Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1472

Published by EMS Press

Mixed-norm of orthogonal projections and analytic interpolation on dimensions of measures

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Abstract. Suppose that μ and ν are compactly supported Radon measures on $\mathbb{R}^d$, $V \in G(d,n)$ is an n-dimensional subspace, and let $\pi_V: \mathbb{R}^d \to V$ denote the orthogonal projection. In this paper, we study the mixed-norm $\int \|\pi^y \mu\|^q_{L^p(G(d,n))}\, d\nu(y)$, where

$$\pi^y \mu(V) := \int_{y+V^\perp} \mu \, d\mathcal{H}^{d-n} = \pi_V \mu(\pi_V y),$$

assuming μ has continuous density. When $n = d-1$ and $p = q$, our result significantly improves a previous result of Orponen on radial projections. We also discuss about consequences including jump discontinuities in the range of p, and m-planes determined by a set of given Hausdorff dimension.

In the proof, we run analytic interpolation not only on p and q, but also on dimensions of measures. This is partially inspired by previous work of Greenleaf and Iosevich on Falconer-type problems. We also introduce a new quantity called s-amplitude, that is crucial for our interpolation and gives an alternative definition of Hausdorff dimension.

1. Introduction

Let $\pi_e(x) = x \cdot e$, $x \in \mathbb{R}^2$, $e \in S^1$, denote the orthogonal projection. In 1954, Marstrand proved in [12] that, for every Borel set $E \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}} E > 1$, the set $\pi_e(E)$ has positive Lebesgue measure for almost all $e \in S^1$. In 1968, Kaufman [8] gave a simple alternative proof of Marstrand projection theorem using Fourier analysis. Moreover, he proved that the induced measure $\pi_e \mu$ on $\pi_e(E)$ has L^2 density for almost all $e \in S^1$, where μ is a Frostman measure on E. Nowadays, orthogonal projection has become a central problem in geometric measure theory, and has been studied actively from different perspectives.

In higher dimensions, we denote orthogonal projections by $\pi_V: \mathbb{R}^d \to V$, where $V \in G(d,n)$ is a n-dimensional subspace of $\mathbb{R}^d$ and $G(d,n)$ denotes the Grassmannian. Also, throughout this paper $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure, and $\gamma_{d,n}$ is the probability measure on $G(d,n)$ induced by the Haar measure θ_d on the orthogonal group.

Mathematics Subject Classification 2020: 28A75 (primary); 42B20 (secondary).
Keywords: orthogonal projection, radial projection, analytic interpolation.

Of course, $\pi_V\mu \in L^2(\mathcal{H}^n|_V)$ is not a necessary condition for $\mathcal{H}^n(\pi_V(E)) > 0$. But for technical reasons, the L^2-method is the most popular approach to this type of problems. We refer to [11, 13] for L^2-approaches to Falconer distance conjecture.

It is not surprising that L^2 does not always help. For $x \neq y$ in $\mathbb{R}^d$, let

$$\pi^y(x) := \frac{x-y}{|x-y|} \in S^{d-1}$$

denote the radial projection. The visibility problem asks that, given $E, F \subset \mathbb{R}^d$, whether there exists $y \in F$ such that $\pi^y(E \setminus \{y\}) \subset S^{d-1}$ has positive surface measure. Although there are L^2-estimates in the literature (e.g., [18]), their geometric consequences are far from being desired. Finally, Orponen [17] proved that if $\operatorname{supp}\mu \cap \operatorname{supp}\nu = \emptyset$, $s > d-1$ and $s + t > 2(d-1)$, then

$$\int \|\pi^y\mu\|^p_{L^p(S^{d-1})}\, d\nu(y) \lesssim I_s(\mu)^{p/2} \cdot I_t(\nu)^{1/2} \tag{1.1}$$

for every

$$1 \leq p < \min\left\{\frac{t}{2(d-1)-s},\, 2 - \frac{t}{d-1}\right\},$$

where I_s denotes the s-energy (see (1.9) below for definition) and $\pi^y\mu$ denotes the push-forward of the weighted measure $c_d|x-y|^{-(d-1)}\, d\mu(x)$ under π^y, which in particular equals

$$\int \mu(y + te)\, dt$$

if μ has continuous density. For applications, there is no difference between this representation and the pushforward of μ itself (we can always assume μ has compact support away from y), but the modified version is more convenient when writing the proof.

Notice that in Orponen's result, $p < 2$ is required, and $p \to 1$ as s, t get close to the critical case $s + t = 2(d-1)$. Both will be improved significantly in this paper.

As a geometric consequence of (1.1), if $E, F \subset \mathbb{R}^d$ are such that $\dim_{\mathcal{H}} E > d-1$ and $\dim_{\mathcal{H}} E + \dim_{\mathcal{H}} F > 2(d-1)$, then

$$\mathcal{H}^{d-1}(\pi^y(E \setminus \{y\})) > 0, \quad \text{for some } y \in F. \tag{1.2}$$

It is known that neither of assumptions $\dim_{\mathcal{H}} E > d-1$, $\dim_{\mathcal{H}} E + \dim_{\mathcal{H}} F > 2(d-1)$ can be relaxed. We refer to [16] for the discussion on the sharpness, and to Example 5.13 in [15] for details of examples. This also implies that both $s > (d-1)$ and $s + t > 2(d-1)$ are necessary for the existence of $p > 1$ in (1.1).

What is more, the estimate (1.1) was quickly introduced into the study of Falconer distance conjecture, playing crucial roles in a couple of recent breakthroughs [7, 9]. This brings more attention to L^p-estimates of projections.

Even more recently, Dabrowski, Orponen, Villa [1] proved that, if μ is a compactly supported measure on $\mathbb{R}^d$ satisfying the s-dimensional Frostman condition

$$\mu(B(x,r)) \lesssim r^s, \quad \text{for all } r > 0,\ x \in \mathbb{R}^d,$$

then

$$\int \|\pi_V\mu\|^p_{L^p(\mathcal{H}^n)}\, d\gamma_{d,n}(V) < \infty, \quad \text{for all } 2 \leq p < 2 + \frac{s-n}{d-s}. \tag{1.3}$$

This is sharp for $s > d-1$, and has applications to Furstenberg sets and discretized sum-product problems.

Estimates (1.1) and (1.3) are closely related, due to Orponen's formula (see Lemma 3.1 in [17] and Lemma 4.17 in [1], with some notation redefined here)

$$\int \|\pi^y \mu\|^p_{L^p(G(d,n))}\, d\nu(y) = \int \|\pi_V \mu\|^p_{L^p(\pi_V \nu)}\, d\gamma_{d,n}(V), \tag{1.4}$$

given μ has continuous density and

$$\pi^y \mu(V) := \int_{y+V^\perp} \mu\, d\mathcal{H}^{d-n} = \pi_V \mu(\pi_V y). \tag{1.5}$$

A remark is that π^y itself is not a projection when $n < d-1$, as a $(d-n)$-plane cannot be determined by two points. For this reason, it is not straightforward to define $\pi^y \mu$ as a measure on $G(d,n)$ when μ is not continuous. We leave the detailed discussion to the beginning of Section 4. This is also why we prefer naming this paper "mixed-norm of orthogonal projections" instead of "radial projections".

Although the proof of (1.4) is not hard (just change variables $\mathbb{R}^d = V \oplus V^\perp$), it is very important. In [17], the proof of (1.1) starts from (1.4); in [1], (1.4) is used to obtain an incidence estimate from (1.3).

In [1], the authors suggest studying the mixed-norm

$$\int \|\pi_V \mu\|^q_{L^p(\mathcal{H}^n)}\, d\gamma_{d,n}(V),$$

especially for $q \leq p$. Their motivation is the following. As a corollary of (1.3), for almost all $V \in G(d,n)$,

$$\pi_V \mu \in L^p(\mathcal{H}^n|_V), \quad \text{for all } p < 2 + \frac{s-n}{d-s}. \tag{1.6}$$

Since both $\operatorname{supp}\mu$ and $G(d,n)$ are compact, one may expect a wider range of p by considering smaller q.

In fact, it is not hard to obtain mixed-norm estimates for $q \leq p$. It follows directly from the classical L^2-estimate (see, e.g., Section 4.1 in [15]) and the Sobolev embedding that

$$\int \|\pi_V \mu\|^2_{L^p(\mathcal{H}^n)}\, d\gamma_{d,n}(V) < \infty, \quad \forall p \begin{cases} < \frac{2n}{2n-s}, & \text{if } n < s \leq 2n, \\ = \infty, & \text{if } s > 2n. \end{cases} \tag{1.7}$$

The range of p in (1.7) is wider than (1.6) if $n < s < 2(d-n)$ or $s > 2n$. Then by analytic interpolation, one can prove

$$\int \|\pi_V \mu\|^q_{L^p(\mathcal{H}^n)}\, d\gamma_{d,n}(V) < \infty$$

for $2 \leq q \leq p < \infty$ and

$$\frac{n}{p} + \frac{2d-2n-s}{q} > d-s. \tag{1.8}$$

Although this interpolation is not the standard Riesz–Thorin because of the mixed-norm, it follows from its vector-valued version (see e.g. Exercise 5.5.2 in [4]), or one can just see our interpolation argument in Section 7 below.

On the other hand, unfortunately, interpolation on p never makes its range wider. In this paper, we shall interpolate also on dimensions of measures, that indeed extend the range of p.

Now we turn to the mixed-norm

$$\int \|\pi^y \mu\|^q_{L^p(G(d,n))}\, d\nu(y).$$

When $q > p$, it is equivalent to consider

$$\int \|\pi^y \mu\|^p_{L^p(G(d,n))}\, f(y)\, d\nu(y), \quad \text{for } \|f\|_{L^{(q/p)'}(\nu)} = 1,$$

so some results follow from the case $p = q$, as $f d\nu$ is also a Frostman measure if the measure ν is (Hölder's inequality); while when $q < p$, due to the lack of Sobolev embedding, there seems no way to reduce it to $q = p$. As we mentioned above, the first step of Orponen's argument on (1.1) is to apply his formula (1.4), whose proof relies on $q = p$. So we need new ideas for $q < p$, even to start.

To state our main theorem, we need the classical s-energy

$$I_s(\mu) := \iint |x-y|^{-s}\, d\mu(x)\, d\mu(y) \tag{1.9}$$

as well as a new quantity $A_s(\mu)$, called the s-amplitude. This new quantity plays a key role in our analytic interpolation, and makes our statement quite clean. See Section 3 for more discussions.

Definition 1.1. For every compactly supported Radon measure μ on $\mathbb{R}^d$ and $0 < s < d$, define the s-amplitude of μ by

$$A_s(\mu) := \sup_{x\in\mathbb{R}^d} \int |x-y|^{-s}\, d\mu(y).$$

Theorem 1.2. *Suppose μ and ν are compactly supported Radon measures on $\mathbb{R}^d$, and let $0 < t < n$, $0 < s, \alpha < d$, and $s + t \geq 2n$. Denote*

$$q_0 := 1 + \frac{s+t-2n}{2(d-\alpha)}.$$

Then, for every

$$1 \leq p < \frac{2nq_0}{n+t} = \frac{2n}{n+t}\left(1 + \frac{s+t-2n}{2(d-\alpha)}\right),$$

we have

(1.10)

$$\int \|\pi^y \mu\|^q_{L^p(G(d,n))}\, d\nu(y) \lesssim_{d,n,p,s,t,\alpha} \begin{cases} I_s(\mu)^{1/2}\cdot A_\alpha(\mu)^{q-1}\cdot I_t(\nu)^{1/2}, & q = q_0, \\ I_s(\mu)^{1/2}\cdot A_\alpha(\mu)^{q-1}\cdot A_{\max\{t,\frac{q}{2q_0}t\}}(\nu), & q > q_0. \end{cases}$$

Furthermore, when $t = n$,

(1.11)

$$\int \|\pi^y \mu\|^q_{L^{q_0}(G(d,n))}\, d\nu(y) \lesssim_{d,n,s,\alpha} \begin{cases} I_s(\mu)^{1/2}\cdot A_\alpha(\mu)^{q-1}\cdot I_n(\nu)^{1/2}, & q = q_0, \\ I_s(\mu)^{1/2}\cdot A_\alpha(\mu)^{q-1}\cdot A_{\max\{n,\frac{q}{2q_0}n\}}(\nu), & q > q_0. \end{cases}$$

With Theorem 1.2, one can compute for what p, q the mixed-norm estimates on Frostman measures μ, ν are finite. We leave the full table of p, q to the end of this section, as it looks quite complicated. In particular, by taking $p = q$ and $n = d - 1$, we can improve the range of p in (1.1) to

$$1 \leq p \leq \frac{2(d-1)}{d-1+t}\Big(1 + \frac{s+t-2(d-1)}{2(d-s)}\Big). \tag{1.12}$$

To see this is an improvement, first notice that the range of p in (1.1) makes sense only if $t \in (2(d-1)-s, d-1)$. Then we fix $s > d-1$ and observe that, as a piecewise linear function in t,

$$\max_{t\in[2(d-1)-s, d-1]}\Big(\min\Big\{\frac{t}{2(d-1)-s}, 2 - \frac{t}{d-1}\Big\}\Big) = \frac{2(d-1)}{3(d-1)-s}.$$

Finally, one can easily check that (1.12) is increasing in $t \in [2(d-1)-s, d-1]$, and therefore by plugging in $t = 2(d-1)-s$, it is

$$\geq \frac{2(d-1)}{3(d-1)-s},$$

as desired. In fact, the improvement of (1.12) over (1.1) is significant: the range of p tends to $[1, \infty]$ as $s \to d$ and equals $[1, \frac{2(d-1)}{d-1+t})$ when $s + t = 2(d-1)$, while (1.1) cannot go beyond $p = 2$, and collapses when $s + t = 2(d-1)$.

