Rend. Sem. Mat. Univ. Padova 151 (2024), 253–263
DOI 10.4171/RSMUP/130

Orders of products of elements and nilpotency of terms in the lower central series and the derived series

Juan Martínez (*)

Abstract – In this paper we prove that if G is a finite group, then the k-th term of the lower central series is nilpotent if and only if for every γ_k-values $x, y \in G$ with coprime orders, either $\pi(o(x)o(y)) \subseteq \pi(o(xy))$ or $o(x)o(y) \leq o(xy)$. We obtain an analogous version for the derived series of finite solvable groups, but replacing γ_k-values by δ_k-values. We will also discuss the existence of normal Sylow subgroups in the derived subgroup in terms of the order of the product of certain elements.

Mathematics Subject Classification (2020) – Primary 20F14; Secondary 20D20.

Keywords – Commutators, element orders, nilpotent group, normal Hall subgroup.

1. Introduction

All groups considered in this paper will be finite. It is an interesting theme to obtain results that relate the structure of a finite group with relationships among the order of products of some elements of the group. In this article we have studied the nilpotency and the existence of normal Hall π-subgroups in a term of the derived series or a term of the lower central series.

B. Baumslag and J. Wiegold proved in [4] that a group G is nilpotent if and only if $o(x)o(y) = o(xy)$ for every pair of elements $x, y \in G$ with $(o(x), o(y)) = 1$. There have been many papers that have generalized the Baumslag–Wiegold theorem.

On the one hand, in [5,10], A. Beltrán, A. Sáez, A. Moretó and the author proved more general versions of this result. Those versions give necessary and sufficient conditions

(*) *Indirizzo dell'A.*: Departament de Matemàtiques, Universitat de València, 46100 Burjassot, Spain; Juan.Martinez-Madrid@uv.es

for the existence of normal Hall π-subgroups and nilpotent Hall π-subgroups in terms of orders of products. More precisely, the hypothesis of these results have the form $\pi(o(x)o(y)) \subseteq \pi(o(xy))$ or $o(x)o(y) \leq o(xy)$, where x is a π-element and y a π'-element. (Recall that if n is a positive integer, $\pi(n)$ is the set of prime divisors of n.) On other hand, in [2, 3, 6] R. Bastos, C. Monetta, J. da Silva and P. Shumyatsky proved some theorems on the nilpotency of a term of the derived group and a term of the lower central series, in terms of orders of products.

Our purpose in this paper is to get new versions of the aforementioned results, but combining them with hypotheses of the type $o(x)o(y) \leq o(xy)$, or $\pi(o(x)o(y)) \subseteq \pi(o(xy))$. To explain them we introduce two families of words. Given $k \geq 1$, the word $\gamma_k = \gamma_k(x_1, \ldots, x_k)$ is defined inductively as

$$\gamma_1 = x_1 \quad \text{and} \quad \gamma_k = [\gamma_{k-1}(x_1, \ldots, x_{k-1}), x_k] = [x_1, \ldots, x_k].$$

Any element of the form $\gamma_k(x_1, \ldots, x_k)$ is called a γ_k*-value* of G. The subgroup of G generated by all of its γ_k-values is the k-th term of the lower central series, which we denote by $\gamma_k(G)$. Analogously, the word $\delta_k = \delta_k(x_1, \ldots, x_{2^k})$ is defined inductively as

$$\delta_0 = x_1 \quad \text{and} \quad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{(k-1)}}), \delta_{k-1}(x_{2^{(k-1)}+1}, \ldots, x_{2^k})].$$

As before, the elements of the form $\delta_k(x_1, \ldots, x_{2^k})$ are called δ_k*-values*. The subgroup of G generated by all of its δ_k-values is the k-th term of the derived series, which we denote by $G^{(k)}$.

In the case of the γ_k-values we have proved a new version of the main result of [2], but replacing the hypothesis $o(x)o(y) = o(xy)$ by either $o(x)o(y) \leq o(xy)$ or $\pi(o(x)o(y)) \subseteq \pi(o(xy))$ for each pair of elements.

