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New hexachordal theorems in metric spaces with a probability measure

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Abstract – The hexachordal theorem is an intriguing combinatorial property of the sets in $\mathbb{Z}/12\mathbb{Z}$, discovered and popularized by the musicologist Milton Babbitt (1916–2011). It has been given several explanations and partial generalizations. Here we enhance how this phenomenon can be understood by giving both a geometrical and a probabilistic perspective.

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1. Introduction

We first state the original hexachordal theorem. This theorem finds its origin in an observation [4] made by the American composer and musicologist Milton Babbitt about the musical intervals appearing in a set of six different notes – called a hexachord – and those in the complementary set with respect to the twelve-tone scale.

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Theorem 1.1 (Babbitt's hexachordal theorem). *Let A be a subset of $\mathbb{Z}/12\mathbb{Z}$ of cardinal 6 and let $A^c = (\mathbb{Z}/12\mathbb{Z}) \setminus A$ denote its complementary set. Then, for every $k \in \mathbb{Z}/12\mathbb{Z}$, the sets*

$$\big\{(x,y) \in A \times A : y - x = k\big\} \quad \text{and} \quad \big\{(x,y) \in A^c \times A^c : y - x = k\big\}$$

have the same cardinal.

In the present paper we introduce a natural probabilistic setting in which we generalize this result to metric spaces. We dare to believe that, as the referee wrote, we have found the correct setting for a "clear and neat final result" that "permits us to derive most (if not all) the previous results". Hereafter, $(\mathcal{X}, d)$ is a separable metric space and μ a Borel σ-finite measure on it. We will refer to such triples $(\mathcal{X}, d, \mu)$ as *metric measure spaces* and *metric probability spaces* if μ is a probability measure. Metric measure spaces are a popular setting in geometric analysis, at least since Gromov's famous Chapter $3\frac{1}{2}$ [8]. For recent contributions see [13] and references therein.

We introduce the constant volume condition on $(\mathcal{X}, d, \mu)$, which provides a sufficient condition for the main result of this paper.

Definition 1.2 (Constant volume condition). A metric measure space $(\mathcal{X}, d, \mu)$ is said to satisfy the *constant volume condition* if there exists a function ρ on $[0, \infty)$ such that for any center $x \in \mathcal{X}$ and radius $r \in [0, \infty)$ the ball $\mathcal{B}(x, r) = \{y \in \mathcal{X} : d(x, y) \leq r\}$ has measure $\rho(r)$. This can also be written as

$$\text{(CVC)} \qquad \forall x, y \in \mathcal{X},\ \forall r \geq 0, \quad \mu(\mathcal{B}(x, r)) = \mu(\mathcal{B}(y, r)).$$

For future development we introduce $\rho_x : r \mapsto \mu(\mathcal{B}(x, r))$ the *volume function of center* x, and $\bar{\rho} := \mu(\mathcal{X})^{-1} \int \rho_x \,\mathrm{d}\mu(x)$ the *mean volume function*.

We can now state our hexachordal theorem for metric probability spaces.

Theorem 1.3 (Hexachordal theorem for metric probability spaces). *Let $(\mathcal{X}, d, \mu)$ be a metric probability space. Assume that it satisfies the constant volume condition. Then, for every Borel set A of μ-measure $1/2$, with notation $A^c = \mathcal{X} \setminus A$, one has*

$$\text{(Hex)} \qquad \mu^2\big\{(x,y) \in A^2 : d(x,y) \in E\big\} = \mu^2\big\{(x,y) \in (A^c)^2 : d(x,y) \in E\big\}$$

for every open subset $E \subset [0, \infty)$, where μ^2 is the product measure $\mu \times \mu$.

Let us show how this theorem specializes to Babbitt's theorem. On the cyclic group $\mathbb{Z}/12\mathbb{Z}$ we consider the distance defined by

$$d(x, y) = \min_{k \in \mathbb{Z}} |x - y + 12k|.$$

Since this formula corresponds to the minimum number of steps ± 1 in $\mathbb{Z}/12\mathbb{Z}$ necessary to move from x to y, the distance d is the classical graph distance, the edges being distributed here exactly between the consecutive numbers of $\mathbb{Z}/12\mathbb{Z}$. By choosing for μ the normalized counting measure on $\mathbb{Z}/12\mathbb{Z}$, i.e. $\mu(A) = \#A/12$ we obtain the following expression for (Hex) in Theorem 1.3:

$$\frac{1}{12^2}\#\big\{(x,y) \in A^2 : d(x,y) \in E\big\} = \frac{1}{12^2}\#\big\{(x,y) \in (A^c)^2 : d(x,y) \in E\big\}.$$