As a remark, the assumption $0 < t < n$ is necessary (see Section 10), so our result is not comparable to (1.3) by taking $\mu = \nu$ and $p = q$. But our method does give an alternative proof of (1.3), with a more delicate upper bound. See also Section 10.

Now we ignore q and focus on when $\pi^y\mu \in L^p(G(d,n))$. By Theorem 1.2, or from the full table in Corollary 1.5, one can obtain the following.

Corollary 1.3. *Suppose μ and ν are compactly supported Radon measures on $\mathbb{R}^d$ satisfying*

$$\mu(B(x,r)) \lesssim r^{s_\mu},\ \nu(B(x,r)) \lesssim r^{s_\nu}, \quad \text{for all } x \in \mathbb{R}^d,\ r > 0,$$

where $0 < s_\nu < n$ and $2n - s_\nu < s_\mu < d$. Then, for every

$$1 \leq p < \begin{cases} \frac{2n}{n+s_\nu}\big(1 + \frac{s_\mu+s_\nu-2n}{2(d-s_\mu)}\big), & \text{if } s_\mu \geq 2d-3n, \\ \frac{2n}{n+\max\{2n-s_\mu,0\}}\Big(1 + \frac{\max\{s_\mu-2n,0\}}{2(d-s_\mu)}\Big), & \text{if } s_\mu < 2d-3n, \end{cases}$$

there exists $y \in \operatorname{supp}\nu$ such that

$$\pi^y\mu \in L^p(G(d,n)).$$

In particular, if $s_\mu > 2n$, then the set of $y \in \mathbb{R}^d$ such that

$$\inf\{p : \pi^y\mu \notin L^p(G(d,n))\} < 2 + \frac{s_\mu - 2n}{d - s_\mu}$$

has Hausdorff dimension 0.

Notice that, near the critical line segment

$$\{(s_\mu, s_\nu) \in (0,d)^2 : s_\mu + s_\nu = 2n,\ 0 < s_\nu < n\},$$

the critical p in Corollary 1.3 equals

$$\frac{2n}{n+s_\nu} = \frac{2n}{3n-s_\mu} > 1.$$

On the other hand, by considering μ, ν on $\mathbb{R}^{n+1} \times \{0\} \subset \mathbb{R}^d$, the sharpness of the visibility problem in $\mathbb{R}^{n+1}$ implies that $s_\mu + s_\nu > 2n$ is necessary for the existence of $p > 1$. Together, it follows that the range of p has jump discontinuities at critical cases. See Figure 1 below: in the shadow, $p > 2n/(3n - s_\mu)$, while in the blank, $p = 1$. This phenomenon cannot be seen from Orponen's previous result (1.1), thus unexpected and surprising. I think there are deep reasons behind it to be explored.

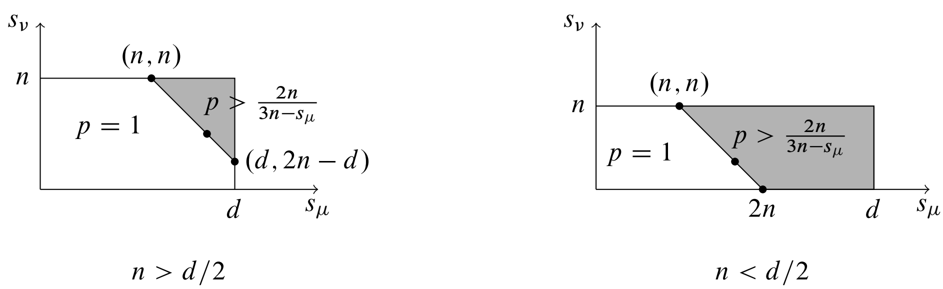

Figure 1. Jump discontinuities of p at $s_\mu + s_\nu = 2n$.

I do not know whether our range of p is sharp, or how far it is from being sharp. It seems very hard to compute L^p-estimates from known geometric examples. I think the first step towards the sharpness is to understand the jump discontinuities along the critical line segment.

Our results also generalize the visibility problem. Notice that not every generalization is nontrivial. For example, one can easily conclude that for every $E, F \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}} E > n$ and $\dim_{\mathcal{H}} E + \dim_{\mathcal{H}} F > 2n$, there exists $y \in F$ such that

$$\gamma_{d,d-n}\{W \in G(d,d-n) : W \cap (E-y) \neq \emptyset\} > 0. \tag{1.13}$$

To see this, just project E and F onto a $(n+1)$-dimensional subspace, preserving their dimensions or having positive Lebesgue measure; then (1.13) follows from the visibility problem in $\mathbb{R}^{n+1}$. See [2] for an application of this trick on Falconer distance conjecture.

To make nontrivial generalizations, we consider the set of m-planes determined by $E - y$, that is,

$$\pi^y(E^m) := \{W \in G(d,m) : W = \mathrm{Span}\{x_1 - y, \dots, x_m - y\} : x_1, \dots, x_m \in E\}.$$

Throughout this paper, when writing

$$W = \mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_m\},$$

vectors $\vec{v}_1, \dots, \vec{v}_m$ are a priori assumed nonzero and linearly independent.

Corollary 1.4. *Suppose* $1 \leq m \leq d-1$ *and let* $E \subset \mathbb{R}^d$ *be a Borel subset.*

(i) *If*

$$\dim_{\mathcal{H}} E > \max\left\{d - \frac{m}{2m-1},\ 3m-d\right\},$$

then

$$\dim_{\mathcal{H}}\{y \in \mathbb{R}^d : \gamma_{d,m}(\pi^y(E^m)) = 0\}$$
$$\leq \max\left\{2(d-m) - \dim_{\mathcal{H}} E,\ \frac{(d-m)(\dim_{\mathcal{H}} E - 2m + m(d-\dim_{\mathcal{H}} E))}{d-m-m(d-\dim_{\mathcal{H}} E)},\ 0\right\}.$$

(ii) *If*

$$\dim_{\mathcal{H}} E > \max\left\{2(d-m),\ d - \frac{2m-d}{m-1}\right\},$$

then

$$\dim_{\mathcal{H}}\{y \in \mathbb{R}^d : \gamma_{d,m}(\pi^y(E^m)) = 0\} = 0.$$

One can check that (i) matches the results above on the visibility problem with $m = 1$. Also (ii) makes sense only if $m > d/2$.

To deal with $(m+1)$-point configurations like Corollary 1.4, people used to consider $(m+1)$-linear estimates. One example is the method in [5,6] that we shall discuss briefly in next section. In our case, multi-linear means $q \in \mathbb{Z}$, but for Corollary 1.4 we need $p = m + \varepsilon$ (see Section 8), whose associated q may not be an integer. This explains why our interpolation between multi-linear estimates gives a better bound than multi-linear estimates themselves. We hope this will shed lights on other multiple configuration problems in the future.

I do not know how sharp Corollary 1.4 is, or what to expect for $m \geq 2$. One could follow the idea in [16] on the sharpness of the visibility problem ($m = 1$), to transfer special cases to orthogonal projections via projective transformations. But then one should check, not only the projected image has positive Lebesgue measure, but also each slice is not contained in a lower dimensional affine subspace. This seems not easy.

Now we give the full table of p, q as the end of the introduction. Since $p, q \to \infty$ as $s_\mu \to d$, our result covers all $p, q \in [1, \infty)$.

Corollary 1.5. *Suppose* μ *and* ν *are compactly supported Radon measures on* $\mathbb{R}^d$ *satisfying*

$$\mu(B(x,r)) \lesssim r^{s_\mu},\ \nu(B(x,r)) \lesssim r^{s_\nu}, \quad \text{for all } x \in \mathbb{R}^d,\ r > 0,$$

where $0 < s_\nu < n$ *and* $2n - s_\nu < s_\mu < d$. *Then*

$$\int \|\pi^y \mu\|^q_{L^p(G(d,n))}\, d\nu(y) < \infty \tag{1.14}$$

if one of the following holds:

- $s_\mu \leq 2d - 3n$, *and*

$$1 \leq p < \frac{2n}{n + \max\{2n - s_\mu, 0\}}\left(1 + \frac{\max\{s_\mu - 2n, 0\}}{2(d-s_\mu)}\right) = \begin{cases} \frac{2n}{3n - s_\mu}, & s_\mu < 2n, \\ 2 + \frac{s_\mu - 2n}{d - s_\mu}, & s_\mu \geq 2n, \end{cases}$$

$$1 \leq q < \frac{2s_\nu}{\max\{2n - s_\mu, 0\}}\left(1 + \frac{\max\{s_\mu - 2n, 0\}}{2(d-s_\mu)}\right) = \begin{cases} \frac{2s_\nu}{2n - s_\mu}, & s_\mu < 2n, \\ \infty, & s_\mu \geq 2n. \end{cases}$$

- $s_\mu \geq 2d - 2n$, *and*
$$1 \leq p < \frac{2n}{n+s_\nu}\Big(1 + \frac{s_\mu + s_\nu - 2n}{2(d-s_\mu)}\Big), \quad 1 \leq q < 2 + \frac{s_\mu + s_\nu - 2n}{d - s_\mu}.$$
- $2d - 3n < s_\mu < 2d - 2n$, *and* p *and* q *lie in the region enclosed by*
$$1 \leq p < \frac{2n}{n+s_\nu}\Big(1 + \frac{s_\mu + s_\nu - 2n}{2(d-s_\mu)}\Big),$$
$$1 \leq q < \frac{2s_\nu}{\max\{2n - s_\mu, 0\}}\Big(1 + \frac{\max\{s_\mu - 2n, 0\}}{2(d-s_\mu)}\Big) = \begin{cases} \frac{2s_\nu}{2n - s_\mu}, & s_\mu < 2n, \\ \infty, & s_\mu \geq 2n, \end{cases}$$
and
$$\frac{s_\mu - 2d + 3n}{1 - \frac{d-s_\mu}{n} p} < n + \frac{2d - 2n - s_\mu}{\frac{d-s_\mu}{s_\nu} q - 1}.$$
In particular, in the case $p = q$,
$$\int \|\pi^y \mu\|_{L^p(G(d,n))}^p \, d\nu(y) < \infty$$

for every
$$1 \leq p < \begin{cases} \frac{2n}{n+s_\nu}\big(1 + \frac{s_\mu+s_\nu-2n}{2(d-s_\mu)}\big), & \text{if } s_\mu \geq 2d - 3n, \\ \frac{2n}{n+\max\{2n-s_\mu,0\}}\big(1 + \frac{\max\{s_\mu-2n,0\}}{2(d-s_\mu)}\big), & \text{if } s_\mu < 2d - 3n. \end{cases}$$

Organization. In Sections 2–4, we discuss in detail about different ingredients in our proof, including analytic interpolation (Section 2), the newly defined s-amplitude and its role in the dimension theory (Section 3), as well as some "trivial" estimates on orthogonal projections (Section 4). In Section 5–9, we prove our theorems and corollaries. In Section 10, we compare between our setup and the estimate (1.3) of Dabrowski–Orponen–Villa, and then give an alternative proof of (1.3) using our method.

Notation. $X \lesssim Y$ means $X \leq CY$ for some constant $C > 0$, $X \approx Y$ means $X \lesssim Y$ and $Y \lesssim X$, and $X \lesssim_\varepsilon Y$ means $X \leq CY$ for some constant $C = C(\varepsilon) > 0$.

