Rend. Sem. Mat. Univ. Padova 151 (2024), 213–251
DOI 10.4171/RSMUP/137

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A local construction of zip period maps of Shimura varieties

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Abstract – Let S be the special fibre of a Shimura variety of Hodge type of good reduction at a fixed place above p. We give a local approach to the construction of the zip period map for S, which is used to define the Ekedahl–Oort strata of S. The method employed is p-adic and group theoretic in nature.

Mathematics Subject Classification (2020) – 11G35.

Keywords – Zip period maps, Shimura varieties, Ekedahl–Oort stratification, G-zips.

1. Introduction

1.1 – *History of zip period maps*

Zip period maps made their debut as background artists during the development of Ekedahl–Oort (henceforth referred to as EO for brevity) stratification theory for Shimura varieties. Initially, the EO stratification was defined by Ekedahl and Oort [29] for the moduli space of principally polarized abelian varieties $\mathcal{A}_g \otimes \mathbb{F}_p$ of dimension g in characteristic $p > 0$ (can be viewed as the Siegel-type Shimura variety) by declaring that two points (A, λ) and (A', λ') over $\overline{\mathbb{F}}_p$ lie in the same stratum if their p-kernels are isomorphic.

Later on, this stratification was extended to PEL-type Shimura varieties in a series of papers [14, 26–28, 35], and to Hodge-type Shimura varieties [34, 40]. The underlying idea is the same as in the Siegel case, i.e., using the isomorphism classes of p-kernels

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of abelian varieties with additional structures. The way of defining and studying these strata evolves over time. Let S be the special fibre of a PEL-type Shimura variety of good reduction at p; it is defined over a finite field κ (say). In order to describe the dimension of the EO strata of S, Wedhorn [36] constructed a sequence of morphisms of stacks over κ (later viewed as a period map in characteristic p),

$$(1.1) \qquad S \longrightarrow \mathsf{BT}_1 \longrightarrow \mathsf{DS}_1,$$

where BT_1 is the stack of BT 1's (i.e., p-kernels of p-divisible groups) with PEL structures and DS_1 is the stack of Dieudonné spaces with PEL structures (i.e., Dieudonné modules associated with BT 1's with extra structure). He also showed that the map $S \to \mathsf{BT}_1$ is smooth and the natural map $\mathsf{BT}_1 \to \mathsf{DS}_1$ given by the crystalline Dieudonné functor is a homeomorphism.

Soon, Moonen and Wedhorn [28] established the theory of F-zips with the underlying idea that an F-zip structure on a vector bundle in characteristic p is like a Hodge structure in characteristic 0. Moreover, they constructed a morphism of κ-stacks

$$(1.2) \qquad S \longrightarrow [G\backslash X_\mu],$$

where X_μ is the moduli of trivialized F-zips with PEL structures (of certain type μ determined by S). Here, the map is given by taking the F-zip associated with the universal BT 1 over S, which is by definition the relative de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S)$ of the universal abelian scheme $\mathcal{A}$ equipped with its F-zip structure. In fact, an F-zip associated with a BT 1 is equivalent to the corresponding Dieudonné space defined in [36] and we get that the map (1.2) is essentially the same as (1.1). Thanks to the analogy of F-zip structures with Hodge structures, they are considered as period maps in characteristic p.

Based on the theory of F-zips, Pink, Wedhorn and Ziegler [30, 31] defined the notions of G-zips (F-zips endowed with a G-structure) and the stack of G-zips of type μ denoted by $G\text{-}\mathsf{Zip}^\mu$. They also showed that the stack $G\text{-}\mathsf{Zip}^\mu$ can be realized as the quotient stack of G_κ by some zip group E_μ. In other words, we have an isomorphism of κ-stacks (see Section 4.1 for the precise definitions and more details)

$$(1.3) \qquad G\text{-}\mathsf{Zip}^\mu \cong [G_\kappa/E_\mu].$$

Suppose now that S is of Hodge type and $p \geq 3$. In order to extend EO stratification to Shimura varieties of Hodge type, Zhang [40] (see also [37]) constructed a map of algebraic stacks (see Section 4.2 for a review of construction),

$$\zeta\colon S \longrightarrow G\text{-}\mathsf{Zip}^\mu,$$

and showed that ζ is smooth. The EO strata of S are defined as geometric fibres of ζ. The strata thus defined are automatically smooth and many properties on these strata are obtained by translating the information of the target stack into that of S via ζ.

We call ζ the *zip period map* for S. There are also other related maps in the literature that can be viewed as variants of ζ , e.g., the perfectly smooth map $\mathrm{Sh}_\mu \to \mathrm{Sht}_\mu^{\mathrm{loc}(2,1)}$ in [38, Remark 7.2.5] (see also [33] for its generalization) and the map $\eta\colon S \to \mathcal{D}_1/\mathcal{K}^\diamond$ in [39, Theorem 8.5.2]. Our aim in the present paper is to give an alternative construction of ζ while avoiding the language of G-zips, which provides a different perspective on the zip period map. To be more precise, the composition of ζ with the isomorphism (1.3) is reconstructed here.

1.2 – *Main results and the strategy of proof*

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum of Hodge type and denote by $\mathcal{S}_\mathsf{K}$ the Kisin–Vasiu integral model of the associated Shimura variety $\mathrm{Sh}_\mathsf{K}(\mathbf{G}, \mathbf{X})$ of level K, which is hyperspecial at p. This hyperspecial condition on K implies that $\mathbf{G}_{\mathbb{Q}_p}$ admits a reductive $\mathbb{Z}_p$-model $\mathcal{G}$, whose special fibre we denote by G. Recall that the integral model $\mathcal{S}_\mathsf{K}$ is a quasi-projective and smooth scheme over $\mathcal{O}$, the localization at some place above p of the ring of integers of the reflex field of $(\mathbf{G}, \mathbf{X})$. Write κ for the residue field of $\mathcal{O}$ and $S := \mathcal{S}_\mathsf{K} \otimes_\mathcal{O} \kappa$. Let $\mu\colon \mathbb{G}_{m,\kappa} \to G_\kappa$ be a representative for the reduction over κ of the $\mathbf{G}(\mathbb{C})$-conjugacy class $[\mu]_\mathbb{C}$ containing the inverses of Hodge cocharacters $\mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_\mathbb{C}$ determined by $(\mathbf{G}, \mathbf{X})$.

Denote by $P_\pm \subseteq G_\kappa$ the opposite parabolic subgroups of G_κ defined by μ, by $U_\pm \subseteq P_\pm$ the corresponding unipotent radicals, and by U_-^σ the base change of U_- along the p-power Frobenius $\sigma\colon \kappa \to \kappa$. The zip group appearing in (1.3) is a smooth algebraic group over κ given by $E_\mu = P_+ \ltimes U_-^\sigma$ (see Section 4.1 for the group action). The following nearly trivial observation turns out to be important to this work: since U_-^σ, as a normal subgroup of E_μ, acts freely on G_κ by right multiplication, by passing to the quotient we obtain a canonical isomorphism of algebraic stacks over κ (Section 5.6):

$$[(G_\kappa/U_-^\sigma)/P_+] \cong [G_\kappa/E_\mu],$$

where G_κ/U_-^σ is represented by a smooth κ-scheme. Hence, giving the zip period map ζ above is equivalent to giving a P_+-torsor, say T, over S and a P_+-equivariant map of κ-schemes $T \to G_\kappa/U_-^\sigma$. The natural candidate for T is the scheme I_+ of trivializations of the Hodge filtration $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S) \supseteq \omega_{\mathcal{A}/S}$, respecting certain tensors that we do not specify in this introduction. We want to emphasize that the parabolic bundle I_+, which is part of the data for the universal G-zip which defines ζ, is crucial for our main results.

THEOREM 1.1. *There exists an (explicitly constructed) morphism of κ-schemes (Theorem 5.7)*

$$\gamma\colon \mathrm{I}_+ \to G_\kappa/U_-^\sigma.$$

The map γ is P_+-equivariant, and hence induces a morphism of algebraic κ-stacks (Theorem 5.9)

$$\eta\colon S \longrightarrow [(G_\kappa/U_-^\sigma)/P_+] \cong [G_\kappa/E_\mu] \cong G\text{-}\mathsf{Zip}^\mu.$$

THEOREM 1.2. *We have a natural 2-isomorphism $\eta \cong \zeta$. Consequently, we give an alternative construction of the zip period map for S.*

The reader is referred to Theorem 6.1 for more details on this. We next describe the construction of γ on geometric points. Set $k = \overline{\mathbb{F}}_p$, an algebraic closure of $\mathbb{F}_p$. From now on we fix a cocharacter $\tilde{\mu}\colon \mathbb{G}_{m,W(\kappa)} \to \mathcal{G}_{W(\kappa)}$ of $\mathcal{G}_{W(\kappa)}$ which lifts μ. A point $\bar{x}^\flat = (\bar{x}, \beta_{\bar{x}}) \in \mathrm{I}_+(k)$ consists of a point $\bar{x} \in S(k)$ and a trivialization

$$\beta_{\bar{x}}\colon [\Lambda_k^* \supseteq \Lambda_k^{*,1}] \cong [\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_x/k) \supseteq \omega_{\mathcal{A}_{\bar{x}}/k}] \cong [\bar{M} \supset \bar{M}_1],$$

respecting tensors on both sides, where $\bar{M}$ and $\bar{M}_1$ denote the reduction modulo p of the contravariant Dieudonné module M of the p-divisible group $\mathcal{A}_{\bar{x}}[p^\infty]$ over k and respectively its Hodge filtration M_1 (Section 2.3). Here, $\Lambda_k^{*,1}$ is the weight 1 subspace of Λ_k^* induced by $\mu_k\colon \mathbb{G}_{m,k} \to G_k$. Let $\mathbb{I}_+$ be the integral model over $\mathcal{S}$ of I_+. The first step of the construction of γ on k-points is to choose a lift $x^\flat = (x, \beta_x) \in \mathbb{I}_+(W(k))$ of $\bar{x}^\flat$ which provides a lift of $\beta_{\bar{x}}$,

$$\beta_x\colon [\Lambda_{W(k)}^* \supseteq \Lambda_{W(k)}^{*,1}] \cong [\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_x/W(k)) \supseteq \omega_{\mathcal{A}_x/W(k)}],$$

and hence a trivialization of M via the canonical isomorphism $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_x/W(k)) \cong M$, and then show that via the trivialization β_x the Frobenius of M admits a uniform decomposition

$$\mathcal{f}_{x^\flat}\, \tilde{\mu}^\sigma(p) \quad \text{with } \mathcal{f}_{x^\flat} \in \mathcal{G}(W(k)),$$

where $\tilde{\mu}^\sigma\colon \mathbb{G}_{m,W(k)} \to \mathcal{G}^\sigma_{W(k)} \cong \mathcal{G}_{W(k)}$ is the base change along $\sigma\colon W(k) \to W(k)$ of $\tilde{\mu}_{W(k)}$. The key point here is that the element $\mathcal{f}_{x^\flat}$ is integral and hence we can take its reduction modulo p, denoted by $\overline{\mathcal{f}_{x^\flat}} \in G(k)$. Then one proceeds by showing that the image of $\overline{\mathcal{f}_{x^\flat}}$ in $G_\kappa/U_-^\sigma(k)$ is independent of lifts $x^\flat$ (Lemma 5.5); we denote it by $\gamma_{\bar{x}^\flat}$. To summarize, the map γ on k-points is given by performing the following operations (Section 5):

$$\begin{aligned} \bar{x}^\flat \in \mathrm{I}_+(k) \xrightarrow{\text{choose } x^\flat} \mathcal{f}_{x^\flat} \in \mathcal{G}(W(k)) &\xrightarrow{\text{mod } p} \overline{\mathcal{f}_{x^\flat}} \in G(k) \\ &\xrightarrow{\text{projection}} \gamma_{\bar{x}^\flat} \in G_\kappa/U_-^\sigma(k). \end{aligned} \tag{1.4}$$

The technical heart of the construction of γ in Theorem 1.1 is to justify the operations in (1.4) and to show that these operations can be performed in a relative sense: for every smooth κ-algebra $\bar{R}$ which (automatically) admits a simple frame (equivalently, a crystalline prism if one prefers) and every point $\bar{x}^\flat \in \mathrm{I}_+(\bar{R})$, we can construct a point $\gamma_{\bar{x}^\flat} \in G_\kappa/U_-^\sigma(\bar{R})$ whose specialization at geometric points coincides with (1.4); see Proposition 5.4. This relative construction relies on relative classifications of p-divisible groups as in [10] and is more complicated in the sense that in the relative setting we need to compare not only different lifts $x^\flat$ as aforementioned, but also different choices of simple frames for $\bar{R}$. The independence of these two different types of choices are proved via matrices calculations; see Section 5.4. Finally, the global map $\gamma\colon \mathrm{I}_+ \to G_\kappa/U_-^\sigma$ is obtained by first constructing it on Zariski opens of I_+ and then gluing the local maps together.

Let us now compare η with ζ. Zhang's construction [40] of the map ζ uses the global geometry in positive characteristic, namely the language of G-zips, but follows the original definition of EO stratification for $\mathcal{A}_g$ in spirit. This is because the stack $G\text{-}\mathsf{Zip}^\mu$ can be viewed as the moduli space of BT 1's, while the universal G-zip for ζ corresponds to the universal BT 1, namely $\mathcal{A}[p]$ over S. In particular, the map ζ is determined by $\mathcal{A}[p]$. We remark that in the biased opinion of the current author, the language of G-zips is somewhat complicated. To witness, a G-zip involves three torsors plus some delicate zip relations. In contrast, we work locally (because of the need for Frobenius lifts) during our construction of η and the process makes use of more group-theoretic arguments. Our endeavor employs properties of the parabolic bundle I_+ and its integral model $\mathbb{I}_+$ to the fullest extent needed. From our construction, the reader may hopefully be able to better see the role that the zip group $E_\mu = P_+ \ltimes U_-^\sigma$ plays in the business of zip period maps.On the other hand, our approach does not start with $\mathcal{A}[p]$ which obscures the dependence of γ (and, hence of η) on $\mathcal{A}[p]$ in the end. Our proof of Theorem 1.2 is not formal, partly because the canonical isomorphism $[G_\kappa/E_\mu] \overset{\text{can}}{\cong} G\text{-}\mathsf{Zip}^\mu$ is not.

This work has some overlap with my Ph.D. thesis [39]. The relationship between these two works will be made transparent in a paper to be written in the near future.

1.3 – *Convention and notation*

Throughout the paper we fix a prime number $p \geq 3$. The Dieudonné crystals (resp. modules) used in this paper are contravariant. Let R be a ring and M an R-module. If $\sigma\colon R \to R$ is a ring endomorphism we write $M^\sigma = \sigma^* M$ for the base change $M \otimes_{R,\sigma} R$. If M is finite locally free, we denote by M^* its dual R-module. Then we have the canonical identification $M^\otimes \overset{\text{can}}{\cong} M^{*,\otimes}$ of R-modules, where $M^\otimes$ is the direct sum of all R-modules obtained from M by applying the operations of taking duals, tensor products, symmetric powers, and exterior powers. Here, as a general convention, the notation "$\overset{\text{can}}{\cong}$" means canonical isomorphism between mathematical objects. For any R-automorphism $f\colon M \cong M$, we have an induced isomorphism $(f^{-1})^*\colon M^* \to M^*$, $a \mapsto f^{-1} \circ a$, and hence a canonical isomorphism of R-group schemes $(\cdot)^\vee\colon \mathrm{GL}(M) \overset{\text{can}}{\cong} \mathrm{GL}(M^*)$, $g \mapsto g^\vee := (g^{-1})^*$. We also use the letter M to denote a Levi subgroup (of some algebraic group), but it will be always clear from the context whether M is a module or an algebraic group. We will use the notation $\bar{x}$ to indicate that $\bar{x}$ itself is in the characteristic p world or is the reduction modulo p of x, assuming that such statements make sense.

For an $\mathbb{F}_p$-algebra $\bar{R}$, we use $\sigma\colon \bar{R} \to \bar{R}$ for the absolute (i.e., p-power) Frobenius of $\bar{R}$. If X is a scheme over $\bar{R}$, we write X^σ for its pull-back along σ and $\sigma\colon X \to X^\sigma$ for the relative Frobenius over $\bar{R}$. In particular, when X is defined over $\mathbb{F}_p$, sometimes we also write $\sigma\colon X \to X$ for the composition of the relative Frobenius of X with the canonical isomorphism $\sigma\colon X^\sigma \overset{\text{can}}{\cong} X$. Similarly, if $f\colon X \to Y$ is a map between objects over $\bar{R}$, we write f^σ for its base change along $\sigma\colon \bar{R} \to \bar{R}$. Now let k be a perfect field and $\mathcal{G}$ a group scheme over $W(k)$, which is defined over $\mathbb{Z}_p$. For a $W(k)$-algebra R with a Frobenius lift $\sigma = \sigma_R\colon R \to R$ over $W(k)$, we often denote by

$$\sigma\colon \mathcal{G}(R) \to \mathcal{G}(R) \tag{1.5}$$

the homomorphism induced by $\sigma\colon R \to R$. Note that σ is only a $\mathbb{Z}_p$-endomorphism, and not a $W(k)$-endomorphism in general. We abuse language and also call (1.5) the "Frobenius" of $\mathcal{G}$. In the case that $\mathcal{G}$ is defined over $\mathbb{F}_p$, this Frobenius coincides with the relative Frobenius mentioned above.

In this paper, we systematically use right actions instead of left or mixed actions for quotient stacks. As an example, the stack $[G_\kappa/E_\mu]$ in this paper corresponds to $[E_\mu \backslash G_\kappa]$ found in the literature.

2. Classification of p-divisible groups (recollection)

Throughout this section we let k be a perfect field of characteristic p and denote by $\sigma\colon W(k) \to W(k)$ its unique ring automorphism inducing the absolute Frobenius of k.

