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On sup-norm bounds part I: Ramified Maaß newforms over number fields

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Abstract. We prove new upper bounds for the sup-norm of Hecke Maaß newforms on GL(2) over a number field. Our newforms are more general than those considered in a recent paper by Blomer, Harcos, Maga, and Milićević: we do not require square-free level. Furthermore, we allow for non-trivial central character. Over the rationals we recover the best bounds obtained by Saha.

Keywords: sup-norm, cusp forms, number fields, amplification.

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1. Introduction

In this paper we prove bounds for the L^∞-norm of automorphic forms on GL_2 which improve upon the local bounds. The problem of establishing such bounds is commonly referred to as the sup-norm problem for GL_2. Just recently in [4] this problem was solved over number fields for square-free level and trivial central character. Previously, in [20], this problem was solved over $\mathbb{Q}$ with arbitrary level and central character. In the present paper we go beyond these ideas. Containing each of them as special cases, our results recover the strongest bounds from both.

The true size of the sup-norm of automorphic forms is still quite mysterious. For conjectures and results concerning the sup-norm problem we refer to [4,20] and the references therein. Let us just say that there are many results towards improved upper bounds for the sup-norm in various settings. However, there are also some results giving lower bounds. On non-compact surfaces these lower bounds can have two sources. First, they can come from the transition region of Whittaker functions. In this case the peak appears far from the so-called bulk of the manifold. Such lower bounds were considered for example in [2, 8, 19]. Second, they can appear in the bulk of the manifold, which also appears in the compact setting. In this case we refer to [7] for more details. The reason for mentioning these two sources for lower bounds is that we will encounter them in some sense in our arguments.

Roughly speaking, our proof will consist of two main parts which are delicate generalizations of their counterparts from [4,20]. First, we will estimate the Whittaker expansion to obtain suitable bounds outside the bulk. Instead of reducing this estimate to second moments of Whittaker functions as in [20], we will use fourth moments. Second, we use the amplification method to obtain good bounds in the bulk. To control the amplifier we have to use a generalization of the Siegel–Walfisz theorem to number fields, obtained in [14].

Next we are going to state and briefly discuss our main theorems. The notation used is mostly standard and will be explained in detail in Section 1.2.

1.1. Statement of results

The theorems we will now state deal with cuspidal Hecke–Maaß newforms $\phi_\circ$ on GL_2 over a number field F. As explained below, for us these will be functions on $\mathrm{GL}_2(\mathbb{A})$ which naturally come from cuspidal automorphic representations. However, to understand the statement of our theorems it suffices to think of them as eigenfunctions on a congruence quotient of the symmetric space $\mathrm{GL}_2(F_\infty)/K_\infty$. Either way we have to introduce some parameters in order to measure the complexity of these functions.

First of all, each archimedean place ν of F will contribute a spectral parameter

$$\lambda_\nu = \begin{cases} \frac{1}{4} + t_\nu^2 & \text{if } \nu \text{ is real,} \\ 1 + 4t_\nu^2 & \text{if } \nu \text{ is complex.} \end{cases}$$

These are the Laplace eigenvalues of $\phi_\circ$ and we define the parameters $T = (T_\nu)_\nu$ by

$$T_\nu = \max(1/2, |t_\nu|).$$

A more geometric invariant is given by the level $\mathfrak{n}$ of $\phi_\circ$. Classically the absolute norm, $\mathcal{N}(\mathfrak{n})$, of this level can roughly be thought of as the volume of the congruence quotient on which $\phi_\circ$ lives. We take $\mathfrak{n}_0^2$ to be the largest square ideal dividing $\mathfrak{n}$ and set $\mathfrak{n}_2 = \mathfrak{n}/\mathfrak{n}_0^2$. Obviously $\mathfrak{n}_2$ is squarefree and we have the decomposition $\mathfrak{n} = \mathfrak{n}_2\mathfrak{n}_0^2$. Finally, we allow $\phi_\circ$ to transform with respect to some possibly non-trivial nebentypus ω of conductor $\mathfrak{m}$ and we set $\mathfrak{m}_1 = \mathfrak{m}/\gcd(\mathfrak{m}, \mathfrak{n}\mathfrak{n}_0)$. We can now state the main theorems of this article.

Theorem 1.1. *Let $\phi_\circ$ be a cuspidal Hecke–Maaß newform of level $\mathfrak{n}$ and spectral parameter $(t_\nu)_\nu$. Then*

$$\frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2} \ll_{F,\varepsilon} (|T|_\infty \mathcal{N}(\mathfrak{n}))^\varepsilon \big(|T|_\infty^{5/12}\mathcal{N}(\mathfrak{n}_2)^{1/3}\mathcal{N}(\mathfrak{n}_0^2)^{1/4}\mathcal{N}(\mathfrak{m}_1)^{1/2} + |T|_{\mathbb{R}}^{1/4}|T|_{\mathbb{C}}^{1/2}\mathcal{N}(\mathfrak{n})^{1/4}\mathcal{N}(\mathfrak{m}_1)^{1/2}\big).$$

Note that if we assume that $\mathfrak{n}$ is square-free, then this bound reads

$$\frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2} \ll_{F,\varepsilon} (|T|_\infty \mathcal{N}(\mathfrak{n}))^\varepsilon \big(|T|_\infty^{5/12}\mathcal{N}(\mathfrak{n})^{1/3} + |T|_{\mathbb{R}}^{1/4}|T|_{\mathbb{C}}^{1/2}\mathcal{N}(\mathfrak{n})^{1/4}\big).$$

Thus, if we further assume the nebentypus to be trivial (i.e. $\mathfrak{m} = 1$), then we recover [4, Theorem 1]. On the other hand, if we take $F = \mathbb{Q}$, $\mathfrak{n} = (N)$ and $\mathfrak{m}_1 = (M_1)$, then our bound reads

$$\frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2} \ll_{F,\varepsilon} (|T|_\infty N)^\varepsilon |T|_\infty^{5/12} N_2^{1/3} N_0^{1/2} M_1^{1/2},$$

which agrees with [20, Theorem 3.2].

However, this theorem fails to meet our expectations for non-totally-real fields F. Therefore, we will prove the following alternative result.

Theorem 1.2. *Let F be a number field with maximal totally real subfield $F^{\mathbb{R}}$ such that $[F : F^{\mathbb{R}}] = m \geq 2$. Further let $\phi_\circ$ be a cuspidal Hecke–Maaß newform of level $\mathfrak{n}$ and spectral parameter $(t_\nu)_\nu$. Then*

$$\frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2} \ll_{F,\varepsilon} (|T|_\infty \mathcal{N}(\mathfrak{n}))^\varepsilon |T|_\infty^{\frac{1}{2}-\frac{1}{8m-4}} \mathcal{N}(\mathfrak{n}_2)^{\frac{1}{2}-\frac{1}{8m-4}} \mathcal{N}(\mathfrak{n}_0^2)^{1/4}\mathcal{N}(\mathfrak{m}_1)^{1/2}.$$

For $\mathfrak{n}$ square-free and with trivial nebentypus this reduces precisely to [4, Theorem 2]. It has no analogue in [20], since it does not apply to $F = \mathbb{Q}$.

In order to put these results into perspective let us recall that the local bound in our setting reads

$$\frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2} \ll_{F,\varepsilon} (|T|_\infty \mathcal{N}(\mathfrak{n}_2\mathfrak{n}_0\mathfrak{m}_1))^{1/2+\varepsilon}. \tag{1.1}$$

Thus both our theorems are sublocal (or subconvex) in the parameters $|T|_\infty$ and $\mathcal{N}(\mathfrak{n}_2)$. In particular, Theorem 1.1 achieves very strong exponents in these two aspects when $|T|_\mathbb{C}$ is not to large. Even though we do not break the convexity barrier in the other aspects, this paper is still the first to consider the sup-norm problem for automorphic forms over number fields in this generality. This also means that to the best of our knowledge this is the first place where the local bound as stated in (1.1) is rigorously established in such generality (see Corollary 3.18 below).

While it seems undisputed that (1.1) is the local bound in the spectral and the square-free level aspect, the story is not as simple for general level and arbitrary nebentypus. An indication that also in general (1.1) is the correct notion of local bound is given over $\mathbb{Q}$ by [17]. Furthermore, one recovers (1.1) by only using the Fourier/Whittaker expansion together with a suitable generating domain (i.e. Corollary 2.7 together with Proposition 3.17). This was already noted in [20] over $\mathbb{Q}$.

In general only the contribution of $\mathfrak{m}_1$ to our results as well as to (1.1) remains disputable. Note that in the highly ramified situation when $\mathfrak{m} = \mathfrak{n}_0^2 = \mathfrak{n}$ we only get[1]

$$\frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2} \ll_{F,\varepsilon,|T|_\infty} \mathcal{N}(\mathfrak{n})^{1/2}.$$

However, in this situation we also have the very strong lower bound

$$\frac{\|\phi_\circ\|_\infty}{\|\phi_\circ\|_2} \gg_{F,\varepsilon,|T|_\infty} \mathcal{N}(\mathfrak{n})^{1/4}$$

derived in [21]. The recent work [10] over $\mathbb{Q}$ shows that this lower bound is essentially sharp. This suggests that one should be able to remove the contribution of $\mathfrak{m}_1$ from our theorems as well as in (1.1). However, doing so in practice over arbitrary number fields without assuming the full Ramanujan–Petersson conjecture and without compromising the quality of the exponents in the other aspects seems to require some new ideas and more hard analysis.

Remark 1.3. In [1], which is the second part of this manuscript, the author applies similar ideas to solve the sup-norm problem for very general newform Eisenstein series over number fields. Furthermore, a slight generalization of Theorem 1.1 can be found in the Bristol PhD thesis of the author, where $\phi_\circ$ is allowed to be a cuspidal Hecke–Hilbert–Maaß newform. This allows for the possibility that $\phi_\circ$ looks like a holomorphic modular form of weight k_ν at several real places ν.

1.2. Set-up and basic definitions

Let F be a number field of degree $n = r_1 + 2r_2$, where r_1 is the number of real embeddings and $2r_2$ is the number of complex embeddings. We denote the norm on $F/\mathbb{Q}$ by $\mathcal{N}$

[1] By direct extrapolation from the undisputed local bound for square-free levels one may think of this as the trivial bound in general.

and let $\mathcal{O}_F$ be the ring of integers in F. We denote a typical ideal in $\mathcal{O}_F$ by $\mathfrak{n}$ and save the letter $\mathfrak{p}$ for prime ideals. Each prime ideal gives rise to a non-archimedean place of F which we also denote by $\mathfrak{p}$. The corresponding (canonically normalized) valuation (used for field elements and ideals interchangeably) will be called $v_{\mathfrak{p}}(\cdot)$ and gives rise to the absolute value $|\cdot|_{\mathfrak{p}} = q_{\mathfrak{p}}^{-v_{\mathfrak{p}}(\cdot)}$, where $q_{\mathfrak{p}} = \mathcal{N}(\mathfrak{p})$. In a similar spirit we use ν for an archimedean place and at the same time for the corresponding embedding $\nu\colon F \to F_\nu$. We put

$$|\cdot|_\nu = |\cdot|^{[F_\nu:\mathbb{R}]}.$$

Here and in the following $|\cdot|$ always denotes the standard absolute value on $\mathbb{R} \subset \mathbb{C}$. If ν is a real place, then $F_\nu = \mathbb{R}$ and we equip it with the standard Lebesgue measure giving mass 1 to the interval $[0, 1]$. On the other hand, if ν is complex, then $F_\nu = \mathbb{C}$ and we use the two-dimensional Lebesgue measure coming from $\mathbb{R}^2$ normalized by $\mathrm{Vol}([0, 1]^2) = 1$.

If $F_{\mathfrak{p}}$ is the local non-archimedean field associated to $\mathfrak{p}$ then we write $\mathfrak{o}_{\mathfrak{p}}$ for its ring of integers and $\varpi_{\mathfrak{p}}$ for its uniformizer. These fields are equipped with two measures. First, the Haar measure $\mu_{\mathfrak{p}}$ on $(F, +)$, which we normalize so that $\mu_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}}) = 1$. Further, we have the Haar measure $\mu_{\mathfrak{p}}^\times$ on $(F^\times, \cdot)$. This will be normalized to satisfy $\mu_{\mathfrak{p}}^\times(\mathfrak{o}_{\mathfrak{p}}^\times) = 1$.

We define $F_\infty = \prod_\nu F_\nu$ and equip it with the modulus $|\cdot|_\infty = \prod_\nu |\cdot|_\nu$. Sometimes we use $|\cdot|_{\mathbb{R}}$ (respectively $|\cdot|_{\mathbb{C}}$) to denote the part of $|\cdot|_\infty$ coming from the real (respectively complex) embeddings only. Let $\mathbb{A}_{\mathrm{fin}}$ denote the finite adeles equipped with the absolute value $|\cdot|_{\mathrm{fin}}$ being the product of all the local absolute values. Note that if $q \in F$ is diagonally embedded in $\mathbb{A}_{\mathrm{fin}}$, then $|q|_{\mathrm{fin}} = \mathcal{N}((q))$. We will also write $\mathcal{N}(q) = \mathcal{N}((q))$ in this case. The usual adele ring is then given by

$$\mathbb{A}_F = F_\infty \times \mathbb{A}_{\mathrm{fin}}$$

and equipped with $|\cdot|_{\mathbb{A}}$ and μ in the usual manner. We also define the set of totally positive field elements F^+ to contain all $x \in F$ such that $x_\nu > 0$ for all real ν. Furthermore, put $F^0(\mathbb{A}_F) = \{a \in \mathbb{A}_F\colon |a|_{\mathbb{A}_F} = 1\}$ and $F_\infty^+ = \mathbb{R}^+ \subset F_\infty$ diagonally. Finally, let $F_{\infty,+}$ denote the set of elements $x \in F_\infty$ such that $x_\nu > 0$ for all real ν.

Further, let us choose ideal representatives $\theta_1, \ldots, \theta_{h_F} \in \hat{\mathcal{O}}_F$, where h_F denotes the class number of F. We write d_F for the discriminant of F and $\mathfrak{d}$ for the different ideal of F. Then by [18, Theorem 2.9] we have $\mathcal{N}(\mathfrak{d}) = |d_F|$. For any ideal $\mathfrak{m}$ we use $[\mathfrak{m}]_{\mathfrak{n}} = \mathfrak{m}/(\mathfrak{m}, \mathfrak{n}^\infty)$ for the coprime-to-$\mathfrak{n}$ part of $\mathfrak{m}$.

Given a character $\chi\colon F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}$ we associate the corresponding analytically normalized L-function

$$\Lambda(s, \chi) = \underbrace{\prod_\nu L_\nu(s, \chi_\nu)}_{=\gamma_\infty(s,\chi)} \underbrace{\prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \chi_{\mathfrak{p}})}_{=L(s,\chi)}.$$

A complete description of the local factors can be found in [16] and reads

$$L_{\mathfrak{p}}(s, \chi_{\mathfrak{p}}) = \begin{cases} (1 - \chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}) q_{\mathfrak{p}}^{-s})^{-1} & \text{if } \chi_{\mathfrak{p}} \text{ is unramified,} \\ 1 & \text{if } \chi_{\mathfrak{p}} \text{ is ramified} \end{cases}$$

in the non-archimedean case and

$$L_\nu(s,\chi_\nu) = \begin{cases} \pi^{-\frac{s+t}{2}}\Gamma\left(\frac{s+t}{2}\right) & \text{if } \nu \text{ is real and } \chi = |\cdot|_\nu^t, \\ \pi^{-\frac{s+t+1}{2}}\Gamma\left(\frac{s+t+1}{2}\right) & \text{if } \nu \text{ is real and } \chi = \operatorname{sgn}(\cdot)|\cdot|_\nu^t, \\ 2(2\pi)^{-(s+t+|l|/2)}\Gamma(s+t+|l|/2) & \text{if } \nu \text{ is complex and } \chi = \arg(\cdot)^l|\cdot|_\nu^t \end{cases}$$

in the archimedean case. If χ is the trivial character, this leads to the Dedekind zeta function and we introduce the shorthand notation $\zeta_\mathfrak{p}(s) = L_\mathfrak{p}(1,s)$ and $\zeta_\mathfrak{n}(s) = \prod_{\mathfrak{p}|\mathfrak{n}}\zeta_\mathfrak{p}(s)$. At the archimedean places we write $L_\nu(1,s) = \Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$ if ν is real and $L_\nu(1,s) = \Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$ otherwise.

Let R be a commutative ring with 1. Typically this will be one of the objects introduced above. Then we set $G(R) = \mathrm{GL}_2(R)$. We will also need the subgroups

$$Z(R) = \left\{ z(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} : r \in R^\times \right\}, \quad A(R) = \left\{ a(r) = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} : r \in R^\times \right\},$$
$$N(R) = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in R \right\}, \quad B(R) = Z(R)A(R)N(R).$$

We use the following compact subgroups of $G(R)$ which depend on the underlying ring R:

$$G(F_\nu) \supset K_\nu = \begin{cases} \mathrm{SU}_2(\mathbb{C}) & \text{if } \nu \text{ is complex}, \\ \mathrm{SO}_2(\mathbb{R}) & \text{if } \nu \text{ is real}, \end{cases}$$
$$G(F_\mathfrak{p}) \supset K_\mathfrak{p} = \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p}),$$
$$K_\infty = \prod_\nu K_\nu.$$

At the non-archimedean places we additionally need the smaller groups

$$K_\mathfrak{p}^0(n) = K_\mathfrak{p} \cap \begin{bmatrix} \mathfrak{o}_\mathfrak{p} & \varpi_\mathfrak{p}^n\mathfrak{o}_\mathfrak{p} \\ \mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p} \end{bmatrix}, \qquad K_{0,\mathfrak{p}}(n) = K_\mathfrak{p} \cap \begin{bmatrix} \mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p} \\ \varpi_\mathfrak{p}^n\mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p} \end{bmatrix},$$
$$K_{1,\mathfrak{p}}(n) = K_\mathfrak{p} \cap \begin{bmatrix} 1+\varpi_\mathfrak{p}^n\mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p} \\ \varpi_\mathfrak{p}^n\mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p} \end{bmatrix}, \quad K_{2,\mathfrak{p}}(n) = K_\mathfrak{p} \cap \begin{bmatrix} \mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p} \\ \varpi_\mathfrak{p}^n\mathfrak{o}_\mathfrak{p} & 1+\varpi_\mathfrak{p}^n\mathfrak{o}_\mathfrak{p} \end{bmatrix}.$$

Globally, we put

$$K_1(\mathfrak{n})_{\text{fin}} = \prod_\mathfrak{p} K_{1,\mathfrak{p}}(v_\mathfrak{p}(\mathfrak{n})), \quad K_1(\mathfrak{n}) = K_\infty \cdot K_1(\mathfrak{n})_{\text{fin}}, \quad K = K_\infty \prod_\mathfrak{p} K_\mathfrak{p}.$$

Further, let

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be the long Weyl element.

Let us briefly describe the measures on the groups we use. Locally we will stick to the measure convention from [19,20]: we use the identifications $N(R) \cong (R,+)$, $A(R) \cong R^\times$, and $Z(R) \cong R^\times$ to transport the measures defined on the local fields to the corresponding groups. Further, we take $\mu_{K_\mathfrak{p}}$ to be the probability Haar measure on $K_\mathfrak{p}$. Globally, we choose the product measure on K. The measures on the adeles and the ideles are given by

$$\mu_{\mathbb{A}_F} = \frac{2^{r_2}}{\sqrt{|d_F|}} \prod_{\nu} \mu_\nu \prod_{\mathfrak{p}} \mu_\mathfrak{p} \quad \text{and} \quad \mu^\times_{\mathbb{A}_F^\times} = \prod_{\nu} \mu^\times_\nu \prod_{\mathfrak{p}} \mu^\times_\mathfrak{p}.$$

This choice implies the volume normalization $\mathrm{Vol}(F\backslash\mathbb{A}_F) = 1$, as can be seen from strong approximation together with [18, Chapter I, Proposition 5.2]. Here it is important to be aware of the convention in [18, p. 211] when identifying the Minkowski space with F_∞. As in the local situation, we use the identifications $N(\mathbb{A}_F) \cong \mathbb{A}_F$ and $A(\mathbb{A}_F) \cong \mathbb{A}_F^\times$ to lift the measures defined above to the groups $N(\mathbb{A}_F)$ and $A(\mathbb{A}_F)$. Finally, we define

$$\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g)\, d\mu(g) \\ = \int_K \int_{\mathbb{A}_F^\times} \int_{N(\mathbb{A}_F)} f(na(y)k)\, d\mu_{N(\mathbb{A}_F)}(n)\, \frac{d\mu^\times_{\mathbb{A}_F^\times}(y)}{|y|_\mathbb{A}}\, d\mu_K(k) \tag{1.2}$$

as in [13].

As mentioned above, we are interested in bounding the sup-norm of Hecke–Maaß newforms over F. In particular, the automorphic forms in question will be spherical at infinity. More precisely, we will study functions

$$\phi \in L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega) \subset L^2(G(F)\backslash G(\mathbb{A}_F), \omega)$$

which are right $K_1(\mathfrak{n})$-invariant, and eigenfunctions of the Casimir element $(C_\nu)_\nu \in \mathcal{U}(\mathfrak{g}_\infty)$ with eigenvalues $(\lambda_\nu)_\nu$. These are automorphic forms in the sense of [6, Section 4.2]. Thus, it is standard procedure to associate a cuspidal automorphic representation[2] π_ϕ to ϕ. As explained in [6, Section 4.6], each cuspidal automorphic representation with central character ω can be (uniquely) realized as a closed invariant subspace of $L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega)$. In this way the problem of estimating the sup-norm of the Maaß newform ϕ is closely linked to the underlying cuspidal automorphic representation π_ϕ. However, the sup-norm itself is only defined for smooth elements in $L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega)$ and it does not make sense in different realizations of π_ϕ. Therefore we will make the following convention.

Convention 1.4. *Let (π, V_π) be a cuspidal automorphic representation with central character ω_π. Then there is an intertwiner $\sigma\colon V_\pi \hookrightarrow L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega)$. Then the* sup-norm *of a K-finite vector $v \in V_\pi$ is defined to be*

$$\|v\|_\infty = \frac{\|\sigma(v)\|_\infty}{\|\sigma(v)\|_2}.$$

[2]We use the definition of an automorphic representation given in [6, Section 4.6]. In particular, irreducibility is included in the definition.

Let us make some remarks concerning this convention.

- Note that this is indeed well defined. First, we observe that by *multiplicity* 1 for GL_2 the intertwiner σ is unique up to scaling. However, the scaling does not matter since we L^2-normalize the image. Secondly, K-finiteness ensures that the L^∞-norm of $\sigma(v)$ is defined.
- This convention may seem unnecessary at first. But it gives us the flexibility to realize π in arbitrary models without changing the fixed cusp form whose sup-norm we want to bound.
- The restriction to K-finite vectors shows that we should actually work with the $G(\mathbb{A}_F)$-module underlying π.

Let us now describe the structure of the cuspidal automorphic representation π, keeping in mind that we are mainly interested in spherical Hecke–Maaß newforms. We write V_π for the representation space of π. First note that since (π, V_π) is a cuspidal automorphic representation it is in particular unitary and admissible. For convenience we assume throughout the text that the central character ω_π of π satisfies $\omega_\pi|_{F_\infty^+} = 1$. This can be achieved without loss of generality by twisting by an unramified character.

By the tensor product theorem [12, Theorem 4] we may assume that

$$\pi = \bigotimes_\nu \pi_\nu \otimes \bigotimes_{\mathfrak{p}} \pi_{\mathfrak{p}},$$

where $(\pi_{\mathfrak{p}}, V_{\pi,\mathfrak{p}})$ (resp. $(\pi_\nu, V_{\pi,\nu})$) is an irreducible representation of $G(F_{\mathfrak{p}})$ (resp. $G(F_\nu)$) with central character $\omega_{\pi,\mathfrak{p}}$ (resp. $\omega_{\pi,\nu}$). Note that this decomposition also preserves the subspaces of K-finite vectors.