2. Preliminaries on analytic interpolation

2.1. The Hadamard three-lines lemma

Suppose f is a bounded analytic function in the strip $\{0 < \operatorname{Re} z < 1\}$, continuous on the boundary, and let
$$M_0 := \sup_{\operatorname{Re} z = 0} |f(z)| < \infty, \quad M_1 := \sup_{\operatorname{Re} z = 1} |f(z)| < \infty.$$
The Hadamard three-lines lemma states that for every $z \in \{0 < \operatorname{Re} z < 1\}$,
$$|f(z)| \leq M_0^{\operatorname{Re} z} \cdot M_1^{1 - \operatorname{Re} z}.$$

There is a higher dimensional version of the Hadamard three-lines lemma. We give the statement and a proof for completeness. One can also see, e.g., Lemma 3.16 in [5].

Lemma 2.1. *Let $f(z_1, \ldots, z_n)$ be an analytic function on $\mathbb{C}^n$ and let $a_1, \ldots, a_k \in \mathbb{R}^n$ be real-valued vectors, $k \geq 2$. Suppose that for every $j = 1, \ldots, k$,*

$$\sup_{(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n) = a_j} |f(z_1, \ldots, z_n)| \leq 1.$$

Then $|f(z_1, \ldots, z_n)| \leq 1$ whenever $(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n) \in \mathbb{R}^n$ lies in the convex hull generated by $a_1, \ldots, a_k$.

Proof. The proof goes by induction in k. When $k = 2$, one can assume $a_1 = (0, \ldots, 0)$ and $a_2 = (1, 0, \ldots, 0)$. Then, by restricting f to the complex line $\mathbb{C} \times \{0\} \cdots \{0\}$, the desired estimate follows from the one-dimensional three-lines lemma. For general k, first by the inductive hypothesis on $k - 1$ we have $|f| \leq 1$ if $(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n)$ lies in the boundary of the convex hull generated by $a_1, \ldots, a_k$. For $(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n)$ lying in the interior of this convex hull, one can find a line segment containing this point with end points in the boundary. Then, by restricting f to this line segment, the desired estimate again follows from the one-dimensional three-lines lemma. ∎

The Hadamard three-lines lemma was first announced in 1890s. It was introduced into harmonic analysis in mid-20th century to bound norms of operators between L^p spaces. These techniques are still widely used today, including Riesz–Thorin interpolation, Stein's interpolation, and many others.

2.2. The Riesz potential

Our analytic interpolation is inspired by both classical ones and the work of Greenleaf–Iosevich [6] and Grafakos–Greenleaf–Iosevich–Palsson [5]. Suppose $\phi \in C_0^\infty(\mathbb{R}^d)$, $\phi \geq 0$, $\int \phi = 1$, define $\phi_\delta(\cdot) := \delta^{-d}\phi(\cdot/\delta)$, and let

$$\mu^\delta(x) := \phi_\delta * \mu(x).$$

Then the Riesz potential,

$$\frac{\pi^{z/2}}{\Gamma(z/2)} |\cdot|^{-d+z} * \mu^\delta(x), \tag{2.1}$$

is a smooth function in $x \in \mathbb{R}^d$, initially defined for $\operatorname{Re} z > 0$, that can be extended to $z \in \mathbb{C}$ by analytic continuation. It is well known that (2.1) has Fourier transform

$$\frac{\pi^{(d-z)/2}}{\Gamma((d-z)/2)} \widehat{\mu^\delta}(\xi) |\xi|^{-z},$$

in the sense of distributions, and in particular (2.1) equals $c_d \mu^\delta$ as a distribution when $z = 0$. We refer to pp. 71 and 192 in [3] for details. Also, when $z \in (0, d)$ is real, the Riesz potential is the same as the fractional Laplacian $(-\Delta)^{-z/2}\mu^\delta$ up to a normalization which is defined by the Fourier inverse of $\widehat{\mu^\delta}(\xi)|\xi|^{-z}$.

In [5, 6], geometric k-point configuration problems are reduced to k-linear forms

$$\Lambda(\mu^\delta, \ldots, \mu^\delta),$$

where μ is a Frostman measure and Λ is a symmetric k-linear form. Take

$$\Phi(z_1,\dots,z_k) := \Lambda\Big(\frac{\pi^{z_1/2}}{\Gamma(z_1/2)}\,|\cdot|^{-d+z_1} * \mu^\delta,\dots,\frac{\pi^{z_k/2}}{\Gamma(z_k/2)}\,|\cdot|^{-d+z_k} * \mu^\delta\Big) \tag{2.2}$$

as an analytic function on $\mathbb{C}^k$. If $\mu(B(x,r)) \lesssim r^{s+\varepsilon}$ for all $r>0$ and $\operatorname{Re} z_j = d-s \in (0,d)$, it follows immediately that

$$\Big|\int |x-y|^{-d+z_j}\,d\mu(y)\Big| \le \int |x-y|^{-s}\,d\mu(y) \lesssim 1. \tag{2.3}$$

Therefore, $|\Phi(z)|$ can be reduced to a bilinear form

$$B\Big(\frac{\pi^{z_{j_1}/2}}{\Gamma(z_{j_1}/2)}\,|\cdot|^{-d+z_{j_1}} * \mu^\delta,\ \frac{\pi^{z_{j_2}/2}}{\Gamma(z_{j_2}/2)}\,|\cdot|^{-d+z_{j_2}} * \mu^\delta\Big),$$

where

$$\operatorname{Re} z_{j_1} = \operatorname{Re} z_{j_2} = -\frac{k(d-s)}{2}.$$

We skip details here. One can read [6] or Section 6 below for a clue.

Since Λ is symmetric, the bilinear form estimate is independent in the choice of j_1, j_2, which means the same estimate holds for $\binom{k}{2}$ non-proportional vectors $(\operatorname{Re} z_1,\dots,\operatorname{Re} z_k)$ whose convex hull contains the origin. Hence estimates of

$$\Phi(0) = \Lambda(\mu^\delta,\dots,\mu^\delta)$$

follow from the Hadamard three-lines lemma.

2.3. Remarks

There are some technical issues about [5,6] to be clarified.

(i) Though (2.3) holds, one cannot conclude that the Riesz potential (2.1) is also $\lesssim 1$, due to the unbounded factor $|\Gamma(z/2)|^{-1}$. To resolve this issue, we shall work with

$$\mu_z^\delta(x) := \psi(x)\cdot e^{z^2}\frac{\pi^{z/2}}{\Gamma(z/2)}\,|\cdot|^{-d+z} * \mu^\delta(x), \tag{2.4}$$

with $\psi \in C_0^\infty(\mathbb{R}^d)$, nonnegative and equal to 1 on $\operatorname{supp}\mu$. The role of e^{z^2} is to control

$$|e^{z^2}\cdot\Gamma^{-1}(z)| \lesssim_{\operatorname{Re} z} 1,$$

as Γ^{-1} is an entire function of order 1. It also guarantees the boundedness of $|\mu_z^\delta(x)|$ in each strip $\{a < \operatorname{Re} z < b\}$, necessary for the Hadamard three-lines lemma. The role of ψ is to ensure the support of $\mu_z^\delta(x)$ is compact.

(ii) When $\operatorname{Re} z \in (0,d)$, it is straightforward that

$$|\mu_z^\delta(x)| \lesssim_{\operatorname{Re} z} \int |x-y|^{-d+\operatorname{Re} z}\,d\mu^\delta(y) \approx \mu_{\operatorname{Re} z}^\delta(x).$$

But we are not convinced that this relation can be extended to general $z \in \mathbb{C}$: the right-hand side is not even guaranteed positive for $\operatorname{Re} z \notin (0, d)$, because it is defined via analytic continuation. In our argument below, we are super careful when taking absolute values of possibly complex-valued μ_z^δ, as we do not know how to compute the Fourier transform of $|\mu_z^\delta|$. Readers can keep an eye on our timing of taking absolute values in Section 6.

3. Energy, amplitude, and dimensions of measures

For every Borel set $E \subset \mathbb{R}^d$, denote by $\mathcal{M}(E)$ the collection of nonzero compactly supported Radon measures on E.

The well-known Frostman lemma implies that, for every Borel subset $E \subset \mathbb{R}^d$ and every $s < \dim_{\mathcal{H}} E$, there exists $\mu \in \mathcal{M}(E)$ such that

$$\mu(B(x,r)) \lesssim r^s, \quad \text{for all } r > 0 \text{ and all } x \in \mathbb{R}^d.$$

In fact,

$$\dim_{\mathcal{H}} E = \sup \left\{ s : \exists\, \mu \in \mathcal{M}(E) \text{ such that } \sup_{x,r} \frac{\mu(B(x,r))}{r^s} < \infty \right\}.$$

We call

$$\sup_{x} \frac{\mu(B(x,r))}{r^s}$$

the Frostman constant of μ (of dimension s).

By direct computation, the above implies that for every $s < \dim_{\mathcal{H}} E$, there exists $\mu \in \mathcal{M}(E)$ such that the s-energy

$$I_s(\mu) := \iint |x-y|^{-s}\, d\mu(x)\, d\mu(y)$$

is finite. Also

$$\dim_{\mathcal{H}} E = \sup\{ s : \exists\, \mu \in \mathcal{M}(E) \text{ such that } I_s(\mu) < \infty \}. \tag{3.1}$$

In dimension theories, s-energy plays an important role due to its Fourier-analytic representation (see, e.g., Section 3.5 in [15])

$$I_s(\mu) = C_{d,s} \int |\hat{\mu}(\xi)|^2\, |\xi|^{-d+s}\, d\xi = C_{d,s}\, \|(-\Delta)^{-(d-s)/4} \mu\|_{L^2}^2.$$

In fact, Kaufman's simple alternative proof of Marstrand projection theorem is just

$$\begin{aligned} \int_{S^1} \|\pi_e \mu\|_{L^2(\mathbb{R})}^2\, d\sigma(e) &= \iint |\widehat{\pi_e \mu}(r)|^2\, dr\, d\sigma(e) = \iint |\hat{\mu}(re)|^2\, dr\, d\sigma(e) \\ &= \int |\hat{\mu}(\xi)|^2 |\xi|^{-1}\, d\xi = C I_1(\mu). \end{aligned} \tag{3.2}$$

So far, in the literature, all estimates on Frostman measures are written in terms of the Frostman constant and the energy . But these are not enough for our analytic interpolation.

For our use, estimates should hold with complex-valued μ, or more precisely, μ_z^δ defined in (2.4). The s-energy works well with μ_z^δ: thanks to its Fourier-analytic representation, we have that, for $\psi \in C_0^\infty$ and μ_z^δ defined as (2.4),

$$I_s(\mu_z^\delta) \lesssim_{\psi,d,s,\operatorname{Re} z} I_{s-2\operatorname{Re} z}(\mu). \tag{3.3}$$

To prove (3.3), notice $\mu_z^\delta \in C_0^\infty$, and therefore

$$\begin{aligned}
I_s(\mu_z^\delta) &= C_{d,s} \int |\widehat{\mu_z^\delta}(\xi)|^2 \, |\xi|^{-d+s} \, d\xi \\
&= C_{d,s} \cdot \left| \frac{e^{z^2} \pi^{(d-z)/2}}{\Gamma((d-z)/2)} \right|^2 \int \left| \int \hat{\mu}(\eta) \, |\eta|^{-z} \, |\hat{\phi}(\delta\eta)| \, \hat{\psi}(\xi-\eta) \, d\eta \right|^2 |\xi|^{-d+s} \, d\xi \\
&\lesssim \iint |\hat{\mu}(\eta)|^2 \, |\eta|^{-2\operatorname{Re} z} \, |\hat{\psi}(\xi-\eta)| \, d\eta \, |\xi|^{-d+s} \, d\xi,
\end{aligned}$$

where the implicit constant depends on ψ, d, s and $\operatorname{Re} z$, but is independent in δ. Then, when $|\xi| > |\eta|/2$,

$$\int_{|\xi|>|\eta|/2} |\hat{\psi}(\xi-\eta)| \, |\xi|^{-d+s} \, d\xi \lesssim |\eta|^{-d+s} \int |\hat{\psi}(\xi-\eta)| \, d\xi \lesssim |\eta|^{-d+s}.$$

When $|\xi| < |\eta|/2$, we have $|\xi-\eta| \gtrsim |\eta|$, and therefore

$$\int_{|\xi|<|\eta|/2} |\hat{\psi}(\xi-\eta)| \, |\xi|^{-d+s} \, d\xi \lesssim_N \int_{|\xi-\eta|\gtrsim|\eta|} |\xi-\eta|^{-N} \, |\xi|^{-d+s} \, d\xi \lesssim_N (1+|\eta|)^{s-N},$$

where the last inequality follows from changing variables $\xi = |\eta|\zeta$. This completes the proof of (3.3).

