_{t_1}^{t_2}\int \Big(2|D^{\frac12}w_n|^2 g'(\tfrac{x}{A}) + 2D^{\frac12}w_n(D^{\frac12}(w_n g'(\tfrac{x}{A})) - D^{\frac12}(w_n)g'(\tfrac{x}{A}))\\ + 2Dw_n w_{nx} Ag(\tfrac{x}{A}) + w_n^2 g'(\tfrac{x}{A}) + w_n^2(Ag(\tfrac{x}{A})Q' - g'(\tfrac{x}{A})Q)\Big)dxdt\\ - 2\int_{t_1}^{t_2}\beta(t)\int x((1-\tfrac{1}{n}\partial_x^2)^{-1}Q)Ag(\tfrac{x}{A})w_n\\ - \frac{2}{n}\int_{t_1}^{t_2}\int((1-\tfrac{1}{n}\partial_x^2)^{-1}Q)(2Q'w_{nx} + w_nQ'')_x Ag(\tfrac{x}{A})w_n.\end{aligned}$$

Then, (3.30) is proved using Lemmas 3 and 14 (see the proof of Lemma 4 for similar arguments) and then passing to the limit as $A \to +\infty$ applying the Lebesgue convergence theorem.

From (3.30), we claim that for any t_1, t_2,

$$\limsup_{n\to+\infty}\int_{t_1}^{t_2}\|w_n(t)\|_{H^{\frac12}}^2 dt < +\infty. \tag{3.31}$$

Proof of (3.31). By Claim 1 (i), we have

$$\begin{aligned}\tfrac{1}{n}\int x(1-\tfrac{1}{n}\partial_x^2)^{-1}(2Q'w_{nx} + w_nQ'')_x w_n\\ =\tfrac{1}{n}\int w_n(1-\tfrac{1}{n}\partial_x^2)^{-1}(2xQ'w_{nxx} + 3xQ''w_{nx} + xQ^{(3)}w_n)\\ - \tfrac{2}{n^2}\int w_n(1-\tfrac{1}{n}\partial_x^2)^{-2}(2Q'w_{nx} + w_nQ'')_{xx} = \mathbf{I} + \mathbf{II}.\end{aligned}$$

As in the proof of Lemma 4 (control of $\mathbf{B}$), we have

$$|\mathbf{I}| \le \frac{C}{n^{\frac14}}\|w_n\|_{H^{\frac12}}\|w_n\|_{L^2}, \quad |\mathbf{II}| \le \frac{C}{n^{\frac12}}\|w_n\|_{L^2}^2. \tag{3.32}$$

From (3.30), (3.27), the L^2 bounds on $w(t)$ and $w_n(t)$ and (3.32) we obtain

$$\int_{t_1}^{t_2}\|w_n(t)\|_{H^{\frac12}}^2 dt \le C|t_2 - t_1| + \sup_t \int |x|w_n^2(t) + \frac{C}{n^{\frac14}}\int_{t_1}^{t_2}\|w_n\|_{H^{\frac12}}^2 dt.$$

For n large enough, we get $\int_{t_1}^{t_2}\|w_n(t)\|_{H^{\frac12}}^2 dt \le C$. Thus (3.31) is proved.

By the well-posedness of the equation of $w(t)$ in $H^{\frac12}$, we obtain $\forall t$, $w(t) \in H^{\frac12}$ and $w_n \to w$ in $H^{\frac12}$. Finally, from (3.29) and (3.32), we obtain (3.26) by passing to the limit as $n \to \infty$ in (3.30).

3.2. Positivity of a quadratic form related to the dual problem

In this section, we prove Proposition 3. The main ingredient is the following result.

Proposition 4. *There exists* $\lambda_0 > 0$ *such that for all* $z \in H^{\frac{1}{2}}$,

$$\int z(xQ)' = 0 \ \Rightarrow$$

$$(\widetilde{\mathcal{L}}z, z) = 2\int |D^{\frac{1}{2}}z|^2 + \int z^2 - \int (xQ' + Q)z^2 \geq \lambda_0 \|z\|^2_{H^{\frac{1}{2}}}. \tag{3.33}$$

Proof of Proposition 4. First, we introduce some notation. Recall that

$$\mathcal{L}f = -\mathcal{H}f' + f - Qf. \tag{3.34}$$

We define $S = (xQ)'$. Note that $S = \frac{d}{dc}Q_{c|c=1}$ and thus by differentiating the equation of Q_c with respect to c, and taking $c = 1$, we find $\mathcal{L}S = -Q$. Observe also that $\mathcal{L}Q = -\mathcal{H}Q' + Q - Q^2 = -\frac{1}{2}Q^2$, by the equation of Q. Now, we set $T = S - Q$. Then, $\mathcal{L}T = -Q + \frac{1}{2}Q^2 = (xQ)' = S$, by using the explicit expression $Q(x) = \frac{4}{1+x^2}$. We compute $\int TS = \int S^2 - \int QS$. Since $S = \mathcal{H}Q' = \frac{1}{2}Q^2 - Q$ (explicit computation), we have $\int S^2 = \int (Q')^2$ and $(Q')^2 = \frac{64x^2}{(1+x^2)^4} = Q^3 - \frac{1}{4}Q^4$, thus

$$\int (Q')^2 = \int Q^3 - \frac{1}{4}\int Q^4 = \int S^2 = \int (\frac{1}{2}Q^2 - Q)^2 = \frac{1}{4}\int Q^4 - \int Q^3 + \int Q^2,$$

we find $\int S^2 = \frac{1}{2}\int Q^2$. Moreover, $\int SQ = -\int xQQ' = \frac{1}{2}\int Q^2$, and so $\int TS = 0$. Finally, $\int TQ = -\int T\mathcal{L}S = -\int S^2$.

In conclusion, we have proved $((.,.)$ denotes the L^2 scalar product):

$$S = \tfrac{1}{2}Q^2 - Q = (xQ)', \ T = S - Q, \ \mathcal{L}Q = -\tfrac{1}{2}Q^2, \ \mathcal{L}S = -Q, \ \mathcal{L}T = S,$$
$$(S, Q) = \tfrac{1}{2}\int Q^2, \ (S, T) = 0, \ (T, Q) = -\int S^2. \tag{3.35}$$

Now, we claim the following.

Lemma 7. *There exists* $\lambda > 0$ *such that, for all* $\varepsilon > 0$, *if* $\int wS_\varepsilon = 0$, *where* $S_\varepsilon = S + \varepsilon Q$, *then* $(\mathcal{L}w, w) \geq 0$ *and* $(\widetilde{\mathcal{L}}w, w) \geq \lambda \|w\|^2_{H^{\frac{1}{2}}}$.

Proof of Lemma 7. Let $T_\varepsilon = T - \varepsilon S$ and $S_\varepsilon = S + \varepsilon Q$, then by (3.35): $\mathcal{L}T_\varepsilon = S_\varepsilon$ and

$$(\mathcal{L}T_\varepsilon, T_\varepsilon) = (S_\varepsilon, T_\varepsilon) = (S, T) + \varepsilon(-(S, S) + (T, Q)) - \varepsilon^2(S, Q) \leq -2\varepsilon(S, S) < 0.$$

Moreover, it is clear that if f_0, λ_0 denote respectively the first eigenfunction and first eigenvalue of $\mathcal{L}$ (see Lemma 15) we have $(S, f_0) = (\mathcal{L}T, f_0) = (T, \mathcal{L}f_0) = \lambda_0(xQ', f_0) \neq 0$, since $f_0 > 0$. Thus, by Lemma E.1 in [29], we obtain the first part of Lemma 7.

Now, we note that since $xQ' > 0$,

$$(3.36)\quad \begin{aligned}(\widetilde{\mathcal{L}}w, w) &= 2\int |D^{\frac12} w|^2 + \int w^2 - \int (xQ'+Q)w^2\\ &\geq 2\int |D^{\frac12} w|^2 + \int w^2 - \int Qw^2 = \int |D^{\frac12} w|^2 + (\mathcal{L}w, w).\end{aligned}$$

Using the inequality $\|w\|_{L^4}^2 \leq C\|w\|_{L^2}\|D^{\frac12}w\|_{L^2}$ (see (A.1)) and Cauchy-Schwarz' inequality, we have, for some constant $C_0 > 0$,

$$\int Qw^2 \leq C\|w^2\|_{L^2} \leq C_0 \int |D^{\frac12} w|^2 + \frac{1}{2}\int w^2.$$

Thus, for $\delta_0 > 0$ such that $2 - C_0\delta_0 > 1 - \frac{\delta_0}{2}$, we have

$$\begin{aligned}(\widetilde{\mathcal{L}}w, w) &\geq (2 - C_0\delta_0)\int |D^{\frac12}w|^2 + (1 - \tfrac12\delta_0)\int w^2 - (1-\delta_0)\int Qw^2\\ &\geq (1-\delta_0)(\mathcal{L}w, w) + \tfrac{\delta_0}{2}\|w\|^2_{H^{\frac12}} \geq \tfrac{\delta_0}{2}\|w\|^2_{H^{\frac12}},\end{aligned}$$

provided $\int wS_\varepsilon = 0$.

Now, we finish the proof of Proposition 4. Let $z \in H^{\frac12}$ be such that $\int zS = \int z(xQ)' = 0$. Let $w = z + aQ$, where $\int wS_\varepsilon = 0$, $0 < \varepsilon < \varepsilon_0$, where ε_0 is to be chosen small enough. In particular, we have

$$\begin{aligned}\int wS_\varepsilon &= \int zS_\varepsilon + a\int QS_\varepsilon = \varepsilon\int zQ + a\int SQ + a\varepsilon\int Q^2\\ &= \varepsilon\int zQ + a\left(\tfrac12 + \varepsilon\right)\int Q^2 = 0,\end{aligned}$$

and so $|a| \leq \frac{2}{\|Q\|_{L^2}}\varepsilon\|z\|_{L^2}$, and $\|w\|_{L^2} \leq 2\|z\|_{L^2}$ for ε_0 small enough. Similarly, we have $\|z\|_{L^2} \leq 2\|w\|_{L^2}$, by possibly choosing a smaller ε_0. By Lemma 7, we obtain

$$\frac{\lambda}{2}\|z\|^2_{H^{\frac12}} \leq \lambda\|w\|^2_{H^{\frac12}} \leq (\widetilde{\mathcal{L}}w, w) = (\widetilde{\mathcal{L}}z, z) + a^2(\widetilde{\mathcal{L}}Q, Q) + 2a(\widetilde{\mathcal{L}}Q, z).$$

For ε_0 small, we get $(\widetilde{\mathcal{L}}z, z) \geq \frac{\lambda}{4}\|z\|^2_{H^{\frac12}}$.

Now, we are in a position to prove Proposition 3.

Proof of Proposition 3. In Proposition 3, we want to prove that for B large and γ small, and for some $\lambda_1 > 0$, for all t,

$$(\widetilde{\mathcal{L}}z(t), z(t)) \geq \lambda_1\|z(t)\|^2_{H^{\frac12}}, \quad \text{for} \quad z(t) = v(t)\sqrt{g'(\frac{x}{(B+t^2)^\alpha})},$$

where $v = (1-\gamma\partial_x^2)^{-1}(\mathcal{L}w)$. Formally, if $B = +\infty$ and $\gamma = 0$, we have $z(t) = v(t) = \mathcal{L}w$ and $0 = \int wQ = -\int w\mathcal{L}S = -\int zS$, and the result follows from Proposition 4. Now, we justify that the result persists for large values of B and small values of γ.

Let $S_{B,\gamma}(t) = (g'(\frac{x}{(B+t^2)^\alpha}))^{-\frac{1}{2}}(S-\gamma S'')$. Then,

$$\mathcal{L}((1-\gamma\partial_x^2)^{-1}(\sqrt{g'(\frac{x}{(B+t^2)^\alpha}})S_{B,\gamma}(t)) = -Q$$

and so $\int S_{B,\gamma}(t)z = -\int wQ = 0$. Now, we control $S_{B,\gamma}(t) - S$:

$$\begin{aligned} S_{B,\gamma}(t) - S &= \sqrt{1+\tfrac{x^2}{(B+t^2)^\alpha}}(S-\gamma S'') - S \\ &= \Big(\sqrt{1+\tfrac{x^2}{(B+t^2)^\alpha}} - 1\Big)S - \gamma\sqrt{1+\tfrac{x^2}{(B+t^2)^\alpha}}S''. \end{aligned}$$

Thus, by elementary estimates and the expression of S, we obtain:

$$|S_{B,\gamma}(t,x) - S(x)| \leq \big(B^{-\frac{\alpha}{2}} + \gamma\big)\frac{1}{1+|x|}.$$

It follows that

$$\left|\int Sz(t)\right| = \left|\int (S - S_{B,\gamma}(t))z(t)\right| \leq \big(B^{-\frac{\alpha}{2}} + \gamma\big)\|z\|_{L^2}.$$

Setting $z = z_1 + aQ$, where $\int z_1 S = 0$ and $|a| \leq \big(B^{-\frac{\alpha}{2}}+\gamma\big)\|z\|_{L^2}$, we conclude the proof of Proposition 3 as at the end of the proof of Proposition 4, for B large enough and γ small enough.

4. Proof of asymptotic stability - Theorem 1

In this section, we first prove that Theorem 2 implies Theorem 1. Then, we prove that Theorem 3 (proved in Section 3) implies Theorem 2.

4.1. Proof of Theorem 1 assuming Theorem 2

We follow the strategy of [16], [17], the main idea being to use monotonicity type arguments (such as Proposition 1) to prove that a limiting solution of (1.1) has uniform decay in space. See also [18] for similar use of monotonicity arguments.

