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Galois equivariant functions on Galois orbits in large p-adic fields

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Abstract – Given a prime number p let $\mathbb{C}_p$ be the topological completion of the algebraic closure of the field of p-adic numbers. Let $O(T)$ be the Galois orbit of a transcendental element T of $\mathbb{C}_p$ with respect to the absolute Galois group. Our aim is to study the class of Galois equivariant functions defined on $O(T)$ with values in $\mathbb{C}_p$. We show that each function from this class is continuous and we characterize the class of Lipschitz functions, respectively the class of differentiable functions, with respect to a new orthonormal basis. Then we discuss some aspects related to analytic continuation for the functions of this class.

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Dedicated to Alexandru Zaharescu on the occasion of his 60th birthday

1. Introduction

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic absolute value $|\cdot|$. Let $O(T)$ denote the Galois orbit of an element $T \in \mathbb{C}_p$ with respect to the Galois group $G = \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$. The class of Galois equivariant functions plays an important role in the study of generating elements for some classes of closed subfields of $\mathbb{C}_p$, see [5], and in the study of p-adic measures on the orbits of elements of $\mathbb{C}_p$,

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see [2]. The problem of characterization of Lipschitz functions in the case of local fields by using various bases has been studied by Amice and Fresnel [6], Barsky [8], Helsmoortel [10], and de Shalit [9]. A study of the $\mathbb{C}_p$-Banach algebra of Lipschitz functions on arbitrary compact subsets of $\mathbb{C}_p$ has been presented by one of the authors in [14]. In this paper, we complement their results for the class of Galois equivariant functions defined on Galois equivariant compacts of the Tate fields. We are interested in studying the class of Galois equivariant functions defined on the Galois orbit of a transcendental element of $\mathbb{C}_p$ and to characterize the class of Lipschitz functions, respectively of differentiable functions, with respect to the orthonormal basis that has been introduced in [3, 12]. Also, the problem of analytic continuation for the functions of this class is considered.

The paper consists of three sections. After this introduction, Section 2 contains notation and some preliminary results. In Section 3, we present the main results of the paper. Precisely, Theorem 3.3 says that a Galois equivariant function f defined on the Galois orbit of a transcendental element of $\mathbb{C}_p$ with values in $\mathbb{C}_p$ is continuous and has a representation as a series with respect to the orthonormal basis mentioned above with coefficients in the field of p-adic numbers. We give sufficient conditions for such a function to be Lipschitz or differentiable. The conditions depend only on the coefficients of the development of f in the orthonormal basis and on a sequence of invariants associated with T. Analogous problems are studied in Theorem 3.5 for G_K-equivariant functions defined on the Galois orbit of T with values in $\mathbb{C}_p$, where K is a finite normal extension of $\mathbb{Q}_p$ and $G_K = \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K)$. In the case when T is a normal element, that is, $\widetilde{\mathbb{Q}_p[T]}$ is a normal extension of $\mathbb{Q}_p$, any continuous function defined on the Galois orbit of T with values in $\mathbb{C}_p$ is a uniform limit of a sequence of functions, which are representable as series in the same orthonormal basis but with coefficients in some finite and normal extensions of $\mathbb{Q}_p$. In the final part of the paper, we study the problem of analytic continuation for the functions from the class of Galois equivariant functions defined on Galois orbits with values in $\mathbb{C}_p$, see Theorem 3.6.

2. Notation and preliminary results

Let p be a prime number and let $\mathbb{Q}_p$ be the field of p-adic numbers. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic valuation v, see [3, 4, 7, 11, 13]. Denote by $|\cdot|$ the p-adic module on $\mathbb{C}_p$, where $|x| = (\frac{1}{p})^{v(x)}$, for any $x \in \mathbb{C}_p$. Let G be the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ endowed with the Krull topology. We know that G is canonically isomorphic to $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$, which is the group of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. In the following, we shall identify these two groups.

For any closed subgroup H of G, we denote

$$\mathrm{Fix}(H) = \{T \in \mathbb{C}_p : \sigma(T) = T \text{ for all } \sigma \in H\}.$$

Then $\mathrm{Fix}(H)$ is a closed subfield of $\mathbb{C}_p$. If $T \in \mathbb{C}_p$, denote

$$H(T) = \{\sigma \in G : \sigma(T) = T\}.$$

Then $H(T)$ is a subgroup of G, and $\mathrm{Fix}(H(T)) = \widetilde{\mathbb{Q}_p[T]}$ (see [3, 4]), which is the closure of the polynomial ring $\mathbb{Q}_p[T]$ in $\mathbb{C}_p$. We say that T is a *topological generic element* of $\widetilde{\mathbb{Q}_p[T]}$. Any closed subfield K of $\mathbb{C}_p$ has a topological generic element, i.e., there exists $T \in K$ such that $K = \widetilde{\mathbb{Q}_p[T]}$, see [3, Theorem 1.2] and [11, Theorem 2].