Though the s-energy is compatible with μ_z^δ, there seems no easy way to deal with the Frostman constant of μ_z^δ, namely

$$\sup_x \frac{|\mu_z^\delta(B(x,r))|}{r^s}.$$

This is why we introduce the s-amplitude defined in Definition 1.1. This definition is very natural. In fact, (3.1), the connection between $\dim_{\mathcal{H}} E$ and $I_s(\mu)$, is built upon $A_s(\mu) < \infty$ (see, for example, Theorem 2.8 in [15]), that immediately implies

$$\dim_{\mathcal{H}} E = \sup\{s : \exists\, \mu \in \mathcal{M}(E),\ A_s(\mu) < \infty\}.$$

However, there seems no further discussion about this quantity in the literature.

Unlike Frostman constant, one can expect the s-amplitude to act on μ_z^δ, due to the semigroup property of the Riesz potential (see, e.g., p. 48 in [10]): heuristically, if $0 < s - \operatorname{Re} z < d$, then

$$\begin{aligned}
A_s\Big(\frac{\pi^{z/2}}{\Gamma(z/2)} |\cdot|^{-d+z} * \mu^\delta\Big) &= \left|\frac{\pi^{z/2}}{\Gamma(z/2)}\right| \cdot \big\| |\cdot|^{-s} * |\cdot|^{-d+z} * \mu^\delta \big\|_{L^\infty} \\
&= \left| \frac{e^{z^2} \pi^{\frac{d+z}{2}} \cdot \Gamma(\frac{d-s}{2}) \cdot \Gamma(\frac{s-z}{2})}{\Gamma(\frac{s}{2}) \cdot \Gamma(\frac{d-z}{2}) \cdot \Gamma(\frac{d-(s-z)}{2})} \right| \cdot \big\| |\cdot|^{-s+z} * \mu^\delta \big\|_{L^\infty} \\
&\le C_{d,s,\operatorname{Re} z} \cdot A_{s-\operatorname{Re} z}(\mu^\delta).
\end{aligned} \tag{3.4}$$

To make this heuristic argument rigorous, some functional analysis is required. For simplicity, we keep (3.4) in mind and write our proof in Section 7 in a slightly different way to avoid tedious discussion.

It is routine to consider μ^δ first and take $\delta \to 0$ at the very end. One can check that all implicit constants below are independent in δ. From now, we write μ for μ^δ for abbreviation, and assume μ has continuous density.

4. Trivial estimates on orthogonal projections

4.1.

The finiteness for $p = q = 1$ is trivial. In fact, for every finite measure μ of continuous density, we have

$$\int \pi^y \mu(V)\, d\gamma_{d,n}(V) \approx 1, \quad \text{for all } y \notin \operatorname{supp} \mu. \tag{4.1}$$

This also implies that for μ of singular support, one can define $\pi^y \mu$ as a measure on $G(d,n)$ by taking the weak limit of a subsequence of $\pi^y \mu^\delta$. However, as we mentioned at the end of the previous section, it is more convenient to take this limit at the very end, after all estimates are proved. So we continue working with μ of continuous density.

For the proof of (4.1), since $\gamma_{d,n}$ is induced by the Haar measure θ_d on the orthogonal group $O(d)$, by its invariance

$$\begin{aligned}
\int \pi^y \mu(V)\, d\gamma_{d,n}(V) &= \int_{O(d)} \int_{\mathbb{R}^n} \mu(y + g \cdot (x', 0))\, dx'\, d\theta_d(g) \\
&= \int_{O(d)} \int_{S^{n-1}} \int_0^\infty \mu(y + g \cdot (r\sigma, 0))\, r^{n-1}\, dr\, d\sigma\, d\theta_d(g) \\
&= \int_{S^{n-1}} \int_0^\infty \Big(\int_{O(d)} \mu(y + g \cdot (r\sigma, 0))\, d\theta_d(g) \Big) r^{n-1}\, dr\, d\sigma \\
&= \frac{|S^{n-1}|}{|S^{d-1}|} \int_{S^{d-1}} \int \mu(y + r\sigma)\, r^{n-1}\, dr\, d\sigma \\
&= \frac{|S^{n-1}|}{|S^{d-1}|} \int \mu(y + x)\, |x|^{n-d}\, dx \approx 1. \qquad (4.2)
\end{aligned}$$

As a consequence, if $\operatorname{supp} \mu$ and $\operatorname{supp} \nu$ are disjoint,

$$\iint \pi^y \mu(V)\, d\gamma_{d,n}(V)\, d\nu(y) \approx 1.$$

4.2.

The trivial estimate (4.2) looks perfect and there seems nothing more to discuss. But here we would like to present another trivial estimate of

$$\iint \pi^y \mu(V)\, d\gamma_{d,n}(V)\, d\nu(y)$$

that inspires our argument below.

Assume both μ and ν have continuous density. Then, by the definition of $\pi^y\mu(V)$ (see (1.5)), with $s+t=2n$ and $s>n$,

$$
\begin{aligned}
&\iint \pi^y\mu(V)\,d\gamma_{d,n}(V)\,d\nu(y)\\
&\quad = \iint \pi_V\mu(u)\cdot\pi_V\nu(u)\,d\mathcal{H}^n(u)\,d\gamma_{d,n}(V)\\
&\quad \le \left(\int_{G(d,n)}\int_{\mathbb{R}^n} |\widehat{\pi_V\mu}(\xi)|^{-n+s}\,d\xi\,d\gamma_{d,n}(V)\right)^{1/2} \|(-\Delta)^{-(n-t)/4}\pi_V\nu\|_{L^2(\mathcal{H}^n\times\gamma_{d,n})}\\
&\quad = C\cdot I_s^{1/2}(\mu)\cdot I_t^{1/2}(\nu). \qquad (4.3)
\end{aligned}
$$

The second line of (4.3) follows because in general, given $\alpha\in(0,n)$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} f\,\bar{g} = \int_{\mathbb{R}^n}\hat{f}\,\bar{\hat{g}} &= \int_{\mathbb{R}^n}\hat{f}(\xi)|\xi|^{\alpha}\,\overline{\hat{g}(\xi)}|\xi|^{-\alpha}\,d\xi\\
&\le \left(\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2|\xi|^{2\alpha}\,d\xi\right)^{1/2}\|(-\Delta)^{-\alpha/2}g\|_{L^2}.
\end{aligned}
\qquad (4.4)
$$

The last line of (4.3) follows because for arbitrary but fixed $V_0\in G(d,n)$,

$$
\begin{aligned}
&\iint |\widehat{\pi_V\mu}(\xi)|^2\,|\xi|^{\alpha}\,d\xi\,d\gamma_{d,n}(V) = \iint |\widehat{\pi_{gV_0}\mu}(\xi)|^2\,|\xi|^{\alpha}\,d\xi\,d\theta_d(g)\\
&\quad = \iint_{gV_0}|\hat{\mu}(\xi)|^2\,|\xi|^{\alpha}\,d\mathcal{H}^n(\xi)\,d\theta_d(g) = \iint_{V_0}|\hat{\mu}(g\xi)|^2\,|\xi|^{\alpha}\,d\mathcal{H}^n(\xi)\,d\theta_d(g),
\end{aligned}
\qquad (4.5)
$$

which equals a constant multiple of $I_{n+\alpha}(\mu)$ by applying polar coordinate on V_0 and integrate in θ_d first as in (4.2).

The estimate (4.3) looks useless, as we already know it is ≈ 1 even if the energy blows up. However, it inspires our proof in Section 5, which is the beginning of this project. We hope that presenting this "trivial" estimate here would help readers have a better understanding of this paper.

5. The case $q=1$

We start our proof with $q=1$. This is also how we discover jumps of p at $s+t=2n$.

Proposition 5.1. *Suppose μ is a complex-valued C_0^∞ function on $\mathbb{R}^d$, ν is a compactly supported measure on $\mathbb{R}^d$, and $0<t<n$. Then*

$$
\int \|\pi^y\mu\|_{L^p(G(d,n))}\,d\nu(y) \lesssim_{d,n,p,t} I_{2n-t}(\mu)^{1/2}\cdot I_t(\nu)^{1/2} \qquad (5.1)
$$

for every

$$
1\le p<\frac{2n}{n+t}.
$$

Furthermore,

$$
\int \|\pi^y\mu\|_{L^1(G(d,n))}\,d\nu(y) \lesssim_{d,n} I_n(\mu)^{1/2}\cdot I_n(\nu)^{1/2}. \qquad (5.2)
$$

By considering $\nu * \phi_\delta \to \nu$, we may assume ν has continuous density. The reason we work with complex-valued μ is for the analytic interpolation in Section 7. It brings extra difficulties, and even the case $p = 1$ is no longer trivial. More precisely, if we follow the argument in Section 4,

$$\begin{aligned}\int \|\pi^y \mu\|_{L^1(G(d,n))}\, d\nu(y) &= \int \int_{G(d,n)} |\pi^y \mu(V)|\, d\gamma_{d,n}(V)\, d\nu(y) \\ &= \iint |\pi_V \mu(u)| \cdot \pi_V \nu(u)\, d\mathcal{H}^n(u)\, d\gamma_{d,n}(V),\end{aligned} \tag{5.3}$$

getting stuck due to the absolute value symbol. The following argument overcomes this obstacle, and eventually proves Proposition 5.1.

Now we start our proof. The endpoint case $t = n$, $p = 1$ follows directly from taking Cauchy–Schwarz of (5.3). So we assume $0 < t < n$ and $1 < p < \frac{2n}{n+t} < 2$.

To estimate $\|\pi^y \mu\|_{L^p(G(d,n))}$, $1 \le p < \infty$, the most natural way is to consider

$$\sup_{\|f\|_{L^{p'}}=1} \left| \int \pi^y \mu(V) \cdot f(V)\, d\gamma_{d,n}(V) \right|.$$

But this setup is not convenient to us due to the integral in $d\nu(y)$ outside the supremum. Instead, we take f to be the maximizer directly, namely,

$$f_y(V) = f(V, y) = \frac{\operatorname{sgn}(\pi^y \mu(V)) \cdot |\pi^y \mu(V)|^{p-1}}{\|\pi^y \mu\|^{p-1}_{L^p(G(d,n))}},$$

with

$$\operatorname{sgn}(\pi^y \mu(V)) := \frac{\pi^y \mu(V)}{|\pi^y \mu(V)|} \cdot \chi_{\pi^y \mu(V) \neq 0}(V, y).$$

Then the mixed-norm in Proposition 5.1 is reduced to

$$\iint \pi^y \mu(V) \cdot f(V, y)\, d\gamma_{d,n}(V)\, d\nu(y) \tag{5.4}$$

with $\|f(\cdot, y)\|_{L^{p'}} = 1$, $\forall\, y$.

We first fix V, parametrize $\mathbb{R}^d$ by $(u, v) \in V \oplus V^\perp$ and integrate in $v \in V^\perp$ first. Since $\pi^y \mu(V) = \pi_V \mu(\pi_V y)$ is a constant in each level set $y \in \pi_V^{-1}(u)$, the integral (5.4) equals

$$\int_{G(d,n)} \int_V \pi_V \mu(u) \Big(\int_{y \in \pi_V^{-1}(u)} f(V, y)\, \nu(y)\, d\mathcal{H}^{d-n}(y) \Big)\, d\mathcal{H}^n(u)\, d\gamma_{d,n}(V). \tag{5.5}$$

Denote

$$F(V, u) := \int_{y \in \pi_V^{-1}(u)} f(V, y)\, \nu(y)\, d\mathcal{H}^{d-n}(y). \tag{5.6}$$

Then, similar to (4.3), the integral (5.5) is reduced to

$$\begin{aligned}&\int_{G(d,n)} \int_V \pi_V \mu(u)\, F(V, u)\, d\mathcal{H}^n(u)\, d\gamma_{d,n}(V) \\ &\qquad \le I^{1/2}_{2n-t}(\mu) \cdot \|(-\Delta_u)^{-(n-t)/4} F\|_{L^2(\mathcal{H}^n \times \gamma_{d,n})}.\end{aligned} \tag{5.7}$$

The first factor is desirable, so it remains to consider the second. Since $0 < t < n$, it follows that

$$\begin{aligned}\|(-\Delta)^{-(n-t)/4}F\|_{L^2}^2 =& \int\int_{\mathbb{R}^n} |\hat{F}(V,\xi)|^2\, |\xi|^{-n+t}\, d\xi\, d\gamma_{d,n}(V) \\ =& C_{d,n,s} \iiint |u-u'|^{-t}\, F(V,u)\, F(V,u')\, du\, du'\, d\gamma_{d,n}(V).\end{aligned} \tag{5.8}$$

By the definition of F in (5.6) and again $\mathbb{R}^d = V \oplus V^{\perp}$, (5.8) equals, up to a multiplicative constant,

$$\iiint |\pi_V(y-y')|^{-t}\, f(V,y)\, d\nu(y)\, f(V,y')\, d\nu(y')\, d\gamma_{d,n}(V). \tag{5.9}$$

Now fix $y \neq y'$ and integrate over $G(d,n)$ first. Since $1 < p < 2$ and $\|f(\cdot,y)\|_{L^{p'}} = 1$ for every y, it follows that

$$\|f(\cdot,y)f(\cdot,y')\|_{L^{p'/2}} \leq 1, \quad \text{for all } y, y',$$

with $1 < p'/2 < \infty$. Therefore, by Hölder's inequality,

$$\begin{aligned}&\int |\pi_V(y-y')|^{-t}\, f(V,y)\, f(V,y')\, d\gamma_{d,n}(V) \\ &\leq \Big(\int |\pi_V(y-y')|^{-t\cdot(p'/2)'}\, d\gamma_{d,n}(V)\Big)^{1/(p'/2)'}.\end{aligned} \tag{5.10}$$

It is well known (see, for example, Theorem 3.12 in [14]) that (5.10) is integrable and

$$\lesssim_{d,n,p,s} |y-y'|^{-t} \quad \text{if } t\cdot(p'/2)' < n.$$

After computation, this is equivalent to

$$p < \frac{2n}{n+t},$$

which completes the proof of Proposition of 5.1.