Let ψ_A be the function defined for $k \in \mathbb{N}$ by

$$\psi_A(k) = \#\big\{(x,y) \in A^2 : d(x,y) = k\big\}$$

and I_A the so-called *interval content* of A defined for $k \in \mathbb{Z}/12\mathbb{Z}$ by

$$I_A(k) = \#\big\{(x,y) \in A^2 : y - x = k\big\}.$$

These two functions count the number of oriented pairs at distance $k \in \mathbb{N}$ and of oriented intervals $k \in \mathbb{Z}/12\mathbb{Z}$, respectively. Consequently, (Hex) can be written as $\psi_A = \psi_{A^c}$. Next, for every A (and A^c), $I_A(k) = \psi_A(k)$ for $k = 0$ and $k = 6$ and, since $(x,y) \in A^2 \Leftrightarrow (y,x) \in A^2$, we also have $I_A(k) = I_A(12-k) = \psi_A(k)/2$ for $k = 1,\ldots,5$. Thus $I_A = I_{A^c}$ holds on the whole $\mathbb{Z}/12\mathbb{Z}$. In the latter we recognize Babbitt's hexachordal theorem.

Since Babbitt's original formulation [4], his hexachordal theorem has been discussed, re-proved and sometimes generalized, many times. Hereafter we distinguish between two types of hexachordal theorems: the metric ones similar to Theorem 1.3, and the general ones in the continuation of the interval content formulation $I_A = I_{A^c}$ by Babbitt. By "general" we mean that the distance d can be replaced by a non-real-valued function, such as the antisymmetric $f\colon (x,y) \mapsto x^{-1} \cdot y$ when $(\mathfrak{X}, \cdot)$ is a group or some symmetric function. Our main theorem (the already-stated Theorem 1.3) falls into the first category since it deals with *metric* probability spaces satisfying the constant volume condition (CVC). In Theorem 4.2 we prove that the metric hexachordal phenomenon is in fact equivalent to the (CVC) when the latter is properly modified. Finally, Theorem 4.5 is a *general* hexachordal theorem where we adapt our theorems for general functions.

Previous literature. Before we proceed with the proofs, let us give a few more comments on the mathematical content, also with respect to the existing literature (more historical comments and relations to music and other domains such as spectroscopy are given in a companion paper [3]). While scanning the literature, we noticed that one simple idea appears more or less clearly behind the proof of most instances of the hexachordal theorems. It is the idea of not only counting the intervals – or sometimes measuring the size of objects that generalize them – between A and A, on the one side, and A^c and A^c on the other side, but also the intervals between A and A^c. This principle is already well explained in Ralph Hartzler Fox's contribution [7] that is possibly the first complete written proof of Babbitt's hexachordal theorem – notice however that completely different, short, interesting proofs of Babbitt's case are possible [2,6]. While Fox's explanation is for abstract discrete sets, one stream of research has been to explore continuous spaces. This is the case [5] for the circle $\mathbb{S}^1$ – extending the discrete circle $\mathbb{Z}/12\mathbb{Z}$ – and [10,11] for the spheres $\mathbb{S}^3$, $\mathbb{S}^7$, among other locally compact groups. A still geometric but discrete result is the full characterization of simple graphs exhibiting the hexachordal property by Althuis and Göbel [1]. It seems to be the only metric theorem in the hexachordal literature. With our probabilistic approach we implement the principle described in [7] to the whole geometric setting, discrete or continuous (or even mixed). Our probabilistic presentation also adapts to the general hexachordal theorem as we show in Section 4, for which some additional examples appear in the companion paper [3].

2. Probabilistic interpretation and proof of Theorem 1.3

Our proof uses a probabilistic writing of (Hex). Let (X, Y) be a pair of $\mathfrak{X}$-valued independent random variables of law μ and $D = d(X, Y)$. Property (Hex) is written

$$\mathbb{P}(X \in A \text{ and } Y \in A \text{ and } D \in E) = \mathbb{P}(X \in A^c \text{ and } Y \in A^c \text{ and } D \in E). \tag{1}$$

Adding $\mathbb{P}(X \in A \text{ and } Y \in A^c \text{ and } D \in E)$ to both sides we see that (Hex) holds if (and only if) one has

$$\mathbb{P}(X \in A \text{ and } D \in E) = \mathbb{P}(Y \in A^c \text{ and } D \in E) \tag{2}$$

for every Borel set $E \subseteq \mathbb{R}$. Hence, for Theorem 1.3 it suffices to prove (2).