Let T be a transcendental element of $\mathbb{C}_p$ and let $O(T) = \{\sigma(T) : \sigma \in G\}$ be the Galois orbit of T. The map $\sigma \rightsquigarrow \sigma(T)$ from G to $O(T)$ is continuous and it defines a homeomorphism from $G/H(T)$ to $O(T)$. Then $O(T)$ is a compact and totally disconnected subspace of $\mathbb{C}_p$ and the group G acts continuously on $O(T)$: if $\sigma \in G$ and $\tau(T) \in O(T)$, then $\sigma \times \tau(T) = (\sigma\tau)(T)$.

In what follows, we recall the notions of distinguished pair and distinguished sequence that were introduced by the first author, Popescu and Zaharescu [3]. For any $\alpha \in \overline{\mathbb{Q}}_p$ we denote by $\dim \alpha = [\mathbb{Q}_p(\alpha) : \mathbb{Q}_p]$ the dimension of α. We say that a pair $(\alpha, \beta) \in \overline{\mathbb{Q}}_p^2$ is a *distinguished pair* if:

(a) $\dim \alpha < \dim \beta$;

(b) if $\gamma \in \overline{\mathbb{Q}}_p$ and $\dim \gamma < \dim \beta$, then $|\beta - \alpha| \le |\beta - \gamma|$;

(c) if $\gamma \in \overline{\mathbb{Q}}_p$ and $\dim \gamma < \dim \alpha$, then $|\beta - \alpha| < |\beta - \gamma|$.

Now, if $(\alpha, \beta) \in \overline{\mathbb{Q}}_p^2$ is a *distinguished pair*, then $\dim \alpha$ is a divisor of $\dim \beta$, see [3, Corollary 1.5] and [12, Remark 3.3].

A sequence $\{\alpha_n\}_{n\ge 0}$ of algebraic elements of $\overline{\mathbb{Q}}_p$ is called a *distinguished sequence* if:

(a) $\alpha_0 \in \mathbb{Q}_p$;

(b) the pair (α_{n-1}, α_n) is distinguished for any $n \ge 1$;

(c) $|\alpha_n - \alpha_{n-1}| = \delta_{n-1} \to 0$, when $n \to \infty$.

It is known that the limit of a distinguished sequence is a transcendental element of $\mathbb{C}_p$ and any transcendental element T of $\mathbb{C}_p$ is a limit of a distinguished sequence. The sequence $\{\delta_n\}_{n\ge 0}$ defined above is strictly decreasing to 0 and it is an invariant of T, that is, it does not depend on the distinguished sequence that converges to T, which means that it depends only on T, see [3, Remark 2.4].

Let f_n be the monic minimal polynomial of α_n over $\mathbb{Q}_p$, where $\{\alpha_n\}_{n\ge 0}$ is a distinguished sequence that converges to a transcendental element T of $\mathbb{C}_p$. Denote

$D_n = \dim \alpha_n = \deg f_n$, for any $n \geq 0$. Also, the sequence $\{D_n\}_{n\geq 0}$ is an invariant of T. Let $\mathcal{A}$ be the subset of $\mathbb{N}^{(\mathbb{N})}$ which consists of elements $\underline{s} = (s_0, s_1, \ldots, s_n, 0, \ldots)$ such that $s_i < \frac{D_{i+1}}{D_i}$, for any $i \geq 0$. On the set $\mathbb{N}^{(\mathbb{N})}$ we have the anti-lexicographic order. More precisely, for any $\underline{s} = (s_0, s_1, \ldots, s_n, \ldots)$ and $\underline{s'} = (s'_0, s'_1, \ldots, s'_n, \ldots)$ in $\mathbb{N}^{(\mathbb{N})}$, we have $\underline{s} < \underline{s'}$ if and only if there exists a natural number k such that $s_i = s'_i$ for any $i > k$ and $s_k < s'_k$. With this order, $\mathbb{N}^{(\mathbb{N})}$ becomes a well-ordered set. We consider $\mathcal{A}$ as a well-ordered set with the induced ordering of $\mathbb{N}^{(\mathbb{N})}$. To any element $\underline{s} = (s_0, s_1, \ldots, s_n, 0, \ldots) \in \mathcal{A}$ we associate the polynomial $H_{\underline{s}} = f_0^{s_0} f_1^{s_1} \cdots f_n^{s_n} \cdots$. Denote

$$M_{\underline{s}} = M_{\underline{s}}(T) = \frac{H_{\underline{s}}(T)}{p^{q_{\underline{s}}}},$$

where $q_{\underline{s}} = [v(H_{\underline{s}}(T))]$ and $[x]$ stands for the integral part of x.

Claim. *Let* $\underline{s} = (s_0, s_1, \ldots, s_n, 0, \ldots) \in \mathcal{A}$. *Then*

$$\deg M_{\underline{s}} = s_0 D_0 + s_1 D_1 + \cdots + s_n D_n < D_{n+1}, \quad \text{for any } n \geq 0.$$

Since $s_i < \frac{D_{i+1}}{D_i}$, we have $s_i D_i < D_{i+1}$, for any $i \geq 0$. Because D_i is a divisor of D_{i+1}, it follows that $D_{i+1} - s_i D_i \geq D_i$, so $D_{i+1} \geq D_i(s_i + 1)$, for any $i \geq 0$.

