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A criterion for cofiniteness of modules

Mohammad Khazaei (*) – Reza Sazeedeh (**)

Abstract – Let A be a commutative noetherian ring, $\mathfrak{a}$ be an ideal of A, and m, n be non-negative integers. Let M be an A-module such that $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq m + n$. We define a class $\mathcal{S}_n(\mathfrak{a})$ of modules and we assume that $H^s_\mathfrak{a}(M) \in \mathcal{S}_n(\mathfrak{a})$ for all $s \leq m$. We show that $H^s_\mathfrak{a}(M)$ is $\mathfrak{a}$-cofinite for all $s \leq m$ if either $n = 1$ or $n \geq 2$ and $\operatorname{Ext}^i_A(A/\mathfrak{a}, H^{t+s-i}_\mathfrak{a}(M))$ is finitely generated for all $1 \leq t \leq n-1$, $i \leq t-1$ and $s \leq m$. If A is a ring of dimension d and $M \in \mathcal{S}_n(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of dimension $\leq d-1$, then we prove that $M \in \mathcal{S}_n(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of A.

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1. Introduction

Throughout this paper A is a commutative noetherian ring, $\mathfrak{a}$ is an ideal of A, m, n are non-negative integer numbers and M is an A-module, unless otherwise stated. Grothendieck [4, Exposé XIII, Conjecture 1.2] conjectured that if M is a finitely generated A-module, then $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_\mathfrak{a}(M))$ is finitely generated, where $H^i_\mathfrak{a}(M)$ is the i-th local cohomology of M with respect to the ideal $\mathfrak{a}$. The concept of cofiniteness of modules was first defined by Hartshorne [5], giving a negative answer to the Grothendieck's conjecture, and later was studied by many other authors [1, 3, 6–10].

An A-module M is said to be $\mathfrak{a}$-*cofinite* if $\operatorname{Supp} M \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all integers $i \geq 0$. In [11], Nazari and Sazeedeh introduced a

(*) *Indirizzo dell'A.*: Department of Mathematics, Urmia University, P.O. Box 165, Urmia, Iran; mohmmad.khazaei@gmail.com

(**) *Indirizzo dell'A.*: Department of Mathematics, Urmia University, P.O. Box 165, Urmia, Iran; rsazeedeh@ipm.ir

criterion for the cofiniteness of modules. Roughly speaking, the criterion estimates how much a module is close to be cofinite. We recall that an A-module M satisfies the condition $\mathrm{P}_n(\mathfrak{a})$ if the following implication holds:

$\mathrm{P}_n(\mathfrak{a})$: *If* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *is finite for all* $i \leq n$ *and* $\mathrm{Supp}\, M \subseteq V(\mathfrak{a})$*, then* M *is* $\mathfrak{a}$*-cofinite.*

We denote by $\mathcal{S}_n(\mathfrak{a})$ the class of all modules satisfying the condition $\mathrm{P}_n(\mathfrak{a})$.

We start this paper by describing and getting some explanations of the class $\mathcal{S}_n(\mathfrak{a})$, especially in the case where $n \leq 2$. We give several examples and results at the beginning of the paper. However, our main aim of this paper is to study the cofiniteness of local cohomology modules when they belong to $\mathcal{S}_n(\mathfrak{a})$ without any condition on the ideal $\mathfrak{a}$ or on the ring A. The first result is devoted to the case $n = 1$ as follows.

THEOREM 1.1 (Theorem 2.10). *If* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *is finitely generated for all* $i \leq m + 1$ *and* $H^i_{\mathfrak{a}}(M) \in \mathcal{S}_1(\mathfrak{a})$ *for all* $i \leq m$*, then* $H^i_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cofinite for all* $i \leq m$*.*

In the case where $\dim A/\mathfrak{a} = 1$, Melkersson [10] showed that $\mathcal{S}_1(\mathfrak{a}) = \text{Mod-}A$ and so the theorem implies that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i whenever $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq n + 1$ where $n = \dim M$.

Moreover, we have the following result for the class $\mathcal{S}_n(\mathfrak{a})$ with $n \geq 2$.

THEOREM 1.2 (Theorem 2.11). *Assume that* m *is a non-negative integer such that* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *is finitely generated for all* $i \leq m + n$ *and* $H^s_{\mathfrak{a}}(M) \in \mathcal{S}_n(\mathfrak{a})$ *for all* $s \leq m$*. If* $\mathrm{Ext}^i_A(A/\mathfrak{a}, H^{t+s-i}_{\mathfrak{a}}(M))$ *is finitely generated for all* $1 \leq t \leq n - 1$*,* $i \leq t - 1$ *and* $s \leq m$*, then* $H^s_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cofinite for all* $s \leq m$*.*

As an application of this theorem, let $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ be finitely generated for all $i \leq m + 2$ and let $H^i_{\mathfrak{a}}(M) \in \mathcal{S}_2(\mathfrak{a})$ for all $i \leq m$. We show that if $\mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \leq m + 1$, then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \leq m$. In case $\dim A/\mathfrak{a} = 2$, where A is a local ring, Bahmanpour et al. [1] showed that $\mathcal{S}_2(\mathfrak{a}) = \text{Mod-}A$ and so this application generalizes [1, Theorem 3.7] and [11, Theorem 3.7]. For the case of $\dim A/\mathfrak{a} = 3$, assume that $\mathrm{depth}(\mathrm{Ann}\, M, A/\mathfrak{a}) > 0$ and $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq 2$. Then we show that $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite if $\mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for $i = 0, 1$.