Proof of Theorem 1.3. Let S be a Borel set of $\mathfrak{X}$ and $r \geq 0$. We have

$$\begin{aligned}
\mathbb{P}(X \in S \text{ and } D \in [0,r]) &= \iint_{\mathfrak{X}\times\mathfrak{X}} \mathbb{1}(x \in S) \cdot \mathbb{1}(d(x,y) \leq r) \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&= \int_S \left(\int_{\mathfrak{X}} \mathbb{1}(d(x,y) \leq r)\,\mathrm{d}\mu(y) \right) \mathrm{d}\mu(x) \\
&= \int_S \mu(\mathcal{B}(x,r))\,\mathrm{d}\mu(x) \\
&= \mu(S) \cdot \rho(r). \qquad (3)
\end{aligned}$$

This proves that X and D are independent random variables, X has law μ (this is not new) and D has cumulative distribution function ρ (see Remark 2.3). Therefore, on the left-hand side of (2), $\mathbb{P}(X \in A \text{ and } D \in E) = \mathbb{P}(X \in A) \times \mathbb{P}(D \in E) = (1/2)\mathbb{P}(D \in E)$. In exactly the same way (or noticing that (X, D) and (Y, D) have the same joint law), we see that Y and D are independent and $\mathbb{P}(Y \in A^c \text{ and } D \in E) = (1/2)\mathbb{P}(D \in E)$. This proves (2) and hence completes the proof. ■

Remark 2.1. We can express (Hex) in a different way in terms of conditional laws. Dividing equation (1) by $\frac{1}{4} = \mathbb{P}((X, Y) \in A^2) = \mathbb{P}((X, Y) \in (A^c)^2)$ we obtain

$$\mathbb{P}(D \in \cdot \mid X \in A \text{ and } Y \in A) = \mathbb{P}(D \in \cdot \mid X \in A^c \text{ and } Y \in A^c).$$

This may be read as follows: provided points X and Y are in A, their distance D is distributed in the same way as it would be if they were in the complementary set.

Remark 2.2. Similarly, $\mathbb{P}(D \in \cdot \mid X \in A) = \mathbb{P}(D \in \cdot \mid Y \in A^c)$ is a version of (2) formulated with conditional laws. The one-line computation

$$\mathbb{P}(D \leq r \mid X \in A) = \mu(A)^{-1} \int_A \underbrace{\mathbb{P}(d(x, Y) \leq r)}_{=\rho_x(r)=\rho(x)} \mathrm{d}\mu(x) = \rho(r),$$

with its counterpart $\mathbb{P}(D \leq r \mid Y \in A^c) = \rho(r)$ (for every $r \geq 0$), constitute an alternative, shorter and more probabilistic proof of Theorem 1.3.

Remark 2.3. Taking $S = \mathfrak{X}$ in (3), for a general $\mathfrak{X}$ without (CVC) we obtain $\mathbb{P}(D \leq r) = \bar{\rho}(r)$ so that $\bar{\rho}$ is the cumulative distribution function of D. The cumulative distribution functions of $d(x, Y)$ and $d(X, y)$ are simply ρ_x and ρ_y. Moreover, under the (CVC) all these functions equal ρ.

Remark 2.4. The random variables X, Y and D are pairwise independent but they are not independent. In particular, for very localized sets S and T, say balls of (small) radius ε, the law $\mathbb{P}(D \in \cdot \mid X \in S \text{ and } Y \in T)$ is a measure concentrated on an interval of length shorter than 4ε, hence different from $\mathbb{P}(D \in \cdot)$.

