On the local structure of the Brill–Noether locus of locally free sheaves on a smooth variety

Donatella Iacono (*) – Elena Martinengo (**)

Abstract – We study the functor Def_E^k of infinitesimal deformations of a locally free sheaf E of $\mathcal{O}_X$-modules on a smooth variety X, such that at least k independent sections lift to the deformed sheaf. We deduce some information on the kth Brill–Noether locus of E, such as the description of the tangent cone at some singular points, of the tangent space at some smooth ones and some links between the smoothness of the functor Def_E^k and the smoothness of some well-known deformation functors and their associated moduli spaces. As a tool for the investigation of Def_E^k, we study infinitesimal deformations of the pairs (E, U), where U is a linear subspace of sections of E. We generalise many classical results concerning the moduli space of coherent systems to the case where E has any rank and X any dimension. This includes a description of its tangent space and the link between smoothness and the injectivity of the Petri map.

1. Introduction

Let X be a smooth projective variety of dimension n and E be a locally free sheaf of $\mathcal{O}_X$-modules on X. We are interested in the deformations of E together with some of its sections.

(*) *Indirizzo dell'A.*: Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via E. Orabona 4, 70125 Bari, Italy; donatella.iacono@uniba.it

(**) *Indirizzo dell'A.*: Dipartimento di Matematica "Giuseppe Peano", Università degli Studi di Torino, via Carlo Alberto 10, 10123 Torino, Italy; elena.martinengo@unito.it

Much is known about the classical case of a line bundle L on a smooth projective curve C. Classical Brill–Noether theory concerns the subvarieties $W_d^k(C)$ of $\mathrm{Pic}^d(C)$ of linear systems on C of degree d and projective dimension at least k or equivalently of line bundles on C of degree d having at least $k+1$ independent sections. Properties of $W_d^k(C)$ like nonemptiness, connectedness, irreducibility and dimension were largely investigated, successfully determined and summarised in [2].

Over the last few years, several generalisations of this problem have been investigated. Much effort has been made to analyse the moduli space of stable vector bundles of fixed rank and degree on a curve, having at least k independent sections. In this context, less is known about dimension, connected components and singular locus; see for instance [14] and references therein for an overview of this case.

To overcome the difficulties of the Brill–Noether theory of vector bundles, the concept of coherent systems was introduced; see [3, 4, 19, 20, 29, 30] and references therein. A coherent system on an algebraic variety is the pair of a vector bundle E of fixed rank n and degree d with a linear subspace U of sections of E of dimension k. There is a notion of stability which allows us to construct the moduli spaces of coherent systems fixing the parameters (n, d, k); see [19, 20]. The relation between these moduli spaces and Brill–Noether theory is obvious: Any vector bundle which occurs as part of a coherent system must have at least a prescribed number of independent sections. Conversely, a vector bundle with a prescribed number of linearly independent sections determines, in a natural way, a coherent system. This defines a forgetful map $(E, U) \to E$ surjecting to the corresponding Brill–Noether locus. This map is an obvious generalisation of the classical projection $G_d^k(C) \to W_d^k(C)$, where $G_d^k(C)$ is the variety that parametrises the linear systems of degree d and projective dimension exactly k on a curve C. In many of the above works, the aim is to deduce as much information as possible on the Brill–Noether loci from the moduli spaces of coherent systems that are easier to describe.

In [16], an infinitesimal study of the moduli space of a coherent system on varieties was carried out, enabling a description of the tangent space and an obstruction space; see also [3, Section 3].

On the other hand, in [28] the authors generalised Brill–Noether theory to line bundles on smooth projective varieties of dimension greater than one. They were able to prove nonemptiness and find the dimension of the Brill–Noether loci of a curve C over a smooth surface X of maximal Albanese dimension, under the hypothesis that a properly defined Brill–Noether number is positive and under some mild additional assumptions.

Finally, the general case of vector bundles on varieties of higher dimension is still quite mysterious. In [8], the authors prove the existence of a Brill–Noether-type stratification of the moduli spaces of stable vector bundles on smooth projective

varieties with fixed Chern classes under the assumption that all the cohomology groups of degree greater than one vanish. Some properties known in the classical Brill–Noether theory for line bundles on curves are expected to hold for the general case of vector bundles on higher-dimensional varieties too. Many of them are known to experts but often there is no reference for them.

In this paper, we are interested in the infinitesimal deformations of a locally free sheaf E of $\mathcal{O}_X$-modules on a smooth variety X that preserve at least a prescribed number of independent sections. As a tool for this analysis we also study infinitesimal deformations of E with a fixed subspace of global sections U. We focus on the local study of the Brill–Noether loci and of the moduli space of local systems on all properties we can predict using deformation theory, and we do not concentrate on their global structure.

It is nowadays almost accepted that the most appropriate way to locally analyse a moduli problem through infinitesimal deformations is via derived algebraic geometry. The philosophy behind this, called Deligne's principle, may be formulated in the following way: every deformation problem on a field of characteristic zero is controlled by a differential graded Lie algebra (dgLa) via the Maurer–Cartan equation and gauge equivalence. A rigorous proof of this philosophy was independently given by Lurie [21] and Pridham [32] via an equivalence of infinity categories between dgLa and the formal moduli problem. The dgLa associated with a certain deformation problem is defined only up to quasi-isomorphism and it encodes much information about the problem. For instance, its first cohomology group coincides with the Zariski tangent space of the moduli space and its second cohomology group is an obstruction space for the problem.

This approach has been successfully applied in many cases such as deformations of locally free sheaves [13], locally free sheaves with prescribed cohomological dimensions [27], coherent sheaves [11], pairs of manifold and coherent sheaves [17].

Inspired by this philosophy and following [26], we find the dgLa controlling infinitesimal deformations of a pair (E, U) as above and are able to recover and generalise some classical results. Later, we deduce from this study and the obvious forgetful maps of functors of deformations, some information about infinitesimal deformations of E such that at least a certain number of independent sections lift. Note that such deformations are basically more difficult to study than the deformations of (E, U), since they do not define a deformation functor and they do not classically fall within the reach of Deligne's principle. The dgLa approach completely describes deformations of the pairs (E, U), for any rank of E and any dimension of X, and provides some information on deformations of E with at least a certain number of independent sections. An approach based on dgl pairs was used in [6] and [5] to analyse the cohomology jump functors. There, the authors extend Deligne's principle to

deformations with cohomology constraints and in [9] this extension is fully proved. The fact that these are not deformation functors is still an obstacle to a full understanding of the problem.

Our main motivation here was to test the power of the derived techniques in this very classical context, where the classical theory has not yet answered all questions, and predict some properties the associated moduli spaces have to satisfy. In particular, using this alternative approach, we are able to show some results, probably expected by the experts. This can be considered a first step to tackle this kind of deformation.

Here, we restrict our attention to a holomorphic locally free sheaf E of $\mathcal{O}_X$-modules on a smooth complex manifold. In [18], we extend these techniques to describe a dgLa that controls deformations of (E, U) on any algebraically closed field of characteristic zero $\mathbb{K}$, using Thom–Whitney constructions. This would offer the possibility to broaden the classical results of Brill–Noether theory over any $\mathbb{K}$. Once we have an explicit description of the dgLa, we aim to apply this powerful approach to investigate formality or abelian homotopy and deduce information about smoothness of the Brill–Noether locus.

The article is organised as follows. For the convenience of the reader, we first collect some basic notions on deformation functors, differential graded Lie algebras and the link between them.

In the second section, we briefly recall the definition of deformations of a locally free sheaf and exhibit the dgLa that controls these deformations following [13].

The third section is finally devoted to the study of deformations of pairs (E, U), where E is a locally free sheaf of $\mathcal{O}_X$-modules on a smooth variety X and U is a linear subspace of its sections. For the basic definitions and the identification of the dgLa that controls these deformations, we follow [26]. Moreover, we are able to generalise some classical results: the condition for a section of a locally free sheaf to be extended to a first-order deformation and the description of the image of the Petri map (Proposition 4.14). We also describe the tangent space to the functor of deformations of (E, U) in the case $U = H^0(E)$ (Corollary 4.15).

In Section 4, we specialise the study of deformations of pairs (E, U) to the case of a smooth curve. In Lemma 5.1, we find two equivalent conditions to the injectivity of the Petri map, which is known to be crucial in the classical study of Brill–Noether loci. In Proposition 5.6, we compute the dimension of the tangent space to the functor of deformations of a pair (E, U) and find equivalent conditions to its smoothness, generalising the classical results concerning the variety $G_d^r(C)$.

Section 5 is devoted to our main aim. Let E be a locally free sheaf E of $\mathcal{O}_X$-modules on a smooth variety X, such that $\dim H^0(X, E) \geq k$. We study infinitesimal deformations of E such that at least k independent sections of E lift. First we define this kind of deformations and observe that the functor Def_E^k associated to them is not

a deformation functor in the sense of Definition 2.2. Theorem 6.3 describes the first-order deformations $\text{Def}_E^k(\mathbb{C}[\varepsilon])$ and the vector space generated by them, suggesting that the locally free sheaves with at least $k + 1$ independent sections are singular points in the moduli space of sheaves with at least k sections. Propositions 6.5 and 6.6 deal with the smoothness of the functor Def_E^k, linking it with other known conditions.

We indicate with $\mathbb{K}$ an algebraic closed field of characteristic zero. We will work often over the field $\mathbb{C}$ of complex numbers. We denote by $\mathbb{K}[\varepsilon]$ the ring of dual numbers, meaning $\varepsilon^2 = 0$.
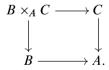
2. Preliminaries on deformation functors

The first part of this section is dedicated to some preliminaries on functors of Artin rings and deformation functors that we will need in the article. In the second part, we introduce the basic definitions of differential graded Lie algebras and the deformation theory associated to them. For a complete presentation of the topics, we refer the reader to [10, 22, 23, 25, 33].