Now, we will prove the claim by induction on n. It is clear that $D_0 = 1$ and $s_0 D_0 = s_0 < D_1$. Let us suppose that $s_0 D_0 + s_1 D_1 + \cdots + s_{n-1} D_{n-1} < D_n$. Then

$$s_0 D_0 + s_1 D_1 + \cdots + s_{n-1} D_{n-1} + s_n D_n < D_n + s_n D_n \leq D_{n+1},$$

which proves our claim.

Let $\underline{s}, \underline{s'} \in \mathcal{A}$ be such that $\underline{s} < \underline{s'}$ with respect to the anti-lexicographic order on $\mathcal{A}$. From the definition of the anti-lexicographic order on $\mathcal{A}$ and the claim we deduce that $\deg M_{\underline{s}} < \deg M_{\underline{s'}}$. We put the elements $M_{\underline{s}}$, $\underline{s} \in \mathcal{A}$, in a sequence $(M_0, M_1, \ldots, M_n, \ldots)$ according to the order of $\mathcal{A}$ and, by [3], we have the following result.

Proposition 2.1. *Let T be a transcendental element of $\mathbb{C}_p$. Then, there exists a family $\{M_n\}_{n\geq 0}$ of polynomials in $\mathbb{Q}_p[T]$, such that:*

(i) $\deg M_n = n$ *for all* $n \geq 0$*;*

(ii) $\frac{1}{p} < |M_n(T)| \leq 1$*;*

(iii) *any element f of the field $\widetilde{\mathbb{Q}_p[T]}$ can be written uniquely as a series in the form $f = \sum_{n\geq 0} a_n M_n$, where $\{a_n\}_{n\geq 0}$ is a sequence of elements of $\mathbb{Q}_p$ such that $\lim_{n\to\infty} a_n = 0$. Moreover, we have $|f| = \sup_{n\geq 0} |a_n M_n(T)|$.*

(iv) *If $K_T = \widetilde{\mathbb{Q}_p[T]} \cap \overline{\mathbb{Q}}_p$, then $\widetilde{K_T} = \widetilde{\mathbb{Q}_p[T]}$ and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_T)$ is canonically isomorphic to $H(T)$.*

A subset $D \subseteq \mathbb{C}_p$ is *G-equivariant* or *Galois equivariant* provided that $\sigma(x) \in D$ for any $\sigma \in G$ and any $x \in D$. An example is the Galois orbit $O(T)$, where $T \in \mathbb{C}_p$. Another example is

$$B[O(T), |p|^{1+\varepsilon}] = \{z \in \mathbb{C}_p : \operatorname{dist}(z, O(T)) \leq |p|^{1+\varepsilon}\}, \quad \text{for any } \varepsilon > 0.$$

A function $f : D \to \mathbb{C}_p$, where D is a Galois equivariant subset of $\mathbb{C}_p$, is called *G-equivariant* or *Galois equivariant* if $f(\sigma x) = \sigma f(x)$, for any $\sigma \in G$ and $x \in D$, see [1,4,5,14]. Finally, let K be a complete subfield of $\mathbb{C}_p$. Denote by $G_K = \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K)$ the Galois group of continuous automorphisms of $\mathbb{C}_p$ over K. If instead of G we consider G_K, then we define in a similar way the notions of G_K-equivariant subset of $\mathbb{C}_p$, respectively of G_K-equivariant function.

3. Main results

Proposition 3.1. *Let T be a transcendental element of $\mathbb{C}_p$ and let $F \in \mathbb{Q}_p[X]$ be a polynomial of degree $n \geq 1$. Then F is a Lipschitz function on the Galois orbit $O(T)$ of T and its Lipschitz constant C_F is bounded by $|F'(T)| \leq C_F \leq \frac{|F(T)|}{|T-\alpha|}$, where the root $\alpha \in \overline{\mathbb{Q}}_p$ of F is the closest to T.*

Proof. It is clear that F is Lipschitz on $O(T)$. Now, for the sake of simplicity, we suppose that F is monic and $F(X) = \prod_{i=1}^{n}(X - \alpha_i)$, where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \overline{\mathbb{Q}}_p$ are all the roots of the polynomial F. Also, we can suppose that

$$|T - \alpha_1| \leq |T - \alpha_2| \leq \cdots \leq |T - \alpha_n|. \tag{1}$$

Denote $F_l(X) = \prod_{i=l+1}^{n}(X - \alpha_i)$, for any $0 \leq l < n$. For any $\sigma \in G \setminus H(T)$ we have

$$\begin{aligned}\frac{F(T) - F(\sigma T)}{T - \sigma T} &= \frac{(T - \alpha_1)F_1(T) - (\sigma T - \alpha_1)F_1(T)}{T - \sigma T} \\ &\quad + \frac{(\sigma T - \alpha_1)F_1(T) - (\sigma T - \alpha_1)F_1(\sigma T)}{T - \sigma T} \\ &= F_1(T) + (\sigma T - \alpha_1)\frac{F_1(T) - F_1(\sigma T)}{T - \sigma T}.\end{aligned}$$