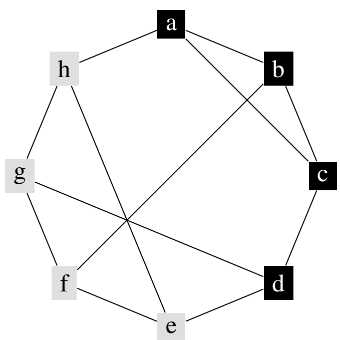
3. Metric probability spaces satisfying the constant volume condition

In this section we give examples of spaces where Theorem 1.3 applies. Since there are nontransitive simple graphs that satisfy (CVC), the hexachordal phenomenon surprisingly happens for them; see Section 3.1. Thus, a fascinating open question remains: Can there be radically different examples, such as Riemannian manifolds of unitary volume in particular? In Section 3.2 we treat of the case of the 2-dimensional manifolds for which the answer is no. More insight should be given in our paper in preparation.

3.1 – *Nontransitive graphs satisfying* (CVC)

The following metric measure spaces are particularly interesting since these are graphs that satisfy (CVC) – and hence (Hex) – but are not transitive. Briefly, in our context *transitive* would mean that for x, x' there exists an isometry f with $f_{\#}\mu = \mu$ and $f(x) = x'$. Example 3.2 is with 7 vertices the smallest possible nontransitive simple graph that satisfies (CVC). During the writing of the present paper we realized that a collection of similar graphs (notably three graphs with 12 vertices) have already been exhibited by Althuis and Göbel [1].

Example 3.1. Consider the finite 3-regular graph depicted on the left-hand side of Figure 1. One can easily confirm that it satisfies the constant volume condition: the balls of radius 0 have cardinal 1, the balls of radius 1 have cardinal 4 and all the larger balls are the whole space, with cardinal 8. However, it is clear that a and h are points of different types: the neighbors of h are disconnected whereas the neighbors b and c of a satisfy $b \sim c$. Consequently, the group of isomorphisms does not act transitively.



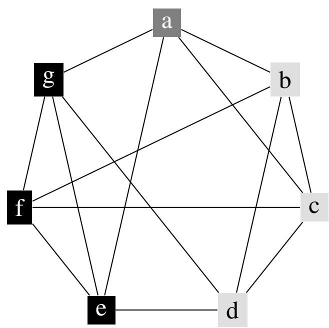


Figure 1. Left: Two points randomly picked in the dark region of the graph have distance distributed equally to that between points picked in the bright region. Conditional upon one or the other region, the random distance D takes values 0, 1 and 2 with probability 4/16, 8/16 and 4/16 respectively – Right: Two points randomly picked in the dark region of the graph have distance distributed equally to that between points picked in the bright region. Vertex a is half bright and half dark.

Example 3.2. The graph on the right-hand side of Figure 1 also satisfies the constant volume condition ($\rho(0) = 1$, $\rho(1) = 5$, $\rho(2) = 7$). With cardinal 7 it has the minimal cardinal for a graph satisfying (CVC) without transitive action of a group of isomorphisms. However, since 7 is an odd number, the hexachordal property is – contrary to Example 3.1 – a trivial statement: subsets A and A^c of cardinal $7/2$ do not exist. Hence Theorem 1.3 is a correct but empty statement. Theorem 4.2 in the next section will give a new turn to this poor conclusion. See the figure caption for some preliminary intuition.

3.2 – *Riemannian surfaces satisfying* (CVC)

In the following we consider connected, complete and separable Riemannian surfaces with their geodesic distance and Riemannian volume.

Proposition 3.3. *Let $(\mathfrak{X}, d, \mu)$ be a connected, complete and separable Riemannian surfaces with its geodesic distance and Riemannian volume such that $\mu(\mathfrak{X}) = 1$. Then it satisfies* (CVC) *if and only if it isomorphic to one of the following metric probability spaces:*

- *a flat torus $\mathbb{R}^2/(\mathbb{Z}u + \mathbb{Z}v)$ with $|\det(u, v)| = 1$,*
- *a Klein bottle (quotient of $\mathbb{R}^2$ through the group generated by a translation and a glide reflection) of volume 1,*
- *the sphere of dimension 2 and radius $1/\sqrt{4\pi}$,*
- *the projective two plane $\mathbb{RP}^2$ obtained from the sphere of radius $1/\sqrt{2\pi}$ when the opposite points are identified.*