6. The case $q = k$

Proposition 5.1 only works when $p < 2$. This is not satisfactory. In this section, we prove results for large p. For technical reasons, both μ and ν have to be positive. See (6.5) in the proof below.

Proposition 6.1. *Suppose μ and ν are compactly supported Radon measures on $\mathbb{R}^d$, and let $k \geq 2$, $0 < t < n$ and $0 < s, \alpha < d$ be real numbers satisfying*

$$s + t = 2n + 2(k-1)(d-\alpha). \tag{6.1}$$

Then

$$\int \|\pi^y \mu\|^k_{L^p(G(d,n))}\, d\nu(y) \lesssim_{d,n,p,k,s,t} I_s(\mu)^{1/2} \cdot A_\alpha(\mu)^{k-1} \cdot I_t(\nu)^{1/2} \tag{6.2}$$

for every

$$1 \le p < \frac{2nk}{n+t}.$$

Furthermore, when $t = n$,

$$\int \|\pi^y \mu\|^k_{L^k(G(d,n))}\, d\nu(y) \lesssim_{d,n,k,s} I_s(\mu)^{1/2} \cdot A_\alpha(\mu)^{k-1} \cdot I_n(\nu)^{1/2}. \tag{6.3}$$

As $G(d,n)$ is compact, we may assume $p \ge k$ and denote $p_k := p/k \ge 1$. Write

$$\int \|\pi^y \mu\|^k_{L^p(G(d,n))}\, d\nu(y) = \int \|(\pi^y \mu)^k\|_{L^{p_k}(G(d,n))}\, d\nu(y).$$

We shall study the p_k-norm of the k-linear form $\prod_{j=1}^k \pi^y \mu_j$, with $\mu_j = \mu$.

If one repeats the argument in Section 5, there is an obstacle that (4.4) does not work for multi-linear forms. To overcome this difficulty, we take ideas from Greenleaf–Iosevich [6] and Grafakos–Greenleaf–Iosevich–Palsson [5], that are already sketched in Section 3. Compared with their multi-linear estimates, our case is more subtle because of the mixed-norm. We should be super careful, especially on the timing of taking absolute values.

Now we start the proof of Proposition 6.1. Again we assume both μ and ν have continuous density. With $\mu_j = \mu$, $j = 1, \dots, k$, similar to Section 5 it suffices to consider

$$\iint \prod_{j=1}^k \pi^y \mu_j(V) \cdot f(V,y)\, d\gamma_{d,n}(V)\, d\nu(y), \tag{6.4}$$

with $\|f(\cdot, y)\|_{L^{p_k'}} = 1$, $\forall\, y$.

Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be nonnegative and equal to 1 on $\operatorname{supp}\mu$, and define

$$\Phi(z_1, \dots, z_k) := \iint \prod_{j=1}^k \pi^y \mu_{z_j}(V) \cdot f(V,y)\, d\gamma_{d,n}(V)\, d\nu(y)$$

to be an analytic function on $\mathbb{C}^k$, where $\mu_z(x)$ is defined as in (2.4). Similar to Section 5, since $\pi_V \mu_z$ is a constant in each level set $\pi_V^{-1}(u)$, one can write $\Phi(z_1, \dots, z_k)$ as

$$\int_{G(d,n)} \int_V \pi_V \mu_{z_{j_0}}(u) \Big(\int_{y \in \pi_V^{-1}(u)} \prod_{j \ne j_0} \pi^y \mu_{z_j}(V)\, f(V,y)\, \nu(y)\, d\mathcal{H}^{d-n}(y) \Big) \times d\mathcal{H}^n(u)\, d\gamma_{d,n}(V),$$

for arbitrary $j_0 \in \{1, \dots, k\}$. Take

$$\operatorname{Re} z_j = d - \alpha \quad \text{for } j \ne j_0, \quad \text{and} \quad \operatorname{Re} z_{j_0} = -\sum_{j \ne j_0} \operatorname{Re} z_j = -(k-1)(d-\alpha).$$

Since μ is positive,

$$(6.5)\quad |\mu_{z_j}(x)| = \left|\psi(x)\cdot \frac{e^{z_j^2}\pi^{z_j/2}}{\Gamma(z_j/2)}\int |x-y|^{-d+z_j}\,d\mu(y)\right| \lesssim_{d,\alpha} A_\alpha(\mu), \quad \text{for all } j\neq j_0.$$

Denote

$$(6.6)\qquad F(V,u) := \int_{y\in\pi_V^{-1}(u)} \prod_{j\neq j_0} \pi^y\mu_{z_j}(V)\, f(V,y)\,\nu(y)\,d\mathcal{H}^{d-n}(y).$$

Then, as in (4.3) and (5.7),

$$(6.7)\qquad \begin{aligned} |\Phi(z_1,\dots,z_k)| &= \left|\iint \pi_V\mu_{z_{j_0}}(u)\,F(V,u)\,du\,dV\right| \\ &\le I_{2n-t}(\mu_{z_{j_0}})^{1/2}\cdot \|(-\Delta)^{-(n-t)/4}F\|_{L^2}. \end{aligned}$$

By (6.1), we have

$$2n-t = s-2(k-1)(d-\alpha) = s+2\,\mathrm{Re}\,z_{j_0},$$

so by (3.3), the square of the first factor is

$$I_{2n-t}(\mu_{z_{j_0}}) \lesssim_t I_{2n-t-2\,\mathrm{Re}\,z_{j_0}}(\mu) = I_s(\mu),$$

as desired.

It remains to estimate the second factor in (6.7). Now we can feel free to take absolute values. By (6.6) and (6.5),

$$(6.8)\qquad |F(V,u)| \lesssim_{d,\alpha,\,\mathrm{diam(supp}\,\mu)} A_\alpha(\mu)^{k-1}\cdot \int_{y\in\pi_V^{-1}(u)} |f(V,y)|\,\nu(y)\,d\mathcal{H}^{d-n}(y).$$

Then the special case (6.3) follows quickly: when $t=n$ and $p_k=1$, we have that $\|f(\cdot,y)\|_{L^\infty}=1$, for all y, and therefore

$$\|F\|_{L^2(\mathcal{H}^n\times\gamma_{d,n})} \lesssim A_\alpha(\mu)^{k-1}\cdot\|\pi_V\nu\|_{L^2(\mathcal{H}^n\times\gamma_{d,n})} \lesssim A_\alpha(\mu)^{k-1}\cdot I_n(\nu)^{1/2},$$

as desired.

We now prove (6.2). When $0<t<n$, as in Section 5, we write $\|(-\Delta)^{-(n-t)/4}F\|_{L^2}^2$ to be

$$(6.9)\qquad \begin{aligned} &\int\int_{\mathbb{R}^n} |\hat F(V,\xi)|^2\,|\xi|^{-n+t}\,d\xi\,d\gamma_{d,n}(V) \\ &\qquad = \iiint |u-u'|^{-t}\,F(V,u)\,\overline{F(V,u')}\,du\,du'\,dV. \end{aligned}$$

By plugging (6.8) into (6.9), we obtain the upper bound

$$A_\alpha^{2(k-1)}(\mu)\cdot\iiint |\pi_V(y-y')|^{-t}\,|f(V,y)|\,d\nu(y)\,|f(V,y')|\,d\nu(y')\,d\gamma_{d,n}(V).$$

The rest is the same as Section 5, and eventually gets to

$$p/k = p_k < \frac{2n}{n+t}.$$

As $j_0 \in \{1, \dots, k\}$ is arbitrary, the estimate holds for k non-proportional vectors $(\operatorname{Re} z_1, \dots, \operatorname{Re} z_k) \in \mathbb{R}^k$ whose sum equals $\vec{0}$. This implies the origin lies in their convex hull. Hence by Lemma 2.1,

$$|\Phi(0)| = \left| \iint \prod_{j=1}^{k} \pi^y \mu_j(V) \cdot f(V,y)\, d\gamma_{d,n}(V)\, d\nu(y) \right| \lesssim I_s(\mu)^{1/2} \cdot A_\alpha(\mu)^{k-1} \cdot I_t(\nu)^{1/2},$$

that completes the proof of Proposition 6.1.

7. Analytic interpolation

The previous sections already constitute an interesting paper. One can even prove a weaker version of Corollary 1.4 from Proposition 6.1 as a geometric application. But if we compute the range of p for

$$\pi^y \mu \in L^p(G(d,n)),$$

there is something strange. Suppose μ and ν are Frostman measures satisfying

$$\mu(B(x,r)) \lesssim r^{s_\mu}, \ \nu(B(x,r)) \lesssim r^{s_\nu}, \quad \text{for all } x \in \mathbb{R}^d \text{ and all } r > 0,$$

with $0 < s_\nu < n$ and $2n - s_\nu < s_\mu < 2n$. It follows from Propositions 5.1 and 6.1 that

$$\pi^y \mu(V) \in L^p(G(d,n)), \quad \text{for some } y \in \operatorname{supp} \nu,$$

for every

$$1 \le p < \max_k \frac{2nk}{3n - s_\mu + 2(k-1)(d-s_\mu)}, \tag{7.1}$$

where k is taken over all positive integers satisfying

$$s_\mu + s_\nu \ge 2n + 2(k-1)(d - s_\mu). \tag{7.2}$$

By solving k from (7.1), we see that when $s_\mu \le 2d - 3n$ we should take k as small as possible, that is, $k = 1$, while when $s_\mu > 2d - 3n$ we should take k as large as possible, that is, by (7.2),

$$k = 1 + \left[\frac{s_\mu + s_\nu - 2n}{2(d - s_\mu)} \right],$$

where $[\cdot]$ denotes the integer part. This means the range of p has jump discontinuities at each $\frac{1}{2}(s_\mu + s_\nu - 2n)/(d - s_\mu) \in \mathbb{Z}_+$. We have seen in the introduction that jumps at $s_\mu + s_\nu - 2n = 0$ do exist. However, there is no evidence to support jumps elsewhere.

We would like to drop the $[\cdot]$ symbol for a wider range of p. As we commented right after (1.8), traditional interpolations only on p and q do not help. To make it, we introduce a new technique that also interpolates dimensions of measures. For technical reasons, one cannot interpolate between the estimates in Proposition 6.1 directly. We have pointed out that both μ and ν there have to be positive. In fact, for the proof of Theorem 1.2, readers can skip Section 6 and read this section directly. However, without Proposition 6.1 and the observation above, there is no way to know what to prove. For later use of this mechanism, one can first obtain an analog of Proposition 6.1 on the scratch paper, and then prove desired estimates by arguments below.

We shall see how s-amplitude helps us, and why we consider s and α to be possibly different in Proposition 6.1.

Proposition 7.1. *Suppose μ and ν are compactly supported Radon measures on $\mathbb{R}^d$, and let $0 < t < n$, $0 < s, \alpha < d$ and $s + t \geq 2n$. Denote*

$$q := 1 + \frac{s+t-2n}{2(d-\alpha)}. \tag{7.3}$$

Then

$$\int \|\pi^y \mu\|^q_{L^p(G(d,n))} \, d\nu(y) \lesssim_{d,n,p,s,t,\alpha} I_s^{1/2}(\mu) \cdot A_\alpha(\mu)^{q-1} \cdot I_t(\nu)^{1/2} \tag{7.4}$$

for every

$$1 \leq p < \frac{2nq}{n+t} = \frac{2n}{n+t}\Big(1 + \frac{s+t-2n}{2(d-\alpha)}\Big). \tag{7.5}$$

Furthermore, when $t = n$,

$$\int \|\pi^y \mu\|^q_{L^q(G(d,n))} \, d\nu(y) \lesssim_{d,n,s,\alpha} I_s(\mu)^{1/2} \cdot A_\alpha(\mu)^{q-1} \cdot I_n(\nu)^{1/2}. \tag{7.6}$$

The case $q = 1$ is already done in Proposition 5.1. So, as $t < n$ and the integrand is compactly supported, we may assume $p > q > 1$.