We give a result on modules whose local cohomology modules are nonzero only in two consecutive numbers. To be more precise, assume that t is a non-negative integer such that $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq n + t + 1$ and $H^i_{\mathfrak{a}}(M) = 0$ for all $i \neq t, t + 1$. Then we show that $H^{t+1}_{\mathfrak{a}}(M) \in \mathcal{S}_n(\mathfrak{a})$ if and only if $H^t_{\mathfrak{a}}(M) \in \mathcal{S}_{n+2}(\mathfrak{a})$.

One of the substantial results in the local cohomology theory and cofiniteness is the change of ring principle. We show that this result holds for $\mathcal{S}_n(\mathfrak{a})$ as well. Let B be a finitely generated A-algebra and let M be a B-module. Then we show that $M \in \mathcal{S}_n(\mathfrak{a})$ if and only if $M \in \mathcal{S}_n(\mathfrak{a}B)$.

We end this paper with the following theorem which is a generalization of [11, Theorem 2.3].

THEOREM 1.3 (Theorem 2.16). *Let A be a ring of dimension d and $M \in \mathcal{S}_n(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of dimension $\leq d-1$ (i.e.* $\dim A/\mathfrak{a} \leq d-1$*). Then $M \in \mathcal{S}_n(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of A.*

2. The main results

We start this section by defining a class of A-modules which has an essential role in this paper. For the basic properties of local cohomology modules, we refer the readers to [2].

DEFINITION 2.1. Let n be a non-negative integer and let M be an A-module. We say that M satisfies the *condition* $\mathrm{P}_n(\mathfrak{a})$ if the following implication holds:

$\mathrm{P}_n(\mathfrak{a})$: *If* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *is finitely generated for all* $i \leq n$ *and* $\mathrm{Supp}\, M \subseteq V(\mathfrak{a})$*, then* M *is* $\mathfrak{a}$*-cofinite.*

We define a class of A-modules as

$$\mathcal{S}_n(\mathfrak{a}) = \{M \in \text{Mod-}A \mid M \text{ satisfies the condition } \mathrm{P}_n(\mathfrak{a})\}.$$

We observe that $\mathcal{S}_0(\mathfrak{a}) \subseteq \mathcal{S}_1(\mathfrak{a}) \subseteq \cdots$. We also say that A *satisfies the condition* $\mathrm{P}_n(\mathfrak{a})$ if $\mathcal{S}_n(\mathfrak{a}) = \text{Mod-}A$.

In the rest of this section, we assume that $\mathfrak{a}$ is an ideal of A, n is a non-negative integer and M is an A-module, unless otherwise stated. In order to describe the class $\mathcal{S}_n(\mathfrak{a})$, we give several examples. The first example shows that the top local cohomology modules lie in $\mathcal{S}_0(\mathfrak{a})$.

EXAMPLE 2.2. Assume that $\mathfrak{a}$ is an arbitrary ideal of A and M is an A-module of dimension d, where $\dim M$ means the dimension of $\mathrm{Supp}\, M$. Then $H^d_{\mathfrak{a}}(M)$ is in $\mathcal{S}_0(\mathfrak{a})$. To be more precise, if $\mathrm{Hom}_A(A/\mathfrak{a}, H^d_{\mathfrak{a}}(M))$ is a finitely generated A-module, then it follows from [11, Theorem 3.11] that $H^d_{\mathfrak{a}}(M)$ is artinian and so, since $\mathrm{Hom}_A(A/\mathfrak{a}, H^d_{\mathfrak{a}}(M))$ has finite length, according to [9, Proposition 4.1], the module $H^d_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite.

The following example identifies some modules in $\mathcal{S}_1(\mathfrak{a})$.

EXAMPLE 2.3. Given an arbitrary ideal $\mathfrak{a}$ of A, by virtue of [1, Lemma 2.2], $M \in \mathcal{S}_1(\mathfrak{a})$ for all modules M with $\dim M \leq 1$. In particular, if $\dim A/\mathfrak{a} = 1$, then it follows from [10, Theorem 2.3] that $\mathcal{S}_1(\mathfrak{a}) = \text{Mod-}A$. Furthermore, if $\dim A = 2$, then it follows from [11, Corollary 2.4] that $\mathcal{S}_1(\mathfrak{a}) = \text{Mod-}A$ for any ideal $\mathfrak{a}$ of A.

For the class $\mathcal{S}_2(\mathfrak{a})$, we have the following example.