We consider a solution $u(t)$ of (1.1) in $H^{\frac{1}{2}}$ which satisfies

$$\|u_0 - Q\|_{H^{\frac{1}{2}}} = \alpha < \alpha_0, \qquad \text{for } \alpha_0 > 0 \text{ small enough.}$$

By the stability property, for all $t \in \mathbb{R}$,

$$\inf_y \|u(t) - Q(. - y)\|_{H^{\frac{1}{2}}} \leq C\alpha.$$

1. *Decomposition of $u(t)$ around the asymptotic soliton.* First, we determine the parameter $c^+ > 0$. It is given by the amount of L^2 norm that remains on the region $x > \frac{t}{10}$ asymptotically as $t \to +\infty$. Let φ be as in (2.6), with $A > 1$ so that Proposition 1 holds. Let

$$c^+ = \frac{1}{\pi \int Q^2} \limsup_{t\to+\infty} \int u^2(t,x)\, \varphi(x - \tfrac{t}{10})\, dx. \tag{4.1}$$

From the stability property, $|c^+ - 1| \leq C\alpha_0$ ($\lim_{+\infty} \varphi = \pi$). Using Lemma 1 to decompose $u(t)$ around Q_{c^+}, we consider the following decomposition of $u(t)$

$$\begin{aligned} &u(t,x) = Q_{c^+}(x - \rho(t)) + \eta(t, x - \rho(t)),\\ &\int Q'_{c^+}\eta(t,x)dx = 0, \quad \sup_t \|\eta(t)\|_{H^{\frac{1}{2}}} \leq K\alpha_0. \end{aligned} \tag{4.2}$$

In what follows, we consider $\alpha_0 > 0$ small enough, so that the following holds (by (2.2)):

$$\forall t, \quad \frac{99}{100} \leq \rho'(t) \leq \frac{101}{100}, \quad \frac{99}{100} \leq c^+ \leq \frac{101}{100}. \tag{4.3}$$

2. *Monotonicity arguments.* We claim the following estimates:

Lemma 8. (Asymptotics on $u(t)$).

$$\forall y_0 > 1, \quad \limsup_{t\to+\infty} \int u^2(t,x)\varphi(x - y_0 - \rho(t))dx \leq \frac{C}{y_0}, \tag{4.4}$$

(4.5)
$$\forall y_0 > 1, \quad \limsup_{t\to+\infty} \int u^2(t,x)(\varphi(x-\rho(t)+\tfrac{t}{10}) - \varphi(x-\rho(t)+y_0))dx \leq \frac{C}{y_0},$$

$$\lim_{t\to+\infty} \int u^2(t,x)(\varphi(x-\rho(t)+\tfrac{19}{20}t) - \varphi(x-\rho(t)+\tfrac{t}{10}))dx = 0, \tag{4.6}$$

$$\lim_{t\to+\infty} \int u^2(t,x)\varphi(x-\rho(t)+\tfrac{t}{10})dx = c^+\pi \int Q^2. \tag{4.7}$$

Proof of Lemma 8. Monotonicity property on the right of the soliton. By (2.7), with $\lambda = \frac{1}{2}$, we have, for all $y_0 > 1$,

$$\int u^2(t,x)\varphi(x - y_0 - \rho(t))dx \leq \int u^2(0,x)\varphi(x - y_0 - \rho(0) - \tfrac{1}{2}t)dx + \frac{C}{y_0}.$$

Since $\lim_{t\to+\infty} \int u^2(0,x)\varphi(x - y_0 - \rho(0) - \frac{1}{2}t)dx = 0$, we obtain (4.4).

Monotonicity property on the left of the soliton. By (2.8), with $\lambda = \frac{19}{20}$ and $x_0 = \frac{19}{20}t'$, we have for all $0 \le t' \le t$,

$$\int u^2(t,x)\varphi(x - \rho(t) + \tfrac{19}{20}t)dx \le \int u^2(t',x)\varphi(x - \rho(t') + \tfrac{19}{20}t')dx + \frac{C}{t'}.$$

It follows that $\int u^2(t,x)\varphi(x - \rho(t) + \frac{19}{20}t)dx$ has a limit as $t \to +\infty$. Set

$$\ell = \lim_{t\to+\infty} \int u^2(t,x)\varphi(x - \rho(t) + \tfrac{19}{20}t)dx, \qquad \ell \ge c^+\pi \int Q^2.$$

Applying (2.8) with $\lambda = \frac{100}{99}(\frac{19}{20} - \frac{1}{1000}) < 1$ and $x_0 = \frac{t}{1000}$, we find

$$\int u^2(t,x)\varphi(x - \rho(t) + \tfrac{19}{20}t)dx \le \int u^2(\tfrac{t}{100},x)\varphi(x - \rho(\tfrac{t}{100}) + \tfrac{t}{1000})dx + \frac{C}{t}.$$

Since

$$\limsup_{t\to+\infty} \int u^2(\tfrac{t}{100},x)\varphi(x - \rho(\tfrac{t}{100}) + \tfrac{1}{10}\tfrac{t}{100})dx \le c^+\pi \int Q^2,$$

we obtain $c^+\pi \int Q^2 = \ell$ and (4.6).

Fix $y_0 > 1$, pick $\lambda = \frac{1}{2}$. Consider $t_2 > t$ and define $t_1 = \frac{4}{5}t_2 + 2y_0$, so that for t large, $t_1 < t_2$. But then, by (2.8),

$$\begin{aligned}\int u^2(t_2,x)\varphi(x-\rho(t_2)+\tfrac{t_2}{10})dx &= \int u^2(t_2,x)\varphi(x-\rho(t_2)+\lambda(t_2-t_2)+y_0)dx\\ &\le \int u^2(t_1,x)\varphi(x-\rho(t_1)+y_0)dx + \frac{C}{y_0}.\end{aligned}$$

In light of (4.6) and the existence of ℓ, (4.5) follows. Thus Lemma 8 is proved.

3. *Construction of a compact limit object.* Let $t_n \to +\infty$. By the uniform bound on $u(t)$ in $H^{\frac{1}{2}}$, there exist $\widetilde{u}_0 \in H^{\frac{1}{2}}$ and a subsequence, still denoted by (t_n), such that

$$u(t_n, . + \rho(t_n)) \rightharpoonup \widetilde{u}_0 \quad \text{in } H^{\frac{1}{2}} \text{ weak as } n \to +\infty.$$

Consider $\widetilde{u}(t)$ the global $H^{\frac{1}{2}}$ solution of (1.1) such that $\widetilde{u}(0) = \widetilde{u}_0$. By (4.2), $\|\widetilde{u}_0 - Q_{c^+}\| \le C\alpha_0$ and thus by the stability property, $\sup_t \inf_y \|\widetilde{u}(t) - Q(. - y)\|_{H^{\frac{1}{2}}} \le C\alpha_0$. Let $\widetilde{\rho}(t)$, $\widetilde{\eta}(t)$ correspond to the decomposition of $\widetilde{u}(t)$ around Q_{c^+} given by Lemma 1.

By Theorem 5 below and Remark 2, for all $t \in \mathbb{R}$, we have

$$\begin{aligned}&u(t_n + t, . + \rho(t_n)) \rightharpoonup \widetilde{u}(t) \quad \text{in } H^{\frac{1}{2}} \text{ weak},\\ &\rho(t_n + t) - \rho(t_n) \to \widetilde{\rho}(t) \quad \text{as } n \to +\infty.\end{aligned}$$

From weak convergence and Lemma 8, we claim the following decay estimate on $\widetilde{u}(t)$:

$$\forall y_0 > 1, \forall t \in \mathbb{R}, \quad \int_{|x|>y_0} \widetilde{u}^2(t, x + \widetilde{\rho}(t))dx \leq \frac{C}{y_0}. \tag{4.8}$$

Indeed, first, from (4.4), for any fixed $y_0 > 1$, $t \in \mathbb{R}$, we have

$$\limsup_{n\to+\infty} \int u^2(t+t_n, x+\rho(t_n))\varphi(x-\rho(t_n+t)+\rho(t_n)-y_0)dx \leq \frac{C}{y_0},$$

and so by weak convergence

$$\int \widetilde{u}^2(t,x)\varphi(x-\widetilde{\rho}(t)-y_0)dx \leq \frac{C}{y_0}.$$

Second, from (4.5), for fixed $t \in \mathbb{R}$,

$$\limsup_{n\to+\infty} \int u^2(t+t_n, x+\rho(t_n))\times$$
$$\times \left(\varphi(x-\rho(t_n+t)+\rho(t_n)+\tfrac{t+t_n}{10}) - \varphi(x-\rho(t_n+t)+\rho(t_n)+y_0)\right)dx \leq \frac{C}{y_0}.$$

Note that for fixed t, y_0, we have

$$\lim_{n\to+\infty} \varphi(x-\rho(t_n+t)+\rho(t_n)+\tfrac{t+t_n}{10}) - \varphi(x-\rho(t_n+t)+\rho(t_n)+y_0)$$
$$= \pi - \varphi(x-\widetilde{\rho}(t)+y_0) = \varphi(-x+\widetilde{\rho}(t)-y_0).$$

Thus, we obtain

$$\int \widetilde{u}^2(t,x)\varphi(-x+\widetilde{\rho}(t)-y_0)dx \leq \frac{C}{y_0}.$$

Finally, from (4.4)–(4.7), for any $y_0 > 1$, we have

$$\lim_{n\to+\infty} \left| \int u^2(t_n,x)(\varphi(x-\rho(t_n)+y_0)-\varphi(x-\rho(t_n)-y_0))dx - c^+\pi \int Q^2 \right| \leq \frac{C}{y_0}.$$

Thus, by L^2_{loc} convergence, for any $y_0 > 1$,

$$\left| \int \widetilde{u}_0^2(x)(\varphi(x+y_0)-\varphi(x-y_0))dx - c^+\pi \int Q^2 \right| \leq \frac{C}{y_0}.$$

Passing to the limit $y_0 \to +\infty$, we obtain

$$\|\widetilde{u}_0\|_{L^2} = \|\widetilde{u}(t)\|_{L^2} = \sqrt{c^+}\|Q\|_{L^2} = \|Q_{c^+}\|_{L^2}.$$

4. *Conclusion by Theorem* 2. From Theorem 2, it follows that for some c_1 close to c^+ and x_1 close to 0, we have

$$\widetilde{u}(t,x) \equiv Q_{c_1}(x - x_1 - c_1 t).$$

But $\|\widetilde{u}(t)\|_{L^2} = \|Q_{c^+}\|_{L^2}$ implies that $c_1 = c^+$. Moreover, $\widetilde{\rho}(0) = 0$ and $\widetilde{u}(0) = Q_{c^+}(x - x_1) = Q_{c^+}(x) + \widetilde{\eta}(0,x)$ where x_1 is small and $\int \widetilde{\eta}(0) Q'_{c^+} = 0$ imply $x_1 = 0$. In conclusion, $\widetilde{u}_0 = Q_{c^+}$.

By a standard argument and (4.4), (4.7), we have obtained

$$u(t, . + \rho(t)) \rightharpoonup Q_{c^+} \quad \text{in } H^{\frac{1}{2}} \text{ weak as } t \to +\infty,$$

$$\lim_{t\to+\infty} \int_{x > \frac{t}{10}} |u(t,x) - Q_{c^+}(x - \rho(t))|^2 dx = 0. \tag{4.9}$$

Thus Theorem 1 is a consequence of Theorem 2.

4.2. Proof of Theorem 2

First, we note that it is sufficient to prove Theorem 2 in the case $\int u_0^2 = \int Q^2$. Indeed, for u_0 satisfying the assumptions of Theorem 2, set

$$c_1 = \frac{\int u_0^2}{\int Q^2} \quad \text{and} \quad \widetilde{u}(t) = \frac{1}{c_1} u\Big(\frac{1}{c_1^2} t, \frac{1}{c_1} x\Big).$$

Then, $|c_1 - 1| \leq C\alpha_0$ and $\widetilde{u}$ satisfies (1.1), $\int \widetilde{u}^2 = \int Q^2$ and $\|\widetilde{u}_0 - Q\|_{H^{\frac{1}{2}}} \leq C\alpha_0$. Thus, by the stability property –see Introduction– for all t, there exists $y(t)$ such that $\sup_t \|\widetilde{u}(t) - Q(. - y(t))\|_{H^{\frac{1}{2}}} \leq C'\alpha_0$. Moreover, $\widetilde{u}(t)$ also satisfies (1.9). If we prove $\widetilde{u}(t,x) = Q(x - t - x_0)$, with $|x_0| \leq C\alpha_0$, the result follows for $u(t)$.

The proof of Theorem 2 is by contradiction. Assume that there exists a sequence $u_n(t)$ of $H^{\frac{1}{2}}$ solutions of (1.1) such that

$$\sup_{t\in\mathbb{R}} \|u_n(t) - Q(. - \rho_n(t))\|_{L^2} \to 0 \quad \text{as } n \to +\infty, \tag{4.10}$$

$$\int u_n^2(0) = \int Q^2, \qquad \eta_n \not\equiv 0, \tag{4.11}$$

$$\forall n, \forall \varepsilon > 0, \exists A_{n,\varepsilon} > 0, \text{ s.t. } \forall t \in \mathbb{R}, \int_{|x| > A_{n,\varepsilon}} u_n^2(t, x + \rho_n(t)) dx < \varepsilon, \tag{4.12}$$

where $\rho_n(t)$ and $\eta_n(t)$ are defined from $u_n(t)$ by Lemma 1. Note that $\int u_n^2(0) = \int Q^2$ implies

$$\forall n, \forall t, \quad \int \eta_n^2(t) = -2 \int \eta_n(t) Q. \tag{4.13}$$

Define

$$0 \not\equiv b_n = \sup_t \|\eta_n(t)\|_{L^2} \to 0 \quad \text{as } n \to +\infty. \tag{4.14}$$

Then, there exists t_n such that $\|\eta_n(t_n)\|_{L^2} \geq \frac{1}{2} b_n$. We set

$$w_n(t,x) = \frac{\eta_n(t_n + t, x)}{b_n}.$$

For such a sequence w_n, we claim the following result.

Proposition 5. (Weak convergence of renormalized solutions). *There exists a subsequence of (w_n) denoted by $(w_{n'})$ and $w \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^\infty(\mathbb{R}, L^2(\mathbb{R}))$ such that*

$$\forall t \in \mathbb{R}, \quad w_{n'}(t) \rightharpoonup w(t) \quad \textit{in } L^2 \textit{ weak as } n \to +\infty.$$

Moreover, $w(t)$ satisfies for some continuous function $\beta(t)$:

$$\begin{aligned} & w_t = (\mathcal{L}w)_x + \beta(t) Q' \quad \textit{on } \mathbb{R} \times \mathbb{R}, \\ & w(0) \neq 0, \quad \int wQ = \int wQ' = 0, \\ & \forall t \in \mathbb{R}, \forall x_0 > 1, \quad \int_{|x| > x_0} w^2(t,x)\,dx \leq \frac{C}{x_0}. \end{aligned}$$

Proposition 5 is in contradiction with Theorem 3. Thus, for $\alpha_0 > 0$ small, for $u(t)$ satisfying the assumptions of Theorem 2, we have $\eta \equiv 0$ so that $\rho'(t) = 1$ (by Lemma 1) and $u(t,x) = Q(x - \rho(0) - t)$, with $|\rho(0)| \leq C\alpha_0$.

Therefore, we are reduced to prove Proposition 5.

Proof of Proposition 5. One can actually prove a strong L^2 convergence result. See the end of the proof.

Note that the main point in Proposition 5 is the fact that $w \neq 0$. This requires a strong convergence in L^2 for some suitable t.