It suffices to consider

$$\iint \pi^y \mu(V) \cdot f(V, y) \, d\gamma_{d,n}(V) \, g(y) \, d\nu(y),$$

with $\|f(\cdot, y)\|_{L^{p'}} = 1$, for all y, and $\|g\|_{L^{q'}(\nu)} = 1$.

For every $z \in \mathbb{C}$, let

$$s_z = s + 2z, \quad \alpha_z = \alpha + z,$$
$$q_z = 1 + \frac{s_z + t - 2n}{2(d - \alpha_z)} = \frac{2(d-\alpha) + s + t - 2n}{2(d - \alpha - z)}, \quad p_z = \frac{p}{q} q_z.$$

Then we make the following observations:

- $s_0 = s, \alpha_0 = \alpha, q_0 = q, p_0 = p$;
- For all $z \in \mathbb{C}$,
$$s_z + t = 2n + 2(q_z - 1)(d - \alpha_z),$$
and particularly,
$$s_{\mathrm{Re}\, z} + t = 2n + 2(k-1)(d - \alpha_{\mathrm{Re}\, z}), \quad \text{if } q_z = k \in \mathbb{Z}_+; \tag{7.7}$$
- both $1/p_z$ and $1/q_z$ are linear in z.

The linearity of $1/p_z$ and $1/q_z$ in z is important to us, because it guarantees that $|e^{1/p_z}| = e^{1/p_{\operatorname{Re} z}}$ and $|e^{1/q_z}| = e^{1/q_{\operatorname{Re} z}}$.

Now take

$$\Phi(z) = \iint \pi^y \mu_z(V) \cdot f_z(V, y)\, d\gamma_{d,n}(V)\, g_z(y)\, d\nu(y)$$

as an analytic function, where μ_z is defined as (2.4),

$$f_z(V, y) := \operatorname{sgn}(f) \cdot |f(V, y)|^{p'(1-1/p_z)} \quad \text{and} \quad g_z(y) := \operatorname{sgn}(g) \cdot |g(y)|^{q'(1-1/q_z)}.$$

Since both $1/p_z$ and $1/q_z$ are linear in z,

$$\begin{aligned} \|f_z(\cdot, y)\|_{L^{(p_{\operatorname{Re} z})'}} &= \left\| |f(\cdot, y)|^{p'(1-1/p_{\operatorname{Re} z})} \right\|_{L^{(p_{\operatorname{Re} z})'}} = 1, \quad \text{for all } y, \\ \|g_z\|_{L^{q'_{\operatorname{Re} z}}} &= \left\| |g|^{q'(1-1/q_{\operatorname{Re} z})} \right\|_{L^{q'_{\operatorname{Re} z}}} = 1. \end{aligned} \tag{7.8}$$

Heuristically, if Proposition 6.1 could be applied to μ_z, it would imply that for each $q_{\operatorname{Re} z} \in \mathbb{Z}_+$,

$$\begin{aligned} |\Phi(z)|^{q_{\operatorname{Re} z}} &\lesssim I^{1/2}_{\operatorname{Re} s_z}(\mu_z) \cdot A_{\operatorname{Re} \alpha_z}(\mu_z)^{q_{\operatorname{Re} z}-1} \cdot I_t(\nu)^{1/2} \\ &\lesssim I_{\operatorname{Re}(s_z - 2z)}(\mu)^{1/2} \cdot A_{\operatorname{Re}(\alpha_z - z)}(\mu)^{q_{\operatorname{Re} z}-1} \cdot I_t(\nu)^{1/2} \\ &= I_s(\mu)^{1/2} \cdot A_\alpha(\mu)^{q_{\operatorname{Re} z}-1} \cdot I_t(\nu)^{1/2}. \end{aligned} \tag{7.9}$$

Recall that $q > 1$ and that $1/q_z$ is linear in z. So $q_{\operatorname{Re} z}$ is monotonic in $\operatorname{Re} z$, and there must exist $a_1 < 0 < a_2$ such that $q_{a_1}, q_{a_2} \in \mathbb{Z}_+$. Therefore one can apply the Hadamard three-lines lemma to the analytic function

$$\frac{\Phi(z)}{I_s(\mu)^{1/(2q_z)} \cdot A_\alpha(\mu)^{1-1/q_z} \cdot I_t(\nu)^{1/(2q_z)}},$$

whose absolute value is $\lesssim 1$ when $\operatorname{Re} z = a_1 < 0$ and $\operatorname{Re} z = a_2 > 0$. Consequently,

$$|\Phi(0)| \lesssim I_s(\mu)^{1/(2q)} \cdot A_\alpha(\mu)^{1-1/q} \cdot I_t(\nu)^{1/(2q)},$$

as desired.

In the following, we make the estimate (7.9) rigorous.

First consider $\operatorname{Re} z = -(s + t - 2n)/2 < 0$. In this case, $q_{\operatorname{Re} z} = 1$. Then by Hölder's inequality, (7.8), Proposition 5.1, and (3.3),

$$\begin{aligned} |\Phi(z)| &\leq \int \|\pi^y \mu_z\|_{L^{p_{\operatorname{Re} z}}(G(d,n))}\, d\nu(y) \lesssim I_{\operatorname{Re} s_z}(\mu_z)^{1/2} \cdot I_t(\nu)^{1/2} \\ &\lesssim_{d,s,\operatorname{Re} z} I_s(\mu)^{1/2} \cdot I_t(\nu)^{1/2}. \end{aligned}$$

We also need estimates with $\operatorname{Re} z > 0$. Since $q_{\operatorname{Re} z} \to \infty$ as $\operatorname{Re} z \to d - \alpha > 0$, there exists $\operatorname{Re} z > 0$ such that $q_{\operatorname{Re} z}$ is an integer $k \geq 2$. Recall we have assumed $p > q > 1$, so $p_{\operatorname{Re} z} > q_{\operatorname{Re} z} = k$. In this case, by Hölder's inequality and (7.8),

$$\begin{aligned} |\Phi(z)|^k &\leq \int \|\pi^y \mu_z\|^k_{L^{p_{\operatorname{Re} z}}(G(d,n))}\, d\nu(y) \\ &= \iint \pi^y \mu_z(V) \cdots \pi^y \mu_z(V) \cdot h(V, y)\, d\gamma_{d,n}(V)\, d\nu(y), \end{aligned}$$

where

$$\|h(\cdot, y)\|_{L^{(p_{\mathrm{Re}\,z}/k)'}} = 1, \quad \text{for all } y.$$

Fix z. Our intuition for the next step is to follow the idea from the previous section to consider

$$\iint \prod_{j=1}^{k} \pi^y(\mu_z)_{w_j}(V) \cdot h(V, y)\, d\gamma_{d,n}(V)\, d\nu(y).$$

With the semigroup property (3.4) in mind, one should expect $(\mu_z)_{w_j} = \mu_{z+w_j}$ up to a complex-valued factor. However, as we explained after (3.4), this requires some functional analysis. To make the proof simpler, instead we directly consider

$$\Psi_z(w_1, \ldots, w_k) := \iint \prod_{j=1}^{k} \pi^y(\mu_{z+w_j})(V) \cdot h(V, y)\, d\gamma_{d,n}(V)\, d\nu(y),$$

that is analytic in $\mathbb{C}^k$, because by our definition of μ_z in (2.4), the integrand is compactly supported. For an arbitrary but fixed $1 \le j_0 \le k$, take

$$\mathrm{Re}\, w_j = d - \mathrm{Re}\,\alpha_z, \quad \text{for } j \ne j_0, \quad \text{and} \quad \mathrm{Re}\, w_{j_0} = -\sum_{j \ne j_0} \mathrm{Re}\, z_j = -(k-1)(d - \mathrm{Re}\,\alpha_z).$$

Then, with

$$F(V, u) := \int_{y \in \pi_V^{-1}(u)} \prod_{j \ne j_0} \pi^y(\mu_{z+w_j})(V)\, h(V, y)\, \nu(y)\, d\mathcal{H}^{d-n}(y),$$

we can write $|\Psi_z(w_1, \ldots, w_k)|$ as

$$\begin{aligned} &\left| \int_{G(d,n)} \int_V \pi_V(\mu_{z+w_{j_0}})(u) \cdot F(V, u)\, d\mathcal{H}^n(u)\, d\gamma_{d,n}(V) \right| \\ &\qquad \le I_{2n-t}(\mu_{z+w_{j_0}})^{1/2} \cdot \|(-\Delta)^{-(n-t)/4} F\|_{L^2} \\ &\qquad \lesssim I_{2n-t-2\,\mathrm{Re}(z+w_{j_0})}(\mu)^{1/2} \cdot \|(-\Delta)^{-(n-t)/4} F\|_{L^2}, \end{aligned}$$

where the last inequality follows from (3.3).

Due to $q_{\mathrm{Re}\, z} = k$, (7.7) and our choice of w_{j_0}, the first factor equals $I_s(\mu)^{1/2}$, as desired.

It remains to estimate the second factor. We shall need the observation that

$$\|\mu_z\|_{L^\infty} \lesssim A_{d-\mathrm{Re}\, z}(\mu), \quad \text{if } \mathrm{Re}\, z \in (0, d), \tag{7.10}$$

where the implicit constant is independent in $\mathrm{Im}\, z$. See (6.5) above for the proof.

When $t = n$, it follows immediately that

$$\|F\|_{L^2(\mathcal{H}^n \times \gamma_{d,n})} \lesssim A_{\mathrm{Re}(\alpha_z - z)}(\mu)^{k-1} \cdot \|\pi_V \nu\|_{L^2(\mathcal{H}^n \times \gamma_{d,n})} \lesssim A_\alpha(\mu)^{k-1} \cdot I_n(\nu)^{1/2},$$

as desired.

When $0 < t < n$,

$$\begin{aligned}\|(-\Delta)^{-(n-t)/4}F\|_{L^2}^2 &= \iiint |u-u'|^{-t}\, F(V,u)\,\overline{F(V,u')}\,du\,du'\,dV\\ &\lesssim A_{\mathrm{Re}(\alpha_z - z)}^{2(k-1)}(\mu)\cdot \iiint |\pi_V(y-y')|^{-t}\,|h(V,y)|\,d\nu(y)\,|h(V,y')|\,d\nu(y')\,d\gamma_{d,n}(V)\\ &= A_\alpha^{2(k-1)}(\mu)\cdot \iiint |\pi_V(y-y')|^{-t}\,|h(V,y)|\,d\nu(y)\,|h(V,y')|\,d\nu(y')\,d\gamma_{d,n}(V).\end{aligned}$$

The rest is the same as Section 5. We omit details. Eventually, it ends up with

$$|\Psi_z(w_1,\dots,w_k)| \lesssim I_s(\mu)^{1/2}\cdot A_\alpha(\mu)^{k-1}\cdot I_t(\nu)^{1/2},$$

where the implicit constant is independent in the choice of j_0. By (3.3) and (7.10), this implicit constant is also independent in $\mathrm{Im}\, w$ and $\mathrm{Im}\, z$. Therefore, by Lemma 2.1, when $q_{\mathrm{Re}\, z} = k \geq 2$,

$$|\Phi(z)|^{q_{\mathrm{Re}\, z}} = |\Psi_z(0)| \lesssim I_s(\mu)^{1/2}\cdot A_\alpha(\mu)^{q_{\mathrm{Re}\, z}-1}\cdot I_t(\nu)^{1/2},$$

where the implicit constant is independent in $\mathrm{Im}\, z$.

Now we have estimates of $|\Phi(z)|$ at hand, for both positive and negative $\mathrm{Re}\, z$. Hence

$$|\Phi(0)|^q \lesssim I_s(\mu)^{1/2}\cdot A_\alpha(\mu)^{q-1}\cdot I_t(\nu)^{1/2}$$

by the Hadamard three-lines lemma again, which completes the proof of Proposition 7.1.

8. Proof of Corollaries 1.3 and 1.4: Optimize the range of p

To finally go from Proposition 7.1 to Theorem 1.2, it remains to broaden the range of q. We leave it to the next section. In fact, Proposition 7.1 is enough for Corollaries 1.3 and 1.4, as we should ignore q for the maximal possible p.