THEOREM A. *Let G be a finite group. Then $\gamma_k(G)$ is nilpotent if and only if for every pair of γ_k-values $x, y \in G$ with $(o(x), o(y)) = 1$, at least one of the following holds.*

(1) $\pi(o(x)o(y)) \subseteq \pi(o(xy))$.

(2) $o(x)o(y) \leq o(xy)$.

It is important to remark that in the converse assertion we admit the existence of pairs of γ_k-values of coprime orders such that $\pi(o(x)o(y)) \subseteq \pi(o(xy))$ but $o(x)o(y) > o(xy)$ or $o(x)o(y) \leq o(xy)$ but $\pi(o(x)o(y)) \nsubseteq \pi(o(xy))$.

In the case of the δ_k-values, Theorem B generalizes the main result of [6] in the same way.

Theorem B. *Let G be a finite solvable group. Then $G^{(k)}$ is nilpotent if and only if for every pair of δ_k-values $x, y \in G$ with $(o(x), o(y)) = 1$, at least one of the following holds.*

(1) $\pi(o(x)o(y)) \subseteq \pi(o(xy))$.

(2) $o(x)o(y) \leq o(xy)$.

Our proofs of Theorems A and B are similar to the proofs in [2,3,6].

In the final section of the paper, we study the following conjecture.

Conjecture. *The following are equivalent:*

(1) $G' = H_\pi \times H_{\pi'}$, *where* $H_\pi \in \mathrm{Hall}_\pi(G')$ *and* $H_{\pi'} \in \mathrm{Hall}_{p'}(G')$.

(2) *For every pair of commutators of $x, y \in G$ such that x is a π-element and y is a π'-element, we have $\pi(o(x)o(y)) \subseteq \pi(o(xy))$.*

We recall that given a set of primes π and $x \in G$ we may define the π-part of x as $x_\pi = x^{o(x)_{\pi'}}$. Analogously, we define the π'-part of G as $x_{\pi'} = x^{o(x)_\pi}$. It is easy to see that x_π is a π-element, $x_{\pi'}$ is a π'-element and $x = x_\pi x_{\pi'}$. The following definition is relevant for the study of the above conjecture.

Definition. Given a prime p, we say that a finite group G is *p-exponential* if the p'-part of every commutator of G is a commutator.

As will be discussed, it seems that given a prime p, "most groups" are p-exponential. We characterize the existence of normal Sylow p-subgroups in p-solvable p-exponential groups.

Theorem C. *Let G be a p-solvable and p-exponential group. Then G' has a normal Sylow p-subgroup if and only if for every pair of commutators $x, y \in G$ such that x is a p-element and y is a p'-element, we have $\pi(o(y)) \subseteq \pi(o(xy))$.*

From this theorem we can deduce that the conjecture holds for $\pi = \{p\}$, when the group is p-exponential and p-solvable.

Corollary D. *Let G be a p-solvable and p-exponential group. Then G' has a normal Sylow p-subgroup and a normal p-complement if and only if for every pair of commutators $x, y \in G$ such that x is a p-element and y is a p'-element, we have $\pi(o(y)o(x)) \subseteq \pi(o(xy))$.*

2. Preliminary results

Most of the results of this section have the same structure. We have a subset of γ_k-values or δ_k-values of G, and a condition concerning the order of the products of these elements. Our results assert that under such a condition, each of our elements commutes with some subgroup that they normalize.