Since we are only interested in automorphic forms which are spherical eigenfunctions of the Casimir operator, we can restrict ourselves to a very particular situation at the archimedean places. Indeed, we will always assume that $\pi_\nu = \chi_1 \boxplus \chi_2$ with

$$\begin{aligned} \chi_j(y) &= |y|_\nu^{it_{\nu,j}} \operatorname{sgn}(y)^{m_\nu} && \text{if } \nu \text{ is real,} \\ \chi_j(re^{i\theta}) &= r^{i2t_{\nu,j}} e^{im_\nu\theta} && \text{else.} \end{aligned}$$

These are principal series representations and the representation space is denoted by

$$V_\pi = \mathcal{B}(\chi_1, \chi_2).$$

We define the invariants

$$t_\nu = (t_{\nu,1} - t_{\nu,2})/2 \quad \text{and} \quad s_\nu = t_{\nu,1} + t_{\nu,2}. \tag{1.3}$$

In particular, $\omega_{\pi,\nu}|_{F_\infty^+} = \prod_\nu |\cdot|_\nu^{s_\nu}$. Thus, the assumption $\omega_\pi|_{F_\infty^+} = 1$ yields $\sum_\nu [F_\nu : \mathbb{R}] s_\nu = 0$. Furthermore, $v \in V_{\pi,\nu}$ is an eigenvector of the Casimir operator with eigenvalue

$$\lambda_\nu = \begin{cases} \frac{1}{4} + t_\nu^2 & \text{if } \nu \text{ is real,} \\ 1 + 4t_\nu^2 & \text{else.} \end{cases}$$

This justifies calling t_ν the spectral parameter of π. Note that in absence of a proof of the Ramanujan–Petersson Conjecture we cannot exclude the case of t_ν being imaginary. However, based on ideas by Kim and Sarnak it has been shown in [3, Theorem 1] that

$$t_\nu \in \mathbb{R} \cup \left[-\tfrac{7}{64}, \tfrac{7}{64}\right] \cdot i.$$

Note that a representation π featuring these types of representations at the archimedean places is spherical. In other words, each representation (π_ν, V_π) contains a K_ν-invariant vector v_ν° which is unique up to scaling. At the non-archimedean places we define $n_\mathfrak{p}$ to be the log-conductor of $\pi_\mathfrak{p}$. More precisely, $n_\mathfrak{p}$ is the smallest non-negative integer such that there exists a vector $v_\mathfrak{p}^\circ \in V_{\pi,\mathfrak{p}}$ which is $K_{1,\mathfrak{p}}(n_\mathfrak{p})$-invariant. This vector is unique up to scaling. Globally, we define the conductor of π to be the ideal $\mathfrak{n} = \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}}$. This is the smallest ideal $\mathfrak{n}$ such that π admits a non-zero $K_1(\mathfrak{n})$-invariant vector. Thus, V_π contains a unique (up to scaling) vector which is $K_1(\mathfrak{n})$-invariant. The vector

$$v^\circ = \bigotimes_\nu v_\nu^\circ \otimes \bigotimes_\mathfrak{p} v_\mathfrak{p}^\circ$$

does the job and we will call it the (global) *new vector*.

With this restrictions on π in place we observe that

$$\phi_\circ = \frac{\sigma(v^\circ)}{\|\sigma(v^\circ)\|_2} \tag{1.4}$$

is a cuspidal Hecke–Maaß newform over F of level $\mathfrak{n}$ and nebentypus ω_π. In particular it is $K_1(\mathfrak{n})$-invariant and has Casimir eigenvalue $(\lambda_\nu)_\nu$. Furthermore, by our convention $\|v^\circ\|_\infty = \|\phi_\circ\|_\infty$. This is exactly the setting in which we will study the sup-norm problem. It is the natural generalization of classical Maaß wave forms on the upper half-plane $\mathbb{H}$.

Note that every $\phi_\circ$ to which the statements of Theorems 1.1 and 1.2 apply is given by $\phi_\circ = \sigma(v^\circ)$ where v° is the global new vector in some cuspidal automorphic representation (π, V_π). This makes the formulation of the theorems precise and concludes this section.

1.3. Guide to the rest of the paper

Let us now briefly overview the rest of the paper. In Section 2 we find a nice generating domain for $G(\mathbb{A}_F)$ which is tailor-made for the transformation behaviour of $\phi_\circ$. Our argument combines the fundamental domain derived in [4] with the action of the Atkin–Lehner operators from [20].

We then move on towards the study of Whittaker functions associated to newforms. This will take up most of Section 3 and culminate in the first upper bounds which are good near the cusps. The main difficulty is to separate the contribution of ramified $\mathfrak{p}$-adic Whittaker functions from the one of the archimedean Whittaker functions. We achieve this by applying a generalized Hölder inequality to the Whittaker expansion. This will lead to fourth moments of Hecke eigenvalues. Finally, we can adapt the estimate from [4] to our setting.

The next step is to define an integral operator which will serve as an approximate spectral projector. Locally, we will combine the test functions from [5] with those from [20] to deal with highly ramified places. This operator will then lead to what is usually called an amplified pre-trace formula. The geometric side of this pre-trace inequality can then be estimated as in [4].

Finally, in Section 5, we will give complete proofs for the theorems stated above.

Convention. As is common in analytic number theory and related areas, we will use the Vinogradov symbols $\ll$ and $\gg$. Since we consider the number field F as fixed we will allow all the implicit constant to depend on F without further notice. Similarly ε will be reserved for some small positive quantity that may change from line to line. All the constants may also depend on ε.

2. The reduction step

In this section we follow [20, Section 3B] to derive a generating domain for

$$Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)/K_1(\mathfrak{n}).$$

We then continue to show that in order to solve the sup-norm problem for the automorphic forms under consideration we only have to bound our functions (and possibly their twists) on very special elements in $G(\mathbb{A}_F)$. The central result of this section is Corollary 2.7 below.

2.1. Local preliminaries

Several steps that are necessary to deal with powerful level rely on local methods. In this section we briefly recall the ingredients needed from [20].

Let $\mathfrak{p}$ be a finite place and let $(\pi_\mathfrak{p}, V_{\pi,\mathfrak{p}})$ denote an admissible, irreducible representation of $F_\mathfrak{p}$. Define $n_\mathfrak{p} = a(\pi_\mathfrak{p})$ to be the log-conductor of $\pi_\mathfrak{p}$. Let $\omega_{\pi,\mathfrak{p}}$ be the central character of $\pi_\mathfrak{p}$ and let $m_\mathfrak{p} = a(\omega_{\pi,\mathfrak{p}})$ be its log-conductor. Of course, we have $n_\mathfrak{p} = v_\mathfrak{p}(\mathfrak{n})$ and $m_\mathfrak{p} = v_\mathfrak{p}(\mathfrak{m}_\mathfrak{p})$ where $\mathfrak{n}$ is the (global) conductor of π and $\mathfrak{m}$ is the (global) conductor of the central character ω_π.

Most local computations rely on the decomposition

$$G(F_\mathfrak{p}) = \bigsqcup_{t\in\mathbb{Z}} \bigsqcup_{0\leq l\leq n_\mathfrak{p}} \bigsqcup_{u\in\mathfrak{o}_\mathfrak{p}^\times/(1+\varpi_\mathfrak{p}^{\min(l,n_\mathfrak{p}-l)}\mathfrak{o}_\mathfrak{p}^\times)} Z(F_\mathfrak{p})N(F_\mathfrak{p}) \underbrace{a(\varpi_\mathfrak{p}^t)wn(\varpi_\mathfrak{p}^{-l}u)}_{=g_{t,l,u}} K_{1,\mathfrak{p}}(n). \tag{2.1}$$

This is [20, (3)] or originally [19, Lemma 2.13]. The decomposition suggests defining the invariants $t_\mathfrak{p}(g)$, $l_\mathfrak{p}(g)$ and $n_{0,\mathfrak{p}}(g)$ in the obvious way by writing

$$g \in Z(F_\mathfrak{p})N(F_\mathfrak{p})g_{t(g),l(g),u}K_{1,\mathfrak{p}}(n_\mathfrak{p})$$

with $u \in \mathfrak{o}^\times/(1+\varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g)}\mathfrak{o}^\times)$. We further define

$$\begin{aligned} n_{0,\mathfrak{p}} &= \lfloor n_\mathfrak{p}/2 \rfloor, \\ n_{1,\mathfrak{p}} &= n_\mathfrak{p} - n_{0,\mathfrak{p}}, \\ m_{1,\mathfrak{p}} &= \max(0, m_\mathfrak{p} - n_{1,\mathfrak{p}}), \\ n_{1,\mathfrak{p}}(g) &= \begin{cases} n_{0,\mathfrak{p}} & \text{if } l(g) \le n_{0,\mathfrak{p}}, \\ n_{1,\mathfrak{p}} & \text{if } l(g) \ge n_{1,\mathfrak{p}}, \end{cases} \\ m_{1,\mathfrak{p}}(g) &= \max(0, n_{0,\mathfrak{p}}(g) - n_\mathfrak{p} + m_\mathfrak{p}). \end{aligned}$$

Obviously we have the following relations to the ideals defined earlier:

$$n_{0,p} = v_\mathfrak{p}(\mathfrak{n}_0), \quad n_{1,\mathfrak{p}} = v_\mathfrak{p}(\mathfrak{n}_2\mathfrak{n}_0), \quad m_{1,\mathfrak{p}} = v_\mathfrak{p}(\mathfrak{m}_1).$$

In Section 3 below we will also encounter the ideal $\mathfrak{m}_1(g) = \prod_{p|\mathfrak{n}} \mathfrak{p}^{m_{1,\mathfrak{p}}(g_\mathfrak{p})}$ such that $m_{1,\mathfrak{p}}(g_\mathfrak{p}) = v_\mathfrak{p}(\mathfrak{m}_1(g))$.

Let us collect some simple results capturing the behaviour of these invariants in crucial situations.

Lemma 2.1. *Let $g \in K_\mathfrak{p} a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}})$. If $n_\mathfrak{p}$ is odd, then*

$$n_{1,\mathfrak{p}}(g) = n_{0,\mathfrak{p}} \iff g \in wK_\mathfrak{p}^0(1)a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}}).$$

If $n_\mathfrak{p}$ is even, then

$$n_{1,\mathfrak{p}}(g) = n_{0,\mathfrak{p}}.$$

Proof. The first part is a consequence of [20, Lemma 2.2 (2)]. The second part holds since for even $n_\mathfrak{p}$ one has $n_{0,\mathfrak{p}} = n_{1,\mathfrak{p}}$. ∎

Lemma 2.2 ([20, Lemma 2.3]). *Let $n_\mathfrak{p}$ be odd. Further take $k \in K_{0,\mathfrak{p}}(1)$ and*

$$\epsilon_\mathfrak{p} \in \left\{1, \begin{pmatrix} 0 & 1 \\ \varpi_\mathfrak{p} & 0 \end{pmatrix}\right\}.$$

Then

$$k\epsilon_\mathfrak{p} w a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}}) = wk' a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}})\epsilon'_\mathfrak{p} z$$

for $k' \in K_\mathfrak{p}^0(1)$, $z \in Z(F_\mathfrak{p})$ and

$$\epsilon'_\mathfrak{p} = \begin{cases} 1 & \text{if } \epsilon_\mathfrak{p} = 1, \\ \begin{pmatrix} 0 & 1 \\ \varpi_\mathfrak{p}^{n_\mathfrak{p}} & 0 \end{pmatrix} & \text{else.} \end{cases}$$

2.2. *The global generating domain*

Our goal is to recreate the argument from [20, Section 3B] coupled with the results from [4, Section 5] to deal with arbitrary number fields. As expect this general setting brings the class group and the unit group into the picture. We start with several definitions. For any ideal $\mathfrak{L}$ in $\mathcal{O}_F$ we define

$$\begin{aligned}
\eta_{\mathfrak{L}} &= \prod_{\mathfrak{p}\mid\mathfrak{L}} \begin{pmatrix} 0 & 1 \\ \varpi_{\mathfrak{p}}^{n_{\mathfrak{p}}} & 0 \end{pmatrix} \prod_{\mathfrak{p}\nmid\mathfrak{L}} 1, \\
h_{\mathfrak{L}} &= \prod_{\mathfrak{p}\mid\mathfrak{L}} a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) \prod_{\mathfrak{p}\nmid\mathfrak{L}} 1, \\
K_{\mathfrak{L}} &= \prod_{\mathfrak{p}\mid\mathfrak{L}} K_{\mathfrak{p}} \prod_{\mathfrak{p}\nmid\mathfrak{L}} \{1\} \subset \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}}), \\
J_{\mathfrak{L}} &= K_{\mathfrak{L}} h_{\mathfrak{L}}, \\
\mathcal{J}_{\mathfrak{L}} &= \{g \in J_{\mathfrak{L}} \colon n_{1,\mathfrak{p}}(g_{\mathfrak{p}}) = n_{0,\mathfrak{p}} \ \forall \mathfrak{p} \mid \mathfrak{L}\}.
\end{aligned}$$

Let us make the following minor observation.

Lemma 2.3. *For $g \in J_{\mathfrak{L}}$ one has*

$$g \in \mathcal{J}_{\mathfrak{L}} \iff g_{\mathfrak{p}} \in wK_{\mathfrak{p}}^0(1)a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) \text{ for all } \mathfrak{p} \mid \mathfrak{L} \text{ with } n_{\mathfrak{p}} \text{ odd.}$$

Proof. Apply Lemma 2.1 for each $\mathfrak{p} \mid \mathfrak{L}$. ■

Corollary 2.4. *For $g_{\mathfrak{p}} \in \mathcal{J}_{\mathfrak{p}}$ and $v \in \mathfrak{o}_{\mathfrak{p}}^{\times}$ we have $a(v)g \in \mathcal{J}_{\mathfrak{p}}$.*

Proof. Obviously $a(v)g_{\mathfrak{p}} \in J_{\mathfrak{p}}$. One then concludes using Lemma 2.3 and the fact that

$$a(v)w = w\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}. \qquad ■$$

In terms of the local invariants we write

$$\mathfrak{n}_0 = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{0,\mathfrak{p}}}, \quad \mathfrak{n}_1 = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{1,\mathfrak{p}}}, \quad \mathfrak{n}_2 = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{1,\mathfrak{p}}-n_{0,\mathfrak{p}}}.$$

Note that $\mathfrak{n}_2$ is square-free and $\mathfrak{n} = \mathfrak{n}_0^2\mathfrak{n}_2$.

Now we want to use the generating domain from [4] for the square-free ideal $\mathfrak{n}_2$. Recall the group

$$K^* = Z(F_\infty)K_\infty \prod_{\mathfrak{p}\nmid\mathfrak{n}_2} Z(F_{\mathfrak{p}})K_{\mathfrak{p}} \prod_{\mathfrak{p}\mid\mathfrak{n}_2} \left\langle K_{0,\mathfrak{p}}(1), \begin{pmatrix} 0 & 1 \\ \varpi_{\mathfrak{p}} & 0 \end{pmatrix} \right\rangle$$

defined in [4, Section 2]. Let $\mathcal{F}(\mathfrak{n}_2)$ be the generating domain for $G(F)\backslash G(\mathbb{A}_F)/K^*$ defined in [4, p. 14]. An element in $\mathcal{F}(\mathfrak{n}_2)$ is of the form

$$\underbrace{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}}_{\in B(F_\infty)} \underbrace{\begin{pmatrix} \theta_i & 0 \\ 0 & 1 \end{pmatrix}}_{=a(\theta_i)},$$

where $|y|_\infty$ is maximal and $\theta_i \in \hat{\mathcal{O}}_F^\times$ is an ideal representative. We will call such matrices *special*. Define

$$\mathcal{F}_{\mathfrak{n}_2} = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : \exists i \in \{1, \dots, h\} \text{ such that } \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} a(\theta_i) \in \mathcal{F}(\mathfrak{n}_2) \right\}.$$

We can write down a generating domain in the spirit of [20, Proposition 3.6].

Proposition 2.5. *For $g \in G(\mathbb{A}_F)$ we find $\mathfrak{L} \mid \mathfrak{n}_2$ and $1 \leq i \leq h_F$ such that*

$$g \in Z(\mathbb{A})G(F)\big(a(\theta_i)\mathcal{J}_{\mathfrak{n}} \times \mathcal{F}_{\mathfrak{n}_2}\big)\eta_{\mathfrak{L}} K_1(\mathfrak{n}).$$

Proof. The proof follows exactly the steps in [20] exploiting the fact that the fundamental domain $\mathcal{F}(\mathfrak{n}_2)$ from [4] is already given adelically. Let $w_{\mathfrak{n}}$ be the diagonal embedding of w in $K_{\mathfrak{n}}$. Then the determinant map

$$w_{\mathfrak{n}} h_{\mathfrak{n}} K_1(\mathfrak{n})_{\mathrm{fin}} h_{\mathfrak{n}}^{-1} w_{\mathfrak{n}}^{-1} \to \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}^\times$$

is surjective. Thus we can apply strong approximation to the element $g h_{\mathfrak{n}}^{-1} w_{\mathfrak{n}}^{-1}$ and find $g_\infty \in G(F_\infty)$ and $i \in \{1, \dots, h_F\}$ such that

$$g \in G(F) g_\infty a(\theta_i) w_{\mathfrak{n}} h_{\mathfrak{n}} K_1(\mathfrak{n}).$$

Using the properties of $\mathcal{F}(\mathfrak{n}_2)$ we write $g_\infty a(\theta_i) = \gamma f z k^*$ with $\gamma \in G(F)$, $zk^* \in K^*$ and $f \in \mathcal{F}(\mathfrak{n}_2)$. By construction of K^* we can assume

$$k_{\mathfrak{p}}^* = \begin{cases} k_{\mathfrak{p}}^* \in K_{\mathfrak{p}} & \text{if } \mathfrak{p} \nmid \mathfrak{n}_2, \\ k_{\mathfrak{p}}' \epsilon_{\mathfrak{p}} \in K_{0,\mathfrak{p}} \epsilon_{\mathfrak{p}} & \text{if } \mathfrak{p} \mid \mathfrak{n}_2, \end{cases}$$
$$k_v^* \in K_v$$

for $\epsilon_{\mathfrak{p}} \in \left\{1, \left(\begin{smallmatrix} 0 & 1 \\ \varpi_{\mathfrak{p}} & 0 \end{smallmatrix}\right)\right\}$. Define

$$\mathfrak{L} = \prod_{\mathfrak{p} \colon \epsilon_{\mathfrak{p}} \neq 1} \mathfrak{p}$$

and note that $\mathfrak{L}$ must divide $\mathfrak{n}_2$ by construction. With this at hand we can write

$$g \in Z(\mathbb{A})G(F) f \underbrace{\prod_{\mathfrak{p} \nmid \mathfrak{n}} k_{\mathfrak{p}}^*}_{\in K_1(\mathfrak{n})} \prod_{\mathfrak{p} \mid \mathfrak{n},\, \mathfrak{p} \nmid \mathfrak{n}_2} k_{\mathfrak{p}}^* w a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) \prod_{\mathfrak{p} \mid \mathfrak{n}_2,\, \mathfrak{p} \nmid \mathfrak{L}} k_{\mathfrak{p}}' w a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})$$
$$\cdot \prod_{\mathfrak{p} \mid \mathfrak{L}} k_{\mathfrak{p}}' \epsilon_{\mathfrak{p}} w a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) K_1(\mathfrak{n}).$$

Let us treat each product appearing above separately. First, we include the product over $\mathfrak{p} \nmid \mathfrak{n}$ into $K_1(\mathfrak{n})$. Next, we notice that if $\mathfrak{p} \mid \mathfrak{n}$ but $\mathfrak{p} \nmid \mathfrak{n}_2$ then $n_{\mathfrak{p}}$ must be even. Since

$k_{\mathfrak{p}}^* w \in K_{\mathfrak{p}}$ we apply Lemma 2.1 to absorb the second product into $\mathcal{J}_{\mathfrak{n}}$. In the remaining two cases, namely $\mathfrak{p} \mid \mathfrak{n}_2$, $n_{\mathfrak{p}}$ must be odd. First, for $\mathfrak{p} \nmid \mathfrak{L}$ we apply Lemma 2.2 to obtain

$$k'_{\mathfrak{p}} w a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) = w \underbrace{\hat{k}_{\mathfrak{p}}}_{\in K_{\mathfrak{p}}^{o}(1)} a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}).$$

It follows from Lemma 2.3 that also the third product is in $\mathcal{J}_{\mathfrak{n}}$. Finally, we use Lemmata 2.2 and 2.3 again to get

$$k'_{\mathfrak{p}} \epsilon_{\mathfrak{p}} w a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) = \underbrace{w \overbrace{\hat{k}_{\mathfrak{p}}}^{K_{\mathfrak{p}}^{0}(1)} a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})}_{\in \mathcal{J}_{\mathfrak{n}}} \widehat{\epsilon'_{\mathfrak{p}}(z)}.$$

Thus,

$$g \in Z(\mathbb{A})G(F) f \mathcal{J}_{\mathfrak{n}} \underbrace{\prod_{\mathfrak{p} \mid \mathfrak{L}} \epsilon'_p}_{=\eta_{\mathfrak{L}}} K_1(\mathfrak{n}).$$

One concludes the proof by writing $f = pa(\theta_i)$ for a special matrix $p \in \mathcal{F}_{\mathfrak{n}_2}$ and some $i \in \{1, \dots, h_F\}$. ∎

2.3. *The action of* $\eta_{\mathfrak{L}}$

The next step is to understand how the matrix $\eta_{\mathfrak{L}}$, for $\mathfrak{L} \mid \mathfrak{n}_2$, acts on the automorphic functions under consideration.

We start by constructing a certain unitary character $\omega_{\pi}^{\mathfrak{L}} = \omega_{\pi,\infty}^{\mathfrak{L}} \prod_{\mathfrak{p}} \omega_{\pi,\mathfrak{p}}^{\mathfrak{L}}$ of $F^\times \backslash \mathbb{A}_F^\times$ with the properties

$$\omega_{\pi,\mathfrak{p}}^{\mathfrak{L}}|_{\mathfrak{o}_{\mathfrak{p}}^\times} = \begin{cases} 1 & \text{if } \mathfrak{p} \mid \mathfrak{L}, \\ \omega_{\pi,\mathfrak{p}}|_{\mathfrak{o}_{\mathfrak{p}}^\times} & \text{if } \mathfrak{p} \nmid \mathfrak{L}, \end{cases}$$

and $\omega_{\pi,\infty}^{\mathfrak{L}}|_{F_{\infty,+}} = 1$. We claim that such a character exists. To see this we apply strong approximation for the open compact subgroup $\widehat{\mathcal{O}_F^\times} = \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}^\times$, which reads

$$F^\times \backslash \mathbb{A}_F^\times = \bigsqcup_{i=1}^{h_F} F_{\infty,+} \theta_i \widehat{\mathcal{O}_F^\times}.$$

But using our requirements together with say $\omega_{\pi}^{\mathfrak{L}}(\theta_i) = 1$ for all i allows us to define such a character on the right hand side of the above equality. It is clear that the resulting character is unitary.

Let us make some observations. Locally, one has

$$\omega_{\pi,\mathfrak{p}}^{-1} \omega_{\pi,\mathfrak{p}}^{\mathfrak{L}}|_{\mathfrak{o}_{\mathfrak{p}}^\times} = \begin{cases} \omega_{\pi,\mathfrak{p}}^{-1}|_{\mathfrak{o}_{\mathfrak{p}}^\times} & \text{if } \mathfrak{p} \mid \mathfrak{L}, \\ 1 & \text{if } \mathfrak{p} \nmid \mathfrak{L}. \end{cases} \tag{2.2}$$

Let (π, V_π) be a cuspidal automorphic representation. We define the twisted representation $(\pi^{\mathfrak{L}}, V_\pi)$ by

$$\pi^{\mathfrak{L}}(g) = \omega_\pi^{-1}\omega_\pi^{\mathfrak{L}}(\det(g))\pi(g).$$

This representation is sometimes denoted by $\pi^{\mathfrak{L}} = (\omega_\pi^{-1}\omega_\pi^{\mathfrak{L}})\pi$. The central character of $\pi^{\mathfrak{L}}$ is $\omega_\pi^{-1}(\omega_\pi^{\mathfrak{L}})^2$ and looks locally like

$$\omega_{\pi,\mathfrak{p}}^{-1}(\omega_{\pi,\mathfrak{p}}^{\mathfrak{L}})^2|_{\mathfrak{o}_\mathfrak{p}^\times} = \begin{cases} \omega_{\pi,\mathfrak{p}}^{-1}|_{\mathfrak{o}_\mathfrak{p}^\times} & \text{if } \mathfrak{p} \mid \mathfrak{L}, \\ \omega_{\pi,\mathfrak{p}}|_{\mathfrak{o}_\mathfrak{p}^\times} & \text{if } \mathfrak{p} \nmid \mathfrak{L}. \end{cases} \tag{2.3}$$

In particular, the log-conductor of the new central character coincides with the log-conductor of ω_π, namely

$$\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{m_\mathfrak{p}}.$$

Further, we note that this twist does not change the spectral data at ∞. Concerning the conductor of $\pi^{\mathfrak{L}}$ we have the following statement, which corresponds to [20, Lemma 3.4].

Lemma 2.6. *For $\mathfrak{L} \mid \mathfrak{n}_2$ the log-conductor of $\pi^{\mathfrak{L}}$ is $\mathfrak{n}$ and*

$$v_\mathfrak{L}^\circ = \pi(\eta_\mathfrak{L})v^\circ \tag{2.4}$$

is a new vector in $\pi^{\mathfrak{L}}$.

Proof. If $\mathfrak{p} \nmid \mathfrak{L}$, then $\pi_\mathfrak{p}^{\mathfrak{L}}$ and $\pi_\mathfrak{p}$ differ only by some unramified twist. This does not change conductor. However, at the places $\mathfrak{p} \mid \mathfrak{L}$ the representation $\pi_\mathfrak{p}^{\mathfrak{L}}$ is equivalent to $\tilde{\pi}_\mathfrak{p}$ up to some unramified twist. Here $\tilde{\pi}_\mathfrak{p}$ denotes the contragredient representation of $\pi_\mathfrak{p}$. Since $a(\pi_\mathfrak{p}) = a(\tilde{\pi}_\mathfrak{p})$ it suffices to show that the vector given in (2.4) has the correct transformation behaviour under $K_1(\mathfrak{n})$.