Example 2.4. Let $\mathfrak{a}$ be an ideal of a local ring A with $\dim A/\mathfrak{a} = 2$. It follows from [1, Theorem 3.5] that $\mathcal{S}_2(\mathfrak{a}) = \text{Mod-}A$. Furthermore, if A is a local ring with $\dim A = 3$, then it follows from [11, Corollary 2.5] that $\mathcal{S}_2(\mathfrak{a}) = \text{Mod-}A$ for any ideal $\mathfrak{a}$ of A.

In the previous example we may have $\mathcal{S}_0(\mathfrak{a}) \neq \text{Mod-}A$ or $\mathcal{S}_1(\mathfrak{a}) \neq \text{Mod-}A$.

Example 2.5. The Hartshorne's example [5, §3] shows that if $\mathfrak{a}$ is an ideal of A with $\dim A/\mathfrak{a} = 2$, then we may have both $\mathcal{S}_0(\mathfrak{a}) \neq \text{Mod-}A$ and $\mathcal{S}_1(\mathfrak{a}) \neq \text{Mod-}A$. More precisely, assume that $A = k[\![x, y, u, v]\!]$, where x, y, u, v are variables and k is a field. Let $\mathfrak{p} = (x, u)$ and $M = A/(xy - uv)$. Then $H^1_{\mathfrak{p}}(M) \notin \mathcal{S}_1(\mathfrak{a})$ and $H^2_{\mathfrak{p}}(M) \notin \mathcal{S}_0(\mathfrak{a})$. Indeed, $H^i_{\mathfrak{p}}(A) = 0$ for all $i \neq 2$ as $\operatorname{depth}(\mathfrak{p}, A) = 2$ and since $\operatorname{depth}(\mathfrak{p}, M) = 1$, we have $\Gamma_{\mathfrak{p}}(M) = 0$; hence we have an exact sequence of modules

$$0 \longrightarrow H^1_{\mathfrak{p}}(M) \longrightarrow H^2_{\mathfrak{p}}(A) \stackrel{xy-uv}{\longrightarrow} H^2_{\mathfrak{p}}(A) \longrightarrow H^2_{\mathfrak{p}}(M) \longrightarrow 0.$$

By [5], the module $H^2_{\mathfrak{p}}(M)$ is not $\mathfrak{p}$-cofinite, and it follows from [7, Proposition 2.5] that $H^2_{\mathfrak{p}}(A)$ is $\mathfrak{p}$-cofinite, hence $H^1_{\mathfrak{p}}(M)$ is not $\mathfrak{p}$-cofinite. This said, we observe that $\operatorname{Ext}^i_A(A/\mathfrak{p}, H^1_{\mathfrak{p}}(M))$ is finitely generated for $i = 0, 1$, so that $H^1_{\mathfrak{p}}(M) \notin \mathcal{S}_1(\mathfrak{a})$; moreover, $\operatorname{Hom}_A(A/\mathfrak{p}, H^2_{\mathfrak{p}}(M))$ is finitely generated, so that $H^2_{\mathfrak{p}}(M) \notin \mathcal{S}_0(\mathfrak{a})$.

For the case $\dim A/\mathfrak{a} = 3$, we have the following result.

Proposition 2.6. *Let A be a local ring with* $\dim A/\mathfrak{a} = 3$, $\operatorname{depth}(\operatorname{Ann} M, A/\mathfrak{a}) > 0$, *and let* $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ *be finitely generated for all $i \leq 2$. If* $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ *is finitely generated for $i = 0, 1$, then $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite.*

Proof. There exists an element $x \in \operatorname{Ann} M$ such that x is an $A/\mathfrak{a}$-sequence; hence $\dim A/xA + \mathfrak{a} = 2$. It follows from [3, Proposition 1] that $\operatorname{Ext}^i_A(A/xA + \mathfrak{a}, M)$ is finitely generated for all $i \leq 2$. On the other hand, using [3, Proposition 2], for each $i \geq 0$, the module $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite if and only if $H^i_{xA+\mathfrak{a}}(M)$ is $xA + \mathfrak{a}$-cofinite. Set $\mathfrak{b} = xA + \mathfrak{a}$ and $\overline{M} = M/\Gamma_{\mathfrak{a}}(M)$. Then there is an exact sequence of modules $0 \to \overline{M} \to E \to N \to 0$ in which E is an injective A-module with $\Gamma_{\mathfrak{a}}(E) = 0$, so that $\Gamma_{\mathfrak{b}}(E) = 0$ as $\mathfrak{a} \subseteq \mathfrak{b}$. By assumption, $\operatorname{Hom}_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(N)) \cong \operatorname{Hom}_A(A/\mathfrak{a}, H^1_{\mathfrak{a}}(M))$ is finitely generated, and since $\mathfrak{a} \subseteq \mathfrak{b}$ and $\Gamma_{\mathfrak{b}}(N) \subseteq \Gamma_{\mathfrak{a}}(N)$, the module $\operatorname{Hom}_A(A/\mathfrak{b}, \Gamma_{\mathfrak{b}}(N))$ is finitely generated. Since $x \in \operatorname{Ann} M$, we have $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{b}}(M)$; hence $\operatorname{Hom}_A(A/\mathfrak{b}, \Gamma_{\mathfrak{b}}(M))$ is finitely generated. Therefore, the isomorphisms $H^1_{\mathfrak{b}}(M) \cong H^1_{\mathfrak{b}}(\overline{M}) \cong \Gamma_{\mathfrak{b}}(N)$ imply that $\mathrm{Hom}_A(A/\mathfrak{b}, H^1_{\mathfrak{b}}(M)) \cong \mathrm{Hom}_A(A/\mathfrak{b}, \Gamma_{\mathfrak{b}}(N))$ is finitely generated. Now, it follows from [11, Theorem 3.7] that $\Gamma_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite, so by the first argument we deduce that $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. ■