Decay estimate. From Proposition 2, we have

$$\begin{aligned} & \int \eta_n^2(t_0,x)(\varphi(x - x_0) - \varphi(-x_0))\,dx \\ & \leq \int \eta_n^2(t,x)(\varphi(x - x_0 - \lambda(t_0 - t)) - \varphi(-x_0 - \lambda(t_0 - t)))\,dx + \frac{Cb_n^2}{x_0}. \end{aligned}$$

Letting $t \to -\infty$ and using (4.12), we obtain, for any $x_0 > 1$,

$$\int \eta_n^2(t_0,x)(\varphi(x - x_0) - \varphi(-x_0))\,dx \leq \frac{Cb_n^2}{x_0}.$$

Similarly, arguing on $\eta_n(-t,-x)$, for any $x_0>1$,

$$\int \eta_n^2(t_0,x)(\varphi(x_0)-\varphi(x+x_0))dx \leq \frac{Cb_n^2}{x_0},$$

which gives, by (3.23), similarly as in the proof of (3.24):

$$\text{(4.15)}\quad \forall x_0>1,\quad \int_{|x|>x_0} \eta_n^2(t,x)dx \leq \frac{Cb_n^2}{x_0} \quad\text{and}\quad \int_{|x|>x_0} w_n^2(t,x)dx \leq \frac{C}{x_0}.$$

Local smoothing estimate on w_n. Let φ be defined in (2.6) for a fixed value of A ($A=1$ for example). Then,

$$\text{(4.16)}\qquad \int_0^1 \int |D^{\frac{1}{2}}(w_n(t,x)\sqrt{\varphi'(x)})|^2 dxdt \leq C.$$

Proof of (4.16). First, we claim the following estimate:

$$\text{(4.17)}\quad \frac{1}{2}\frac{d}{dt}\int \eta_n^2\varphi \leq -\frac{1}{2}\int |D^{\frac{1}{2}}(\eta_n\sqrt{\varphi'})|^2 + C\int \eta_n^2 \leq -\frac{1}{2}\int |D^{\frac{1}{2}}(\eta_n\sqrt{\varphi'})|^2 + Cb_n^2.$$

Thus, by integration,

$$\text{(4.18)}\qquad \begin{aligned} &\forall t\in\mathbb{R},\quad \int_t^{t+1}\int |D^{\frac{1}{2}}(\eta_n\sqrt{\varphi'})|^2 dxdt \leq Cb_n^2 \quad\text{and}\\ &\qquad\qquad \int_t^{t+1}\int |D^{\frac{1}{2}}(w_n\sqrt{\varphi'})|^2 dxdt \leq C. \end{aligned}$$

Now, we justify (4.17). Using direct computations, Lemma 3, (2.2) and then $|\int \eta_n^3\varphi'| \leq C\int |\eta|^3 \leq C\int \eta^2$ (by (A.1)), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\int \eta_n^2\varphi \\ &= -\int (\mathcal{L}\eta_n - \tfrac{1}{2}\eta_n^2)(\eta_{nx}\varphi + \eta_n\varphi') + (\rho_n'-1)\int Q'\eta_n\varphi - \frac{1}{2}(\rho_n'-1)\int \eta_n^2\varphi' \\ &= \int ((\mathcal{H}\eta_{nx})\eta_{nx}\varphi + (\mathcal{H}\eta_{nx})\eta_n\varphi') - \frac{1}{2}\int \eta_n^2\varphi' + \frac{1}{2}\int \eta_n^2(-Q'\varphi + Q\varphi') \\ &\quad + \frac{1}{3}\int \eta_n^3\varphi' + (\rho_n'-1)\int Q'\eta_n\varphi - \frac{1}{2}(\rho_n'-1)\int \eta_n^2\varphi' \\ &\leq \int (\mathcal{H}\eta_{nx})\eta_n\varphi' + C\int \eta_n^2. \end{aligned}$$

Using (A.3) and then (A.1), we have

$$
\begin{aligned}
-\int (\mathcal{H}\eta_{nx})\eta_n\varphi' &= \int \eta_n D(\eta_n\varphi') \\
&= \int \eta_n\sqrt{\varphi'}D(\eta_n\sqrt{\varphi'}) + \int \eta_n\left(D(\eta_n\varphi') - \sqrt{\varphi'}D(\eta_n\sqrt{\varphi'})\right) \\
&\geq \int |D^{\frac12}(\eta_n\sqrt{\varphi'})|^2 - C\|\eta_n\|_{L^2}\|D(\eta_n\varphi') - \sqrt{\varphi'}D(\eta_n\sqrt{\varphi'})\|_{L^2} \qquad (4.19)\\
&\geq \int |D^{\frac12}(\eta_n\sqrt{\varphi'})|^2 - C\|\eta_n\|_{L^2}\|\eta_n\sqrt{\varphi'}\|_{L^4}\|D\sqrt{\varphi'}\|_{L^4} \\
&\geq \frac12\int |D^{\frac12}(\eta_n\sqrt{\varphi'})|^2 - C\|\eta_n\|_{L^2}^2.
\end{aligned}
$$

(Note that we have used $\|D\sqrt{\varphi'}\|_{L^4} < +\infty$.) Thus (4.17) is proved.

Compactness in L^2 for some time. From the equation of η_n and (2.2), it follows that

$$
\frac{d}{dt}\int \eta_n^2 = -\frac12\int Q'\eta_n^2 + (\rho_n' - 1)\int Q'\eta_n \quad \text{and so} \quad \left|\frac{d}{dt}\int \eta_n^2\right| \leq C_0\int \eta_n^2.
$$

In particular, by the definition of t_n, $\forall t\in[0,1]$, $\int \eta_n^2(t+t_n) \geq e^{-C_0}b_n^2$ and so

$$
\forall t\in[0,1],\quad \|w_n(t)\|_{L^2} \geq e^{-\frac12 C_0} = \delta > 0. \qquad (4.20)
$$

It follows from (4.16) that for all n, there exists $\tau_n\in[0,1]$ such that $\int |D^{\frac12}(w_n(\tau_n)\sqrt{\varphi'})|^2 \leq C$. Thus, there exists a subsequence of (w_n) (still denoted by (w_n)) and $s_0\in[0,1]$, $W\in H^{\frac12}$ such that

$$
w_n(\tau_n)\sqrt{\varphi'} \rightharpoonup W \quad \text{in } H^{\frac12} \text{ weak}, \qquad \tau_n\to s_0 \quad \text{as } n\to+\infty.
$$

But (by possibly extracting a further subsequence), there exists $w_{s_0}\in L^2$ such that

$$
\tau_n\to s_0,\quad w_n(\tau_n)\rightharpoonup w_{s_0} \quad \text{in } L^2 \text{ weak as } n\to+\infty.
$$

It follows that $W = w_{s_0}\sqrt{\varphi'}$. Since $\sqrt{\varphi'}>0$ on $\mathbb{R}$, we get

$$
w_n(\tau_n)\to w_{s_0} \quad \text{in } L^2_{loc} \text{ as } n\to+\infty.
$$

By (4.15) and (4.20), we finally get

$$
w_n(\tau_n)\to w_{s_0} \quad \text{in } L^2 \text{ as } n\to+\infty, \qquad \int w_{s_0}Q' = 0, \quad w_{s_0}\neq 0. \qquad (4.21)
$$

Note also that from (4.13) and $\int \eta_n Q' = 0$, we have

$$
\int w_{s_0}Q = \int w_{s_0}Q' = 0. \qquad (4.22)
$$

Weak convergence for all time. Consider $\widetilde{w}(t) \in C(\mathbb{R}, L^2(\mathbb{R}))$ the unique solution of

$$\widetilde{w}_t = (\mathcal{L}\widetilde{w})_x \quad \text{on } \mathbb{R}\times\mathbb{R}, \quad \widetilde{w}(s_0) = w_{s_0}, \quad \text{on } \mathbb{R}.$$

(It is clear by a standard energy estimate and regularization arguments that the corresponding Cauchy problem is well-posed in L^2).

Now, to obtain weak convergence, we need to remove some terms from the equation of w_n, following some arguments in [16], Lemma 8 and beginning of proof of Lemma 11. We write

$$\begin{aligned} w_{nt} &= (\mathcal{L}w_n - \tfrac{b_n}{2} w_n^2)_x + \frac{1}{b_n}(\rho_n' - 1)(Q + b_n w_n)_x \\ &= (\mathcal{L}w_n)_x - \tfrac{b_n}{2}(w_n^2)_x + \beta_n Q' + b_n F_n' + b_n \widetilde{\beta}_n (w_n)_x, \end{aligned}$$

where

$$\beta_n = \frac{1}{\int (Q')^2} \int w_n \mathcal{L}(Q''), \quad \widetilde{\beta}_n = \frac{1}{b_n}(\rho_n' - 1), \quad F_n = \frac{1}{b_n}(\widetilde{\beta}_n - \beta_n) Q.$$

Set $\widetilde{w}_n(t) = w_n(t) - Q' \int_{\tau_n}^t \beta_n(s) ds$. Then, the equation of $\widetilde{w}_n(t)$ writes

$$\widetilde{w}_{nt} = (\mathcal{L}\widetilde{w}_n)_x - \tfrac{b_n}{2}(w_n^2)_x + b_n F_n' + b_n \widetilde{\beta}_n (\widetilde{w}_n)_x + b_n \widetilde{\beta}_n Q'' \int_{\tau_n}^t \beta_n(s) ds.$$

We claim the following weak convergence result.

Lemma 9. *For all* $t \in \mathbb{R}$,

$$\widetilde{w}_n(t) \rightharpoonup \widetilde{w}(t) \quad \text{in } L^2 \text{ weak.}$$

Assuming this lemma, from (2.5), we have, for all t,

$$\widetilde{\beta}_n(t) \to \widetilde{\beta}(t) = \frac{1}{\int (Q')^2} \int \widetilde{w} \mathcal{L}(Q''), \quad \int_{\tau_n}^t \widetilde{\beta}_n(s) ds \to \int_{s_0}^t \widetilde{\beta}(s) ds.$$

Set $w(t) = \widetilde{w}(t) + Q' \int_{s_0}^t \widetilde{\beta}(s) ds$. Then, $w(t)$ solves

$$w_t = (\mathcal{L}w)_x + \widetilde{\beta} Q',$$

and $w(s_0) = w_{s_0} \neq 0$. Moreover, for all $t \in \mathbb{R}$,

$$w_n(t) \rightharpoonup w(t) \quad \text{in } L^2 \text{ weak.}$$

Finally, from (4.13) and $\int \eta_n Q' = 0$, we have $\int w(t) Q = \int w(t) Q' = 0$, and by weak convergence and (4.15), we have

$$\forall x_0 > 1, \forall t, \quad \int_{|x| > x_0} w^2(t,x) dx \leq \frac{C}{x_0}.$$

Thus, we are reduced to prove Lemma 9.

Proof of Lemma 9. Set

$$G_{1,n} = -\frac{1}{2}w_n^2, \quad G_{2,n} = F_n + \widetilde{\beta}_n \widetilde{w}_n + \widetilde{\beta}_n Q' \int_{\tau_n}^t \beta_n(s)ds, \quad G_n = G_{1,n} + G_{2,n}.$$

Observe that

$$\|G_{1,n}\|_{L^1} + \|G_{2,n}\|_{L^2} \leq C(t), \quad \text{with } C(t) \text{ bounded on bounded intervals.}$$

Let $T \in \mathbb{R}$. By $\sup_t \|w_n(t)\|_{L^2} \leq C$ and the expression of $\widetilde{w}_n$, we have $\sup_{[-T,T]} \|\widetilde{w}_n(t)\|_{L^2} \leq C_T$.

Let $g \in C_0^\infty(\mathbb{R})$ and let v solve the problem

$$\begin{cases} \partial_t v = \mathcal{L}(v_x) \\ v_{|t=T} = g. \end{cases}$$

Then

$$\begin{aligned}
&\int (\widetilde{w}_n - \widetilde{w})(T)g(x)dx - \int (w_n(\tau_n) - w(\tau_n))v(\tau_n)dx \\
&\quad = \int_{\tau_n}^T \int \partial_t((\widetilde{w}_n - \widetilde{w})(t)v(t,x))dxdt \\
&\quad = \int_{\tau_n}^T \int ((\mathcal{L}\widetilde{w}_n)_x - (\mathcal{L}\widetilde{w})_x + b_n(G_n)_x)v(t,x) + (\widetilde{w}_n - \widetilde{w})(\mathcal{L}v)_x dxdt \\
&\quad = -b_n \int_{\tau_n}^T \int G_n v_x(t,x)dxdt.
\end{aligned}$$

The energy method gives

$$\|v\|_{L^\infty([\tau_n,T],L^2(\mathbb{R}))} + \|v_x\|_{L^\infty([\tau_n,T]\times\mathbb{R})} + \|v_x\|_{L^\infty([\tau_n,T],L^2(\mathbb{R}))} \leq C.$$

Moreover, by continuity of $t \mapsto w(t)$ in L^2,

$$\begin{aligned}
&\lim_{n\to+\infty} \int (w_n(\tau_n) - w(\tau_n))v(\tau_n)dx \\
&= \lim_{n\to+\infty} \int (w_n(\tau_n) - w(s_0))v(\tau_n)dx + \lim_{n\to+\infty} \int (w(s_0) - w(\tau_n))v(\tau_n)dx = 0.
\end{aligned}$$

Thus,

$$\widetilde{w}_n(T) \rightharpoonup \widetilde{w}(T) \quad \text{as } n \to +\infty.$$

and the proof of Lemma 9 is concluded.

Alternate proof by strong L^2 convergence for all time. Now, we use Theorem 6 in Section 6 to prove strong L^2 convergence of the sequence $(w_n(t))$ for all t.

Let $T > 0$. Set

$$\begin{aligned}\zeta_n(t,x) &= w_n(t, x-\rho_n(t)+\rho_n(0)) \\ &\quad + \frac{1}{b_n}[Q(x-(\rho_n(t)-\rho_n(0))) - Q(x-t)],\end{aligned} \tag{4.23}$$

so that

$$\begin{aligned}u_n(t, x+\rho_n(0)) &= Q(x-\rho_n(t)+\rho_n(0)) + b_n w_n(t, x-\rho_n(t)+\rho_n(0)) \\ &= Q(x-t) + b_n\zeta_n(t,x),\end{aligned}$$

and ζ_n satisfies

$$\begin{gathered}(\zeta_n)_t = (-\mathcal{H}(\zeta_n)_x - Q(x-t)\zeta_n)_x - \frac{b_n}{2}(\zeta_n^2)_x, \\ \|\zeta_n(t)\|_{L^2} \leq C_T, \quad \forall t \in [-T,T].\end{gathered}$$

Indeed, since $|\rho_n'(t) - 1| \leq C\|\eta_n\|_{L^2} \leq Cb_n$, we have

$$|\rho_n(t) - \rho_n(0) - t| \leq Cb_n|t|, \tag{4.24}$$

and the estimate on ζ_n follows.