2.1 – *Theory of deformation functors*

DEFINITION 2.1. A *functor of Artin rings* is a covariant functor $F\colon \text{Art}_{\mathbb{K}} \to \text{Set}$, such that $F(\mathbb{K}) = *$, where $*$ is the one point set, Set denotes the category of sets in a fixed universe and $\text{Art}_{\mathbb{K}}$ the category of local Artinian $\mathbb{K}$-algebras with residue fields $\mathbb{K}$.

Consider the following diagram whose objects and arrows are in $\text{Art}_{\mathbb{K}}$:

$$
\begin{array}{ccc}
B \times_A C & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & A.
\end{array}
$$

Applying a functor $F\colon \text{Art}_{\mathbb{K}} \to \text{Set}$, we get a map

$$\eta\colon F(B \times_A C) \to F(B) \times_{F(A)} F(C).$$

DEFINITION 2.2. A functor of Artin rings F is called a *deformation functor* if it satisfies the following conditions:

- η is surjective whenever $B \to A$ is surjective,
- η is an isomorphism whenever $A = \mathbb{K}$.

The functor F is called *homogeneous* if η is an isomorphism, whenever $B \to A$ is surjective.

The name comes from the fact that most functors arising by deforming geometric objects are deformation functors and some of them are actually homogeneous.

REMARK 2.3. Our definition of a deformation functor follows [22]. The first condition here is the classical Schlessinger condition (H1) in [33], while the second is slightly more restrictive than the (H2) of [33]. We assume these conditions because they guarantee good properties for tangent spaces and obstruction theory as stated in Proposition 2.4 and Theorems 2.12 and 2.15. For more details see [10, Example 6.8].

PROPOSITION 2.4. *Let F be a deformation functor. The set $t_F = F(\mathbb{K}[\varepsilon])$ has a natural structure of a $\mathbb{K}$-vector space. If $\varphi \colon F \to G$ is a morphism of deformation functors, the induced map $\varphi \colon t_F \to t_G$ is linear.*

DEFINITION 2.5. Let F be a deformation functor. The vector space $t_F = F(\mathbb{K}[\varepsilon])$ is called the *tangent space* to F.

DEFINITION 2.6. A morphism of functors of Artin rings $\varphi \colon F \to G$ is called *smooth* if for every surjection $B \to A$ in $\mathrm{Art}_\mathbb{K}$, the map

$$F(B) \to G(B) \times_{G(A)} F(A)$$

is also surjective.

A functor of Artin rings F is *smooth* if the morphism $F \to *$ is smooth, i.e. if $F(B) \to F(A)$ is surjective for every surjective morphism $B \to A$ in $\mathrm{Art}_\mathbb{K}$.

REMARK 2.7. Note that if $\varphi \colon F \to G$ is smooth, then the induced map $F(A) \to G(A)$ is surjective for all $A \in \mathrm{Art}_\mathbb{K}$.

PROPOSITION 2.8. *Let $\varphi \colon F \to G$ be a smooth morphism of functors of Artin rings. Then, F is smooth if and only if G is smooth.*

PROOF. Let $B \to A$ be a surjection in $\mathrm{Art}_\mathbb{K}$.

($\Rightarrow$) Consider the commutative diagram

$$
\begin{array}{ccc}
F(B) & \twoheadrightarrow & F(A) \\
\downarrow{\scriptstyle \varphi_B} & & \downarrow{\scriptstyle \varphi_A} \\
G(B) & \longrightarrow & G(A)
\end{array}
$$

in which the vertical arrows are surjective by smoothness of φ and the upper horizontal arrow is surjective by smoothness of F. Thus the lower arrow is surjective too and G is smooth.

($\Longleftarrow$) Let $f_A \in F(A)$. Since G is smooth, there exists an element $g_B \in G(B)$ that lifts $\varphi_A(f_A) \in G(A)$. By smoothness of φ, the element $(f_A, g_B) \in F(A) \times_{G(A)} G(B)$ has a pre-image $f_B \in F(B)$ that ensures the surjectivity of $F(B) \to F(A)$ and then the smoothness of F. $\blacksquare$

We now introduce the notion of obstruction theory that is crucial in the study of deformations.

By a *small extension* in $\mathrm{Art}_{\mathbb{K}}$ we mean an exact sequence

$$e: 0 \to J \to B \xrightarrow{\varphi} A \to 0$$

where $\varphi: B \to A$ is a morphism in $\mathrm{Art}_{\mathbb{K}}$ and J is an ideal of B annihilated by the maximal ideal m_B. In particular, J is a finite-dimensional vector space over $B/m_B = \mathbb{K}$.

DEFINITION 2.9. Let F be a functor of Artin rings. An *obstruction theory* (V, v_e) for F is the data of a k-vector space V, called an *obstruction space*, and for every small extension in $\mathrm{Art}_{\mathbb{K}}$,

$$e: 0 \to J \to B \xrightarrow{\varphi} A \to 0$$

of an obstruction map $v_e: F(A) \to V \otimes_{\mathbb{K}} J$ satisfying the following properties:

- If $a \in F(A)$ can be lifted to $F(B)$ then $v_e(a) = 0$.
- (*Base change*) For every morphism $\alpha: e_1 \to e_2$ of small extensions, i.e. for every commutative diagram

(2.1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \downarrow{\alpha_J} & & \downarrow{\alpha_B} & & \downarrow{\alpha_A} & & \\
0 & \longrightarrow & J_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0,
\end{array}
$$

we have $v_{e_2}(\alpha_A(a)) = (\mathrm{Id}_V \otimes \alpha_J)(v_{e_1}(a))$ for every $a \in F(A_1)$.

An obstruction theory (V, v_e) for F is called *complete* if the converse of the first condition holds, i.e. the lifting exists if and only if the obstruction vanishes.

REMARK 2.10. Note that if F is smooth then all the obstruction maps are trivial. The inverse holds if the obstruction theory is complete.

Clearly if F admits a complete obstruction theory then it admits infinitely many; it is in fact sufficient to embed V in a bigger vector space. One of the main interests is to look for the smallest complete obstruction theory.

DEFINITION 2.11. A *morphism of obstruction* theories $(V, v_e) \to (W, w_e)$ is a linear map $\varphi \colon V \to W$, such that $w_e = \varphi(v_e)$ for every small extension e. An obstruction theory (O_F, ob_e) for F is called *universal* if for every obstruction theory (V, v_e) for F there exists a unique morphism $(O_F, \mathrm{ob}_e) \to (V, v_e)$.

THEOREM 2.12 ([10, Theorem 3.2]). *Let F be a deformation functor. Then there exists a universal obstruction theory (O_F, ob_e) for F. Moreover, the universal obstruction theory is complete and every element of the vector space O_F is of the form $\mathrm{ob}_e(a)$ for some small extension*

$$e \colon 0 \to \mathbb{K} \to B \to A \to 0$$

and some $a \in F(A)$.

In the following, we will also need the notion of relative obstruction theory.

DEFINITION 2.13. Let $\varphi \colon F \to G$ be a morphism of functors of Artin rings and suppose G to be a deformation functor. A *relative obstruction theory* (V, v_e) for φ is the data of

- a $\mathbb{K}$-vector space V, called an *obstruction space*,
- for every small extension in $\mathrm{Art}_{\mathbb{K}}$,

$$e \colon 0 \to J \to B \xrightarrow{\varphi} A \to 0$$

of an obstruction map $v_e \colon F(A) \times_{G(A)} G(B) \to V \otimes_{\mathbb{K}} J$ satisfying the following properties:

(1) If $(a, \beta) \in F(A) \times_{G(A)} G(B)$ can be lifted to $F(B)$ then $v_e(a, \beta) = 0$.

(2) (*Base change*) For every morphism $\alpha \colon e_1 \to e_2$ of small extensions, i.e. for every commutative diagram as (2.1), the following diagram is also commutative:

$$
\begin{array}{ccc}
F(A_1) \times_{G(A_1)} G(B_1) & \xrightarrow{v_{e_1}} & V \times_{\mathbb{K}} J_1 \\
\downarrow{\scriptstyle (\alpha_A, \alpha_B)} & & \downarrow{\scriptstyle \mathrm{Id}_V \otimes \alpha_J} \\
F(A_2) \times_{G(A_2)} G(B_2) & \xrightarrow{v_{e_2}} & V \times_{\mathbb{K}} J_2.
\end{array}
$$

A relative obstruction theory is called *complete* if the converse of the first condition holds, i.e. the lifting exists if and only if the obstruction vanishes.

REMARK 2.14. Note that if $\varphi\colon F \to G$ is smooth then all the relative obstruction maps are trivial. The inverse holds if the relative obstruction theory is complete.

THEOREM 2.15 ([10, Theorem 3.2]). *Let $\varphi\colon F \to G$ be a morphism of deformation functors. Then there exists a unique universal relative obstruction theory for φ.*

THEOREM 2.16 ([22, Proposition 2.17]). *Let $\varphi\colon F \to G$ be a morphism of deformation functors and $\varphi'\colon (V, v_e) \to (W, we)$ be a compatible morphism between obstruction theories. If (V, v_e) is complete, $\varphi'\colon V \to W$ is injective and $t_\varphi\colon t_F \to t_G$ is surjective, then φ is smooth.*

2.2 – *Differential graded Lie algebras and deformation functors*

DEFINITION 2.17. A *differential graded Lie algebra*, briefly a *dgLa*, is the data $(L, d, [\,,\,])$, where $L = \bigoplus_{i\in\mathbb{Z}} L^i$ is a $\mathbb{Z}$-graded vector space over $\mathbb{K}$, $d\colon L^i \to L^{i+1}$ is a linear map, such that $d \circ d = 0$, and $[\,,\,]\colon L^i \times L^j \to L^{i+j}$ is a bilinear map, such that

- $[\,,\,]$ is graded skewsymmetric, i.e. $[a, b] = -(-1)^{\deg a \deg b}[b, a]$,

- $[\,,\,]$ verifies the graded Jacobi identity, i.e.

$$[a, [b, c]] = [[a, b], c] + (-1)^{\deg a \deg b}[b, [a, c]],$$

- $[\,,\,]$ and d verify the graded Leibniz rule, i.e. $d[a, b] = [da, b] + (-1)^{\deg a}[a, db]$,

for every a, b and c homogeneous.