Proposition 2.1. *Let G be a finite group and let π be a set of primes such that for every pair of γ_k-values $x, y \in G$, with x a π-element and y a π'-element, we have either $o(x)o(y) \leq o(xy)$ or $\pi(o(x)o(y)) \subseteq \pi(o(xy))$. If x is a γ_k-value and a π-element, and N is a π'-group which is normalized by x, then $[x, N] = 1$.*

Proof. Let $y \in N$. Since x normalizes N, $[x, y] \in N$. Hence, x and $[x, y]$ are γ_k-values which are respectively a π-element and a π'-element. Now, by our hypothesis there are two possible cases. Suppose first that

$$o(x)o([x, y]) \leq o(x[x, y]) = o(y^{-1}xy) = o(x).$$

Then $o([x, y]) = 1$, so $[x, y] = 1$. Therefore, we may assume that

$$\pi(o(x)o([x, y])) \subseteq \pi(o(x[x, y])) = \pi(o(y^{-1}xy)) = \pi(o(x)).$$

This implies $[x, y] = 1$ again. So, in any of the two possible cases we have $[x, y] = 1$, for every $y \in N$. Thus, $[x, N] = 1$, as we claimed. ∎

Corollary 2.2. *Let G be a finite group such that for every pair of γ_k-values $x, y \in G$ with $(o(x), o(y)) = 1$ we have either $o(x)o(y) \leq o(xy)$ or $\pi(o(x)o(y)) \subseteq \pi(o(xy))$. If x is a γ_k-value, and N is a subgroup normalized by x such that $(o(x), |N|) = 1$, then $[x, N] = 1$.*

Proof. It suffices to apply the previous proposition to the set $\pi = \pi(o(x))$. ∎

Proposition 2.3. *Let G be a finite group and let π be a set of primes such that for every pair of γ_k-values $x, y \in G$, with x a π-element and y a π'-element, we have $\pi(o(y)) \subseteq \pi(o(xy))$. If x is a γ_k-value and a π-element, and N is a π'-group normalized by x, then $[x, N] = 1$.*

Proof. Let $y \in N$. As before, $[x, y] \in N$. So, we may apply our hypothesis to x and $[x, y]$. Thus, $\pi(o([x, y])) \subseteq \pi(o(x[x, y])) = \pi(o(y^{-1}xy)) = \pi(o(x))$. This implies that $[x, y] = 1$. It follows that $[x, N] = 1$, as desired. ∎

Now we will prove analogous versions of the previous results for δ_k-values.

Proposition 2.4. *Let G be a finite group and let π be a set of primes such that for every pair of δ_k-values $x, y \in G$, with x is a π-element and y is a π'-element, we have either $o(x)o(y) \leq o(xy)$ or $\pi(o(x)o(y)) \subseteq \pi(o(xy))$. If x is a δ_k-value and a π-element, and N is a π'-group normalized by x, then $[x, N] = 1$.*

Proof. Let $y \in N$. Since x normalizes N, we have that $[x, y] \in N$ and so $[y, x, x] = [[y, x], x] \in N$. Hence, x^{-1} and $[y, x, x]$ are δ_k-values which are respectively a π-element and a π'-element. Moreover,

$$[y, x, x]x^{-1} = [[y, x], x]x^{-1} = ((x^{-1})^{[y,x]}x)x^{-1} = (x^{-1})^{[y,x]},$$

which is a conjugate of x^{-1}. Now, by our hypothesis there are two possible cases, and reasoning as in Proposition 2.1 we have $[y, x, x] = 1$ in both cases. Thus, $[N, x, x] = 1$. Then, by [7, Theorem 5.3.6], we have $[N, x] = 1$, as desired. ■

Corollary 2.5. *Let G be a finite group such that for every pair of γ_k-values $x, y \in G$ with $(o(x), o(y)) = 1$ we have either $o(x)o(y) \leq o(xy)$ or $\pi(o(x)o(y)) \subseteq \pi(o(xy))$. If x is a δ_k-value, and N is a subgroup normalized by x such that $(o(x), |N|) = 1$, then $[x, N] = 1$.*

Proof. As in Corollary 2.2, it is enough to apply the previous proposition to the set $\pi = \pi(o(x))$. ■

Proposition 2.6. *Let G be a finite group and let π be a set of primes such that for every pair of δ_k-values $x, y \in G$, with x a π-element and y a π'-element, we have $\pi(o(y)) \subseteq \pi(o(xy))$. If x is a δ_k-value and a π-element, and N is a π'-group normalized by x, then $[x, N] = 1$.*

Proof. As in Proposition 2.4, given $y \in N$ we consider $[y, x, x] \in N$. Hence, arguing as before, we have that $\pi(o([y, x, x])) \subseteq \pi(o([y, x, x]x^{-1})) = \pi(o((x^{-1})^{[y,x]})) = \pi(o(x))$. Thus, $[N, x, x] = 1$. Applying [7, Theorem 5.3.6] again, we have that $[N, x] = 1$. ■

We observe that a similar argument could work for other families of words, for example to the γ_k^*-values or the δ_k^*-values, introduced in [12]. Therefore, by a similar argument as above, we could obtain analogous results as in [1], but using our hypotheses.