It follows

$$\left|\frac{F(T) - F(\sigma T)}{T - \sigma T}\right| \leq \max\left\{|F_1(T)|, |T - \sigma^{-1}\alpha_1| \cdot \left|\frac{F_1(T) - F_1(\sigma T)}{T - \sigma T}\right|\right\}. \tag{2}$$

We consider F_1 instead of F in the left-hand side of (2), so we have

$$\left|\frac{F_1(T) - F_1(\sigma T)}{T - \sigma T}\right| \leq \max\left\{|F_2(T)|, |T - \sigma^{-1}\alpha_2| \cdot \left|\frac{F_2(T) - F_2(\sigma T)}{T - \sigma T}\right|\right\}. \tag{3}$$

By (2) and (3), we deduce

$$\left|\frac{F(T)-F(\sigma T)}{T-\sigma T}\right| \leq \max\Big\{|F_1(T)|, |F_2(T)|\cdot|T-\sigma^{-1}\alpha_1|, \\ |T-\sigma^{-1}\alpha_1|\cdot|T-\sigma^{-1}\alpha_2|\cdot\left|\frac{F_2(T)-F_2(\sigma T)}{T-\sigma T}\right|\Big\}. \tag{4}$$

By using an iteration process, as in (2)–(4), we see that

$$\left|\frac{F(T)-F(\sigma T)}{T-\sigma T}\right| \leq \max\{|F_1(T)|, |F_2(T)|\cdot|T-\sigma^{-1}\alpha_1|, \dots, \\ |T-\sigma^{-1}\alpha_1|\cdot|T-\sigma^{-1}\alpha_2|\cdot\ldots\cdot|T-\sigma^{-1}\alpha_{n-1}|\}. \tag{5}$$

From inequalities (1) and (5), it is clear that

$$\frac{1}{|F(T)|}\cdot\left|\frac{F(T)-F(\sigma T)}{T-\sigma T}\right| \leq \frac{1}{|T-\alpha_1|},$$

therefore $C_F \leq \frac{|F(T)|}{|T-\alpha_1|}$. When σT tends to T in $|\frac{F(T)-F(\sigma T)}{T-\sigma T}| \leq C_F$, we have $|F'(T)| \leq C_F$, and this completes the proof of the proposition. ■

REMARK 3.2. Let T be a transcendental element of $\mathbb{C}_p$, which is Lipschitz (that is, $\lim_{\varepsilon\to 0}\frac{\varepsilon}{|N(T,\varepsilon)|} = 0$, where $N(T,\varepsilon)$ is the number of open balls of radius ε that cover the Galois orbit of T), and let $\{\alpha_n\}_{n\geq 0}$ be a distinguished sequence that converges to T. Let $f_n = \mathrm{Irr}_{\mathbb{Q}_p}(\alpha_n)$ be the minimal polynomial of α_n over $\mathbb{Q}_p$. We know that

$$\lim_{n\to\infty}\frac{|f_n'(T)|}{|f_n(T)|} = \infty$$

(see [1,4]). So, by Proposition 3.1, we have

$$\lim_{n\to\infty}\frac{C_{f_n}}{|f_n(T)|} = \infty,$$

which means that C_{f_n} is growing faster to infinity than $|f_n(T)|$.

THEOREM 3.3. *Let T be a transcendental element of $\mathbb{C}_p$. We have:*

(i) *The set of Galois equivariant $\mathbb{C}_p$-valued functions defined on $O(T)$ coincides with the set $\mathcal{C}_G(O(T),\mathbb{C}_p)$ of continuous Galois equivariant $\mathbb{C}_p$-valued functions defined on $O(T)$, and is isomorphic with $\widetilde{\mathbb{Q}_p[T]}$ as $\mathbb{Q}_p$-algebra.*

(ii) *Let $f : O(T) \to \mathbb{C}_p$ be a Galois equivariant function and let $f = \sum_{n\geq 0} a_n M_n$ be its representation as in Proposition 2.1, where $\{a_n\}_{n\geq 0}$ is a sequence of elements of $\mathbb{Q}_p$ such that $\lim_{n\to\infty} a_n = 0$. If the sequence $\{\frac{|a_n|}{|T-\gamma_n|}\}_{n\geq 0}$ is uniformly bounded, where γ_n is a root of M_n, which is the closest to T, then f is Lipschitz. Moreover, if $\lim_{n\to\infty}\frac{|a_n|}{|T-\gamma_n|} = 0$, then f is differentiable on $O(T)$.*

Proof. (i) Let f be a $\mathbb{C}_p$-valued function defined on $O(T)$ that is Galois equivariant. We have $f(\sigma T) = \sigma f(T) = f(T)$ for any $\sigma \in H(T) = \{\sigma \in G : \sigma(T) = T\}$. It follows that $f(T) \in \mathrm{Fix}(H(T)) = \widetilde{\mathbb{Q}_p[T]}$. The function f is clearly continuous if $f(T) \in \mathbb{Q}_p[T]$ and every other $f \in \widetilde{\mathbb{Q}_p[T]}$ is a uniform limit of such functions. In fact, $f(T) = \sum_{n\geq 0} a_n M_n(T)$ by Proposition 2.1 and then $f(x) = \sum_{n\geq 0} a_n M_n(x)$, for any $x \in O(T)$. The isomorphism of $\mathbb{Q}_p$-algebras is given by the map $f \rightsquigarrow f(T)$.