DEFINITION 2.18. Let $(L, d_L, [\,,\,]_L)$ and $(M, d_M, [\,,\,]_M)$ be two dgLas. A *morphism of dgLas $\varphi\colon L \to M$* is a degree-zero linear morphism that commutes with the brackets and the differentials.

A *quasi-isomorphism* of dgLas is a morphism of dgLas that induces an isomorphism in cohomology.

Let L be a differential graded Lie algebra. Then there is a deformation functor $\mathrm{Def}_L\colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$ canonically associated to it, as follows:

DEFINITION 2.19. For all $(A, \mathfrak{m}_A) \in \mathrm{Art}_{\mathbb{K}}$, we define

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(A)}{\sim_{\mathrm{gauge}}},$$

where

$$\mathrm{MC}_L(A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \mid dx + \tfrac{1}{2}[x, x] = 0 \right\}$$

and the gauge action is the action of $\exp(L^0 \otimes \mathfrak{m}_A)$ on $\mathrm{MC}_L(A)$, given by

$$e^a * x = x + \sum_{n=0}^{+\infty} \frac{([a,-])^n}{(n+1)!}([a,x]-da).$$

We recall that the tangent to the deformation functor Def_L is the first cohomology space $H^1(L)$ of the dgLa L. Moreover, a complete obstruction theory for the functor Def_L can be naturally defined and its obstruction space is the second cohomology space $H^2(L)$ of the dgLa L.

If the functor of deformations of a geometrical object $\mathcal{X}$ is isomorphic to the deformation functor associated to a dgLa L, then we say that L controls the deformations of $\mathcal{X}$.

By definition, any morphism $\varphi \colon L \to M$ induces a morphism $\varphi \colon \mathrm{Def}_L \to \mathrm{Def}_M$ that is an isomorphism whenever φ is a quasi-isomorphism.
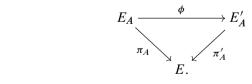
3. Deformation of locally free sheaves

Let X be a smooth projective variety of dimension n and E a locally free sheaf of $\mathcal{O}_X$-modules on X. First of all we recall some notions about the deformations of the locally free sheaf E.

DEFINITION 3.1. Let A be a local Artinian $\mathbb{K}$-algebra with residue field $\mathbb{K}$. An *infinitesimal deformation* of E over A is a locally free sheaf E_A of $\mathcal{O}_X \otimes A$-modules over $X \times \mathrm{Spec}\, A$, with a morphism $\pi_A \colon E_A \to E$ such that the obvious restriction of scalars $\pi_A \colon E_A \otimes_A \mathbb{K} \to E$ is an isomorphism. The deformation will be indicated with (E_A, π_A), or E_A for short.

Two such deformations E_A and E'_A are *isomorphic* if there exists an isomorphism ϕ of sheaves of $\mathcal{O}_X \otimes A$-modules that makes the following diagram commutative:

(3.1)

$$\begin{array}{ccc} E_A & \xrightarrow{\ \ \phi\ \ } & E'_A \\ & \searrow{\scriptstyle \pi_A} \quad \swarrow{\scriptstyle \pi'_A} & \\ & E. & \end{array}$$

The *functor* of infinitesimal deformations of E is

$$\mathrm{Def}_E \colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}.$$

It is classically known that Def_E is a deformation functor. Moreover, its tangent space is $t_{\mathrm{Def}_E} = \mathrm{Def}_E(\mathbb{K}[\varepsilon]) = H^1(X, \mathrm{End}(E))$ and the obstructions are contained in $H^2(X, \mathrm{End}(E))$; see for example [34].

Let E be a locally free sheaf of $\mathcal{O}_X$-modules on X and let End E be the locally free sheaf of its endomorphisms. Over the ground field $\mathbb{C}$, consider

$$A_X^{0,*}(\text{End } E) := \bigoplus_{i=0}^{n} A_X^{0,i}(\text{End } E) := \bigoplus_{i=0}^{n} \Gamma(X, \mathcal{A}_X^{0,i}(\text{End } E)),$$

the graded vector space of global sections of the sheaf of differential forms with values on the sheaf End E. The Dolbeault differential on forms and the bracket defined as the wedge product on forms and the composition of endomorphisms induce a structure of dgLa on it. It is well known that this dgLa is the one that controls the deformations of E.

PROPOSITION 3.2 ([13, Theorem 1.1.1]). *The dgLa $A_X^{0,*}(\text{End } E)$ controls deformations of E. The isomorphism of functors is given, for all $A \in \text{Art}_{\mathbb{C}}$, by*

$$\Psi \colon \text{Def}_{A_X^{0,*}(\text{End } E)}(A) \to \text{Def}_E(A),$$

$$x \to \ker(\bar{\partial} + x).$$

In particular, the tangent space to Def_E is $\text{Def}_E(\mathbb{C}[\varepsilon]) = H^1(X, \text{End } E)$ and the obstructions to deformations are contained in $H^2(X, \text{End } E)$, which fits in the classical picture.

REMARK 3.3. If the ground field is any algebraically closed field of characteristic zero, instead of Dolbeault forms, the associated dgLa is defined via the Thom–Whitney complex associated with the sheaf of endomorphisms End E (see [12]).

4. Deformation of locally free sheaves with a fixed subspace of sections

Let E be a locally free sheaf of O_X-modules on a smooth projective variety X and fix a subspace $U \subseteq H^0(X, E)$. In this section we study infinitesimal deformations of E which preserve the subspace U. We point out that in the literature such a pair (E, U) is called a *local system* of type ($n = \text{rk } E, d = \deg E, k = \dim U$). Deformations of local systems, stability conditions for them and the concerned moduli space are studied in [4, 16, 19, 20].

We start with some definitions and results from [26, 27].

DEFINITION 4.1. Let A be a local Artinian $\mathbb{K}$-algebra with residue field k. An *infinitesimal deformation* of the pair (E, U) over A is the data (E_A, π_A, i_A) of

- a deformation (E_A, π_A) of E over A,
- a morphism $i_A \colon U \otimes A \to H^0(E_A)$,

such that the following diagram commutes:

(4.1)
$$\begin{array}{ccc} U \otimes A & \xrightarrow{\;i_A\;} & H^0(E_A) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_A} \\ U & \xhookrightarrow{\;i\;} & H^0(E). \end{array}$$

Two such deformations (E_A, π_A, i_A), (E'_A, π'_A, i'_A) are *isomorphic* if there exist an isomorphism $\phi\colon E_A \to E'_A$ of sheaves of $\mathcal{O}_X \otimes A$-modules, such that $\pi'_A \circ \phi = \pi_A$ as in diagram (3.1), and an isomorphism $\psi\colon U \otimes A \to U \otimes A$ that make the following diagram commutative:

(4.2)
$$\begin{array}{ccc} U \otimes A & \xrightarrow{\;i_A\;} & H^0(E_A) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} \\ U \otimes A & \xrightarrow{\;i'_A\;} & H^0(E'_A). \end{array}$$

Note that this implies that ϕ induces an isomorphism $\phi\colon i_A(U \otimes A) \to i'_A(U \otimes A)$.

The *functor* of infinitesimal deformations of (E, U) is

$$\mathrm{Def}_{(E,U)}\colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set},$$

which associates with every $A \in \mathrm{Art}_{\mathbb{K}}$ a deformation (E_A, π_A, i_A) as defined above. In the following, we will often shorten the notation of such a deformation to (E_A, i_A).

PROPOSITION 4.2. *The functor* $\mathrm{Def}_{(E,U)}\colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$ *defined above is a deformation functor.*

PROOF. First observe that $\mathrm{Def}_{(E,U)}(\mathbb{K}) = \{(E, i)\}$, where $i\colon U \to H^0(E)$ is the inclusion and so $\mathrm{Def}_{(E,U)}$ is a functor of Artin rings.

To prove it is a deformation functor, we verify the two conditions of Definition 2.2:

- Let $B \to A$ and $C \to A$ be two morphisms of Artin rings; suppose the first one to be surjective. We have to prove that

$$\eta\colon \mathrm{Def}_{(E,U)}(B \times_A C) \to \mathrm{Def}_{(E,U)}(B) \times_{\mathrm{Def}_{(E,U)}(A)} \mathrm{Def}_{(E,U)}(C)$$

is surjective. Let $((E_B, i_B), (E_C, i_C)) \in \mathrm{Def}_{(E,U)}(B) \times_{\mathrm{Def}_{(E,U)}(A)} \mathrm{Def}_{(E,U)}(C)$ and let (E_A, i_A) be the deformation over A to which both reduce. It is classically known (see for example [33, Proposition 3.2] for the line bundle case) that $\widetilde{E} := E_B \times_{E_A} E_C$ is a locally free sheaf of $\mathcal{O}_{B \times_A C}$-modules that deforms E and which reduces to E_B and E_C over B and C, respectively. By hypothesis, $i_B \otimes_B \mathrm{Id}_A =$

$i_C \otimes_C \mathrm{Id}_A = i_A \colon U \otimes A \to H^0(E_A)$, which means that U is a subspace of sections of E that lift to E_B and E_C. Thus there exists $\tilde{\imath} := i_B \times i_C \colon U \otimes (B \times_A C) \to H^0(\tilde{E})$ and $(\tilde{E}, \tilde{\imath}) \in \mathrm{Def}_{(E,U)}(B \times_A C)$ proves the surjectivity of η.