To prove the results on the derived subgroup we will need the focal subgroup theorem [9, Theorem 5.21]. This theorem asserts that if $H \in \mathrm{Hall}_\pi(G)$, then $H \cap G'$ is generated by the commutators lying in H. To prove the result in the derived series we will need the next generalization of the focal subgroup theorem, which was proved in [6] by P. Shumyatsky and J. da Silva.

Lemma 2.7. *Let G be a finite solvable group and let $p \in \pi(G)$. Then for every $P \in \mathrm{Syl}_p(G)$ and every $k \geq 1$, the group $P \cap G^{(k)}$ is generated by δ_k-values of order a power of p.*

3. Proof of Theorems A and B

Our purpose in this section is to prove Theorems A and B. We begin stating two lemmas. The first one is [2, Lemma 3].

Lemma 3.1. *Let G be a metanilpotent finite group and let $p \in \pi(G)$. If x is a p-element such that $[x, O_{p'}(F(G))] = 1$, then $x \in F(G)$.*

The second lemma is a property of coprime actions, which is a direct corollary of [7, Theorem 5.3.6].

Lemma 3.2. *Let G be a finite group, let p a prime and let P a p-subgroup of G. Then, for every x p'-element normalizing P, it holds that*

$$[P, x] = [P, {}_{k-1}x],$$

where $[P, {}_{k-1}x]$ stands for $[P, \underbrace{x, \dots, x}_{(k-1)\text{-times}}]$.

It is also important to recall the concept of *Turull tower*, which was introduced in [13]. Given a subgroup H of G, we say that H is a tower of height h if $H = P_1 \cdots P_h$, where the groups P_i are p_i-groups (each p_i is a prime number) and the following two conditions hold:

(1) P_i normalizes P_j for all $i < j$.

(2) $[P_i, P_{i-1}] = P_i$ for all $i \in \{2, \dots, h\}$.

It was shown in [13] that a solvable finite group G has Fitting height at least h if and only if G admits a tower of height h. Now, we proceed to prove Theorem A in the solvable case.

Theorem 3.3. *Let G be a solvable group such that for every pair of γ_k-values $x, y \in G$ with $(o(x), o(y)) = 1$, we have either $o(x)o(y) \leq o(xy)$ or $\pi(o(x)o(y)) \subseteq \pi(o(xy))$. Then $\gamma_k(G)$ is nilpotent.*

Proof. Let h be the Fitting height of G. If $h = 1$, then G is nilpotent and the result follows trivially.

Suppose first that $h \geq 3$. Let $P_1 \cdots P_h$ be a tower of height h. Then,

$$P_3 = [P_3, [P_2, P_1]].$$

On the other hand, by Lemma 3.2, for every $x \in P_1$ we have $[P_2, x] = [P_2,_{k-1} x]$. Thus, $P_2 = [P_2, P_1]$ is generated by γ_k-values that are p_2-elements normalizing P_3. Then, applying Corollary 2.2, we get $1 = [P_3, P_2] = P_3$, which is a contradiction.

So, we may assume that $h = 2$. Hence, G is metanilpotent. If $G/F(G)$ has nilpotency class less or equal than $k-1$, then $\gamma_k(G) \leq F(G)$, so it is nilpotent. Suppose now that $G/F(G)$ has nilpotency class at least k. Then, there exists a Sylow subgroup of $G/F(G)$ that has nilpotency class at least k. Therefore, there exists a Sylow subgroup P of G and a γ_k-value x of elements of P such that $x \notin F(G)$. On the other hand, by Corollary 2.2, $[O_{p'}(F(G)), x] = 1$, and by Lemma 3.1, $x \in F(G)$. Such a contradiction completes the proof. ∎

Now, we will work to prove Theorem A in the general case. We will need two more lemmas. The next result is [2, Lemma 4].