2.1 – *Existence of simple frames*

Lemma 2.1. *Let $\bar{R}$ be a k algebra which Zariski locally admits a finite p-basis (*[10, Definition 1.1.1] *or* [4, Definition 1.1.1]*). The following hold:*

(1) *There exists a p-complete flat $W(k)$-algebra R lifting $\bar{R}$ (i.e., $R/pR \cong \bar{R}$), which is formally smooth over $W(k)$ with respect to the p-adic topology. Such an R is unique up to (nonunique) isomorphisms and we call it a* lift *of $\bar{R}$.*

(2) *There is a ring endomorphism* $\sigma = \sigma_R\colon R \to R$ *lifting the absolute Frobenius of* $\bar{R}$*, which is compatible with* $\sigma\colon W(k) \to W(k)$*. We call it a* Frobenius lift *of* R *over* $W(k)$*.*

(3) *Let* $\bar{R}$*,* R *be as above and* $\bar{A}$ *an étale* $\bar{R}$ *algebra. Then there exists a formally étale* R*-algebra* A *(for the* p*-adic topology), unique up to unique isomorphism, such that* A *lifts* $\bar{A}$ *and the structure ring homomorphism* $R \to A$ *lifts the structure homomorphism* $\bar{R} \to \bar{A}$*. Moreover, every Frobenius lift* $\sigma_R\colon R \to R$ *of* R *over* $W(k)$ *extends uniquely to a Frobenius lift* $\sigma_A\colon A \to A$ *of* A *over* $W(k)$*.*

(4) *Let* (R, σ) *be as above. If* $\mathfrak{m}$ *is a maximal idea of* R*, then* σ *extends uniquely to a Frobenius lift of the* $\mathfrak{m}$*-adic completion* $\widehat{R}_{\mathfrak{m}}$ *of* R*, which is a lift of the* $\mathfrak{m}$*-adic completion of* $\bar{R}$*.*

Proof. Items (1) and (2) are special cases of [17, Lemma 2.1] (take $I = (p)$) and (3) is a special case of the first part of [17, Lemma 2.5]. For (4), note first that $\sigma(\mathfrak{m}) \subseteq \mathfrak{m}$. This follows from the fact that $\mathfrak{m}$ contains p, and the fact that the morphism $\operatorname{Spec} \bar{R} \to \operatorname{Spec} \bar{R}$, induced by the absolute Frobenius of $\bar{R}$, is the identity on topological spaces. Hence we can define $\sigma_{\widehat{R}_{\mathfrak{m}}}\colon \widehat{R}_{\mathfrak{m}} \to \widehat{R}_{\mathfrak{m}}$ by sending an element $(r_i)_i \in \varprojlim_i R/\mathfrak{m}^i = \widehat{R}_{\mathfrak{m}}$ to $(\sigma(r_i))_i \in \widehat{R}_{\mathfrak{m}}$. ∎

Example 2.2 ([4, §1.1.2]). The main examples of $\bar{R}$ in our later applications are the following:

- $\bar{R}$ is a perfect k-algebra (the empty p-basis case). In this case, the unique simple frame of $\bar{R}$ (up to unique isomorphism) is given by $(W(\bar{R}), \sigma)$.
- $\bar{R}$ is a smooth k-algebra of finite type. Here, Zariski locally $\bar{R}$ indeed admits a finite p-basis: Zariski locally $\bar{R}$ is étale over some polynomial algebra $\bar{A} = k[x_1, \ldots, x_n]$, which has the standard p-basis $\{x_1, \ldots, x_n\}$; then the image of this p-basis in $\bar{R}$ forms a p-basis of $\bar{R}$, as the relative Frobenius map $\bar{A} \otimes_{\sigma, \bar{A}} \bar{R} \to \bar{R}$, $a \otimes r \mapsto ar^p$ is an isomorphism (hence the Frobenius $\sigma\colon \bar{R} \to \bar{R}$ can be identified with the canonical ring map $\bar{R} \to \bar{A} \otimes_{\sigma, \bar{A}} \bar{R}$, $r \mapsto 1 \otimes r$).

Definition 2.3. Let $\bar{R}$ be as in Lemma 2.1. A *simple frame* of $\bar{R}$, relative to $W(k)$, is a pair $\underline{R} = (R, \sigma)$, where R is a lift of $\bar{R}$ and $\sigma\colon R \to R$ is a Frobenius lift of R.

Remark 2.4. A simple frame (R, σ) over $W(k)$ of $\bar{R}$ is the same thing as a crystalline prism over the base prism $(W(k), \sigma)$ in the sense of Bhatt and Scholze [6]. To say the same in fancier language, simple frames of $\bar{R}$ should perhaps be termed *(crystalline) prismatic charts* of $\bar{R}$.

2.2 – *Classification of p-divisible groups over $\bar{R}$*

Let $\bar{R}$ be as in Lemma 2.1 and $\underline{R}$ a simple frame of $\bar{R}$. Till the end of this section, we assume further that

$\bar{R}$ is as in Example 2.2.

As preparation for later sections, in this subsection we review results on the classification of p-divisible groups over $\bar{R}$ (and over R in Section 2.3), in terms of linear algebra data over the simple frame (R, σ).

We denote by $\widehat{\Omega}_R$ the module of p-adically continuous differentials of R, i.e.,

$$\widehat{\Omega}_R := \varprojlim_n \Omega^1_{(R/p^n R)/W(k)}.$$

It is a finite projective R-module due to the existence of a finite p-basis of $\bar{R}$. We denote the category of Dieudonné modules with connections by $\mathbf{DM}(\underline{R}, \nabla)$. Here, a *Dieudonné module with connection* (or simply a *Dieudonné module*) over $\underline{R}$ (or simply over R when σ is chosen) is a tuple $(M, \mathrm{F}, \mathrm{V}, \nabla_M)$, where M is a finite locally free R-module and $\mathrm{F}\colon M^\sigma \to M$, $\mathrm{V}\colon M \to M^\sigma$ are maps between R-modules such that

$$\mathrm{F} \circ \mathrm{V} = p \cdot \mathrm{id}_{M^\sigma}, \quad \mathrm{V} \circ \mathrm{F} = p \cdot \mathrm{id}_M, \tag{2.1}$$

and where $\nabla_M\colon M \to M \otimes_R \widehat{\Omega}_R$ is an integrable topologically quasi-nilpotent connection over the p-adically continuous derivation $d_R\colon R \to \widehat{\Omega}_R$ of R, with respect to which F is horizontal, i.e., $\nabla_M \circ \mathrm{F} = (\mathrm{F} \otimes \mathrm{id}_{\widehat{\Omega}_R}) \circ \sigma^*(\nabla_M)$.

For a p-divisible group $\bar{H}$ over $\bar{R}$, we denote by $\mathbb{D}^*(\bar{H})$ the Dieudonné crystal[1] of $\bar{H}$ as in [3], which coincides with the construction in [22] up to duality: to be precise, our $\mathbb{D}^*(\bar{H})$ here corresponds to the Dieudonné crystal $\mathbb{D}(\bar{H}^*)$ in [22], with $\bar{H}^*$ the dual p-divisible group of $\bar{H}$. Following the usual convention, we write $\mathbb{D}^*(\bar{H})(R)$ for the evaluation of $\mathbb{D}^*(\bar{H})$ at the canonical PD-thickening $R \twoheadrightarrow \bar{R}$. By functoriality of the formation of Dieudonné crystals, we have $\mathbb{D}^{*,\sigma}(\bar{H}) = \mathbb{D}^*(\bar{H}^\sigma)$, where $\mathbb{D}^{*,\sigma}(\bar{H})$ is the pull-back along $\sigma\colon \bar{R} \to \bar{R}$ of $\mathbb{D}^*(\bar{H})$. Consequently, we have a canonical isomorphism $\mathbb{D}^{*,\sigma}(\bar{H})(R) \cong \mathbb{D}^*(\bar{H})^\sigma$ of R-modules. The Frobenius $\bar{H} \to \bar{H}^\sigma$ and Verschiebung $\bar{H} \to \bar{H}^\sigma$ induce morphism of crystals,

$$\mathrm{F}\colon \mathbb{D}^*(\bar{H}^\sigma) \to \mathbb{D}^*(\bar{H}), \quad \mathrm{V}\colon \mathbb{D}^*(\bar{H}) \to \mathbb{D}^*(\bar{H}^\sigma),$$

(1) The superscript $*$ in $\mathbb{D}^*(\bar{H})$ is used to indicate that our Dieudonné crystal here is contravariant.

such that $\mathrm{F} \circ \mathrm{V} = p \cdot \mathrm{id}_{\mathbb{D}^*(\bar{H})}$ and $\mathrm{V} \circ \mathrm{F} = p \cdot \mathrm{id}_{\mathbb{D}^*(\bar{H}^\sigma)}$. Evaluating at the thickening $R \twoheadrightarrow \bar{R}$ we obtain R-linear maps F, V for $\mathbb{D}^*(\bar{H})(R)$, just like an object in $\mathbf{DM}(\underline{R}, \nabla)$ satisfying (2.1). Denote by $(\mathbf{BT}/\bar{R})$ the category of p-divisible groups over $\bar{R}$. The following classification result is known.

Remark 2.5. If $H = \bar{\mathcal{A}}[p^\infty]$ for some abelian scheme $\bar{\mathcal{A}}$ over $\bar{R}$ (the case we are mostly concerned with in later applications), we have the canonical isomorphism of Dieudonné crystals ([3, Proposition 3.3.7, Théorème 2.5.6])

$$\mathbb{D}^*(\bar{H}) \cong \mathbb{D}^*(\bar{\mathcal{A}}) \cong \mathbf{R}^1\pi_{\mathrm{CRIS},*}\mathcal{O}_{\bar{\mathcal{A}}}^{\mathrm{cris}},$$

where $\pi\colon \bar{\mathcal{A}} \to \operatorname{Spec} \bar{R}$ is the structure morphism. It follows then that we have the following canonical isomorphism of R-modules, which is Frobenius equivariant:

$$\mathrm{H}^1_{\mathrm{cris}}(\bar{\mathcal{A}}/R) \cong \mathbb{D}^*(\bar{H})(R).$$

Theorem 2.6. *For any p-divisible group $\bar{H}$ over $\bar{R}$, there exists a natural connection $\nabla_M\colon M \to M \otimes_R \widehat{\Omega}_R$ for $M = \mathbb{D}^*(\bar{H})(R)$ such that the tuple*

$$\underline{M} = (M, \mathrm{F}, \mathrm{V}, \nabla_M)$$

is an object in $\mathbf{DM}(\underline{R}, \nabla)$. Moreover, such an assignment gives an equivalence of categories between $(\mathbf{BT}/\bar{R})$ and $\mathbf{DM}(\underline{R}, \nabla)$.

Proof. If $\bar{R}$ is a perfect k-algebra, this is an unpublished result of Gabber relying on a result of Berthelot [2], where the case of a perfect discrete valuation ring is dealt with; see also [20, 32] for different proofs. In this case, the connection can even be suppressed in the definition of a Dieudonné module. If $\bar{R}$ is a smooth k-algebra of finite type, this follows from [10, Theorem 4.1.1, Definition 2.3.4, Proposition 2.4.8]: indeed, since $\bar{R}$ satisfies [10, §1.3.1.1] by [10, Example (1.3.2.1)], we have that $\mathfrak{X} = \operatorname{Spec} \bar{R}$ satisfies the hypothesis of [10, Theorem 4.1.1] by [10, Example (2.4.7.2)]. ■

2.3 – *Classification of p-divisible groups over R*

The setting is the same as in the previous Section 2.2. Now we start with a p-divisible group H over R and write $\bar{H} = H \otimes_R \bar{R}$. For the dual p-divisible group H^*, in [22, Corollary IV.1.14] a universal extension $0 \to \omega_H \to \mathrm{E}(H^*) \to H^* \to 0$ is constructed; taking $\underline{\mathrm{Lie}}$ (following the notation in [22, IV.1.14]), we get an exact sequence of locally free R-modules

$$0 \to \omega_H \to \underline{\mathrm{Lie}}(\mathrm{E}(H^*)) \to \underline{\mathrm{Lie}}(H^*) \to 0,$$

where ω_H is the sheaf of invariant differentials of H. Moreover, it follows from the construction of $\mathbb{D}^*(\bar{H})$ that we have a canonical isomorphism $\underline{\mathrm{Lie}}(\mathrm{E}(H^*)) \cong \mathbb{D}^*(\bar{H})(R)$ of R-modules ([22, §IV.2.5.4], see also [3, Corollaire 3.3.5]) and thus we can identify them. Similarly we have an exact sequence for $\bar{H}$,

$$0 \to \omega_{\bar{H}} \to \mathbb{D}^*(\bar{H})(\bar{R}) \to \underline{\mathrm{Lie}}(\bar{H}^*) \to 0.$$

Here we stress that the submodule $\omega_H \subseteq \mathbb{D}^*(\bar{H})(R)$ is a locally direct summand of $\mathbb{D}^*(\bar{H})(R)$ which lifts the locally direct summand $\omega_{\bar{H}} \subseteq \mathbb{D}^*(\bar{H})(\bar{R})$. Let $\underline{M} \in (\mathbf{BT}/\bar{R})$ be the Dieudonné module corresponding to $\bar{H}$. Write $\bar{\mathrm{F}}\colon \bar{M}^\sigma \to \bar{M}$ for the reduction modulo p of F. Set $\bar{M}^1 := \omega_{\bar{H}}$, called the *Hodge filtration* of $\bar{M}$. Then we have the relation

$$\bar{M}^{1,\sigma} = \mathrm{Ker}(\bar{\mathrm{F}}) \subseteq \bar{M}^\sigma. \tag{2.2}$$

Denote by $(\mathbf{BT}/R)$ the common category of p-divisible groups over Spec R and over Spf R (justified by [10, Lemma 2.4.4]) and the category of tuples $(M, M^1, \mathrm{F}, \mathrm{V}, \nabla_M)$ by $\mathbf{AFDM}(\underline{R}, \nabla)$, where $(M, \mathrm{F}, \mathrm{V}, \nabla_M)$ is an object in $\mathbf{DM}(\underline{R}, \nabla)$ and $M^1 \subseteq M$ is a locally direct summand, lifting the locally direct summand $\bar{M}^1 \subset \bar{M}$. Morphisms are obvious ones. We call objects in $\mathbf{AFDM}(\underline{R}, \nabla)$ *admissibly filtered Dieudonné modules over* $\underline{R}$ (or simply, over R when σ is chosen); cf. [22, Definition V.1.4].

Remark 2.7. For the purpose of future reference, we recall the following well-known comparison results, which underline the crystalline Dieudonné theory. If $\mathcal{A}$ is an abelian scheme over R, with $\bar{\mathcal{A}}$ its pull-back to $\bar{R}$, we have a canonical isomorphism of R-modules ([1, Corollaire V.2.3.7], also cf. [5, Summary 7.26.3])

$$\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/R) \overset{\mathrm{can}}{\cong} \mathrm{H}^1_{\mathrm{cris}}(\bar{\mathcal{A}}/R).$$

Moreover, we have the following canonical isomorphism of filtered R-modules

$$(\mathbb{D}^*(\bar{H})(R) \supseteq \omega_H) \overset{\mathrm{can}}{\cong} (\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/R) \supseteq \omega_{\mathcal{A}}).$$

Theorem 2.8. *The assignment* $G \mapsto (\mathbb{D}^*(\bar{H})(R), \omega_H, \mathrm{F}, \mathrm{V}, \nabla_M)$ *gives a category equivalence between* $(\mathbf{BT}/R)$ *and* $\mathbf{AFDM}(\underline{R}, \nabla)$.

Proof. To lift a p-divisible group $\bar{H}$ over $\bar{R}$ to R is the same thing as lifting its dual $\bar{H}^*$ to R; the assertion follows from the combination of Theorem 2.6 and Grothendieck–Messing deformation theory ([22, Theorem V.1.6]), which in our setting says that lifting $\bar{H}^*$ to R is equivalent to lifting the locally direct summand $\omega_{\bar{H}} \subseteq \mathbb{D}^*(\bar{H})(\bar{R})$ to a locally direct summand of $\mathbb{D}^*(\bar{H})(R)$. ∎

2.4 – *Base change along simple frames*

Let $\bar{R}'$ be as in Example 2.2 and $\underline{R}' = (R', \sigma')$ be a simple frame of $\bar{R}'$ over $W(k)$. Let $f\colon \underline{R} \to \underline{R}'$ be a morphism of simple frames over $W(k)$ (i.e., a map $f\colon R \to R'$ of $W(k)$-algebras compatible with Frobenius lifts). Then we have the following commutative diagrams induced by base change along f in the obvious sense:

$$\begin{array}{ccc} (\mathbf{BT}/\bar{R}) & \xrightarrow{\cong} & \mathbf{DM}(\underline{R}, \nabla) \\ \downarrow & & \downarrow \\ (\mathbf{BT}/\bar{R}') & \xrightarrow{\cong} & \mathbf{DM}(\underline{R}', \nabla), \end{array} \qquad \begin{array}{ccc} (\mathbf{BT}/R) & \xrightarrow{\cong} & \mathbf{AFDM}(\underline{R}, \nabla) \\ \downarrow & & \downarrow \\ (\mathbf{BT}/R') & \xrightarrow{\cong} & \mathbf{AFDM}(\underline{R}', \nabla). \end{array} \tag{2.3}$$

2.5 – *Partially divided Frobenius*

The setting is the same as in the previous two Sections 2.2 and 2.3. Let $(M, M^1, \mathrm{F}, \mathrm{V}, \nabla_M)$ be an object in $\mathbf{AFDM}(\underline{R}, \nabla)$. Assume now that the submodule $M^1 \subset M$ is a (not just locally) direct summand of M. Let $M = M^1 \oplus M^0$ be a decomposition of M into R-submodules; such a decomposition is called a *normal decomposition* of $\underline{M}$ (or simply of M). Define the maps

$$\Gamma := \frac{1}{p} \cdot \mathrm{F}|_{M^{1,\sigma}} \oplus \mathrm{F}|_{M^{0,\sigma}}, \quad \mathrm{f} := p \cdot \mathrm{id}_{M^{1,\sigma}} \oplus \mathrm{id}_{M^{0,\sigma}}, \tag{2.4}$$

so that we have $\mathrm{F} = \Gamma \circ \mathrm{f}$. We will call Γ the *partially divided Frobenius* of M with respect to the normal decomposition $M = M^1 \oplus M^0$. The next lemma describes the most important property of Γ, with the point being that a normal decomposition of M enables us to decompose F as the composition of an integral part Γ with a rational part f. Such a decomposition is important for later applications.