- Now let $A = \mathbb{K}$. We have to prove that η is bijective. The surjectivity is done. Suppose now that $(\hat{E}, \hat{\imath}) \in \mathrm{Def}_{(E,U)}(B \times_\mathbb{K} C)$ is another deformation of (E, U) sent to $((E_B, i_B), (E_C, i_C))$ under η. Since $\hat{E}$ and $\tilde{E}$ both reduce to E_B, E_C, E over B, C and k respectively, it is classically known (see [33, Proposition 3.2]) that they are isomorphic. Now note that $\hat{\imath} \colon U \otimes (B \times_\mathbb{K} C) \to H^0(\hat{E})$ is completely determined by its reductions over B and C, which are respectively $\hat{\imath} \otimes_{B \times_\mathbb{K} C} B = i_B$ and $\hat{\imath} \otimes_{B \times_\mathbb{K} C} C = i_C$. Thus $\hat{\imath}$ and $\tilde{\imath}$ have to coincide. ∎

There is a natural transformation of functors

$$\mathrm{Def}_{(E,U)} \to \mathrm{Def}_E$$

that associates, with every deformation of the pair (E, U) over $A \in \mathrm{Art}_\mathbb{K}$, the deformation of the sheaf E over A forgetting the deformed space of sections.

LEMMA 4.3. *The relative obstruction theory of the natural transformation* $\mathrm{Def}_{(E,U)}$ $\to \mathrm{Def}_E$ *is contained in* $\mathrm{Hom}(U, H^1(X, E))$.

PROOF. Let $0 \to J \to B \to A \to 0$ be a small extension. Let

$$((E_A, i_A), E_B) \in \mathrm{Def}_{(E,U)}(A) \times_{\mathrm{Def}_E(A)} \mathrm{Def}_E(B),$$

thus E_A is a deformation of E over A that lifts to a deformation E_B over B. Consider the exact sequence

$$0 \to E \otimes J \to E_B \to E_A \to 0,$$

which induces the exact sequence in cohomology

$$0 \to H^0(E) \otimes J \to H^0(E_B) \to H^0(E_A) \xrightarrow{\delta} H^1(E) \otimes J \to \cdots.$$

Note that a section $s \in H^0(E_A)$ lifts to a section of E_B if and only if its image under the boundary map δ in $H^1(X, E) \otimes J$ is zero. Thus the obstructions of $\mathrm{Def}_{(E,U)}$ relative to Def_E are contained in $\mathrm{Hom}(U, H^1(X, E)) \otimes J$ and the obstruction theory is complete. ∎

From now on, the base field will be $\mathbb{C}$. Consider the complex of sheaves of differential forms on X with values in the sheaf E with the Dolbeault differential

$$0 \to \mathcal{A}_X^{0,0}(E) \xrightarrow{\bar{\partial}} \mathcal{A}_X^{0,1}(E) \xrightarrow{\bar{\partial}} \mathcal{A}_X^{0,2}(E) \xrightarrow{\bar{\partial}} \cdots$$

and the sheaf $\mathcal{H}om(\mathcal{A}_X^{0,*}(E), \mathcal{A}_X^{0,*}(E))$ of homomorphisms of this complex. Note that the graded vector space of global sections of the sheaf $\mathcal{H}om(\mathcal{A}_X^{0,*}(E), \mathcal{A}_X^{0,*}(E))$ is the same as the graded vector space of homomorphisms of the complex of global sections

$$0 \to A_X^{0,0}(E) \xrightarrow{\bar{\partial}} A_X^{0,1}(E) \xrightarrow{\bar{\partial}} A_X^{0,2}(E) \xrightarrow{\bar{\partial}} \cdots.$$

We denote it as $\text{Hom}^*(A_X^{0,*}(E), A_X^{0,*}(E))$.

As always, when one considers the homomorphism of a complex, one can endow $\text{Hom}^*(A_X^{0,*}(E), A_X^{0,*}(E))$ with an obvious structure of dgLa using as bracket the wedge product on forms and the composition of homomorphism and as differential the bracket with the differential of the complex. The dgLa $\text{Hom}^*(A_X^{0,*}(E), A_X^{0,*}(E))$ controls the deformation of the complex $(A_X^{0,*}(E), \bar{\partial})$, as proved in [24, Section 4].

Note that there exists an inclusion of dgLas

(4.3) $$\phi\colon A_X^{0,*}(\text{End } E) \to \text{Hom}^*(A_X^{0,*}(E), A_X^{0,*}(E)),$$

defined for $\omega \cdot f \in A_X^{0,p}(\text{End } E)$ and $\eta \cdot s \in A_X^{0,q}(E)$ as

$$\phi(\omega \cdot f)(\eta \cdot s) = \omega \wedge \eta \cdot f(s) \in A_X^{0,p+q}(E).$$

It is easy to see that the elements in $A_X^{0,*}(\text{End } E)$ correspond to the morphisms of the complex $A_X^{0,*}(E)$, which are $A_X^{0,*}$-linear. Moreover, the Maurer–Cartan elements of $A_X^{0,*}(\text{End } E)$ which are equivalent to zero in $\text{Hom}^*(A_X^{0,*}(E), A_X^{0,*}(E))$ under the inclusion ϕ correspond to the deformations of E that preserve the dimension of the cohomology spaces $H^i(X, E)$ for every index i, as proved in [24, Lemma 4.1].

Next, consider the complex

$$Q_U\colon 0 \to U \xhookrightarrow{i} A_X^{0,0}(E) \xrightarrow{\bar{\partial}} A_X^{0,1}(E) \xrightarrow{\bar{\partial}} A_X^{0,2}(E) \xrightarrow{\bar{\partial}} \cdots,$$

where U is in degree -1. We define the graded vector space

$$D_U = \left\{ f \in \text{Hom}^*(Q_U, Q_U) \mid f|_{A_X^{0,*}(E)} \in A_X^{0,*}(\text{End } E) \right\}.$$

For any element $f \in D_U^j$, we use the notation $f = (f_{-1}, f_i)$, where $f_{-1}\colon U \to A_X^{0,j-1}(E)$ and $f_i \in A_X^{0,j}(\text{End } E)$. Endowed with the same differential and bracket as $\text{Hom}^*(Q_U, Q_U)$, D_U is a dgLa. In particular, the tangent space to Def_{D_U} is $\text{Def}_{D_U}(\mathbb{C}[\varepsilon]) = H^1(D_U)$ and the obstructions to deformations are contained in $H^2(D_U)$.

Consider the morphism

$$r\colon D_U \to A_X^{0,*}(\text{End } E),$$

which associates with any $f = (f_{-1}, f_i) \in D_U$ the element $f_i \in A_X^{0,*}(\text{End } E)$. By definition, it is a morphism of dgLas and it is clearly surjective. Denoting by $M^* = \ker r = \{f \in D_U \mid f|_{A_X^{0,*}(E)} = 0\}$, we have the short exact sequence of dgLas

$$0 \to M^* \to D_U \to A_X^{0,*}(\text{End } E) \to 0,$$

which induces the following exact sequence in cohomology:

$$0 \to H^0(M^*) \to H^0(D_U) \to H^0(A_X^{0,*}(\text{End } E))$$

$$\to H^1(M^*) \to H^1(D_U) \to H^1(A_X^{0,*}(\text{End } E))$$

(4.4) $$\to H^2(M^*) \to H^2(D_U) \to H^2(A_X^{0,*}(\text{End } E)) \to \cdots.$$

Since $A_X^{0,*}(\text{End } E)$ is the Dolbeault resolution of the sheaf $\text{End } E$, there are isomorphisms $H^j(A_X^{0,*}(\text{End } E)) \cong H^j(X, \text{End } E)$ for all $j \geq 0$. Note that as a dg vector space M^* is isomorphic to $\text{Hom}^*(U, Q_U)$, where U is considered to be a dg-vector space concentrated in degree -1, thus $H^0(M^*) \cong \text{Hom}(U, H^{-1}(Q_U)) = 0$, $H^1(M^*) \cong \text{Hom}(U, H^0(X, E)/U)$ and $H^j(M^*) \cong \text{Hom}(U, H^{j-1}(X, E))$ for $j \geq 2$.

Therefore the long exact sequence (4.4) becomes

$$0 \to H^0(D_U) \to H^0(X, \text{End } E)$$

$$\to \text{Hom}(U, H^0(X, E)/U) \to H^1(D_U) \to H^1(X, \text{End } E)$$

(4.5) $$\xrightarrow{\alpha} \text{Hom}(U, H^1(X, E)) \xrightarrow{\beta} H^2(D_U) \xrightarrow{\gamma} H^2(X, \text{End } E) \to \cdots,$$

where the map α is the restriction to U of the morphism induced in cohomology by the inclusion ϕ defined in (4.3).

Note that the dgLa morphism $r \colon D_U \to A_X^{0,*}(\text{End } E)$ induces a natural transformation of functors

$$\text{Def}_{D_U} \to \text{Def}_{A_X^{0,*}(\text{End } E)}.$$

LEMMA 4.4. *A complete relative obstruction theory of the natural transformation* $\text{Def}_{D_U} \to \text{Def}_{A_X^{0,*}(\text{End } E)}$ *is contained in* $\text{Hom}(U, H^1(X, E))$.

PROOF. Let $0 \to J \to B \to A \to 0$ be a small extension. Let

$$x = ((x_{-1}, x_i), \tilde{x}_i) \in \text{Def}_{D_U}(A) \times_{\text{Def}_{A_X^{0,*}(\text{End } E)}(A)} \text{Def}_{A_X^{0,*}(\text{End } E)}(B),$$

thus $x_i \in \text{MC}_{A_X^{0,*}(\text{End } E)}(A)$ lifts to $\tilde{x}_i \in \text{MC}_{A_X^{0,*}(\text{End } E)}(B)$.