Lemma 3.4. *Let G be a finite group with $G = G'$ and let $p \in \pi(G)$. Then G is generated by γ_k-values whose order is a power of some prime different from p.*

The following lemma is [11, Proposition 2.8]. Its proof relies on Thompson's classification of the minimal simple groups. We recall that a minimal simple group is a nonabelian simple group such that every proper subgroup is solvable.

Lemma 3.5. *Let G be a minimal simple group. Then G has a subgroup H such that $H = A \rtimes T$, where A is an elementary abelian 2-group, T has odd order, and $A = [A, T]$.*

Finally, we are ready to complete the proof of Theorem A.

Proof of Theorem A. Suppose that G is a counterexample of minimal order. By Theorem 3.3, G cannot be solvable. Moreover, by the minimality of G, every proper subgroup of G is solvable. In particular, it follows that $G = G'$. First, we will show that G is quasisimple.

Let R be the solvable radical. We know that G/R is nonabelian simple, so it is enough to prove that $Z(G) = R$. By Lemma 3.4, given $p \in \pi(G)$, we have that G is generated by γ_k-values of order a power of q, with $q \neq p$. Let $P \in \mathrm{Syl}_p(F(G))$. By Corollary 2.2, we have $[P, x] = 1$ for every γ_k-value x which is a p'-element. It follows that $P \leq Z(G)$ for every $P \in \mathrm{Syl}_p(F(G))$ and for every $p \in \pi(G)$. Hence, $F(G) \leq Z(G)$. On the other hand, $\gamma_k(R)$ is nilpotent, so that $\gamma_k(R) \leq F(G) = Z(G)$.

Thus, $[x_1, \ldots, x_k, x_{k+1}] = 1$. Hence, R is nilpotent, and therefore $R = F(G) = Z(G)$. Thus $G/Z(G)$ is a nonabelian simple group.

Furthermore, all proper subgroups of $G/Z(G)$ are solvable and hence $G/Z(G)$ is a minimal simple group. By the previous lemma, $G/Z(G)$ has a subgroup $H/Z(G) = A/Z(G) \rtimes T/Z(G)$, where $A/Z(G)$ is an elementary abelian 2-group, $T/Z(G)$ is odd, and $A/Z(G) = [A/Z(G), T/Z(G)]$. By Lemma 3.2,

$$A/Z(G) = [A/Z(G),_{k-1} T/Z(G)].$$

Now, let $P \in \mathrm{Syl}_p(A)$. Then, there exists some $a \in P$ such that $x = [a,_{k-1} t] \in A \setminus Z(G)$. Thus, x is a γ_k-value, which is a 2-element and $x^2 \in Z(G)$. Applying the Baer–Suzuki theorem (see [7, Theorem 3.8.2]), there exists $g \in G$ such that $[x, g]$ has odd order. Then

$$1 = [x^2, g] = [x, g]^x [x, g]$$

or, equivalently, $[x, g]^x = [x, g]^{-1}$. Finally, x is a γ_k-value normalizing $\langle [x, g] \rangle$, and $(o(x), |\langle [x, g] \rangle|) = 1$. Thus, applying Corollary 2.2, x centralizes $[x, g]$, so $[x, g]^{-1} = [x, g]^x = [x, g]$. This contradicts that $[x, g]$ has odd order. ■

Next, we prove Theorem B.

Proof of Theorem B. Suppose that G is a counterexample with $G^{(k)}$ as small as possible. Since G is solvable, $|G^{(k)}| > |G^{(k+1)}|$, and by the minimality of G we conclude that $G^{(k+1)}$ is nilpotent. Then, $G^{(k)}$ is metanilpotent.