We proceed place by place. For $\mathfrak{p} \nmid \mathfrak{L}$ and v there is nothing to do. For $\mathfrak{p} \mid \mathfrak{L}$ we calculate

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{=k_\mathfrak{p} \in K_{1,\mathfrak{p}}(n_\mathfrak{p})} \begin{pmatrix} 0 & 1 \\ \varpi_\mathfrak{p}^{n_\mathfrak{p}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varpi_\mathfrak{p}^{n_\mathfrak{p}} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} d & c\varpi_\mathfrak{p}^{-n_\mathfrak{p}} \\ \varpi_\mathfrak{p}^{n_\mathfrak{p}} b & a \end{pmatrix}}_{=k'_\mathfrak{p} \in K_{0,\mathfrak{p}}(n_\mathfrak{p})}.$$

It is easy to verify that $k'_\mathfrak{p} z(\det(k_\mathfrak{p}))^{-1} \in K_{1,\mathfrak{p}}(n_\mathfrak{p})$. Therefore, using (2.2) and (2.3) we have

$$\begin{aligned} \pi_\mathfrak{p}^{\mathfrak{L}}(k_\mathfrak{p})v_{\mathfrak{L},\mathfrak{p}}^\circ &= \omega_{\pi,\mathfrak{p}}^{-1}(\det(k_\mathfrak{p}))\pi_\mathfrak{p}(k_\mathfrak{p}[\eta_\mathfrak{L}]_\mathfrak{p})v_\mathfrak{p}^\circ \\ &= \omega_{\pi,\mathfrak{p}}^{-1}(\det(k_\mathfrak{p}))\underbrace{\pi_\mathfrak{p}(z(\det(k_\mathfrak{p})))}_{=\omega_{\pi,\mathfrak{p}}(\det(k_\mathfrak{p}))}\pi_\mathfrak{p}([\eta_\mathfrak{L}]_\mathfrak{p})\underbrace{\big[\pi_\mathfrak{p}\big(\underbrace{z(\det(k_\mathfrak{p})^{-1})k'_\mathfrak{p}}_{\in K_{1,\mathfrak{p}}(n_\mathfrak{p})}\big)v_\mathfrak{p}^\circ\big]}_{=v_\mathfrak{p}^\circ} \\ &= \pi_\mathfrak{p}([\eta_\mathfrak{L}]_\mathfrak{p})v_\mathfrak{p}^\circ = v_{\mathfrak{L},\mathfrak{p}}^\circ. \end{aligned}$$

∎

Observe that $(\pi^{\mathfrak{L}}, V_\pi)$ is also a cuspidal automorphic representation. Furthermore, an intertwiner, $\sigma^{\mathfrak{L}}$, to $L^2_0(G(F)\backslash G(\mathbb{A}_F), \omega_\pi^{-1}(\omega_\pi^{\mathfrak{L}})^2)$ is given by

$$[\sigma^{\mathfrak{L}}(v)](g) = \omega_\pi^{-1}\omega_\pi^{\mathfrak{L}}(\det(g))[\sigma(v)](g).$$

This leads us to the definition of the twisted newform $\phi_\circ^{\mathfrak{L}} = \sigma^{\mathfrak{L}}(v_{\mathfrak{L}}^\circ)$. We immediately observe that

$$\phi_\circ(g\eta_{\mathfrak{L}}) = \omega_\pi(\omega_\pi^{\mathfrak{L}})^{-1}(\det(g))\phi_\circ^{\mathfrak{L}}(g),$$

giving us exactly the ingredient we needed to understand the action of $\eta_{\mathfrak{L}}$ on $\phi_\circ$. We derive the following corollary.

Corollary 2.7. *If $\phi_\circ$ is the newform associated to a cuspidal automorphic representation (π, V_π) then*

$$\sup_{g\in \mathrm{GL}_2(\mathbb{A})} |\phi_\circ(g)| \leq \sup_{\mathfrak{L}\mid\mathfrak{n}_2} \sup_{1\leq i\leq h_F} \sup_{g\in\mathcal{J}_{\mathfrak{n}}\times\mathcal{F}_{\mathfrak{n}_2}} |\phi_\circ^{\mathfrak{L}}(a(\theta_i)g)|. \tag{2.5}$$

Thus, we have reduced the sup-norm problem for the newform $\phi_\circ$ to bounding the newforms $\phi_\circ^{\mathfrak{L}}$ on very special matrices. In the following we will fix an arbitrary $\mathfrak{L}\mid\mathfrak{n}_2$, write $\phi = \phi_\circ^{\mathfrak{L}}$ and bound ϕ on $a(\theta_i)(\mathcal{J}_{\mathfrak{n}} \times \mathcal{F}_{\mathfrak{n}_2})$.

3. Bounds via Whittaker expansions

In this section we consider the Whittaker expansion of cusp forms. This will lead to a first upper bound for the newform $\phi_\circ$, which is sufficient when the variable is sufficiently close to the cusps. The main result is Proposition 3.17 below.

Throughout this section let (π, V_π) be a cuspidal automorphic representation with new vector $v^\circ \in V_\pi$ and associated newform $\phi_\circ = \sigma(v^\circ)$. Recall that the (arithmetic) conductor of π was denoted by $\mathfrak{n} = \mathfrak{n}_2\mathfrak{n}_0^2$. Furthermore, the conductor of the central character was given by $\mathfrak{m}$ and had $\mathfrak{m}_1 = \mathfrak{m}/\gcd(\mathfrak{m}, \mathfrak{n}_2\mathfrak{n}_0)$. In this section we will further encounter the ideal

$$\mathfrak{m}_1(g) = \prod_{p\mid\mathfrak{n}} \mathfrak{p}^{m_{1,\mathfrak{p}}(g_\mathfrak{p})},$$

where $m_{1,\mathfrak{p}}(g_\mathfrak{p})$ was defined in Section 2.1. Note that $\mathfrak{m}_1(g) \mid \mathfrak{m}_1$ for all g. Without loss of generality we assume that $\phi_\circ$ is L^2-normalized. Further, we fix $g \in \mathcal{J}_{\mathfrak{n}}$ and $n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}$.

3.1. The Whittaker expansion of cusp forms

Let $\psi = \prod_\nu \psi_\nu \prod_\mathfrak{p} \psi_\mathfrak{p}$ be the standard additive character of $\mathbb{A}_F$ as defined in [19]. Recall

$$\psi_\nu(x) = \begin{cases} e(x) & \text{if } \nu \text{ is real,} \\ e(x+\overline{x}) & \text{if } \nu \text{ is complex,} \end{cases}$$

with $e(x) = e^{2\pi i x}$. Further, note that the conductor of ψ is $\mathfrak{d}^{-1}$.

Having fixed the additive character we define the corresponding global Whittaker function

$$W_{\phi_\circ}(g) = \frac{2^{r_2}}{\sqrt{d_F}} \int_{F\backslash \mathbb{A}_F} \phi_\circ(n(x)g)\psi(-x)\, d\mu_{\mathbb{A}_F}(x).$$

We want to factor this global function into a product of local functions each of which matches the ones studied in [20]. Therefore, we define the shifted local character $\psi'_{\mathfrak{p}} = \psi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{-v_{\mathfrak{p}}(\mathfrak{d})}\cdot)$. This local additive character has conductor $\mathfrak{o}_{\mathfrak{p}}$. Further, if $\omega_{\pi,\mathfrak{p}}(\varpi_{\mathfrak{p}}) = |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{ia_{\mathfrak{p}}}$, we define $\pi'_{\mathfrak{p}} = |\cdot|_{\mathfrak{p}}^{-ia_{\mathfrak{p}}/2}\pi_{\mathfrak{p}}$. The purpose of this twist is that the central character $\omega'_{\pi_{\mathfrak{p}}}$ of $\pi'_{\mathfrak{p}}$ is trivial on the uniformizer. Now let $W_{\mathfrak{p}}$ be the Whittaker new vector associated to the representation $\pi'_{\mathfrak{p}}$ with respect to the character $\psi'_{\mathfrak{p}}$, normalized by $W_{\mathfrak{p}}(1) = 1$. These are exactly the Whittaker functions studied in [19, 20]. At infinity we take the local Whittaker function W_ν to be the Whittaker vector associated to v_ν° normalized by $\langle W_\nu, W_\nu\rangle = 1$. This matches the situation in [4]. Having defined these local functions we achieve the factorization

$$W_{\phi_\circ}(g) = c_{\phi_\circ} \underbrace{\prod_\nu W_\nu(g_\nu)}_{=W_\infty(g_\infty)} \prod_{\mathfrak{p}} |\det(g_{\mathfrak{p}})|_{\mathfrak{p}}^{ia_{\mathfrak{p}}/2} W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})})g_{\mathfrak{p}}).$$

The translation in the finite part comes from the shift in the local additive characters as explained in [19, Remark 2.11]. The constant $c_{\phi_\circ}$ comes from our renormalization of the local functions.

For $1 \le i \le h_F$ and $g \in \mathcal{J}_{\mathfrak{n}}$ we have the well known Whittaker expansion

$$\begin{aligned}&\phi_\circ(a(\theta_i)gn(x)a(y)) \\ &\quad = c_{\phi_\circ} \sum_{q\in F^\times} \prod_{\mathfrak{p}} |q\theta_i \det(g_{\mathfrak{p}})|_{\mathfrak{p}}^{ia_{\mathfrak{p}}/2} W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)g_{\mathfrak{p}})W_\infty(a(q)n(x)a(y)). \qquad (3.1)\end{aligned}$$

For convenience we split the local terms into the archimedean part W_∞, the unramified part

$$\lambda_{\mathrm{ur}}(q) = \prod_{\mathfrak{p}\nmid\mathfrak{n}} W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)),$$

and the ramified part

$$\lambda_{\mathfrak{n}}(q) = \prod_{\mathfrak{p}|\mathfrak{n}} W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)g_{\mathfrak{p}}).$$

We also collect all the unramified twists together and write

$$\eta(q) = \prod_{\mathfrak{p}} |q\theta_i \det(g_{\mathfrak{p}})|_{\mathfrak{p}}^{ia_{\mathfrak{p}}/2}.$$

Since $|\eta(q)| = 1$ this factor does not influence any of the upcoming estimates.

Let us continue by gathering some properties of $\lambda_{\mathfrak{n}}$ and λ_{ur}. First, we recall the following standard result.

Lemma 3.1. *If $\mathfrak{p} \nmid \mathfrak{n}$, then there are unramified characters $\chi_{1,\mathfrak{p}}$ and $\chi_{2,\mathfrak{p}}$ such that $\pi'_{\mathfrak{p}} = \chi_{1,\mathfrak{p}} \boxplus \chi_{2,\mathfrak{p}}$. In this case*

$$W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)) = \begin{cases} 0 & \text{if } v_{\mathfrak{p}}(\theta_i q) + v_{\mathfrak{p}}(\mathfrak{d}) < 0, \\ q_{\mathfrak{p}}^{-(v_{\mathfrak{p}}(\theta_i q)+v_{\mathfrak{p}}(\mathfrak{d}))/2} \dfrac{\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{v_{\mathfrak{p}}(\theta_i q)+v_{\mathfrak{p}}(\mathfrak{d})+1} - \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{v_{\mathfrak{p}}(\theta_i q)+v_{\mathfrak{p}}(\mathfrak{d})+1}}{\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}}) - \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})} & \text{if } v_{\mathfrak{p}}(\theta_i q) + v_{\mathfrak{p}}(\mathfrak{d}) \geq 0. \end{cases}$$

Proof. This follows from [9, Theorems 4.6.4, 4.6.5]. ■

We can extract the following fact about the support of unramified coefficients.

Corollary 3.2. *If $\lambda_{\mathrm{ur}}(q) \neq 0$, then $v_{\mathfrak{p}}(q) \geq -v_{\mathfrak{p}}(\mathfrak{d}) - v_{\mathfrak{p}}(\theta_i)$ for all $\mathfrak{p} \nmid \mathfrak{n}$.*

In order to describe the unramified coefficients in terms of more or less well known quantities we quickly introduce the Hecke operators. For $\mathfrak{p} \nmid \mathfrak{n}$ and $k \in \mathbb{N}$ define

$$X_{\mathfrak{p},k} = \{m \in \mathrm{Mat}_2(\mathfrak{o}_{\mathfrak{p}}) \colon v_{\mathfrak{p}}(\det(m)) = k\}.$$

The local new vector $v_{\mathfrak{p}}^{\circ}$ is an eigenvector of the operator $\pi_{\mathfrak{p}}(\mathbb{1}_{X_{\mathfrak{p},k}})$ and we denote its eigenvalue by $\lambda(\mathfrak{p}^k)$. For any ideal $\mathfrak{a}$ coprime to $\mathfrak{n}$ we define the global Hecke operator by

$$T(\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} \pi_{\mathfrak{p}}(\mathbb{1}_{X_{\mathfrak{p},v_{\mathfrak{p}}(\mathfrak{a})}}).$$

It is clear that the global new vector v°, and therefore also the newform $\phi_{\circ}$, is an eigenvector of this operator with eigenvalue

$$\lambda(\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} \lambda(\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}).$$

We can now make a connection between λ_{ur} and the Hecke eigenvalues $\lambda(\cdot)$. At this point let us remark that we follow the normalization of [9, Section 4.6] which differs from the one used in [4, 20].

Lemma 3.3. *We have*

$$\lambda_{\mathrm{ur}}(q) = \frac{\lambda\left(\frac{(q)\theta_i\mathfrak{d}}{[(q)\theta_i\mathfrak{d}]_{\mathfrak{n}}}\right)}{\mathcal{N}\left(\frac{(q)\theta_i\mathfrak{d}}{[(q)\theta_i\mathfrak{d}]_{\mathfrak{n}}}\right)}.$$

Proof. The proof proceeds locally by showing

$$\lambda(\mathfrak{p}^k) = q_{\mathfrak{p}}^k W_{\mathfrak{p}}(a(\varpi^k)) \quad \text{for } \mathfrak{p} \nmid \mathfrak{n}.$$

This can be done by induction using [9, Propositions 4.6.4, 4.6.6] and Lemma 3.1. ■

Next let us inspect the support of $\lambda_{\mathfrak{n}}$.

Lemma 3.4 ([20, Lemma 3.11]). *If* $\lambda_{\mathfrak{n}}(q) \neq 0$ *and* $g \in \mathcal{J}_{\mathfrak{n}}$, *then*

$$v_{\mathfrak{p}}(q) \geq -v_{\mathfrak{p}}(\theta_i) - v_{\mathfrak{p}}(\mathfrak{d}) - n_{0,\mathfrak{p}} - m_{1,\mathfrak{p}}(g_p) \quad \text{for all } \mathfrak{p} \mid \mathfrak{n}.$$

Proof. Since $g \in \mathcal{J}_{\mathfrak{n}}$ it follows that $g_{\mathfrak{p}} \in K_{\mathfrak{p}} a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})$ and $n_{1,\mathfrak{p}}(g_{\mathfrak{p}}) = n_{0,\mathfrak{p}}$. But $W_{\mathfrak{p}}(a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{d})}\theta_i q)g_{\mathfrak{p}}) \neq 0$ so that [20, Proposition 2.11 (1)] implies[3]

$$v_{\mathfrak{p}}(\theta_i q) + v_{\mathfrak{p}}(\mathfrak{d}) \geq -n_{1,\mathfrak{p}}(g_p) - m_{1,\mathfrak{p}}(g_p).$$

Note that we have used Corollary 2.4 to include $a(v')$ into $g_{\mathfrak{p}}$ for $v' \in \mathfrak{o}_{\mathfrak{p}}^{\times}$ where $\theta_i q = v'\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\theta_i q)}$. ∎

Later on it will make sense to view $\lambda_{\mathfrak{n}}$ as a locally constant function on the adeles in an obvious way. It will then be crucial to determine sets on which this function is constant.

Lemma 3.5 ([20, Lemma 3.12]). *Let* $\mathfrak{p} \mid \mathfrak{n}$, $g \in \mathcal{J}_{\mathfrak{n}}$ *and* $u_1, u_2 \in \mathfrak{o}_{\mathfrak{p}}^{\times}$ *such that* $u_1 - u_2 \in \varpi_{\mathfrak{p}}^{n_{0,\mathfrak{p}}(g)}\mathfrak{o}_{\mathfrak{p}}$. *Then*

$$|W_{\pi_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^k u_1)g_{\mathfrak{p}})| = |W_{\pi_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^k u_2)g_{\mathfrak{p}})|.$$

Proof. The proof of this minor lemma goes back to the decomposition (2.1) and the fact that the absolute value of $W_{\pi_{\mathfrak{p}}}$ on $ZNg_{t,l,u}K_{1,\mathfrak{p}}(n_{\mathfrak{p}})$ is determined by $g_{t,l,u}$.

First, by (2.1) we can write

$$g_{\mathfrak{p}} = zng_{t,l,u}\gamma \quad \text{for } \gamma \in K_{1,\mathfrak{p}}(n_{\mathfrak{p}}).$$

Then one observes that

$$a(\varpi_{\mathfrak{p}}^k u_1)g_{\mathfrak{p}} = zn' g_{t+k,l,uu_1^{-1}}\gamma'.$$

By doing the same for u_2 we observe that the claimed equality follows when

$$[uu^{-1}] = [uu_2^{-1}] \in \mathfrak{o}_{\mathfrak{p}}^{\times}/(1+\varpi_{\mathfrak{p}}^{n_0(g_{\mathfrak{p}})}\mathfrak{o}_{\mathfrak{p}}).$$

The last condition leads to $u_1 - u_2 \in \varpi_{\mathfrak{p}}^{n_0(g_{\mathfrak{p}})}\mathfrak{o}_{\mathfrak{p}}$. ∎

Combining the support properties from Lemma 3.4 and Corollary 3.2 we derive

$$|\phi_\circ(a(\theta_i)gn(x)a(y))| \leq |c_{\phi_\circ}| \sum_{q \in \imath^{-1}} |\lambda_{\mathrm{ur}}(q)\lambda_{\mathfrak{n}}(q)W_\infty(a(qy))| \tag{3.2}$$

from (3.1), where

$$\imath = \mathfrak{n}_0\mathfrak{m}_1(g)\mathfrak{d}\prod_{\mathfrak{p}}\mathfrak{p}^{v_{\mathfrak{p}}(\theta_i)}, \quad \mathfrak{m}_1(g) = \prod_{\mathfrak{p}}\mathfrak{p}^{m_{1,\mathfrak{p}}(g_{\mathfrak{p}})}. \tag{3.3}$$

It is easy to deal with the constant $c_{\phi_\circ}$.

[3] Note that in the notation of [20] we have $q(g_{\mathfrak{p}}) = n_{0,\mathfrak{p}} + m_{1,\mathfrak{p}}(g_{\mathfrak{p}})$.

Lemma 3.6. *We have*

$$c_{\phi_\circ} \ll (\mathcal{N}(\mathfrak{n})|T|_\infty)^\varepsilon.$$

Proof. As in [19] we observe

$$c_{\phi_\circ}^2 \ll L^{-1}(1,\pi,\mathrm{Ad})^{-1} \prod_\nu \langle W_\nu, W_\nu\rangle^{-1} = L(1,\pi,\mathrm{Ad})^{-1}.$$

It is a well known fact (see [15] for the corresponding result over $\mathbb{Q}$) that $L(1,\pi,\mathrm{Ad}) \gg (\mathcal{N}(\mathfrak{n})|T|_\infty)^\varepsilon$. Thus,

$$c_{\phi_\circ} \ll (\mathcal{N}(\mathfrak{n})|T|_\infty)^\varepsilon. \qquad \blacksquare$$

Before continuing we fix a parameter $R = (R_\nu)_\nu$ and define the box

$$B(R) = \prod_\nu \{\xi_\nu \in F_\nu \colon |\xi_\nu| \leq R_\nu\}.$$

This box will be used to truncate the Whittaker expansion. We will mostly use $R_\nu \asymp T_\nu/y_\nu$ except in Section 3.4 below, where we allow arbitrary R.

Applying the Hölder inequality to (3.2) together with $1 = |q|_{\mathbb{A}_F} = |q|_{\mathrm{fin}}|q|_\infty = \mathcal{N}(q)^{-1}|q|_\infty$ yields

$$|\phi_\circ(a(\theta_i)gn(x)a(y))| \leq |c_{\phi_\circ}| \underbrace{\Big(\sum_{q\in\iota^{-1}\cap B(R)} |q|_\infty^{-2}|W_\infty(a(qy))|^4\Big)^{1/4}}_{=S_1(R)} \cdot \underbrace{\Big(\sum_{q\in\iota^{-1}\cap B(R)} \mathcal{N}(q)^{2/3}|\lambda_{\mathrm{ur}}(q)\lambda_{\mathfrak{n}}(q)|^{4/3}\Big)^{3/4}}_{=S_2(R)} + |c_{\phi_\circ}|\mathcal{E}(R), \tag{3.4}$$

with

$$S_1(R) = \Big(\sum_{q\in\iota^{-1}\cap B(R)} |q|_\infty^{-2}|W_\infty(a(qy))|^4\Big)^{1/4},$$
$$S_2(R) = \Big(\sum_{q\in\iota^{-1}\cap B(R)} \mathcal{N}(q)^{2/3}|\lambda_{\mathrm{ur}}(q)\lambda_{\mathfrak{n}}(q)|^{4/3}\Big)^{3/4},$$
$$\mathcal{E}(R) = \sum_{q\in\iota^{-1},\, q\notin B(R)} |\lambda_{\mathrm{ur}}(q)\lambda_{\mathfrak{n}}(q)W_\infty(qy)|.$$

We will estimate each of these three quantities in the upcoming subsections.

3.2. *Counting field elements in boxes*

This subsection is concerned with estimating the number of field elements in different adelic boxes. These estimates will be needed in order to estimate $S_1(R)$, $S_2(R)$, and $\mathcal{E}(R)$.

We start by considering some archimedean boxes. The following argument is almost completely taken from [4]. Take

$$R_\nu = \frac{T_\nu + T_\nu^{1/3+\varepsilon}}{2\pi|y_\nu|} \asymp \frac{T_\nu}{y_\nu}$$

and recall the ideal $\imath$ from (3.3). Further, fix $a \in \imath$ such that

$$\mathcal{N}(\imath) \leq \mathcal{N}((a)) \leq (2/\pi)^{r_1}\sqrt{|d_F|}\,\mathcal{N}(\imath). \tag{3.5}$$

This is possible by [18, Lemma 6.2]. In particular, $a\imath^{-1} \subset \mathcal{O}_F$.

Define

$$I_\nu(k_\nu) = \begin{cases} \{\xi_\nu \in F_\nu^\times : k_\nu|a|R_\nu < |\xi_\nu| \leq (k_\nu+1)|a|R_\nu\} & \text{if } k_\nu \geq 1, \\ \left\{\xi_\nu \in F_\nu^\times : |\xi_\nu| \leq |a|R_\nu,\ -k_\nu \leq \left||\xi_\nu| - \frac{|a|T_\nu}{2\pi|y_\nu|}\right| < -k_\nu+1\right\} & \text{else.} \end{cases} \tag{3.6}$$

For $\underline{\underline{k}} \in \mathbb{Z}^{\sharp\{\nu\}}$ let $I(\underline{\underline{k}}) = \prod_\nu I_\nu(k_\nu)$.

Let us start by establishing a simple but crucial property of these sets.

Lemma 3.7. *If $k_\nu < -\lfloor |a|R_\nu \rfloor$ then $I_\nu(k_\nu) = \emptyset$.*

Proof. Suppose $k_\nu < -\lfloor |a|R_\nu \rfloor$. We consider two cases. First, let $|\xi_\nu| > \frac{|a|T_\nu}{2\pi|y_\nu|}$. Then the two inequalities in the definition of $I_\nu(\cdot)$ yield

$$\frac{|a|T_\nu}{2\pi|y_\nu|} + \lfloor |a|R_\nu \rfloor < |\xi_\nu| \leq |a|R_\nu.$$

But the set of such ξ_ν is empty. Secondly, we assume $|\xi_\nu| \leq \frac{|a|R_\nu}{2\pi}$. This gives

$$|\xi_\nu| < \frac{|a|T_\nu}{2\pi|y_\nu|} - \lfloor |a|R_\nu \rfloor < 0,$$

which is also impossible. ■

We also need good estimates for $\sharp(I(\underline{\underline{k}}) \cap a\imath^{-1})$. These are obtained by a standard volume argument. Let us start with some prerequisites.

Choose a fundamental set $\mathcal{P}$ for the lattice $a\imath^{-1} \subset F_\infty$. Without loss of generality we can assume $0 \in \mathcal{P}$. Let D be the diameter of $\mathcal{P}$. It is an elementary fact (see [18]) that

$$\mathrm{Vol}(\mathcal{P}) \sim \mathcal{N}((a))\mathcal{N}(\imath^{-1}) \approx 1.$$

Further, we define

$$J_\nu(k_\nu) = \begin{cases} \{\xi_\nu \in F_\nu : k_\nu|a|R_\nu - D < |\xi_\nu| \leq (k_\nu+1)|a|R_\nu + D\} & \text{if } k_\nu \geq 1, \\ \left\{\xi_\nu \in F_\nu : -k_\nu - D \leq \left||\xi_\nu| - \frac{|a|T_\nu}{2\pi|y_\nu|}\right| < -k_\nu + 1 + D\right\} & \text{else,} \end{cases}$$

and $J(\underline{\underline{k}}) = \prod_\nu J_\nu(k_\nu)$.