Choose a lifting $\tilde{x}_{-1} \in \text{Hom}(U, A_X^{0,0}(E)) \otimes B$ of x_{-1} and define $\tilde{x} = (\tilde{x}_{-1}, \tilde{x}_i)$. The relative obstruction of x is the class of $\text{ob}(x) = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in H^2(D_U) \otimes J$. Tensoring the sequence in (4.5) with J, we get the exact sequence

$$\cdots \to \text{Hom}(U, H^1(X, E)) \otimes J \to H^2(D_U) \otimes J \xrightarrow{\gamma} H^2(X, \text{End } E) \otimes J \to \cdots.$$

Since the element $\text{ob}(x)$ goes to zero under the map γ, the relative obstruction $\text{ob}(x)$ is contained in $\text{Hom}(U, H^1(X, E)) \otimes J$.

The defined obstruction is complete. Indeed, if there exists a lifting $\tilde{x} \in \text{Def}_{D_U}(B)$ of x, it satisfies the Maurer–Cartan equation, thus $\text{ob}(x) = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] = 0$. ∎

The following proposition is one of the main results of this section.

PROPOSITION 4.5 ([26, Corollary 4.1.14]). *The dgLa D_U controls deformations of the pair (E, U). The isomorphism of functors is given, for all $A \in \text{Art}_{\mathbb{C}}$, by*

$$\Phi: \text{Def}_{D_U}(A) \to \text{Def}_{(E,U)}(A),$$
$$x \to (\ker(\bar{\partial} + x_0), \text{Id} + x_{-1}).$$

PROOF. For completeness and clearness we write out the proof here. We leave to the reader the classically known calculations for the isomorphism of the functors of Proposition 3.2. We divide the proof into two steps.

First step: The natural transformation of functors Φ is well defined. Let $x = (x_{-1}, x_i) \in D_U^1 \otimes m_A$ be a Maurer–Cartan element and prove that it defines a deformation of the pair (E, U). It is a classical fact that $E_A := \ker(\bar{\partial} + x_0)$ with the map $\pi_A := \text{Id} \otimes \pi$ defines a locally free sheaf that is a deformation of the sheaf E (Proposition 3.2). The map $i_A := \text{Id} + x_{-1}$ fits in diagram (4.1), in particular $i_A(U \otimes A) \subset H^0(X, E_A)$. Indeed,

$$(\bar{\partial} + x_0) \circ (\text{Id} + x_{-1})|_{U \otimes A} = \bar{\partial} \circ \text{Id} + \bar{\partial} \circ x_{-1} + x_0 \circ \text{Id} + x_0 \circ x_{-1}$$
$$= 0 + (\bar{\partial} \circ x_{-1} + x_0 \circ \text{Id}) + x_0 \circ x_{-1}$$
$$= (dx)_{-1} + \frac{1}{2}[x, x]_{-1} = 0,$$

since $U \subset H^0(X, E)$ and $x \in \text{MC}_{D_U}(A)$. Then the maps i_A and π_A make diagram (4.1) commutative. Indeed, since $x_{-1} \in D_U \otimes m_A$,

$$\pi_A \circ i_A|_{U \otimes A} = (\text{Id} \otimes \pi) \circ (\text{Id} + x_{-1})|_{U \otimes A}$$
$$= (\text{Id} \otimes \pi)|_{U \otimes A} + (\text{Id} \otimes \pi) \circ x_{-1} = \pi + 0 = i \circ \pi.$$

Moreover, the morphism above is well defined on deformation functors. Let $x, y \in \text{MC}_{D_U}(A)$ be two gauge-equivalent elements via $z \in D_U^0 \otimes m_A$, i.e. $e^z * x = y$. For

$i \geq 0$, the elements $e^{z_i} \colon A_X^{0,i}(E) \otimes A \to A_X^{0,i}(E) \otimes A$ define an isomorphism of degree zero and, as classically known, the gauge relation is equivalent to the commutativity

$$\bar{\partial} + y_0 = e^{[z_i, -]}(\bar{\partial} + x_0) = e^{z_{i+1}} \circ (\bar{\partial} + x_0) \circ e^{-z_i}.$$

Thus $\phi := e^{-z_0}$ defines an isomorphism between the deformed sheaves $\ker(\bar{\partial} + x_0)$ and $\ker(\bar{\partial} + y_0)$.

Similarly, the element $\psi := e^{z_{-1}} \colon U \otimes A \to U \otimes A$ defines an isomorphism and the gauge relation is equivalent to the commutativity of the diagram (4.2). Indeed,

$$
\begin{aligned}
y_{-1} = e^z * x_{-1} &= x_{-1} + \sum_{n=0}^{+\infty} \frac{([z, -])^n}{(n+1)!} ([z, x]_{-1} - (dz)_{-1}) \\
&= x_{-1} + \sum_{n=0}^{+\infty} \frac{([z, -])^n}{(n+1)!} ([z, x]_{-1} + [z, \mathrm{Id}]_{-1}) \\
&= x_{-1} + \sum_{n=1}^{+\infty} \frac{([z, -])^n}{n!} (\mathrm{Id} + x_{-1}) \\
&= \sum_{n=0}^{+\infty} \frac{([z, -])^n}{n!} (\mathrm{Id} + x_{-1}) - \mathrm{Id} = e^{[z, -]}(\mathrm{Id} + x_{-1}) - \mathrm{Id},
\end{aligned}
$$

(4.6)

where we use $(dz)_{-1} = i \circ z_{-1} - z_0 \circ i = -[z, \mathrm{Id}]_{-1}$. Thus

$$\mathrm{Id} + y_{-1} = e^{[z, -]}(\mathrm{Id} + x_{-1}) = e^{z_0} \circ (\mathrm{Id} + x_{-1}) \circ e^{-z_{-1}},$$

as we wanted.

Second step: Φ is an isomorphism of functors. First, we consider the injectivity of $\Phi(A)$ for every $A \in \mathrm{Art}_{\mathbb{C}}$. Suppose that $x = (x_{-1}, x_i)$ and $y = (y_{-1}, y_i) \in \mathrm{MC}_{D_U}(A)$ induce isomorphic deformations $(\ker(\bar{\partial} + x_0), \mathrm{Id} + x_{-1})$ and $(\ker(\bar{\partial} + y_0), \mathrm{Id} + y_{-1})$ via the isomorphisms (ϕ, ψ), as in Definition 4.1.

It is classical to lift ϕ to an isomorphism of the form e^z, with $z \in A^{0,0}(\mathrm{End}\, E) \otimes m_A$ and to get the following commutative diagram:

(4.7)

$$
\begin{array}{ccccccc}
0 & \longrightarrow & U \otimes A & \xrightarrow{\mathrm{Id} + x_{-1}} & \ker(\bar{\partial} + x_0) & \xrightarrow{\;\;i\;\;} & A_X^{(0,0)}(E) \otimes A & \xrightarrow{\bar{\partial} + x_0} & \cdots \\
& & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi = e^z} & & \downarrow{\scriptstyle e^z} & & \\
0 & \longrightarrow & U \otimes A & \xrightarrow{\mathrm{Id} + y_{-1}} & \ker(\bar{\partial} + y_0) & \xrightarrow{\;\;i\;\;} & A_X^{(0,0)}(E) \otimes A & \xrightarrow{\bar{\partial} + y_0} & \cdots.
\end{array}
$$

The isomorphism ψ is of the form e^w, with $w \in \mathrm{Hom}(U, U) \otimes m_A$ too, because it is the identity on the residue field. Thus there exists an element $t = (w, z) \in D_U^0 \otimes m_A$,

such that e^t is an isomorphism that makes diagram (4.7) commutative. It is an easy calculation, similar to (4.6), to see that the commutativity is equivalent to the gauge relation $y = e^t * x$.

Moreover, by the following Proposition 4.6, the morphism of functors Φ is smooth, thus $\Phi(A)$ is surjective, for all $A \in \mathrm{Art}_{\mathbb{C}}$.   ∎

PROPOSITION 4.6. *The morphism of functors* $\Phi \colon \mathrm{Def}_{D_U} \to \mathrm{Def}_{(E,U)}$ *defined in Proposition* 4.5 *is smooth.*

PROOF. Let $0 \to J \to B \to A \to 0$ be a small extension. Let $x = (x_{-1}, x_i) \in \mathrm{Def}_{D_U}(A)$ and $\Phi(x) = (E_A, i_A) \in \mathrm{Def}_{(E,U)}(A)$. The smoothness of Φ is equivalent to saying that x lifts to an element $\tilde{x} \in \mathrm{Def}_{D_U}(B)$ if and only if (E_A, i_A) lifts to a pair $(E_B, i_B) \in \mathrm{Def}_{(E,U)}(B)$. One direction is obvious.

For the other one, we recall that the morphism of functors $\Psi \colon \mathrm{Def}_{A^{0,*}(\mathrm{End}\,E)} \to \mathrm{Def}_E$, defined in Proposition 3.2, is smooth. Thus it is enough to show that the relative obstruction theories of Lemmas 4.3 and 4.4 are isomorphic via the correspondence between the Dolbeault and Čech cohomologies.

As in Lemma 4.4, let

$$x = ((x_{-1}, x_i), \tilde{x}_i) \in \mathrm{Def}_{D_U}(A) \times_{\mathrm{Def}_{A^{0,*}_X(\mathrm{End}\,E)}(A)} \mathrm{Def}_{A^{0,*}(\mathrm{End}\,E)}(B)$$

and let $\mathrm{ob}(x) \in \mathrm{Hom}(U, H^1(X, E)) \otimes J$ be its obstruction. Observe that here $H^1(X, E)$ is the Dolbeault cohomology group and let us find the element in Čech cohomology that corresponds to $\mathrm{ob}(x)$. For every $s \in U \otimes A$, $\mathrm{ob}(x)(s) \in H^1(X, E) \otimes J$. This class is represented by a closed element in $A^{0,*}_X(E) \otimes J$, denoted again by $\mathrm{ob}(x)(s)$, which is then locally exact. Therefore, there exist an open cover $\mathcal{W} = \{W_i\}$ of X and $\tau_i(s) \in A^{0,0}_{W_i}(E) \otimes J$, such that $\bar{\partial}\tau_i(s) = \mathrm{ob}(x)(s)|_{W_i}$. Define on $W_i \cap W_j$ the elements $\sigma_{ij}(s) = \tau_i(s) - \tau_j(s)$; they are Čech cocycles and their class $[\{\sigma_{ij}(s)\}_{ij}] \in H^1(X, E) \otimes J$ defines the corresponding element $\mathrm{ob}(x)(s)$ in Čech cohomology.