Let P be a Sylow p-subgroup of $G^{(k)}$. By Lemma 2.7, P is generated by δ_k-values that are p-elements. Each of these generators is a δ_k-value that normalizes $O_{p'}(F(G^{(k)}))$ and has order coprime with $|O_{p'}(F(G^{(k)}))|$. Then, applying Lemma 2.5, $[O_{p'}(F(G^{(k)})), x] = 1$, and since $G^{(k)}$ is metanilpotent, applying Lemma 3.1 we have that $x \in F(G)$.

Then, $P \leq F(G)$, for every p-Sylow subgroup P of $G^{(k)}$. Thus, $G^{(k)} \leq F(G^{(k)})$, and hence $G^{(k)}$ is nilpotent, a contradiction. ■

4. Normal Hall π-subgroups of the derived subgroup

In this section, we prove Theorem C and Corollary D. The "if" part of both results is trivial and hence we will only prove the "only if" part. We begin with a result that implies that the conjecture stated in the introduction is true in the case when $O_\pi(G') = 1$ and G is π-separable.

Theorem 4.1. *Let π be a set of primes and G be a π-separable group with $O_\pi(G') = 1$. Suppose that for every pair of commutators $x, y \in G$, such that x is a π-element and y is a π'-element, we have $\pi(o(y)) \subseteq \pi(o(xy))$. Then no prime in π divides $|G'|$.*

Proof. Let $p \in \pi$ and let $P \in \mathrm{Syl}_p(G')$. By the focal subgroup theorem, P is generated by $\{x_1, \dots, x_n\}$, where the x_i are commutators that are p-elements. Hence, every x_i is a commutator that normalizes $O_{\pi'}(G')$ and $(o(x_i), |O_{\pi'}(G')|) = 1$. By Proposition 2.3, $x_i \in C_{G'}(O_{\pi'}(G'))$ for every x_i. Hence, we have that $P \leq C_{G'}(O_{\pi'}(G'))$.

On the other hand, $O_\pi(G') = 1$ and G' is π-separable, hence by [8, Lemma 1.2.3], $C_{G'}(O_{\pi'}(G')) \subseteq O_{\pi'}(G')$. Consequently, $P \leq O_{\pi'}(G')$ with $p \in \pi$. Thus, $P = 1$ as desired. ∎

Now, we proceed to prove Theorem C.

Proof of Theorem C. Let $N = O_p(G')$ and let xN, yN be commutators of G/N such that xN is a p-element and yN is a p'-element. Then, x is a commutator of G that is a p-element. On the other hand, y is also a commutator (but not necessarily a p'-element) and since G is p-exponential we have that $y_{p'}$ is a commutator and a p'-element. In addition, we have that $yN = y_{p'}N$. By our hypothesis, we have that

$$\pi(o(yN)) = \pi(o(y_{p'}N)) = \pi(o(y_{p'})) \subseteq \pi(o(x)o(y_{p'})) \subseteq \pi(o(xy_{p'})).$$

Using that N is a p-subgroup and that $xy_{p'}N = xyN$, we have that

$$\pi(o(yN)) \subseteq \pi(o(xy_{p'}N)) = \pi(o(xyN)).$$

Thus, our hypothesis also holds in G/N.

Now we consider $\overline{G} = G/N$ and use the bar convention. Notice that $O_p(\overline{G}') = 1$, $\overline{G}$ is p-solvable and $\pi(o(y)) \subseteq \pi(o(xy))$ for every commutators x, y of $\overline{G}$, where x is a p-element and y is a p'-element. By the previous theorem, p cannot divide $|\overline{G}'|$. Thus, N is a normal Sylow p-subgroup of G', as desired. ∎

Using Theorem C and the Schur–Zassenhaus theorem, we can now prove Corollary D.

Proof of Corollary D. We observe that all the hypotheses of Theorem C are satisfied. Thus, G' has a normal Sylow p-subgroup.