Proof. Let $(\mathfrak{X}, d, \mu)$ be, as in the statement, a Riemannian surface that satisfies the constant volume condition. At any point $x \in \mathfrak{X}$ one has

$$\mu(\mathcal{B}(x, r)) =_{r \to 0^+} \pi r^2 (1 - \kappa(x) r/24) + o(r^3),$$

where $\kappa(x)$ is the curvature at x. It follows that $\kappa(x) = \lim_{r\to 0} 24(\pi r^2 - \rho(r))/r$, where $\rho(r) = \mu(\mathcal{B}(x, r))$ is independent of x. Therefore $\mathfrak{X}$ has constant curvature. Hence, up to multiplying d by $\sqrt{|\kappa|}$ (if $\kappa \neq 0$), the universal cover of $\mathfrak{X}$ is one of the three simply connected "space forms": Euclidean space (of curvature 0), the hyperbolic plane (curvature -1) and the sphere (curvature 1). For zero and negative curvature we find tori, Klein bottles, spheres and the projective plane. Moreover, the right scaling is enforced by $\mu(\mathfrak{X}) = 1$. Conversely, since the isometry group acts transitively on these spaces and the isometries preserve the Riemannian volume, we see that (CVC) is satisfied.

For negative curvature let us prove that (CVC) is not satisfied. It is well known that the small balls have the same volume as the balls of radius r of its universal cover (the hyperbolic plane up to a metric scaling) that we denote by $\tilde{\rho}(r)$. However, in the compact case, if x is on the systole (the shortest closed geodesic curve of length ℓ) and x' is not, there will be $\varepsilon > 0$ such that $\mu(\mathcal{B}(x, \varepsilon + \ell/2)) < \tilde{\rho}(\varepsilon + \ell/2) = \mu(\mathcal{B}(x', \varepsilon + \ell/2))$. The strict inequality is due to the cut-locus phenomenon on the systole: balls of center x and radius $> \ell/2$ overlap. In the noncompact case there is not necessarily a systole but another argument is possible. For some $x_0 \in \mathcal{X}$ let r_0 be such that $\mu(\mathcal{B}(x_0, r_0)) = \tilde{\rho}(r_0)$. Then, since $\mathcal{X}$ is not bounded, there exist infinitely many disjoint balls of radius r_0 and centers $(x_n)_{n\in\mathbb{N}}$. Since $\sum_n \mu(\mathcal{B}(x_n, r_0)) \leq 1$ we obtain a contradiction with the (CVC). ■

4. Full characterization of the spaces satisfying the hexachordal property

In this last section we show that (CVC) is not far from being a necessary and sufficient condition for the hexagonal property (Hex). To obtain this equivalence we (i) observe that sets of measure zero have no incidence in the hexachordal property and introduce for this (CVC′), (ii) carefully avoid the logical trap explained in Example 3.2 by introducing (Hex′). This being done we obtain Theorem 4.2. In Theorem 4.5 we give a second generalization that connects our work with previous group-theoretic [10, 12] or abstract [7] interpretations of the hexachordal theorem.

4.1 – *Full characterization for metric probability spaces*

For our full characterizations of Theorems 4.2 and 4.5 we introduce the concept of *balanced decomposition*. It is an appropriate answer to the problem described in Example 3.2. Similar concepts are to be found in the literature in the *weights* of [5] and the bounded functions of [10].

Definition 4.1. Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a probability space. We call balanced decomposition of μ any pair (μ_0, μ_1) of probability measures such that $2\mu = \mu_0 + \mu_1$. Note that μ_0 and μ_1 can be identified with functions of density smaller than or equal to 2.

We can now state our full characterization of spaces that satisfy (Hex′), i.e. (Hex) generalized as suggested in Example 3.2.

Theorem 4.2 (Characterization for metric probability spaces). *Let $(\mathcal{X}, d, \mu)$ be a metric probability space. The following properties are equivalent:*

(CVC′) *There exists a set $\mathcal{X}' \subseteq \mathcal{X}$ of full measure for μ such that the constant volume condition is satisfied on $(\mathcal{X}', d, \mu)$.*

(Ind) *For any independent random variables X and Y of law μ and $D = d(X,Y)$, the random variables X, Y and D are pairwise independent.*

(Hex′) *For every balanced decomposition (μ_0, μ_1) of μ and two random triples $(X_i, Y_i, D_i)_{i=0,1}$, where for every i, (X_i, Y_i) is a pair of independent random variables of law μ_i and $D_i = d(X_i, Y_i)$, we have the equality on distributions*

$$\mathbb{P}(D_0 \in \cdot) = \mathbb{P}(D_1 \in \cdot).$$

Remark 4.3. We recover Theorem 1.3 as follows: The constant volume condition implies (CVC′) (take $\mathcal{X}' = \mathcal{X}$ for example). Hence (Hex′) is satisfied for any balanced decomposition, in particular for (μ_A, μ_{A^c}), where A has measure $1/2$ and μ_A is defined by $\mu_A = \mu(A)^{-1}\mu(A \cap \cdot)$. This directly corresponds to (Hex) in Theorem 1.3, up to a factor 4.