As in Lemma 4.3, let

$$((E_A, i_A), E_B) \in \mathrm{Def}_{(E,U)}(A) \times_{\mathrm{Def}_E(A)} \mathrm{Def}_E(B),$$

where $\Phi(x) = (E_A, i_A)$ and $E_B = \Psi(\tilde{x}_i)$. For every $s \in U \otimes A$, the obstruction to lift $i_A(s) \in H^0(E_A)$ to a section of E_B lives in $H^1(X, E) \otimes J$ and is given by $\delta(i_A)(s)$, where δ is the coboundary map

$$\cdots \to H^0(E_B) \to H^0(E_A) \xrightarrow{\delta} H^1(E) \otimes J \to \cdots .$$

Recall that the construction of the coboundary map is obtained by chasing the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \check{C}^0(\mathcal{W}, E) \otimes J & \longrightarrow & \check{C}^0(\mathcal{W}, E_B) & \longrightarrow & \check{C}^0(\mathcal{W}, E_A) & \longrightarrow & 0 \\
& & \downarrow{\check{\delta}} & & \downarrow{\check{\delta}} & & \downarrow{\check{\delta}} & & \\
0 & \longrightarrow & \check{C}^1(\mathcal{W}, E) \otimes J & \longrightarrow & \check{C}^1(\mathcal{W}, E_B) & \longrightarrow & \check{C}^1(\mathcal{W}, E_A) & \longrightarrow & 0.
\end{array}
$$

The element $(i_A)(s) = \{i_A(s)|_{W_i}\}_i \in H^0(E_A)$ can be lifted to an element $i_A(s)|_{W_i} - \tau_i(s) \in \check{C}^0(\mathcal{W}, E_B)$. Applying the Čech differential to it, we get $\check{\delta}(i_A(s)|_{W_i} - \tau_i(s)) = \{i_A(s)|_{W_i} - \tau_i(s) - i_A(s)|_{W_j} + \tau_j(s)\}_{ij} = \{\tau_i(s) - \tau_j(s)\}_{ij} = \{\sigma_{ij}(s)\}_{ij}$. As we state, the two obstructions coincide. ∎

As a direct consequence of Proposition 4.5, we get the following result, already obtained in [16, Théoreme 3.12]. See also [4, Proposition 3.4] for the curve case.

COROLLARY 4.7. *The tangent space to* $\mathrm{Def}_{(E,U)}$ *is* $H^1(D_U)$ *and all obstructions are contained in* $H^2(D_U)$.

REMARK 4.8. If the ground field is any algebraically closed field of characteristic zero, in the same spirit as for deformations of the sheaf E (see Remark 3.3), we expect to define a dgLa that controls deformations of (E, U) using the Thom–Whitney complex associated with the sheaf of homomorphism of a suitable complex of sheaves [18].

In the following, we briefly focus on smoothness of the forgetful morphism $r: \mathrm{Def}_{(E,U)} \to \mathrm{Def}_E$. The following corollary is a direct consequence of Lemmas 4.3 and 4.4. Otherwise, it can be obtained by applying Theorem 2.16 to the exact sequence (4.5).

COROLLARY 4.9. *If* $\mathrm{Hom}(U, H^1(E)) = 0$, *the forgetful morphism of functors* $r: \mathrm{Def}_{(E,U)} \to \mathrm{Def}_E$ *is smooth.*

REMARK 4.10. By Proposition 2.8, the smoothness of the forgetful morphism $r: \mathrm{Def}_{(E,U)} \to \mathrm{Def}_E$ implies equivalence between the smoothness of the two functors Def_E and $\mathrm{Def}_{(E,U)}$.

COROLLARY 4.11. *If the map* $\alpha: H^1(X, \mathrm{End}\, E) \to \mathrm{Hom}(U, H^1(X, E))$ *that appears in* (4.5) *is surjective, then the forgetful morphism* $r: \mathrm{Def}_{(E,U)} \to \mathrm{Def}_E$ *is smooth.*

Proof. Let $0 \to J \to B \to A \to 0$ be a small extension in $\mathrm{Art}_\mathbb{C}$ and consider

$$x = ((x_{-1}, x_i), \tilde{x}_i) \in \mathrm{Def}_{D_U}(A) \times_{\mathrm{Def}_{A_X^{0,*}(\mathrm{End}\,E)}(A)} \mathrm{Def}_{A_X^{0,*}(\mathrm{End}\,E)}(B).$$

Since x_i lifts to $\tilde{x}_i$, from the diagram of obstruction theories

$$
\begin{array}{ccc}
\mathrm{Def}_{D_U}(A) & \xrightarrow{\;\;\mathrm{ob}\;\;} & H^2(D_U) \otimes J \\
\downarrow & & \downarrow{\scriptstyle \gamma} \\
\mathrm{Def}_{A_X^{0,*}(\mathrm{End}\,E)}(A) & \xrightarrow{\;\;\mathrm{ob}\;\;} & H^2(\mathrm{End}\,E) \otimes J,
\end{array}
$$

we get that the relative obstruction to lift x to an element in $\mathrm{Def}_{(E,U)}(B)$ is contained in $\ker \gamma$. This kernel is trivial: indeed, looking at (4.5), the surjectivity of the map $\alpha \colon H^1(X, \mathrm{End}\,E) \to \mathrm{Hom}(U, H^1(X, E))$ implies that the morphism $\gamma \colon H^2(D_U) \to H^2(\mathrm{End}\,E)$ is injective. ∎

Remark 4.12. The condition $\mathrm{Hom}(U, H^1(E)) = 0$ is equivalent to the surjectivity of the map $\alpha \colon H^1(X, \mathrm{End}\,E) \to \mathrm{Hom}(U, H^1(X, E))$. Indeed, by Corollary 4.11, if α is surjective, then r is smooth and also the map $H^1(D_U) \to H^1(X, \mathrm{End}\,E)$ on the tangent spaces of the functors is surjective. By the exact sequence (4.5), the map α is actually the zero map and so $\mathrm{Hom}(U, H^1(X, E)) = 0$.

The other implication is obvious.

Corollary 4.13. *In the notation above, we have*

$$\dim t_{\mathrm{Def}_{(E,U)}} \geq \dim t_{\mathrm{Def}_E} - k \cdot \dim H^1(X, E),$$

where k is the dimension of $U \subseteq H^0(X, E)$.

Proof. By the long exact sequence (4.5),

$$
\begin{aligned}
\cdots \to \mathrm{Hom}(U, H^0(X, E)/U) &\to H^1(D_U) \xrightarrow{\beta} H^1(X, \mathrm{End}\,E) \\
&\xrightarrow{\alpha} \mathrm{Hom}(U, H^1(X, E)) \to \cdots,
\end{aligned}
$$

we have

$$
\begin{aligned}
\dim t_{\mathrm{Def}_{(E,U)}} = \dim H^1(D_U) &\geq \dim \mathrm{Im}\,\beta = \dim \ker \alpha \\
&= \dim H^1(X, \mathrm{End}\,E) - \dim \mathrm{Im}\,\alpha \\
&\geq \dim H^1(X, \mathrm{End}\,E) - \dim \mathrm{Hom}(U, H^1(X, E)) \\
&= \dim t_{\mathrm{Def}_E} - k \cdot \dim H^1(X, E). \qquad \blacksquare
\end{aligned}
$$

Using our description of deformations via dgLas, we can generalise a classical result. Fix a section $s \in H^0(X, E)$; the morphism ϕ of (4.3) induces in cohomology the cup product

$$- \cup s \colon H^1(X, \mathrm{End}\, E) \to H^1(X, E),$$

where $a \cup s = \alpha(a)(s)$ for every $a \in H^1(X, \mathrm{End}\, E)$.

PROPOSITION 4.14. *Let E be a locally free sheaf over a projective variety X. A section $s \in H^0(X, E)$ can be extended to a section of a first-order deformation of E associated to an element $a \in H^1(X, \mathrm{End}\, E)$ if and only if $a \cup s = 0 \in H^1(X, E)$.*

PROOF. Let $s \in H^0(X, E)$ be a section and define $U = \langle s \rangle$. Recalling our descriptions via dgLas of the first-order deformations given after Proposition 3.2 and in Corollary 4.7, we can rewrite the exact sequence (4.5) as

$$\cdots H^1(D_U) = \mathrm{Def}_{(E,U)}(\mathbb{C}[\varepsilon]) \xrightarrow{r} H^1(X, \mathrm{End}\, E)$$
$$= \mathrm{Def}_E(\mathbb{C}[\varepsilon]) \xrightarrow{\alpha} \mathrm{Hom}(U, H^1(X, E)) \cdots.$$

The section s can be extended to a deformation associated to $a \in H^1(X, \mathrm{End}\, E) = \mathrm{Def}_E(\mathbb{C}[\varepsilon])$ if and only if $a \in \mathrm{Im}\, r$. Since $\mathrm{Im}\, r = \ker \alpha$ we have the required description. ■

The same result is classically known for line bundles over a curve (see [2, Lemma, p. 186]) and for line bundles over a projective variety (see [34, Proposition 3.3.4]).

