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# On rank one and Weyl–von Neumann theorem for multiplicative perturbations of unitary operators

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**Abstract.** For multiplicative perturbations of unitary operators, it is presented a version of Weyl–von Neumann theorem and a sufficient conditions for generic (in the intensity parameter) singular continuous spectrum under unitary rank one perturbations.

## 1. Introduction

We are interested in the spectral properties of multiplicative perturbations

$$U \mapsto UX \tag{1.1}$$

of unitary operators $U$, on a (complex and infinite-dimensional) Hilbert separable space $\mathscr{H}$, with also unitary perturbing $X$. This is a *right* perturbation, and $U \mapsto XU$ is a *left* one.

The main physical motivation comes from time $\tau$-periodically kicked quantum Hamiltonians ($A$ and $B$ are self-adjoint operators)

$$A + B \sum_{j \in \mathbb{Z}} \delta(t - \tau n)$$

whose Floquet operator, from just before a kick to just before the next one, is $e^{-i\tau A} e^{-iB}$; see, for instance, [3]. In (1.1), one immediately identifies $U = e^{-i\tau A}$ and $X = e^{-iB}$.

In a previous work [1], the present authors have shown that there is no nontrivial generalization of the multiplicative version of Birman–Krein theorem [2] on preservation of absolutely continuous spectrum under certain perturbations. The original version of Birman–Krein is for additive perturbations, but from this, the multiplicative version follows; that is, the absolutely continuous parts of the unitary operators $U$ and $UX$ (or $XU$) are unitarily equivalent if $X = \mathbf{1} + W$ with trace class $W$.

In this note, we present multiplicative versions of two important known results for additive self-adjoint perturbations. First, a version of Weyl–von Neumann theorem [6] and, second, a version of a result on the generic presence of singular continuous spectrum for rank one perturbations due to del Rio, Makarov, and Simon [7].

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Recall that, according to Weyl–von Neumann theorem, given a self-adjoint operator $A$ and $\varepsilon > 0$, there is a self-adjoint operator $S$ with Hilbert–Schmidt norm $\|S\|_{\mathrm{HS}} < \varepsilon$ such that $A + S$ has pure point spectrum. Our conclusion will be similar (see Theorem 2.2): given a unitary operator $U$, there exists another unitary operator $X = 1 + W$ with $\|W\|_{\mathrm{HS}} < \varepsilon$ so that the perturbation $UX$ (or $XU$) has pure point spectrum.

The other set of results culminate in the following (let $\sigma(A)$ denote the spectrum of the linear operator $A$). Given a singular (i.e., with no absolutely continuous spectrum) unitary operator $U$, with $\{e^{it} \mid a < t < b\} \subset \sigma(U)$, and a unitary rank one perturbation $X_\lambda = e^{i\lambda P_\phi}$, $0 \le \lambda < 2\pi$, with $P_\phi$ the projection onto the one-dimensional subspace generated by the cyclic vector $\phi$, then, for generic (i.e., dense $G_\delta$ set) of intensities $\lambda$s, the perturbed operator $UX_\lambda$ has purely singular continuous spectrum in $\{e^{it} \mid a < t < b\}$. This will be a consequence of Theorem 3.5.

Section 2 presents general remarks on multiplicative perturbations of unitary operators, then Theorem 2.2 and its proof. In Section 3, after a suitable preparation, one finds Theorem 3.5 and its proof.

## 2. Multiplicative perturbations

If we have a unitary operator $X$, it is convenient to write it in the form $X = e^{iY}$, with $Y$ a bounded self-adjoint operator. Then,

$$X = e^{iY} = \sum_{j=0}^{\infty} \frac{(iY)^j}{j!} = \mathbf{1} + \sum_{j=1}^{\infty} \frac{(iY)^j}{j!} = \mathbf{1} + W,$$

where $W = \sum_{j=1}^{\infty} \frac{(iY)^j}{j!}$. Thus, we can write

$$UX = U(\mathbf{1} + W) = U + UW.$$

**Remark 2.1.** If the operator $X = \mathbf{1} + W$ is unitary, one has

$$\mathbf{1} = (\mathbf{1} + W)(\mathbf{1} + W)^* = \mathbf{1} + W + W^* + W^*W,$$
$$\mathbf{1} = (\mathbf{1} + W)^*(\mathbf{1} + W) = \mathbf{1} + W^* + W + WW^*;$$

then $W^* + W + WW^* = W^* + W + W^*W = 0$, and it follows that

$$W^*W = WW^*,$$

so $W$ is a normal operator. But this condition is not sufficient for $X = \mathbf{1} + W$ to be unitary; for example, if $W = \pm\mathbf{1}$, then $X$ would not be unitary (it is necessary that $\sigma(W) \subset \{e^{it} - 1 \mid t \in \mathbb{R}\}$).

Our main result in this section is the following theorem.