Remark 4.4. If X and Y are independent of law μ, since d is symmetric we have equality of laws $(X, D) = (X, d(X,Y)) \sim (Y, d(Y,X)) = (Y, D)$. Therefore, to satisfy (Ind) it suffices that X and D are independent. The symmetry condition is also sufficient in the setting of the upcoming Theorem 4.5. If X and Y have the same law and both (X,Y), (X,F) are independent pairs (where $F = f(X,Y)$ with f symmetric), the last pair (Y,F) is independent. As can be easily checked, the same happens when f is antisymmetric, in the sense there exists an involution i with $f(y,x) = i(f(x,y))$.

Proof of Theorem 4.2. The beginning of the proof of Theorem 1.3 is the implication (CVC) $\Rightarrow$ (Ind). The reader can check that it also readily constitutes a proof of (CVC′) $\Rightarrow$ (Ind) too. We use that $x \mapsto \mu(\mathcal{B}(x,r))$ is equal to $\bar{\rho}(r)$ in all points x apart from a set of empty measure. Let us now prove (Ind) $\Rightarrow$ (CVC′). For every $r \geq 0$ we set $S_r^- = \{x \in \mathcal{X} \mid \rho_x(r) < \bar{\rho}(r)\}$ and $S_r^+ = \{x \in \mathcal{X} \mid \rho_x(r) > \bar{\rho}(r)\}$. Recall from Remark 2.3 that $\bar{\rho}$ is the cumulative distribution function of D and ρ_x that of $d(x,Y)$. Suppose by contradiction that $\mu(S_r^-) > 0$. Thus

$$\begin{aligned}
\mathbb{P}(X \in S_r^- \text{ and } D \in [0,r]) &= \iint \mathbb{1}(x \in S_r^-) \cdot \mathbb{1}(d(x,y) \leq r)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&= \int_{S_r^-} \left(\int \mathbb{1}(d(x,y) \leq r)\,\mathrm{d}\mu(y) \right) \mathrm{d}\mu(x) \\
&= \int_{S_r^-} \mu(\mathcal{B}(x,r))\,\mathrm{d}\mu(x) < \mu(S_r^-) \cdot \bar{\rho}(r),
\end{aligned}$$

which shows that X and D are not independent, a contradiction. Therefore $\mu(S_r^-) = 0$ and similarly $\mu(S_r^+) = 0$. We deduce that $\bigcup_{r \geq 0,\, r \in \mathbb{Q}} (S_r^- \cup S_r^+)$ has μ-measure zero.

If we denote the complementary set by $\mathfrak{X}'$, we obtain $\bar{\rho}(r) = \rho_x(r)$ for every $x \in \mathfrak{X}'$ and $r \in \mathbb{Q}$. This extends to every $r \in \mathbb{R}_+$ because cumulative distribution functions are right continuous. Hence (CVC′) is satisfied.

We have proved (CVC′) ⇔ (Ind) and will be ready after we prove (Ind) ⇔ (Hex′). We postpone this proof to Theorem 4.5, because considering that d is symmetric and measurable on $\mathfrak{X} \times \mathfrak{X}$, this theorem states a result that includes (Ind) ⇔ (Hex′). Its proof is also independent from the rest of Theorem 4.2. ■

4.2 – *Full characterization for general spaces and groups*

Here we replace d by a general function f that neither needs to be real valued nor symmetric. Typically, f is an "interval" antisymmetric function defined by $f(x, y) = x^{-1} \cdot y$ as in Corollary 4.6. This is the original music-theoretical point of view of Babbitt and Lewin [9].