On the one hand, Theorem 6 applied to $\frac{1}{C_T}\zeta_n$ for n large enough (so that b_n is small enough) implies that $t \in [-T,T] \mapsto \zeta_n(t) \in L^2$ is equicontinuous in n.

On the other hand, from (4.16), we have

$$\int_{[-T,T]}\int \left|D^{\frac{1}{2}}(\zeta_n(t,x)\sqrt{\varphi'(x)})\right|^2 dxdt \leq C_T,$$

and the decay property (4.15) also holds for $\zeta(t)$ on $[-T,T]$ with constant depending on T. In particular, there exists $N \subset [-T,T]$ of zero Lebesgue measure such that for all $t \in [-T,T] \setminus N$, $\int |D^{\frac{1}{2}}(\zeta_n(t,x)\sqrt{\varphi'(x)})|^2 dxdt < +\infty$. Now, we choose a dense and countable subset I of $[-T,T]$ such that for all $t \in I$, $\int |D^{\frac{1}{2}}(\zeta_n(t,x)\sqrt{\varphi'(x)})|^2 dxdt < +\infty$. Arguing as in the proof of (4.21), and using a diagonal argument, there exists a subsequence of (ζ_n) which we will still denote by (ζ_n) such that for any $t \in I$, $\zeta_n(t) \to \zeta(t)$ in L^2 strong as $n \to +\infty$. Using the equicontinuity, we obtain

$$\forall t \in [-T,T], \quad \zeta_n(t) \to \zeta(t) \quad \text{in } L^2 \text{ strong as } n \to +\infty. \tag{4.25}$$

By (4.24) and $|\rho_n' - 1| \leq Cb_n$, we may also assume that for the same subsequence

$$\forall t \in [-T,T], \quad \frac{1}{b_n}(\rho_n(t) - \rho_n(0) - t) \to \kappa(t). \tag{4.26}$$

Now, we deduce from (4.23), (4.25) and (4.26) that

$$\forall t \in [-T,T],\ w_n(t) \to w(t) = \eta(t,.+t) + \kappa(t)Q' \text{ in } L^2 \text{ strong as } n \to +\infty.$$

4.3. Proof of Remark 1

Let $u(t)$ be a solution satisfying the assumptions of Theorem 1. Let c^+, $\rho(t)$ and $\eta(t)$ be defined as in the proof of Theorem 1. In particular, by (4.9), we have

$$\lim_{t\to+\infty} \int_{x>\frac{t}{10}-\rho(t)} |\eta(t,x)|^2 dx = 0. \tag{4.27}$$

To prove (1.11), we use the identity (2.34) on η, where $\varphi = \frac{\pi}{2} + \arctan(\frac{x}{A})$, $A > 1$ large enough be to defined later:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \eta^2\varphi = &\int(\mathcal{H}\eta_x)\eta\varphi' + \int(\mathcal{H}\eta_x)\eta_x\varphi - \frac{1}{2}\int\eta^2\varphi' + \frac{1}{2}\int\eta^2(-Q'\varphi + Q\varphi')\\ &+\frac{1}{3}\int\eta^3\varphi' + (\rho'-1)\int Q'\eta\varphi - \frac{1}{2}(\rho'-1)\int\eta^2\varphi'.\end{aligned}$$

We claim that for A large enough and α_0 small enough, for $C > 0$ independent of A,

$$\frac{1}{2}\frac{d}{dt}\int\eta^2\varphi \leq \int(\mathcal{H}\eta_x)\eta\varphi' + C\int\frac{\eta^2}{1+x^2}. \tag{4.28}$$

Indeed, by Lemma 3, we have $\int(\mathcal{H}\eta_x)\eta_x\varphi \leq \frac{C}{A}\int\eta^2\varphi'$. By the definition of Q, $\int\eta^2(-Q'\varphi + Q\varphi') \leq C\int\frac{\eta^2}{1+x^2}$. By (2.30) (note that the constant in (2.30) is independent of A) $|\int\eta^3\varphi'| \leq C\alpha_0\int\eta^2\varphi'$. Finally, the last two terms are controlled using (2.2), so that (4.28) is proved for A large enough, α_0 small enough.

Now, we use (4.19) on η. We obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\eta^2\varphi\\ &\leq -\int|D^{\frac{1}{2}}(\eta\sqrt{\varphi'})|^2 + C\|\eta\|_{L^2}\|\eta\sqrt{\varphi'}\|_{L^4}\|D\sqrt{\varphi'}\|_{L^4} + C\int\frac{\eta^2}{1+x^2}.\end{aligned} \tag{4.29}$$

Note that for $A>1$, we have $\frac{1}{1+x^2}\leq C\varphi$ on $\mathbb{R}$. Let $t_0>0$. Integrating the above estimate on $[t_0,t_0+1]$, we get

$$\int_{t_0}^{t_0+1}\int |D^{\frac12}(\eta\sqrt{\varphi'})|^2 \leq C \sup_{t\in[t_0,t_0+1]}\left(\int \eta^2(t)\varphi + C\|\eta\sqrt{\varphi'}\|_{L^4}\|D\sqrt{\varphi'}\|_{L^4}\right).$$

On the other hand, by (A.3), we have

$$\begin{aligned}\int |D^{\frac12}\eta|^2\varphi' &\leq 2\int |D^{\frac12}(\eta\sqrt{\varphi'})|^2 + 2\int |(D^{\frac12}\eta)\sqrt{\varphi'} - D^{\frac12}(\eta\sqrt{\varphi'})|^2\\ &\leq 2\int |D^{\frac12}(\eta\sqrt{\varphi'})|^2 + C\|\eta\|_{L^4}^2\|D^{\frac12}\sqrt{\varphi'}\|_{L^4}^2\\ &\leq 2\int |D^{\frac12}(\eta\sqrt{\varphi'})|^2 + C\|D^{\frac12}\sqrt{\varphi'}\|_{L^4}^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\int_{t_0}^{t_0+1}\int |D^{\frac12}\eta|^2\varphi' dt \leq C &\sup_{t\in[t_0,t_0+1]}\left(\int \eta^2(t)\varphi + C\|\eta\sqrt{\varphi'}\|_{L^4}\|D\sqrt{\varphi'}\|_{L^4}\right)\\ &+ C\|D^{\frac12}\sqrt{\varphi'}\|_{L^4}^2.\end{aligned}$$

We have $\|D^{\frac12}\sqrt{\varphi'}\|_{L^4}^2\leq CA^{-\frac32}$, $\|D\sqrt{\varphi'}\|_{L^4}\leq CA^{-\frac54}$ and

$$\|\eta\sqrt{\varphi'}\|_{L^4}\leq \|\eta\|_{L^8}\|\sqrt{\varphi'}\|_{L^8}\leq CA^{-\frac38}.$$

Therefore,

$$\int_{t_0}^{t_0+1}\int |D^{\frac12}\eta(t,x)|^2\frac{dxdt}{1+(\frac{x}{A})^2}\leq A\sup_{t\in[t_0,t_0+1]}\left(\int \eta^2(t)\varphi\right) + CA^{-\frac12}.$$

We now choose A depending on t_0:

$$A=A_{t_0}=\min\left(\frac{\sqrt{t_0}}{2}, \left(\sup_{t\in[t_0,t_0+1]}\int_{x\geq \frac{t}{10}-\rho(t)}\eta^2(t,x)dx\right)^{-\frac12}\right).$$

For this choice of A_{t_0}, we have $\lim_{t_0\to+\infty}A_{t_0}=+\infty$ and, since $\frac{t}{10}-\rho(t)\leq -\frac{t}{2}$,

$$A\sup_{t\in[t_0,t_0+1]}\left(\int \eta^2(t)\varphi\right)\leq CA\sup_{t\in[t_0,t_0+1]}\left(\int_{x\geq\frac{t}{10}-\rho(t)}\eta^2(t)\right)+\frac{CA}{t_0}\leq CA^{-1}.$$

so that $\lim_{t_0\to+\infty}A\sup_{t\in[t_0,t_0+1]}\left(\int\eta^2(t)\varphi\right)=0$. It follows that

$$\lim_{t_0\to+\infty}\int_{t_0}^{t_0+1}\int |D^{\frac12}\eta(t,x)|^2\frac{dxdt}{1+x^2}=0.$$

5. Multi-soliton case

Using the previous arguments and the strategy of [21] for the gKdV equation, we obtain the following result concerning multi-soliton solutions of (1.1).

Theorem 4 (Asymptotic stability of a sum of decoupled solitons). *Let $N \geq 1$ and $0 < c_1^0 < \ldots < c_N^0$. There exist $L_0 > 0$, $A_0 > 0$ and $\alpha_0 > 0$ such that if $u_0 \in H^{\frac{1}{2}}$ satisfies for some $0 \leq \alpha < \alpha_0$, $L \geq L_0$,*

$$\left\| u_0 - \sum_{j=1}^{N} Q_{c_j^0}(.-y_j^0) \right\|_{H^{\frac{1}{2}}} \leq \alpha \quad \text{where } \forall j \in \{2,\ldots,N\},\ y_j^0 - y_{j-1}^0 \geq L, \tag{5.1}$$

and if $u(t)$ is the solution of (1.1) corresponding to $u(0) = u_0$, then there exist $\rho_1(t), \ldots, \rho_N(t)$ such that the following hold

(a) *Stability of the sum of N decoupled solitons.*

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - \rho_j(t)) \right\|_{H^{\frac{1}{2}}} \leq A_0 \left(\alpha + \frac{1}{L} \right). \tag{5.2}$$

(b) *Asymptotic stability of the sum of N solitons. There exist $c_1^+, \ldots, c_N^+$, with $|c_j^+ - c_j^0| \leq A_0 \left(\alpha + \frac{1}{L}\right)$, such that*

$$\forall j, \quad u(t, . + \rho_j(t)) \rightharpoonup Q_{c_j^+} \quad \text{in } H^{\frac{1}{2}} \text{ weak as } t \to +\infty, \tag{5.3}$$

$$\left\| u(t) - \sum_{j=1}^{N} Q_{c_j^+}(.-\rho_j(t)) \right\|_{L^2(x \geq \frac{1}{10} c_1^0 t)} \to 0, \quad \rho_j'(t) \to c_j^+ \text{ as } t \to +\infty. \tag{5.4}$$

Recall that the Benjamin-Ono equation admits explicit multi-soliton solution. We denote by $U_N(x; c_j, y_j)$ the explicit family of N-soliton profiles, see e.g. [22, formula (1.7)] and Appendix A (see also references in [22]). We obtain the following corollary of the above Theorem and the continuous dependence of the solution in $H^{\frac{1}{2}}$.

Let $N \geq 1$, $0 < c_1^0 < \cdots < c_N^0$ and set

$$d_N(u) = \inf \left\{ \|u - U_N(.; c_j^0, y_j)\|_{H^{\frac{1}{2}}},\ y_j \in \mathbb{R} \right\}.$$

Corollary 1. (Asymptotic stability in $H^{\frac{1}{2}}$ of multi-solitons). *For all $\delta > 0$, there exists $\alpha > 0$ such that if $d_N(u_0) \leq \alpha$ then for all $t \in \mathbb{R}$, $d_N(u(t)) \leq \delta$.*

Recall that a result of stability in H^1 of double solitons for the BO equation was proved by variational methods in [23]. See also [22] for stability related results.

5.1. Sketch of the stability argument [31]

For the reader's convenience, we now sketch the proof of the stability argument for one soliton (see statement in the Introduction). Let $u(t)$ be an $H^{\frac{1}{2}}$ solution of (1.1) such that $u(0)$ is close to Q in $H^{\frac{1}{2}}$. Let $c^+ > 0$ be close to 1 such that $\int u^2(0) = c^+ \int Q^2$. We use Lemma 1 on $u(t)$ around Q_{c^+} so that $\eta(t,x) = u(t, x+\rho(t)) - Q_{c^+}(x)$ satisfies $\int \eta(t) Q'_{c^+} = 0$ and by L^2 conservation $\int \eta(t) Q_{c^+} = -\frac{1}{2}\int \eta^2(t)$.

We define the functional

$$\mathcal{G}(u(t)) = E(u(t)) + c^+ \int u^2(t). \tag{5.5}$$

Observing that $\mathcal{G}(u(t)) = \mathcal{G}(u(0))$ and so expanding $u(t)$ in $\mathcal{G}(u(t))$, we obtain

$$(\mathcal{L}_{c^+}\eta(t), \eta(t)) + O(\eta^3(t)) = (\mathcal{L}_{c^+}\eta(0), \eta(0)) + O(\eta^3(0))$$

where $\mathcal{L}_{c^+}\eta = -\mathcal{H}\eta_x + c^+\eta - Q_{c^+}\eta$. By the positivity property of $\mathcal{L}_{c^+}$, (property (A.10) of $\mathcal{L}$ and a scaling argument), we then obtain

$$\|\eta(t)\|_{H^{\frac{1}{2}}} \le C\|\eta(0)\|_{H^{\frac{1}{2}}}.$$

Note that $\left|\int \eta Q_{c^+}\right| \le C\|\eta\|_{L^2}^2$ replaces the orthogonality condition $\int \eta Q_{c^+} = 0$.

5.2. Sketch of the proof of Theorem 4

The proof is the same as the proof of Theorem 1 in [21].

First, we recall four lemmas (corresponding to Lemmas 1–4 in [21]) which are the main tools in proving Theorem 4.

Lemma 10. (Decomposition of the solution). *There exist $L_1, \alpha_1, K_1 > 0$ such that the following is true. If for $L > L_1$, $0 < \alpha < \alpha_1$, $t_0 > 0$,*

$$\sup_{0\le t\le t_0} \left(\inf_{y_j > y_{j-1}+L} \left\{ \left\| u(t,.) - \sum_{j=1}^N Q_{c_j^0}(. - y_j) \right\|_{H^{\frac{1}{2}}} \right\} \right) < \alpha,$$

then there exist unique C^1 functions $c_j : [0,t_0] \to (0,+\infty)$, $\rho_j : [0,t_0] \to \mathbb{R}$, such that

$$\eta(t,x) = u(t,x) - R(t,x)$$

$$\textit{where} \quad R(t,x) = \sum_{j=1}^N R_j(t,x), \quad R_j(t,x) = Q_{c_j(t)}(x - \rho_j(t)),$$

satisfies the following orthogonality conditions

$$\forall j, \forall t \in [0,t_0], \quad \int R_j(t)\eta(t) = \int (R_j(t))_x \eta(t) = 0.$$

Moreover, there exists $C > 0$ *such that* $\forall t \in [0, t_0]$,

$$\|\eta(t)\|_{H^{\frac{1}{2}}} + \sum_{j=1}^{N} |c_j(t) - c_j^0| \leq C\alpha,$$

$$\forall j,\ \left|c_j'(t)\right| + \left|\rho_j'(t) - c_j(t)\right| \leq C\left(\|\eta(t)\|_{L^2} + \frac{1}{L}\right).$$

Remark 5. In the rest of the argument, the modulation in the scaling parameter for all time (i.e. the introduction of $c_j(t)$) is not necessary. Indeed, modulation at $t = 0$ would be sufficient since we deal with the subcritical case. However, we have preferred to introduce this modulation to match the strategy of [21].