Lemma 3.8. *The volume of $J_\nu(k_\nu)$ is given by*

$$\mathrm{Vol}(J_\nu(k_\nu)) = \begin{cases} 2|a|R_\nu + 4D & \text{if } \nu \text{ is real and } k_\nu \geq 1, \\ 4(1+2D) & \text{if } \nu \text{ is real and } k_\nu \leq 0, \\ \pi(2k_\nu+1)|a|R_\nu(|a|R_\nu + 2D) & \text{if } \nu \text{ is complex and } k_\nu \geq 1, \\ 2\frac{|a|T_\nu}{y_\nu}(1+2D) & \text{if } \nu \text{ is complex and } k_\nu \leq 0. \end{cases}$$

Proof. The proof is an elementary volume calculation. ■

As a consequence of Minkowski theory we can choose $\mathcal{P}$ such that $D \ll \mathcal{N}(a\iota^{-1})^{1/n} \ll_F 1$. Therefore, it is clear that

$$\mathrm{Vol}(J(\underline{\underline{k}})) \ll \prod_\nu f_\nu(k_\nu), \quad f_\nu(k_\nu) = \begin{cases} \frac{|a|T_\nu}{y_\nu} + 1 & \text{if } \nu \text{ is real and } k_\nu \geq 1, \\ 1 & \text{if } \nu \text{ is real and } k_\nu \leq 0, \\ k_\nu\big(\frac{|a|T_\nu}{y_\nu} + 1\big)^2 & \text{if } \nu \text{ is complex and } k_\nu \geq 1, \\ \frac{|a|T_\nu}{y_\nu} + 1 & \text{if } \nu \text{ is complex and } k_\nu \leq 0. \end{cases} \tag{3.7}$$

We are now ready to count points in our boxes.

Lemma 3.9. *One has*

$$\sharp(a\iota^{-1} \cap I(\underline{\underline{k}})) \ll \prod_\nu f_\nu(k_\nu).$$

Proof. By the construction of $\mathcal{P}$ we have

$$\sharp(a\iota^{-1} \cap I(\underline{\underline{k}})) = \frac{\mathrm{Vol}(\bigcup_{q \in a\iota^{-1} \cap I(\underline{\underline{k}})}(q + \mathcal{P}))}{\mathrm{Vol}(\mathcal{P})} \leq \frac{\mathrm{Vol}(J(\underline{\underline{k}}))}{\mathrm{Vol}(\mathcal{P})}.$$

One concludes using the calculations above. ■

For the estimation of $S_2(R)$ we need to count field elements with strong non-archimedean restrictions. We will be able to reduce this problem to [4, Lemma 6].

We will now define certain subsets of the adeles depending on tuples

$$\underline{\underline{k}} \in \mathbb{Z}^{\sharp\{\mathfrak{p}|\mathfrak{n}\}} \quad \text{and} \quad [\underline{\underline{u}}] \in \prod_{\mathfrak{p}|\mathfrak{n}} \mathfrak{o}_\mathfrak{p}^\times/(1 + \varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g)}\mathfrak{o}_\mathfrak{p}).$$

In our applications the tuples $\underline{\underline{k}}$ will usually lie in

$$\mathbb{Z}^\mathfrak{n} = \prod_{\mathfrak{p}|\mathfrak{n}} \{k_\mathfrak{p} \in \mathbb{Z} \colon k_\mathfrak{p} \geq -v_\mathfrak{p}(\iota)\}.$$

For such $\underline{\underline{k}}$ and $[\underline{\underline{u}}]$ we define the sets

$$\begin{aligned} \mathbb{A}_{\mathrm{fin}}^\iota &= \{a \in \mathbb{A}_{\mathrm{fin}} \colon v_\mathfrak{p}(a_\mathfrak{p}) \geq -v_\mathfrak{p}(\iota)\}, \\ C^\iota(\underline{\underline{k}}) &= \{a \in \mathbb{A}_{\mathrm{fin}}^\iota \colon v_\mathfrak{p}(a_\mathfrak{p}) = k_\mathfrak{p} \ \forall \mathfrak{p} \mid \mathfrak{n}\}, \\ C^\iota(\underline{\underline{k}}, [\underline{\underline{u}}]) &= \{a \in C^\iota(\underline{\underline{k}}) \colon a_\mathfrak{p} = \varpi_\mathfrak{p}^{k_\mathfrak{p}} a'_\mathfrak{p} \text{ with } [a'_\mathfrak{p}] = [u_\mathfrak{p}] \in \mathfrak{o}_\mathfrak{p}^\times/(1 + \varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}}\mathfrak{o}_\mathfrak{p}) \ \forall \mathfrak{p} \mid \mathfrak{n}\}. \end{aligned}$$

It will be useful to know the volumes of these sets.

Lemma 3.10. *For* $\underline{\underline{k}} \in \mathbb{Z}^{\mathfrak{n}}$ *and* $[\underline{\underline{u}}], [\underline{\underline{u'}}] \in \prod_{\mathfrak{p}|\mathfrak{n}} \mathfrak{o}_{\mathfrak{p}}^{\times}/(1+\varpi_{\mathfrak{p}}^{n_{0,\mathfrak{p}}(g_{\mathfrak{p}})}\mathfrak{o}_{\mathfrak{p}})$ *we have*

$$\begin{aligned} \mathrm{Vol}(\mathbb{A}_{\mathrm{fin}}^{\iota}, d\mu_{\mathrm{fin}}) &= \mathcal{N}(\iota), \\ \mathrm{Vol}(C^{\iota}(\underline{\underline{k}}), d\mu_{\mathrm{fin}}) &= \frac{\mathcal{N}(\iota)}{\mathcal{N}([\iota]_{\mathfrak{n}})}\zeta_{\mathfrak{n}}(1)\prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{-k_{\mathfrak{p}}}, \\ \mathrm{Vol}(C^{\iota}(\underline{\underline{k}}, [\underline{\underline{u}}]), d\mu_{\mathrm{fin}}) &= \mathrm{Vol}(C^{\iota}(\underline{\underline{k}}, [\underline{\underline{u'}}]), d\mu), \\ \mathrm{Vol}(C^{\iota}(\underline{\underline{k}}, [\underline{\underline{u}}]), d\mu_{\mathrm{fin}}) &= \frac{\mathcal{N}(\iota)}{\mathcal{N}([\iota]_{\mathfrak{n}})}\prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{-k_{\mathfrak{p}}-n_{0,\mathfrak{p}}(g)}. \end{aligned} \tag{3.8}$$

Proof. This is a standard adelic volume computation done place by place. The key facts we use are $\mu_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}}^{\times}) = \zeta_{\mathfrak{p}}(1)^{-1}$, $\mu_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{r}\mathfrak{o}_{\mathfrak{p}}) = q_{\mathfrak{p}}^{-r}$, and that both $\mu_{\mathfrak{p}}$ and $\mu_{\mathfrak{p}}^{\times}$ are Haar measures for $\mathfrak{o}_{\mathfrak{p}}^{\times}$. ∎

Finally, we are ready to prove the following counting result.

Lemma 3.11. *For* $\underline{\underline{k}} \in \mathbb{Z}^{\mathfrak{n}}$ *we have*

$$\sharp((a\iota^{-1}\backslash\{0\}) \cap B(R)C^{\iota}(\underline{\underline{k}}, [\underline{\underline{u}}])) \ll F_R(\underline{\underline{k}}) = 1 + \frac{|R|_{\infty}\mathcal{N}(\iota)}{\mathcal{N}(\mathfrak{n}_0(g))\mathcal{N}([\iota]_{\mathfrak{n}})}\prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{-k_{\mathfrak{p}}}$$

uniformly in $[\underline{\underline{u}}] \in \prod_{\mathfrak{p}|\mathfrak{n}} \mathfrak{o}_{\mathfrak{p}}^{\times}/(1+\varpi_{\mathfrak{p}}^{n_{0,\mathfrak{p}}(g)}\mathfrak{o}_{\mathfrak{p}})$*. Moreover, for* $\prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{k_{\mathfrak{p}}} > |R|_{\infty}\mathcal{N}(\iota^{-1}[\iota]_{\mathfrak{n}})$ *there is no* $q \in (F^{\times} \cap B(R)C^{\iota}(\underline{\underline{k}}))\backslash\{0\}$.

Proof. Let us call the set we want to count S. If S is empty we have nothing to show. Thus, take $q_0 \in S$. Now define the shifted set $S' = \frac{1}{q_0}S - 1$. Any $x \in S'$ satisfies

$$\begin{aligned} |x|_{\nu} &\le 2\left|\frac{R}{q_0}\right|_{\nu} && \text{for all } \nu, \\ |x|_{\mathfrak{p}} &\le \left|\frac{\varpi_{\mathfrak{p}}^{-v_{\mathfrak{p}}(\iota)}}{q_0}\right|_{\mathfrak{p}} && \text{for all } \mathfrak{p} \nmid \mathfrak{n}, \\ |x|_{\mathfrak{p}} &\le |\varpi_{\mathfrak{p}}^{n_{0,\mathfrak{p}}(g_{\mathfrak{p}})}|_{\mathfrak{p}} && \text{for all } \mathfrak{p} \mid \mathfrak{n}. \end{aligned}$$

Define the idele s by $s_{\nu} = 2^{1/[F_{\nu}:\mathbb{R}]}\frac{R}{q_0}$ and

$$s_{\mathfrak{p}} = \begin{cases} \varpi_{\mathfrak{p}}^{-v_{\mathfrak{p}}(\iota)}/q_0 & \text{if } \mathfrak{p} \nmid \mathfrak{n}, \\ \varpi_{\mathfrak{p}}^{n_{0,\mathfrak{p}}(g_{\mathfrak{p}})} & \text{else.} \end{cases}$$

After noting that $0 \in S'$ we conclude that

$$\sharp S \le 1 + \sharp\{x \in F^{\times}: |x|_{\nu} \le |s|_{\nu} \text{ and } |x|_{\mathfrak{p}} \le |s|_{\mathfrak{p}}\}.$$

To estimate the last set we use [4, Lemma 7]. This yields

$$\sharp S \le 1 + |s|_{\mathbb{A}_F}.$$

We are left with calculating the adelic norm of s. This is done using

$$\prod_{\nu} |q_0|_\nu^{-1} \prod_{\mathfrak{p}\nmid\mathfrak{n}} |q_0|_{\mathfrak{p}}^{-1} = \prod_{\mathfrak{p}|\mathfrak{n}} |q_0|_{\mathfrak{p}} = \prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{-k_{\mathfrak{p}}}.$$

To prove the second part we suppose $\prod_{\mathfrak{p}|\mathfrak{n}} q_{\mathfrak{p}}^{k_{\mathfrak{p}}} > |R|_\infty \mathcal{N}(\imath^{-1}[\imath]_{\mathfrak{n}})$. Let us define the ideal $\mathfrak{m} = \prod_{\mathfrak{p}|\mathfrak{n}} \mathfrak{p}^{k_{\mathfrak{p}}}$. Then in order to have $q \in C^{\imath}(\underline{k})$ one needs $\mathcal{N}((q)) \geq \mathcal{N}(\mathfrak{m})\imath[\imath]_{\mathfrak{n}}^{-1}$. But for $q \in B(R)$ we require $|q|_\infty \leq |R|_\infty$. Now we conclude by obtaining

$$1 = |q|_{\mathbb{A}} = |q|_\infty |q|_{\text{fin}} = \frac{|q|_\infty}{\mathcal{N}((q))} \leq \frac{|R|_\infty \mathcal{N}([\imath]_{\mathfrak{n}})}{\mathcal{N}(\mathfrak{m}\imath)} < 1. \qquad \blacksquare$$

Roughly the same reasoning applies to elements of $\imath^{-1} \cap B(R)$.

Corollary 3.12. *If $|R|_\infty < \mathcal{N}(\imath)^{-1}$, then $\imath^{-1} \cap B(R) = \{0\}$.*

3.3. *The sum $S_1(R)$*

In this section we will treat the sum S_1. Due to the transition region of the archimedean Whittaker function this argument requires

$$R_\nu = \frac{T_\nu + T_\nu^{1/3+\varepsilon}}{2\pi|y_\nu|} \asymp \frac{T_\nu}{y_\nu}. \tag{3.9}$$

Note that in view of Corollary 3.12 the sum S_1 is empty if $|R|_\infty < \mathcal{N}(\imath)^{-1}$. Therefore, we assume

$$|R|_\infty \asymp \left|\frac{T}{y}\right|_\infty \gg \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{-1}$$

throughout this section. Let us fix $a \in \imath$ such that (3.5) holds.

Lemma 3.13. *For $R = (R_\nu)_\nu$ as in (3.9) we have*

$$S_1(R) \ll |y|_\infty^{1/2} |T|_\infty^{-1/2} \prod_\nu \left(|T_\nu|_\nu^{1/6} + |a|_\nu^{1/4} \left|\frac{T_\nu}{y_\nu}\right|_\nu^{1/4} \right)^{1+\varepsilon}.$$

The proof will be in the spirit of [4, Section 8]. We start by expressing the archimedean Whittaker functions explicitly in terms of the K-Bessel function. We have

$$|W_\nu(a(\xi_\nu))| = \begin{cases} \dfrac{|\xi_\nu|_\nu^{1/2} |K_{it_\nu}(2\pi|\xi_\nu|)|}{|\Gamma(1/2+it_\nu)\Gamma(1/2-it_\nu)|^{1/2}} & \text{if } \nu \text{ is real,} \\[2ex] \dfrac{|\xi_\nu|_\nu^{1/2} |K_{i2t_\nu}(4\pi|\xi_\nu|)|}{|\Gamma(1+i2t_\nu)\Gamma(1-i2t_\nu)|^{1/2}} & \text{else.} \end{cases}$$

This holds as in [4, p. 26]. One notes that the Gamma factors are due to the L^2-normalization in the archimedean Whittaker model.

By Stirling's approximation one finds $|\Gamma(1/2+it_\nu)| \gg e^{-\pi/2t_\nu}$ and $|\Gamma(1+2it)| \gg T_\nu^{1/2}e^{-\pi t_\nu}$. Thus using [22, (3.1)] one derives

$$W_\nu(a(qy)) \ll |qy|_\nu^{1/2}\frac{T_\nu^{1/2}}{|T_\nu|_\nu^{1/2}}\min\big(T_\nu^{-1/3},(t_\nu|y_\nu|)^{-1/4}\big||q|-T_\nu/2\pi|y_\nu|\big|^{-1/4}\big).$$

We define

$$h_\nu(k_\nu)=\min\left(T_\nu^{1/6},\left|\frac{aT_\nu}{y_\nu k_\nu}\right|^{1/4}\right)$$

and observe that for $k_\nu\le 0$ and $q\in I_\nu(k_\nu)$ we have

$$W_\nu(a(qa^{-1}y)) \ll |qa^{-1}y|_\nu^{1/2}|T_\nu|_\nu^{-1/2}h_\nu(k_\nu). \tag{3.10}$$

But for $q\in B(R)$ we also have $|qa^{-1}|_\nu\le|R_\nu|_\nu\asymp|\frac{T_\nu}{y_\nu}|_\nu$ and hence $|qa^{-1}y_\nu|_\nu\ll|T_\nu|_\nu$. And thus

$$W_\nu(a(qa^{-1}y_\nu))\ll h_\nu(k_\nu).$$

Finally, we are ready to estimate $S_1(R)$.

Proof of Lemma 3.13. First, we shift the sum by a. This gives

$$S_1^4=|a|_\infty^2\sum_{q\in a\imath^{-1}\cap B(|a|R)}|q|_\infty^{-2}|W_\infty(a(qa^{-1}y))|^4.$$

Then we partition $B(|a|R)$ using the boxes defined in (3.6). In each box we exploit (3.10) to get

$$S_1^4\ll|a|_\infty^2\sum_{\substack{\underline{\underline{k}}\in\mathbb{Z}^{\sharp\{\nu\}}\\ -\lfloor aR_\nu\rfloor\le k_\nu\le 0}}\sharp(I(\underline{\underline{k}})\cap a\imath^{-1})\prod_\nu|a^{-1}y_\nu|_\nu^2|T_\nu|_\nu^{-2}h_\nu(k_\nu)^4.$$

Inserting the result from Lemma 3.9 yields

$$S_1^4\ll|y|_\infty^2|T|_\infty^{-2}\prod_\nu\sum_{k_\nu=0}^{\lfloor|a|R_\nu\rfloor}h_\nu(-k_\nu)^4f_\nu(-k_\nu).$$

To estimate the remaining sums we use ideas from [4, Section 8] and treating each archimedean place ν separately.

We start with ν real and obtain

$$\sum_{k_\nu=0}^{\lfloor|a|R_\nu\rfloor}h_\nu(-k_\nu)^4f_\nu(-k_\nu)=T_\nu^{2/3}+\sum_{k_\nu=1}^{\lfloor|a|R_\nu\rfloor}\frac{|a|T_\nu}{|y_\nu|k_\nu}$$
$$\ll\left(|T_\nu|_\nu^{2/3}+\frac{|a|T_\nu}{|y_\nu|}\right)^{1+\varepsilon}.$$

Similarly one treats the complex places:

$$\sum_{k_\nu=0}^{\lfloor |a|R_\nu \rfloor} h_\nu(-k_\nu)^4 f_\nu(-k_\nu) \leq T_\nu^{2/3}\left(|a|\frac{T_\nu}{|y_\nu|}+1\right) + \sum_{k_\nu=1}^{\lfloor |a|R_\nu \rfloor}\left(|a|\frac{T_\nu}{|y_\nu|}+1\right)\frac{|a|T_\nu}{|y_\nu|k_\nu}$$
$$\ll \left(|a|\frac{T_\nu}{|y_\nu|}+1\right)\left(T_\nu^{2/3}+|a|\frac{T_\nu}{|y_\nu|}\right)^{1+\varepsilon}$$
$$\ll \left(T_\nu^{2/3}+|a|\frac{T_\nu}{|y_\nu|}\right)^{2+\varepsilon} \ll \left(T_\nu^{4/3}+|a|^2\frac{T_\nu^2}{y_\nu^2}\right)^{1+\varepsilon}.$$

Putting everything together gives the desired estimate. ■

Corollary 3.14. *If*

$$|y_\nu|_\nu \asymp |a^3 T_\nu|_\nu^{\log|y|_\infty/\log|a^3T|_\infty} \tag{3.11}$$

for all ν, then

$$S_1(R) \ll |y|_\infty^{1/2}|T|_\infty^{-1/2}\left(|T|_\infty^{1/6}+\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4}\left|\frac{T}{y}\right|_\infty^{1/4}\right)^{1+\varepsilon}.$$

Proof. We consider two cases. First, assume $|y|_\infty \leq |a^3T|_\infty^{1/3}$. Then the balancing assumption implies $|y_\nu|_\nu \ll |a|_\nu|T_\nu|_\nu^{1/3}$ for all ν. Therefore, we have

$$\left|\frac{aT_\nu}{y_\nu}\right|_\nu^{1/4} \gg |T_\nu|_\nu^{1/6}.$$

Secondly, if $|y|_\infty \geq |a^3T|_\infty^{1/3}$, one argues analogously to obtain

$$\left|\frac{aT_\nu}{y_\nu}\right|_\nu^{1/4} \ll |T_\nu|_\nu^{1/6}.$$

Recalling that $|a|_\infty \ll \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))$ completes the proof. ■

3.4. The sum $S_2(R)$

In this section we will estimate the sum $S_2(R)$ by reducing it to well known averages of Hecke eigenvalues and local Whittaker functions. In view of estimating the error $\mathcal{E}(R)$ in the next section it will be useful to allow general parameters R.

Lemma 3.15. *For arbitrary $R=(R_\nu)_\nu \in (\mathbb{R}_{>0})^{\sharp\{\nu\}}$ we have*

$$S_2(R) \ll (|T|_\infty\mathcal{N}(\mathfrak{n}))^\varepsilon |R|_\infty^{1/4+\varepsilon}\left(\frac{\mathcal{N}(\mathfrak{n}_0)^{1/4}}{\mathcal{N}(\mathfrak{m}_1(g))^{1/4}}+|R|_\infty^{1/2}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4}\right).$$

Proof. We start by defining

$$I(\mathfrak{a}) = \{q \in \iota^{-1}\colon |q|_\nu \leq |R_\nu|_\nu,\ (q)=\mathfrak{a}\}.$$

Using [4, Corollary 1] we observe that

$$\sharp I(\mathfrak{a}) \ll_\varepsilon |R|_\infty^\varepsilon \mathcal{N}(\mathfrak{a})^{-\varepsilon}. \tag{3.12}$$

In particular, if $\mathcal{N}(\mathfrak{a}) \gg |R|_\infty$, then $I(\mathfrak{a})$ must be empty.

By Lemma 3.3 we have

$$S_2(R)^{4/3} = \sum_{\substack{\mathfrak{a}\subset\imath^{-1} \\ \mathcal{N}(\mathfrak{a})\ll|R|_\infty}} \mathcal{N}(\mathfrak{a})^{2/3}\mathcal{N}([\mathfrak{a}\imath]_{\mathfrak{n}})^{4/3}\frac{\left|\lambda\left(\frac{\mathfrak{a}\imath}{[\mathfrak{a}\imath]_{\mathfrak{n}}}\right)\right|^{4/3}}{\mathcal{N}(\mathfrak{a}\imath)^{4/3}} \sum_{q\in I(\mathfrak{a})} |\lambda_{\mathfrak{n}}(q)|^{4/3}$$

$$= \mathcal{N}(\imath)^{-2/3} \sum_{\substack{\mathfrak{a}_1|\mathfrak{n}^\infty \\ \mathcal{N}(\mathfrak{a}_1)\ll\mathcal{N}(\imath)|R|_\infty}} \mathcal{N}(\mathfrak{a}_1)^{2/3} \sum_{\substack{(\mathfrak{a}_2,\mathfrak{n})=1 \\ \mathcal{N}(\mathfrak{a}_2)\ll\frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{a}_1)}}} \frac{|\lambda(\mathfrak{a}_2)|^{4/3}}{\mathcal{N}(\mathfrak{a}_2)^{2/3}} \sum_{q\in I(\imath^{-1}\mathfrak{a}_1\mathfrak{a}_2)} |\lambda_{\mathfrak{n}}(q)|^{4/3}.$$

At this stage we apply Hölder to the $\mathfrak{a}_2$-sum. This yields

$$S_2(R)^{4/3} = \mathcal{N}(\imath)^{-2/3} \sum_{\substack{\mathfrak{a}_1|\mathfrak{n}^\infty \\ \mathcal{N}(\mathfrak{a}_1)\ll\mathcal{N}(\imath)|R|_\infty}} \mathcal{N}(\mathfrak{a}_1)^{2/3} S_{\mathrm{ur}}\left(\frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{a}_1)}\right)^{1/3}$$
$$\cdot\left(\sum_{\substack{(\mathfrak{a}_2,\mathfrak{n})=1 \\ \mathcal{N}(\mathfrak{a}_2)\ll\frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{a}_1)}}} \left(\sum_{q\in I(\imath^{-1}\mathfrak{a}_1\mathfrak{a}_2)} |\lambda_{\mathfrak{n}}(q)|^{4/3}\right)^{3/2}\right)^{2/3}.$$

Here

$$S_{\mathrm{ur}}(X) = \sum_{\substack{(\mathfrak{a},\mathfrak{n})=1 \\ \mathcal{N}(\mathfrak{a})\leq X}} \frac{|\lambda(\mathfrak{a})|^4}{\mathcal{N}(\mathfrak{a})^2}.$$

Before we continue, it is important to recall that our Hecke operators are differently normalized than the ones in [4, 15, 20]. It is well known that

$$S_{\mathrm{ur}}(X) \ll (|T|_\infty \mathcal{N}(\mathfrak{n}))^\varepsilon X^{1+\varepsilon}.$$

This was proved in [15] over $\mathbb{Q}$.