The following result establishes a relation between the classes $\mathcal{S}_n(\mathfrak{a})$ and $\mathcal{S}_n(\mathfrak{p}_i)$, where $\mathfrak{p}_i$ are the minimal prime ideals of $\mathfrak{a}$ for $1 \le i \le t$.

Proposition 2.7. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal prime ideals of $\mathfrak{a}$ and let M be an A-module with* $\mathrm{Supp}(M) \subseteq V(\mathfrak{p}_1 + \cdots + \mathfrak{p}_t)$*. If $M \in \mathcal{S}_n(\mathfrak{p}_i)$ for each $1 \le i \le t$, then $M \in \mathcal{S}_n(\mathfrak{a})$.*

Proof. Clearly, $\mathrm{Supp}\, M \subseteq V(\mathfrak{p}_i) \subseteq V(\mathfrak{a})$. Now assume that $\mathrm{Ext}^j_A(A/\mathfrak{a}, M)$ is finitely generated for all $1 \le j \le n$. It follows from [3, Proposition 1] that $\mathrm{Ext}^j_A(A/\mathfrak{p}_i, M)$ is finitely generated for all $1 \le j \le n$ and all $1 \le i \le t$; hence, by assumption, $\mathrm{Ext}^j_A(A/\mathfrak{p}_i, M)$ is finitely generated for all $j \ge 0$ and all $1 \le i \le t$. Then it follows from [3, Corollary 1] that $\mathrm{Ext}^j_A(A/\mathfrak{a}, M)$ is finitely generated for all $j \ge 0$. ■

Proposition 2.8. *Let $x \in \mathfrak{a}$ and M be an A-module with* $\mathrm{Supp}\, M \subseteq V(\mathfrak{a})$ *such that* $(0 :_M x), M/xM \in \mathcal{S}_1(\mathfrak{a})$*. Then $M \in \mathcal{S}_2(\mathfrak{a})$.*

Proof. Assume that $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ is finitely generated for all $i \le 2$. Applying the functor $\mathrm{Hom}_A(A/\mathfrak{a}, -)$ to the exact sequences of modules $0 \to (0 :_M x) \to M \to xM \to 0$ and $0 \to xM \to M \to M/xM \to 0$ it is straightforward to see that $\mathrm{Ext}^i_A(A/\mathfrak{a}, (0 :_M x))$ is finitely generated for $i = 0, 1$; since $(0 :_M x) \in \mathcal{S}_1(\mathfrak{a})$, we conclude that $(0 :_M x)$ is $\mathfrak{a}$-cofinite. This implies that $\mathrm{Ext}^i_A(A/\mathfrak{a}, M/xM)$ is finitely generated for $i = 0, 1$, and since $M/xM \in \mathcal{S}_1(\mathfrak{a})$, we conclude that M/xM is $\mathfrak{a}$-cofinite. It now follows from [9, Corollary 3.4] that M is $\mathfrak{a}$-cofinite. ■

For the local cohomology modules of a finitely generated A-module of dimension 3 we have the following result.

Proposition 2.9. *If M is a finitely generated A-module of dimension 3 such that $H^2_{\mathfrak{a}}(M) \in \mathcal{S}_0(\mathfrak{a})$, then $H^1_{\mathfrak{a}}(M) \in \mathcal{S}_2(\mathfrak{a})$.*

Proof. Assume that $\mathrm{Ext}^i_A(A/\mathfrak{a}, H^1_{\mathfrak{a}}(M))$ is finitely generated for $i \le 2$. We may assume that $\Gamma_{\mathfrak{a}}(M) = 0$ and so $\mathfrak{a}$ contains a non-zerodivisor x of M. Applying the functor $\Gamma_{\mathfrak{a}}(-)$ to the exact sequence $0 \to M \xrightarrow{x.} M \to M/xM \to 0$ we get the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M/xM) \longrightarrow H^1_{\mathfrak{a}}(M) \xrightarrow{x.} H^1_{\mathfrak{a}}(M) \longrightarrow H^1_{\mathfrak{a}}(M/xM) \longrightarrow H^2_{\mathfrak{a}}(M) \xrightarrow{x.} \cdots.$$

Since $\mathrm{Ext}^i_A(A/\mathfrak{a}, H^1_{\mathfrak{a}}(M))$ is finitely generated for $i \leq 2$, it is straightforward to see that $\mathrm{Hom}_A(A/\mathfrak{a}, H^2_{\mathfrak{a}}(M))$ is finitely generated, so the assumption implies that $H^2_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. On the other hand, $\dim M/xM = 2$, so it follows by [9, Proposition 5.1] that $H^2_{\mathfrak{a}}(M/xM)$ and $H^3_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite; hence, by virtue of [7, Proposition 2.5], the module $H^1_{\mathfrak{a}}(M/xM)$ is $\mathfrak{a}$-cofinite. Now, it is straightforward to show that $H^1_{\mathfrak{a}}(M)/xH^1_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite, thus it follows from [9, Corollary 3.4] that $H^1_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. ∎

The following result is the first main theorem about cofiniteness of local cohomology modules lying in $\mathcal{S}_1(\mathfrak{a})$.