This result can be reinterpreted in terms of some special maps and it can be seen as a generalisation of [2, Proposition 4.2 (i)], [14, Section 2] and [28, Section 4.3]. In the spirit of [28, Section 4.3], we define a generalisation of the Petri map – we will properly introduce it in the next section – as the map induced by the cup product:

$$\mu_0 \colon H^0(X, E) \otimes H^0(X, K_X \otimes E^*) \to H^0(X, K_C \otimes E \otimes E^*),$$

where K_X is the canonical bundle of X, E^* is the dual bundle of E and the map

$$\alpha_n \colon H^1(X, \mathrm{End}\, E) \otimes H^{n-1}(\mathcal{O}_X) \to H^n(\mathrm{End}\, E)$$

is given by the cup product. Proposition 4.14 can be stated as saying that for all $\sigma \in H^1(X, \mathrm{End}\, E) \otimes H^{n-1}(\mathcal{O}_X)$ and for all $\psi \in H^0(X, E) \otimes H^0(X, K_X \otimes E^*)$ the following cup product vanishes:

$$\alpha_n(\sigma) \cup \mu_0(\psi) = 0,$$

or equivalently that $\alpha_n(H^1(X, \mathrm{End}\, E) \otimes H^{n-1}(\mathcal{O}_X)) \subset H^n(\mathcal{O}_X)$ is orthogonal to $\mathrm{Im}\, \mu_0 \subset H^0(X, K_C \otimes E \otimes E^*)$.

In the particular case of deformations of pairs $(E, H^0(E))$, the exact sequence (4.5) splits:

$$0 \to H^1(D_U) \to H^1(X, \operatorname{End} E) \xrightarrow{\alpha} \operatorname{Hom}(H^0(X, E), H^1(X, E)) \to \cdots.$$

Thus the tangent space $t_{\operatorname{Def}_{(E, H^0(E))}} = H^1(D_U)$ can be identified with the kernel of the morphism $\alpha \colon H^1(X, \operatorname{End} E) \to \operatorname{Hom}(H^0(E), H^1(X, E))$.

COROLLARY 4.15. *The tangent space to the deformations of the pair $(E, H^0(E))$ can be identified with*

$$t_{\operatorname{Def}_{(E, H^0(E))}} = \left\{ a \in H^1(X, \operatorname{End} E) \mid a \cup s = 0 \; \forall s \in H^0(X, E) \right\}.$$

In the case of line bundles the description of the tangent space above is also given in [28, Proposition 3.3 (i)].

5. Deformations of the pair (E, U) over a curve

In this section, we restrict our attention to curves, i.e. we fix a smooth projective curve C of genus g and we study deformations of the pair (E, U), where E is a locally free sheaf of rank n and degree d on C and $U \subseteq H^0(C, E)$ is a subspace of sections of dimension k.

First suppose $E = L$ to be a line bundle on C. The Petri map, introduced first by Petri in [31] and studied deeply in [1, 2], is classically defined as the map induced by the cup product:

$$\mu_0 \colon U \otimes H^0(C, K_C \otimes L^*) \to H^0(C, K_C),$$

where K_C denotes the canonical sheaf of C and L^* the dual line bundle of L. In [4, 14, 28] a generalisation of μ_0 to the case of a vector bundle E is introduced as the map induced by the cup product:

$$\mu_0 \colon U \otimes H^0(C, K_C \otimes E^*) \to H^0(C, K_C \otimes E \otimes E^*),$$

where E^* is the dual of the vector bundle E.

Classically for line bundles and also in the successive generalisations [4, 14, 28], the Petri map plays a role in the study of the smoothness of the deformations of the pair (E, U) over a curve C. We aim to recover and generalise these kinds of results.

Consider the sequence (4.5); in the case of curves, it reduces to

$$0 \to H^0(D_U) \to H^0(C, \operatorname{End} E) \to \operatorname{Hom}(U, H^0(C, E)/U)$$

$$(5.1) \quad \to H^1(D_U) \to H^1(C, \operatorname{End} E) \xrightarrow{\alpha} \operatorname{Hom}(U, H^1(C, E)) \to H^2(D_U) \to 0.$$

So we are able to recover [4, Proposition 3.4 (i)].

LEMMA 5.1. *In the above notation, the following conditions are equivalent:*

- $H^2(D_U) = 0$,
- *the map α is surjective,*
- $\mathrm{Hom}(U, H^1(E)) = 0$,
- *the Petri map μ_0 is injective.*

PROOF. The equivalence between the first two conditions follows from the exact sequence (5.1). The equivalence between the second and the third is Remark 4.12. Finally, the second and the last condition are equivalent because α and μ_0 are dual maps. Indeed, using Serre duality, $H^1(C, \mathrm{End}\, E)^* \cong H^0(C, K_C \otimes E \otimes E^*)$ and

$$(\mathrm{Hom}(U, H^1(C, E)))^* \cong (U^* \otimes H^1(C, E))^* \cong U \otimes H^1(C, E)^*$$
$$\cong U \otimes H^0(C, K_C \otimes E^*). \qquad \blacksquare$$

Aiming to link these conditions with the smoothness of the functor of deformations of (E, U), we prove the following result.

LEMMA 5.2. *In the above notation,*

$$h^1(D_U) = h^2(D_U) + h^0(D_U) + k\chi(E) - \chi(\mathrm{End}\, E) - k^2,$$

where $\chi(E)$ and $\chi(\mathrm{End}\, E)$ denote the Euler characteristics of E and $\mathrm{End}\, E$ respectively.

PROOF. From the above exact sequence (5.1), we obtain that

$$h^0(D_U) - h^0(\mathrm{End}\, E) + k \cdot (h^0(E) - k) - h^1(D_U) + h^1(\mathrm{End}\, E)$$
$$- k \cdot h^1(E) + h^2(D_U) = 0;$$

therefore

$$h^1(D_U) = h^2(D_U) + h^0(D_U) + k \cdot (h^0(E) - h^1(E)) + h^1(\mathrm{End}\, E)$$
$$- h^0(\mathrm{End}\, E) - k^2$$
$$= h^2(D_U) + h^0(D_U) + k \cdot \chi(E) - \chi(\mathrm{End}\, E) - k^2. \qquad \blacksquare$$

REMARK 5.3. Let E be a vector bundle of rank n and degree d on a curve C of genus g. Then $\chi(E) = d + n(1 - g)$ (see [15, p. 154]), and then $\chi(\mathrm{End}\, E) = n^2(1 - g)$. Therefore

$$k\chi(E) - \chi(\mathrm{End}\, E) - k^2 = k(d + n(1 - g)) - n^2(1 - g) - k^2$$
$$= k(d + n(1 - g)) + n^2(g - 1) - k^2.$$

Then, as in [4, Definition 2.7] and [14, Definition 2.1], we can introduce the Brill–Noether number.

DEFINITION 5.4. Let E be a vector bundle of rank n and degree d on a curve C of genus g and let U be a subspace of sections of dimension k. The *Brill–Noether number* is

$$\beta(n, d, k) = n^2(g - 1) - k(k - d + n(g - 1)) + 1.$$

REMARK 5.5. This number is a generalisation to vector bundles of the well-known *Brill–Noether* number ρ for the data of a degree d line bundle over a curve of genus g with a subspace of sections of dimension k:

$$\rho = g - k(g - d + k),$$

defined in [1, 2]. As for the classical case of ρ, β gives an estimate of the dimension of the Brill–Noether loci in the corresponding moduli spaces.

We are now ready to prove our main result of this section. It generalises [2, Proposition 4.1], which for a line bundle L on a curve connects the injectivity of the Petri map with the smoothness of the deformations of the pair (L, U) and calculates the dimension of the concerned moduli space in the smooth case.

PROPOSITION 5.6. *Let E be a vector bundle of rank n and degree d on the curve C of genus g and let U be a subspace of sections of dimension k. Then the tangent space to deformations of the pair (E, U) has dimension*

$$\beta(n, d, k) - 1 + h^0(D_U) + h^2(D_U).$$

Moreover, the functor $\mathrm{Def}_{(E,U)}$ is smooth and its tangent space has dimension $\beta(n, d, k) - 1 + h^0(D_U)$ if and only if $H^2(D_U) = 0$ if and only if the Petri map is injective.

PROOF. The tangent space to the deformations of the pair (E, U) is $H^1(D_U)$. Then, according to Lemma 5.2 and Remark 5.3, the dimension of it is given by

$$\begin{aligned} h^1(D_U) &= h^2(D_U) + h^0(D_U) + k\chi(E) - \chi(\mathrm{End}\, E) - k^2 \\ &= h^2(D_U) + h^0(D_U) + k(d + n(1 - g)) + n^2(g - 1) - k^2 \\ &= h^2(D_U) + h^0(D_U) - 1 + \beta(n, d, k). \end{aligned}$$

As already pointed out in Lemma 5.1, the Petri map is injective if and only if its dual map α is surjective, which is equivalent to the condition that $H^2(D_U) = 0$. Since the obstructions to deform the pair (E, U) are contained in $H^2(D_U)$, if it vanishes,

then the functor $\mathrm{Def}_{(E,U)}$ is smooth and the dimension of the tangent space is easily calculated by the above formula. For the other direction, the condition on the dimension of the tangent space implies that $H^2(D_U) = 0$. ∎

REMARK 5.7. Proposition 5.6 is analogous to [4, Proposition 3.10]. In this article, the authors focus their attention on the moduli space of coherent systems from the global point of view. In order to construct a moduli space, they need a suitable notion of stability. The stability conditions they define imply that $h^0(D_U) = 1$. Thus the formula for the dimension of the tangent space reduces to $\beta(n, d, k) + h^2(D_U)$ (see [4, Lemma 3.5]).

6. Deformations of a locally free sheaf and some of its sections

In this section we consider a locally free sheaf E of $\mathcal{O}_X$-modules on a smooth projective variety X, such that $\dim H^0(X, E) \geq k$ and study infinitesimal deformations of E such that at least k independent sections of E lift to the deformed locally free sheaf.