Proof of Corollary 1.3. Since we need

$$q_0 = 1 + \frac{s+t-2n}{2(d-\alpha)} \geq 1,$$

the assumption $s+t \geq 2n$ is required. Also, for the finiteness of $I_s(\mu)$ $A_\alpha(\mu)$ and $I_t(\nu)$, we need $0 < s, \alpha < s_\mu$ and $0 < t < s_\nu$. Therefore, the maximal p equals, up to the end point,

$$\sup_{\substack{0\leq s,\alpha\leq s_\mu\\ 0\leq t\leq s_\nu\\ s+t\geq 2n}} \frac{2n}{n+t}\Big(1+\frac{s+t-2n}{2(d-\alpha)}\Big). \tag{8.1}$$

The assumptions $0 < s_\nu < n$ and $2n - s_\nu < s_\mu < d$ ensure the supremum is well defined, namely not taken over an empty set. As parameters lie in a compact set, the supreme can be attained, say at s_0, α_0, t_0.

Notice that, for every fixed t, the critical p is an increasing function in s and α. Therefore $s_0 = \alpha_0 = s_\mu$. To find t_0, write

$$\frac{2n}{n+t}\Big(1+\frac{s_\mu+t-2n}{2(d-s_\mu)}\Big) = \frac{n}{d-s_\mu}\Big(1+\frac{2d-3n-s_\mu}{n+t}\Big).$$

To make this quantity large, one can see that, when $s_\mu \geq 2d-3n$, we should take t_0 as large as possible, namely $t_0 = s_\nu$; and when $s_\mu < 2d-3n$, we should take t_0 as small as possible, namely $t_0 = \max\{0, 2n-s_\mu\}$. This completes the proof of Corollary 1.3. ∎

Proof of Corollary 1.4. When $m=1$, it matches previous result on the visibility problem, so assume $m \geq 2$.

Let $F \subset \mathbb{R}^d$ denote the exceptional set. First we may assume $E \cap F = \emptyset$. This is because for every $\varepsilon > 0$, one can find disjoint compact subsets $E_1, E_2 \subset E$ satisfying $\dim_{\mathcal{H}} E_1, \dim_{\mathcal{H}} E_2 > \dim_{\mathcal{H}} E - \varepsilon$ (see, e.g., p. 59 and Theorem 8.13 in [14]), and then consider $\pi_y(E_1^m)$ if $y \in E_2$ and $\pi_y(E_2^m)$ if $y \in E_1$.

Next we point out that F is Borel. This follows because one can define $\gamma_{d,m}$ by counting the number of almost disjoint δ-"cubes" in the covering (like the Lebesgue measure), which implies the set $\{y \in \mathbb{R}^d \setminus E : \gamma_{d,m}(\pi^y(E^m)) > \varepsilon\}$ is Borel for every $\varepsilon > 0$.

Now we consider Frostman measures μ and ν on E and F, of exponents s_μ and s_ν, respectively. It suffices to show that when s_ν is large enough, then there exists $y \in F$ such that $\gamma_{d,m}(\pi^y(E^m)) > 0$, contradiction.

To prove our result, we shall consider the push-forward measure induced by the map

$$\Phi_y(x_1,\dots,x_m) = \mathrm{Span}\{x_1-y,\dots,x_m-y\} \in G(d,m).$$

However, the map Φ_y is not well defined for arbitrary $x_1,\dots,x_m$, because the vectors $x_1-y,\dots,x_m-y$ may be linearly dependent. Fortunately, as in either case of Corollary 1.4 one has $\dim_{\mathcal{H}} E > d-1$, we may assume $\dim_{\mathcal{H}} E > d-1$. Then there exist compact subsets $E_1,\dots,E_m \subset E$ such that $\mu(E_i) > 0$, for all i, and no m-tuple $(x_1,\dots,x_m) \in E_1 \times \cdots \times E_m$ lies in a m-dimensional affine subspace. This guarantees the map Φ_y is well defined on $E_1 \times \cdots \times E_m$, for all $y \in F$. We can also conclude that Φ_y has no critical point on $E_1 \times \cdots \times E_m$. This is convenient to us when changing variables later. For its proof, just consider the action of the orthogonal group on a neighborhood of $E_1 \times \cdots \times E_m$. From this point of view, if there exists a critical point in this neighborhood, then every point in this neighborhood is critical, which implies Φ_y is a constant map, contradiction.

Denote $\mu_i = \mu|_{E_i}$ and define a measure $\Phi_y(\mu_1 \times \cdots \times \mu_m)$ on $\pi^y(E^m)$ by

$$\int f(W)\, d\Phi_y(\mu_1 \times \cdots \times \mu_m)(W) = \int \cdots \int f(\Phi_y(x_1,\dots,x_m))\, d\mu_1(x_1)\dots d\mu_m(x_m).$$

Then, to show that $\gamma_{d,m}(\pi^y(E^m)) > 0$, it suffices to show the measure $\Phi_y(\mu_1 \times \cdots \times \mu_m)$ has $L^{\tilde{p}}$ density for some $\tilde{p} > 1$.

By considering $\mu * \phi_\delta \to \mu$, we may assume μ has continuous density. Since Φ_y is regular and $E_1 \times \cdots \times E_m$ is compact, by the co-area formula we have

$$\Phi_y(\mu_1 \times \cdots \times \mu_m)(W) \approx \int_{\Phi_y^{-1}(W)} \mu_1 \cdots \mu_m \, d\mathcal{H}^{m^2} = \prod_{i=1}^m \int_{y+W} \mu_i \, d\mathcal{H}^m \leq |\pi^y \mu(W^\perp)|^m.$$

Here $\approx$ holds because Φ_y has no critical point.

Therefore, to show $\Phi_y(\mu_1 \times \cdots \times \mu_m)$ has $L^{\tilde{p}}$ density, it suffices to show

$$\int |\pi^y \mu(W^\perp)|^{m\tilde{p}} \, d\gamma_{d,m}(W) = \int |\pi^y \mu(V)|^{m\tilde{p}} \, d\gamma_{d,d-m}(V) < \infty.$$

Denote $p = m\tilde{p}$.

To proceed, one could apply the multi-linear estimates from Proposition 6.1. However, as we already observed at the beginning of Section 7, the range of p would have jumps. This means one has to make s_μ and s_ν large enough to ensure the range of p jumps across m. In other words, the q associated to $p = m$ may not be an integer. This explains why our interpolation between multi-linear estimates gives a better bound than multi-linear estimates themselves. We hope it will shed lights on other multiple configuration problems in the future.

To finish the proof, we invoke Corollary 1.3 with $n = d - m$, and check when the range of p covers $m = d - n$. Recall $0 < s_\nu < n$ and $s_\mu + s_\nu > 2n$ are always required.

We first prove (i) in Corollary 1.4.

Since $s_\mu > 2d - 3n$, we need to solve

$$\frac{2n}{n + s_\nu}\left(1 + \frac{s_\mu + s_\nu - 2n}{2(d - s_\mu)}\right) > d - n.$$

This can be reduced to

$$(n + s_\nu)(n - (d - n)(d - s_\mu)) > n(s_\mu - 2d + 3n). \tag{8.2}$$

Since $n + s_\nu < 2n$ is required, (8.2) has a solution only if

$$n - (d - n)(d - s_\mu) > \frac{1}{2}(s_\mu - 2d + 3n) > 0,$$

equivalent to

$$s_\mu > d - \frac{d - n}{2(d - n) - 1} = d - \frac{m}{2m - 1}.$$

Then we solve for s_ν from (8.2) to obtain

$$s_\nu > \frac{n(s_\mu - 2(d - n) + (d - n)(d - s_\mu))}{n - (d - n)(d - s_\mu)} = \frac{(d - m)(s_\mu - 2m + m(d - s_\mu))}{d - m - m(d - s_\mu)}, \tag{8.3}$$

which completes the proof of (i) in Corollary 1.4. One can check that the right-hand side of (8.3) is non-negative unless

$$s_\mu > d - \frac{d - 2n}{d - n - 1} = d - \frac{2m - d}{m - 1}.$$

Now we turn to (ii) in Corollary 1.4.

When $s_\mu > 2d - 3n$, it follows directly from taking the right-hand side of (i) to be 0. So we assume $s_\mu < 2d - 3n$. Since $s_\mu > 2n$, by the second part of Corollary 1.3, we need to solve

$$2 + \frac{s_\mu - 2n}{d - s_\mu} > d - n,$$

which is equivalent to

$$s_\mu > d - \frac{d-2n}{d-n-1} = d - \frac{2m-d}{m-1},$$

that completes the proof.

There is one case we did not discuss, that is when $s_\mu < 2d - 3n$ and $s_\mu < 2n$. This is because in this case we need

$$\frac{2n}{3n - s_\mu} > d - n,$$

that has a solution only if $n = d - 1$, namely $m = 1$, already ruled out. ∎

9. Proof of Theorem 1.2 and Corollary 1.5: Broaden the range of q

Since both $G(d,n)$ and supp ν are compact, for every pair p, q in Proposition 7.1, the estimate also holds for smaller p, q. So it remains to prove results for large q. Theorem 1.2 is a summary of Proposition 7.1 and the following.

Proposition 9.1. *Suppose μ and ν are compactly supported Radon measures on $\mathbb{R}^d$, and let $0 < t < n$, $0 < \alpha < d$ and $2n - t \le s < d$. Denote*

$$q_0 := 1 + \frac{s+t-2n}{2(d-\alpha)}.$$

Then

$$\int \|\pi^y \mu\|^q_{L^p(G(d,n))}\, d\nu(y) \lesssim_{d,n,p,s,t} I_s^{1/2}(\mu) \cdot A_\alpha(\mu)^{q-1} \cdot A_{\max\{t, \frac{q}{2q_0} t\}}(\nu) \tag{9.1}$$

for every

$$q > q_0 \quad \text{and} \quad 1 \le p < \frac{2nq_0}{n+t} = \frac{2n}{n+t}\left(1 + \frac{s+t-2n}{2(d-\alpha)}\right).$$

Furthermore, when $t = n$,

$$\int \|\pi^y \mu\|^q_{L^q(G(d,n))}\, d\nu(y) \lesssim_{d,n,q,s} I_s(\mu)^{1/2} \cdot A_\alpha(\mu)^{q-1} \cdot A_{\max\{t, \frac{q}{2q_0} t\}}(\nu). \tag{9.2}$$

The proof goes by analyzing those right-hand sides in Proposition 7.1. We do not need to treat $t = n$ and $0 < t < n$ separately.

It suffices to consider

$$\int \|\pi^y \mu\|^{q_0}_{L^p(G(d,n))}\, g(y)\, d\nu(y), \quad \text{for } \|g\|_{L^{(q/q_0)'}(\nu)} = 1.$$

When $q < 2q_0$, we treat g as $g \in L^2$. So we may assume $q \ge 2q_0$. By Proposition 7.1 it is bounded above by

$$I_s^{1/2}(\mu) \cdot A_\alpha(\mu)^{q-1} \cdot I_t(|g|\, d\nu)^{1/2}.$$

Now it remains to estimate

$$I_t(|g|\,d\nu) = \iint |y-y'|^{-t}\,|g(y)|\,d\nu(y)\,|g(y')|\,d\nu(y').$$

Let $r := q/(2q_0) \geq 1$. Then the relation

$$\frac{1}{r} = 1 - \Big(\frac{1}{(q/q_0)'} - \frac{1}{q/q_0}\Big)$$

is satisfied. Denote

$$K(y,y') = |y-y'|^{-t}$$

as the kernel. By definition of A_s, we have

$$\Big(\int |K(y,y')|^r\,d\nu(y)\Big)^{1/r} = \Big(\int |K(y,y')|^r\,d\nu(y)\Big)^{1/r} \leq A_{\frac{q}{2q_0}t}(\nu)^{1/r}.$$

Then, by Hölder's inequality, the fact $\|g\|_{L^{(q/q_0)'}(\nu)} = 1$, and Young's inequality,

$$\iint K(y,y')\,|g(y)|\,d\nu(y)\,|g(y')|\,d\nu(y') \leq \Big\| \int K(y,\cdot)\,|g(y)|\,d\nu(y)\Big\|_{L^{q/q_0}(\nu)} \leq A_{\frac{qt}{2q_0}}(\nu)^{1/r},$$

as desired.

Proof of Corollary 1.5. It follows directly from Theorem 1.2.

We first consider the special case $p=q$. Observe from Theorem 1.2 that lifting q from q_0 to $2q_0$ does not change the finiteness. Then, since

$$p < \frac{2nq_0}{n+t} \leq 2q_0,$$

it follows that lifting q from q_0 to p does not influence the optimal p, that is Corollary 1.3.