Expanding $u(t)$ in the energy conservation and using $E(Q_c) = c^2 E(Q)$, we have

Lemma 11. *There exists* $C > 0$ *such that in the context of Lemma* 10, $\forall t \in [0, t_0]$,

$$\left|E(Q)\sum_{j=1}^{N}\left[c_j^2(t)) - c_j^2(0)\right] + \frac{1}{2}\int (\eta_x \mathcal{H}\eta - R\eta^2)(t)\right|$$
$$\leq C\left(\|\eta(0)\|_{H^{\frac{1}{2}}}^2 + \|\eta(t)\|_{H^{\frac{1}{2}}}^3 + \frac{1}{L}\right).$$

We consider φ defined as in (2.6), with A large enough, and we set $\forall j \in \{2, \ldots, N\}$,

$$\mathcal{I}_j(t) = \int u^2(t,x)\varphi(x - m_j(t))dx, \quad m_j(t) = \frac{1}{2}(\rho_{j-1}(t) + \rho_j(t)).$$

Then, proceeding as in the proof of Proposition 1, we obtain the following.

Lemma 12. *There exists* $C > 0$ *such that in the context of Lemma* 10,

$$\forall j \in \{2, \ldots, N\},\ \forall t \in [0, t_0], \quad \mathcal{I}_j(t) - \mathcal{I}_j(0) \leq \frac{C}{L}.$$

Finally, setting $c(t,x) = c_1(t) + \sum_{j=2}^{N}(c_j(t) - c_{j-1}(t))\varphi(x - m_j(t))$, and proceeding as in the proof of Propositions 3 and 4, we have

Lemma 13. *There exists* $\lambda > 0$ *such that in the context of Lemma* 10,

$$\forall t \in [0, t_0], \quad \mathcal{G}_N(t) := \int \eta_x \mathcal{H}\eta + c(t,x)\eta^2 - Q\eta^2 \geq \lambda \|\eta(t)\|_{H^{\frac{1}{2}}}^2.$$

Recall that the introduction of the functional $\mathcal{G}_N(t)$ for the problem of stability of multi-soliton solutions is justified as follows. For the stability of one soliton, the suitable functional is $\mathcal{G}(u(t))$ defined in (5.5). For the case of N solitons, we introduce the functional $\mathcal{G}_N(t)$ which is approximately $E(u(t)) + c_j(0)\int u^2(t)$ around the soliton Q_{c_j}. Then, we observe (using the energy conservation and Lemma 11) that this quantity is almost decreasing. This is sufficient to conclude the stability argument for several solitons. We now sketch the argument. We refer to [21], Section 3 for more details in the stability proof.

Sketch of the proof of the stability. Let

$$\mathcal{V}_{A_0}(L,\alpha) = \left\{ u \in H^{\frac{1}{2}};\ \inf_{y_j - y_{j-1} \geq L} \left\| u - \sum_{j=1}^{N} Q_{c_j^0}(.-y_j) \right\|_{H^{\frac{1}{2}}} \leq A_0 \left(\alpha + \frac{1}{L} \right) \right\}.$$

Part (a) of Theorem 4 is a consequence of the following proposition and continuity arguments.

Proposition 6. *There exist $A_0 > 0$, $L_0 > 0$ and $\alpha_0 > 0$ such that, for all $u_0 \in H^{\frac{1}{2}}$, if*

$$\left\| u_0 - \sum_{j=1}^{N} Q_{c_j^0}(.-y_j^0) \right\|_{H^{\frac{1}{2}}} \leq \alpha,$$

where $L \geq L_0$, $0 < \alpha < \alpha_0$, $y_j^0 - y_{j-1}^0 + L$, and if for $t^* > 0$,

$$\forall t \in [0, T^*], \quad u(t) \in \mathcal{V}_{A_0}(L,\alpha),$$

where $u(t)$ is the solution of (1.1), then

$$\forall t \in [0, T^*], \quad u(t) \in \mathcal{V}_{\frac{1}{2}A_0}(L,\alpha).$$

The proof of Proposition 6 is exactly the same as the proof of Proposition 1 in [21], using Lemmas 10–13. In particular, we first prove

$$\text{(5.6)} \quad \forall t \in [0, t^*], \quad \sum_{j=1}^{N} |c_j(t) - c_j(0)| \leq C_1 \left(\|\eta(t)\|^2_{H^{\frac{1}{2}}} + \|\eta(0)\|^2_{H^{\frac{1}{2}}} + \frac{1}{L} \right),$$

and then

$$\text{(5.7)} \qquad \|\eta(t)\|^2_{H^{\frac{1}{2}}} \leq C_2 \left(\|\eta(0)\|^2_{H^{\frac{1}{2}}} + \frac{1}{L} \right),$$

where C_1, $C_2 > 0$ are independent of A_0, and we then conclude by using the decomposition of $u(t)$ is terms of $\eta(t)$ and $R(t)$.

Note that in proving (5.6), we make use of the following algebraic fact:

$$E(Q_c) = c^2 E(Q), \quad \int Q_c^2 = c \int Q^2, \quad E(Q) = -\frac{1}{2}\int Q^2.$$

The last formula is easily obtained from the equation of Q multiplying by Q and then by xQ' and using $\int(\mathcal{H}Q')(xQ') = 0$. This allows us to prove the following estimate

$$\left| E(Q)\sum_{j=1}^{N}(c_j(t) - c_j(0)) + \int Q^2 \sum_{j=1}^{N}\{c_j(0)\,(c_j(t) - c_j(0))\}\right| \leq C\sum_{j=1}^{N}|c_j(t) - c_j(0)|^2 .$$

which is the analogue of (44) in [21].

The proof of part (b) of Theorem 4 is exactly the same as in [21], Section 4, using Theorem 2, the monotonicity arguments (Proposition 1) and Theorem 5. It follows closely the proof of Theorem 1 in the present paper.

The proof of Corollary 1 is omitted since it is the same as the proof of Corollary 1 in [21].

6. Weak convergence and well-posedness results

6.1. Weak convergence

Theorem 5. (Weak continuity of the BO flow map). *Let (u_n) be a sequence of global $H^{\frac{1}{2}}$ solutions of equation (1.1). Assume that $u_n(0) \rightharpoonup u_0$ in $H^{\frac{1}{2}}$ weak and let $u(t)$ be the solution of (1.1) corresponding to $u(0) = u_0$. Then, for all $t \in \mathbb{R}$, $u_n(t) \rightharpoonup u(t)$ in $H^{\frac{1}{2}}$ weak.*

Proof of Theorem 5. Let $u_{0,n} = u_n(0)$. It is sufficient to prove the result for $T \in [0,1]$.

Step 1. H^2case. Here, we assume $u_{0,n} \rightharpoonup u_0$ in H^2. Let $w_n = u_n - u$. The equation for w_n is

$$\begin{cases} w_{nt} + \mathcal{H}(w_n)_{xx} + u_n w_{nx} + u_x w_n = 0 \\ w_n(0) = \psi_n, \quad \psi_n = u_{0,n} - u_0. \end{cases} \tag{6.1}$$

Fix $t = T$, $g \in C_0^\infty(\mathbb{R})$. For a function $\widetilde{u}$ to be determined, we consider the solution $v(t)$ of

$$\begin{cases} v_t + \mathcal{H}v_{xx} + (\widetilde{u}v)_x - u_x v = 0, \\ v(T) = g. \end{cases}$$

Then

$$\int w_n(T,x)g(x)dx - \int \psi_n(x)v(0,x)dx = \int_0^T \int w_{nt}(t)v(t) + \int_0^T \int w_n(t)v_t$$
$$= \mathbf{I} + \mathbf{II}.$$

$$\mathbf{I} = \int_0^T \int w_n(\mathcal{H}v_{xx} + (u_n v)_x - u_x v), \quad \mathbf{II} = -\int_0^T \int w_n(\mathcal{H}v_{xx} + (\widetilde{u}v)_x - u_x v)$$

so that

$$\int w_n(T,x)g(x)dx - \int \psi_n(x)v(0,x)dx = \int_0^T \int w_n((u_n - \widetilde{u})v)_x$$
$$= -\int_0^T \int w_{nx}(u_n - \widetilde{u})v.$$

We can assume, after passing to a subsequence, that $u_n - \widetilde{u} \to 0$ in $L^2_{loc}(\mathbb{R} \times [0,T])$. Next, we will show that given $\varepsilon > 0$, there exists $R > 0$ such that

$$\left| \int_0^T \int_{|x|>R} w_{nx}(u_n - \widetilde{u})v \right| \leq \varepsilon, \quad \text{uniformly in } n.$$

In fact, since $\|w_{nx}\|_{L^\infty} \leq C$, $\sup_t \|v\|_{L^2} \leq C$ and $\sup_t \|u_n - \widetilde{u}\|_{L^2} \leq C$, the claim is clear.

But then, $\mathbf{I} + \mathbf{II} \to 0$ as $n \to +\infty$. We only needed $\sup_t \|v\|_{L^2} \leq C$, which needs $\widetilde{u}_x \in L^\infty$, $u_x \in L^\infty$, which are both clear. (We use the energy method to bound v.)

Step 2. General case. Fix N large and define $u^N_{0,n}$ such that $\widehat{u^N_{0,n}}(\xi) = \mathbf{1}_{[-N,N]}(\xi)\widehat{u}_{0,n}(\xi)$, where $\mathbf{1}_I$ is the characteristic function of I. Note that

$$\|u^N_{0,n} - u_{0,n}\|^2_{L^2} = \int_{|\xi|\geq N} \left|\widehat{u^N_{0,n}}(\xi)\right|^2 \leq \frac{1}{N}\|u_{0,n}\|^2_{H^{\frac{1}{2}}} \leq \frac{C}{N},$$

so that $u^N_{0,n} \to u_{0,n}$ in L^2 as $N \to +\infty$, uniformly in n.

Fix $g \in C_0^\infty$, $T \in \mathbb{R}$, $\varepsilon > 0$. The proof of the L^2 continuity of the flow map (see [11]) shows that

$$\sup_{t\in[0,1]} \|u^N(t) - u(t)\|_{L^2} \leq C\|u^N_0 - u_0\|_{L^2},$$

$$\sup_{t\in[0,1]} \|u^N_n(t) - u_n(t)\|_{L^2} \leq C\|u^N_{0,n} - u_{0,n}\|_{L^2}$$

for some universal constant $C > 0$. We fix N such that

$$\left| \int (u_n(T) - u(T))g - \int (u^N_n(T) - u^N(T))g \right| \leq \frac{\varepsilon}{2}, \quad \text{uniformly in } n.$$

But, for fixed N, we let $n \to +\infty$, and use step 1 and the proof is concluded.

6.2. Well-posedness result for the nonlinear BO equation with potential

In this subsection, for $0 < b < b_0$, b_0 small, we consider the IVP

$$(6.2) \quad \begin{cases} v_t = (-\mathcal{H}v_x)_x - (Q(x-t)v)_x - \frac{b}{2}(v^2)_x = 0 & \text{on } [-T,T] \times \mathbb{R}, \\ v(t=0,x) = v_0(x) & \text{on } \mathbb{R}. \end{cases}$$

The well-posedness of the Cauchy problem in L^2 for this equation is clear from [11] since $u(t,x) = Q(x-t) + bv(t,x)$ satisfies the BO equation. Our main concern is a result of equicontinuity of the map $t \mapsto v(t)$ in L^2 with respect to b. To establish such a result we follow the strategy of [11] on equation (6.2), using the special form of Q and keeping track of the dependency in b.

Theorem 6. (a) *Let $v_0 \in H^\infty$. Then, there exists $T = T(Q) > 0$ and a unique solution $v = S_b^\infty(v_0)$ of (6.2) in $[-T,T]$, $v \in C([-T,T], H^\infty)$.*

(b) *There exists a constant C, independent of b such that*

$$(6.3) \qquad \sup_{t\in[-T,T]} \|v(t)\|_{H^2} \le C\|v_0\|_{H^2}.$$

(c) *The mapping S_b^∞ extends uniquely to a continuous mapping $S_b^0 : L^2 \to C([-T,T], L^2)$, and there exists C, independent of b such that*

$$(6.4) \qquad \sup_{t\in[-T,T]} \|v(t)\|_{L^2} \le C\|v_0\|_{L^2}.$$

Moreover, given $v_0 \in L^2$, $\|v_0\|_{L^2} \le 2$, for any $\varepsilon > 0$, there exits $\delta = \delta(v_0,\varepsilon) > 0$ (δ independent of b) such that for any $v_1 \in L^2$, $\|v_1\|_{L^2} \le 2$,

$$(6.5) \quad \|v_0 - v_1\|_{L^2} \le \delta \quad \Rightarrow \quad \sup_{t\in[-T,T]} \|S_b^0(v_0)(t) - S_b^0(v_1)(t)\|_{L^2} \le \varepsilon.$$

Finally, there exists $\widetilde{\delta} = \widetilde{\delta}(v_0,\varepsilon) > 0$ (independent of b) such that for any $t,t' \in [-T,T]$,

$$(6.6) \qquad |t-t'| \le \widetilde{\delta} \quad \Rightarrow \quad \|S_b^0(v_0)(t) - S_b^0(v_0)(t')\|_{L^2} \le \varepsilon.$$

Reduction of the proof. For $0 < \lambda \ll 1$, consider $v_\lambda(t,x) = \lambda v(\lambda^2 t, \lambda x)$. Then v_λ solves

$$(6.7) \quad \begin{cases} (v_\lambda)_t = (-\mathcal{H}(v_\lambda)_x)_x - (\lambda Q(\lambda x - \lambda^2 t)v_\lambda)_x - \frac{b}{2}(v_\lambda^2)_x = 0 & \text{on } [-T,T]\times\mathbb{R}, \\ v_\lambda(t=0,x) = v_{0,\lambda}(x) \ \text{ on } \mathbb{R}, \quad v_{0,\lambda}(x) = \lambda v_0(\lambda x). \end{cases}$$

Define

$$Q_\lambda(t,x) = \lambda Q(\lambda x - \lambda^2 t).$$

Then the proof of Theorem 6 reduces to prove that for the following (IVP)

$$(6.8)\qquad \begin{cases} v_t = (-\mathcal{H}v_x)_x - (Q_\lambda(t,x)v)_x - \frac{b}{2}(v^2)_x = 0 \quad \text{on } [-1,1]\times\mathbb{R}, \\ v(t=0,x) = v_0(x) \quad \text{on } \mathbb{R}, \quad \|v_0\|_{L^2} \le \lambda^{\frac{1}{2}}, \end{cases}$$

we have

Theorem 7. *There exists b_0, $\lambda > 0$ small enough such that if $0 < b < b_0$, the following hold*

(a) *Assume $v_0 \in H^\infty$. Then, there exists a unique solution $v = S_b^\infty(v_0)$ of* (6.8) *in $[-1,1]$, $v \in C([-1,1], H^\infty)$.*

(b) *There exists a constant C, independent of b such that*

$$(6.9)\qquad \sup_{t\in[-1,1]} \|v(t)\|_{H^2} \le C\|v_0\|_{H^2}.$$

(c) *The mapping S_b^∞ extends uniquely to a continuous mapping $S_b^0 : L^2 \to C([-1,1], L^2)$, and there exists C, independent of b such that*

$$(6.10)\qquad \sup_{t\in[-1,1]} \|v(t)\|_{L^2} \le C\|v_0\|_{L^2}.$$

Moreover, given $v_0 \in L^2$, $\|v_0\|_{L^2} \le \lambda^{\frac{1}{2}}$, for any $\varepsilon > 0$, there exits $\delta = \delta(v_0, \varepsilon) > 0$ (δ independent of b) such that for any $v_1 \in L^2$, $\|v_1\|_{L^2} \le \lambda^{\frac{1}{2}}$,

$$(6.11)\qquad \|v_0 - v_1\|_{L^2} \le \delta \quad \Rightarrow \quad \sup_{t\in[-1,1]} \|S_b^0(v_0)(t) - S_b^0(v_1)(t)\|_{L^2} \le \varepsilon.$$

Finally, there exists $\widetilde{\delta} = \widetilde{\delta}(v_0, \varepsilon) > 0$ (independent of b) such that for any $t, t' \in [-1,1]$,

$$(6.12)\qquad |t - t'| \le \widetilde{\delta} \quad \Rightarrow \quad \|S_b^0(v_0)(t) - S_b^0(v_0)(t')\|_{L^2} \le \varepsilon.$$

The proof of Theorem 7 is based on the following three propositions.