(ii) Let $C > 0$ be a positive constant such that $\frac{|a_n|}{|T-\gamma_n|} \leq C$, for any $n \geq 0$. We have

$$\frac{f(T) - f(\sigma T)}{T - \sigma T} = \sum_{n\geq 0} a_n \cdot \frac{M_n(T) - M_n(\sigma T)}{T - \sigma T}, \quad \text{for any } \sigma \in G \setminus H(T). \tag{6}$$

By Propositions 2.1 and 3.1 we find that

$$\frac{|M_n(T) - M_n(\sigma T)|}{|T - \sigma T|} \leq \frac{1}{|T - \gamma_n|}.$$

Therefore, by the hypothesis and (6),

$$\frac{|f(T) - f(\sigma T)|}{|T - \sigma T|} \leq \sup_{n\geq 0} \frac{|a_n|}{|T - \gamma_n|} \leq C,$$

which means that f is Lipschitz and its Lipschitz constant is $\leq C$.

Again, by Propositions 2.1 and 3.1, we have

$$|M_n'(T)| \leq \frac{|M_n(T)|}{|T - \gamma_n|} \leq \frac{1}{|T - \gamma_n|}.$$

Then, by the hypothesis, we know that the series $\sum_{n\geq 0} a_n M_n'(T)$ is convergent and, by a classical argument of analysis, we infer that f is differentiable and $f'(x) = \sum_{n\geq 0} a_n M_n'(x)$, for any $x \in O(T)$. This completes the proof of the theorem. ■

Remark 3.4. Let $f : O(T) \to \mathbb{C}_p$ be a Galois equivariant function and let $f = \sum_{n\geq 0} a_n M_n$ be its representation as in Theorem 3.3. For any $n \geq 0$ there exists $\underline{s}(n) \in \mathcal{A}$ such that

$$M_n(T) = M_{\underline{s}(n)}(T) = \frac{H_{\underline{s}(n)}(T)}{p^{q_{\underline{s}(n)}}},$$

where $q_{\underline{s}(n)} = [v(H_{\underline{s}(n)}(T))]$. By preliminary results, the map $n \to \underline{s}(n)$ from $\mathbb{N}$ to $\mathcal{A}$ is bijective and strictly increasing with respect to the order defined on $\mathcal{A}$. We have $f = \sum_{n\geq 0} a_{\underline{s}(n)} M_{\underline{s}(n)}$, where $a_{\underline{s}(n)} = a_n$. Let us suppose that

$$\underline{s}(n) = (s_0(n), s_1(n), \ldots, s_{k_n}(n), 0, \ldots).$$

We deduce that the roots of M_n are the same with the roots of

$$H_{\underline{s}(n)} = f_0^{s_0(n)} f_1^{s_1(n)} \cdots f_{k_n}^{s_{k_n}(n)}.$$

So, by preliminary results, the closest root to T of M_n is $\gamma_n = \alpha_{k_n}$ and thus we have $|T - \gamma_n| = |T - \alpha_{k_n}| = \delta_{k_n}$. Moreover, the following properties hold:

(i) The sufficient conditions for a Galois equivariant function on the orbit of T to be Lipschitz (respectively differentiable), see Theorem 3.3 (ii), depend only on the coefficients of the development of f in the orthonormal basis $\{M_n\}_{n\geq 0}$ and on the sequence of invariants $\{\delta_n\}_{n\geq 0}$ of T. More precisely, if the sequence $\{\frac{|a_n|}{\delta_{k_n}}\}_{n\geq 0}$ is uniformly bounded, then f is Lipschitz on $O(T)$ and, if $\lim_{n\to\infty} \frac{|a_n|}{\delta_{k_n}} = 0$ then f is differentiable on $O(T)$.

(ii) If T is an integral transcendental element of $\mathbb{C}_p$ and the sequence $\{|a_n| p^{q_{\underline{s}(n)}}\}_{n\geq 0}$ is upper bounded, then f has a unique Galois equivariant analytic continuation to $B[O(T), |p|^{1+\varepsilon}]$, for any $\varepsilon > 0$, see [5, Theorem 3.1].