THEOREM 2.10. *If* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *is finitely generated for all* $i \leq m+1$ *and* $H^i_{\mathfrak{a}}(M) \in \mathcal{S}_1(\mathfrak{a})$ *for all* $i \leq m$, *then* $H^i_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cofinite for all* $i \leq m$.

PROOF. We proceed by induction on m. If $m = 0$, then the isomorphism

$$\mathrm{Hom}_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \cong \mathrm{Hom}_A(A/\mathfrak{a}, M)$$

and the exact sequence $0 \to \mathrm{Ext}^1_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}^1_A(A/\mathfrak{a}, M)$ imply that the module $\mathrm{Ext}^i_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is finite for $i \leq 1$; moreover, since $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}_1(\mathfrak{a})$, we deduce that $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. Now, suppose $m > 0$ and that the result has been proved for all values $< m$. Considering $\bar{M} = M/\Gamma_{\mathfrak{a}}(M)$, there is an exact sequence of modules $0 \to \bar{M} \to E \to N \to 0$ in which E is injective and $\Gamma_{\mathfrak{a}}(E) = 0$. The case $m = 0$ implies that $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite, so that $\mathrm{Ext}^i_A(A/\mathfrak{a}, \bar{M})$ is finitely generated for all $i \leq m+1$. Thus, the isomorphism $\mathrm{Ext}^i_A(A/\mathfrak{a}, N) \cong \mathrm{Ext}^{i+1}_A(A/\mathfrak{a}, \bar{M})$ for all $i \geq 0$ implies that $\mathrm{Ext}^i_A(A/\mathfrak{a}, N)$ is finitely generated for all $i \leq m$; furthermore, $H^i_{\mathfrak{a}}(N) \cong H^{i+1}_{\mathfrak{a}}(M) \in \mathcal{S}_1(\mathfrak{a})$ for all $i \leq m-1$. Now, the induction hypothesis implies that $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all $i \leq m-1$, and the isomorphism $H^i_{\mathfrak{a}}(N) \cong H^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$ forces $H^i_{\mathfrak{a}}(M)$ to be $\mathfrak{a}$-cofinite for all $i \leq m$. ∎

We now extend the above theorem for the class $\mathcal{S}_n(\mathfrak{a})$, where $n \geq 2$.

THEOREM 2.11. *Assume that* m *is a non-negative integer such that* $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *is finitely generated for all* $i \leq m+n$ *and* $H^s_{\mathfrak{a}}(M) \in \mathcal{S}_n(\mathfrak{a})$ *for all* $s \leq m$. *If*

$$\mathrm{Ext}^i_A(A/\mathfrak{a}, H^{t+s-i}_{\mathfrak{a}}(M))$$

is finitely generated for all $1 \leq t \leq n-1$, $i \leq t-1$ *and* $s \leq m$, *then* $H^s_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cofinite for all* $s \leq m$.

Proof. We proceed by induction on m. Assume that $m = 0$ and consider the exact sequences

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow \bar{M} \longrightarrow E_0 \longrightarrow M_1 \longrightarrow 0, \tag{1'}$$

$$\vdots$$

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M_{i-1}) \longrightarrow M_{i-1} \longrightarrow \overline{M_{i-1}} \longrightarrow 0 \tag{i}$$

$$0 \longrightarrow \overline{M_{i-1}} \longrightarrow E_{i-1} \longrightarrow M_i \longrightarrow 0. \tag{i'}$$

In view of the exact sequence (1) and the fact that $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}_n(\mathfrak{a})$, it suffices to show that $\mathrm{Ext}^i_A(A/\mathfrak{a}, \bar{M})$ is finitely generated for all $i \leq n-1$. Fix $i \leq n-1$. The case $i = 0$ is clear. For the case $i = 1$, in view of (1′), the module $\mathrm{Ext}^1_A(A/\mathfrak{a}, \bar{M}) \cong \mathrm{Hom}_A(A/\mathfrak{a}, M_1) \cong \mathrm{Hom}_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M_1)) \cong \mathrm{Hom}_A(A/\mathfrak{a}, H^1_{\mathfrak{a}}(M))$ is finitely generated by assumption (consider $t = 1$ and $i = 0$). For $1 < i \leq n-1$, using (1′), we have an isomorphism $\mathrm{Ext}^i_A(A/\mathfrak{a}, \bar{M}) \cong \mathrm{Ext}^{i-1}_A(A/\mathfrak{a}, M_1)$. Now, using the exact sequence (2), it suffices to show that $\mathrm{Ext}^{i-1}_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M_1))$ and $\mathrm{Ext}^{i-1}_A(A/\mathfrak{a}, \overline{M_1})$ are finitely generated. The module $\mathrm{Ext}^{i-1}_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M_1)) \cong \mathrm{Ext}^{i-1}_A(A/\mathfrak{a}, H^1_{\mathfrak{a}}(M))$ is finitely generated by assumption (replace t by $i-1$ and i by $i-2$). Continuing this way, we have to prove that $\mathrm{Ext}^1_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M_{i-1}))$ and $\mathrm{Ext}^1_A(A/\mathfrak{a}, \overline{M_{i-1}})$ are finitely generated. Using the above exact sequence, we obtain