Now we consider general p,q. We need to find all $p,q \in [1,\infty)$ such that, there exist s, α and t satisfying

$$0<s, \quad \alpha < s_\mu, \quad 0 < \max\Big\{t, \frac{q}{2q_0}t\Big\} < s_\nu, \quad s+t \geq 2n \quad \text{and} \quad p < \frac{2nq_0}{n+t},$$

where

$$q_0 := 1 + \frac{s+t-2n}{2(d-\alpha)}.$$

Since both p,q are increasing in s and α, up to the end point we may take $s=\alpha=s_\mu$ for large p,q. Then conditions above are equivalent to

$$\max\{2n-s_\mu, 0\} < t < s_\nu,$$

and

$$\begin{aligned} p &< \frac{2n}{n+t}\Big(1+\frac{s_\mu+t-2n}{2(d-s_\mu)}\Big) = \frac{n}{d-s_\mu}\Big(1+\frac{2d-3n-s_\mu}{n+t}\Big),\\ q &< \frac{2s_\nu}{t}\Big(1+\frac{s_\mu+t-2n}{2(d-s_\mu)}\Big) = \frac{s_\nu}{d-s_\mu}\Big(1+\frac{2d-2n-s_\mu}{t}\Big). \end{aligned} \tag{9.3}$$

When $s_\mu \le 2d - 3n$, both p, q are decreasing in t, so for large p, q, we should take t as small as possible, namely $t \searrow \max\{2n - s_\mu, 0\}$. Therefore, the range of p, q is

$$p < \frac{2n}{n + \max\{2n - s_\mu, 0\}}\Big(1 + \frac{\max\{s_\mu - 2n, 0\}}{2(d - s_\mu)}\Big) = \begin{cases} \frac{2n}{3n - s_\mu}, & s_\mu < 2n, \\ 2 + \frac{s_\mu - 2n}{d - s_\mu}, & s_\mu \ge 2n, \end{cases}$$

$$q < \frac{2s_\nu}{\max\{2n - s_\mu, 0\}}\Big(1 + \frac{\max\{s_\mu - 2n, 0\}}{2(d - s_\mu)}\Big) = \begin{cases} \frac{2s_\nu}{2n - s_\mu}, & s_\mu < 2n, \\ \infty, & s_\mu \ge 2n. \end{cases}$$

When $s_\mu \ge 2d - 2n$, both p, q are increasing in t, so for large p, q we should take t as large as possible, that is, $t \nearrow s_\nu$. Therefore, the range of p, q is

$$p < \frac{2n}{n + s_\nu}\Big(1 + \frac{s_\mu + s_\nu - 2n}{2(d - s_\mu)}\Big), \quad q < 2 + \frac{s_\mu + s_\nu - 2n}{d - s_\mu}.$$

When $2d - 3n < s_\mu < 2d - 2n$, p is increasing in t, while q is decreasing in t, so there is a relation between p and q. By solving for t from (9.3), we end up with the region enclosed by

$$1 \le p < \frac{2n}{n + s_\nu}\Big(1 + \frac{s_\mu + s_\nu - 2n}{2(d - s_\mu)}\Big),$$

$$1 \le q < \frac{2s_\nu}{\max\{2n - s_\mu, 0\}}\Big(1 + \frac{\max\{s_\mu - 2n, 0\}}{2(d - s_\mu)}\Big) = \begin{cases} \frac{2s_\nu}{2n - s_\mu}, & s_\mu < 2n, \\ \infty, & s_\mu \ge 2n, \end{cases}$$

and

$$\frac{s_\mu - 2d + 3n}{1 - \frac{d - s_\mu}{n} p} < n + \frac{2d - 2n - s_\mu}{\frac{d - s_\mu}{s_\nu} q - 1}. \qquad \blacksquare$$

10. An alternative proof of Dabrowski–Orponen–Villa

When $\mu = \nu$ and $p = q$, Orponen's formula says

$$\int \|\pi^y \mu\|^p_{L^p(G(d,n))}\, d\mu(y) = \int \|\pi_V \mu\|^{p+1}_{L^{p+1}(\mathcal{H}^n)}\, d\gamma_{d,n}(V).$$

Then it is natural to ask whether Proposition 7.1 covers (1.3). The answer is unfortunately no, due to the constraint $0 < t < n$. In general, this condition $0 < t < n$ cannot be relaxed. Technically, we need (5.8) and (6.9) to come back to the physical space and integrate in V. Geometrically, if our results hold for some $t > n$, then s is allowed to be $< n$, contradicting the visibility problem in $\mathbb{R}^{n+1}$.

Despite this, it does not mean our method is not strong enough. In fact, when $\mu = \nu$, there are extra symmetries. With ideas from previous sections, it is straightforward to recover (1.3), with a more delicate upper bound.

Proposition 10.1. *Suppose μ is a compactly supported measure on $\mathbb{R}^d$, and let $0 < s$, $\alpha < d$ and $2 \le q < \infty$ be such that*

$$s = n + (q - 2)(d - \alpha). \tag{10.1}$$

Then

$$\int \|\pi_V \mu\|_{L^q(\mathcal{H}^n)}^q \, d\gamma_{d,n}(V) \lesssim_{d,n,q} I_s(\mu) \cdot A_\alpha(\mu)^{q-2}. \tag{10.2}$$

Moreover, for every $2 \le q < p \le \infty$,

$$\int \|\pi_V \mu\|_{L^p(\mathcal{H}^n)}^q \, d\gamma_{d,n}(V) \lesssim_{d,n,p,q} I_s(\mu) \cdot A_\alpha(\mu)^{q-2}, \tag{10.3}$$

with

$$s = n + (q-2)(d-\alpha) + n\Big(1 - \frac{q}{p}\Big). \tag{10.4}$$

When $s = \alpha$, the condition (10.1) becomes

$$q = 2 + \frac{s-n}{d-s} = \frac{2d-n-s}{d-s},$$

that coincides with (1.3).

There are two ways to obtain (10.3) from (10.2): run interpolation on p, q with (1.7) as in the introduction, or invoke the Sobolev embedding directly:

$$\int \|\pi_V \mu\|_{L^p(\mathcal{H}^n)}^q \, d\gamma_{d,n}(V) \lesssim \int \|(-\Delta)^{\frac{n}{2}(1/q-1/p)} \pi_V \mu\|_{L^q(\mathcal{H}^n)}^q \, d\gamma_{d,n}(V) \tag{10.5}$$

and then run the argument below. We leave details to readers. From now, we only consider the case $p = q$.

As the idea is already explained clearly in Section 7, we decide to skip computations on the scratch paper and present the rigorous proof directly. Readers can follow the explanation in Section 7 to figure out where the exponents below come from.

Now we start the proof. It suffices to consider

$$\iint \pi_V \mu(u) \cdot f(V,u) \, d\mathcal{H}^n(u) \, d\gamma_{d,n}(V), \quad \text{for } \|f\|_{L^{q'}} = 1.$$

For every $z \in \mathbb{C}$, let

$$s_z = s + 2z, \quad \alpha_z = \alpha + z \quad \text{and} \quad q_z = 2 + \frac{s_z - n}{d - \alpha_z} = \frac{2(d-\alpha) + s - n}{d - \alpha - z}.$$

Observe that

- $s_0 = s$, $\alpha_0 = \alpha$, $q_0 = q$;
- For all $z \in \mathbb{C}$,
$$s_z = n + (q_z - 2)(d - \alpha_z),$$
and particularly,
$$s_{\mathrm{Re}\, z} = n + 2(k-1)(d - \alpha_{\mathrm{Re}\, z}), \quad \text{if } q_z = 2k \in 2\mathbb{Z}_+; \tag{10.6}$$
- $1/q_z$ is linear in z.

Now take

$$\Phi(z) = \iint \pi_V \mu_z(u) \cdot f_z(V, u)\, d\mathcal{H}^n(u)\, d\gamma_{d,n}(V)$$

as an analytic function, where μ_z is defined as (2.4), and

$$f_z(V, y) := \operatorname{sgn}(f) \cdot |f(V, y)|^{q'(1-1/q_{\operatorname{Re} z})}.$$

Since $1/q_z$ is linear in z,

$$\|f_z\|_{L^{(q_{\operatorname{Re} z})'}} = \left\| |f|^{q'(1-1/q_{\operatorname{Re} z})} \right\|_{L^{(q_{\operatorname{Re} z})'}} = 1.$$

When $\operatorname{Re} z = -(s-n)/2 < 0$, we have $q_{\operatorname{Re} z} = 2$. Then by Cauchy–Schwarz, the classical L^2-estimate of orthogonal projections, (3.3) and (10.6), it follows that

$$|\Phi(z)|^2 \le \|\pi_V \mu_z\|_{L^2}^2 = I_n(\mu_z) \lesssim I_s(\mu).$$

For positive $\operatorname{Re} z$, notice that $q_{\operatorname{Re} z} \to \infty$ as $\operatorname{Re} z \to d - \alpha > 0$. Therefore there exists $\operatorname{Re} z > 0$ such that $q_{\operatorname{Re} z} = 2k \ge 4$ is an even integer. Then

$$|\Phi(z)|^{2k} \le \int \|\pi_V \mu_z\|_{L^{2k}}^{2k}\, d\gamma_{d.n}(V) = \iint \pi_V \mu_z \cdot \pi_V \overline{\mu_z} \cdots \pi_V \mu_z \cdot \pi_V \overline{\mu_z}.$$

By our definition of μ_z in (2.4), one can easily see that $\overline{\mu_z} = \mu_{\bar z}$.
Fix this z and let

$$\Psi_z(w_1, \dots, w_{2k}) := \iint \pi_V(\mu_{z+w_1}) \cdot \pi_V(\mu_{\bar z + w_2}) \cdots \pi_V(\mu_{z+w_{2k-1}}) \cdot \pi_V(\mu_{\bar z + w_{2k}}),$$

that is analytic in $(w_1, \dots, w_{2k}) \in \mathbb{C}^{2k}$ as z is fixed and the integrand is compactly supported. For arbitrary but fixed $j_1, j_2 \in \{1, \dots, 2k\}$, take

$$\operatorname{Re} w_{j_1} = \operatorname{Re} w_{j_2} = -(k-1)(d - \operatorname{Re} \alpha_z) \quad \text{and} \quad \operatorname{Re} w_j = d - \operatorname{Re} \alpha_z, \quad \text{for } j \ne j_1, j_2.$$

Then, similar to (6.5) and (7.10), for every $j \ne j_1, j_2$,

$$\|\mu_{z+w_j}\|_{L^\infty}, \|\mu_{\bar z + w_j}\|_{L^\infty} \lesssim_{\operatorname{Re} z, \operatorname{Re} w_j} A_{d-\operatorname{Re}(z+w_j)}(\mu) = A_\alpha(\mu).$$

Therefore, $|\Psi_z(w_1, \dots, w_{2k})|$ is bounded above by

$$\begin{aligned}&\|\pi_V(\mu_{z+w_{j_1}})\|_{L^2} \cdot \|\pi_V(\mu_{\bar z + w_{j_2}})\|_{L^2} \cdot A_\alpha^{2(k-1)}(\mu) \\ &\quad \lesssim I^{1/2}_{n-2\operatorname{Re}(z+w_{j_1})}(\mu) \cdot I^{1/2}_{n-2\operatorname{Re}(z+w_{j_2})}(\mu) \cdot A_\alpha^{2(k-1)}(\mu) = I_s(\mu) \cdot A_\alpha^{2(k-1)}(\mu).\end{aligned}$$

As j_1 and j_2 are arbitrary, by Lemma 2.1,

$$|\Phi(z)|^{q_{\operatorname{Re} z}} = |\Psi_z(0)| \lesssim I_s(\mu) \cdot A_\alpha^{q_{\operatorname{Re} z} - 2}(\mu), \tag{10.7}$$

for every even integer $q_{\operatorname{Re} z} \ge 4$, where the implicit constant is independent in $\operatorname{Im} z$.

Now we have estimates of $|\Phi(z)|$ for both positive and negative $\operatorname{Re} z$, with implicit constants independent in $\operatorname{Im} z$. Hence, by the Hadamard three-lines lemma,

$$|\Phi(0)|^q = |\Psi_z(0)| \lesssim I_s(\mu) \cdot A_\alpha^{q-2}(\mu),$$

that completes the proof.

Acknowledgments. We thank the anonymous referee for careful reading and helpful suggestions on the manuscript.

Funding. This work was partially supported by the National Natural Science Foundation of China grant 12131011.

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Received November 8, 2022; revised November 11, 2023.

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