We now use Jensen's inequality exploiting the fact that the q-sum is short by (3.12). This yields

$$S_2(R)^{4/3} \ll (|T|_\infty|R|_\infty\mathcal{N}(\mathfrak{n}))^\varepsilon \mathcal{N}(\imath)^{-1/3}|R|_\infty^{1/3}$$
$$\cdot \sum_{\substack{\mathfrak{a}_1|\mathfrak{n}^\infty \\ \mathcal{N}(\mathfrak{a}_1)\ll\mathcal{N}(\imath)|R|_\infty}} \mathcal{N}(\mathfrak{a}_1)^{1/3+\varepsilon}\left(\sum_{\substack{(\mathfrak{a}_2,\mathfrak{n})=1 \\ \mathcal{N}(\mathfrak{a}_2)\ll\frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{a}_1)}}} \sum_{q\in I(\imath^{-1}\mathfrak{a}_1\mathfrak{a}_2)} |\lambda_{\mathfrak{n}}(q)|^2\right)^{2/3}. \tag{3.13}$$

We will continue to analyze the $\mathfrak{a}_2$-sum. For the sake of notation we define

$$S_{\mathrm{ram}} = \sum_{\substack{(\mathfrak{a}_2,\mathfrak{n})=1 \\ \mathcal{N}(\mathfrak{a}_2)\ll\frac{\mathcal{N}(\imath)|R|_\infty}{\mathcal{N}(\mathfrak{a}_1)}}} \sum_{q\in I(\imath^{-1}\mathfrak{a}_1\mathfrak{a}_2)} |\lambda_{\mathfrak{n}}(q)|^2. \tag{3.14}$$

In order to use the notation from Section 3.2 we set

$$\underline{k}(\mathfrak{a}_1\iota^{-1}) = (v_\mathfrak{p}(\mathfrak{a}_1\iota^{-1}))_{\mathfrak{p}|\mathfrak{n}} \in \mathbb{Z}^{\mathfrak{n}}.$$

By the local definition of $\lambda_\mathfrak{n}$ we can view it as a function on $\mathbb{A}_{\mathrm{fin}}^{\iota}$. Lemma 3.5 implies that this function is constant on $C^\iota(\underline{k},[\underline{u}])$ for $\underline{k}\in\mathbb{Z}^{\mathfrak{n}}$ and $[\underline{u}]\in\prod_{\mathfrak{p}|\mathfrak{n}}\mathfrak{o}_\mathfrak{p}^\times/(1+\varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g)}\mathfrak{o}_\mathfrak{p})$. Therefore, we have

$$\begin{aligned} S_{\mathrm{ram}} &= \sum_{q\in\iota^{-1}\cap B(R)C^\iota(\underline{k}(\mathfrak{a}_1\iota^{-1}))} |\lambda_\mathfrak{n}(q)|^2 \\ &= \sum_{[\underline{u}]\in\prod_{\mathfrak{p}|\mathfrak{n}}\mathfrak{o}_\mathfrak{p}^\times/(1+\varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g)}\mathfrak{o}_\mathfrak{p})} \sum_{q\in\iota^{-1}\cap B(R)C^\iota(\underline{k}(\mathfrak{a}_1\iota^{-1}),[\underline{u}])} |\lambda_\mathfrak{n}(q)|^2 \\ &= \sum_{[\underline{u}]\in\prod_{\mathfrak{p}|\mathfrak{n}}\mathfrak{o}_\mathfrak{p}^\times/(1+\varpi_\mathfrak{p}^{n_{0,\mathfrak{p}}(g)}\mathfrak{o}_\mathfrak{p})} \frac{\sharp(\iota^{-1}\cap B(R)C^\iota(\underline{k}(\mathfrak{a}_1\iota^{-1}),[\underline{u}]))}{\mathrm{Vol}(C^\iota(\underline{k}(\mathfrak{a}_1\iota^{-1}),[\underline{u}]),\,d\mu)} \\ &\qquad\qquad \cdot \int_{C^\iota(\underline{k}(\mathfrak{a}_1\iota^{-1}),[\underline{u}])} |\lambda_\mathfrak{n}(q)|^2\,d\mu_{\mathrm{fin}}(q). \end{aligned}$$

Using (3.8) and Lemma 3.11 reveals

$$S_{\mathrm{ram}} \ll \frac{\mathcal{N}(\mathfrak{a}_1)\mathcal{N}(\mathfrak{n}_0(g))}{\mathcal{N}(\iota)} F_R(\underline{k}(\mathfrak{a}_1\iota^{-1})) \int_{C^\iota(\underline{k}(\mathfrak{a}_1\iota^{-1}))} |\lambda_\mathfrak{n}(q)|^2\,d\mu_{\mathrm{fin}}(q). \tag{3.15}$$

The integral appearing here can be estimated using the local result [20, Proposition 2.11 (2)], which in our set-up reads

$$\int_{\mathfrak{o}_\mathfrak{p}^\times} |W_\mathfrak{p}(a(\varpi_\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{d})+v_\mathfrak{p}(\theta_i)+k_\mathfrak{p}}q)g_\mathfrak{p})|^2\,d\mu_\mathfrak{p}^\times(q) \ll q_\mathfrak{p}^{-(v_\mathfrak{p}(\mathfrak{d})+v_\mathfrak{p}(\theta_i)+k_\mathfrak{p}+n_{0,\mathfrak{p}}+m_{1,\mathfrak{p}}(g_\mathfrak{p}))/2}$$

for $g_\mathfrak{p}\in K_\mathfrak{p}a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}})$ and $n_{1,\mathfrak{p}}(g_\mathfrak{p})=n_{0,\mathfrak{p}}$. Note that in the situation at hand the condition on $g_\mathfrak{p}$ is satisfied since we are assuming $g\in\mathcal{J}_\mathfrak{n}$. With this at hand the estimate proceeds as follows:

$$\begin{aligned} &\int_{C^\iota(\underline{k})} |\lambda_\mathfrak{n}(q)|^2\,d\mu_{\mathrm{fin}}(q) \\ &\quad= \prod_{\mathfrak{p}\nmid\mathfrak{n}} \int_{\varpi_\mathfrak{p}^{-v_\mathfrak{p}(\iota)}\mathfrak{o}_\mathfrak{p}} 1\,d\mu_\mathfrak{p} \prod_{\mathfrak{p}|\mathfrak{n}} \int_{\varpi_\mathfrak{p}^{k_\mathfrak{p}}\mathfrak{o}_\mathfrak{p}^\times} |W_\mathfrak{p}(a(\varpi_\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{d})}\theta_i q)g_\mathfrak{p})|^2\,d\mu_\mathfrak{p}(q) \\ &\quad= \frac{\mathcal{N}(\iota)}{\mathcal{N}([\iota]_\mathfrak{n})}\zeta_\mathfrak{n}(1)^{-1}\prod_{\mathfrak{p}|\mathfrak{n}} q_\mathfrak{p}^{-k_\mathfrak{p}} \int_{\mathfrak{o}_\mathfrak{p}^\times} |W_\mathfrak{p}(a(\varpi_\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{d})+v_\mathfrak{p}(\theta_i)+k_\mathfrak{p}}q)g_\mathfrak{p})|^2\,d\mu_\mathfrak{p}^\times(q) \\ &\quad\ll \mathcal{N}(\mathfrak{n})^\varepsilon \frac{\mathcal{N}(\iota)}{\mathcal{N}([\iota]_\mathfrak{n})}\zeta_\mathfrak{n}(1)^{-1}\prod_{\mathfrak{p}|\mathfrak{n}} q_\mathfrak{p}^{-(v_\mathfrak{p}(\mathfrak{d})+v_\mathfrak{p}(\theta_i)+n_{0,\mathfrak{p}}+m_{1,\mathfrak{p}}(g_\mathfrak{p})+3k_\mathfrak{p})/2} \\ &\quad= \mathcal{N}(\mathfrak{n})^\varepsilon \frac{\mathcal{N}(\iota)}{\mathcal{N}([\iota]_\mathfrak{n})^{3/2}}\zeta_\mathfrak{n}(1)^{-1}\prod_{\mathfrak{p}|\mathfrak{n}} q_\mathfrak{p}^{-3k_\mathfrak{p}/2}. \end{aligned}$$

Inserting this estimate in our expression for S_{ram} we get

$$S_{\mathrm{ram}} \ll \zeta_{\mathfrak{n}}(1)^{-1}\mathcal{N}(\mathfrak{n}_0(g))\mathcal{N}(\mathfrak{a}_1)^{-1/2}F_R(\underline{k}(\mathfrak{a}_1\iota^{-1})).$$

The result from Lemma 3.11 yields

$$S_{\mathrm{ram}} \ll \zeta_{\mathfrak{n}}(1)^{-1}\left(\frac{\mathcal{N}(\mathfrak{n}_0(g))}{\mathcal{N}(\mathfrak{a}_1)^{1/2}} + \frac{|R|_\infty\mathcal{N}(\iota)}{\mathcal{N}(\mathfrak{a}_1)^{3/2}}\right).$$

From (3.13) we deduce

$$\begin{aligned} S_2(R) \ll (|T|_\infty|R|_\infty\mathcal{N}(\mathfrak{n}))^{\varepsilon}\frac{|R|_\infty^{1/4}}{\mathcal{N}(\iota)^{1/4}}\left(\sqrt{\mathcal{N}(\mathfrak{n}_0(g))} + \sqrt{|R|_\infty\mathcal{N}(\iota)}\right) \\ \cdot\Bigg(\sum_{\substack{\mathfrak{a}_1|\mathfrak{n}^\infty \\ \mathcal{N}(\mathfrak{a}_1)\ll\mathcal{N}(\iota)|R|_\infty}} \mathcal{N}(\mathfrak{a}_1)^{\varepsilon}\Bigg)^{4/3}. \end{aligned}$$

In the end we note that by the Rankin trick we have

$$\sum_{\substack{\mathfrak{a}_1|\mathfrak{n}^\infty \\ \mathcal{N}(\mathfrak{a}_1)\ll\mathcal{N}(\iota)|R|_\infty}} \mathcal{N}(\mathfrak{a}_1)^{\varepsilon} \ll \mathcal{N}(\mathfrak{n})^{\varepsilon}|R|_\infty^{\varepsilon}.$$ ∎

3.5. *The error $\mathcal{E}$*

For R as in (3.9) we will roughly prove that the error is always absorbed in the main-contribution. More precisely, we have the following lemma.

Lemma 3.16. *Under the balancing assumption*

$$|y_\nu|_\nu \asymp |a^3T_\nu|_\nu^{\log|y|_\infty/\log|a^3T|_\infty} \quad \textit{for all } \nu$$

and with R as in (3.9) we have

$$\begin{aligned} \mathcal{E} \ll (|T|_\infty|R|_\infty\mathcal{N}(\mathfrak{n}))^{\varepsilon} \\ \cdot\left(|T|_\infty^{1/6}\mathcal{N}(\mathfrak{n}_0)^{1/2} + |T|_\infty^{1/6}|R|_\infty^{1/4}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4} + |R|_\infty^{1/2}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/2}\right). \end{aligned}$$

Proof. For $S \subset \{\nu\}$ and $\underline{k} \in \mathbb{N}^{\sharp S}$ we define

$$\begin{aligned} R'(\underline{k}) &= \begin{cases} (k_\nu+1)R_\nu & \text{if } \nu \in S, \\ R_\nu & \text{else,} \end{cases} \\ I_S(\underline{k}) &= \prod_{\nu\in S} I_\nu(k_\nu), \\ B_S(R) &= \prod_{\nu\notin S}\{\xi_\nu \in F^\times \colon |\xi_\nu| \le R_\nu\}. \end{aligned}$$

For $k_\nu \ge 1$ we define

$$h_\nu(k_\nu) = e^{-\pi|y_\nu|R_\nu k_\nu}.$$

By the exponential decay of the K-Bessel function we have the bound

$$|a^{-1}q|_{\nu}^{-2}|W_{\nu}(a(a^{-1}qy_{\nu}))|^4 \ll k_{\nu}^{-2}R_{\nu}^{-2}h_{\nu}(k_{\nu})^4 \tag{3.16}$$

for $q \in I_{\nu}(k_{\nu})$ and $k_{\nu} \geq 1$. We now decompose $\mathcal{E}$ as follows:

$$\mathcal{E} \leq \sum_{\emptyset \neq S \subset \{\nu\}} \sum_{\underline{\underline{k}} \in \mathbb{N}^{\sharp S}} \Big(\sum_{q \in a\iota^{-1} \cap I_S(\underline{\underline{k}}) \times B_S(|a|R)} |a^{-1}q|_{\infty}^{-2}|W_{\infty}(a^{-1}qy)|^4 \Big)^{1/4} S_2(R'(\underline{\underline{k}})).$$

Again we included the shift by a only in the archimedean part. Note that by Corollary 3.12 below the sum $S_2(R'(\underline{\underline{k}})) = 0$ if

$$\prod_{\nu \in S} |k_{\nu}+1|_{\nu}|R|_{\infty} < \mathcal{N}(\iota)^{-1}.$$

We can add the condition

$$\prod_{\nu \in S} |k_{\nu}+1|_{\nu} \geq |R|_{\infty}^{-1}\mathcal{N}(\iota)^{-1}$$

to the sum over $\underline{\underline{k}}$.

First, note that Lemma 3.15 is general enough to deal with the non-archimedean part of the sum. To deal with the archimedean part we use the same approach as in Section 3.3. In particular, with (3.10) and (3.16) we have

$$\begin{aligned}
&\sum_{q \in a\iota^{-1} \cap I_S(\underline{\underline{k}}) \times B_S(|a|R)} |qa^{-1}|_{\infty}^{-2}|W_{\infty}(a(qa^{-1}y))|^4 \\
&\quad = \sum_{\substack{\underline{\underline{k}}^c \\ -\lfloor |a|R_{\nu}\rfloor \leq k_{\nu} \leq 0 \,\forall \nu \notin S}} \sum_{q \in a\iota^{-1} \cap I(\underline{\underline{k}} \times \underline{\underline{k}}^c)} |qa^{-1}|_{\infty}^{-2}|W_{\infty}(a(qa^{-1}y))|^4 \\
&\quad \ll |R|_{\infty}^{-2} \prod_{\nu \in S} |k_{\nu}|_{\nu}^{-2} h_{\nu}(k_{\nu})^4 f_{\nu}(k_{\nu}) \prod_{\nu \notin S} \sum_{\substack{\underline{\underline{k}}^c \\ -\lfloor |a|R_{\nu}\rfloor \leq k_{\nu} \leq 0 \,\forall \nu \ni S}} h_{\nu}(k_{\nu})^4 f_{\nu}(k_{\nu}) \\
&\quad \ll |R|_{\infty}^{-2} \prod_{\nu \in S} |k_{\nu}|_{\nu}^{-2} h_{\nu}(k_{\nu})^4 f_{\nu}(k_{\nu}) \prod_{\nu \notin S} (|T_{\nu}|_{\nu}^{2/3} + |aR_{\nu}|_{\nu})^{1+\varepsilon}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\mathcal{E} \ll &\left(\left| \frac{T}{y} \right|_{\infty} \mathcal{N}(\mathfrak{n}) \right)^{\varepsilon} \\
&\cdot \sum_{\emptyset \neq S \subset \{\nu\}} \sum_{\substack{\underline{\underline{k}} \in \mathbb{N}^{\sharp S} \\ \prod_{\nu \in S} |k_{\nu}+1|_{\nu} \geq |R|_{\infty}^{-1}\mathcal{N}(\iota)^{-1}}} \left(\frac{|R|_{\infty}^{-1/4}\mathcal{N}(\mathfrak{n}_0)^{1/4}}{\mathcal{N}(\mathfrak{m}_1(g))^{1/4}} + |R|_{\infty}^{1/4}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4} \right) \\
&\cdot \prod_{\nu \notin S} (|T_{\nu}|_{\nu}^{1/6} + |aR_{\nu}|_{\nu}^{1/4}) \prod_{\nu \in S} h_{\nu}(k_{\nu}) f_{\nu}(k_{\nu})^{1/4}.
\end{aligned}$$

Inserting the definition of f_ν from (3.7) and using the balancing assumption as in the proof of Corollary 3.14 yields

$$\begin{aligned}\mathcal{E} \ll \sum_{\emptyset\neq S\subset\{\nu\}} \sum_{\substack{\underline{\underline{k}}\in\mathbb{N}^{\sharp S}\\ \prod_{\nu\in S}|k_\nu+1|_\nu \geq |R|_\infty^{-1}\mathcal{N}(\iota)^{-1}}} &\Bigg[\Bigg(\frac{|R|_\infty^{-1/4}\mathcal{N}(\mathfrak{n}_0)^{1/4}}{\mathcal{N}(\mathfrak{m}_1(g))^{1/4}} \\ &+ |R|_\infty^{1/4}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4}\Bigg)^{1+\varepsilon} \left(|T|_\infty^{1/6} + |R|_\infty^{1/4}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4}\right)\prod_{\nu\in S} k_\nu^{[F_\nu:\mathbb{R}]-1}h_\nu(k_\nu)\Bigg] \\ \ll (|T|_\infty|R|_\infty\mathcal{N}(\mathfrak{n}))^{\varepsilon} & \\ \cdot \sum_{\emptyset\neq S\subset\{\nu\}} \sum_{\substack{\underline{\underline{k}}\in\mathbb{N}^{\sharp S}\\ \prod_{\nu\in S}|k_\nu+1|_\nu \geq |R|_\infty^{-1}\mathcal{N}(\iota)^{-1}}} &\Bigg[\Bigg(|R|_\infty^{-1/4}\frac{|T|_\infty^{1/6}\mathcal{N}(\mathfrak{n}_0)^{1/4}}{\mathcal{N}(\mathfrak{m}_1(g))^{1/4}} + \mathcal{N}(\mathfrak{n}_0)^{1/2} \\ &+ |T|_\infty^{1/6}|R|_\infty^{1/4}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4} + |R|_\infty^{1/2}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/2}\Bigg)\prod_{\nu\in S} k_\nu^{[F_\nu:\mathbb{R}]-1}h_\nu(k_\nu)\Bigg].\end{aligned}$$

Finally, we use the condition in the $\underline{\underline{k}}$-sum to remove the factor $|R|^{-1/4}$. We drop any unnecessary condition on $\underline{\underline{k}}$ and end up with

$$\begin{aligned}\mathcal{E} &\ll (|T|_\infty|R|_\infty\mathcal{N}(\mathfrak{n}))^{\varepsilon} \\ &\quad\cdot \left(|T|_\infty^{1/6}\mathcal{N}(\mathfrak{n}_0)^{1/2} + |T|_\infty^{1/6}|R|_\infty^{1/4}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4} + |R|_\infty^{1/2}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/2}\right) \\ &\quad\cdot \sum_{\emptyset\neq S\subset\{\nu\}}\sum_{\underline{\underline{k}}\in\mathbb{N}^{\sharp S}} |k_\nu|^{[F_\nu:\mathbb{R}]/2-1/4}h_\nu(k_\nu).\end{aligned}$$

Due to the exponential decay of $h_\nu(k_\nu)$ for positive k_ν it is no problem to estimate the remaining sums by

$$\sum_{\emptyset\neq S\subset\{\nu\}}\sum_{\underline{\underline{k}}\in\mathbb{N}^{\sharp S}} |k_\nu|^{[F_\nu:\mathbb{R}]/2-1/4}h_\nu(k_\nu) \ll 1.$$

This completes the proof. ■

3.6. *The final Whittaker bound*

Now we have all the pieces together to prove an upper bound for $\phi_\circ$ via its Whittaker expansion.

Proposition 3.17. *Let $\phi_\circ = \sigma(v^\circ)$ for some cuspidal automorphic representation (π, V_π) with new vector v°. For $g \in \mathcal{J}_\mathfrak{n}$ and R as in (3.9) we have*

$$\begin{aligned}&|\phi_\circ(a(\theta_i)gn(x)a(y))| \\ &\quad\ll (|T|_\infty|y|_\infty^{-1}\mathcal{N}(\mathfrak{n}))^{\varepsilon} \\ &\qquad\cdot\left(|T|_\infty^{1/6}\mathcal{N}(\mathfrak{n}_0)^{1/2} + |T|_\infty^{1/6}|R|_\infty^{1/4}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4} + |R|_\infty^{1/2}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/2}\right).\end{aligned}$$

Proof. As in [4, (8.7)] we can assume that y is balanced in the sense of (3.11).

Next we note that if $|R|_\infty < \mathcal{N}(\iota)^{-1}$, it follows from Corollary 3.12 and (3.4) that

$$|\psi(a(\theta_i)gn(x)a(y))| \leq |c_{\phi_\circ}|\mathcal{E}.$$

In this case we get the desired bound from Lemmas 3.6 and 3.16.

Finally, for $|R|_\infty \geq \mathcal{N}(\iota)^{-1}$ the main contribution will obviously come from $S_1S_2(R)$, since the error is controlled by Lemma 3.16. As in [4, (8.7)], we can assume that y is balanced in the sense of (3.11) so that we can use Corollary 3.14 and Lemma 3.15 to show

$$\begin{aligned} S_1S_2(R) &\ll \left(\left|\frac{T}{y}\right|_\infty \mathcal{N}(\mathfrak{n})\right)^{\varepsilon} |R|_\infty^{-1/4}\left(|T|_\infty^{1/6} + \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4}\left|\frac{T}{y}\right|_\infty^{1/4}\right)^{1+\varepsilon} \\ &\quad \cdot \left(\frac{\mathcal{N}(\mathfrak{n}_0)^{1/4}}{\mathcal{N}(\mathfrak{m}_1(g))^{1/4}} + |R|_\infty^{1/2}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4}\right) \\ &\ll \left(\left|\frac{T}{y}\right|_\infty \mathcal{N}(\mathfrak{n})\right)^{\varepsilon}\Bigg(|R|_\infty^{-1/4}\frac{|T|_\infty^{1/6}\mathcal{N}(\mathfrak{n}_0)^{1/4}}{\mathcal{N}(\mathfrak{m}_1(g))^{1/4}} + \mathcal{N}(\mathfrak{n}_0)^{1/2} \\ &\qquad + |T|_\infty^{1/6}|R|_\infty^{1/4}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/4} + |R|_\infty^{1/2}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{1/2}\Bigg). \end{aligned}$$

Using $|R|_\infty^{-1} \leq \mathcal{N}(\iota)^{-1} \ll \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1(g))^{-1}$ to get rid of the factor $|R|_\infty^{-1/4}$ concludes the proof. ∎

Corollary 3.18. *Let $\phi_\circ = \sigma(v^\circ)$ for some cuspidal automorphic representation (π, V_π) with new vector v°. Then*

$$\frac{\phi_\circ(g)}{\|\phi_\circ\|_2} \ll (|T|_\infty\mathfrak{n}_2\mathfrak{n}_0\mathfrak{m}_1)^{1/2+\varepsilon} \quad \textit{for all } g \in \mathrm{GL}_2(\mathbb{A}).$$

Proof. First we note that by Corollary 2.7 it is enough to consider $\phi_\circ$ on elements of the form $a(\theta_i)gn(x)a(y)$ for $g \in \mathcal{J}_\mathfrak{n}$ and $n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}$. The result now follows from Proposition 3.17 upon noting that $|R|_\infty \asymp |T|_\infty/|y|_\infty$ and $|y|_\infty \gg \mathcal{N}(\mathfrak{n}_2)^{-1}$. ∎

4. Bounds in the bulk

After having obtained estimates for automorphic forms near the cusps we have to determine their size in the bulk. Note that the term bulk is used somewhat informally and stands for a part of the generating domain $\bigsqcup_{i=1}^{h_F} a(\theta_i)\mathcal{J}_\mathfrak{n} \times \mathcal{F}_{\mathfrak{n}_2}$ containing the region where the Whittaker expansion fails to deliver the desired bounds. To make this somewhat more precise, let us remark that later in the proof of Theorem 1.1 the bulk is understood to be

$$\{a(\theta_i)g'h_\mathfrak{n}n(x)a(y) \in a(\theta_i)\mathcal{J}_\mathfrak{n} \times \mathcal{F}_{\mathfrak{n}_2} \colon i = 1, \ldots, h_F \text{ and } |y|_\infty \leq |T|_\infty^{1/3}\mathcal{N}(\mathfrak{n}_2)^{-1/3}\}.$$

Since $\mathcal{J}_\mathfrak{n}$ is compact and we can assume y is balanced, this can really be thought of as some bounded piece of the generating domain justifying the term bulk.

The strategy we will pursue in this section is based on the so called amplification method. More precisely, we will define an integral operator which approximates a spectral projector on a certain subspace of $L^2(G(F)\backslash G(\mathbb{A}_F))$ related to the automorphic form under consideration. A geometric estimation of the kernel will yield the desired estimate.

Let (π, V_π) be a cuspidal automorphic representation with new vector v° and associated newform $\phi_\circ = \sigma(v^\circ)$. The (arithmetic) conductor of π is given by $\mathfrak{n} = \mathfrak{n}_2\mathfrak{n}_0^2$. As earlier, we write $\mathfrak{m}$ for the conductor of the central character ω_π of π and set $\mathfrak{m}_1 = \mathfrak{m}/\gcd(\mathfrak{m}, \mathfrak{n}_2\mathfrak{n}_0)$.

Throughout this section we fix a square-free ideal $\mathfrak{q}$ such that all the units that are quadratic residues modulo $\mathfrak{q}$ are indeed contained in $(\mathcal{O}_F^\times)^2$. We will further assume that $(\mathfrak{q}, \mathfrak{n}) = 1$. Later on we will see how to construct such an ideal with the additional important property $\mathcal{N}(q) \ll (\log L)^A$ for some positive constant A. This constant will only depend on the field F and therefore we may allow all implicit constant in this section to depend on A.

4.1. *Amplification and the spectral expansion*

Let $\phi = \phi_\circ^{\mathfrak{L}} = \sigma^{\mathfrak{L}}(v_{\mathfrak{L}}^\circ)$. By Corollary 2.7 it is enough to consider $\phi(g)$ for

$$g = a(\theta_i)g'n(x)a(y) \quad \text{for } n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2},\ g' = kh_{\mathfrak{n}} \in \mathcal{J}_{\mathfrak{n}}.$$

Therefore, we further define $\phi' = \phi(\cdot\, h_{\mathfrak{n}})$. This function is $K_1'(\mathfrak{n}) = h_{\mathfrak{n}}K_1(\mathfrak{n})h_{\mathfrak{n}}^{-1}$-invariant and can be considered as an element of the Hilbert space

$$L^2(X) = L^2(G(F)\backslash G(\mathbb{A}_F)/K_1'(\mathfrak{n}), \omega_\pi) \subset L^2(G(F)\backslash G(\mathbb{A}_F)).$$

Furthermore, we put $w^\circ = \pi^{\mathfrak{L}}(h_{\mathfrak{n}})v_{\mathfrak{L}}^\circ$. Then $\phi' = \sigma^{\mathfrak{L}}(w^\circ)$. We will bound ϕ' on elements $g = a(\theta_i)g'n(x)a(y)$ with $g' \in K_{\mathfrak{n}}h_{\mathfrak{n}}^{-1}$ and $n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}$.