$$\mathrm{Ext}^1_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M_{i-1})) \cong \mathrm{Ext}^1_A(A/\mathfrak{a}, H^{i-1}_{\mathfrak{a}}(M)),$$

and so it is finitely generated by assumption (replace t by i and i by 1). On the other hand, in view of (i′), $\mathrm{Ext}^1_A(A/\mathfrak{a}, \overline{M_{i-1}}) \cong \mathrm{Hom}_A(A/\mathfrak{a}, M_i) \cong \mathrm{Hom}_A(A/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M_i)) \cong \mathrm{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated by assumption (replace t by i and i by 0). Suppose $m > 0$ and that the result has been proved for all values $< m$. By the inductive hypothesis, $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite, so that $\mathrm{Ext}^i_A(A/\mathfrak{a}, \bar{M})$ is finitely generated for all $i \leq m+n$; so, in view of (1′), the module $\mathrm{Ext}^i_A(A/\mathfrak{a}, M_1) \cong \mathrm{Ext}^{i+1}_A(A/\mathfrak{a}, \bar{M})$ is finitely generated for all $i \leq m+n-1$. On the other hand, for all $1 \leq t \leq n-1$, $i \leq t-1$ and $s \leq m$, the modules

$$\mathrm{Ext}^i_A(A/\mathfrak{a}, H^{t+s-1-i}_{\mathfrak{a}}(M_1)) \cong \mathrm{Ext}^i_A(A/\mathfrak{a}, H^{t+s-i}_{\mathfrak{a}}(M))$$

are finitely generated and hence, using the inductive hypothesis, $H^s_{\mathfrak{a}}(M) \cong H^{s-1}_{\mathfrak{a}}(M_1)$ is $\mathfrak{a}$-cofinite for all $s \leq m$. ■

The following corollary is a generalization of [11, Theorem 3.7] and [1, Theorem 3.7] without any conditions on A and $\mathfrak{a}$.

Corollary 2.12. *Let* $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ *be finitely generated for all* $i \leq m+2$ *and let* $H^i_{\mathfrak{a}}(M) \in \mathcal{S}_2(\mathfrak{a})$ *for all* $i \leq m$. *If* $\operatorname{Hom}_A(A/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ *is finitely generated for all* $i \leq m+1$, *then* $H^i_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-cofinite for all* $i \leq m$.

Proof. The proof follows by the previous theorem considering $n = 2$. ∎

When the local cohomology modules of a module are nonzero only in two consecutive numbers, we have the following result.

Proposition 2.13. *Let* t *be a non-negative integer such that* $H^i_{\mathfrak{a}}(M) = 0$ *for all* $i \neq t, t+1$ *and let* $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ *be finitely generated for all* $i \leq n+t+1$. *Then* $H^{t+1}_{\mathfrak{a}}(M) \in \mathcal{S}_n(\mathfrak{a})$ *if and only if* $H^t_{\mathfrak{a}}(M) \in \mathcal{S}_{n+2}(\mathfrak{a})$.

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_A(A/\mathfrak{a}, H^q_{\mathfrak{a}}(M)) \Longrightarrow \operatorname{Ext}^{p+q}_A(A/\mathfrak{a}, M).$$

For each p, consider the sequence

$$E_2^{p-2,t+1} \xrightarrow{d_2^{p-2,t+1}} E_2^{p,t} \xrightarrow{d_2^{p,t}} E_2^{p+2,t-1}.$$

By assumption, $E_2^{p+2,t-1} = 0$ therefore we have $E_3^{p,t} = \operatorname{Ker} d_2^{p,t}/\operatorname{Im} d_2^{p-2,t+1} = \operatorname{Coker} d_2^{p-2,t+1}$. Now consider the sequence

$$E_3^{p-3,t+2} \xrightarrow{d_3^{p-3,t+2}} E_3^{p,t} \xrightarrow{d_3^{p,t}} E_3^{p+3,t-2}.$$