Proposition 7. *Assume $v_0 \in H^\infty$, then there exists $T = T(\|v_0\|_{H^2})$ and a unique solution v of* (6.8) *in $(-T,T)$. Also, for any $\sigma \ge 2$,*

$$(6.13)\qquad \sup_{t\in(-T,T)} \|u(t)\|_{H^\sigma} \le C(\sigma, \|v_0\|_\sigma, \sup_{t\in(-T,T)} \|v(t)\|_{H^2}).$$

In particular, the constant C is independent of b, ($0 < b < b_0$) and $\lambda < 1$.

Proposition 7 is a consequence of the energy method, taking into account that

$$\|\partial_x Q_\lambda\|_{L^1((-1,1),L^\infty_x)} \le C.$$

Proposition 8. *For λ small enough, we have that if $T \in (0,1]$, $\|v_0\|_{L^2} \le \lambda^{\frac{1}{2}}$, $v = S^\infty(v_0) \in C((-T,T),H^\infty)$ is a solution, then*

$$\sup_{t\in[-T,T]} \|v(t)\|_{H^2} \le C\|v_0\|_{H^2},$$

where C is independent of b $(0 < b < b_0)$.

Proposition 9. *For $v_0 \in H^\infty$, $N \in [1,\infty)$, $\|v_0\|_{L^2} \le \lambda^{\frac{1}{2}}$, let $\widehat{v_0^N}(\xi) = \mathbf{1}_{[-N,N]}(\xi)\hat{v}_0(\xi)$, $v_0^N \in H^\infty$. Then,*

$$\sup_{t\in(-1,1)} \|S_b^\infty(v_0)(t) - S_b^\infty(v_0^N)(t)\|_{L^2} \le C\|v_0 - v_0^N\|_{L^2},$$
$$\sup_{t\in(-1,1)} \|S_b^\infty(v_0)(t)\|_{L^2} \le C\|v_0\|_{L^2}.$$

where C is independent of b $(0 < b < b_0)$.

Proof of Theorem 7 from Propositions 7, 8 and 9. First, note that Propositions 7 and 8 clearly give (a) and (b) in Theorem 7. Let us turn to the proof of (c): it suffices to show first that if $v_{0,n} \in H^\infty$, $\lim_{n\to+\infty} v_{0,n} = v_0$ in L^2, the sequence $S_b^\infty(v_{0,n})$ is Cauchy in $C([-1,1],L^2)$. Let $\varepsilon > 0$ be given. We want to show that there exists M_ε (independent of b) such that

$$m,\ n \ge M_\varepsilon \quad \Rightarrow \quad \sup_{t\in[-1,1]} \|S_b^\infty(v_{0,n})(t) - S_b^\infty(v_{0,m})(t)\|_{L^2} \le \varepsilon.$$

Observe that

$$\|v_{0,n} - v_{0,n}^N\|_{L^2} \le \|v_0 - v_0^N\|_{L^2} + \|v_0 - v_{0,n}\|_{L^2}.$$

Hence, we can fix $N = N(\varepsilon, v_0)$ large and M_ε^1 large such that $\|v_{0,n}-v_{0,n}^N\|_{L^2} \le \frac{\varepsilon}{4C}$, for $n \ge M_\varepsilon^1$, where C is the constant in Proposition 9 ($\|v_{0,n}\|_{L^2} \le \lambda^{\frac{1}{2}}$). Then, by Proposition 9, for $n \ge M_\varepsilon^1$,

$$\sup_{t\in[-1,1]} \|S_b^\infty(v_{0,n})(t) - S_b^\infty(v_{0,n}^N)(t)\|_{L^2} \le \frac{\varepsilon}{4}.$$

It remains to estimate $\sup_{t\in[-1,1]}\|S_b^\infty(v_{0,n}^N)(t)-S_b^\infty(v_{0,m}^N)(t)\|_{L^2}$. But energy estimates for the difference equation give

$$\begin{aligned}
&\sup_{t\in[-1,1]}\|S_b^\infty(v_{0,n}^N)(t)-S_b^\infty(v_{0,m}^N)(t)\|_{L^2}\\
&\leq \|v_{0,n}^N-v_{0,m}^N\|_{L^2}\exp\left(C\int_{-1}^1\|\partial_x(S_b^\infty(v_{0,n}^N))(t)\|_{L^\infty_x}+C\|\partial_x(S_b^\infty(v_{0,m}^N))(t)\|_{L^\infty_x}\right)\\
&\leq \|v_{0,n}-v_{0,m}\|_{L^2}\exp\left(C\sup_{|t|<1}\|S_b^\infty(v_{0,n}^N)(t)\|_{H^2}+C\sup_{|t|<1}\|S_b^\infty(v_{0,m}^N)(t)\|_{H^2}\right)\\
&\leq \|v_{0,n}-v_{0,m}\|_{L^2}\exp\left(CN^2\|v_{0,n}\|_{L^2}+CN^2\|v_{0,n}\|_{L^2}\right)\\
&\leq \|v_{0,n}-v_{0,m}\|_{L^2}\exp(CN^2)\leq\frac{\varepsilon}{2},
\end{aligned}$$

for n, m large (we have used the estimate of Proposition 8). Also, by Proposition 9, we have $\sup_{t\in(-1,1)}\|S_b^\infty(v_{0,n})(t)\|_{L^2}\leq C$. Thus, we obtain the unique extension S_b^0 and (6.10) holds.

To check (6.11), fix v_0, $\|v_0\|_{L^2}\leq\lambda^{\frac{1}{2}}$, let $\varepsilon>0$ be given. With C as in Proposition 9, find N ($N=N(\varepsilon,v_0)$) so large that $\|v_0-v_0^N\|_{L^2}\leq\frac{\varepsilon}{8C}$. Now find $\delta_1=\delta_1(\varepsilon,v_0)$ so small that if $\|v_0-v_1\|_{L^2}\leq\delta_1$, then $\|v_1-v_1^N\|_{L^2}\leq\frac{\varepsilon}{4C}$. We have

$$\begin{aligned}
\sup_{t\in[-1,1]}\|S_b^0(v_0)(t)-S_b^0(v_1)(t)\|_{L^2}&\leq\sup_{t\in[-1,1]}\|S_b^0(v_0)(t)-S_b^0(v_0^N)(t)\|_{L^2}\\
&+\sup_{t\in[-1,1]}\|S_b^0(v_1)(t)-S_b^0(v_1^N)(t)\|_{L^2}+\sup_{t\in[-1,1]}\|S_b^0(v_1^N)(t)-S_b^0(v_0^N)(t)\|_{L^2}.
\end{aligned}$$

By Proposition 9, the first two terms are smaller than $\frac{\varepsilon}{2}$. For the last one, we again use the energy estimate and get, as before

$$\sup_{t\in[-1,1]}\|S_b^0(v_1^N)(t)-S_b^0(v_0^N)(t)\|_{L^2}\leq C\|v_1-v_0\|_{L^2}\exp(CN^2),$$

using Propositions 8 and 9 and (6.11) follows.

For (6.12), first find $N=N(\varepsilon,v_0)$ so large that $\|v_0-v_0^N\|_{L^2}\leq\frac{\varepsilon}{4C}$, where C is as in Proposition 9. Then,

$$\sup_{t\in[-1,1]}\|S_b^0(v_0)(t)-S_b^0(v_0^N)(t)\|_{L^2}\leq\frac{\varepsilon}{4}$$

and we are reduced to showing, for N fixed that if $|t-t'|\leq\widetilde{\delta}$, then $\|S_b^0(v_0^N)(t)-S_b^0(v_0^N)(t')\|_{L^2}\leq\frac{\varepsilon}{2}$.

Let $f(t) = \|S_b^0(v_0^N)(t)\|_{L^2}^2$. The energy method, combined with Proposition 8 shows that $|f'(t)| \le f(0)\exp(CN^2)$. But then, for $|t-t'| \le \widetilde{\delta}_1$, $|f(t)-f(t')| \le \frac{\varepsilon}{4}$. But

$$\|S_b^0(v_0^N)(t) - S_b^0(v_0^N)(t')\|_{L^2}^2 = f(t) + f(t') - 2\int S_b^0(v_0^N)(t).S_b^0(v_0^N)(t')dx$$
$$= f(t') - f(t) + 2\int S_b^0(v_0^N)(t)[S_b^0(v_0^N)(t) - S_b^0(v_0^N)(t')]dx.$$

Let $v^N(t) = S_b^0(v_0^N)(t)$. The second term equals

$$2\int v^N(t)\int_{t'}^t \partial_s v^N(s)dsdx$$
$$= 2\int_{t'}^t \int v_N(t)[-\mathcal{H}\partial_x^2 v^N(s) - (Q_\lambda v^N)_x - \frac{b}{2}((v^N)^2)_x(s)]dxds.$$

But by Proposition 8, $\sup_{t\in[-1,1]}\|v^N(t)\|_{H^2} \le C\|v_0^N\|_{H^2} \le CN^2$. Thus, the second term is controlled by $C|t-t'|N$, and the proof is complete, provided we prove Propositions 8 and 9.

Proof of Propositions 8 and 9. *Step 1.* Assume $v_0 \in H^\infty$, $\|v_0\|_{H^2} \le M$ and $0 < T \le 2$, $v = S_b^\infty(t)$ exists in $[-T,T]$. Then, there exist $\lambda_0 = \lambda_0(M)$, $b_0 = b_0(M)$ such that for $0<\lambda<\lambda_0$, $0 \le b < b_0$, we have

$$\sup_{t\in[-T,T]}\|v(t)\|_{H^2} \le 2\|v_0\|_{H^2}. \tag{6.14}$$

Proof of (6.14). Note that $\|\partial_x^k Q\lambda\|_{L^\infty} \le C_k\lambda^{k+1}$. Let $f(t) = \|v(t)\|_{H^2}^2$. The standard energy method shows that

$$|f'(t)| \le C(\lambda_0^2 + b_0\|\partial_x v(t)\|_{L_x^\infty})f(t) \le (\lambda_0^2 + b_0(f(t))^{\frac{1}{2}})f(t).$$

Integrating the ODE gives the result.

As a corollary, we obtain under the circumstances of Step 1 that v exists in $(-1,1)$ and

$$\sup_{t\in[-2,2]}\|v(t)\|_{H^2} \le 2\|v_0\|_{H^2}.$$

Step 2. From now on, we will follow closely [11]. Some of the ideas used before were developed in a forthcoming paper [8]. We have now reduced everything to *a priori* estimates. We will change notation slightly to match [11]. We then study the problem

$$\begin{cases} u_t + \mathcal{H}u_{xx} + (Q_\lambda u)_x + b(\frac{1}{2}u^2)_x = 0 & (t,x)\in(-1,1)\times\mathbb{R}, \\ u_{|t=0} = \phi, \quad \|\phi\|_{L^2} \le \lambda^{\frac{1}{2}}, \end{cases} \tag{6.15}$$

We use the notation P_{low}, $P_{\pm\text{high}}$ as in [11]:

$$P_{\text{low}} \text{ defined by the Fourier multiplier } \xi \to \mathbf{1}_{[-2^{10},2^{10}]}(\xi);$$

$$P_{\pm\text{high}} \text{ defined by the Fourier multiplier } \xi \to \mathbf{1}_{[2^{10},\infty)}(\pm\xi);$$

$$P_{\pm} \text{ defined by the Fourier multiplier } \xi \to \mathbf{1}_{[0,\infty)}(\pm\xi).$$

Let $\phi_0 = P_{\text{low}}\phi \in H^\infty$, real-valued, $\|\phi_0\|_{H^2} \le 2^{20} = M$. We choose λ_0, b_0 as in Step 1 and its corollary, so that Proposition 7 and these results gives, with $u_0^{(1)} = S_b^\infty(\phi_0)(t)$ that

$$\sup_{t\in[-2,2]} \|\partial_t^{\sigma_1}\partial_x^{\sigma_2} u_0^{(1)}\|_{L^2_x} \le C_{\sigma_1,\sigma_2}\|\phi\|_{L^2}, \quad \sigma_i \ge 0.$$

Let $\widetilde{u} = u - u_0^{(1)}$. The equation for $\widetilde{u}$ is

$$\begin{cases} \widetilde{u}_t + \mathcal{H}\widetilde{u}_{xx} + (Q_\lambda \widetilde{u})_x + b(u_0^{(1)}\widetilde{u})_x + b(\tfrac{1}{2}\widetilde{u}^2)_x = 0, \\ \widetilde{u}_{|t=0} = P_{+\text{high}}\phi + P_{-\text{high}}\phi. \end{cases} \tag{6.16}$$

Let now $u_0(t,x) = Q_\lambda(t,x) + bu_0^{(1)}(t,x)$. Then

$$\sup_{t\in[-2,2]} \|\partial_t^{\sigma_1}\partial_x^{\sigma_2} u_0\|_{L^2_x} \le C_{\sigma_1,\sigma_2}(\lambda_0^{\frac{1}{2}} + b_0).$$

We now want to construct U_0 similarly to [11], with the following properties $\partial_x U_0(t,x) = \frac{1}{2}u_0(t,x)$, $U_0(0,0) = 0$ and

$$\sup_{t\in[-2,2]} \|\partial_t^{\sigma_1}\partial_x^{\sigma_2} U_0(t,.)\|_{L^2_x} \le C_{\sigma_1,\sigma_2}(\lambda_0^{\frac{1}{2}} + b_0)$$

where $\sigma_1, \sigma_2 \ge 0$, $(\sigma_1,\sigma_2) \ne (0,0)$.