Next we define the kernel function which will be used to construct the approximate spectral projector mentioned earlier. We do this place by place and immediately give some basic properties.

Let ν be an archimedean place. Here, since the vector w_ν° is spherical, it is enough to consider functions f_ν that factor through the point pair invariant u_ν. If ν is real, let $\mathcal{H}^2 \subset \mathbb{C}$ be the usual upper half-plane and put $i_\nu = i$. In this case $u_\nu\colon \mathcal{H}^2 \times \mathcal{H}^2 \to \mathbb{R}_{\geq 0}$ is given by

$$u_\nu(z_1, z_2) = \frac{|z_1 - z_2|}{2\Im(z_1)\Im(z_2)} \quad \text{for } z_1, z_2 \in \mathcal{H}^2.$$

For ν complex we consider the upper half-space

$$\mathcal{H}^3 = \{z + yj \in \mathbb{H} \colon z \in \mathbb{C},\ y > 0\}.$$

This space is viewed as a subspace of the Hamiltonian quaternions $\mathbb{H}$ equipped with the standard norm $\|\cdot\|$. We fix $i_\nu = j$ serving as central point. Now the point pair invariant $u_\nu\colon \mathcal{H}^3 \times \mathcal{H}^3 \to \mathbb{R}_{\geq 0}$ is given by

$$u_\nu(z_1, z_2) = \frac{\|z_1 - z_2\|}{2\Im(z_1)\Im(z_2)} \quad \text{for } z_1, z_2 \in \mathcal{H}^3.$$

Finally, the archimedean test function at ν is defined as

$$f_\nu(g_\nu) = k_\nu(u_\nu(g_\nu.i_\nu, i_\nu)),$$

for k_ν as in [4, Lemma 10]. By uniqueness of the spherical vector we have

$$R(f_\nu)w_\nu^\circ = c_\nu(\pi_\nu)w_\nu^\circ.$$

The number $c_\nu(\pi_\nu)$ is positive and depends only on the equivalence class of π_ν and is given by the spherical transform of f_ν at π_ν. By a suitable parametrization of spherical representations of $G(F_\nu)$ one relates this to the classical Selberg/Harish-Chandra transform of k_ν. Therefore, we have

$$c_\nu(\pi_\nu) \gg 1 \tag{4.1}$$

by [4, Lemma 10].

For $\mathfrak{p} \mid \mathfrak{n}$ we would like to choose the new vector matrix coefficient

$$\Phi_{\pi_\mathfrak{p}}(g_\mathfrak{p}) = \frac{\langle v_\mathfrak{p}^\circ, \pi_\mathfrak{p}(g_\mathfrak{p})v_\mathfrak{p}^\circ\rangle}{\langle v_\mathfrak{p}^\circ, v_\mathfrak{p}^\circ\rangle}$$

of $\pi_\mathfrak{p}$ as test function. However, since this function fails to have nice support properties in general, we must modify it by forcing its support to lie in a convenient compact set. To be precise, we define

$$\Phi'_{\pi'_\mathfrak{p}}(g_\mathfrak{p})\mathbb{1}_{ZK_\mathfrak{p}^\circ}(g)\Phi_{\pi'_\mathfrak{p}}(a(\varpi_\mathfrak{p}^{-n_{1,\mathfrak{p}}})g_\mathfrak{p}a(\varpi_\mathfrak{p}^{n_{1,\mathfrak{p}}}))$$

as in [20, Section 2F]. Note that besides chopping off parts of the matrix coefficient we conjugated the variable to facilitate the change from the new vector $v_\mathfrak{p}^\circ$ to its translate $w_\mathfrak{p}^\circ$. The test function is finally defined by

$$f_\mathfrak{p}(g_\mathfrak{p}) = |\det(g_\mathfrak{p})|^{ia_\mathfrak{p}/2}\Phi'_{\pi'_\mathfrak{p}}(g_\mathfrak{p}).$$

By construction (see [20, Proposition 2.13]) there is $\delta_{\pi_p} > 0$ such that

$$R_\mathfrak{p}(f_\mathfrak{p})w_\mathfrak{p}^\circ = \int_{Z(F_\mathfrak{p})\backslash G(F_\mathfrak{p})} f_\mathfrak{p}(g)\pi_\mathfrak{p}(g)w_\mathfrak{p}^\circ \, d\mu_\mathfrak{p}(g) = \delta_{\pi'_\mathfrak{p}}w_\mathfrak{p}^\circ \quad \text{and} \quad \delta_{\pi'_\mathfrak{p}} \gg q_\mathfrak{p}^{-n_{1,\mathfrak{p}}-m_{1,\mathfrak{p}}},$$

where $n_{1,\mathfrak{p}} = \lceil v_\mathfrak{p}(\mathfrak{n})/2\rceil$ and $m_{1,\mathfrak{p}} = \max(0, v_\mathfrak{p}(\mathfrak{m}) - n_{1,\mathfrak{p}})$ as in Section 2.1. The action of $R_\mathfrak{p}(f_\mathfrak{p})$ on elements of generic representations is described in [20, Corollary 2.16]. Furthermore, in Appendix A we compute this action on one non-generic one-dimensional representation which may appear in the discrete spectrum. Let us remark that

$$|f_\mathfrak{p}(g)| \leq 1 \quad \text{for all } g \in G(F_\mathfrak{p}),$$

$$\operatorname{supp}(f_\mathfrak{p}) = \begin{cases} Z(F_\mathfrak{p})K_\mathfrak{p} & \text{if } n_\mathfrak{p} \text{ is even,} \\ Z(F_\mathfrak{p})K_\mathfrak{p}^0(1) & \text{else.} \end{cases}$$

For $\mathfrak{p} \mid \mathfrak{q}$ define

$$\tilde{K}_{0,\mathfrak{p}}(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{0,\mathfrak{p}}(1) \colon a - d \in \varpi_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}} \right\}.$$

Then we put

$$f_{\mathfrak{p}}(g_{\mathfrak{p}}) = \begin{cases} \mathrm{Vol}(Z(\mathfrak{o}_{\mathfrak{p}}) \backslash \tilde{K}_{0,\mathfrak{p}}(1))^{-1} \omega_{\pi_{\mathfrak{p}}}^{-1}(z) & \text{if } g_{\mathfrak{p}} = zk \in Z(F_{\mathfrak{p}}) \tilde{K}_{0,\mathfrak{p}}(1), \\ 0 & \text{else.} \end{cases}$$

Note that because $(\mathfrak{n}, \mathfrak{q}) = 1$, the central $\omega_{\pi_{\mathfrak{p}}}$ is unramified for $\mathfrak{p} \mid \mathfrak{q}$. This makes $f_{\mathfrak{p}}$ well defined. Since $w_{\mathfrak{p}}^{\circ}$ is $K_{\mathfrak{p}}$-fixed, we see that

$$\begin{aligned} R_{\mathfrak{p}}(f_{\mathfrak{p}}) w_{\mathfrak{p}}^{\circ} &= \int_{Z(F_{\mathfrak{p}}) \backslash G(F_{\mathfrak{p}})} f_{\mathfrak{p}}(g) \pi_{\mathfrak{p}}(g) w_{\mathfrak{p}}^{\circ} \, d\mu_{\mathfrak{p}}(g) \\ &= \mathrm{Vol}(Z(\mathfrak{o}_{\mathfrak{p}}) \backslash \tilde{K}_{0,\mathfrak{p}}(1))^{-1} \int_{Z(\mathfrak{o}_{\mathfrak{p}}) \backslash \tilde{K}_{0,\mathfrak{p}}(1)} \omega_{\pi_{\mathfrak{p}}}(z)^{-1} \pi_{\mathfrak{p}}(zk) w_{\mathfrak{p}}^{\circ} \, d\mu_{\mathfrak{p}}(zk) = w_{\mathfrak{p}}^{\circ}. \end{aligned}$$

We also have the estimate

$$|f_{\mathfrak{p}}| \leq [K_{\mathfrak{p}} : \tilde{K}_{0,\mathfrak{p}}(1)] \ll q_{\mathfrak{p}}^{2+\varepsilon}.$$

We will treat the remaining places all at once. Set $S_{\mathrm{ur}} = \{\mathfrak{p} \colon (\mathfrak{p}, \mathfrak{q}\mathfrak{n}) = 1\}$ and define the unramified Hecke algebra

$$\mathcal{H}_{\mathrm{ur}} = \left\langle \left\{ \kappa_{\mathrm{ur}} = \bigotimes_{\mathfrak{p} \in S_{\mathrm{ur}}} \kappa_{\mathfrak{p}} \colon \kappa_{\mathfrak{p}} \in \mathcal{C}_c^{\infty}(G(F_{\mathfrak{p}}), \omega_{\pi_{\mathfrak{p}}}),\ \kappa_{\mathfrak{p}}(K_{\mathfrak{p}} g K_{\mathfrak{p}}) = \kappa_{\mathfrak{p}}(g) \right\} \right\rangle_{\mathbb{C}}.$$

This is a commutative algebra by [9, Theorem 4.6.1]. To an integral ideal $\mathfrak{c}$ we associate the special element

$$\kappa_{\mathfrak{c}} = \bigotimes_{\mathfrak{p} \in S_{\mathrm{ur}}} \kappa_{\mathfrak{p}, v_{\mathfrak{p}}(\mathfrak{c})} \in \mathcal{H}_{\mathrm{ur}}$$

where

$$\kappa_{\mathfrak{p},k}(g) = \begin{cases} \omega_{\pi_{\mathfrak{p}}}(z)^{-1} & \text{for } g = z \ \in Z(F_{\mathfrak{p}}) X_{\mathfrak{p},k}, \\ 0 & \text{else.} \end{cases}$$

This is well defined since the central character is unramified at the places under consideration. The function $\kappa_{\mathfrak{p},k}$ is constructed such that $\pi(\mathbb{1}_{X_{\mathfrak{p},k}}) = R(\kappa_{\mathfrak{p},k})$. Therefore, for $w_{\mathrm{ur}}^{\circ} = \bigotimes_{\mathfrak{p} \in S_{\mathrm{ur}}} w_{\mathfrak{p}}^{\circ}$ we have

$$R(\kappa_{\mathfrak{c}}) w_{\mathrm{ur}}^{\circ} = \lambda(\mathfrak{c}) w_{\mathrm{ur}}^{\circ}.$$

Fix a large parameter L such that $\mathcal{N}(q) \ll (\log L)^A$ for some constant A. We define the sets

$$\begin{aligned} \mathcal{P}_{\mathfrak{q}} &= \{\mathfrak{a} \colon \mathfrak{a} = (\alpha) \text{ for } \alpha \in F_{+}^{\times} \cap (1 + \mathfrak{q})\}, \\ \mathcal{J}(\mathfrak{q}) &= \{\mathfrak{a} \colon (\mathfrak{a}, \mathfrak{q}) = 1\}, \\ \mathcal{P}(L) &= \{\alpha \in \mathcal{O}_F \colon (\alpha) \in \mathcal{P}_{\mathfrak{q}} \text{ is a prime ideal}, \mathcal{N}(\alpha) \in [L, 2L],\ ((\alpha), \mathfrak{n}) = 1\}/\sim. \end{aligned}$$

In the last definition we write $\alpha \sim \beta$ for the equivalence relation $(\alpha) = (\beta)$. We identify $\mathcal{P}(L)$ with a suitable fundamental domain for $\sim$. We can arrange that $\alpha_\nu \asymp L^{[F:\mathbb{Q}]}$ for all ν and all $\alpha \in \mathcal{P}(L)$.

We need a lower bound for $\sharp\mathcal{P}(L)$. Since we cannot assume that $\mathfrak{q}$ is fixed (it might depend on $\mathfrak{n}$), we need a stronger argument than in [4]. The following variation of the generalized Siegel–Walfisz theorem does the job.

Lemma 4.1. *If $\mathcal{N}(\mathfrak{q}) \ll (\log L)^A$ for some positive constant A, then*

$$\frac{L}{\mathcal{N}(\mathfrak{q})\log L} \ll_{F,A} \sharp\mathcal{P}(L) \ll_{F,A} \frac{L}{\log L}. \tag{4.2}$$

This is a very lazy estimate but it uses some heavy machinery, so we will sketch the proof.

Proof. Let $Cl_F^{\mathfrak{q}} = \mathcal{J}(\mathfrak{q})/\mathcal{P}_{\mathfrak{q}}$ be the ray class group. The explicit formula [16, VI, Theorem 1] for the cardinality of $Cl_F^{\mathfrak{q}}$ implies

$$\sharp Cl_F^{\mathfrak{q}} \ll \mathcal{N}(\mathfrak{q}).$$

For our purposes this is enough.

The statement follows immediately from [14, Korollar 1.3]. ∎

Remark 4.2. We could also work with the weaker assumption $\mathcal{N}(\mathfrak{q}) \ll_\varepsilon \mathcal{N}(\mathfrak{n})^\varepsilon$. In this case we can still obtain a good lower bound for $\sharp\mathcal{P}(L)$ using a version of Linnik's theorem over number fields.

To $\alpha \in \mathcal{O}_F$ we associate the numbers

$$x_\alpha = \begin{cases} \overline{\lambda((\alpha))}/|\lambda((\alpha))| & \text{if } \lambda((\alpha)) \neq 0, \\ 0 & \text{else.} \end{cases}$$

Finally, we define the unramified test function to be

$$f_{\mathrm{ur}} = \Bigg(\sum_{\alpha\in\mathcal{P}(L)} \frac{x_\alpha\kappa_\alpha}{\sqrt{\mathcal{N}(\alpha)}}\Bigg)\Bigg(\sum_{\alpha\in\mathcal{P}(L)} \frac{x_\alpha\kappa_\alpha}{\sqrt{\mathcal{N}(\alpha)}}\Bigg)^* + \Bigg(\sum_{\alpha\in\mathcal{P}(L)} \frac{x_{\alpha^2}\kappa_{\alpha^2}}{\sqrt{\mathcal{N}(\alpha^2)}}\Bigg)\Bigg(\sum_{\alpha\in\mathcal{P}(L)} \frac{x_{\alpha^2}\kappa_{\alpha^2}}{\sqrt{\mathcal{N}(\alpha^2)}}\Bigg)^*.$$

Here $*$ indicates the adjoint operator. Note that it is essential to include the adjoint operator in order to guarantee that the operator associated to f_{ur} is positive. Indeed, the operator $R(f_{\mathrm{ur}})$ is positive and satisfies

$$R(f_{\mathrm{ur}})w_{\mathrm{ur}}^\circ = \underbrace{\Bigg[\Bigg(\sum_{\alpha\in\mathcal{P}(L)} \frac{|\lambda((\alpha))|}{\sqrt{\mathcal{N}(\alpha)}}\Bigg)^2 + \Bigg(\sum_{\alpha\in\mathcal{P}(L)} \frac{|\lambda((\alpha^2))|}{\sqrt{\mathcal{N}(\alpha^2)}}\Bigg)^2\Bigg]}_{=c_{\mathrm{ur}}>0} w_{\mathrm{ur}}^\circ = c_{\mathrm{ur}}w_{\mathrm{ur}}^\circ.$$

Using [9, Propositions 4.6.4, 4.6.6] together with (4.2) and arguing as in [4, (9.17)] one gets

$$c_{\mathrm{ur}} \gg \frac{L^2}{\mathcal{N}(\mathfrak{q})^2(\log L)^2}. \tag{4.3}$$

On the other hand, we can linearize f_{ur} to obtain

$$f_{\mathrm{ur}} = \sum_{\alpha \in \mathcal{O}_F} y_\alpha \frac{\kappa_\alpha}{\sqrt{\mathcal{N}(\alpha)}}. \tag{4.4}$$

The coefficients y_α are very similar in spirit to the coefficients w_m in [4, (9.16)]. Indeed,

$$y_\alpha = \begin{cases} \sum_{\alpha' \in \mathcal{P}(L)} (|x_{\alpha'}|^2 \omega_{\pi(\alpha')}^{-1}(\varpi_{(\alpha')}) + |x_{\alpha'^2}|^2 \omega_{\pi(\alpha')}^{-1}(\varpi_{(\alpha')}^2)) & \text{if } \alpha = 1, \\ x_{\alpha_1}\overline{x_{\alpha_2}} + \delta_{\alpha_1=\alpha_2}\omega_{\pi(\alpha_1)}^{-1}(\varpi_{(\alpha_1)}) x_{\alpha_1^2}\overline{x_{\alpha_2^2}} & \text{if } \alpha = \alpha_1\alpha_2 \text{ for } \alpha_1, \alpha_2 \in \mathcal{P}(L), \\ x_{\alpha_1^2}\overline{x_{\alpha_2^2}} & \text{if } \alpha = \alpha_1^2\alpha_2^2 \text{ for } \alpha_1, \alpha_2 \in \mathcal{P}(L) \\ 0 & \text{else.} \end{cases}$$

Thus, most importantly we have

$$|y_\alpha| \ll \begin{cases} L & \text{if } \alpha = 1, \\ 1 & \text{if } \alpha = \alpha_1^j \alpha_2^j \text{ for some } j = 1, 2 \text{ and } \alpha_1, \alpha_2 \in \mathcal{P}(L), \\ 0 & \text{else.} \end{cases}$$

One compares this to [19, p. 29] and [4, p. 27] and notes similarity.

Combining everything we define

$$f = \bigotimes_{v} f_v \otimes \bigotimes_{\mathfrak{p}|\mathfrak{q}\mathfrak{n}} f_{\mathfrak{p}} \otimes f_{\mathrm{ur}}.$$

Associated to this function is the integral operator

$$R(f)\colon L^2(G(F)\backslash G(\mathbb{A}_F), \omega_\pi) \to L^2(G(F)\backslash G(\mathbb{A}_F), \omega_\pi),$$
$$\phi \mapsto \left[x \mapsto \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g)\phi(gx)\, dg \right].$$

In particular, we have

$$R(f)\phi' = \sigma^{\mathfrak{L}}\left(\int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g)\pi^{\mathfrak{L}}(g) w^\circ \, dg \right) = c_{\mathrm{ur}} \prod_v c_v(\pi_v) \prod_{\mathfrak{p}|\mathfrak{n}} \delta_{\pi'_{\mathfrak{p}}} \phi'.$$

The corresponding automorphic kernel is given by

$$K_f(g_1, g_2) = \sum_{\gamma \in Z(F)\backslash G(F)} f(g_1^{-1}\gamma g_2).$$

The spectral expansion of K_f will enable us to bound the sup-norm of ϕ' in terms of the geometric definition of K_f. Let us work out the spectral expansion in detail.

We decompose

$$K_f = K_{\mathrm{cusp}} + K_{\mathrm{sp}} + K_{\mathrm{cont}}. \tag{4.5}$$

First, we deal with the cuspidal part.

Lemma 4.3. *For any $g \in G(\mathbb{A}_F)$ we have*

$$0 \leq \frac{L^{2-\varepsilon}}{\mathcal{N}(\mathfrak{q})^2 \mathcal{N}(\mathfrak{n}_1)\mathcal{N}(\mathfrak{m}_1)} |\phi'(g)|^2 \ll K_{\mathrm{cusp}}(g,g),$$

where $\mathfrak{m}_1 = \prod_{\mathfrak{p}|\mathfrak{n}} \mathfrak{p}^{m_{1,\mathfrak{p}}}$.

Proof. We begin by fixing a basis $\mathcal{B}_{\mathrm{cusp}}$ for $L^2_0(X)$ containing ϕ' and consisting of $R(f)$-eigenfunctions. This is possible by a standard multiplicity argument. For $\Psi \in \mathcal{B}_{\mathrm{cusp}}$ let $c(\Psi)$ be the associated $R(f)$-eigenvalue. Then we obtain

$$K_{\mathrm{cusp}}(h,g) = \sum_{\Psi \in \mathcal{B}_{\mathrm{cusp}}} \langle K_{\mathrm{cusp}}(\cdot, g), \Psi\rangle_{L^2(X)} \Psi(h) = \sum_{\Psi \in \mathcal{B}_{\mathrm{cusp}}} \overline{c(\Psi)\Psi(g)}\, \Psi(h).$$

We can choose $\mathcal{B}_{\mathrm{cusp}}$ in such a way that for each Ψ there is a cuspidal automorphic representation (π_Ψ, V_Ψ) and $\Psi = \sigma_\Psi(v)$ for some pure tensor $v \in V_\Psi$. Then we have

$$c(\Psi) = \delta_\Psi c_{\Psi,\mathrm{ur}} \prod_\nu c_\nu(\pi_{\Psi,\nu}).$$

By [20, Corollary 2.16] we have $\delta_\Psi \in \{0, \delta_\pi\}$. In particular, $\delta_\Psi \geq 0$. At the archimedean places positivity of $c_\nu(\pi_{\Psi,\nu})$ is ensured by the definition of k_ν. Finally, also c_{ur} must obviously be positive since $R(f_{\mathrm{ur}})$ is a positive operator. Therefore, $c(\Psi) \geq 0$ for all $\Psi \in \mathcal{B}_{\mathrm{cusp}}$. An explicit lower bound for $c(\phi')$ follows from (4.3), (4.1), and [20, Proposition 2.13]. We then conclude by dropping all unnecessary terms. ∎

The argument for the continuous part is quite similar. We obtain the following result.

Lemma 4.4. *For $g \in G(\mathbb{A}_F)$ one has*

$$K_{\mathrm{cont}}(g,g) \geq 0.$$

Proof. Using the theory of Eisenstein series we have the expansion

$$K_{\mathrm{cont}}(h,g) = \frac{1}{4\pi} \sum_{\Psi_1,\Psi_2 \in \mathcal{B}_{\tilde{\mathbf{H}}}} \int_{-\infty}^{\infty} \langle R(f)\Psi_2(iy), \Psi_1(iy)\rangle_{\tilde{\mathbf{H}}(iy)} E_{\Psi_1}(iy,h)\overline{E_{\Psi_2}(iy,g)}\, dy \tag{4.6}$$

(see [13, (5.21)]). Let us briefly recall the notation. We define the space

$$\tilde{\mathbf{H}}(s) = \Bigg\{\Psi\colon G(\mathbb{A}_F) \to \mathbb{C}\colon \Psi\left[\begin{pmatrix} \alpha a u & x \\ 0 & \beta a v\end{pmatrix} g\right] = \omega_\pi(a) \left|\frac{u}{v}\right|_\infty^{s+1/2} \Psi(g)$$
$$\text{for } \alpha,\beta \in F^\times, a \in \mathbb{A}_F^\times, u,v \in F_\infty^+, \text{ and}$$
$$\int_K \int_{F^\times \backslash F^0(\mathbb{A}_F)} |\Psi(a(y)k)|^2\, dy\, d\mu_K(k) < \infty\Bigg\}.$$

This defines a representation $(\pi_s, \tilde{\mathbf{H}}(s))$ of $G(\mathbb{A}_F)$ where $G(\mathbb{A}_F)$ acts by right translation. For $s \in i\mathbb{R}$ an inner product is given by

$$\langle \Psi_1, \Psi_2 \rangle_{\tilde{\mathbf{H}}(s)} = \int_{\mathbb{A}_F^\times} \int_K \Psi_1(a(y)k)\overline{\Psi_2(a(y)k)}\, d\mu_{\mathbb{A}_F^\times}^\times(y)\, d\mu_K(k).$$

We can also view $\tilde{\mathbf{H}}(s)$ as a trivial holomorphic fiber bundle over $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}(0)$. For $\phi \in \tilde{\mathbf{H}}$ we define $\Psi(s) = \Psi \cdot H(\cdot)^s \in \tilde{\mathbf{H}}(s)$, where

$$H\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k\right) = \left|\frac{a}{b}\right|_{\mathbb{A}_F} \quad \text{for all } k \in K$$

is naturally defined via the Iwasawa decomposition of $G(\mathbb{A}_F)$. Further, to $\Psi \in \tilde{\mathbf{H}}$ we associate the Eisenstein series by

$$E_\Psi(s, g) = \sum_{\gamma \in B(F)\backslash G(F)} [\Psi(s)](\gamma g)$$

for $\Re(s) > 1/2$ and extended to $s \in \mathbb{C}$ by analytic continuation. The sum in (4.6) is taken over an orthonormal basis $\mathcal{B}_{\tilde{\mathbf{H}}}$ for $\tilde{\mathbf{H}}$.

As earlier, it is no problem to choose this basis to consist of $R(F)$-eigenfunctions. For $\Psi \in \mathcal{B}_{\tilde{\mathbf{H}}}$ we denote the corresponding $R(f)$-eigenvalue by $c_\Psi(0)$. Note that then also $\Psi(s)$ is an $R(f)$-eigenfunction but the eigenvalue may depend on s. Thus by putting $h = g$ we obtain

$$K_{\mathrm{cont}}(g, g) = \frac{1}{4\pi} \sum_{\Psi \in \mathcal{B}_{\tilde{\mathbf{H}}}} \int_{-\infty}^{\infty} c_\Psi(iy)|E_\Psi(iy, g)|^2\, dy.$$

We can now argue as before using the construction of f to show that $c_\Psi(s) \geq 0$ for all Ψ. This concludes the proof. ■

Finally, we treat the residual part of the spectrum.