By the Schur–Zassenhaus theorem, $G' = PH$, where P is a normal Sylow p-subgroup and H is a p-complement. By the focal subgroup theorem, H is generated by commutators that are p'-elements. Clearly, these generators normalize P and have orders coprime with $|P|$. We also have that $\pi(o(x)) \subseteq \pi(o(xy))$ for every x, y commutators in G, where x is a p-element and y is a p'-element. By Proposition 2.6, each of the generators of H centralizes P. Thus, H centralizes P and hence $G' = P \times H$, where $P \in \mathrm{Syl}_p(G')$ and $H \in \mathrm{Hall}_{p'}(G')$. ∎

Finally, we remark that we have used GAP [14] to find non-p-exponential groups, for the primes $p \in \{2, 3, \ldots, 19\}$. All examples of non-$p$-exponential groups that we have found are non-p-exponential for $p = 2$. Moreover, all of them have nontrivial center and nonabelian Sylow 2-subgroups. The smallest non-p-exponential groups have order $2^3 \cdot 3^3$.

Acknowledgments – I thank the referee for carefully reading the paper and many suggestions that have helped improve the readability of the paper. I also thank Alexander Moretó for suggesting me this subject, and for helpful conversations.

Funding – This work was done while the author was supported by Becas de Colaboración of Ministerio de Educación y Formación Profesional.

References

[1] R. Bastos – C. Monetta, Coprime commutators in finite groups. *Comm. Algebra* **47** (2019), no. 10, 4137–4147. Zbl 1472.20043 MR 3975992

[2] R. Bastos – C. Monetta – P. Shumyatsky, A criterion for metanilpotency of a finite group. *J. Group Theory* **21** (2018), no. 4, 713–718. Zbl 1402.20032 MR 3819548

[3] R. Bastos – P. Shumyatski, A sufficient condition for the nilpotency of a commutator subgroup. *Sibirsk. Mat. Zh.* **57** (2016), no. 5, 978–980. Zbl 1358.20014 MR 3604560

[4] B. Baumslag – J. Wiegold, A sufficient condition for nilpotency in a finite group. 2014, arXiv:1411.2877.

[5] A. Beltrán – A. Sáez, Existence of normal Hall subgroups by means of orders of products. *Math. Nachr.* **292** (2019), no. 4, 720–723. Zbl 1445.20015 MR 3937612

[6] J. da Silva Alves – P. Shumyatsky, On nilpotency of higher commutator subgroups of a finite soluble group. *Arch. Math. (Basel)* **116** (2021), no. 1, 1–6. Zbl 1501.20002 MR 4203614

[7] D. Gorenstein, *Finite groups*. Second edn., Chelsea Publishing Co., New York, 1980. Zbl 0185.05701 MR 569209

[8] P. Hall – G. Higman, On the p-length of p-soluble groups and reduction theorems for Burnside's problem. *Proc. London Math. Soc. (3)* **6** (1956), 1–42. Zbl 0073.25503 MR 72872

[9] I. M. Isaacs, *Finite Group Theory*. Amer. Math. Soc. Providence, Rhode Island, 2008. Zbl 1169.20001 MR 2426855

[10] J. Martínez – A. Moretó, Variations on results on orders of products in finite groups. *Proc. Roy. Soc. Edinburgh Sect. A* **151** (2021), no. 5, 1443–1449. Zbl 07400626 MR 4313568

[11] C. Monetta – A. Tortora, A nilpotency criterion for some verbal subgroups. *Bull. Aust. Math. Soc.* **100** (2019), no. 2, 281–289. Zbl 1444.20022 MR 4001545

[12] P. Shumyatsky, Commutators of elements of coprime orders in finite groups. *Forum Math.* **27** (2015), no. 1, 575–583. Zbl 1311.20021 MR 3334073

[13] A. Turull, Fitting height of groups and of fixed points. *J. Algebra* **86** (1984), no. 2, 555–566. Zbl 0526.20017 MR 732266

[14] The GAP group, GAP Groups, Algorithms, and programming, version 4.8.6, 2016, http://www.gap-system.org.

Manoscritto pervenuto in redazione il 16 dicembre 2021.