Lemma 2.9. *The map Γ defined in* (2.4) *is an isomorphism of R-modules.*

Proof. Let us first note that Γ is surjective: indeed, from (2.2) we obtain the equality displayed below, which implies $\mathrm{Im}\,(\Gamma) = M$:

$$\mathrm{F}^{-1}(pM) = \pi^{-1}(\bar{M}^{1,\sigma}) = M^{1,\sigma} + pM^{\sigma} = M^{1,\sigma} \oplus pM^{0,\sigma},$$

where $\pi\colon M^{\sigma} \to \bar{M}^{\sigma}$ is the canonical reduction modulo p map.

It is enough to show that for every maximal ideal $\mathfrak{m}$ of R, the pull-back to $\widehat{R}_{\mathfrak{m}}$ of Γ is an isomorphism. To ease notation, write $A = \widehat{R}_{\mathfrak{m}}$ and let (A, σ) be the unique simple frame of the $\mathfrak{m}$-adic completion of $\bar{R}$, induced by (R, σ) as in Lemma 2.1. By functoriality as discussed above (2.3), the base change along $(R, \sigma) \to (A, \sigma)$

of $\underline{M}$ is equal to the admissibly filtered Dieudonné module of $H \otimes_R A$ if H is the p-divisible group over R corresponding to $\underline{M}$. So we are reduced to showing the Γ map over (A, σ) corresponding to $H \otimes_R A$ and the decomposition $M_A = M_A^1 \oplus M_A^0$ is an isomorphism. But as A is local, the source and target of $\Gamma\colon M^\sigma \to M$ are free A-modules of the same rank; by Nakayama's lemma the assertion follows from the surjectivity of Γ. ■

We remark that in later sections we will not use the full power of the classification results Theorems 2.6 and 2.8. We only need the fact that given a p-divisible group over $\bar{R}$ (over R), one can associate an object in $\mathbf{DM}(\underline{R}, \nabla)$ (in $\mathbf{AFDM}(\underline{R}, \nabla)$) with it and such an association is compatible with base change of simple frames.

3. Good reduction of Shimura varieties of Hodge type

3.1 – *Shimura varieties of Hodge type*

Let $\mathbf{G}$ be a (connected) reductive group over $\mathbb{Q}$ and $\mathbf{X}$ a $\mathbf{G}(\mathbb{R})$ conjugacy class of homomorphisms

$$h\colon \mathbb{S} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \mathbf{G}_{\mathbb{R}}$$

of algebraic groups over $\mathbb{R}$, such that $(\mathbf{G}, \mathbf{X})$ is a Shimura datum in the sense that they satisfy axioms (2.1.1.1)–(2.1.1.3) of [12, §2.1.1]. Suppose that V is a finite-dimensional $\mathbb{Q}$-vector space with a perfect alternating pairing ψ and write $\mathrm{GSp} = \mathrm{GSp}(V, \psi)$ for the corresponding group of symplectic similitudes. Then we get the most important example of a Shimura datum $(\mathrm{GSp}, \mathbf{S}^{\pm})$, with $\mathbf{S}^{\pm}$ the Siegel double space, which is defined to be the set of homomorphisms $\mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}$ such that

(1) the $\mathbb{C}^{\times}$ action on $V_{\mathbb{R}}$ gives rise to a Hodge structure of type $(-1, 0)$ and $(0, -1)$;

(2) $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

In this paper, we consider a Shimura datum $(\mathbf{G}, \mathbf{X})$ of Hodge type, i.e., there exists an embedding of Shimura data $(\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathrm{GSp}, \mathbf{S}^{\pm})$ for some $(\mathrm{GSp}, \mathbf{S}^{\pm})$. Let $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f)$ be an open compact subgroup such that $\mathsf{K}_p \subseteq \mathbf{G}(\mathbb{Q}_p)$ is a hyperspecial subgroup and that $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ is sufficiently small (hence is neat). The condition that K_p is hyperspecial means that there is a reductive group $\mathscr{G}$ over $\mathbb{Z}_{(p)}$, which we fix from now on, such that $\mathsf{K}_p = \mathscr{G}(\mathbb{Z}_p)$. The condition that K^p is sufficiently small guarantees that the double quotient

$$\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})_{\mathbb{C}} := \mathbf{G}(\mathbb{Q})\backslash \mathbf{X} \times \mathbf{G}(\mathbb{A}_f)/\mathsf{K}$$

has the structure of a *smooth* quasi-projective complex variety by a theorem of Baily–Borel. Results of Shimura, Deligne, Milne, and others imply that, up to an isomorphism,

$\mathrm{Sh}_{\mathsf{K}}(\mathbf{G},\mathbf{X})_{\mathbb{C}}$ has a unique quasi-projective smooth model $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G},\mathbf{X})$ over the reflex field E of $(\mathbf{G},\mathbf{X})$. The reflex field E only depends on the Shimura datum $(\mathbf{G},\mathbf{X})$. For $(\mathrm{GSp},\mathbf{S}^{\pm})$, the reflex field is $\mathbb{Q}$.

3.2 – *Integral canonical models*

As explained in [18, Lemma 2.3.1, §2.3.2], for a given Shimura datum $(\mathbf{G},\mathbf{X})$ with embedding $(\mathbf{G},\mathbf{X}) \hookrightarrow (\mathrm{GSp},\mathbf{S}^{\pm})$, using Zarhin's trick we may modify (V,ψ) so that there exists a $\mathbb{Z}_{(p)}$-lattice Λ of V with the following properties: (1) the pairing ψ induces a perfect $\mathbb{Z}_{(p)}$-pairing on Λ, still denoted by ψ; (2) the embedding $\mathbf{G} \to \mathrm{GSp}$ is induced by an embedding $\mathcal{G} \hookrightarrow \mathrm{GSp}(\Lambda,\psi)$ of reductive group schemes over $\mathbb{Z}_{(p)}$. From now on, we fix such an embedding and accordingly the modified embedding of Shimura data $(\mathbf{G},\mathbf{X}) \hookrightarrow (\mathrm{GSp},\mathbf{S}^{\pm})$. Set $\widetilde{\mathsf{K}}_p = \mathrm{GSp}(\mathbb{Z}_p)$. By [18, Lemma 2.1.2] there exists an open compact subgroup $\widetilde{\mathsf{K}}^p \subseteq \mathrm{GSp}(\mathbb{A}_f)$ containing K^p such that ι induces an embedding of Shimura varieties over E,

$$\mathrm{Sh}_{\mathsf{K}} \hookrightarrow \mathrm{Sh}_{\widetilde{\mathsf{K}}} \otimes_{\mathbb{Q}} E.$$

Moreover, if $\widetilde{\mathsf{K}}^p$ is sufficiently small, $\mathrm{Sh}_{\widetilde{\mathsf{K}}}$ has a quasi-projective smooth model over $\mathbb{Z}_{(p)}$, denoted by $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_{\mathsf{K}}$, which has an explicit moduli interpretation as described in [19, §1.3.4]. In what follows we always assume that K^p and $\widetilde{\mathsf{K}}^p$ are sufficiently small, and we will also fix a $\mathbb{Z}$-lattice $\Lambda_{\mathbb{Z}}$ of the $\mathbb{Z}_{(p)}$-module Λ such that $\Lambda_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$ is $\widetilde{\mathsf{K}}$-stable. The choice of such a $\mathbb{Z}$-lattice allows one to describe the scheme $\tilde{\mathcal{S}}$ as the moduli space of polarized abelian varieties (not just up to prime to p-isogeny). In particular, it comes with a universal abelian scheme, denoted by $\mathcal{A}$.

Fix a place v of E above p. Denote by $\mathcal{O}_{E,(v)}$ the localization at v of the ring of integers $\mathcal{O}_E$ of E. Denote by $\mathcal{S} = \mathcal{S}_{\mathsf{K}}(\mathbf{G},\mathbf{X})$ the normalization of the schematic closure of Sh_{K} in $\tilde{\mathcal{S}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$. Recall that we have the assumption $p \geq 3$. The following theorem is now well known and is independently due to Vasiu and Kisin.

Theorem 3.1. *The scheme $\mathcal{S}$ is smooth over $\mathcal{O}_{E,(v)}$ and is the integral canonical model over $\mathcal{O}_{E,(v)}$ of* Sh_{K}.

Strictly speaking, *integral canonical model* refers to a tower of models $\{\mathcal{S}_{\mathsf{K}}\}_{\mathsf{K}^p}$ over $\mathcal{O}_{E,(v)}$ for the tower $\{\mathrm{Sh}_{\mathsf{K}}\}_{\mathsf{K}^p}$, with $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ and K^p varying; see [23, §2] for its precise meaning. Here we abuse language since from this very moment we fix K, till the end of this paper.

In particular, we obtain a finite morphism $\varepsilon\colon \mathcal{S} \to \tilde{\mathcal{S}}$ of schemes over $\mathcal{O}_{E,(v)}$. We call the pull-back to $\mathcal{S}$ of $\mathcal{A}$ the *universal abelian scheme* of $\mathcal{S}$, still denoted by $\mathcal{A}$. Write κ for the residue field of $\mathcal{O}_{E,(v)}$ and $S = S_{\mathsf{K}}$ for the special fibre of $\mathcal{S}$. In particular,

S is a quasi-projective smooth scheme over κ, coming with a universal abelian scheme $\mathcal{A} = \mathcal{A}_\kappa$.

In fact, the existence of the hyperspecial subgroup K_p implies that E is unramified at p ([24, Corollary 4.7]), and hence we have $\mathcal{O}_{E,v} = W(\kappa)$, where $\mathcal{O}_{E,v}$ is the completion of $\mathcal{O}_{E,(v)}$ with respect to its maximal ideal. In what follows, we will mainly work over $W(\kappa)$ or over κ. We will use the same notation for the base change to $W(\kappa)$ of those objects defined over $O_{E,(v)}$ (e.g., the integral model $\mathcal{S}$).

3.3 – *Reduction of Hodge cocharacters and their Frobenius twists*

As shown in [18, Proposition 1.3.2], the $\mathbb{Z}_{(p)}$-reductive group scheme $\mathcal{G}$ can be realized as the schematic stabilizer of a finite set of tensors $(s_\alpha)_\alpha \subseteq \Lambda^{\otimes} = (\Lambda^*)^{\otimes}$; i.e., for any $\mathbb{Z}_{(p)}$-algebra R,

$$\mathcal{G}(R) = \big\{g \in \mathrm{GL}(\Lambda^*_R) \mid g(s_{\alpha,R}) = s_{\alpha,R}\ \forall\alpha\big\},$$

where $s_{\alpha,R} \in (\Lambda^*_R)^{\otimes}$ denotes the tensor induced by s_α. Here, for the functoriality consideration later, we view $\mathcal{G}$ as a reductive $\mathbb{Z}_{(p)}$-subgroup scheme of $\mathrm{GL}(\Lambda^*)$ via the dual representation $\mathrm{GL}(\Lambda) \overset{\mathsf{can}}{\cong} \mathrm{GL}(\Lambda^*)$,

$$\iota\colon \mathcal{G} \hookrightarrow \mathrm{GSp}(\Lambda,\psi) \hookrightarrow \mathrm{GL}(\Lambda) \overset{\mathsf{can}}{\cong} \mathrm{GL}(\Lambda^*). \tag{3.1}$$

Write G for the special fibre of $\mathcal{G}$. It is a (connected) reductive group over $\mathbb{F}_p$.

For any $h \in \mathbf{X}$, there is an associated Hodge cocharacter $\nu_h\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ which can be described as follows. For any $\mathbb{C}$-algebra R, we have $R \otimes_{\mathbb{R}} \mathbb{C} = R \times c^*(R)$, where c denotes complex conjugation. Then, on R-points, ν_h is given by

$$R^\times \hookrightarrow R^\times \times c^*(R)^\times = (R \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{S}(R) \xrightarrow{h} \mathbf{G}_{\mathbb{C}}(R),$$

where the first inclusion is given by $x \in R^\times \mapsto (x,1)$. The unique $\mathbf{G}(\mathbb{C})$-conjugacy class in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}, \mathbf{G}_{\mathbb{C}})$ which contains the *inverses* of all the ν_h is denoted by $[\mu]_{\mathbb{C}}$. Let $\mathsf{Z} = \mathcal{H}om_{\mathbb{Z}_{(p)}}(\mathbb{G}_{m,\mathbb{Z}_{(p)}}, \mathcal{G})$ be the fppf sheaf of cocharacters, and $\mathsf{Ch} = \mathcal{G}\backslash\mathsf{Z}$ the fpqc quotient sheaf of Z by the adjoint action of $\mathcal{G}$. By [13, Chapter XI, Corollary 4.2], the sheaf Z is represented by a smooth separated scheme over $\mathbb{Z}_{(p)}$, and it is shown in [40, Proposition 3.2.1] that Ch is represented by a disjoint union of connected finite étale schemes over $\mathbb{Z}_{(p)}$. Moreover, it is shown in [40] that the $\mathbb{C}$-point of Ch corresponding to the conjugacy class $[\mu]_{\mathbb{C}}$ descends to a $W(\kappa)$-point Ch. We call the resulting κ-point of Ch the *reduction over κ of* $[\mu]_{\mathbb{C}}$ and denote it by $[\mu]_\kappa$. In fact, the conjugacy class $[\mu]_\kappa$ admits a representative

$$\mu\colon \mathbb{G}_{m,\kappa} \to G_\kappa. \tag{3.2}$$

We choose such a representative μ but note that it is the $G(\kappa)$-conjugacy class $[\mu]_\kappa$ that is canonically determined by the Shimura datum $(\mathbf{G}, \mathbf{X})$. We define a Frobenius twist of μ,

$$\sigma(\mu)\colon \mathbb{G}_{m,\kappa} \overset{\mathsf{can}}{\cong} \mathbb{G}_{m,\kappa}^{\sigma} \xrightarrow{\mu^\sigma} G_\kappa^\sigma \overset{\mathsf{can}}{\cong} G_\kappa,$$

where, following our notational convention, $\mu^\sigma\colon \mathbb{G}_{m,\kappa}^\sigma \to G_\kappa^\sigma$ is the base change $\sigma^*\mu$ of μ along the absolute Frobenius $\sigma\colon \kappa \to \kappa$ of κ. In what follows, we will mostly identify μ^σ and $\sigma(\mu)$.

Every element $h \in \mathbf{X}$ defines a Hodge decomposition $V_{\mathbb{C}} = V^{(-1,0)} \oplus V^{(0,-1)}$ via the embedding $\mathbf{X} \hookrightarrow \mathbf{S}^{\pm}$. By definition of $\mathbf{S}^{\pm}$, $\nu_h(z)$ acts on $V^{(-1,0)}$ through multiplication by z and on $V^{(0,-1)}$ as the identity. In particular, ν_h is of weight 1 and 0, and hence $\mu\colon \mathbb{G}_{m,\kappa} \to G_\kappa$ is of weight -1 and 0. Since the scheme Z is smooth, there exists a $\mathcal{G}_{W(\kappa)}$-valued cocharacter $\tilde{\mu}\colon \mathbb{G}_{m,W(\kappa)} \to \mathcal{G}_{W(\kappa)}$, which lifts $\mu\colon \mathbb{G}_{m,\kappa} \to G_\kappa$. From now on we fix such a lift $\tilde{\mu}$, and also define a Frobenius twist of $\tilde{\mu}$,

$$\sigma(\tilde{\mu})\colon \mathbb{G}_{m,W(\kappa)} \overset{\mathsf{can}}{\cong} \mathbb{G}_{m,W(\kappa)}^{\sigma} \xrightarrow{\tilde{\mu}^\sigma} \mathcal{G}_{W(\kappa)}^\sigma \overset{\mathsf{can}}{\cong} \mathcal{G}_{W(\kappa)},$$

where $\tilde{\mu}^\sigma\colon \mathbb{G}_{m,W(\kappa)}^\sigma \to \mathcal{G}_{W(\kappa)}^\sigma$ is the base change $\sigma^*\tilde{\mu}$ of $\tilde{\mu}$ along the Frobenius $\sigma\colon W(\kappa) \to W(\kappa)$. Again, in what follows, we will mostly identify $\sigma(\tilde{\mu})$ and $\tilde{\mu}^\sigma$. Clearly $\mu^\sigma = \sigma(\mu)$ is the reduction modulo p of $\tilde{\mu}^\sigma = \sigma(\tilde{\mu})$. The cocharacter $\tilde{\mu}$ induces the weight decomposition of $\Lambda_{W(\kappa)}$ and $\Lambda^*_{W(\kappa)}$,

$$\Lambda_{W(\kappa)} = \Lambda^0_{W(\kappa)} \oplus \Lambda^{-1}_{W(\kappa)}, \quad \Lambda^*_{W(\kappa)} = \Lambda^{*,0}_{W(\kappa)} \oplus \Lambda^{*,1}_{W(\kappa)}. \tag{3.3}$$

3.4 – *Some group-theoretic preliminaries*

To begin with, we introduce some subgroup schemes of $\mathcal{G}_{W(\kappa)}$ that are induced by $\tilde{\mu}$. Denote by $\mathcal{P}_+ = \mathcal{P}_\mu \subseteq \mathcal{G}_{W(\kappa)}$ the scheme-theoretic stabilizer of the filtration $\Lambda_{W(\kappa)} \supseteq \Lambda^{-1}_{W(\kappa)}$ (equivalently, of the filtration $\Lambda^*_{W(\kappa)} \supseteq \Lambda^{*,1}_{W(\kappa)}$ via dual representations). It is a parabolic subgroup scheme of $\mathcal{G}_{W(\kappa)}$. Similarly, we denote by $\mathcal{P}_- = \mathcal{P}_{\mu^{-1}} \subseteq \mathcal{G}_{W(\kappa)}$ the opposite subgroup scheme of $\mathcal{P}_+$. Write $\mathcal{U}_\pm = \mathcal{U}_\pm(\mu) \subseteq \mathcal{P}_\pm$ for the corresponding unipotent radicals, and $\mathcal{M} = \mathcal{P}_+ \cap \mathcal{P}_-$ for the common Levi subgroup scheme of $\mathcal{P}_-$ and $\mathcal{P}_+$. Note that $\mathcal{M}$ is also the centralizer in $\mathcal{G}_{W(\kappa)}$ of $\tilde{\mu}$.