Since $Q_\lambda(t,x) = \frac{4\lambda}{1+(\lambda x-\lambda^2 t)^2}$, we set $U_0^{(2)}(t,x) = 2\arctan(\lambda x - \lambda^2 t)$. We next recall the equation $u_0^{(1)}(t,x)$ verifies:

$$\partial_t(\tfrac{1}{2}u_0^{(1)}) + \mathcal{H}\partial_x^2(\tfrac{1}{2}u_0^{(1)}) + \partial_x(Q_\lambda \tfrac{1}{2}u_0^{(1)}) + b\partial_x((\tfrac{1}{2}u_0^{(1)})^2) = 0.$$

We then define first $U_0^{(1)}(t,0)$ by $U_0^{(1)}(0,0) = 0$ and

$$\partial U_0^{(1)}(t,0) + \mathcal{H}\partial_x(\tfrac{1}{2}u_0^{(1)}(t,0)) + Q_\lambda(t,0)\tfrac{1}{2}u_0^{(1)}(t,0) + b(\tfrac{1}{2}u_0^{(1)}(t,0))^2 = 0.$$

We then construct $U_0^{(1)}(t,x)$ by $\partial_x U_0^{(1)}(t,x) = \frac{1}{2}u_0^{(1)}(t,x)$. Notice that $U_0^{(1)}$ is real-valued. Using the equation for $u_0^{(1)}$, we have

$$\partial_x\left(\partial_t U_0^{(1)} + \mathcal{H}\partial_x^2 U_0^{(1)} + Q_\lambda \partial_x U_0^{(1)} + b(\partial_x U_0^{(1)})^2\right) = 0 \quad \text{on } \mathbb{R}\times[-2,2].$$

But then, on $\mathbb{R} \times [-2,2]$, we have

$$\partial_t U_0^{(1)}(t,x) + \frac{1}{2}\mathcal{H}\partial_x u_0^{(1)}(t,x) + Q_\lambda(t,x)\frac{1}{2}u_0^{(1)}(t,x) + \frac{b}{4}(u_0^{(1)}(t,x))^2.$$

We then define $U_0(t,x) = bU_0^{(1)}(t,x) + U_0^{(2)}(t,x)$, and all our properties hold. We recall that

$$\begin{cases} \widetilde{u}_t + \mathcal{H}\widetilde{u}_{xx} + (u_0\widetilde{u})_x + b(\frac{1}{2}\widetilde{u}^2)_x = 0, \\ \widetilde{u}_{|t=0} = P_{+\text{high}}\phi + P_{-\text{high}}\phi. \end{cases} \tag{6.17}$$

We now proceed as in Section 2 of [11]. We define

$$P_{+\text{high}}\widetilde{u} = e^{-iU_0}w_+, \quad P_{-\text{high}}\widetilde{u} = e^{iU_0}w_- \quad \text{and} \quad P_{\text{low}}\widetilde{u} = w_0.$$

Applying $P_{+\text{high}}$, $P_{-\text{high}}$, P_{low} to the above equation and using the definitions above, we have (we write the equation for w_+, the one for w_- is analogous, the one for w_0 will be written later). Following the argument in [11], one gets:

$$\begin{aligned}
&(w_+)_t + \mathcal{H}\partial_x^2 w_+ = -\frac{b}{2}e^{iU_0}P_{+\text{high}}\partial_x((e^{-iU_0}w_+ + e^{iU_0}w_- + w_0)^2) \\
&- e^{-iU_0}P_{+\text{high}}\partial_x(u_0(e^{iU_0}w_- + w_0)) + e^{iU_0}(P_{-\text{high}} + P_{\text{low}})(e^{iU_0}u_0\partial_x w_+) \\
&+ 2iP_-\partial_x^2 w_+ - e^{iU_0}P_{+\text{high}}(\partial_x(u_0 e^{-iU_0}w_+)) + iw_+\left[(U_0)_t - i(U_0)_{xx} - ((U_0)_x)^2\right],
\end{aligned}$$

and so after more calculations, we get

$$\begin{aligned}
(w_+)_t + \mathcal{H}\partial_x^2 w_+ = & -\frac{b}{2}e^{iU_0}P_{+\text{high}}\partial_x((e^{-iU_0}w_+ + e^{iU_0}w_- + w_0)^2) \\
& - e^{-iU_0}P_{+\text{high}}\left[\partial_x(u_0 P_{-\text{high}}(e^{iU_0}w_-) + u_0 P_{\text{low}}(w_0))\right] \\
& + e^{iU_0}(P_{-\text{high}} + P_{\text{low}})\left[\partial_x(u_0 P_{+\text{high}}(e^{-iU_0}w_+))\right] \\
& + 2iP_-\left[\partial_x^2(e^{iu_0}P_{+\text{high}}(e^{-iU_0}w_+))\right] \\
& + iw_+\left[(U_0)_t + \mathcal{H}\partial_x^2 U_0 + (\partial_x U_0)^2 + iP_+\partial_x U_0\right],
\end{aligned}$$

We recall $\partial_x U_0^{(2)} = \frac{1}{2}Q_\lambda$ and that Q_λ solves $\partial_t Q_\lambda + \mathcal{H}\partial_x^2 Q_\lambda + \partial_x(\frac{1}{2}Q_\lambda^2) = 0$ or $\partial_t U_0^{(2)} + \mathcal{H}\partial_x^2 U_0^{(2)} = -\frac{1}{4}Q_\lambda^2$ and $\partial_t U_0^{(1)} + \mathcal{H}\partial_x^2 U_0^{(1)} = -Q_\lambda \partial U_0^{(1)} - b(\partial_x U_0^{(1)})^2$. Hence, $\partial_t U_0 + \mathcal{H}\partial_x^2 U_0 + (\partial_x U_0)^2 = 0$ and we get $\partial w_+ + \mathcal{H}w_+ = E_+(w_+, w_-, w_0)$, where E_+ is defined as in [11, p. 756], except that the first term is multiplied now by b. The equation for w_- and E_- is similar. The equation for w_0 writes

$$\partial_t(P_{\text{low}}\widetilde{u}) + \mathcal{H}\partial_x^2 P_{\text{low}}\widetilde{u} + P_{\text{low}}\partial_x(u_0\widetilde{u}) + \tfrac{b}{2}P_{\text{low}}\partial_x((\widetilde{u})^2) = 0,$$

where $\widetilde{u} = e^{-iU_0}w_+ + e^{iU_0}w_- + w_0$.

Next, we note that, with $\delta = (\lambda_0^{\frac{1}{2}} + b_0)$, the estimates (10.19) in [11] hold. Because of this and the form of E_+, E_-, E_0, just as in Proposition 10.5 in [11], we have

$$\begin{aligned}\|\psi(t)(\mathbf{E}(\mathbf{w}) - \mathbf{E}(\mathbf{w}'))\|_{N^\sigma} \leq{}& Cb_0\|\mathbf{w} - \mathbf{w}'\|_{F^\sigma}(\|\mathbf{w}\|_{F^0} + \|\mathbf{w}'\|_{F^0}) \\ &+ Cb_0\|\mathbf{w} - \mathbf{w}'\|_{F^0}(\|\mathbf{w}\|_{F^\sigma} + \|\mathbf{w}'\|_{F^\sigma}) + C\delta\|\mathbf{w} - \mathbf{w}'\|_{F^\sigma}.\end{aligned}$$

Note that $\mathbf{w} = (w_+, w_-, w_0)$ and

$$\mathbf{E}(\mathbf{w}) = (E_+(w_+, w_-, w_0), E_-(w_+, w_-, w_0), E_0(w_+, w_-, w_0))$$

as in [11]. The rest of the notation (the norm $\|.\|_{N^\sigma}$ and the function ψ) is also taken from [11]. We have a slightly different formula for E_0, but (10.27) in [11] gives the estimate in our case also.

We then construct a solution to

$$\begin{cases} \mathbf{v}_t + \mathcal{H}\mathbf{v}_{xx} = \mathbf{E}(\mathbf{v}) & \text{on } \mathbb{R} \times [-\frac{5}{4}, \frac{5}{4}], \\ \mathbf{v}(0) = \Phi, \end{cases}$$

as in (10.32)-(10.37) in [11]. Note that (10.35) and $\|v(\Phi) - v(\Phi')\|_{F^0([-\frac{5}{4},\frac{5}{4}])} \leq C\|\Phi - \Phi'\|_{\widetilde{H}^0}$ hold here too. Next, with

$$\Phi = (\phi_+, \phi_-, \phi_0) = (e^{iU_0(0,.)}P_{+\text{high}}\phi, e^{-iU_0(0,.)}P_{-\text{high}}\phi, 0),$$

$\Phi \in \widetilde{H}^{20}$, by Lemma 10.1 in [11].

We next show $(w_+, w_-, w_0) = \mathbf{v}(\Phi)$ in $\mathbb{R} \times [-1, 1]$. This is as in [11]. Proposition 8, and the second estimate in Proposition 9 now follow from the bounds on $\mathbf{v}(\Phi)$ i.e. (10.35). For Proposition 9, note that for N large, U_0 corresponding to ϕ and to ϕ_N defined by $\hat{\phi}_N = \mathbf{1}_{[-N,N]}(\xi)\hat{\phi}(\xi)$ are the same. We then have

$$u(t, x) = u_0^{(1)} + u - u_0^{(1)} = u_0^{(1)} + \widetilde{u} = u_0^{(1)} + e^{-iU_0}w_+ + e^{iU_0}w_- + w_0$$

and similarly,

$$u^N(t, x) = u_0^{(1)} + u^N - u_0^{(1)} = u_0^{(1)} + e^{-iU_0}w_+^N + e^{iU_0}w_-^N + w_0^N.$$

Hence,

$$\begin{aligned}\sup_{t\in[-1,1]} \|u(t, .) - u^N(t, .)\|_{L^2} &\leq \sup_{t\in[-1,1]} \|w(t) - w^N(t)\|_{L^2} \\ &\leq C\|\psi(t)[w - w^N]\|_{F^0} \leq C\|\phi - \phi^N\|_{L^2}\end{aligned}$$

as desired, giving Proposition 9.

A. Appendix

First, we recall the following inequalities:

Lemma 14.

$$\forall 2 \le p < +\infty, \quad \|f\|_{L^p} \le C_p \|f\|_{L^2}^{\frac{2}{p}} \|D^{\frac{1}{2}} f\|_{L^2}^{\frac{p-2}{p}}, \tag{A.1}$$

$$\|D(fg) - gDf\|_{L^2} \le C\|f\|_{L^4}\|Dg\|_{L^4}, \tag{A.2}$$

$$\|D^{\frac{1}{2}}(fg) - gD^{\frac{1}{2}} f\|_{L^2} \le C\|f\|_{L^4}\|D^{\frac{1}{2}} g\|_{L^4}. \tag{A.3}$$

Recall that (A.1) is the Gagliardo-Nirenberg inequality, which follows from complex interpolation and Sobolev embedding.

Estimate (A.2) is due to Calderón [6], see also Coifman and Meyer [7], formula (1.1).

Estimate (A.3) is a consequence of Theorem A.8 in [13] for functions depending only on x, with the following choice of parameters: $\alpha = \frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$, $p = 2$, $p_1 = p_2 = 4$.

A.1. Proof of (2.15)

We claim that for a function $u(x)$ fixed in $H^2(\mathbb{R})$

$$\int_{y=0} \partial_y(U^2)\Phi = -2 \iint_{\mathbb{R}^2_+} |\nabla U|^2 \Phi + \int_{y=0} U^2 \partial_y \Phi \tag{A.4}$$

where $U(x, y)$ is the harmonic extension of $u(x)$ in $\mathbb{R}^2_+$ and $\Phi(x, y)$ is defined in (2.14).

First, we observe that

$$U, \nabla U \in L^\infty(\mathbb{R}^2_+) \quad \text{and} \quad \sup_{y>0} |U(x, y)| \to 0 \text{ as } |x| \to +\infty. \tag{A.5}$$

Indeed, from [26], Theorem 1, p. 62, we have $\sup_{y>0} |U(x, y)| \le Mu(x)$, where $Mu(x)$ is the maximal function of u (see [26] Chapter 1), and similarly, $\sup_{y>0} |\partial_x U(x, y)| \le Mu_x(x)$, $\sup_{y>0} |\partial_y U(x, y)| \le M(\mathcal{H}u_x)(x)$. Moreover, from [26] Theorem 1, p. 5, since $u, u_x, \mathcal{H}u_x \in H^1 \subset L^\infty$, we obtain $Mu, Mu_x, M(Hu_x) \in L^\infty$. Finally, since $u \in H^1$, we have $|u(x)| \to 0$ as $|x| \to +\infty$, which implies by the definition of the maximal function (see [26, page 4]) that $Mu(x) \to 0$ as $|x| \to +\infty$. Thus (A.5) is proved.

Let $R > 0$. We use the Green formula on $D_R^+ = \{(x, y) \in \mathbb{R}^2_+ \mid x^2 + y^2 < R^2\}$. Let $\Gamma_R^+ = \{(x, y) \in \mathbb{R}^2_+ \mid x^2 + y^2 = R^2\}$ and $I_R = (x, 0) \mid x \in [-R, R]$.