Since $E_3^{p-3,t+2}$ and $E_3^{p+3,t-2}$ are subquotients of $E_2^{p-3,t+2}$ and $E_2^{p+3,t-2}$ respectively, the assumption implies that

$$E_3^{p-3,t+2} \overset{d_3^{p-3,t+2}}{=\!=\!=\!=} E_3^{p+3,t-2} = 0,$$

so that $E_4^{p,t} = E_3^{p,t}$. Continuing this manner, we deduce that $E_3^{p,t} = E_\infty^{p,t}$; hence there is an exact sequence of modules

$$(\dagger) \qquad E_2^{p-2,t+1} \longrightarrow E_2^{p,t} \longrightarrow E_\infty^{p,t} \longrightarrow 0.$$

Using a similar argument, we have the exact sequence of modules

$$(\ddagger) \qquad 0 \longrightarrow E_\infty^{p,t+1} \longrightarrow E_2^{p,t+1} \longrightarrow E_2^{p+2,t}.$$

As $E_\infty^{p,t}$ and $E_\infty^{p,t+1}$ are subquotients of $\operatorname{Ext}^{p+t}_A(A/\mathfrak{a}, M)$ and $\operatorname{Ext}^{p+t+1}_A(A/\mathfrak{a}, M)$ respectively, they are finitely generated for all $p \leq n$ by assumption. Assume that

$H_{\mathfrak{a}}^{t+1}(M) \in \mathcal{S}_n(\mathfrak{a})$ and that $\operatorname{Ext}_A^p(A/\mathfrak{a}, H_{\mathfrak{a}}^t(M))$ is finitely generated for all $p \leq n+2$. The exact sequence (‡) implies that $\operatorname{Ext}_A^p(A/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(M))$ is finitely generated for all $p \leq n$, whence $H_{\mathfrak{a}}^{t+1}(M)$ is $\mathfrak{a}$-cofinite. It now follows from (†) that $H_{\mathfrak{a}}^t(M)$ is $\mathfrak{a}$-cofinite, so that $H_{\mathfrak{a}}^t(M) \in \mathcal{S}_{n+2}(\mathfrak{a})$. The converse is obtained by a similar argument. ■

EXAMPLE 2.14. Let k be a field of characteristic 0, and let $R = K[X_{ij}]$ for $1 \leq i \leq 2$ and $1 \leq j \leq 3$. Let $\mathfrak{p}$ be the height two prime ideal generated by the 2×2 minors of the matrix (X_{ij}). As $\mathfrak{p}$ is generated by three elements and A is a domain, $H_{\mathfrak{p}}^i(A) = 0$ for all $i \neq 2, 3$. Since $\operatorname{Hom}_A(A/\mathfrak{p}, H_{\mathfrak{p}}^3(A))$ is not finitely generated, we have $H_{\mathfrak{p}}^3(A) \in \mathcal{S}_0(\mathfrak{p})$, hence the previous proposition implies that $H_{\mathfrak{p}}^2(A) \in \mathcal{S}_2(\mathfrak{p})$.

We show that the change of ring principle holds for $\mathcal{S}_n(\mathfrak{a})$.

PROPOSITION 2.15. *Let B be a finitely generated A-algebra and let M be a B-module. Then $M \in \mathcal{S}_n(\mathfrak{a})$ if and only if $M \in \mathcal{S}_n(\mathfrak{a}B)$.*

PROOF. It is clear that $\operatorname{Supp}_A M \subseteq V(\mathfrak{a})$ if and only if $\operatorname{Supp}_B M \subseteq V(\mathfrak{a}B)$. Assume that $\operatorname{Ext}_B^i(B/\mathfrak{a}B, M)$ is finitely generated for all $0 \leq i \leq n$. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}_B^p(\operatorname{Tor}_q^R(B, A/\mathfrak{a}), M) \Longrightarrow H^{p+q} = \operatorname{Ext}_A^{p+q}(A/\mathfrak{a}, M).$$

By assumption, $E_2^{p,0}$ is finitely generated for all $0 \leq p \leq n$; moreover, since $\operatorname{Supp}_B \operatorname{Tor}_q^A(B, A/\mathfrak{a}) \subseteq V(\mathfrak{a}B)$ for all $q \geq 0$, we deduce by [3, Proposition 1] that $E_2^{p,q}$ is finitely generated for all $0 \leq p \leq n$ and all $q \geq 0$. For any $r > 2$, the B-module $E_r^{p,q}$ is a subquotient of $E_{r-1}^{p,q}$, so an easy induction yields that $E_r^{p,q}$ is finitely generated for all $r \geq 2$, $0 \leq p \leq n$ and all $q \geq 0$, whence $E_\infty^{p,q}$ is finitely generated for all $0 \leq p \leq n$ and all $q \geq 0$. For any $0 \leq t \leq n$, there is a finite filtration

$$0 = \Phi^{t+1} H^t \subset \Phi^t H^t \subset \cdots \subset \Phi^1 H^t \subset \Phi^0 H^t \subset H^t$$