The next lemma will become useful in later sections; it can also be seen easily using the embedding $\mathcal{G}_{W(\kappa)} \hookrightarrow \mathrm{GL}_{2g,W(\kappa)}$ to be discussed right after this lemma.

LEMMA 3.2. *Let A be a flat $W(\kappa)$-algebra such that $\bar{A} := A/pA \neq 0$. Then we have*

$$\tilde{\mu}(p)\mathcal{P}_+(A)\tilde{\mu}(p)^{-1} \subseteq \mathcal{G}(A), \quad \tilde{\mu}(p)\mathcal{U}_+(A)\tilde{\mu}(p)^{-1} \subseteq \mathrm{K}_1(\mathcal{G})(A),$$

with $\mathrm{K}_1(\mathcal{G})(A) := \{g \in \mathcal{G}(A) \mid \bar{g} = 1 \in G(\bar{A})\}$, *where* $\bar{g}$ *denotes the image of* g *in* $G(\bar{A})$ *under the canonical reduction map* $\mathcal{G}(A) \to G(\bar{A})$.

Proof. Recall the dynamic descriptions of $\mathcal{P}_+$ and $\mathcal{U}_+$ (see for example [9, §2.1]):

$$\mathcal{P}_+(A) = \left\{g \in \mathcal{G}(A) \mid \lim_{t\to 0} \tilde{\mu}(t) g \tilde{\mu}(t)^{-1} \text{ exists}\right\},$$
$$\mathcal{U}_+(A) = \left\{g \in \mathcal{P}^+(A) \mid \lim_{t\to 0} \tilde{\mu}(t) g \tilde{\mu}(t)^{-1} = 1\right\},$$

where the condition $\lim_{t\to 0} \tilde{\mu}(t) g \tilde{\mu}(t)^{-1}$ exists means that the homomorphism of A-group schemes

$$f_{\tilde{\mu},g} \colon \mathbb{G}_{m,A} \to \mathcal{G}_A, \quad t \mapsto \tilde{\mu}(t) g \tilde{\mu}(t)^{-1},$$

extends to a morphism of A-schemes $F_{\tilde{\mu},g} \colon \mathbb{G}_{a,A} \to \mathcal{G}_A$, while the condition $\lim_{t\to 0} \tilde{\mu}(t) g \tilde{\mu}(t)^{-1} = 1$ requires further that $F_{\tilde{\mu},g}(0) = 1 \in \mathcal{G}(A)$. Now let $g \in \mathcal{P}_+(A)$. Since $p \in \mathbb{G}_m(A[\frac{1}{p}]) \cap \mathbb{G}_a(A)$ (A is p-torsion-free), one finds that

$$\tilde{\mu}(p) g \tilde{\mu}(p)^{-1} = f_{g,\tilde{\mu}}(p) = F_{\tilde{\mu},g}(p) \in G(A).$$

If, moreover, $g \in \mathcal{U}_+(A)$, the functoriality of $F_{\tilde{\mu},g}$ for the canonical projection $A \to \bar{A}$, viewed as a map between $W(\kappa)$-algebras, implies

$$\overline{\tilde{\mu}(p) g \tilde{\mu}(p)^{-1}} = \overline{F_{\tilde{\mu},g}(p)} = 1. \qquad \blacksquare$$

For later applications, we fix an embedding of $\mathcal{G}_{W(\kappa)}$ into $\mathrm{GL}_{2\mathrm{g},W(\kappa)}$ as follows: choose a $W(\kappa)$-basis

$$v_1, \ldots, v_{\mathrm{g}}, v_{\mathrm{g}+1}, \ldots, v_{2\mathrm{g}} \in \Lambda^*_{W(\kappa)}$$

such that the first g elements above lie in $\Lambda^{*,1}_{W(\kappa)}$ and the remaining ones lie in $\Lambda^{*,0}_{W(\kappa)}$. Then, by sending an element $h \in \mathrm{GL}(\Lambda^*_{W(\kappa)})$ to the matrix $X_h \in \mathrm{GL}_{2\mathrm{g},W(\kappa)}$ such that

$$h(v_1, \ldots, v_{2\mathrm{g}}) = (v_1, \ldots, v_{2\mathrm{g}}) X_h,$$

we obtain an isomorphism of $W(\kappa)$-group schemes between $\mathrm{GL}(\Lambda^*_{W(\kappa)})$ and $\mathrm{GL}_{2\mathrm{g},W(\kappa)}$. Hence, from (3.1) we obtain an embedding of $\mathcal{G}_{W(\kappa)}$ into $\mathrm{GL}_{2\mathrm{g},W(\kappa)}$, as $W(\kappa)$-reductive group schemes,

$$\iota \colon \mathcal{G}_{W(\kappa)} \hookrightarrow \mathrm{GSp}(\Lambda, \psi)_{W(\kappa)} \hookrightarrow \mathrm{GL}(\Lambda^*_{W(\kappa)}) \cong \mathrm{GL}_{2\mathrm{g},W(\kappa)}, \tag{3.4}$$

and accordingly a cocharacter $\tilde{\mu}' := \iota \circ \tilde{\mu}$ of $\mathrm{GL}_{2\mathrm{g},W(\kappa)}$. In particular, for every $W(\kappa)$-algebra R such that $p \in \mathbb{G}_m(R) = R^\times$, we have

$$\tilde{\mu}'(p) = \begin{pmatrix} p\mathrm{I}_{\mathrm{g}} & \\ & \mathrm{I}_{\mathrm{g}} \end{pmatrix} \in \mathrm{GL}_{2\mathrm{g}}(R).$$

We denote by $\mathscr{P}'_{\pm}$, $\mathscr{U}'_{\pm}$, $\mathscr{M}'$ the counterparts of $\mathscr{P}_{\pm}$, $\mathscr{U}_{\pm}$, $\mathscr{M}$ respectively for the cocharacter $\tilde{\mu}'$ of $\mathrm{GL}_{2g,W(\kappa)}$. Clearly, these subgroups can be described explicitly in term of matrices; for example $\mathscr{U}'_{-} \subseteq \mathscr{G}'_{W(\kappa)}$ consists of matrices of the form

$$\begin{pmatrix} \mathrm{I_g} & \\ * & \mathrm{I_g} \end{pmatrix},$$

where $*$ denotes a $\mathrm{g} \times \mathrm{g}$ matrix. It is a general fact that we have (see [8, Proposition 4.1.10] for example)

$$\mathscr{P}_{\pm} = \mathscr{P}'_{\pm} \cap \mathscr{G}, \quad \mathscr{U}_{\pm} = \mathscr{U}'_{\pm} \cap \mathscr{G}, \quad \mathscr{M} = \mathscr{M}' \cap \mathscr{G}. \tag{3.5}$$

We will see that the embedding ι in (3.4) will enable us to reduce some group-theoretic arguments in later sections to much easier problems like multiplying 2×2 block matrices.

3.5 – *Tensors on* $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{S})$

For all $i \geq 0$, write $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{A}/\mathcal{S}) := \mathbf{R}^i \pi_*(\Omega^{\bullet}_{\mathcal{A}/\mathcal{S}})$ for the ith relative de Rham cohomology of $\mathcal{A}$ over $\mathcal{S}$, where $\pi\colon \mathcal{A} \to \mathcal{S}$ is the structure morphism. As is shown in [3, Proposition 2.5.2] (generalizing the well-known case where the base is a field to the case where the base is an arbitrary scheme), for all $i \geq 0$ (resp. all $r, s \geq 0$), the $\mathcal{O}_{\mathcal{S}}$-module $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{A}/\mathcal{S})$ (resp. $\mathbf{R}^s \pi_*(\Omega^r_{\mathcal{A}/\mathcal{S}})$) is locally free and its formation commutes with arbitrary base change. Moreover, the Hodge–de Rham spectral sequence

$${}_{\mathrm{H}}\mathrm{E}^{r,s} = \mathbf{R}^s \pi_*(\Omega^r_{\mathcal{A}/\mathcal{S}}) \implies \mathrm{H}^{r+s}_{\mathrm{dR}}(\mathcal{A}/\mathcal{S})$$

degenerates at the E_1-page. In particular, we have an exact sequence of locally free $\mathcal{O}_{\mathcal{S}}$-modules

$$0 \to \omega_{\mathcal{A}/\mathcal{S}} \to \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{S}) \to \mathbf{R}^1 \pi_* \mathcal{O}_{\mathcal{S}} \to 0,$$

where the Hodge filtration $\omega_{\mathcal{A}/\mathcal{S}} = \pi_* \Omega^1_{\mathcal{A}/\mathcal{S}}$ is of rank g and $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{S})$ is of rank 2g.

For typographical reasons, in this and the next subsections we write $\mathcal{V}_{\mathrm{dR}}$ for $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{S})$. Below we explain the so-called (*integral*) *de Rham tensors* on $\mathcal{V}_{\mathrm{dR}}$. We will need these tensors to define interesting torsors over $\mathcal{S}$ in Section 3.7.

The $\mathbb{Q}$-representation V of $\mathbf{G}$ coming from the embedding $\mathbf{G} \hookrightarrow \mathrm{GSp}(V, \psi)$ gives rise to a $\mathbb{Q}$-local system $\mathcal{V}_{B,\mathbb{Q}} = \mathbf{R}^1 \pi^{\mathrm{an}}_* \mathbb{Q}$ on $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}$. Below we first explain how the tensors $(s_\alpha)_\alpha \subseteq V^{\otimes}$ that cut out $\mathbf{G}$ inside $\mathrm{GL}(V^*)$ induce global sections on $\mathcal{V}_B^{\otimes}$; cf. [18, §2.2] and [7, §2.3]. We write

$$\widetilde{\mathrm{Sh}_{\mathsf{K}}} = X \times \mathbf{G}(\mathbb{A}_f^p)/\mathsf{K}, \quad \widetilde{\mathrm{Sh}_{\mathsf{K}'}} = \mathbf{S}^{\pm} \times \mathrm{GSp}(\mathbb{A}_f^p)/\mathsf{K}'.$$

Then the canonical projection $\widetilde{\mathrm{Sh}_{\mathsf{K}}} \to \mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}$ (resp. $\widetilde{\mathrm{Sh}_{\mathsf{K}'}} \to \mathrm{Sh}_{\mathsf{K}',\mathbb{C},\mathrm{an}}$) makes $\widetilde{\mathrm{Sh}_{\mathsf{K}}}$ a $\mathbf{G}(\mathbb{Q})$-torsor over $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}$ (resp. $\widetilde{\mathrm{Sh}_{\mathsf{K}'}}$ a $\mathrm{GSp}(\mathbb{Q})$-torsor over $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}',\mathbb{C}}$). To make distinctions, we write $\mathcal{A}'$ for the universal (analytic) abelian variety over $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}',\mathbb{C}}$ with $\pi'\colon \mathcal{A}' \to \mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}',\mathbb{C}}$ the structure map. Then we know that the isogeny class of $\mathcal{A}'$ corresponds to the dual of the $\mathbb{Q}$-local system $\mathbf{R}^1\pi'_*\mathbb{Q}$ (viewed as a variation of Hodge structure over $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}',\mathbb{C}}$), which in turn corresponds to the constant $\mathbb{Q}$-local system V over the cover $\widetilde{\mathrm{Sh}_{\mathsf{K}'}}$, together with the structure morphism $\mathrm{GSp}(\mathbb{Q}) \to \mathrm{GL}(V)$. Clearly we have the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Sh}_{\mathsf{K}}} & \longrightarrow & \widetilde{\mathrm{Sh}_{\mathsf{K}'}} \\ \downarrow & & \downarrow \\ \mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}} & \longrightarrow & \mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}',\mathbb{C}}, \end{array}$$

where the top horizontal map is equivariant with respect to the group homomorphism $\mathbf{G}(\mathbb{Q}) \to \mathrm{GSp}(\mathbb{Q})$. Hence, the variation of Hodge structure $\mathcal{V}_{B,\mathbb{Q}}$ over $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}$, which corresponds to the isogeny class of $\mathcal{A}$, also corresponds to the constant $\mathbb{Q}$-local system V^* over the cover $\widetilde{\mathrm{Sh}_{\mathsf{K}}}$, together with the representation $\mathbf{G}(\mathbb{Q}) \hookrightarrow \mathrm{GSp}(\mathbb{Q}) \hookrightarrow \mathrm{GL}(V) \overset{\mathrm{can}}{\cong} \mathrm{GL}(V^*)$. Now it is clear that the set of tensors $(s_\alpha)_\alpha \subseteq V^{\otimes}$ gives rise to a set of global sections (call them *Betti tensors*),

$$(s_{\alpha,B})_\alpha \subseteq \Gamma(\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}, \mathcal{V}^{\otimes}_{B,\mathbb{Q}}). \tag{3.6}$$

By the Riemann–Hilbert correspondence [11], we have the equivalences of tensor categories

$$\mathrm{Loc}_{\mathbb{C}}(\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}) \xrightarrow[\cong]{(\cdot)\otimes\mathcal{O}_{\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}}} \mathrm{VBIC}(\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}) \xleftarrow[\cong]{(\cdot)^{\mathrm{an}}} \mathrm{VBIC}(\mathrm{Sh}_{\mathsf{K},\mathbb{C}})^{\mathrm{reg}},$$

where $\mathrm{Loc}_{\mathbb{C}}(\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}})$ denotes the tensor category of $\mathbb{C}$-local systems over $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}}$, $\mathrm{VBIC}(\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}})$ (resp. $\mathrm{VBIC}(\mathrm{Sh}_{\mathsf{K},\mathbb{C}})^{\mathrm{reg}}$) denotes the tensor category of holomorphic (resp. algebraic) vector bundles with integrable connections (resp. with integrable connections, with regular singularities at infinity). Under these category equivalences, the $\mathbb{C}$-local system $\mathcal{V}_{B,\mathbb{C}} := \mathcal{V}_{B,\mathbb{Q}} \otimes \mathbb{C}$ corresponds to the vector bundle $\mathcal{V}_{\mathrm{dR},\mathbb{C}} = \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathrm{Sh}_{\mathsf{K},\mathbb{C}})$ over $\mathrm{Sh}_{\mathsf{K},\mathbb{C}}$ and we have a parallel isomorphism of analytic vector bundles over $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}}$,

$$\epsilon\colon \mathcal{V}_{B,\mathbb{C}} \otimes \mathcal{O}_{\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K},\mathbb{C}}} \cong \mathcal{V}^{\mathrm{an}}_{\mathrm{dR},\mathbb{C}},$$

where the left-hand side is equipped with trivial connections. Hence, by transport of structure, we obtain from the Betti tensors (3.6) our desired horizontal global sections (call them *de Rham tensors*),

$$(s_{\alpha,\mathrm{dR}})_\alpha \subseteq \Gamma(\mathrm{Sh}_{\mathsf{K},\mathbb{C}}, \mathcal{V}_{\mathrm{dR},\mathbb{C}}^{\otimes}), \tag{3.7}$$

such that $\epsilon(s_{\alpha,B}) = s_{\alpha,\mathrm{dR}}^{\mathrm{an}}$, with $s_{\alpha,\mathrm{dR}}^{\mathrm{an}} \in \Gamma(\mathrm{Sh}_{\mathsf{K},\mathbb{C}}^{\mathrm{an}}, (\mathcal{V}_{\mathrm{dR},\mathbb{C}}^{\mathrm{an}})^{\otimes})$ understood. Here, note that although $\mathcal{V}_{B,\mathbb{C}}^{\otimes}$ does not live inside $\mathrm{Loc}_{\mathbb{C}}(\mathrm{Sh}_{\mathsf{K},\mathbb{C}}^{\mathrm{an}})$, each Betti tensor $s_{\alpha,B}$ lies in some direct summand of $\mathcal{V}_{B,\mathbb{C}}^{\otimes}$ which does live inside $\mathrm{Loc}_{\mathbb{C}}(\mathrm{Sh}_{\mathsf{K},\mathbb{C}}^{\mathrm{an}})$.

Proposition 3.3 ([18, Lemma 2.2.1, Corollary 2.3.9]). *Each of the de Rham tensors $s_{\alpha,\mathrm{dR}}$ in (3.7) descends to $\mathcal{O}_{E,(v)}$; i.e., there exist (necessarily unique) horizontal global sections*

$$(s_{\alpha,\mathrm{dR}})_\alpha \subseteq \Gamma(\mathcal{S}, \mathcal{V}_{\mathrm{dR}}^{\otimes}),$$

whose restriction on $\mathrm{Sh}_{\mathsf{K},\mathbb{C}}$ are the tensors in (3.7).

We call the global sections $s_{\alpha,\mathrm{dR}}$ obtained here (*integral*) *de Rham tensors*.

3.6 – *Crystalline nature of integral de Rham tensors*

In this subsection we remark on a consequence of Proposition 3.3 concerning the property of integral tensors $s_{\alpha,\mathrm{dR}}$ being horizontal, since it will be needed later.