Lemma 4.5. *As long as $\mathfrak{n}_0 \neq \mathcal{O}_F$ we have*

$$K_{\mathrm{sp}}(h, g) = 0$$

for any $g, h \in G(\mathbb{A}_F)$. Otherwise, we still have $K_{\mathrm{sp}}(g, g) \geq 0$ and the only contribution comes from characters $\chi^2 = \omega_\pi$ with $a(\chi_\mathfrak{p}) \leq 1$ at $\mathfrak{p} \mid \mathfrak{q}\mathfrak{n}_1[\mathfrak{n}_1]_{\mathfrak{n}_0}^{-1}$ and $a(\chi_\mathfrak{p}) = 0$ otherwise.

Proof. We start from the spectral expansion of K_{sp}. This reads

$$K_{\mathrm{sp}}(h, g) = \frac{1}{\mathrm{Vol}(Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F))} \times \sum_{\chi^2 = \omega_\pi} \chi(\det(h))\overline{\chi(\det(g))} \int_{Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x)\chi(\det(x)))\, dx.$$

Since the character χ factors and also f is almost a pure tensor, the last integral factors into the product of the local integrals

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} f_{\mathfrak{p}}(g)\chi_{\mathfrak{p}}(\det(g))\,dg \quad \text{if } \mathfrak{p} \mid \mathfrak{n}\mathfrak{q}$$

and the unramified part $I_{\text{ur}}(\chi_{\text{ur}})$. By Lemma A.1 it is clear that $I_{\text{ur}}(\chi_{\text{ur}}) \geq 0$. The lemma follows from the evaluation of the integral $I_{\mathfrak{p}}(\chi_{\mathfrak{p}})$ given in Lemmata A.2 and A.3. ■

By combining the last three lemmata with the definition of K_f one concludes

$$|\phi'|^2 \ll_\varepsilon L^{-2+\varepsilon}\mathcal{N}(\mathfrak{q})^2\mathcal{N}(\mathfrak{n}_1\mathfrak{m}_1) \sum_{\gamma\in Z(F)\backslash G(F)} |f(g^{-1}\gamma g)|. \tag{4.7}$$

This gives an upper bound for ϕ' in terms of the geometry of $G(F)$ and the test function f. We will estimate this further in the next section.

4.2. Estimating the geometric expansion

In this subsection we prove an upper bound for ϕ' which is good in the bulk. This will be done by estimating the right hand side of (4.7).

Proposition 4.6. *Assume that* $(\mathfrak{q},\mathfrak{n}) = 1$ *and* $\mathcal{N}(\mathfrak{q}) \ll (\log \mathcal{N}(\mathfrak{n}))^A$ *for a constant* A. *If*

$$g = a(\theta_i)g'n(x)a(y) \quad \textit{with } g' \in K_{\mathfrak{n}}h_{\mathfrak{n}}^{-1} \textit{ and } n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2},$$

then

$$\begin{aligned}|\phi'(g)|^2 \ll (|T|_\infty\mathcal{N}(\mathfrak{n}))^\varepsilon\mathcal{N}(\mathfrak{q})^{4+\varepsilon}\big(&|T|_\infty^{5/6}\mathcal{N}(\mathfrak{n}_1)^{2/3}\mathcal{N}(\mathfrak{n}_0)^{1/3}\mathcal{N}(\mathfrak{m}_1)\\ &+|T|_{\mathbb{R}}^{1/2}|T|_{\mathbb{C}}\mathcal{N}(\mathfrak{n}_1)^{1/2}\mathcal{N}(\mathfrak{n}_0)^{1/2}\mathcal{N}(\mathfrak{m}_1)\\ &+|T|_\infty^{1/2}\mathcal{N}(\mathfrak{n}_1)\mathcal{N}(\mathfrak{m}_1)|y|_\infty\big).\end{aligned} \tag{4.8}$$

The only thing we will have to do is exploiting the support properties of f and reducing the estimate to the counting problem solved in [4]. Comparing this result to [4, Theorem 1] and [20, Theorem 3.2] shows that the exponents here are indeed as one would expect.

Proof of Proposition 4.6. To save ink we put

$$k(u(\gamma P, P)) = \prod_\nu k_\nu(u_\nu(\gamma_\nu P_\nu, P_\nu)) \quad \text{with } P_\nu = n(x_\nu)a(y_\nu).i_\nu.$$

Inserting the linearization of f_{ur} given in (4.4) into (4.7) yields

$$\begin{aligned}|\phi'(g)|^2 \ll L^{-2+\varepsilon}\mathcal{N}(\mathfrak{n}_1\mathfrak{m}_1) &\sum_{0\neq\alpha\in\mathcal{O}_F} \frac{|y_\alpha|}{\sqrt{\mathcal{N}(\alpha)}}\\ &\cdot \sum_{\gamma\in Z(F)\backslash G(F)} \Big|\kappa_\alpha \prod_{\mathfrak{p}|\mathfrak{q}\mathfrak{n}} f_{\mathfrak{p}}(g'^{-1}a(\theta_i^{-1})\gamma a(\theta_i)g')\Big||k(u(\gamma P, P))|.\end{aligned}$$

Let us analyze the support of $f_{\mathfrak{p}}$ and $\kappa_{(\alpha)}$ place by place. At this point we will also exploit the special structure of g.

First, note that if $\mathfrak{p} \nmid \mathfrak{n}$ we have $g'_{\mathfrak{p}} = 1$. This case consists of two subcases. Namely,

$$a(\theta_i^{-1})\gamma a(\theta_i) \in \begin{cases} Z(F_{\mathfrak{p}})\tilde{K}_{0,\mathfrak{p}}(1) & \text{if } \mathfrak{p} \mid \mathfrak{q}, \\ Z(F_{\mathfrak{p}})K_{\mathfrak{p}}a(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\alpha)})K_{\mathfrak{p}} & \text{else.} \end{cases}$$

If $\mathfrak{p} \mid \mathfrak{n}$, then we use Lemma 2.3 to see that $g'_{\mathfrak{p}} \in \omega K_{\mathfrak{p}}^0(1)$ if $p \mid \mathfrak{n}_2$ and $g'_{\mathfrak{p}} K_{\mathfrak{p}}$ otherwise. Using the support property of $f_{\mathfrak{p}}$ we conclude that

$$a(\theta_i^{-1})\gamma a(\theta_i) \in \begin{cases} Z(F_{\mathfrak{p}})K_{\mathfrak{p}} & \text{if } \mathfrak{p} \nmid \mathfrak{n}_2, \\ Z(F_{\mathfrak{p}})\underbrace{\omega K_{\mathfrak{p}}^0(1)\omega^{-1}}_{=K_{0,\mathfrak{p}}(1)} & \text{if } \mathfrak{p} \mid \mathfrak{n}_2. \end{cases}$$

It is straightforward to choose a suitable representative for $\gamma \in Z(F)\backslash \mathrm{GL}_2(F)$ such that we arrive at the analogue of [4, (9.20)]. In our case this reads

$$|\phi'(g)|^2 \ll_{\varepsilon} \mathcal{N}(\mathfrak{q})^{2+\varepsilon} L^{-2+\varepsilon} \mathcal{N}(\mathfrak{n}_1\mathfrak{m}_1) \sum_{0 \neq \alpha \in \mathcal{O}_F} \frac{|y_\alpha|}{\sqrt{\mathcal{N}(\alpha)}} \sum_{\gamma \in \Gamma(i,\alpha)} |k(u(\gamma P, P))|$$

with

$$\Gamma(i,\alpha) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \colon a,d \in \mathcal{O}_F,\ a-d \in \mathfrak{q},\ b \in \theta_i \mathcal{O}_F,\ c \in \theta_i^{-1}\mathfrak{n}\mathfrak{q},\ ad-bc = \alpha \right\}.$$

Since our coefficients y_α have the same properties as the corresponding w_m in [4], we can replicate the argument from [4, pp. 35–36]. One quickly sees that this argument does not produce any new $\mathfrak{q}$-dependence. We arrive at

$$|\phi'(g)|^2 \ll_{\varepsilon} \mathcal{N}(\mathfrak{q})^{2+\varepsilon} L^{\varepsilon} \mathcal{N}(\mathfrak{n}_1\mathfrak{m}_1) \cdot \sum_{\substack{\underline{k} \in \mathbb{Z}^{\sharp\{\nu\}} \\ T_\nu^{-2} \leq \delta_\nu = 2^{k_\nu} \leq 4}} \frac{|T|_\infty^{1/2}}{|\delta|_\infty^{1/4}} \left(\frac{M(L,0,\delta)}{L} + \frac{M(L,1,\delta)}{L^3} + \frac{M(L,2,\delta)}{L^4} \right) \tag{4.9}$$

for

$$M(L,j,\delta) = \sum_{\alpha_1,\alpha_2 \in \mathcal{P}(L)} \sharp\{\gamma \in \Gamma(i, \alpha_1^j \alpha_2^j) \colon u_\nu(\gamma_\nu P_\nu, P_\nu) \leq \delta_\nu \text{ for all } \nu\}.$$

This is analogous to [4, (9.24)].

The last sum contains $\ll_{\varepsilon} |T|_\infty^{\varepsilon}$ terms, so we can estimate it trivially. Further, we note that $|T_\nu|_\nu^{-2} \leq |\delta_\nu|_\nu \ll 1$. This allows us to use the bounds for $M(L,j,\delta)$ as summarized in [4, p. 50] to estimate[4]

$$|\phi'(g)|^2 \ll (L|T|_\infty \mathcal{N}(\mathfrak{n}))^\varepsilon \mathcal{N}(\mathfrak{q})^{2+\varepsilon} \mathcal{N}(\mathfrak{n}_1 \mathfrak{m}_1) \cdot \left(\frac{|T|_\infty}{L} + |T|_\infty^{1/2} |y|_\infty + \frac{L^2 |T|_\infty}{\mathcal{N}\mathfrak{n}_2} + \frac{|T|_{\mathbb{R}}^{1/2} |T|_{\mathbb{C}}}{(\mathcal{N}\mathfrak{n}_2)^{1/2}} \right).$$

Choosing $L = |T|_\infty^{1/6} (\mathcal{N}\mathfrak{n}_2)^{1/3} \mathcal{N}(\mathfrak{n})^\varepsilon$ and noting that $\mathfrak{n}_2 \mathfrak{n}_0 = \mathfrak{n}_1$ leads to (4.8). Note that we include the factor $\mathcal{N}(\mathfrak{n})^\varepsilon$ in the definition of L to ensure that $\mathcal{N}(\mathfrak{q}) \ll_\varepsilon (\log L)^A$ follows from the assumption $\mathcal{N}(\mathfrak{q}) \ll_\varepsilon (\log \mathcal{N}(\mathfrak{n}))^A$. ∎

As in [4], we can give another estimate for non-totally-real number fields.

Proposition 4.7. *Let $C \leq \mathcal{N}(\mathfrak{q}) \ll (\log \mathcal{N}(\mathfrak{n}))^A$, where C is an explicitly computable constant depending only on the field F. Further, let $F^{\mathbb{R}}$ be the maximal totally real subfield of F and let $m = [F : F^{\mathbb{R}}] \geq 2$. Then*

$$|\phi'(g)|^2 \ll (|T|_\infty \mathcal{N}(\mathfrak{n}))^\varepsilon \mathcal{N}(\mathfrak{q})^{4+\varepsilon} \mathcal{N}(\mathfrak{n}_1 \mathfrak{m}_1) |T|_\infty \cdot \left(|T|_\infty^{\frac{-1}{4m-4}} + (|T|_\infty \mathcal{N}\mathfrak{n}_2)^{\frac{-1}{4m-2}} + \frac{|y|_\infty}{|T|_\infty^{1/2}} \right).$$

Proof. One uses the second list in [4, p. 37] together with (4.9). This yields

$$|\phi'(g)|^2 \ll (|T|_\infty L \mathcal{N}(\mathfrak{n}))^\varepsilon \mathcal{N}(\mathfrak{q})^{4+\varepsilon} \mathcal{N}(\mathfrak{n}_1 \mathfrak{m}_1) \cdot \left(\frac{|T|_\infty}{L} + |T|_\infty^{1/2} \left(|y|_\infty + L^{2m-3} + \frac{L^{2m-2}}{\mathcal{N}(\mathfrak{n}_2)^{1/2}} \right) \right).$$

Using $L = \mathcal{N}(\mathfrak{n})^\varepsilon \min(2^n C_0 |T|_\infty^{\frac{1}{4m-4}}, (\mathcal{N}(\mathfrak{n}_2)|T|_\infty)^{1/4m-2})$ completes the proof.[5] ∎

5. The endgame

In this section we put all the pieces together to prove the theorems stated at the beginning.

5.1. Constructing the ideal $\mathfrak{q}$

The section on amplification depends on the existence of a square-free ideal $\mathfrak{q}$ which eliminates certain technicalities coming from the unit group of F. Here we will show that one can indeed construct $\mathfrak{q}$ with the desired properties.

Lemma 5.1. *There is an absolute constant $A > 0$ depending only on F such that for any $\mathfrak{n}$ there is an ideal $\mathfrak{q}$ satisfying the following two properties.*

[4] As mentioned in [4], the results counting the elements in $M(L, j, \delta)$ totally ignore the conditions depending on $\mathfrak{q}$. Therefore, using these bounds does not generate any new $\mathfrak{q}$-dependence.

[5] The constant $2^n C_0$ ensures that L is not too small.

- $C \leq \mathcal{N}(\mathfrak{q}) \ll (\log \mathcal{N}(\mathfrak{n}))^A$, *where* C *is the absolute constant from Proposition* 4.7.
- *If* x *is a quadratic residue modulo* $\mathfrak{q}$ *then* $x \in (\mathcal{O}_F^\times)^2$.

Proof. For $u \in \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2$ non-trivial, we look at the quadratic extension $F(\sqrt{u}) : F$. Its Galois group is abelian and consists of two elements, say $\mathrm{Gal}(F(\sqrt{u})|F) = \{1, \sigma_u\}$. Since we are dealing with a quadratic extension, we know that a prime $\mathfrak{p}$ of F is inert in $F(\sqrt{u})$ if and only if the Artin symbol satisfies

$$\left(\frac{F(\sqrt{u}) : F}{\mathfrak{p}}\right) = \sigma_u.$$

Thus the Chebotarev set

$$P_{F(\sqrt{u})|F}(\sigma_u) = \left\{\mathfrak{p} \text{ unramified in } F(\sqrt{u}) \colon \left(\frac{F(\sqrt{u}) : F}{\mathfrak{p}}\right) = \sigma_u\right\}$$

contains exactly all the primes of F that are inert in $F(\sqrt{u})$. It is standard that u is not a square modulo $\mathfrak{p}$ for any $\mathfrak{p} \in P_{F(\sqrt{u})|F}(\sigma_u)$. Therefore, we want to define

$$\mathfrak{q} = \prod_{\substack{u \in \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \\ [u] \neq [1]}} \mathfrak{p}_u$$

for suitably chosen $\mathfrak{p}_u \in P_{F(\sqrt{u})|F}(\sigma_u)$. The rest of the proof concerns the choice of $\mathfrak{p}_u$.

To do so we make several definitions. First, we set

$$[\mathfrak{n}]_u = \prod_{\substack{\mathfrak{p}|\mathfrak{n} \\ \mathfrak{p} \in P_{F(\sqrt{u})|F}(\sigma_u)}} \mathfrak{p}.$$

Further, we enumerate $P_{F(\sqrt{u})|F}(\sigma_u) = \{\mathfrak{p}_{u,1}, \mathfrak{p}_{u,2}, \dots\}$ so that $\mathcal{N}(\mathfrak{p}_{u,1}) \leq \mathcal{N}(\mathfrak{p}_{u,2}) \leq \cdots$.

Consider two cases. First, if $\mathfrak{p}_{u,1} \nmid [\mathfrak{n}]_u$ then we take $\mathfrak{p}_u = \mathfrak{p}_{u,1}$. By a version of Linnik's theorem for Chebotarev sets [23] we have

$$\mathcal{N}(\mathfrak{p}_u) \ll_F 1.$$

Second, we consider the worst case

$$[\mathfrak{n}]_u = \mathfrak{p}_{u,1} \cdot \ldots \cdot \mathfrak{p}_{u,k-1}.$$

Here we define $\mathfrak{p}_u = \mathfrak{p}_{u,k}$. It is clear that we only need to show $\mathcal{N}(\mathfrak{p}_u) \ll (\log \mathcal{N}([\mathfrak{n}]_u))^A$. But this follows from elementary calculations using Chebotarev's density theorem [18, Theorem (13.4), Chapter VII].

It is obvious that we can assume $C \leq \mathcal{N}(\mathfrak{q})$. ∎

5.2. *Proof of the main theorems*

In this subsection let $\phi_\circ$ be an L^2-normalized Maaß newform. In other words, it corresponds to a new vector in some cuspidal automorphic representation (π, V_π).

Proof of Theorem 1.1. By Corollary 2.7 it is enough to consider $\phi(g) = \phi_\circ^{\mathfrak{L}}(g)$ for some $\mathfrak{L} \mid \mathfrak{n}$. Further, we fix $1 \leq i \leq h_F$ and restrict ourselves to $g = a(\theta_i)g'h_{\mathfrak{n}}n(x)a(y)$ with $n(x)a(y) \in \mathcal{F}_{\mathfrak{n}_2}$ and $g'h_{\mathfrak{n}} \in \mathcal{J}_{\mathfrak{n}}$.

If $|y_\infty| > |T|^{1/3}\mathcal{N}(\mathfrak{n}_2)^{-1/3}$, then Proposition 3.17 yields

$$|\phi(g)|^2 \ll_{F,\varepsilon} (|T|_\infty \mathcal{N}(\mathfrak{n}))^\varepsilon \big(|T|_\infty^{1/3}\mathcal{N}(\mathfrak{n}_0) + |T|_\infty^{2/3}\mathcal{N}(\mathfrak{n}_2)^{1/3}\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)\big).$$

Finally, we deal with $|y|_\infty \leq |T|_\infty^{1/3}\mathcal{N}(\mathfrak{n}_2)^{-1/3}$. Recall that by Lemma 5.1 there is an ideal $\mathfrak{q}$ which satisfies the conditions needed in order to apply Proposition 4.6. We conclude that

$$|\phi(g)|^2 \ll |T|_\infty^\varepsilon \mathcal{N}(\mathfrak{n})^\varepsilon \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)\big(|T|_\infty^{5/6}\mathcal{N}(\mathfrak{n}_2)^{2/3} + |T|_{\mathbb{R}}^{1/2}|T|_{\mathbb{C}}\mathcal{N}(\mathfrak{n}_2)^{1/2}\big). \qquad \blacksquare$$

Next we consider fields that are not totally real. Therefore, we can find a maximal, totally real subfield $F^{\mathbb{R}}$. Put $m = [F : F^{\mathbb{R}}] \geq 2$.

Proof of Theorem 1.2. We start by choosing $\mathfrak{q}$ suitably and using Corollary 2.7 to reduce the problem as far as possible. Observe that for $|y|_\infty > |T|_\infty^{1/4}$ the estimate in Proposition 3.17 gives the upper bound $\mathcal{N}(\mathfrak{n})^\varepsilon \mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)^{1/2}|T|_\infty^{3/8+\varepsilon}$. Therefore, by using Proposition 4.7, we obtain the uniform bound

$$\frac{\sigma(v^\circ)(g)}{\|\sigma(v^\circ)\|_2} \ll_{F,\varepsilon} (\mathcal{N}(\mathfrak{n}_2)\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)|T|_\infty)^{1/2+\varepsilon}\big(|T|_\infty^{-\frac{1}{8m-8}} + (|T|_\infty\mathcal{N}(\mathfrak{n}_2))^{-\frac{1}{8m-4}}\big).$$

If $|T|_\infty^{-\frac{1}{8m-8}} \geq \mathcal{N}(\mathfrak{n}_2)^{-1/4}$, we can use Theorem 1.1 to get a better bound. This leads to

$$\frac{\sigma(v^\circ)(g)}{\|\sigma(v^\circ)\|_2} \ll_{F,\varepsilon} (\mathcal{N}(\mathfrak{n}_2)\mathcal{N}(\mathfrak{n}_0\mathfrak{m}_1)|T|_\infty)^{1/2+\varepsilon} \cdot \big(\min(|T|_\infty^{-\frac{1}{8m-8}}, \mathcal{N}(\mathfrak{n}_2)^{-1/4}) + (|T|_\infty\mathcal{N}(\mathfrak{n}_2))^{-\frac{1}{8m-4}}\big).$$

One concludes by interpolation as in [4]. $\blacksquare$

Appendix A. Evaluation of some integrals

In this appendix we will evaluate local integrals that appear in the residual part ot the spectral expansion. More precisely, we will calculate the integral

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} f_{\mathfrak{p}}(g)\chi_{\mathfrak{p}}(\det(g))\,dg \qquad \text{(A.1)}$$

for all $f_{\mathfrak{p}}$ defined in Section 4.1.

First, we consider

$$f_{\mathfrak{p}}(g) = \kappa_{\mathfrak{p},k}(g) = \begin{cases} \omega_\pi(z)^{-1} & \text{for } g = z \in Z(F)X_{\mathfrak{p},k}, \\ 0 & \text{else,} \end{cases}$$

for some k.

Lemma A.1. *For $k \geq 0$ we have*

$$\int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} \kappa_{\mathfrak{p},k}(g)\chi_{\mathfrak{p}}(\det(g))\,dg = \begin{cases} \chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^k)\operatorname{Vol}(X_{\mathfrak{p},k}) & \text{if } \chi_{\mathfrak{p}} \text{ is unramified,} \\ 0 & \text{else.} \end{cases}$$

Proof. The calculation for unramified $\chi_{\mathfrak{p}}$ is straightforward, so assume that $\chi_{\mathfrak{p}}$ is ramified. Write $X_{\mathfrak{p},k} = \bigsqcup_i \alpha_i K_{\mathfrak{p}}$. Then we clearly have

$$\int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} \kappa_{\mathfrak{p},k}(g)\chi_{\mathfrak{p}}(\det(g))\,dg = \sum_i \chi_{\mathfrak{p}}(\det(\alpha_i)) \int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} \chi_{\mathfrak{p}}(\det(g))\mathbb{1}_{K_{\mathfrak{p}}}(g)\,dg.$$

We now use our choice of Haar measure and the fact that $\mathbb{1}_{K_{\mathfrak{p}}}(n(x)a(y)k) = \mathbb{1}_{\mathfrak{o}_{\mathfrak{p}}}(x)\mathbb{1}_{\mathfrak{o}_{\mathfrak{p}}^\times}(y)$ to obtain

$$\int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} \kappa_{\mathfrak{p},k}(g)\chi_{\mathfrak{p}}(\det(g))\,dg = \sum_i \chi_{\mathfrak{p}}(\det(\alpha_i)) \int_{\mathfrak{o}_{\mathfrak{p}}} \int_{K_{\mathfrak{p}}} \kappa_{\mathfrak{p},k}(k) \int_{\mathfrak{o}_{\mathfrak{p}}^\times} \chi_{\mathfrak{p}}(y)\,d\mu^\times(y)\,d\mu_{K_{\mathfrak{p}}}(k)\,d\mu(x) = 0.$$

This concludes the proof. ∎

Second, we look at

$$f_{\mathfrak{p}}(g) = \begin{cases} \operatorname{Vol}(Z(\mathfrak{o}_{\mathfrak{p}})\backslash \tilde{K}_{0,\mathfrak{p}}(1))^{-1}\omega_\pi(z)^{-1} & \text{if } g = zk \in Z(F_{\mathfrak{p}})\tilde{K}_{0,\mathfrak{p}}(1), \\ 0 & \text{else.} \end{cases}$$

Lemma A.2. *For a quadratic character $\chi_{\mathfrak{p}}$ and ω_π unramified we have*

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \begin{cases} 1 & \text{if } a(\chi_{\mathfrak{p}}) \leq 1, \\ 0 & \text{else.} \end{cases}$$

Proof. We first observe that for each $g \in \tilde{K}_{0,\mathfrak{p}}(1)$ we have $\det(g) \in (\mathfrak{o}_{\mathfrak{p}}^\times)^2 + \varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}}$. Thus, if $a(\chi_{\mathfrak{p}}) \leq 1$ then $\chi_{\mathfrak{p}}(g) = 1$ for all $g \in \tilde{K}_{0,\mathfrak{p}}(1)$.