Let $K \subset \mathbb{C}_p$ be a field that is a finite and normal extension of $\mathbb{Q}_p$ of degree s. Let us suppose that $K = \mathbb{Q}_p(\alpha)$, where $\alpha \in \overline{\mathbb{Q}_p}$ is an algebraic element of degree s over $\mathbb{Q}_p$. We recall that $G_K = \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K)$ is the Galois group of continuous automorphisms of $\mathbb{C}_p$ that fix the field K, which is canonically isomorphic to $\mathrm{Gal}(\overline{K}/K)$, where $\overline{K}$ is an algebraic closure of K. We have that $G = \bigcup_{i=1}^{s} G_K \sigma_i$, where $\{\sigma_i\}_{1\leq i\leq s}$ is a system of representatives for G/G_K. For any transcendental element T of $\mathbb{C}_p$ we consider $O(T) = \bigcup_{i=1}^{s} O_i(T)$, the decomposition of $O(T)$ into homeomorphic domains of transitivity with respect to the action of G_K, where $O_i(T) = \{\sigma\sigma_i(T) : \sigma \in G_K\}$. Let us consider the set

$$\mathcal{H}_K(T) = \big\{f : O(T) \to \mathbb{C}_p : f \text{ is a } G_K\text{-equivariant function}\big\}.$$

(By definition, $f : O(T) \to \mathbb{C}_p$ is G_K-equivariant if $f(\sigma z) = \sigma f(z)$, for any $\sigma \in G_K$ and any $z \in O(T)$.) It is easy to see that any $f \in \mathcal{H}_K(T)$ can be defined in a canonical way by a set of functions $\{f_i\}_{1\leq i\leq s}$, where $f_i : O_i(T) \to \mathbb{C}_p$ is G_K-equivariant, for any $1 \leq i \leq s$, which means that $f_i(\sigma\sigma_i(T)) = \sigma f_i(\sigma_i(T))$, for any $\sigma \in G_K$. If

$$H_K(T) = \{\sigma \in G_K : \sigma(T) = T\},$$

then H_K is a subgroup of G_K and $\mathrm{Fix}(H_K(T)) = \widetilde{K[T]}$. With f as above, we have that $f_i(\sigma\sigma_i(T)) = \sigma f_i(\sigma_i(T)) = f_i(\sigma_i(T))$, for any $\sigma \in H_K(\sigma_i(T)) \leq G_K$ and any $1 \leq i \leq s$. Then

$$f_i(\sigma_i(T)) \in \mathrm{Fix}(H_K(\sigma_i(T))) = \widetilde{K[\sigma_i(T)]}.$$

Two cases may appear:

(a) $K \subset \widetilde{\mathbb{Q}_p[T]}$, then $\widetilde{K[T]} = \widetilde{\mathbb{Q}_p[T]}$, or

(b) $K \not\subset \widetilde{\mathbb{Q}_p[T]}$.

In case (b) we have

$$\widetilde{K[T]} = \widetilde{\mathbb{Q}_p[T]}(\alpha) = \sum_{j=1}^{q} e_j \widetilde{\mathbb{Q}_p[T]},$$

where the sum is direct, $q = [\widetilde{K[T]} : \widetilde{\mathbb{Q}_p[T]}] \leq [K : \mathbb{Q}_p]$, and $\{e_1, e_2, \dots, e_q\} \subseteq \{1, \alpha, \alpha^2, \dots, \alpha^{s-1}\}$ is a basis of $\widetilde{K[T]}$ over $\widetilde{\mathbb{Q}_p[T]}$. In both cases we deduce, via Proposition 2.1, that

$$f_i(z) = \sum_{n\geq 0} \Bigl(\sum_{j=1}^{q} e_j a_n^{(j)}(i)\Bigr) M_n(z), \quad \text{for any } z \in O_i(T),$$

where $a_n^{(j)}(i) \in \mathbb{Q}_p$ is such that $\lim_{n\to\infty} a_n^{(j)}(i) = 0$ for any $1 \leq j \leq q$ and any $1 \leq i \leq s$. We conclude that any $f \in \mathcal{H}_K(T)$ has the representation

$$f(z) = \sum_{n\geq 0} \Bigl(\sum_{j=1}^{q} e_j a_n^{(j)}\Bigr) M_n(z), \quad \text{for any } z \in O(T), \tag{7}$$

where $a_n^{(j)} \in \mathbb{Q}_p$ is such that $\lim_{n\to\infty} a_n^{(j)} = 0$ for any $1 \leq j \leq q$. (We note here that if $z \in O_i(T)$, then $f(z) = f_i(z)$, and for any $1 \leq j \leq q$ we have $a_n^{(j)} = a_n^{(j)}(i)$.) From the proof of [3, Proposition 6.1], we have that $M_n(T)M_m(T) = \sum_{k\geq 0} b_k M_k(T)$ for any $m, n \geq 0$, with $b_k \in \mathbb{Z}_p$ for any $k \geq 0$, and this shows us how the product works in the K-algebra $\mathcal{H}_K(T)$, via the representation (7).