Let R be a p-complete flat $W(\kappa)$-algebra and $x, y\colon \operatorname{Spec} R \to \mathcal{S}$ morphisms of $W(\kappa)$-schemes which are congruent modulo p. Write $\mathcal{V}_{\mathrm{dR},x}$, $s_{\alpha,\mathrm{dR},x}$ for the pull-backs to R along x of $\mathcal{V}_{\mathrm{dR}}$, $s_{\alpha,\mathrm{dR}}$ respectively, and similarly for $\mathcal{V}_{\mathrm{dR},y}$, $s_{\alpha,\mathrm{dR},y}$. It is a well-known fact that the vector bundle $\mathcal{V}_{\mathrm{dR}}$ on $\mathcal{S}$ has an F-crystal structure in the sense of [15] and the Gauss–Manin connection on $\mathcal{V}_{\mathrm{dR}}$ provides a canonical isomorphism of R-modules (see for example [15, §(1.2)]), $\varepsilon(x,y)\colon \mathcal{V}_{\mathrm{dR},x} \cong \mathcal{V}_{\mathrm{dR},y}$. Since $s_{\alpha,\mathrm{dR}}$ is horizontal, we have

$$\varepsilon(x,y)(s_{\alpha,\mathrm{dR},x}) = s_{\alpha,\mathrm{dR},y}.$$

3.7 – *Torsors over Shimura varieties*

For simplicity, from now on we write s instead of $(s_\alpha)_\alpha$; similarly we simply write s_{dR} instead of $(s_{\alpha,\mathrm{dR}})_\alpha$. For a $W(\kappa)$-morphism $x\colon \operatorname{Spec} R \to \mathcal{S}$, we also write $s_{\mathrm{dR},R}$ for $s_{\mathrm{dR},x}$ (i.e., the pull-back of s_{dR} along x) if the structure morphism x is understood. However, in order to keep notation suggestive, we still write $\mathrm{H}_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{S})$ instead of $\mathcal{V}_{\mathrm{dR}}$. Now we are ready to define two $\mathcal{S}$-schemes below, which will play important roles later:

$$\begin{aligned} \mathbb{I} &:= \mathrm{Isom}_{\mathcal{O}_{\mathcal{S}}}([\Lambda^*_{W(\kappa)}, s_{W(\kappa)}] \otimes \mathcal{O}_{\mathcal{S}}, [\mathrm{H}_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{S}), s_{\mathrm{dR}}]), \\ \mathbb{I}_+ &:= \mathrm{Isom}_{\mathcal{O}_{\mathcal{S}}}([\Lambda^*_{W(\kappa)} \supseteq \Lambda^{*,1}_{W(\kappa)}, s_{W(\kappa)}] \otimes \mathcal{O}_{\mathcal{S}}, [\mathrm{H}_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{S}) \supseteq \omega_{\mathcal{A}/\mathcal{S}}, s_{\mathrm{dR}}]). \end{aligned}$$

Unwinding the definition, for every $W(\kappa)$-algebra R, a point $x^\flat \in \mathbb{I}_+(R)$ consists of a pair (x, β_x), where $x \in \mathcal{S}(R)$ corresponds to a morphism $x\colon \operatorname{Spec} R \to \mathcal{S}$ of $W(\kappa)$-schemes, and

$$\beta_x\colon (\Lambda_R^* \supseteq \Lambda_R^{*,1}) \cong (\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_x/R) \supseteq \omega_x)$$

is an isomorphism of R-modules, which maps s_R to $s_{\mathrm{dR},R}$ termwise. Here, following our notational convention, we denote by $\mathcal{A}_x$, ω_x, $s_{\mathrm{dR},R}$ the pull-back to R along x of $\mathcal{A}$, $\omega_{\mathcal{A}/\mathcal{S}}$, s_{dR} respectively. We have similar descriptions for points $x \in \mathbb{I}(R)$ by omitting filtrations in β_x.

Clearly, $\mathcal{G}$ (resp. $\mathcal{P}_+$) naturally acts on $\mathbb{I}$ (resp. $\mathbb{I}_+$) on the right, freely and transitively. To be precise, the action of a section $h \in \mathcal{G}(R)$ (resp. $h \in \mathcal{P}_+(R)$) on $\mathbb{I}(R)$ (resp. on $\mathbb{I}_+(R)$) is given by

$$x^\flat \cdot h = (x, \beta_x) \cdot h = (x, \beta_x h).$$

Lemma 3.4. *The scheme $\mathbb{I}_+$ (resp. $\mathbb{I}$) is a $\mathcal{P}_+$-torsor (resp. $\mathcal{G}$-torsor) over $\mathcal{S}$.*

Proof. We only show here the assertion for $\mathbb{I}_+$ as the assertion for $\mathbb{I}$ can be shown in the same way; or maybe better, it follows from the fact that $\mathbb{I}$ is the push-forward of $\mathbb{I}_+$ along the homomorphism $\mathcal{P}_+ \hookrightarrow \mathcal{G}$.

Since $\mathbb{I}_+$ is an $\mathcal{S}$-scheme of finite presentation and the action of $\mathcal{P}_+$ on $\mathbb{I}_+$ is free and transitive, it suffices to show that $\mathbb{I}_+$ is faithfully flat over $\mathcal{S}$. In other words, we need to show that for each closed point s of $\mathcal{S}$, the pull-back of $\mathbb{I}_+$ to $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{S},s}$ along the natural map $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{S},s} \to \mathbb{I}_+$, denoted by $\mathbb{I}_{+,\hat{s}}$, is a $\mathcal{P}_+$-torsor over $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{S},s}$. As we already know that $\mathbb{I}_+$ is a $\mathcal{P}_+$-torsor over Sh_K when restricted to the generic fibre Sh_K of $\mathcal{S}$ (as it is so after a further base change to $\mathrm{Sh}_{\mathsf{K},\mathbb{C}}$), we may assume that s is a closed point in the special fibre of $\mathcal{S}$. But in that case, it has been adequately dealt with in [40, Lemma 2.3.2]. ■

4. The zip period map ζ for S

In this section, we review the zip period map $\zeta\colon S \to G\text{-}\mathsf{Zip}^\mu$ constructed by Zhang [40], following [37, 40].

4.1 – *The stack of G-zips*

Let $\mu\colon \mathbb{G}_{m,\kappa} \to G_\kappa$ be as in (3.2), and M, $U_\pm \subseteq P_\pm$ the special fibres of the algebraic groups $\mathcal{M}, \mathcal{U}_\pm \subseteq \mathcal{P}_\pm$ defined in Section 3.4. Recall our notational convention in Section 1.3: for a subgroup $H \subseteq G_\kappa$, we write $H^\sigma \subseteq G_\kappa^\sigma \overset{\text{can}}{\cong} G_\kappa$ for its base change along $\sigma\colon \kappa \to \kappa$.

Definition 4.1 ([31, Definition 3.1]). Let T be a scheme over κ. A *G-zip of type μ* over T is a quadruple $\underline{I} = (I, I_+, I_-, \iota)$ consisting of a right G-torsor I over T, a P_+-torsor $I_+ \subseteq I$ and P_-^σ-torsor $I_- \subseteq I$, and an isomorphism of M^σ-torsors:

$$\iota\colon I_+^\sigma/U_+^\sigma \cong I_-/U_-^\sigma.$$

A morphism $\underline{I} \to \underline{I}' = (I', I_+', I_-', \iota')$ of G-zips of type μ over T consists of a G-equivariant morphism $I \to I'$ which sends I_+ to I_+' and I_- to I_-', and which is compatible with the isomorphisms ι and ι'. The category of G-zips over all κ-schemes form an algebraic stack over κ.

For the cocharacter μ there is an associated group scheme $E_\mu \subseteq P_+ \times P_-^\sigma$, called the *zip group* of μ, which is given on points of a κ-scheme T by

$$E_\mu(T) = \big\{(u_+ m, u_- \sigma(m)) \mid m \in M(T),\ u_+ \in U_+(T),\ u_- \in U_-^\sigma(T)\big\}.$$

Here we use the decomposition of κ-groups $P_+ = U_+ \rtimes M$, $P_- = U_- \rtimes M$. Clearly, we have an isomorphism of κ-group schemes

$$U_+ \rtimes M \ltimes U_- \cong E_\mu, \quad (u_+, m, u_-) \mapsto u_+ m u_-,$$

where we omit describing the group law of the left-hand side. In particular, E_μ is a smooth connected linear algebraic group over κ. Consider its right action on G_κ by

$$g \cdot (p_+, p_-) = p_+^{-1} g p_- = m^{-1} u_+^{-1} g u_- \sigma(m). \tag{4.1}$$

With respect to this action one can form the quotient stack $[G_\kappa/E_\mu]$ over κ. Here we use the right action, while in [30, 31, 40] the left action is used, but apparently the resulting stacks $[E_\mu \backslash G_\kappa]$ and $[G_\kappa/E_\mu]$ are canonically isomorphic.

Theorem 4.2 ([31, Proposition 3.11, Corollary 3.12]). *The stacks $G\text{-}\mathsf{Zip}^\mu$ and $[G_\kappa/E_\mu]$ are naturally isomorphic. They are smooth algebraic stacks of dimension* 0 *over κ.*

4.2 – *The universal G-zip over S*

In this subsection we give definitions of those torsors appearing in the universal G-zip constructed in [40] and refer to [40] for more details.

For the relative de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S)$, apart from the well-known Hodge filtration $\omega_{\mathcal{A}/S} \subseteq \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S)$, there is another filtration

$$\bar{\omega}_{\mathcal{A}/S} := \mathbf{R}^1\pi_*\mathcal{H}^0(\Omega^\bullet_{\mathcal{A}/S}),$$

called the *conjugate filtration* of $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S)$, fitting into the short exact sequence

$$0 \to \overline{\omega}_{\mathcal{A}/S} \to \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S) \to \pi_*\mathcal{H}^1(\Omega^\bullet_{\mathcal{A}/S}) \to 0,$$

of locally free $\mathcal{O}_S$-modules. This short exact sequence is a particular consequence of the degeneration at E_2-page of the conjugate spectral sequence [2]

$$_{\mathrm{conj}}\mathrm{E}_2^{r,s} := \mathbf{R}^r\pi_*\mathcal{H}^s(\Omega^\bullet_{\mathcal{A}/S}) \implies \mathrm{H}^{r+s}_{\mathrm{dR}}(\mathcal{A}/S).$$

As discussed in [28, §7.1–§7.5], Cartier isomorphisms ([28, equation (7.4)]) induce the following direct-summand-wise isomorphism of $\mathcal{O}_S$-modules:

$$\delta\colon \omega^\sigma_{\mathcal{A}/S} \oplus (\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S)/\omega_{\mathcal{A}/S})^\sigma \cong (\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S)/\overline{\omega}_{\mathcal{A}/S}) \oplus \overline{\omega}_{\mathcal{A}/S}. \tag{4.2}$$

We call the direct-summand-wise isomorphism δ the *zip isomorphism* associated with the universal abelian scheme $\mathcal{A}$ over S. The tuple $(\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S), \omega_{\mathcal{A}/S}, \overline{\omega}_{\mathcal{A}/S}, \delta)$ is an "F-zip" in the terminology of [28]. We call it the *universal F-zip* over S. The universal G-zip over S, to be defined below, should be viewed as the universal F-zip over S with a G-structure.

Remark 4.3. The zip isomorphism δ above can also be constructed using crystalline Dieudonné theory, without explicit reference to Cartier isomorphisms, as is done in [40]. Indeed, there are canonical isomorphism of $\mathcal{O}_S$-modules $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S) \cong \mathbb{D}^*(\mathcal{A}[p^\infty])_S \cong \mathbb{D}^*(\mathcal{A})_S$, where $\mathbb{D}^*(\mathcal{A})_S$ is the restriction on S_{Zar} (namely, the Zariski site of S) of the Dieudonné crystal $\mathbb{D}^*(\mathcal{A})$ associated to $\mathcal{A}$ ([3, Définition 2.5.7]). Under this canonical isomorphism, the Hodge filtrations on each side coincide ([3, Proposition 2.5.8]) and the conjugate filtration $\overline{\omega}_{\mathcal{A}/S}$ is equal to $\mathrm{Ker}(V\colon \mathbb{D}^*(\mathcal{A})_S \to \mathbb{D}^*(\mathcal{A})^\sigma_S)$. Then one can proceed to construct δ in the same way as described in Section 6.1 below.

Write $\Lambda^*_\kappa = \Lambda^*_0 \oplus \Lambda^*_{-1}$ for the weight decomposition of Λ^*_κ induced by the *inverse* of μ^σ. Due to the canonical isomorphism $\Lambda^*_\kappa \overset{\mathrm{can}}{\cong} \Lambda^{*,\sigma}_\kappa$, such a decomposition can be described in a different way: if $\Lambda^*_\kappa = \Lambda^{*,0} \oplus \Lambda^{*,1}$ is the weight decomposition of Λ^*_κ induced by μ as in (3.3), we have

$$\Lambda^*_0 = \mathrm{can}^{-1}(\Lambda^{*,0,\sigma}), \quad \Lambda^*_{-1} = \mathrm{can}^{-1}(\Lambda^{*,1,\sigma}). \tag{4.3}$$

(2) The degeneration of the conjugate spectral sequence at E_2-page follows from that of the Hodge–de Rham spectral sequence at E_1-page; see for example [16, Proposition 2.3.2].

Here, $\Lambda^{*,0,\sigma} := (\Lambda^{*,0})^{\sigma}$, and similarly for $\Lambda^{*,1,\sigma}$. Then P_{-}^{σ} is the schematic stabilizer in G_{κ} of the filtration $\Lambda_0^* \subseteq \Lambda_{\kappa}^*$. Now we come to the definitions of the following κ-schemes:

$$\begin{aligned} \mathrm{I} &:= \mathrm{Isom}_{\mathcal{O}_S}([\Lambda_{\kappa}^*, s_{\kappa}] \otimes \mathcal{O}_S, [\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S), s_{\mathrm{dR}}]), \\ \mathrm{I}_+ &:= \mathrm{Isom}_{\mathcal{O}_S}([\Lambda_{\kappa}^* \supseteq \Lambda^{*,1}, s_{\kappa}] \otimes \mathcal{O}_S, [\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S) \supseteq \omega_{\mathcal{A}/S}, s_{\mathrm{dR}}]), \\ \mathrm{I}_- &:= \mathrm{Isom}_{\mathcal{O}_S}([\Lambda_0^* \subseteq \Lambda_{\kappa}^*, s_{\kappa}] \otimes \mathcal{O}_S, [\overline{\omega}_{\mathcal{A}/S} \subseteq \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S), s_{\mathrm{dR}}]). \end{aligned}$$

It is clear that I and I_+ are respectively the special fibres of $\mathbb{I}$ and $\mathbb{I}_+$ here (for more, see Section 3.7). The group G_{κ} (resp. P_+, resp. P_{-}^{σ}) acts on I (resp. I_+, resp. I_-) on the right as in (4.1).

Theorem 4.4 ([40, Theorems 3.4.1, 4.1.2]). *The following hold true:*

(1) *The scheme* I *(resp.* I_+, *resp.* I_-*) is a* G_{κ}*-torsor (resp.* P_+*-torsor, resp.* P_{-}^{σ}*-torsor) over* S.

(2) *The direct-summand-wise isomorphism* δ *in* (4.2) *induces an isomorphism*

$$\iota\colon \mathrm{I}_+^{\sigma}/U_+^{\sigma} \cong \mathrm{I}_-/U_-^{\sigma}.$$

Hence, the tuple $\underline{I} := (\mathrm{I}, \mathrm{I}_+, \mathrm{I}_-, \iota)$ *is a* G*-zip of type* μ *over* S*, inducing a morphism of algebraic stacks over* κ*,*

$$\zeta\colon S \longrightarrow G\text{-}\mathsf{Zip}^{\mu} \cong [G_{\kappa}/E_{\mu}].$$

(3) *The map* ζ *is a smooth map of* κ*-stacks.*

We will call the tuple $\underline{I} = (\mathrm{I}, \mathrm{I}_+, \mathrm{I}_-, \iota)$ the *universal* G*-zip over* S and ζ the *zip period map for* S.

The ultimate goal of [40] was to define and study EO strata of S via the zip period map ζ. Our interest, however, is in the map ζ itself. That being said, for the reader's curiosity, we end this section by giving the definition of EO strata for $S_{\overline{\mathbb{F}}_p}$ and some quick remarks. See [40] for more properties of these EO strata.

Definition 4.5 ([40, Definition 4.1.1]). Set $k = \overline{\mathbb{F}}_p$, an algebraic closure of $\mathbb{F}_p$. For a geometric point $w\colon \mathrm{Spec}\, k \to G\text{-}\mathsf{Zip}^{\mu}$, the *EO stratum* of S_k associated to w, denoted by S_k^w, is defined to be the fibre of w under the zip period map $\zeta_k\colon S_k \to G\text{-}\mathsf{Zip}^{\mu}{}_k$.

Merely by definition of ζ, being a morphism of algebraic stacks, and the property of $[G_{\kappa}/E_{\mu}]$ being a zero-dimensional stack, one learns that each S_k^w is a locally closed subscheme of S_k. Moreover, the smoothness of ζ implies that each S_k^w is automatically a smooth κ-scheme.

5. Construction of $\gamma\colon \mathrm{I}_+ \to G_\kappa/U_-^\sigma$

The main goal of this section is to construct a morphism of κ-schemes $\gamma\colon \mathrm{I}_+ \to G_\kappa/U_-^\sigma$ and to deduce from it a morphism of κ-stacks, $\eta\colon S \to [G_\kappa/E_\mu]$. The comparison of η with ζ will be given in Section 6. Here, G_κ/U_-^σ is the quotient fpqc sheaf of G_κ by the U_-^σ-action via right multiplication. It is represented by a scheme, smooth separated of finite type over κ, and the canonical projection $G_\kappa \to G_\kappa/U_-^\sigma$ is smooth; see for example [25, Propositions 7.15, 7.17].