Let us now assume $a(\chi_{\mathfrak{p}}) = b > 1$. Since $\chi_{\mathfrak{p}} \circ \det$ is trivial on

$$K_{1,\mathfrak{p}}^1(b) = \begin{pmatrix} 1 + \varpi_{\mathfrak{p}}^b \mathfrak{o}_{\mathfrak{p}} & \mathfrak{o}_{\mathfrak{p}} \\ \varpi_{\mathfrak{p}}^b \mathfrak{o}_{\mathfrak{p}} & 1 + \varpi_{\mathfrak{p}}^b op_{\mathfrak{p}} \end{pmatrix} \cap K_{\mathfrak{p}},$$

we will start by writing down explicit representatives for $\tilde{K}_{0,\mathfrak{p}}(1)/K_{1,\mathfrak{p}}^1(b)$. We obtain

$$\tilde{K}_{0,\mathfrak{p}}(1) = \bigsqcup_{\substack{a \in \mathfrak{o}_{\mathfrak{p}}^\times/(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}}) \\ b,a',d' \in \mathfrak{o}_{\mathfrak{p}}/\varpi_{\mathfrak{p}}^{b-1}\mathfrak{o}_{\mathfrak{p}}}} \begin{pmatrix} a + \varpi a' & 0 \\ \varpi b & a + \varpi d' \end{pmatrix} K_{1,\mathfrak{p}}^1(b).$$

Therefore,

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \sum_{\substack{a\in\mathfrak{o}_{\mathfrak{p}}^{\times}/(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}})\\ b,a',d'\in\mathfrak{o}_{\mathfrak{p}}/\varpi_{\mathfrak{p}}^{b-1}\mathfrak{o}_{\mathfrak{p}}}} \chi_{\mathfrak{p}}(a+\varpi_{\mathfrak{p}}a')\chi_{\mathfrak{p}}(a+\varpi_{\mathfrak{p}}d') = 0. \qquad \blacksquare$$

After this warm-up we come to the most interesting case. We consider the truncated matrix coefficient which served as a test function for $\mathfrak{p} \mid \mathfrak{n}$. Recall that the new vector matrix coefficient of a unitary, generic representation $\pi_{\mathfrak{p}}'$ can be written as

$$\Phi_{\pi_{\mathfrak{p}}'}(g) = \langle W_{\pi_{\mathfrak{p}}'}, \pi_{\mathfrak{p}}'(g)W_{\pi_{\mathfrak{p}}'}\rangle,$$

where $W_{\pi_{\mathfrak{p}}'}$ is the Whittaker new vector. The function we need to look at is then given by

$$f_{\mathfrak{p}}(g) = \Phi_{\pi_{\mathfrak{p}}'}'(g) = \begin{cases} \Phi_{\pi_{\mathfrak{p}}'}(a(\varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}})ga(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})) & \text{if } g\in ZK_{\mathfrak{p}}^0, \\ 0 & \text{else,}\end{cases}$$

with

$$K_{\mathfrak{p}}^0 = \begin{cases} K_{\mathfrak{p}} & \text{if } n \text{ is even,}\\ K_{\mathfrak{p}}^0(1) & \text{if } n \text{ is odd.}\end{cases}$$

We can compute the following.

Lemma A.3. *If* $\chi_{\mathfrak{p}}^2 = \omega_{\pi_{\mathfrak{p}}}$, *then*

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = 0,$$

unless $a(\pi_{\mathfrak{p}}) = 1$, *in which case the integral may be non-zero but still* $I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) \geq 0$.

Using the finite Fourier coefficients $c_{t,l}(\mu)$ for the Whittaker new vector $W_{\pi_{\mathfrak{p}}'}$, defined in [2, (1.6)], we can prove the following nice formula.

Lemma A.4. *We have*

$$\Phi_{\pi_{\mathfrak{p}}'}(n(x)g_{t,l,1}) = \sum_{m\in\mathbb{Z}} W_{\pi_{\mathfrak{p}}'}(a(\varpi_{\mathfrak{p}}^m)) \sum_{\mu\in\mathfrak{X}_l} \overline{c_{t+m,l}(\mu)} G(-\varpi_{\mathfrak{p}}^m x, \omega_{\pi_{\mathfrak{p}}'}\mu).$$

Proof. First we use the definition of Φ_{π}. We arrive at

$$\begin{aligned}\Phi_{\pi_{\mathfrak{p}}'}(n(x)g_{t,l,1}) &= \langle W_{\pi_{\mathfrak{p}}'}, \pi_{\mathfrak{p}}'(n(x)g_{t,l,1})W_{\pi_{\mathfrak{p}}'}\rangle \\ &= \int_{F_{\mathfrak{p}}^{\times}} W_{\pi_{\mathfrak{p}}'}(a(y))\overline{W_{\pi_{\mathfrak{p}}'}(a(y)n(x)g_{t,l,1})}\, d\mu^{\times}(y) \\ &= \sum_{m\in\mathbb{Z}} W_{\pi_{\mathfrak{p}}'}(a(\varpi_{\mathfrak{p}}^m)) \int_{\mathfrak{o}_{\mathfrak{p}}^{\times}} \omega_{\pi_{\mathfrak{p}}'}(v)\overline{W_{\pi_{\mathfrak{p}}'}(a(\varpi_{\mathfrak{p}}^m v)n(x)g_{t,l,1})}\, d\mu^{\times}(v).\end{aligned}$$

It is straightforward to check that

$$a(\varpi_{\mathfrak{p}}^m v)n(x)g_{t,l,1} = n(\varpi_{\mathfrak{p}}^m vx)g_{t+m,l,v^{-1}} \underbrace{\begin{pmatrix}1 & 0\\ 0 & v\end{pmatrix}}_{\in K_{1,\mathfrak{p}}(n_{\mathfrak{p}})}.$$

Inserting this together with [19, (11)] and the definition of the Gauß sum completes the proof. ∎

Proof of Lemma A.3. Put $b = \max(a(\chi_{\mathfrak{p}}), n_{\mathfrak{p}})$. Then $\chi_{\mathfrak{p}} \circ \det$ and $\Phi_{\pi_{\mathfrak{p}}}$ are bi-$K_{1,\mathfrak{p}}(b)$-invariant. Further, we recall

$$\Phi'_{\pi'_{\mathfrak{p}}}(g) = \mathbb{1}_{ZK^{\circ}_{\mathfrak{p}}}(g)\Phi_{\pi'_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}})ga(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})).$$

Thus a simple change of variables yields

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \int_{Z(F_{\mathfrak{p}})\backslash G(F_{\mathfrak{p}})} \chi_{\mathfrak{p}}(\det(g))\Phi_{\pi'_{\mathfrak{p}}}(g)\mathbb{1}_{ZK^{\circ}_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})ga(\varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}}))\, dg.$$

It is easy to check that $\mathbb{1}_{ZK^{\circ}_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}}) \bullet a(\varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}}))$ is bi-$K_{0,\mathfrak{p}}(b)$-invariant. Therefore the whole integrand is bi-$K_{0,\mathfrak{p}}(b)$-invariant, so that we can use [11, Lemma 3.2.4]. This yields

$$\begin{aligned} I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \sum_{l=0}^{b} c_l \sum_{t\in\mathbb{Z}} q_{\mathfrak{p}}^{t+l} \int_{F_{\mathfrak{p}}} &\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{t})\Phi_{\pi'_{\mathfrak{p}}}(n(x)g_{t,l,1}) \\ &\cdot \mathbb{1}_{ZK^{\circ}_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})n(x)g_{t,l,1}a(\varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}}))\, d\mu_{\mathfrak{p}}(x) \end{aligned}$$

for some positive constants c_l. We remark that since $\omega_{\pi'_{\mathfrak{p}}}$ is trivial on the uniformizer, so is $\chi_{\mathfrak{p}}$. Next we will investigate which restrictions on x, l, and t are imposed by the characteristic function (up to the centre). One checks that

$$a(\varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}})n(x)g_{t,l,1}a(\varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}}) = z \cdot \begin{pmatrix} \varpi_{\mathfrak{p}}^{k}x & \varpi_{\mathfrak{p}}^{n_{1,\mathfrak{p}}-l+k}x - \varpi_{\mathfrak{p}}^{t+n_{1,\mathfrak{p}}+k} \\ \varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}+k} & \varpi_{\mathfrak{p}}^{k-l} \end{pmatrix}.$$

Here we use the centre to force all coefficients to be in $\mathfrak{o}_{\mathfrak{p}}$. This holds for

$$k \geq \max(n_{1,\mathfrak{p}}, l, -v_{\mathfrak{p}}(x), -v_{\mathfrak{p}}(x) - n_{1,\mathfrak{p}} + l)$$

and suitable t. But we also need to make sure that the determinant is in $\mathfrak{o}_{\mathfrak{p}}^{\times}$. This implies $t + 2k = 0$.

We now consider $n_{\mathfrak{p}}$ to be even. In this case $K^{\circ}_{\mathfrak{p}} = K_{\mathfrak{p}}$ and we get the conditions

$$k = n_{1,\mathfrak{p}}, \quad t = -2n_{1,\mathfrak{p}}, \quad l \leq n_{1,\mathfrak{p}}, \quad -v_{\mathfrak{p}}(x) \leq n_{1,\mathfrak{p}}. \tag{A.2}$$

After inserting the formula from Lemma A.4 for the matrix coefficient we obtain

$$\begin{aligned} I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \sum_{l=0}^{n_{1,\mathfrak{p}}} c_l q_{\mathfrak{p}}^{l-2n_{1,\mathfrak{p}}} \sum_{m\in\mathbb{Z}} W_{\pi'_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^{m})) \sum_{\mu\in\mathfrak{X}_l} \overline{c_{t+m}(\mu)} \\ \cdot \int_{\varpi_{\mathfrak{p}}^{-n_1}\mathfrak{o}_{\mathfrak{p}}} G(-\varpi_{\mathfrak{p}}^{m}x, \omega_{\pi'_{\mathfrak{p}}}\mu)\, d\mu(x). \end{aligned} \tag{A.3}$$

Inserting the evaluation of the Gauß sum given in [19] together with character orthogonality shows that most of the integrals vanish. We are left with

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = \sum_{l=a(\omega_{\pi'_{\mathfrak{p}}})}^{n_{1,\mathfrak{p}}} c_l q^{l-2n_1} \sum_{m\in\mathbb{Z}} W_{\pi'_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^m))\overline{c_{m-2n_{1,\mathfrak{p}},l}(\omega_{\pi'_{\mathfrak{p}}}^{-1})} \sum_{t\geq 0} q_{\mathfrak{p}}^{-t+n_{1,\mathfrak{p}}} \cdot \int_{\mathfrak{o}_{\mathfrak{p}}^\times} G(-\varpi_{\mathfrak{p}}^{m+t-n_1}x,1)\,d\mu(x).$$

We have to consider different cases. First, we deal with representations that satisfy $L(s,\pi_{\mathfrak{p}}) = 1$. In this case using [19, (6)–(7)] yields

$$\begin{aligned} I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) &= \sum_{l=a(\omega_{\pi'_{\mathfrak{p}}})}^{n_{1,\mathfrak{p}}} c_l q_{\mathfrak{p}}^{l-2n_{1,\mathfrak{p}}} \overline{c_{-2n_1,l}(\omega_{\pi'_{\mathfrak{p}}}^{-1})} \sum_{t\geq 0} q_{\mathfrak{p}}^{n_{1,\mathfrak{p}}-t} \int_{\mathfrak{o}_{\mathfrak{p}}^\times} G(\varpi_{\mathfrak{p}}^{t-n_{1,\mathfrak{p}}}v,1)\,d\mu(v) \\ &= \sum_{l=a(\omega_{\pi'_{\mathfrak{p}}})}^{n_{1,\mathfrak{p}}} c_l q_{\mathfrak{p}}^{l-2n_{1,\mathfrak{p}}} \overline{c_{-2n_1,l}(\omega_{\pi'_{\mathfrak{p}}}^{-1})} \Big[\sum_{t\geq n_{1,\mathfrak{p}}} q_{\mathfrak{p}}^{n_1-t}\zeta_{F_{\mathfrak{p}}}(1) - 1\Big] = 0. \end{aligned}$$

Second, we consider the case $\pi'_{\mathfrak{p}} = \chi_1 \boxplus \chi_2$ with $a(\chi_1) > a(\chi_2) = 0$. In this case we have $a(\omega_{\pi'_{\mathfrak{p}}}) = a(\chi_1) = n_{\mathfrak{p}} > 0$. Recall that we are considering $n_{\mathfrak{p}}$ even. Thus, $a(\omega_{\pi'_{\mathfrak{p}}}) > n_{1,\mathfrak{p}} \geq 1$. We conclude that $I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = 0$ since the l-sum is empty. Let us remark that $\pi'_{\mathfrak{p}} = \chi St$ for χ unramified has conductor 1 and therefore does not need to be considered yet.

We have checked that $I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = 0$ for $n_{\mathfrak{p}}$ even by considering all necessary types of $\pi'_{\mathfrak{p}}$. Now let us turn to $n_{\mathfrak{p}}$ is odd. In this case $K_{\mathfrak{p}}^\circ = K_{\mathfrak{p}}^0(1)$ and additionally to (A.2) the characteristic function forces $v_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{2n_{1,\mathfrak{p}}-l}x - 1) \geq 1$. This implies

$$l = n_{1,\mathfrak{p}} \quad \text{and} \quad x \in \varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}}(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}}).$$

Analogously to (A.3) we get

$$\begin{aligned} I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) &= c_{n_{1,\mathfrak{p}}} q_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}} \sum_{m\in\mathbb{Z}} W_{\pi'_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^m)) \sum_{\mu\in\mathfrak{X}_{n_{1,\mathfrak{p}}}} \overline{c_{-2n_{1,\mathfrak{p}}+m,n_{1,\mathfrak{p}}}(\mu)} \\ &\qquad \cdot \int_{\varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}}(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}})} G(-\varpi_{\mathfrak{p}}^m x, \omega_{\pi'_{\mathfrak{p}}}\mu)\,d\mu(x) \\ &= c_{n_{1,\mathfrak{p}}} \sum_{m\in\mathbb{Z}} W_{\pi_{\mathfrak{p}}}(a(\varpi_{\mathfrak{p}}^m)) \sum_{\substack{\mu\in\mathfrak{X}_{n_1} \\ a(\mu\omega_{\pi'_{\mathfrak{p}}})\leq 1}} \overline{c_{-2n_{1,\mathfrak{p}}+m,n_{1,\mathfrak{p}}}(\mu)} \\ &\qquad \cdot \int_{(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}})} G(-\varpi_{\mathfrak{p}}^{m-n_1}x, \omega_{\pi'_{\mathfrak{p}}}\mu)\,d\mu(x) \end{aligned}$$

In the last step we have again used the Gauß sum evaluation [19, (6)] and orthogonality of characters to remove all μ with $a(\mu\omega_{\pi'_{\mathfrak{p}}}) > 1$.

We have to consider different cases again. First, let us look at $\pi_{\mathfrak{p}}$ with $L(s,\pi_{\mathfrak{p}})=1$. In this case $n_{\mathfrak{p}}>2$, since we assume $n_{\mathfrak{p}}$ is odd. By [19, (7)] we get

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = c_{n_{1,\mathfrak{p}}} \sum_{\substack{\mu\in\mathfrak{X}_{n_1}\\ a(\mu\omega_{\pi'_{\mathfrak{p}}})\leq 1}} \overline{c_{-2n_{1,\mathfrak{p}},n_{1,\mathfrak{p}}}(\mu)} \int_{(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}})} \underbrace{G(-\varpi_{\mathfrak{p}}^{-n_{1,\mathfrak{p}}}x,\omega_{\pi'_{\mathfrak{p}}}\mu)}_{=0}\, d\mu(x) = 0.$$

Second, let $\pi_{\mathfrak{p}}=\chi_1\boxplus\chi_2$ with $a(\chi_1)>a(\chi_2)=0$. If $n_{\mathfrak{p}}=a(\chi_1)>1$, we immediately have $a(\omega_{\pi'_{\mathfrak{p}}}\mu)>n_{1,\mathfrak{p}}$ for all $\mu\in\mathfrak{X}_{n_{1,\mathfrak{p}}}$. Thus, in these cases $I_{\mathfrak{p}}(\chi_{\mathfrak{p}})=0$. So we can assume $1=n_{\mathfrak{p}}=n_{1,\mathfrak{p}}=a(\chi_1)$. Using [19, (6)–(7)] we have the identity

$$\begin{aligned} I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) &= c_1 \operatorname{Vol}(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}},\mu)\\ &\cdot\Big(\sum_{\substack{\mu\in\mathfrak{X}_1\\ \mu\neq\omega_{\pi'_{\mathfrak{p}}}^{-1}}} \overline{c_{-2,1}(\mu)}\zeta_{F_{\mathfrak{p}}}(1)q_{\mathfrak{p}}^{-1/2}\varepsilon(1/2,\omega_{\pi'_{\mathfrak{p}}}^{-1}\mu^{-1})\omega_{\pi'_{\mathfrak{p}}}(-1)\mu(-1)\\ &+\sum_{m\geq 1}\chi_1(\varpi_{\mathfrak{p}}^m)q_{\mathfrak{p}}^{-m/2}\overline{c_{-2+m,1}(\omega_{\pi'_{\mathfrak{p}}}^{-1})} - \zeta_{F_{\mathfrak{p}}}(1)q_{\mathfrak{p}}^{-1}\overline{c_{-2,1}(\omega_{\pi'_{\mathfrak{p}}}^{-1})}\Big). \end{aligned}$$

Here $\varepsilon(1/2,\omega_{\pi_{\mathfrak{p}'}}^{-1}\mu^{-1})$ denotes the standard GL_1 ε-factor; see for example [19, p. 5] for more details. Inserting the expressions for $c_{t,1}(\cdot)$ given in [2, Lemma 2.3] yields

$$\begin{aligned} I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) &= c_1 \operatorname{Vol}(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}},d\mu)\omega_{\pi'_{\mathfrak{p}}}(-1)\\ &\cdot\Big(\sum_{\mu\neq\omega_{\pi'_{\mathfrak{p}}}^{-1}}\zeta_{F_{\mathfrak{p}}}(1)^2q_{\mathfrak{p}}^{-1}+\sum_{m\geq 1}q_{\mathfrak{p}}^{-m}+\zeta_{F_{\mathfrak{p}}}(1)^2q_{\mathfrak{p}}^{-2}\Big)\\ &= c_1 \operatorname{Vol}(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}},d\mu)\omega_{\pi'_{\mathfrak{p}}}(-1)\\ &\cdot\big(\zeta_{F_{\mathfrak{p}}}(1)^2q_{\mathfrak{p}}^{-1}(q_{\mathfrak{p}}-2)+\zeta_{F_{\mathfrak{p}}}(1)q_{\mathfrak{p}}^{-1}+\zeta_{F_{\mathfrak{p}}}(1)^2q_{\mathfrak{p}}^{-2}\big). \end{aligned}$$

Observe $\omega_{\pi'_{\mathfrak{p}}}(-1)=\chi_{\mathfrak{p}}(-1)^2=1$ and deduce that $I_{\mathfrak{p}}(\chi_{\mathfrak{p}})\geq 0$.

This leaves us with the last case $\pi'_{\mathfrak{p}}=\chi St$ for unramified χ. Note that in this case $\omega_\pi=\chi_{\mathfrak{p}}^2=1$ since we have assumed $\omega_{\pi'_{\mathfrak{p}}}(\varpi)=1$. Thus we are dealing with $\pi_{\mathfrak{p}}=St$ and we have $a(\pi'_{\mathfrak{p}})=n_{\mathfrak{p}}=n_{1,\mathfrak{p}}=1$. We obtain

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = c_1\sum_{m\geq 0}q_{\mathfrak{p}}^{-m}\sum_{\mu\in\mathfrak{X}_1}\overline{c_{m-2,1}(\mu)}\int_{1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}}} G(-\varpi_{\mathfrak{p}}^{m-1}x,\mu)\,d\mu(x).$$

Evaluating the Gauß sum reveals

$$\begin{aligned} I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = c_1 \operatorname{Vol}(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}})\Big(&\sum_{a(\mu)=1}\zeta_{F_{\mathfrak{p}}}(1)q_{\mathfrak{p}}^{-1/2}\epsilon(1/2,\mu^{-1})\mu(-1)\overline{c_{-2,1}(\mu)}\\ &+\sum_{m\geq 1}q_{\mathfrak{p}}^{-m}\overline{c_{m-2,1}(1)}-\zeta_{F_{\mathfrak{p}}}(1)q_{\mathfrak{p}}^{-1}\overline{c_{-2,1}(1)}\Big). \end{aligned}$$

Using the evaluation of $c_{t,l}(\cdot)$ given in [2, Lemma 2.1] one obtains

$$I_{\mathfrak{p}}(\chi_{\mathfrak{p}}) = c_1 \operatorname{Vol}(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}})\Big(\sum_{a(\mu)=1} \zeta_{F_{\mathfrak{p}}}(1)^2 q_{\mathfrak{p}}^{-1} + \sum_{m\geq 1} q_{\mathfrak{p}}^{-2m} + \zeta_{F_{\mathfrak{p}}}(1)^2 q_{\mathfrak{p}}^{-2}\Big)$$
$$= c_1 \operatorname{Vol}(1+\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}})\big(\zeta_{F_{\mathfrak{p}}}(1)^2 q_{\mathfrak{p}}^{-1}(q_{\mathfrak{p}}-2) + q_{\mathfrak{p}}^{-2}\zeta_{F_{\mathfrak{p}}}(2) + \zeta_{F_{\mathfrak{p}}}(1)^2 q_{\mathfrak{p}}^{-2}\big) > 0.$$

This was the last case to consider and the proof is complete. ∎

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References

[1] Assing, E.: On sup-norm bounds part II: $GL(2)$ Eisenstein series. Forum Math. **31**, 971–1006 (2019) Zbl 1462.11035 MR 3975671

[2] Assing, E.: On the size of p-adic Whittaker functions. Trans. Amer. Math. Soc. **372**, 5287–5340 (2019) Zbl 1471.11168 MR 4014277

[3] Blomer, V., Brumley, F.: On the Ramanujan conjecture over number fields. Ann. of Math. (2) **174**, 581–605 (2011) Zbl 1322.11039 MR 2811610

[4] Blomer, V., Harcos, G., Maga, P., Milićević, D.: The sup-norm problem for $GL(2)$ over number fields. J. Eur. Math. Soc. **22**, 1–53 (2020) Zbl 1442.11083 MR 4046009

[5] Blomer, V., Harcos, G., Milićević, D.: Bounds for eigenforms on arithmetic hyperbolic 3-manifolds. Duke Math. J. **165**, 625–659 (2016) Zbl 1339.11062 MR 3474814

[6] Borel, A., Jacquet, H.: Automorphic forms and automorphic representations. In: Automorphic forms, representations and L-functions (Corvallis, OR, 1977), Part 1, Proc. Sympos. Pure Math. 33, Amer. Math. Soc., Providence, RI, 189–207 (1979) Zbl 0414.22020 MR 0546598

[7] Brumley, F., Marshall, S.: Lower bounds for Maass forms on semisimple groups. Compos. Math. **156**, 959–1003 (2020) Zbl 1437.11069 MR 4089374

[8] Brumley, F., Templier, N.: Large values of cusp forms on GL_n. Selecta Math. (N.S.) **26**, article no. 63, 70 pp. (2020) Zbl 1444.11097 MR 4150475

[9] Bump, D.: Automorphic forms and representations. Cambridge Stud. Adv. Math. 55, Cambridge University Press, Cambridge (1997) Zbl 0911.11022 MR 1431508

[10] Comtat, F.: Optimal sup norm bounds for newforms on GL_2 with maximally ramified central character. Forum Math. **33**, 1–16 (2021) Zbl 1469.11065 MR 4193473

[11] Corbett, A.: Period integrals and L-functions in the theory of automorphic forms. PhD thesis (2017)

[12] Flath, D.: Decomposition of representations into tensor products. In: Automorphic forms, representations and L-functions (Corvallis, OR, 1977), Part 1, Proc. Sympos. Pure Math. 33, Amer. Math. Soc., Providence, RI, 179–183 (1979) Zbl 0414.22019 MR 0546596

[13] Gelbart, S., Jacquet, H.: Forms of GL(2) from the analytic point of view. In: Automorphic forms, representations and L-functions (Corvallis, OR, 1977), Part 1, Proc. Sympos. Pure Math. 33, Amer. Math. Soc., Providence, RI, 213–251 (1979) Zbl 0409.22013 MR 0546600

[14] Hinz, J. G.: Eine Erweiterung des nullstellenfreien Bereiches der Heckeschen Zetafunktion und Primideale in Idealklassen. Acta Arith. **38**, 209–254 (1980/81) Zbl 0368.12006 MR 0602191

[15] Hoffstein, J., Lockhart, P.: Coefficients of Maass forms and the Siegel zero. Ann. of Math. (2) **140**, 161–181 (1994) Zbl 0814.11032 MR 1289494

[16] Lang, S.: Algebraic number theory. 2nd ed., Grad. Texts in Math. 110, Springer, New York (1994) Zbl 0811.11001 MR 1282723

[17] Marshall, S.: Local bounds for L^p norms of Maass forms in the level aspect. Algebra Number Theory **10**, 803–812 (2016) Zbl 1382.35175 MR 3519096

[18] Neukirch, J.: Algebraic number theory. Grundlehren Math. Wiss. 322, Springer, Berlin (1999) Zbl 0956.11021 MR 1697859

[19] Saha, A.: Large values of newforms on GL(2) with highly ramified central character. Int. Math. Res. Notices **2016**, 4103–4131 Zbl 1404.11055 MR 3544630

[20] Saha, A.: Hybrid sup-norm bounds for Maass newforms of powerful level. Algebra Number Theory **11**, 1009–1045 (2017) Zbl 1432.11044 MR 3671430

[21] Templier, N.: Large values of modular forms. Cambridge J. Math. **2**, 91–116 (2014) Zbl 1307.11062 MR 3272013

[22] Templier, N.: Hybrid sup-norm bounds for Hecke–Maass cusp forms. J. Eur. Math. Soc. **17**, 2069–2082 (2015) Zbl 1376.11030 MR 3372076

[23] Zaman, A.: Bounding the least prime ideal in the Chebotarev density theorem. Funct. Approx. Comment. Math. **57**, 115–142 (2017) Zbl 1427.11123 MR 3704230