**Theorem 2.2.** *Let $U$ be a unitary operator in $\mathcal{H}$. Given $\varepsilon > 0$, there exists a unitary operator $X = \mathbf{1} + W$, with $\|W\|_{\mathrm{HS}} < \varepsilon$, such that the perturbed operator*

$$U \mapsto UX$$

*has pure point spectrum. It also holds true for left perturbations $U \mapsto XU$. (Left and right perturbations are in general different.)*

First, we prove a Weyl–von Neumann version for additive perturbations of unitary operators.

**Theorem 2.3.** *Given a unitary operator $U$ and $\varepsilon > 0$, there exists a unitary operator $V$ on $\mathcal{H}$ with pure point spectrum such that*

$$\|U - V\|_{\mathrm{HS}} < \varepsilon.$$

*Proof.* Write the unitary operator $U = e^{iT}$, with $T$ self-adjoint and bounded; by the usual Weyl–von Neumann result for self-adjoint operators, there exists a bounded self-adjoint operator $B$ with $\|B\|_{\mathrm{HS}} < \varepsilon$ and $T + B$ is pure point. It follows that $V = e^{i(T+B)}$ is unitary and pure point.

The next ingredient is a version of the Duhamel formula [6]

$$V - U = e^{i(T+B)} - e^{iT} = -i \int_0^1 e^{iT(1-u)} B e^{iu(T+B)}\, \mathrm{d}u.$$

By using the inequality

$$\|TS\|_{\mathrm{HS}} \le \|T\| \|S\|_{\mathrm{HS}}, \tag{2.1}$$

it follows that

$$\begin{aligned} \|V - U\|_{\mathrm{HS}} &\le \int_0^1 \|e^{iT(1-u)} B e^{iu(T+B)}\|_{\mathrm{HS}}\, \mathrm{d}u \\ &\le \int_0^1 \|e^{iT(1-u)}\| \, \|B\|_{\mathrm{HS}} \, \|e^{iu(T+B)}\|\, \mathrm{d}u \\ &\le \|B\|_{\mathrm{HS}} < \varepsilon. \end{aligned}$$

This completes the proof since $V$ is a pure point operator. ■

*Proof of Theorem* 2.2. By Theorem 2.3, given $0 < \delta < 1$, there exists a unitary and pure point operator $V$ such that $Q = U - V$ satisfies $\|Q\|_{\mathrm{HS}} < \delta$. Thus,

$$U = V + Q = V(\mathbf{1} + V^{-1}Q);$$

by inequality (2.1), $\|V^{-1}Q\|_{\mathrm{HS}} < \delta < 1$, and it follows that $(\mathbf{1} + V^{-1}Q)$ is invertible (in norm).

Write $X = (\mathbf{1} + V^{-1}Q)^{-1}$; hence,

$$UX = V,$$

concluding that $X$ is unitary (since $U$ and $V$ are). By writing the operator $X$ as the series

$$X = \mathbf{1} + \sum_{j=1}^{\infty} (-V^{-1}Q)^j,$$

one finds that

$$\|X - \mathbf{1}\|_{\mathrm{HS}} \le \sum_{j=1}^{\infty} \|V^{-1}Q\|_{\mathrm{HS}}^j \le \sum_{j=1}^{\infty} \delta^j = \frac{\delta}{1-\delta}.$$

To complete the proof, it is enough to pick $\delta$ such that $\frac{\delta}{1-\delta} < \varepsilon$ and identify

$$W = \sum_{j=1}^{\infty} (-V^{-1}Q)^j.$$

For a left perturbation

$$U \mapsto XU,$$

it is enough to consider

$$U = V + Q = (\mathbf{1} + QV^{-1})V,$$

with $X = (\mathbf{1} + QV^{-1})^{-1}$, and identify $W = \sum_{j=1}^{\infty} (-QV^{-1})^j$. ■

## 3. Unitary rank one perturbations

Let $\phi$ be a normalized vector in $\mathscr{H}$ that is cyclic for the unitary operator $U$, that is, the closure

$$\overline{\mathrm{Lin}\{U^j\phi \mid j \in \mathbb{Z}\}} = \mathscr{H},$$

$U^0 = \mathbf{1}$. Let $P_\phi(\cdot) = \langle \phi, \cdot \rangle \phi$ (which is self-adjoint and idempotent) denote the projection onto the subspace generated by $\phi$, and for real $\lambda$ consider

$$X_\lambda := e^{i\lambda P_\phi} = \mathbf{1} + W$$

with

$$W\xi = (e^{i\lambda} - 1)\langle \phi, \xi \rangle \phi = (e^{i\lambda} - 1) P_\phi(\xi).$$

In fact,

$$e^{i\lambda P_\phi} = \sum_{j=0}^{\infty} \frac{(i\lambda P_\phi)^j}{j!} = \mathbf{1} + \sum_{j=1}^{\infty} \frac{(i\lambda P_\phi)^j}{j!} \overset{P_\phi = P_\phi^2}{=} \mathbf{1} + (e^{i\lambda} - 1) P_\phi.$$

Note that there is a periodicity in the intensity parameter $\lambda$, and it suffices to consider $0 \le \lambda < 2\pi$.