These deformations correspond to the infinitesimal deformations of the locally free sheaf E induced by an infinitesimal deformation of a pair (E, U), for some subspace $U \subseteq H^0(X, E)$ with $\dim U = k$. In other words, they are the deformations in the image of the forgetful maps of functors:

$$r_U \colon \mathrm{Def}_{(E,U)} \to \mathrm{Def}_E,$$

for some $U \subseteq H^0(X, E)$, with $\dim U = k$. We denote this subfunctor of Def_E by Def_E^k. More explicitly, we give the following definition.

DEFINITION 6.1. Let E be a locally free sheaf of $\mathcal{O}_X$-modules on a smooth projective variety X, such that $h^0(X, E) \geq k$. Let $\mathrm{Gr}(k, H^0(E))$ be the grassmannian of all subspaces of $H^0(X, E)$ of dimension k. We define the functor $\mathrm{Def}_E^k \colon \mathrm{Art}_{\mathbb{K}} \to \mathrm{Set}$ that associates with every $A \in \mathrm{Art}_{\mathbb{K}}$ the set

$$\mathrm{Def}_E^k(A) = \bigcup_{U \in \mathrm{Gr}(k, H^0(E))} r_U(\mathrm{Def}_{(E,U)}(A)),$$

and call it the *functor of deformations of E with at least k sections*.

REMARK 6.2. In the case $h^0(X, E) = k$, all sections are required to lift to the deformed locally free sheaf and the functor Def_E^k is in one-to-one correspondence via the forgetful morphism with the functor $\mathrm{Def}_{(E,H^0(E))}$, analysed at the end of Section 4.

Thus our study of $\text{Def}_{(E, H^0(E))}$ applies completely to it and in particular Def_E^k is in this case a deformation functor.

In general, the functor Def_E^k is a functor of Artin rings, but unfortunately, it is not a deformation functor. Indeed, by definition, if $\mathbb{K}$ is the ground field, we have

$$\text{Def}_E^k(\mathbb{K}) = \bigcup_{U \in \text{Gr}(k, H^0(E))} r_U(\text{Def}_{(E,U)}(\mathbb{K})) = \{E\},$$

since each of the functors $\text{Def}_{(E,U)}$ is of Artin rings.

Now consider two morphisms of Artin rings $B \to A$ and $C \to A$ and suppose one of them to be surjective. The map

$$\eta \colon \text{Def}_E^k(B \times_A C) \to \text{Def}_E^k(B) \times_{\text{Def}_E^k(A)} \text{Def}_E^k(C)$$

will not in general be surjective. Indeed, let $(E_B, E_C) \in \text{Def}_E^k(B) \times_{\text{Def}_E^k(A)} \text{Def}_E^k(C)$ and let U and V be subspaces of sections of E that lift to E_B and to E_C respectively, such that $U \cap V$ has maximal dimension and suppose $\dim U \cap V < k$. Then the existence of a lift of (E_B, E_C) in $\text{Def}_E^k(B \times_A C)$ will contradict the maximality of $\dim U \cap V$.

From now on, we restrict ourselves to the field of complex numbers $\mathbb{C}$. Even if the description of the locus $\text{Def}_E^k(A)$ for $A \in \text{Art}_\mathbb{C}$ is still quite mysterious, we can explicitly determine the first-order deformations and the vector space they generate.

THEOREM 6.3. *In the above notation, if $h^0(X, E) = k$, the tangent space to the deformation functor Def_E^k is*

$$t_{\text{Def}_E^k} = \text{Def}_E^k(\mathbb{C}[\varepsilon]) = \{a \in H^1(X, \text{End}\, E) \mid a \cup s = 0 \ \forall s \in H^0(X, E)\}.$$

If instead $h^0(X, E) \geq k + 1$, the first-order deformations of E with at least k sections are described by the cone

$$\text{Def}_E^k(\mathbb{C}[\varepsilon]) = \{v \in H^1(X, \text{End}\, E) \mid \exists U \in \text{Gr}(k, H^0(E)) \text{ such that } v \cup s = 0 \ \forall s \in U\}$$

and the vector space generated by it, which we call the tangent space to Def_E^k, is

$$t_{\text{Def}_E^k} = H^1(X, \text{End}\, E).$$

PROOF. As already noticed, in the case $h^0(X, E) = k$, the functor Def_E^k is in one-to-one correspondence with the functor $\text{Def}_{(E, H^0(E))}$ and the tangent space is described in Corollary 4.15 to be

$$t_{\text{Def}_E^k} \cong t_{\text{Def}_{(E, H^0(E))}} = \{a \in H^1(X, \text{End}\, E) \mid a \cup s = 0 \ \forall s \in H^0(X, E)\}.$$

If $h^0(X, E) \geq k + 1$, by definition,

$$\mathrm{Def}_E^k(\mathbb{C}[\varepsilon]) = \bigcup_{U \in \mathrm{Gr}(k, H^0(E))} r_U(\mathrm{Def}_{(E,U)}(\mathbb{C}[\varepsilon])).$$

For each $U \in \mathrm{Gr}(k, H^0(E))$, we calculate the image of the tangent space to deformations of the pair (E, U) using the exact sequence (4.5):

$$\cdots \to H^1(X, D_U) \xrightarrow{r_U} H^1(X, \mathrm{End}\, E) \xrightarrow{\alpha_U} \mathrm{Hom}(U, H^1(X, E)) \cdots.$$

Thus

$$r_U(\mathrm{Def}_{(E,U)}(\mathbb{C}[\varepsilon])) = \ker \alpha_U = \left\{ v \in H^1(X, \mathrm{End}\, E) \;\middle|\; v \cup s = 0 \; \forall s \in U \right\}$$

and the first statement is proved.

For the second statement, we have to prove that the vector space generated by $\mathrm{Def}_E^k(\mathbb{C}[\varepsilon])$ is the whole space $H^1(X, \mathrm{End}\, E)$. One inclusion is obvious. For the other one, it is enough to prove that, for all $s \in H^0(X, E)$ a nonzero section and for all $w \in H^1(X, E)$, there exists an element $v \in \mathrm{Def}_E^k(\mathbb{C}[\varepsilon])$ such that $v(s) = w$. Since $\dim H^0(X, E) \geq k + 1$, it is always possible to find a subspace $U \in \mathrm{Gr}(k, H^0(E))$, such that $s \notin U$ and to build the matrix of v. ∎

REMARK 6.4. This theorem generalises the classical results for line bundles on curves [2, Proposition 4.2] and line bundles on a smooth projective varieties [28, Proposition 3.3]. Our explicit description of the tangent space is also a particular case of that of the Zariski tangent space to the cohomology jump functors considered in [5, Theorem 1.7] using dgl pairs.

Moreover, our result is coherent with the well-known fact that the locally free sheaves E such that $h^0(X, E) \geq k + 1$ are contained in the singular locus of the moduli space of locally free sheaves with at least k independent sections. That is classically obtained by defining that moduli space as a determinantal variety (see [2, Proposition 4.2], [4, Theorem 2.8], [8, Corollary 2.8], etc.)

In the setting of deformation functors, the next step after the description of the tangent space is the study of an obstruction space. As is well known, in the deformation functors case both spaces have a meaning in terms of the corresponding moduli space. Unfortunately, our functor Def_E^k is not a deformation functor (see Remark 6.2). However, Definition 2.6 holds for Def_E^k and in the following we try to get some geometrical information linked to its smoothness.

PROPOSITION 6.5. *As above, let E be a locally free sheaf of $\mathcal{O}_X$-modules on the projective variety X, such that $h^0(X, E) \geq k$. If there exists a $U \in \mathrm{Gr}(k, H^0(X, E))$*

such that $\mathrm{Hom}(U, H^1(X, E)) = 0$ *or, in an equivalent way, such that the map* $\alpha_U \colon H^1(X, \mathrm{End}\, E) \to \mathrm{Hom}(U, H^1(X, E))$ *that appears in* (4.5) *is surjective, then*

$$\mathrm{Def}_E \text{ is smooth} \Leftrightarrow \mathrm{Def}_{(E,U)} \text{ is smooth} \Leftrightarrow \mathrm{Def}_E^k \text{ is smooth.}$$

PROOF. From Corollaries 4.9 and 4.11, the two equivalent hypotheses imply that the forgetful morphism r_U is smooth. Then the first equivalence is a direct consequence of Remark 4.10. With regard to the second equivalence, since the obstruction is complete, each $E_A \in \mathrm{Def}_E^k(A)$ comes from a pair $(E_A, i_A) \in \mathrm{Def}_{(E,U)}(A)$ for every $A \in \mathrm{Art}_{\mathbb{K}}$. The above argument obviously implies equivalence between the smoothness of $\mathrm{Def}_{(E,U)}$ and of Def_E^k. ∎

PROPOSITION 6.6. *In the above notation, if there exists a* $U \in \mathrm{Gr}(k, H^0(E))$ *such that* $H^2(D_U) = 0$, *then both functors* $\mathrm{Def}_{(E,U)}$ *and* Def_E^k *are smooth.*

PROOF. Since $H^2(D_U) = 0$, the functor $\mathrm{Def}_{(E,U)}$ is smooth and relative obstruction to r_U is zero, thus r_U is smooth too. These two properties ensure that Def_E^k is smooth too. ∎

REMARK 6.7. In general, the hypothesis $H^2(D_U) = 0$ implies strictly that α_U is surjective. Since for a curve they are both equivalent to the injectivity of the Petri map (see Lemma 5.1), Proposition 6.6 ensures that on a curve C, if there exists $U \in \mathrm{Gr}(k, H^0(E))$ such that the Petri map $\mu_0 \colon U \otimes H^0(C, K_C \otimes E^*) \to H^0(C, K_C \otimes E^* \otimes E)$ is injective, then both functors $\mathrm{Def}_{(E,U)}$ and Def_E^k are smooth. See [7, Proposition 2.1] for a similar result; there the authors assume the injectivity of the Petri map for every $U \in \mathrm{Gr}(k, H^0(E))$.

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