Then:

$$\int_{\Gamma_R^+\cup I_R} \partial_n(U^2)\Phi = -\iint_{D_R^+} (\Delta U^2)\Phi + \iint_{D_R^+} U^2\Delta\Phi + \int_{\Gamma_R^+\cup I_R} U^2\partial_n\Phi$$
$$= -2\iint_{D_R^+} |\nabla U|^2\Phi + \int_{\Gamma_R^+\cup I_R} U^2\partial_n\Phi, \tag{A.6}$$

where ∂_n denotes the inward normal derivative since $\Delta\Phi = 0$ and $\Delta U^2 = 2|\nabla U|^2$. Therefore, we only have to prove the following convergence results:

$$\lim_{R\to+\infty}\int_{\Gamma_R^+} \partial_n(U^2)\Phi{=}0,\ \lim_{R\to+\infty}\int_{I_R} \partial_n(U^2)\Phi = \int_{y=0} \partial_y(U^2)\Phi{=}2\int(\mathcal{H}u_x)u\varphi' \tag{A.7}$$
$$\lim_{R\to+\infty}\int_{\Gamma_R^+} U^2\partial_n\Phi{=}0,\quad \lim_{R\to+\infty}\int_{I_R} U^2\partial_n\Phi = \int_{y=0} U^2\partial_y\Phi = \int u^2(\mathcal{H}\varphi''). \tag{A.8}$$

The limits $\lim_{R\to+\infty}\int_{-R}^R(\mathcal{H}u_x)u\varphi' = \int(\mathcal{H}u_x)u\varphi'$ and

$$\lim_{R\to+\infty}\int_{-R}^{R} u^2(\mathcal{H}\varphi'') = \int u^2(\mathcal{H}\varphi'')$$

are clear since $u \in H^1$. Next, from the expression of $\Phi(x, y)$ in (2.14), we have $\Phi(x, y) \le C(1 + y)R^{-2}$ on Γ_R^+. Therefore, from (A.5), ($d\sigma$ denotes the unit lenght element on Γ_R^+)

$$\begin{aligned}\int_{\Gamma_R^+} |\partial_n(U^2)\Phi| &\le \frac{1}{R^2}\int_{\Gamma_R^+} |\nabla U||U|(1 + y)d\sigma\\
&\le \frac{C}{R^2}\int_{\Gamma_R^+\cap\{|x|\le\sqrt{R}\}} (1 + y)d\sigma + C \sup_{|x|>\sqrt{R},y>0} |U(x, y)|\\
&\le \frac{C}{\sqrt{R}} + C \sup_{|x|>\sqrt{R},y>0} |U(x, y)|\end{aligned}$$

and so (A.7) is proved. Estimate (A.8) is proved similarly and is easier since $\partial_y\Phi$ has more decay than Φ.

A.2. Proof of (2.30)

In the proof of (2.30), the time t is fixed, so we set $y_0 = x_0 + \lambda(t_0 - t)$.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\chi = 1$ on $[0, 1]$, $\chi = 0$ on $(-\infty, -1] \cap [2, +\infty)$ and $\chi \le 1$ on $\mathbb{R}$. Let $\chi_n(x) = \chi(x - n)$. Then, by the

Gagliardo-Nirenberg inequality (A.1), we obtain

$$\int |\eta|^3\varphi'(x-y_0) \le \sum_{n\in\mathbb{Z}} \int_n^{n+1} |\eta|^3\varphi'(x-y_0) \le \sum_{n\in\mathbb{Z}} \left(\int |\eta|^3\chi_n^3\right) \sup_{[n-y_0,n+1-y_0]} \varphi'$$
$$\le \sum_{n\in\mathbb{Z}} \left(\int |D^{\frac{1}{2}}(\eta\chi_n)|^2\right)^{\frac{1}{2}} \left(\int (\eta\chi_n)^2\right) \sup_{[n-y_0,n+1-y_0]} \varphi'.$$

By Lemma 14 and (2.2), we get

$$\|D^{\frac{1}{2}}(\eta\chi_n)\|_{L^2} \le C\|(D^{\frac{1}{2}}\eta)\chi_n\|_{L^2} + C\|\eta\|_{L^4}\|D^{\frac{1}{2}}\chi_n\|_{L^4} \le C\|\eta\|_{H^{\frac{1}{2}}} \le C\alpha_0.$$

Thus,

$$\int |\eta|^3\varphi'(x-y_0) \le C\alpha_0 \sum_{n\in\mathbb{Z}} \left(\int (\eta\chi_n)^2\right) \sup_{[n-y_0,n+1-y_0]} \varphi' \le C\alpha_0 \int \eta^2\varphi'(x-y_0)$$

by the properties of χ and the following elementary remark:

$$\forall y\in\mathbb{R}, \quad \sup_{[y,y+4]} \varphi' \le C \inf_{[y,y+4]} \varphi'. \tag{A.9}$$

Note that the constant C is independent of A, for $A > 1$.

A.3. Properties of the operator $\mathcal{L}$

We recall from [29]–[31] and [3] the following properties of $\mathcal{L}$ (recall $\mathcal{L}\eta = -\mathcal{H}\eta_x + \eta - Q\eta$).

Lemma 15. *The operator $\mathcal{L}$ is self-adjoint on L^2 and satisfies the following properties.*

(i) *The operator $\mathcal{L}$ has exactly one negative eigenvalue λ_0 of multiplicity 1 with corresponding eigenfunction f_0, which can be chosen so that $f_0 > 0$.*

(ii) $\operatorname{Ker}\mathcal{L} = \operatorname{span}\{Q'\}$.

(iii) *There exists $\lambda > 0$ such that, for all $z \in H^{\frac{1}{2}}$,*

$$(z,Q) = (z,Q') = 0 \quad \Rightarrow \quad (\mathcal{L}z,z) \ge \lambda(z,z). \tag{A.10}$$

Remark 6. Recall from Bennett et al. ([3], Appendix B) that the spectrum of $\mathcal{L}$ is completely understood. Indeed, the operator $\mathcal{L}$ has exactly four eigenvalues, $\lambda_0 = -\frac{1}{2}(1+\sqrt{5})$, 0, $\frac{1}{2}(-1+\sqrt{5})$, 1 and a continuous spectrum $[1,+\infty)$.

Now, we sketch a proof of Lemma 15 using general arguments from [29]–[31].

Sketch of proof. One easily checks that $\mathcal{L}Q' = 0$ (differentiate the equation of $Q(x+x_0)$ with respect to x_0 and take $x_0 = 0$), and that $\mathcal{L}f_0 = -\lambda_0 f_0$, where $f_0 = Q + \frac{1}{4}(1+\sqrt{5})Q^2$ (by (3.35)). Moreover, the proof of (i) follows from the variational characterization of Q, see Proposition 4.2 of [31]. Recall that $\frac{d}{dc}\int Q_c^2 = \int Q^2 > 0$ (subcriticality) implies that $\inf\{(\mathcal{L}f,f);\ (f,Q)=0, \|f\|_{L^2}=1\}=0$ (see proof of Proposition 5.1 in [31] and Proposition 3.1 in [30]).

Now, we give a new proof for (ii). Let $f \in L^2$ be such that $\mathcal{L}f = 0$. First, we remark that $f \in H^s$, for all $s \geq 0$. Moreover, by similar estimates as in [1], we have $|f(x)| \leq \frac{C}{1+x^2}$. Integrating $\mathcal{L}f = 0$ on $\mathbb{R}$, we obtain $\int(f - fQ) = 0$. But, we also have $(f,Q) = -(f,\mathcal{L}S) = -(\mathcal{L}f,S) = 0$ (see (3.35)). Thus, $\int f = 0$ and we can define $g(x) = \int_{-\infty}^{x} f(s)ds \in L^2$, which satisfies $\mathcal{L}(g') = 0$. Let now $\widetilde{g} = g - aQ$ be such that $(\widetilde{g},Q) = 0$ and $\mathcal{L}(\widetilde{g}') = 0$. From (3.12) and (3.36), we obtain $\int |D^{\frac{1}{2}}\widetilde{g}|^2 + (\mathcal{L}\widetilde{g},\widetilde{g}) \leq 0$. But, since $(\widetilde{g},Q) = 0$, we have $(\mathcal{L}\widetilde{g},\widetilde{g}) \geq 0$. Thus, $\int |D^{\frac{1}{2}}\widetilde{g}|^2 = 0$ and $\widetilde{g} \equiv 0$, so that $g = aQ$ and $f = aQ'$.

Finally, we sketch the proof of (iii), which follows from the arguments of the proof of Proposition 2.9 in [29] (see also Section 6, example 4 in [31]). By contradiction, assuming that

$$\inf\{(\mathcal{L}f,f);\ (f,Q) = (f,Q') = 0,\ \|f\|_{L^2} = 1\} = 0,$$

and using compactness arguments as in Proposition 2.9 in [29], we obtain the existence of $f \in H^{\frac{1}{2}}$, $\lambda,\beta,\gamma \in \mathbb{R}$ (Lagrange multipliers) such that

$$(\mathcal{L}f,f) = 0, \quad (\mathcal{L}-\lambda)f = \beta Q + \gamma Q', \quad (f,Q) = (f,Q') = 0, \quad \|f\|_{L^2} = 1.$$

But, taking the scalar product by f, we find $\lambda = 0$. Then, taking the scalar product by Q', we find $\gamma = 0$. Taking the scalar product with S (see (3.35)), using $(S,Q) = \frac{1}{2}(Q,Q)$ and $\mathcal{L}(S) = -Q$, we find $\beta = 0$, so that $\mathcal{L}f = 0$ and $(f,Q') = 0$. This implies $f = 0$ by (ii), a contradiction.

References

[1] Amick, C. J. and Toland, J. F.: Uniqueness and related analytic properties for the Benjamin-Ono equation—a nonlinear Neumann problem in the plane. *Acta Math.* **167** (1991), 107–126.

[2] Benjamin, T. B.: Internal waves of permanent form in fluids of great depth. *Journal of Fluid Mechanics* **29** (1967), 559–592.

[3] Bennett, D. P., Brown, R. W., Stansfield, S. E., Stroughair, J. D. and Bona, J. L.: The stability of internal solitary waves. *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 351–379.

[4] Bona, J. L., Souganidis, P. E. and Strauss, W. A.: Stability and instability of solitary waves of Korteweg-de Vries type. *Proc. Roy. Soc. London Ser. A* **411** (1987), 395–412.

[5] Burq, N. and Planchon, F.: On well-posedness for the Benjamin-Ono equation. *Math. Ann.* **340** (2008), no. 3, 497–542.

[6] Calderón, A.-P.: Commutators of singular integral operators. *Proc. Nat. Acad. Sci. U.S.A.* **53** (1965), 1092–1099.

[7] Coifman, R. R. and Meyer, Y.: On commutators of singular integrals and bilinear singular integrals. *Trans. Amer. Math. Soc.* **212** (1975), 315–331.

[8] Herr, S., Ionescu, A. D., Kenig, C. E. and Koch, H.: Global solutions to dispersive nonlinear equations. Preprint.

[9] Ginibre, J. and Velo, G.: Commutator expansions and smoothing properties of generalized Benjamin-Ono equations. *Ann. Inst. H. Poincaré Phys. Théor.* **51** (1989), 221–229.

[10] Gustafson, S., Takaoka, H. and Tsai, T.-P.: Stability in $H^{\frac{1}{2}}$ of the sum of K solitons for the Benjamin-Ono equation. *J. Math. Phys.* **50** (2009), no. 1, 013101, 14 pp.

[11] Ionescu, A. D. and Kenig, C. E.: Global well-posedness of the Benjamin-Ono equation in low-regularity spaces. *J. Amer. Math. Soc.* **20** (2007), 753–798.

[12] Kato, T.: On the Cauchy problem for the (generalized) Korteweg-de Vries equation. In *Studies in applied mathematics*, 93–128. Adv. Math. Suppl. Stud. **8**, Academic Press, New York, 1983.

[13] Kenig, C. E. , Ponce, G. and Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.* **46** (1993), 527–620.

[14] Martel, Y.: Linear problems related to asymptotic stability of solitons of the generalized KdV equations. *SIAM J. Math. Anal.* **38** (2006), 759–781.

[15] Martel, Y. and Merle, F.: Instability of solitons for the critical generalized Korteweg-de Vries equation. *Geom. Funct. Anal.* **11** (2001), 74–123.

[16] Martel, Y. and Merle, F.: A Liouville theorem for the critical generalized Korteweg-de Vries equation. *J. Math. Pures Appl. (9)* **79** (2000), 339–425.

[17] Martel, Y. and Merle, F.: Asymptotic stability of solitons for subcritical generalized KdV equations. *Arch. Ration. Mech. Anal.* **157** (2001), 219–254.

[18] Martel, Y. and Merle, F.: Asymptotic stability of solitons of the subcritical gKdV equations revisited. *Nonlinearity* **18** (2005), no. 1, 55–80.

[19] Martel, Y. and Merle, F.: Asymptotic stability of solitons of the gKdV equations with a general nonlinearity. *Math. Ann.* **341** (2008), 391–427.

[20] Martel, Y. and Merle, F.: Refined asymptotics around soliton for gKdV equations. *Discrete Contin. Dyn. Syst.* **20** (2008), 177–218.

[21] Martel, Y., Merle, F. and Tsai, T.-P.: Stability and asymptotic stability in the energy space of the sum of N solitons for the subcritical gKdV equations. *Comm. Math. Phys.* **231** (2002), 347–373.

[22] Matsuno, Y.: The Lyapunov stability of the N-soliton solutions in the Lax hierarchy of the Benjamin-Ono equation. *J. Math. Phys.* **47** (2006), 103505, 13pp.

[23] Neves, A. and Lopes, O.: Orbital stability of double solitons for the Benjamin-Ono equation. *Comm. Math. Phys.* **262** (2006), 757–791.

[24] Pego, R. L. and Weinstein, M. I.: Asymptotic stability of solitary waves. *Comm. Math. Phys.* **164** (1994), 305–349.

[25] Ponce, G.: Smoothing properties of solutions to the Benjamin-Ono equation. In *Analysis and partial differential equations*, 667–679. Lecture Notes in Pure and Appl. Math. **122**. Dekker, New York, 1990.

[26] Stein, E.: *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series **30**. Princeton Univ. Press, Princeton, 1970.

[27] Tao, T.: Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$. *J. Hyperbolic Differ. Equ.* **1** (2004), 27–49.

[28] Toland, J. F.: The Peierls-Nabarro and Benjamin-Ono equations. *J. Funct. Anal.* **145** (1997), 136–150.

[29] Weinstein, M. I.: Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.* **16** (1985), 472–491.

[30] Weinstein, M. I.: Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.* **39** (1986), 51–67.

[31] Weinstein, M. I.: Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation. *Comm. Partial Differential Equations* **12** (1987), 1133–1173.

Recibido: 2 de enero de 2008
Revisado: 22 de mayo de 2008

Carlos E. Kenig
Department of Mathematics, University of Chicago
5734 University Ave.
Chicago, Illinois 60637-1514, USA
cek@math.uchicago.edu

Yvan Martel
Université de Versailles Saint-Quentin-en-Yvelines, Mathématiques
45, Av. des Etats-Unis
78035 Versailles cedex, France
martel@math.uvsq.fr

The first author is partly supported by the NSF grant DMS-0456583. This work was initiated when the second author was visiting the University of Chicago. He would like to thank the Department of Mathematics for its hospitality. The second author is partly supported by the Agence Nationale de la Recherche (ANR ONDENONLIN).