Theorem 4.5 (Characterization for abstract probability spaces). *Let $(\mathfrak{X}, \mathcal{F}, \mu)$ be a probability space and f a measurable symmetric function into a measured space $(\mathcal{M}, \mathfrak{M})$. The following properties are equivalent:*

(Ind) *For any independent random variables X and Y of law μ and $F = f(X, Y)$, the random variables X, Y and F are pairwise independent*

(Hex′) *For every balanced decomposition (μ_0, μ_1), considering the triples (X_0, Y_0, F_0) and (X_1, Y_1, F_1), where for $i = 0, 1$ the pair (X_i, Y_i) is made of independent random variables of law μ_i and $F_i = f(X_i, Y_i)$, we have equality of both distributions, $\mathbb{P}(F_0 \in \cdot) = \mathbb{P}(F_1 \in \cdot)$ as measures on $\mathcal{M}$.*

(Hex″) *For any balanced decompositions (μ_0, μ_1) and (ν_0, ν_1), where for $i = 0, 1$, X_i has law μ_i, Y_i has law ν_i and $F_i = f(X_i, Y_i)$, we have equality of both distributions $\mathbb{P}(F_0 \in \cdot) = \mathbb{P}(F_1 \in \cdot)$.*

Moreover, if f is no longer supposed to be symmetric, (Ind) ⇔ (Hex″) *still holds, as does* (Hex″) ⇒ (Hex′).

Proof. To complete the proof of Theorem 4.2 we first establish in parts (1) and (2) of the present proof the two implications of (Ind) ⇔ (Hex′) in the case where f is symmetric. For the remainder, notice that (Hex″) ⇒ (Hex′) is obvious since (Hex″) corresponds to a generalization of (Hex′), where the relation $\mu_i = \nu_i$ is relaxed. In part (3) we will finish with the equivalence (Ind) ⇔ (Hex″) by briefly adapting the scheme drawn up in (1) and (2).

(1) (Ind) $\Leftrightarrow$ (Hex′). Let us fix some measurable $E \subseteq \mathcal{M}$ and (μ_0, μ_1) a balanced decomposition of μ. We first prove

$$\mathbb{P}(f(x, Y) \in E) = \mathbb{P}(F \in E) \tag{4}$$

for μ-a.e. $x \in \mathfrak{X}$. This follows from the fact that these two functions have the same integral on the measurable sets S in $\mathfrak{X}$. We have indeed

$$\begin{cases} \displaystyle\int_S \mathbb{P}(f(x, Y) \in E)\,\mathrm{d}\mu(x) = \mathbb{P}(X \in S,\ \underbrace{f(X, Y)}_{F} \in E), \\ \displaystyle\int_S \mathbb{P}(F \in E)\,\mathrm{d}\mu(x) = \mathbb{P}(X \in S) \cdot \mathbb{P}(F \in E). \end{cases}$$

Equality follows from (Ind). Integrating (4) with respect to μ_0 (which is absolutely continuous with respect to μ) we obtain $B_E(\mu_0, \mu) = B_E(\mu, \mu)$, where B_E is the bilinear function defined by $B_E\colon (\alpha, \beta) \mapsto \iint \mathbb{1}(f(x, y) \in E)\,\mathrm{d}\alpha(x)\,\mathrm{d}\beta(y)$. Note now that $f(x, Y) = f(Y, x)$ and that these random variables also have the same law as $f(X, x)$. Therefore, $\mathbb{P}(f(X, y) \in E) = \mathbb{P}(F \in E)$ for μ-a.e. $y \in \mathfrak{X}$. Similarly to before, we deduce $B_E(\mu, \mu) = B_E(\mu, \mu_1)$. Finally, subtracting $B_E(\mu_0, \mu_1)$ on each extreme side of $B_E(\mu_0, 2\mu) = 2B_E(\mu, \mu) = B_E(2\mu, \mu_1)$ we get

$$B_E(\mu_0, \mu_0) = B_E(\mu_1, \mu_1) \quad \text{for every measurable } E \subseteq \mathcal{M}. \tag{5}$$

Translated with random variables it is exactly (Hex′).

(2) (Hex′) $\Rightarrow$ (Ind). For this implication, it is sufficient to prove

$$\mathbb{P}(X \in S \text{ and } F \in E) = \mathbb{P}(X \in S) \cdot \mathbb{P}(F \in E)$$

for every measurable $E \subseteq \mathcal{M}$ and $S \subseteq \mathfrak{X}$ with $\mu(S) \geq 1/2$. For sets S of probability less than $1/2$, the independence relation is obtained through the complementary set $\mathfrak{X} \setminus S$. We fix S and E. Let μ_0 be $\mu(S)^{-1}\mu(\cdot \cap S)$ such that $(\mu_0, 2\mu - \mu_0)$ is a balanced decomposition of μ. Starting back from (5), adding $B_E(\mu_0, \mu_1)$ we obtain back $B_E(\mu_0, \mu) = B_E(\mu, \mu_1) = B_E(\mu_1, \mu) = B_E(\mu, \mu)$, where we use the symmetry of f in the second equality and the bilinearity in the third one. In probabilistic terms we have obtained

$$\mu(S)^{-1}\mathbb{P}(X \in S \text{ and } F \in E) = \mathbb{P}(F \in E),$$

which is exactly the equation wanted, since $\mu(S) = \mathbb{P}(X \in S)$.