We know that the monomials $\{M_n(X)\}_{n\geq 0}$ are linearly independent over $\overline{\mathbb{Q}}_p$ and, since T is transcendental over $\overline{\mathbb{Q}}_p$, we have that the set $\{M_n(T)\}_{n\geq 0}$ is linearly independent over $\overline{\mathbb{Q}}_p$. Now, let $F \in \overline{\mathbb{Q}}_p[X]$ be a polynomial. By applying several times the theorem of division of polynomials, as in the proof of [3, Theorem 6.3], we have that $F = \sum_{i\geq 0} a_i M_i$, with $a_i \in \overline{\mathbb{Q}}_p$, for any $i \geq 0$. We know that $\overline{\mathbb{Q}}_p[X]$ is dense in $\mathcal{C}(O(T), \mathbb{C}_p)$, which is the set of all continuous functions defined on the Galois orbit of T with values in $\mathbb{C}_p$. By this, we have that the space generated by the monomials $M_i(T)$ over $\overline{\mathbb{Q}}_p$ is also dense in $\mathcal{C}(O(T), \mathbb{C}_p)$. We remark that if K is a finite extension of $\mathbb{Q}_p$, $K \neq \mathbb{Q}_p$, and f has the representation $f(T) = \sum_{n\geq 0} a_n M_n(T)$, with $a_n \in K$, it cannot be unique, even though the set $\{M_n(T\}_{n\geq 0}$ is linearly independent over $\overline{\mathbb{Q}}_p$, because $\widetilde{\mathbb{Q}_p[T]}$ could contain algebraic elements from $K \setminus \mathbb{Q}_p$.

We collect the above results, via Theorem 3.3, in the following theorem.

Theorem 3.5. *Let T be a transcendental element of $\mathbb{C}_p$ and let $K \subset \mathbb{C}_p$ be a field that is a finite and normal extension of $\mathbb{Q}_p$. We have:*

(i) *The set $\mathcal{H}_K(T)$ of G_K-equivariant $\mathbb{C}_p$-valued functions defined on $O(T)$ coincides with the set $\mathcal{C}_{G_K}(O(T), \mathbb{C}_p)$ of continuous functions defined on $O(T)$ with values* *in* $\mathbb{C}_p$*, which are* G_K*-equivariant. In the case* $K \subset \widetilde{\mathbb{Q}_p[T]}$*, we have that* $\mathcal{H}_K(T)$ *is isomorphic with* s *copies of* $\widetilde{\mathbb{Q}_p[T]}$ *as* $\mathbb{Q}_p$*-algebras, and in the case* $K \not\subset \widetilde{\mathbb{Q}_p[T]}$*, we have that* $\mathcal{H}_K(T)$ *is isomorphic, as a vector space, with a product of* sq *copies of* $\widetilde{\mathbb{Q}_p[T]}$*, where*

$$q = \left[\widetilde{K[T]} : \widetilde{\mathbb{Q}_p[T]}\right] \leq s = [K : \mathbb{Q}_p].$$

(ii) *Let* $f : O(T) \to \mathbb{C}_p$ *be a* G_K*-equivariant function and let*

$$f(z) = \sum_{n \geq 0} \left(\sum_{j=1}^{q} e_j a_n^{(j)} \right) M_n(z)$$

be its representation as in (7)*. If we assume that the sequence*

$$\left\{ \frac{\max_{1 \leq j \leq q} |a_n^{(j)}|}{|T - \gamma_n|} \right\}_{n \geq 0}$$

is uniformly bounded, where γ_n *is a root of* M_n*, which is the closest to* T*, then* f *is Lipschitz. If* $\lim_{n\to\infty}(\max_{1\leq j\leq q} |a_n^{(j)}|)/|T - \gamma_n| = 0$*, then* f *is differentiable on* $O(T)$*.*

(iii) *If* T *is normal, that is,* $\mathbb{Q}_p \subset \widetilde{\mathbb{Q}_p[T]}$ *is a normal extension, then* $\bigcup_{n=1}^{\infty} \mathcal{H}_{K_n}(T)$ *is dense in* $\mathcal{C}(O(T), \mathbb{C}_p)$*, where*

$$\mathbb{Q}_p \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset \widetilde{\mathbb{Q}_p[T]}$$

is a tower of normal and finite fields extensions of $\mathbb{Q}_p$*, such that*

$$\bigcup_{n=1}^{\infty} K_n = \widetilde{\mathbb{Q}_p[T]} \cap \overline{\mathbb{Q}}_p.$$

In what follows we study some aspects related to the problem of analytic continuation for the functions from the class of Galois equivariant functions defined on $O(T)$, where T is an integral transcendental element of $\mathbb{C}_p$.

Let $F : O(T) \to \mathbb{C}_p$ be a Galois equivariant function with the property that $F(T) \in \widetilde{\mathbb{Z}_p[T]}$. By [5, Theorem 3.1], we know that F has a unique Galois equivariant analytic continuation to $B[O(T), |p|^{1+\varepsilon}]$, which is, for any $\varepsilon > 0$, also denoted by F. Moreover, if $\alpha \in \overline{\mathbb{Q}}_p \cap B[T, |p|^{1+\varepsilon}]$, then we have

$$F(z) = \sum_{n=0}^{\infty} a_n(\alpha)(z - \alpha)^n, \quad \text{for any } z \in B[T, |p|^{1+\varepsilon}], \tag{8}$$