5.1 – *Trivialized Frobenius*

For the purpose of constructing our map γ, in this section we will consider κ-algebras $\bar{R}$ as in Example 2.2 (letting k equal κ there), i.e., we require that

(5.1) $\qquad$ *$\bar{R}$ is either perfect or smooth of finite type over κ.*

Let $\bar{x}^\flat = (\bar{x}, \beta_{\bar{x}}) \in \mathrm{I}_+(\bar{R})$ be an $\bar{R}$-point of I_+ (cf. Section 3.7). Let $\underline{R} = (R, \sigma)$ be a simple frame of $\bar{R}$ (which exists by Lemma 2.1) and $x^\flat = (x, \beta_x) \in \mathbb{I}_+(R)$ a lift of $\bar{x}^\flat$. Here, the existence of $x^\flat$ follows from the smoothness of $\mathbb{I}_+$ over $W(\kappa)$.

By Theorem 2.8, the p-divisible group $\mathcal{A}_x[p^\infty]$ corresponds to an object in $\mathbf{AFDM}(\underline{R}, \nabla)$, namely an admissibly filtered Dieudonné module over $\underline{R} = (R, \sigma)$,

$$\underline{M} = (M, \mathrm{F}, \mathrm{V}, \nabla_M, M_x^1), \quad \text{with } M = \mathbb{D}^*(\mathcal{A}_x[p^\infty])(R),$$

where, with the simple frame $\underline{R}$ fixed, the Dieudonné module $(M, \mathrm{F}, \mathrm{V}, \nabla_M)$ is determined by $\mathcal{A}_{\bar{x}}[p^\infty]$, and hence by $\bar{x}$, while the admissible filtration $M_x^1 \subseteq M$ depends on the lift x of $\bar{x}$. We use the following notation:

$$\mathrm{F} = \mathrm{F}_x = \mathrm{F}_{\bar{x}}\colon M^\sigma \to M.$$

By Remark 2.7, we have the canonical isomorphism of filtered R-modules

$$(M \supseteq M_x^1) \overset{\text{can}}{\cong} (\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_x/R) \supseteq \omega_x).$$

For this reason we identify them and this identification equips M with a set of tensors $s_{\mathrm{dR},R} \in M^\otimes$. With this identification we view β_x as a trivialization of the filtered module $(M \supseteq M_x^1)$,

$$\beta_x\colon (\Lambda_R^* \supseteq \Lambda_R^{*,1}) \cong (M \supseteq M_x^1).$$

Note that since Λ^* is a free $\mathbb{Z}_{(p)}$-module, we have canonical isomorphisms $\varepsilon\colon (\Lambda_R^*, s_R) \overset{\text{can}}{\cong} (\sigma^*\Lambda_R^*, \sigma^* s_R)$. By transport of structure, we obtain the *trivialized Frobenius*

$$\mathrm{F}_{x^\flat} = \beta_x^{-1}\mathrm{F}\sigma(\beta_x)\colon \Lambda_R^* \to \Lambda_R^*,$$

where we set $\sigma(\beta_x)\colon \Lambda_R^* \to M^\sigma$ to be $\beta_x^\sigma \varepsilon$. For an element $h \in \mathcal{P}_+(R)$, we have

$$\mathrm{F}_{x^\flat \cdot h} = h^{-1}\mathrm{F}_{x^\flat}\sigma(h), \tag{5.2}$$

by definition of the action of $\mathcal{P}_+(R)$ on $\mathbb{I}_+(R)$ (Section 3.7). Here, $\sigma(h)$ is defined as

$$\sigma(h) := \varepsilon^{-1}h^\sigma\varepsilon;$$

this coincides with our notational convention (1.5). Sometimes, we simply identify $\sigma(h)$ and h^σ by suppressing the canonical isomorphism ε above. Clearly, for an element $x^\flat \in \mathbb{I}(R)$, we can define $\mathrm{F}_{x^\flat}$ in the same way.

5.2 – *Frobenius invariance of tensors*

The setting in this subsection is the same as in the previous subsection.

Lemma 5.1. *For an element $x^\flat = (x, \beta_x) \in \mathbb{I}(R)$, the Frobenius $\mathrm{F}_{x^\flat}$ defined above preserves tensors s_R termwise. In particular, we have*

$$\mathrm{F}_{x^\flat} \in \mathcal{G}\Big(R\Big[\frac{1}{p}\Big]\Big).$$

Proof. This follows from the next lemma and the definition of $\mathrm{F}_{x^\flat}$. ■

Lemma 5.2. *The Frobenius $\mathrm{F}\colon M^\sigma \to M$, after inverting p, sends $\sigma_R^* s_{\mathrm{dR},R}$ to $s_{\mathrm{dR},R}$ termwise.*

Proof. For any maximal idea $\mathfrak{m}$ of R, by Lemma 2.1 the Frobenius lift $\sigma\colon R \to R$ induces a simple frame $(\widehat{R}_\mathfrak{m}, \sigma)$ and a homomorphism of simple frames $(R, \sigma) \to (\widehat{R}_\mathfrak{m}, \sigma)$. Note that $\widehat{R}_\mathfrak{m}$ is necessarily p-complete (since $\mathfrak{m}$ contains p). Hence, it suffices to show the lemma after base change to $\widehat{R}_\mathfrak{m}$ for all $\mathfrak{m}$. In particular, we may assume that R is a local ring.

Let $s' \in \mathcal{S}$ be the image of the closed point of $\operatorname{Spec} R$, which necessarily lies in the special fibre $S \subseteq \mathcal{S}$. Let $s \in S$ be a closed point which is a specialization of s'. Then the morphism $x\colon \operatorname{Spec} R \to \mathcal{S}$ factors through the canonical embedding $s\colon \operatorname{Spec} A \to \mathcal{S}$, where $A := \widehat{\mathcal{O}}_{\mathcal{S},s}$ is the complete local ring of $\mathcal{S}$ at s. Choose a $W(k)$-isomorphism $A \cong W(k)[\![X_1, \dots, X_r]\!]$ and consider the Frobenius lift $\sigma_A\colon A \to A$ of A given by sending each X_i to its pth power. Write $\underline{N} := (N, \mathrm{F}_N, \mathrm{V}_N, \nabla_N)$ for the Dieudonné module over (A, σ_A) of $\mathcal{A}_s[p^\infty]$. Then the induced de Rham tensor $s_{\mathrm{dR},A} \in N^\otimes$ is horizontal.

We claim that $s_{\mathrm{dR},A}$ is also Frobenius invariant. Prior to showing the claim, let us note that the claim implies our lemma. Indeed, if we let $f\colon A \to R$ denote the structure morphism, then M is canonically isomorphic to the pull-back $f^*N = N \otimes_{A,f} R$. If we identify this canonical isomorphism, then the Frobenius F_M is equal to

$$M^\sigma \overset{\text{can}}{\cong} \sigma_R^* f^* N \overset{\epsilon}{\cong} f^* \sigma_A^* N \xrightarrow{f^*\mathrm{F}_N} f^* N \overset{\text{can}}{\cong} M,$$

where the isomorphism $\sigma_R^* f^* N \overset{\epsilon}{\cong} f^* \sigma_A^* N$ is provided by the integrable connection ∇_N (note that $\sigma_R \circ f$ and $f \circ \sigma_A$ become the same after modulo p); in fact, due to our choice of free coordinates X_i, it is possible to give an explicit formula for ϵ as in [18, §1.5]. Then, since $s_{\mathrm{dR},A}$ is horizontal, one sees that ϵ sends $\sigma_R^* f^* s_{\mathrm{dR},A}$ to $f^*\sigma_A^* s_{\mathrm{dR},A}$ (this can also be seen from the explicit expression of ϵ). Therefore, it only remains for us to show the claim.

We proceed following [18, Proposition 2.3.5]. Write B for the adapted deformation ring R_{G_W} in [18, Proposition 2.3.5] of the p-divisible group $\mathcal{A}_{\bar{s}}[p^\infty]$ over $k := k(s)$, where we use $\bar{s}\colon \operatorname{Spec} k \to S$ to denote the special fibre of s. Again, the $W(\kappa)$-algebra B is isomorphic to some power series ring over $W(k)$ and we equip it with a Frobenius lift $\sigma_B\colon B \to B$ by sending free coordinates to their pth powers. Write $\underline{L} := (L, \mathrm{F}_L, \mathrm{V}_L, \nabla_L)$ for the Dieudonné module over (B, σ_B) that corresponds to the universal p-divisible group over B; the construction of $\underline{L}$ is explained in [18, §1.5.4]. By construction, the Dieudonné module L comes with *Frobenius-invariant* tensors, which we denote by s_{cris}. We know from the proof of [18, §1.5.4] (see also [21, Theorem 3.3.12] for more details) that there exists a $W(\kappa)$-algebra homomorphism (in fact an isomorphism) $g\colon B \to A$ such that the tuple $(\underline{N}, s_{\mathrm{dR},A})$ is obtained as the pull-back along g of the tuple $(\underline{L}, s_{\mathrm{cris}})$, except that one has to use the integrable connection ∇_L, as we did for ∇_N, to deal with the possible incompatibility of Frobenius lifts between σ_A and σ_B. Again using the fact that s_{cris} is horizontal, we conclude that $s_{\mathrm{dR},A}$ is also Frobenius invariant. ■

5.3 – *Trivialized partially divided Frobenius*

Let $\bar{R}$, $\underline{R}$, $\bar{x}^\flat$, and $x^\flat$ be as in Section 5.1. For the next lemma, which plays an important role for our construction of $\gamma\colon \mathrm{I}_+ \to G_\kappa/U_-^\sigma$ below, one may recall the characters $\tilde{\mu}$ and $\tilde{\mu}^\sigma = \sigma(\tilde{\mu})$ in Section 3.3 and the embedding $\iota\colon \mathcal{G}_{W(\kappa)} \hookrightarrow \mathrm{GL}(\Lambda^*)$ there.

Lemma 5.3. *For every $x^\flat \in \mathbb{I}_+(R)$ we have $\mathrm{F}_{x^\flat} \in \mathcal{G}(R)\tilde{\mu}^\sigma(p) \subseteq \mathcal{G}(R[\frac{1}{p}])$; i.e., we have*

$$\mathrm{F}_{x^\flat} = \mathcal{f}_{x^\flat}\, \tilde{\mu}^\sigma(p)$$

for some (necessarily unique) element $\mathcal{f}_{x^\flat} \in \mathcal{G}(R)$.

Proof. The weight decomposition $\Lambda_R^* = \Lambda_R^{*,1} \oplus \Lambda_R^{*,0}$ given by $\tilde{\mu}$ (see (3.3)), induces via β_x a normal decomposition $M = M^1 \oplus M^0$ of M. Then, by Section 2.5, we have the decomposition

$$\mathrm{F}_x = \Gamma_{x^\flat} \circ \mathrm{f}_{x^\flat},$$

with $\Gamma_{x^\flat}$ and $\mathrm{f}_{x^\flat}$ defined as in (2.4). Note that by Lemma 2.9, the partially divided Frobenius $\Gamma_{x^\flat}$ is an isomorphism of R-modules. Now we have

$$\mathrm{F}_{x^\flat} = \beta_x^{-1}\mathrm{F}_{\bar{x}}\sigma(\beta_x) = \beta_x^{-1}(\Gamma_{x^\flat}\mathrm{f}_{x^\flat})\sigma(\beta_x) = \mathfrak{f}_{x^\flat}(\sigma(\beta_x)^{-1}\mathrm{f}_{x^\flat}\sigma(\beta_x)), \tag{5.3}$$

where $\mathfrak{f}_{x^\flat}$ is defined as

$$\mathfrak{f}_{x^\flat} := \beta_x^{-1}\Gamma_{x^\flat}\sigma(\beta_x).$$

Clearly, we have $\mathfrak{f}_{x^\flat} \in \mathrm{GL}(\Lambda_R^*)$. Unwinding the definition of $\tilde{\mu}^\sigma$ in Section 3.3 we see that

$$\sigma(\beta_x)^{-1}\mathrm{f}_{x^\flat}\sigma(\beta_x) = \tilde{\mu}^\sigma(p).$$

Now, equality (5.3) becomes $\mathrm{F}_{x^\flat} = \mathfrak{f}_{x^\flat}\,\tilde{\mu}^\sigma(p)$, and hence by Lemma 5.1 we have

$$\mathfrak{f}_{x^\flat} \in \mathcal{G}\Big(R\Big[\frac{1}{p}\Big]\Big) \cap \mathrm{GL}(\Lambda_R^*) = \mathcal{G}(R). \qquad \blacksquare$$

We will call $\mathfrak{f}_{x^\flat} \in \mathcal{G}(R)$ the *trivialized partially divided Frobenius* attached to $x^\flat$, with respect to the simple frame $\underline{R} = (R, \sigma)$.

5.4 – *Local version of γ*

We continue our discussion in the setting of the previous subsection. With the simple frame $\underline{R}$ fixed, for an element $x^\flat \in \mathbb{I}_+(R)$, as usual we write $\overline{\mathfrak{f}_{x^\flat}} \in G(\bar{R})$ for the reduction modulo p of the trivialized partially divided Frobenius $\mathfrak{f}_{x^\flat}$. Denote by $\gamma_{\bar{x}^\flat} \in G_\kappa/U_-^\sigma(\bar{R})$ the image of $\overline{\mathfrak{f}_{x^\flat}}$ along the canonical projection $G(\bar{R}) \to G_\kappa/U_-^\sigma(\bar{R})$. The notation $\gamma_{\bar{x}^\flat}$ is justified by the following proposition.

Proposition 5.4. *The element $\gamma_{\bar{x}^\flat} \in G_\kappa/U_-^\sigma(\bar{R})$ is determined by $\bar{x}^\flat$, i.e., it is independent of the choice of lifts $x^\flat$ and the choice of simple frames $\underline{R}$ of $\bar{R}$. In particular, it induces a morphism of κ-schemes,*

$$\gamma_{\bar{x}^\flat}\colon \operatorname{Spec} \bar{R} \to G_\kappa/U_-^\sigma.$$

Proof. Let $\underline{R}' = (R', \sigma')$ be another simple frame of $\bar{R}$ and $y^\flat = (y, \beta_y) \in \mathbb{I}_+(R')$ another lift of $\bar{x}^\flat$. Denote by $\mathfrak{f}_{y^\flat,\sigma'} \in \mathcal{G}(R')$ be the trivialized partially divided Frobenius attached to $y^\flat$, with respect to $\underline{R}'$. We will compare below the two elements $\overline{\mathfrak{f}_{x^\flat}}, \overline{\mathfrak{f}_{y^\flat,\sigma'}} \in G(\bar{R})$.

Note first that we may assume $R = R'$. Indeed, by Lemma 2.1 we can choose an isomorphism of $W(\kappa)$-algebras $\varepsilon\colon R' \cong R$ whose reduction modulo p is $\mathrm{id}_{\bar{R}}$. This isomorphism induces another frame structure,

$$(R, \sigma') := \varepsilon^*(R', \sigma')$$

on R. Denote by $\int_{\varepsilon^*(y^\flat),\sigma'}$ the trivialized partially divided Frobenius of $\varepsilon^*(y^\flat) \in \mathbb{I}_+(R)$, with respect to simple frames (R, σ'). Then clearly we have $\overline{\int_{y^\flat,\sigma'}} = \overline{\int_{\varepsilon^*(y^\flat),\sigma'}}$.