(3). We follow part (1) and obtain that $x \mapsto \mathbb{P}(f(x,Y) \in E)$ and $y \mapsto \mathbb{P}(f(X,y) \in E)$ are almost surely constant of value $\mathbb{P}(F \in E)$ on $(\mathfrak{X}, \mu)$. It follows that

$$B_E(\mu_0, \nu_0 + \nu_1) = 2B_E(\mu_0, \mu) = 2B_E(\mu, \nu_1) = B_E(\mu_0 + \mu_1, \nu_1)$$

for every balanced decomposition (μ_0, μ_1) and (ν_0, ν_1). Subtracting $B(\mu_0, \nu_1)$ we obtain $B_E(\mu_0, \nu_0) = B_E(\mu_1, \nu_1)$ which proves the first implication. For the second one, from $B_E(\mu_0, \nu_0) = B_E(\mu_1, \nu_1)$ we obtain back $B_E(\mu_0, \nu) = B_E(\mu, \nu_1)$ for every $\mu_0 \leq 2\mu$ and $\nu_1 \leq 2\mu$ (these inequalities correspond to the conditions that $(\mu_0, 2\mu - \mu_0)$ and $(2\mu - \nu_1, \nu_1)$ are balanced decompositions). Choosing $\mu_0 = \mu(S)^{-1}\mu(\cdot \cap S)$ and $\nu_1 = \mu$ we can reconnect with the proof in (2). ■

In the next corollary we stress that Theorem 4.5 applies to “intervals” $(x, y) \mapsto x^{-1} \cdot y$ on locally compact Hausdorff topological groups. We present this corollary in the slightly larger setting of separable topological groups with bi-invariant Haar measure. Note that such a bi-invariant Haar measure exists when there exists a left-invariant measure μ: if X and Y are independent of laws μ and μ', respectively where μ' is right invariant (as for instance $\mu_{-1}\colon E \mapsto \mu(E^{-1})$), one can check that $Y \cdot X\colon \Omega \to \mathfrak{X}$ is measurable, is both left and right invariant and has laws μ and μ'. Therefore, $\mu = \mu'$ so that there exists a unique Haar measure and it is bi-invariant.

Corollary 4.6 (Separable topological groups). *Let $(\mathfrak{X}, \cdot)$ be a separable topological group with a left- and right-invariant probability measure μ. Then, for every balanced decomposition (μ_0, μ_1) of μ and (X_i, Y_i) independent random variables of law μ_i, $i = 0, 1$, the law of $X_0 \cdot Y_0$ equals the law of $X_1 \cdot Y_1$. Moreover, the same equality holds for $(X_i)^{-1} \cdot Y_i$.*

Proof. Property (Ind) is clearly satisfied as a consequence of the left- and right-invariance of μ. Therefore, Theorem 4.5 applies and we have (Hex′) for the function $f(x, y) = x \cdot y$ (which does not have to be symmetric). Since μ is invariant for $x \mapsto x^{-1}$, the pairwise independence of $(X, Y, X^{-1} \cdot Y)$ follows from the pairwise independence of $(X', Y, X' \cdot Y)$ with $X' = X^{-1}$. Again, Theorem 4.5 applies and we obtain (Hex′). ■

As it appears in the literature [7] and in Section 3, the hexachordal phenomenon should not hastily be associated to regular structures like groups or transitive spaces. For instance, the nonassociative set of octonions of modulus 1 was mentioned in [10] (it is homeomorphic to $\mathbb{S}^7$). In [7, 12] the authors recognize that the sufficient property of the Cayley table of a group that permits the hexachordal phenomenon to show up is that it is a Latin square: every symbol occurs exactly once in each row and exactly once in each column. In particular, (Ind) is satisfied. We plan to provide further examples and counterexamples related to our theorems in the paper in preparation.

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