where $a_n(\alpha) \in \mathbb{Z}_p[\alpha]$. We see that $F(T)$ can be represented as a convergent power series in $T - \alpha$ with coefficients in $\mathbb{Z}_p[\alpha]$. Denote $P_n(T) = H_{\underline{s}(n)}(T)$, see Remark 3.4

for notation. By Proposition 2.1 and the fact that T is integral, we see that $P_n(T)$ is a polynomial with integral coefficients of degree n. Developing $(T-\alpha)^n$ with respect to the polynomials $P_n(T) \in \mathbb{Z}_p[T]$, $n \geq 0$, we find, by (8), that

$$F(T) = \sum_{n=0}^{\infty} b_n(\alpha) P_n(T), \tag{9}$$

where $b_n(\alpha) \in \mathbb{Z}_p[\alpha]$, so that

$$F(\sigma T) = \sigma F(T) = \sum_{n \geq 0} b_n(\alpha) P_n(\sigma T), \quad \text{for any } \sigma \in H(\alpha),$$

where $H(\alpha) = \{\sigma \in G : \sigma(\alpha) = \alpha\}$. Choosing $\sigma_n \in H(\alpha)$, $n \geq 1$, such that $\sigma_n T \to T$, we conclude, by the identity theorem in p-adic fields, that

$$F(z) = \sum_{n \geq 0} b_n(\alpha) P_n(z), \quad \text{for any } z \in B[T, |p|^{1+\varepsilon}]. \tag{10}$$

The convergence of the series on the right-hand side of (10) follows from the equality $\lim_{n\to\infty} |P_n(z)| = 0$, which is a consequence of the fact that the number of the roots of the polynomial $P_n(z)$ into the ball $B[T, |p|^{1+\varepsilon}]$ tends to infinity as soon as $n \to \infty$. Also, this is an argument for the fact that the series on the right-hand side of (9) is convergent.

Now let $F : B[O(T), |p|^{1+\varepsilon}] \to \mathbb{C}_p$ be an analytic function that is Galois equivariant and such that (8) holds. Denote $d = \deg \alpha$. Of course (9) is also true and the restriction of F to $O(T)$ is not necessarily in $\widetilde{\mathbb{Z}_p[T]}$. From (9) we have $b_n(\alpha) \in \mathbb{Z}_p[\alpha]$, so

$$b_n(\alpha) = \sum_{i=0}^{d-1} b_n^{(i)} \alpha^i,$$

with $b_n^{(i)} \in \mathbb{Z}_p$ for any $n \geq 0$ and any $0 \leq i \leq d-1$. Then we derive

$$F(T) = \sum_{i=0}^{d-1} G_i(T) \alpha^i, \tag{11}$$

where $G_i(T) = \sum_{n=0}^{\infty} b_n^{(i)} P_n(T) \in \widetilde{\mathbb{Z}_p[T]}$, for any $0 \leq i \leq d-1$. Again, by [5, Theorem 3.1], we see that $G_i(T)$ has a unique Galois equivariant analytic continuation to $B[O(T), |p|^{1+\varepsilon}]$, which we denote by $\overline{G_i}$. Denote

$$\overline{F}(z) = \sum_{i=0}^{d-1} \overline{G_i}(z) \alpha^i, \quad \text{for } z \in B[O(T), |p|^{1+\varepsilon}].$$

Then, by (11), we know that $F(\sigma T) = \sigma F(T) = \sum_{i=0}^{d-1} G_i(\sigma T)\alpha^i$, so that $F(\sigma T) = \overline{F}(\sigma T)$, for any $\sigma \in H(\alpha)$. Again, choosing $\sigma_n \in H(\alpha)$, $n \geq 1$, such that $\sigma_n T \to T$, we conclude, by the identity theorem in p-adic fields, that $F(z) = \overline{F}(z)$, for any $z \in B[T, |p|^{1+\varepsilon}]$. Consequently, we can finally summarize in the following theorem what has been developed so far.

THEOREM 3.6. *Let T be an integral transcendental element of $\mathbb{C}_p$ and let ε be a positive real number. Let $F\colon B[O(T), |p|^{1+\varepsilon}] \to \mathbb{C}_p$ be a Galois equivariant analytic function such that*

$$F(z) = \sum_{n=0}^{\infty} a_n(\alpha)(z-\alpha)^n, \quad \textit{for any } z \in B[T, |p|^{1+\varepsilon}],$$

where $\alpha \in \overline{\mathbb{Q}}_p \cap B[T, |p|^{1+\varepsilon}]$, $d = \deg \alpha$, and $a_n(\alpha) \in \mathbb{Z}_p[\alpha]$, for any $n \geq 0$. Then, for any $0 \leq i \leq d-1$, there exists $G_i(T) \in \widetilde{\mathbb{Z}_p[T]}$, which has a unique Galois equivariant analytic continuation to $B[O(T), |p|^{1+\varepsilon}]$ denoted also by G_i, such that

$$F(z) = \sum_{i=0}^{d-1} G_i(z)\alpha^i, \quad \textit{for any } z \in B[T, |p|^{1+\varepsilon}].$$

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