From now on, we assume $R = R'$ and denote by $\int_{y^\flat} \in \mathscr{G}(R)$ the trivialized partially divided Frobenius attached to $y^\flat$, with respect to $\underline{R}$. Now the proposition follows from the combination of Lemma 5.5 below which compares $\overline{\int_{x^\flat}}$ and $\overline{\int_{y^\flat}}$, and Lemma 5.6 below which compares $\overline{\int_{y^\flat}}$ and $\overline{\int_{y^\flat,\sigma'}}$. ■

Lemma 5.5. *With the simple frame $\underline{R}$ fixed, if $y^\flat = (y, \beta_y) \in \mathbb{I}_+(R)$ is another lift of $\bar{x}^\flat$, then there exists $u_- \in U_-(\bar{R})$, such that following holds in $G(\bar{R})$:*

$$\overline{\int_{x^\flat}} = \overline{\int_{y^\flat}} \cdot \sigma(u_-).$$

Proof. Recall that by Section 3.6 we have the canonical parallel isomorphism

$$\varepsilon(x, y)\colon \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_x/R) \overset{\mathrm{can}}{\cong} \mathbb{D}^*(\mathcal{A}_{\bar{x}})(R) \overset{\mathrm{can}}{\cong} \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_y/R),$$

which carries $s_{\mathrm{dR},x}$ to $s_{\mathrm{dR},y}$. Hence, for our purpose, we may assume $x = y$; note however that this does *not* mean[3] that $\beta_x = \beta_y$. Denote by $\mathbb{I}_x$ the trivial $\mathscr{G}$-torsor over Spec R, obtained as the pull-back to Spec R of $\mathbb{I}$ along $x\colon \mathrm{Spec}\, R \to \mathcal{S}$ and view β_x, β_y as elements in $\mathbb{I}_x(R)$. Write $h := \beta_y^{-1} \circ \beta_x \in \mathscr{G}(R)$. By (5.2) we have

$$\int_{x^\flat} = h^{-1} \int_{y^\flat} \cdot \sigma(\tilde{\mu}(p) h \tilde{\mu}(p)^{-1}).$$

Hence, it suffices to show that

$$\tilde{\mu}(p) h \tilde{\mu}(p)^{-1} \in \mathscr{G}(R), \quad \overline{\tilde{\mu}(p) h \tilde{\mu}(p)^{-1}} \in U_-(\bar{R}).$$

To show these we use the embedding $\iota\colon \mathscr{G}_{W(\kappa)} \hookrightarrow \mathrm{GL}_{2g,W(\kappa)}$ introduced in Section 3.4. Since we have $\tilde{\mu}(p) h \tilde{\mu}(p)^{-1} \in \mathscr{G}(R[\frac{1}{p}])$, in order to show that it lies in $\mathscr{G}(R)$, it suffices to show that it lies in $\mathrm{GL}_{2g,W(\kappa)}(R)$. Moreover, by (3.5), in order to show $\overline{\tilde{\mu}(p) h \tilde{\mu}(p)^{-1}} \in U_-(\bar{R})$, we may replace $\tilde{\mu}$ by the induced cocharacter $\tilde{\mu}' = \iota \circ \tilde{\mu}$

(3) One may instead write β'_x for β_y in the discussion below, but for typographical reasons we choose not to do so.

of $\mathrm{GL}_{2g,W(\kappa)}$. Inside $\mathrm{GL}_{2g}(R[\frac{1}{p}])$, $\tilde{\mu}'(p)$ and h (note that $\bar{h}=1$) are represented by matrices of the respective forms

$$\begin{pmatrix} p\mathrm{I}_g & \\ & \mathrm{I}_g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{I}_g + pA & pB \\ pC & \mathrm{I}_g + pD \end{pmatrix}, \tag{5.4}$$

where A, B, C, D are $g \times g$ matrices with entries in R. Now our problems become trivial due to the discussion at the end of Section 3.4:

$$\begin{aligned} &\begin{pmatrix} p\mathrm{I}_g & \\ & \mathrm{I}_g \end{pmatrix} \begin{pmatrix} \mathrm{I}_g + pA & pB \\ pC & \mathrm{I}_g + pD \end{pmatrix} \begin{pmatrix} p\mathrm{I}_g & \\ & \mathrm{I}_g \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathrm{I}_g + pA & p^2B \\ C & \mathrm{I}_g + pD \end{pmatrix}. \end{aligned} \tag{5.5}$$

∎

Lemma 5.6. *Fix a lift $x^\flat \in \mathbb{I}_+(R)$ of $\bar{x}^\flat$ and let $\underline{R}' = (R, \sigma')$ be another simple frame of $\bar{R}$. Then there exists an element $u_- \in U_-^\sigma(\bar{R})$, such that*

$$\overline{\int_{x^\flat}} = \overline{\int_{x^\flat,\sigma'}} \cdot u_-.$$

Proof. As in the proof of Lemma 5.2, the connection ∇_M provides a canonical parallel isomorphism $\iota\colon \sigma^*M \to \sigma'^*M$ such that $\mathrm{F} = \mathrm{F}' \circ \iota$ given the fact that σ and σ' become the same after modulo p. One can see by direct computation that

$$\int_{x^\flat} = \int_{x^\flat,\sigma'} (\tilde{\mu}^\sigma(p) h \tilde{\mu}^\sigma(p)^{-1}),$$

where $h := \sigma'(\beta_x)^{-1}\iota\sigma(\beta_x) \in \mathcal{G}(R)$, and the superscript "$\sigma$" in $\tilde{\mu}^\sigma$ certainly refers to the Frobenius lift $\sigma\colon W(\kappa) \to W(\kappa)$. Again we use the embedding $\iota\colon \mathcal{G}_{W(\kappa)} \hookrightarrow \mathrm{GL}_{2g,W(\kappa)}$ in Section 3.4, but in a twisted manner. To be precise, the pull-back of ι along $\sigma\colon W(\kappa) \to W(\kappa)$ induces another embedding

$$\sigma(\iota)\colon \mathcal{G} \overset{\text{can}}{\cong} \mathcal{G}^\sigma \xrightarrow{\iota^\sigma} \mathrm{GL}^\sigma_{2g,W(\kappa)} \overset{\text{can}}{\cong} \mathrm{GL}_{2g,W(\kappa)}.$$

Exactly as in the proof of Lemma 5.5, it suffices to show $\overline{\tilde{\mu}^\sigma(p) h \tilde{\mu}^\sigma(p)^{-1}} \in U_-^{\prime,\sigma}(\bar{R})$, with $U_-^{\prime,\sigma} \subseteq \mathrm{GL}_{2g,\kappa}$ the counterpart of $U_-^\sigma \subseteq G_\kappa$ for the cocharacter

$$\mathbb{G}_{m,\kappa} \overset{\text{can}}{\cong} \mathbb{G}^\sigma_{m,\kappa} \xrightarrow{\mu^{\prime,\sigma}} \mathrm{GL}^\sigma_{2g,\kappa} \overset{\text{can}}{\cong} \mathrm{GL}_{2g,\kappa}.$$

Via the embedding $\sigma(\iota)$, $\tilde{\mu}^\sigma(p)$ and h are represented inside $\mathrm{GL}_{2g}(R[\frac{1}{p}])$ by matrices of the same forms as in (5.4) respectively, and $U_-^{\prime,\sigma}(\bar{R})$ consists of matrices of the form

$$\begin{pmatrix} \mathrm{I}_g & \\ * & \mathrm{I}_g \end{pmatrix}.$$

At the end, we have to perform the same calculation as in (5.5). ∎

5.5 – *The global map* $\gamma\colon \mathrm{I}_+ \to G_\kappa/U_-^\sigma$ *via gluing*

In this subsection we apply Proposition 5.4 to construct the global map $\gamma\colon \mathrm{I}_+ \to G_\kappa/U_-^\sigma$. For this we take a Zariski affine open covering $\{\bar{x}_i^\flat\colon \operatorname{Spec} \bar{R}_i \hookrightarrow \mathrm{I}_+\}$ of I_+. Since each $\bar{R}_i$ is smooth over κ, we can apply Proposition 5.4 and obtain morphisms of κ-schemes,

$$\gamma_i = \gamma_{\bar{x}_i^\flat}\colon \operatorname{Spec} \bar{R}_i \to G_\kappa/U_-^\sigma. \tag{5.6}$$

Theorem 5.7. *The maps* γ_i *defined in* (5.6) *glue to a map of* κ*-schemes,*

$$\gamma\colon \mathrm{I}_+ \longrightarrow G_\kappa/U_-^\sigma.$$

Proof. Since I_+ is quasi-projective (hence separated), the intersection of $\operatorname{Spec} \bar{R}_i$ and $\operatorname{Spec} \bar{R}_j$ is again affine. Denote it by $\operatorname{Spec} \bar{R}_{ij}$. We need to show γ_i and γ_j restrict to the same map on $\operatorname{Spec} \bar{R}_{ij}$ for all i, j. But since $\bar{R}_{ij}$ is again a smooth κ-algebra, this follows from the next lemma, Lemma 5.8. ∎

Lemma 5.8. *Given two* κ*-algebras* $\bar{R}$, $\bar{R}'$ *satisfying* (5.1), *and a morphism of* κ*-schemes* $\bar{x}^\flat\colon \operatorname{Spec} \bar{R} \to \mathrm{I}_+$, *then for any morphism of* κ*-schemes* $\xi\colon \operatorname{Spec} \bar{R}' \to \operatorname{Spec} \bar{R}$, *we have*

$$\gamma_{\bar{x}^\flat \circ \xi} = \gamma_{\bar{x}^\flat} \circ \xi.$$

Proof. This is immediate from our construction in Proposition 5.4 if there exists a homomorphism of simple frames $f\colon (R,\sigma) \to (R',\sigma)$ which lifts the structure map $\bar{R} \to \bar{R}'$. In general we do not know whether such an f always exists; below we proceed by reducing the general case to cases where f does exist.

Note first that for each $\mathbb{F}_p$-algebra $\bar{A}$ which Zariski locally admits a p-basis, the absolute Frobenius map $\sigma\colon \bar{A} \to \bar{A}$ is faithfully flat (the local existence of a p-basis implies that $\bar{A}$ as an $\bar{A}$-module via σ, is locally free). Consequently, the canonical ring homomorphism

$$\bar{A} \to \bar{A}_{\mathrm{perf}} := \varinjlim_{\sigma\colon \bar{A} \to \bar{A}} \bar{A}$$

is faithfully flat. In particular, horizontal arrows in the commutative diagram

$$\begin{array}{ccc} G_\kappa/U_-^\sigma(\bar{R}) & \longrightarrow & G_\kappa/U_-^\sigma(\bar{R}_{\mathrm{perf}}) \\ \downarrow & & \downarrow \\ G_\kappa/U_-^\sigma(\bar{R}') & \longrightarrow & G_\kappa/U_-^\sigma(\bar{R}'_{\mathrm{perf}}) \end{array}$$

are injective. As the formation of $W(\cdot)$ is functorial, we are now reduced to showing that $\gamma_{\bar{x}^\flat \circ \pi} = \gamma_{\bar{x}^\flat} \circ \pi$ with $\pi\colon \operatorname{Spec} \bar{R}_{\mathrm{perf}} \to \operatorname{Spec} \bar{R}$ being the canonical morphism.

This is clear as soon as we realize that there is a homomorphism of simple frames over $W(\kappa)$ given by

$$(R,\sigma) \to (R,\sigma)_{\mathrm{perf}} := (\widehat{R_{\mathrm{perf}}},\sigma) \cong (W(\bar{R}_{\mathrm{perf}}),\sigma),$$

which lifts the structure map $\bar{R} \to \bar{R}_{\mathrm{perf}}$ (also, see [39, Lemma 6.12] for another construction). Here, R_{perf} is defined as the colimit perfection,

$$R_{\mathrm{perf}} := \varinjlim_{\sigma: R \to R} R$$

and $\widehat{R_{\mathrm{perf}}}$ is the p-completion of R_{perf}. We clearly have $R_{\mathrm{perf}}/pR_{\mathrm{perf}} = \bar{R}_{\mathrm{perf}}$. The Frobenius lift $\sigma_R\colon R \to R$ induces a Frobenius lift σ on R_{perf} (and, hence on $\widehat{R_{\mathrm{perf}}}$ too) and we have our homomorphism $(R,\sigma) \to (R,\sigma)_{\mathrm{perf}}$ of simple frames written above. In fact, R_{perf} is p-torsion-free and the simple frame $(R_{\mathrm{perf}},\sigma)$, viewed as a crystalline prism (recall Remark 2.4), is nothing but the perfection of the prism (R,σ) in the sense of Bhatt and Scholze [6, Corollary 2.31 and Lemma 3.9]. The isomorphism of simple frames $(R,\sigma)_{\mathrm{perf}} \cong (W(\bar{R}_{\mathrm{perf}}),\sigma)$ has, thereby, been justified. ■

5.6 – *The zip period map η*

The natural embedding $U_-^\sigma \hookrightarrow E_\mu$ realizes U_-^σ as a normal subgroup of E_μ. Via this embedding U_-^σ acts on G_κ by right multiplication. Passing to the quotient, we obtain an action of $P_+ = E_\mu/U_-^\sigma$ on G_κ/U_-^σ given on local sections by $g \cdot p_+ = p_+^{-1} g\sigma(m)$, where $p_+ = u_+ m$, with $u_+ \in U_+$ and $m \in M$. Denote by $[(G_\kappa/U_-^\sigma)/P_+]$ the resulting quotient algebraic stack over κ. Since the action of U_-^σ on G_κ is free, the canonical projection $G_\kappa \to G_\kappa/U_-^\sigma$ induces a canonical isomorphism of algebraic stacks over κ,

$$[G_\kappa/E_\mu] \cong [(G_\kappa/U_-^\sigma)/P_+].$$

THEOREM 5.9. *The map γ is equivariant with respect to the actions of P_+ on I_+ and on G_κ/U_-^σ, and hence induces a morphism of algebraic stacks over κ,*

$$\eta\colon S \cong \mathrm{I}_+/P_+ \to [(G_\kappa/U_-^\sigma)/P_+] \cong [G_\kappa/E_\mu] \cong G\text{-}\mathsf{Zip}^\mu.$$

PROOF. We need to show the commutativity of the following diagram of κ-schemes:

$$\begin{array}{ccc} \mathrm{I}_+ \times_\kappa P_+ & \xrightarrow{\gamma \times \mathrm{id}_{P_+}} & (G_\kappa/U_-^\sigma) \times_\kappa P_+ \\ \downarrow & & \downarrow \\ \mathrm{I}_+ & \xrightarrow{\gamma} & G_\kappa/U_-^\sigma, \end{array}$$

where vertical arrows are given by P_+-actions. Since $\mathrm{I}_+ \times_\kappa P_+$ is geometrically reduced, it suffices to check the commutativity on k-points for an algebraically closed field extension k of κ. Note first that for any $\bar{x}^\flat \in \mathrm{I}_+(k)$, by Lemma 5.8, we have $\gamma(\bar{x}^\flat) = \gamma_{\bar{x}^\flat}$. For any k-point $(\bar{x}^\flat, \bar{p}_+)$ of $\mathrm{I}_+ \times_\kappa P_+$, take a $W(\kappa)$-point $(x^\flat, p_+)$ of $\mathbb{I}_+ \times_{W(\kappa)} \mathcal{P}_+$, which lifts $(\bar{x}^\flat, \bar{p}_+)$. Then $x^\flat \cdot p_+$ is a lift of $\bar{x}^\flat \cdot \bar{p}_+$. Applying the construction in Section 5.4, we obtain an element $\mathfrak{f}_{x^\flat \cdot p_+} \in \mathcal{G}(W(\kappa))$. A direct calculation using the relation (5.2) gives

$$\mathfrak{f}_{x^\flat \cdot p_+} = p_+^{-1} \mathfrak{f}_{x^\flat}(\tilde{\mu}^\sigma(p)\sigma(p_+)\tilde{\mu}^\sigma(p)^{-1}) = p_+^{-1} \mathfrak{f}_{x^\flat}\, \sigma(\tilde{\mu}(p)u_+\tilde{\mu}(p)^{-1})\sigma(m),$$

where $p_+ = u_+ m$, with $u_+ \in \mathcal{U}_+(W(k))$ and $m \in \mathcal{M}(W(k))$, and where for the second "=" one uses the fact that m commutes with $\tilde{\mu}(p)$ and the trivial fact that $\tilde{\mu}^\sigma(p) = \sigma(\tilde{\mu}(p))$. But by Lemma 3.2, the element $\tilde{\mu}(p)u_+\tilde{\mu}(p)^{-1} \in \mathcal{G}(W(k)[\frac{1}{p}])$ actually lies in $\mathcal{G}(W(k))$ and we have $\overline{\tilde{\mu}(p)u_+\tilde{\mu}(p)^{-1}} = 1 \in G(\bar{R})$. ■

6. Comparison of η with ζ

In this section we show that the map $\eta\colon S \to G\text{-Zip}^\mu$ constructed in Theorem 5.9 coincides with the map $\zeta\colon S \to G\text{-Zip}^\mu$ in [40], in the sense that they are naturally 2-isomorphic. The strategy is to show that there is a natural isomorphism between their corresponding objects in the groupoid $[G_\kappa/E_\mu](S)$.

6.1 – *Zip isomorphisms associated with Dieudonné modules*

As preparation for the next subsection, we let $\bar{R}$ be as in Section 5.1 and $\underline{R} = (R, \sigma)$ a simple frame of it. Take a point $\bar{x} \in S(\bar{R})$ and denote by $\underline{M} = (M, \mathrm{F}, \mathrm{V}, \nabla)$ the Dieudonné module over $\underline{R}$ that is associated with the p-divisible group $\mathcal{A}_{\bar{x}}[p^\infty]$. Write $\bar{\mathrm{F}}\colon \bar{M}^\sigma \to \bar{M}$, $\bar{\mathrm{V}}\colon \bar{M} \to \bar{M}^\sigma$ for the reduction modulo p of F, V respectively; note however that $\bar{M}$, $\bar{\mathrm{F}}$, $\bar{\mathrm{V}}$ are independent of the choice of $\underline{R}$, as they can be obtained by taking evaluation at the trivial PD thickening $\bar{R} \xrightarrow{\mathrm{id}} \bar{R}$ of the Dieudonné crystal $\mathbb{D}^*(\mathcal{A}_{\bar{x}})[p^\infty]$; see Section 2.2. Then the relations $\mathrm{F} \circ \mathrm{V} = p \cdot \mathrm{id}_M$ and $\mathrm{V} \circ \mathrm{F} = p \cdot \mathrm{id}_{M^\sigma}$ give rise to an exact sequence of $\bar{R}$-modules

$$\bar{M}^\sigma \xrightarrow{\bar{\mathrm{F}}} \bar{M} \xrightarrow{\bar{\mathrm{V}}} \bar{M}^\sigma \xrightarrow{\bar{\mathrm{F}}} \bar{M}.$$

And hence canonical isomorphisms $\bar{\mathrm{F}}\colon \bar{M}^\sigma/\operatorname{Ker}(\bar{\mathrm{F}}) \xrightarrow{\cong} \operatorname{Ker}(\bar{\mathrm{V}})$, $[\bar{\mathrm{V}}]\colon \bar{M}/\operatorname{Ker}(\bar{\mathrm{V}}) \xrightarrow{\cong} \operatorname{Ker}(\bar{\mathrm{F}})$; combining them, we obtain a canonical direct-summand-wise isomorphism of $\bar{R}$-modules

$$\delta\colon \operatorname{Ker}(\bar{\mathrm{F}}) \oplus \bar{M}^\sigma/\operatorname{Ker}(\bar{\mathrm{F}}) \xrightarrow{[\bar{\mathrm{V}}]^{-1} \oplus \bar{\mathrm{F}}} \bar{M}/\operatorname{Ker}(\bar{\mathrm{V}}) \oplus \operatorname{Ker}(\bar{\mathrm{V}}). \tag{6.1}$$

We call δ above the *zip isomorphism* associated with the Dieudonné module $\bar{M}$. Now we make a connection to the zip isomorphism we defined in (4.2) (cf. Remark 4.3). Let $\bar{M}I \subseteq \bar{M}$ be the Hodge filtration of $\bar{M}$ as introduced in Section 2.3. As recalled in (2.2), we have $\bar{M}^{1,\sigma} = \mathrm{Ker}(\bar{\mathrm{F}}) \subseteq \bar{M}^{\sigma}$. We identify the following canonical isomorphism:

$$(\bar{M} \supseteq \bar{M}^1) \overset{\text{can}}{\cong} (\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\bar{x}}/\bar{R}) \supseteq \omega_{\bar{x}}).$$