such that $\Phi^p H^t / \Phi^{p+1} H^t \cong E_\infty^{p,t-p}$, where $0 \leq p \leq t$. Since $E_\infty^{p,t-p}$ is finitely generated for all $0 \leq p \leq t$ and $0 \leq t \leq n$, we deduce that H^t is finitely generated for all $0 \leq t \leq n$, and since $M \in \mathcal{S}_n(\mathfrak{a})$, we deduce that M is $\mathfrak{a}$-cofinite. Consequently, using [3, Proposition 2], the module M is $\mathfrak{a}B$-cofinite. Now, assume $M \in \mathcal{S}_n(\mathfrak{a}B)$ and that $\operatorname{Ext}_A^i(A/\mathfrak{a}, M)$ is finitely generated for all $0 \leq i \leq n$. By induction on $i \leq n$, we show that $\operatorname{Ext}_B^i(B/\mathfrak{a}B, M)$ is a finitely generated B-module. For $i = 0$, we have $\operatorname{Hom}_B(B/\mathfrak{a}B, M) \cong \operatorname{Hom}_A(A/\mathfrak{a}, M)$ is finitely generated. Now, assume $i > 0$ and that the result has been proved for all values smaller than $i \leq n$. This means that $E_2^{p,0} = \operatorname{Ext}_B^p(B/\mathfrak{a}B, M)$ is finitely generated for all $0 \leq p < i$. Since $\operatorname{Supp}_B \operatorname{Tor}_q^A(B, A/\mathfrak{a}) \subseteq V(\mathfrak{a}B)$, we conclude that $E_2^{p,q}$ is finitely generated for all

$0 \leq p < i$ and all q. The exact sequence $E_2^{i-2,1} \to E_2^{i,0} \to E_3^{i,0} \to 0$ and the inductive hypothesis imply that $E_2^{i,0}$ is finitely generated if $E_3^{i,0}$ is finitely generated. Continuing this manner, we deduce that $E_2^{i,0}$ is finitely generated if $E_\infty^{i,0}$ is finitely generated. Now, the filtration

$$0 = \Phi^{i+1}H^i \subset \cdots \subset \Phi^1 H^i \subset \Phi^0 H^i \subset H^i$$

is such that $E_\infty^{i,0} \cong \Phi^i H^i/\Phi^{i+1}H^i = \Phi^i H^i$ is a submodule of $H^i = \operatorname{Ext}_A^i(A/\mathfrak{a}, M)$; hence it is finitely generated. Therefore, $\operatorname{Ext}_B^i(B/\mathfrak{a}B, M)$ is finitely generated for all $0 \leq i \leq n$; eventually, since $M \in \mathcal{S}_n(\mathfrak{a}B)$, we deduce that M is $\mathfrak{a}B$-cofinite, and it follows from [3, Proposition 2] that M is $\mathfrak{a}$-cofinite. ■

The following result is a generalization of [11, Theorem 2.2].

THEOREM 2.16. *Let A be a ring of dimension d and $M \in \mathcal{S}_n(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of dimension $\leq d-1$ (i.e.* $\dim A/\mathfrak{a} \leq d-1$*). Then $M \in \mathcal{S}_n(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of A.*

PROOF. Assume that $\mathfrak{a}$ is an arbitrary ideal of A such that $\operatorname{Supp} M \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}_A^i(A/\mathfrak{a}, M)$ is finitely generated for all $i \leq n$. We can choose a positive integer t such that $(0 :_A \mathfrak{a}^t) = \Gamma_\mathfrak{a}(A)$. Put $\bar{A} = A/\Gamma_\mathfrak{a}(A)$ and $\bar{M} = M/(0 :_M \mathfrak{a}^t)$, the latter being an $\bar{A}$-module. By [11, Lemma 2.1], the module $(0 :_M \mathfrak{a}^t)$ is finitely generated; hence for any ideal $\mathfrak{b}$ of A it is clear that $M \in \mathcal{S}_n(\mathfrak{b})$ if and only if $\bar{M} \in \mathcal{S}_n(\mathfrak{b})$. Taking $\bar{\mathfrak{a}}$ as the image of $\mathfrak{a}$ in $\bar{A}$, we have $\Gamma_{\bar{\mathfrak{a}}}(\bar{A}) = 0$. Thus, $\bar{\mathfrak{a}}$ contains an $\bar{A}$-regular element so that $\dim A/\mathfrak{a} + \Gamma_\mathfrak{a}(A) = \dim \bar{A}/\bar{\mathfrak{a}} \leq d-1$. Now the assumption implies that $M \in \mathcal{S}_n(\mathfrak{a} + \Gamma_\mathfrak{a}(A))$, and the previous arguments yield $\bar{M} \in \mathcal{S}_n(\mathfrak{a} + \Gamma_\mathfrak{a}(A))$. Using the rings homomorphism $A \to \bar{A}$, it follows from Proposition 2.15 that $\bar{M}$ lies in $\mathcal{S}_n(\mathfrak{a})$. In view of the exact sequence $0 \to (0 :_M \mathfrak{a}^t) \to M \to \bar{M} \to 0$, the assumption on M implies that $\operatorname{Ext}_A^i(A/\mathfrak{a}, \bar{M})$ is finitely generated for all $i \leq n$; hence $\bar{M}$ is $\mathfrak{a}$-cofinite. Now the previous exact sequence implies that M is $\mathfrak{a}$-cofinite. ■

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