Write $\bar{M}_0 := \mathrm{Ker}(\bar{\mathrm{V}}) = \mathrm{Im}(\bar{\mathrm{F}}) \subseteq \bar{M}$. Under the canonical isomorphism $\bar{M} \overset{\text{can}}{\cong} \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\bar{x}}/\bar{R})$, $\bar{M}_0$ corresponds to the conjugate filtration $\bar{\omega}_{\bar{x}}$ of $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\bar{x}}/\bar{R})$. We also identify the canonical isomorphism

$$(\bar{M}_0 \subseteq \bar{M}) \overset{\text{can}}{\cong} (\bar{\omega}_{\bar{x}} \subseteq \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\bar{x}}/\bar{R})).$$

With these identifications, the zip isomorphism (6.1) is nothing but the pull-back to $\bar{R}$ along $\bar{x}$ of the zip isomorphism (4.2). In what follows we write δ in the form

$$\delta\colon \bar{M}^{1,\sigma} \oplus \bar{M}^{\sigma}/\bar{M}^{1,\sigma} \xrightarrow{[\bar{\mathrm{V}}]^{-1} \oplus \bar{\mathrm{F}}} \bar{M}/\bar{M}_0 \oplus \bar{M}_0. \tag{6.2}$$

6.2 – *A canonical* 2*-isomorphism between* η *and* ζ

Under the isomorphism $G\text{-}\mathsf{Zip}^{\mu} \cong [G_{\kappa}/E_{\mu}]$, the universal G-zip in Section 4.2, $\underline{I}$, corresponds to an E_{μ}-torsor $\mathcal{Z} = \mathcal{Z}_{\zeta}$ over S, together with an E_{μ}-equivariant map $\tilde{\zeta}\colon \mathcal{Z} \to G_{\kappa}$. The E_{μ}-torsor $\mathcal{Z}$ is given by the pull-back of the canonical projection $\mathrm{I}_- \to \mathrm{I}_-/U_-^{\sigma}$ of S-morphism along the S-morphism

$$\mathrm{I}_+ \xrightarrow{\sigma} \mathrm{I}_+^{\sigma} \to \mathrm{I}_+^{\sigma}/U_+^{\sigma} \xrightarrow{\iota} \mathrm{I}_-/U_-^{\sigma}.$$

The map $\tilde{\zeta}\colon \mathcal{Z} \to G_{\kappa}$ is given by sending a local section $(\bar{x}^{\flat}, {}^{\flat}\bar{x})$ of $\mathcal{Z} \subseteq \mathrm{I}_+ \times_S \mathrm{I}_-$ to

$$\tilde{\zeta}(\bar{x}^{\flat}, {}^{\flat}\bar{x}) := \beta_{\bar{x}}^{-1} \circ \theta_{\bar{x}},$$

which is a local section of $G_{\kappa} \subseteq \mathrm{GL}(\Lambda_{\kappa}^*)$. Here, ${}^{\flat}\bar{x} = (\bar{x}, \theta_{\bar{x}})$ is a local section of I_-, with the same underlying point $\bar{x}$ as that of $\bar{x}^{\flat}$. On the other hand, under the isomorphism $[(G_{\kappa}/U_-^{\sigma})/P_+] \cong [G_{\kappa}/E_{\mu}]$, the P_+-equivariant map corresponds to an E_{μ}-torsor $\mathcal{Z}_{\eta}$ over S, given by the pull-back of $\gamma\colon \mathrm{I}_+ \to G_{\kappa}/U_-^{\sigma}$ along the canonical projection $G_{\kappa} \to G_{\kappa}/U_-^{\sigma}$, together with an E_{μ}-equivariant map $\tilde{\eta}\colon \mathcal{Z}_{\eta} \to G_{\kappa}$ given by the canonical projection from $\mathcal{Z}_{\eta}$ to G_{κ}. The right E_{μ}-action on $\mathcal{Z}_{\eta}$ is given by

$$(\bar{x}^{\flat}, g) \cdot (p_+, p_-) = (\bar{x}^{\flat} \cdot p_+, p_+^{-1} g p_-).$$

Theorem 6.1. *There is a natural isomorphism $\mathcal{Z} \cong \mathcal{Z}_{\eta}$ of E_{μ}-torsors over S. In other words, the two morphisms of κ-algebraic stacks ζ and η are 2-isomorphic.*

Proof. Note that it is enough to show the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\tilde{\xi}} & G_\kappa \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow \\ \mathrm{I}_+ & \xrightarrow{\gamma} & G_\kappa / U_-^\sigma. \end{array}$$

This is because, once it is shown, one sees readily that the induced morphism $\mathcal{Z} \to \mathcal{Z}_\eta$ of S-schemes, given on local sections by

$$(\bar{x}^\flat, {}^\flat\bar{x}) \longmapsto (\bar{x}^\flat, \beta_{\bar{x}}^{-1} \circ \theta_{\bar{x}}),$$

is E_μ-equivariant, and hence is a morphism of E_μ-torsors over S, and hence is automatically an isomorphism.

Clearly the problem is local on $\mathcal{Z}$. Let $\bar{z} = (\bar{x}^\flat, {}^\flat\bar{x})\colon \operatorname{Spec} \bar{R} \hookrightarrow \mathcal{Z}$ be an affine open of $\mathcal{Z}$. Since in the discussion below, the underlying point $\bar{x} \in S(\bar{R})$ is fixed, to ease notation we may write $\beta_{\bar{x}} \in \mathrm{I}_+(\bar{R})$ instead of $(\bar{x}, \beta_{\bar{x}}) \in \mathrm{I}_+(\bar{R})$, and similarly for points in $\mathrm{I}_-(\bar{R})$. We need to show that the image of $\tilde{\xi}(\beta_{\bar{x}}, \theta_{\bar{x}}) = \beta_{\bar{x}}^{-1} \circ \theta_{\bar{x}} \in G(\bar{R})$ in $G_\kappa / U_-^\sigma(\bar{R})$ coincides with $\gamma(\beta_{\bar{x}})$. Again since $\mathcal{Z}$ is a smooth κ-scheme, it satisfies condition (5.1). Choose a simple frame (R, σ) for $\bar{R}$ and a lift $x^\flat \in \mathbb{I}_+(R)$ for $\bar{x}^\flat$. It follows from Lemma 5.8 and Proposition 5.4 that $\gamma(\beta_{\bar{x}})$ is equal to the image in $G_\kappa / U_-^\sigma(\bar{R})$ of

$$\mathscr{f}_{x^\flat} = \beta_x^{-1} \Gamma_{x^\flat} \sigma(\beta_x) \in \mathscr{G}(R).$$

Set

$$\theta'_x := \Gamma_{x^\flat} \sigma(\beta_x)\colon (\Lambda_R^*, s_R) \to (M, s_{\mathrm{dR},R}),$$

$$\theta'_{\bar{x}} := \overline{\theta'_x} = \overline{\Gamma_{x^\flat}} \sigma(\beta_{\bar{x}}).$$

Lemma 6.2. *We have* $(\beta_{\bar{x}}, \theta'_{\bar{x}}) \in \mathcal{Z}(\bar{R})$.

Before showing Lemma 6.2, let us note the following: it implies Theorem 6.1. Indeed, if Lemma 6.2 is shown, then by definition of a G-zip, $\theta'_{\bar{x}}$ and $\theta_{\bar{x}}$ have the same image in $\mathrm{I}_-/U_-^\sigma(\bar{R})$, as they both correspond to the image of $\beta_{\bar{x}}$ under the isomorphism $\iota\colon \mathrm{I}_+^\sigma / U_+^\sigma(\bar{R}) \cong \mathrm{I}_-/U_-^\sigma(\bar{R})$. Hence we have $\theta'_{\bar{x}} = \theta_{\bar{x}} \cdot u_-$ for some $u_- \in U_-^\sigma(\bar{R})$, and hence the following equality holds:

$$\overline{\mathscr{f}_{x^\flat}} = \tilde{\xi}(\beta_{\bar{x}}, \theta'_{\bar{x}}) = \tilde{\xi}(\beta_{\bar{x}}, \theta_{\bar{x}}) u_- \in G(\bar{R}),$$

which implies that the image of $\tilde{\xi}(\beta_{\bar{x}}, \theta_{\bar{x}})$ in $G_\kappa / U_-^\sigma(\bar{R})$ is equal to $\gamma(\beta_{\bar{x}})$, as desired.

Proof of Lemma 6.2. We first show $\theta'_{\bar{x}} \in \mathrm{I}_-(\bar{R})$. Note that by our discussion in Section 6.1, the subset $\mathrm{I}_-(\bar{R}) \subseteq \mathrm{I}(\bar{R})$ consists of elements $\theta_{\bar{x}} \in \mathrm{I}(\bar{R})$, which carries the direct summand $\Lambda^*_{0,\bar{R}}$ of $\Lambda^*_{\bar{R}}$ isomorphically onto the conjugate filtration $\bar{M}_0$ of $\bar{M}$.

Using the notation in Lemma 5.3, the normal decomposition $M = M^1 \oplus M^0$ induces a decomposition $\bar{M} = \bar{M}^1 \oplus \overline{M^0}$ of $\bar{M}$, and hence a decomposition $\bar{M}^\sigma = \bar{M}^{1,\sigma} \oplus \sigma^*(\overline{M^0})$ of $\bar{M}^\sigma$. With this decomposition, we have $\bar{M}_0 = \bar{\mathrm{F}}(\sigma^*(\overline{M^0}))$. From this equality we see that the direct summand of M,

$$M_0 := \theta'_x(\Lambda^{*,0}) = \Gamma_{x^\flat}(M^{0,\sigma}) = \mathrm{F}(M^{0,\sigma}),$$

is a lift of the conjugate filtration $\bar{M}_0$ of $\bar{M}$ and we have $\theta'_{\bar{x}}(\Lambda^*_{0,\bar{R}}) = \mathrm{F}(\sigma^*(\overline{M^0}))$. In other words, $\theta'_{\bar{x}} \in \mathrm{I}_-(\bar{R})$.

To finish the proof, we still need to show that the image of $\beta_{\bar{x}}$ in $\mathrm{I}_+^\sigma/U_+^\sigma(\bar{R})$ coincides with the image of $\theta'_{\bar{x}}$ in $\mathrm{I}_-/U_-^\sigma(\bar{R})$, via the isomorphism $\iota\colon \mathrm{I}_+^\sigma/U_+^\sigma(\bar{R}) \cong \mathrm{I}_-/U_-^\sigma(\bar{R})$. Denote by $\mu'\colon \mathbb{G}_{m,\kappa} \xrightarrow{\mu} G_\kappa \hookrightarrow \mathrm{GL}(\Lambda^*_\kappa)$ the cocharacter of $\mathrm{GL}(\Lambda^*_\kappa)$ induced by μ, as in Section 3.4. Then we can form the κ-stack $\mathrm{GL}(\Lambda^*_\kappa)\text{-}\mathsf{Zip}^{\mu'}$. By forgetting tensors everywhere in $\underline{I}$, we obtain a $\mathrm{GL}(\Lambda^*_\kappa)$-zip $\underline{\mathrm{I}}' = (\mathrm{I}', \mathrm{I}'_+, \mathrm{I}'_-, \iota')$. Then, by functoriality of the formation of G-zips, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{I}_+^\sigma/U_+^\sigma(\bar{R}) & \xrightarrow{\iota} & \mathrm{I}_-/U_-^\sigma(\bar{R}) \\ \downarrow & & \downarrow \\ \mathrm{I}'^{\sigma}_+/U'^{\sigma}_+(\bar{R}) & \xrightarrow{\iota'} & \mathrm{I}'_-/U'^{\sigma}_-(\bar{R}), \end{array}$$

where the vertical arrows are injective: this can be seen by working fppf locally and using the fact (see (3.5))

$$\mathrm{Cent}_{\mathrm{GL}(\Lambda^*_\kappa)}(\mu') \cap G_\kappa = \mathrm{Cent}_{G_\kappa}(\mu).$$

Hence, it remains to show that the images of $\beta_{\bar{x}}$ and $\theta'_{\bar{x}}$ match via ι'; that is, we are reduced to the case $G_\kappa = \mathrm{GL}(\Lambda^*_\kappa)$.

Let us now unwind the definition of ι for $G_\kappa = \mathrm{GL}(\Lambda^*_\kappa)$. In this special case, the set $\mathrm{I}_+^\sigma/U_+^\sigma(\bar{R})$ can be realized as the set of equivalence classes in $\mathrm{I}_+(\bar{R})$ with equivalence relations given by declaring $\beta_1, \beta_2 \in \mathrm{I}_+^\sigma(\bar{R})$ equivalent if

$$\mathrm{gr}(\beta_1) = \mathrm{gr}(\beta_2)\colon (\Lambda^*_{\bar{R}}/\Lambda^{*,1}_{\bar{R}})^\sigma \oplus \Lambda^{*,1,\sigma}_{\bar{R}} \cong (\bar{M}/\bar{M}^1)^\sigma \oplus \bar{M}^{1,\sigma}.$$

The set $\mathrm{I}_-/U_-^\sigma(\bar{R})$ may similarly be realized as the set of equivalence classes in $\mathrm{I}_-(\bar{R})$ using the relation which declares $\theta_1, \theta_2 \in \mathrm{I}_-(\bar{R})$ to be equivalent if

$$\mathrm{gr}(\theta_1) = \mathrm{gr}(\theta_2)\colon \Lambda^*_{0,\bar{R}} \oplus \Lambda^*_{\bar{R}}/\Lambda^*_{0,\bar{R}} \cong \bar{M}_0 \oplus \bar{M}/\bar{M}_0.$$

Our map ι is given by sending the equivalence class of $\beta \in \mathrm{I}_+^\sigma(\bar{R})$ to the unique equivalence class containing $\theta \in \mathrm{I}_-(\bar{R})$ with $\mathrm{gr}(\theta)$ equal to the composition of

$$\begin{aligned}\Lambda^*_{0,\bar{R}} \oplus \Lambda^*_{\bar{R}}/\Lambda^*_{0,\bar{R}} &\cong \Lambda^*_{\bar{R}}/\Lambda^*_{-1,\bar{R}} \oplus \Lambda^*_{-1,\bar{R}} \\ &\overset{\text{can}}{\cong} (\Lambda^*_{\bar{R}}/\Lambda^{*,1}_{\bar{R}})^\sigma \oplus \Lambda^{*,1,\sigma}_{\bar{R}} \\ &\xrightarrow{\beta^\sigma} (\bar{M}/\bar{M}^1)^\sigma \oplus \bar{M}^{1,\sigma}\end{aligned}$$

with the zip isomorphism $(\bar{M}/\bar{M}^1)^\sigma \oplus \bar{M}^{1,\sigma} \xrightarrow{\delta} \bar{M}_0 \oplus \bar{M}/\bar{M}_0$ defined in (6.2). Here, the isomorphism $\overset{\text{can}}{\cong}$ is being induced by (4.3). Up to all these kinds of identifications described above, the map ι is actually given by

$$\mathrm{gr}(\theta) \mapsto \delta \circ \mathrm{gr}(\sigma(\theta)).$$

It now remains for us to verify that $\delta = \mathrm{gr}(\overline{\Gamma_{x^\flat}})$ which, in turn, is the same as checking the commutativity of the diagram below (with vertical arrows being canonical projections)

$$\begin{array}{ccc} M^{1,\sigma} \oplus M^{0,\sigma} & \xrightarrow{\Gamma_{x^\flat}} & M_{-1} \oplus M_0 \\ \downarrow & & \downarrow \\ \bar{M}^{1,\sigma} \oplus (\bar{M}/\bar{M}^1)^\sigma & \xrightarrow{\delta} & \bar{M}/\bar{M}_0 \oplus \bar{M}_0, \end{array} \tag{6.3}$$

where M_{-1} is defined to be $\Gamma_{x^\flat}(M^{1,\sigma})$. Insofar as the commutativity of (6.3) is concerned, we only need to check that for every element $m \in M^{1,\sigma}$, we have $[\bar{\mathrm{V}}]^{-1}(\bar{m}) = \overline{\Gamma_{x^\flat}(m)}$. Note that the image $[\bar{\mathrm{V}}]^{-1}(\bar{m})$ is the unique element $\bar{n} \in \bar{M}/\bar{M}_0$ such that $\bar{\mathrm{V}}(\bar{n}) = \bar{m}$. We also have $\bar{\mathrm{V}}(\overline{\Gamma_{x^\flat}(m)}) = \overline{(V \circ \Gamma_{x^\flat})(m)} = \bar{m}$. This finishes the proof of Lemma 6.2 and of Theorem 6.1 as well. ■

Acknowledgments – The main idea of this work has its origin in my Ph.D. thesis [39], during the course of which I received help from many people, including Fabrizio Andreatta, Bas Edixhoven, Ulrich Goertz, Bart de Smit, and Torsten Wedhorn; their help continues to contribute in the context of this work. Hence, it is my pleasure to express my gratitude to all of them once again. I thank Chao Zhang for his encouragement and for answering many of my questions over the years. I thank Hansheng Diao, Lei Fu, Xiaoyu Su, and Liang Xiao for helpful discussions on various subjects that contribute to this work. Last but not the least, I wish to thank Lovy Singhal heartily for his assistance with TeXing and issues of English grammar.

Funding – This work began during the author's postdoctoral days at Yau Mathematical Sciences Center, Tsinghua University and continued during his visit to the

Chinese Academy of Sciences. I thank both of these institutions as well as my mentors Zongbin Chen and Xu Shen. This work was partially supported by Chen's grant *Tsinghua University Initiative Research Funding 2019Z07L01007* and Shen's grant *National Key R&D Program of China 2020YFA0712600*.

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Manoscritto pervenuto in redazione il 25 novembre 2021.