Now, we have the multiplicative rank one perturbation

$$U_\lambda = UX_\lambda = U(\mathbf{1} + (e^{i\lambda} - 1)P_\phi). \tag{3.1}$$

To simplify statements, denote by $\mu^\lambda$ the spectral measure of the pair $(U_\lambda, \phi)$. Note that since $\phi$ is cyclic for $U$, it is also cyclic for $U_\lambda$ for all $\lambda \in \mathbb{R}$.

We are interested in relating the perturbation (3.1) to the Cauchy transform $F(z)$ of a Borel measure $\mu$ on $[0, 2\pi)$, defined for complex numbers $|z| \neq 1$ as

$$F(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \mathrm{d}\mu(t).$$

In case of $\mu^\lambda$, we denote $F_\lambda(z) = \langle \phi, (U_\lambda + z\mathbf{1})(U_\lambda - z\mathbf{1})^{-1}\phi \rangle$.

Important results (see [9]) that relate nontangential limits of this transform to the singular $\mu_s^\lambda$ and absolutely continuous parts of $\mu^\lambda$ are summarized in the following theorem.

**Theorem 3.1.** (1) *The limit* $\lim_{r\to 1} F_\lambda(re^{it})$ *exists for Lebesgue a.e.* $t \in [0, 2\pi)$, *and if*

$$\mathrm{d}\mu^\lambda(t) = f(t)\frac{\mathrm{d}t}{2\pi} + \mathrm{d}\mu_s^\lambda(t),$$

*then* $f(t) = \Re(F_\lambda(e^{it}))$.

(2) $t_0$ *is an eigenvalue of* $U_\lambda$ *if and only if*

$$\lim_{r\to 1}(1 - r)\Re(F_\lambda(re^{it_0})) \neq 0,$$

*and, in general,* $\lim_{r\uparrow 1}(1 - r)\Re(F_\lambda(re^{it_0})) = \mu^\lambda(\{t_0\})$.

(3) $\mu_s^\lambda$ *is supported on* $\{t \mid \lim_{r\uparrow 1} F_\lambda(re^{it}) = \infty\}$.

Consider now the Borel transform $R_\lambda(z)$ associated with the unitary operator $U_\lambda$, which is given by

$$R_\lambda(z) = \langle \phi, (U_\lambda - z\mathbf{1})^{-1}\phi \rangle = \int_0^{2\pi} \frac{\mathrm{d}\mu^\lambda(t)}{e^{it} - z},$$

and it has a simple relation to the Cauchy transform $F_\lambda(z) = 1 + 2zR_\lambda(z)$. After taking expectation values, with $\phi$, of the second resolvent identity, one obtains a unitary analog of the so-called Aronszajn–Krein formula; that is,

$$R_\lambda(z) = \frac{R_0(z)}{e^{i\lambda} + z(e^{i\lambda} - 1)R_0(z)},$$

which is [4, equation (9)] (note that we have a rather different notation from [4]). This is also interesting since one may obtain results for the perturbed operator $U_\lambda$ from asymptotic limits of $R_0(z)$, as in Proposition 3.2 ahead, where, from this formula, one has conditions for the divergence of $R_\lambda$ in terms of $R_0$ only.

By following Combescure [4], introduce $B(x)$ and $G(x)$ by

$$B(x) = \left[\int_0^{2\pi} \mathrm{d}\mu(t)\Big(\sin^2\Big(\frac{x - t}{2}\Big)\Big)^{-1}\right]^{-1} = \frac{1}{G(x)}.$$

For a general unitary operator $V$, with cyclic vector $\phi$ and spectral measure $\nu$, we have corresponding quantities $B_V(z)$ and $G_V(z)$ (just integrate with respect to $\nu$). With such notation, [4, Proposition 1] implies the following proposition.

**Proposition 3.2.** *Let $\lambda \neq 0$. Then, $\mathrm{d}\mu^\lambda$ has an atom at the point $x \in [0, 2\pi)$ if and only if $B(x) \neq 0$ (i.e., $G(x) < \infty$) and*

$$\lim_{\varepsilon \to 0} e^{i(x+i\varepsilon)} R_0(e^{i(x+i\varepsilon)}) = \frac{e^{i\lambda}}{1 - e^{i\lambda}}$$

*or, equivalently,*

$$\lim_{\varepsilon \to 0} F_0(e^{i(x+i\varepsilon)}) = i \cot\left(\frac{\lambda}{2}\right).$$

**Remark 3.3.** Given the relation $F_\lambda(z) = 1 + 2zR_\lambda(z)$, with $z = e^{i(x+i\varepsilon)}$, if

$$\lim_{\varepsilon \to 0} \frac{F_0(e^{i(x+i\varepsilon)}) - 1}{2} = \frac{e^{i\lambda}}{1 - e^{i\lambda}},$$

then

$$\begin{aligned} \lim_{\varepsilon \to 0} F_0(e^{i(x+i\varepsilon)}) &= \frac{2e^{i\lambda}}{1 - e^{i\lambda}} + 1 = \frac{1 + e^{i\lambda}}{1 - e^{i\lambda}} \\ &= \frac{e^{-i\frac{\lambda}{2}} + e^{i\frac{\lambda}{2}}}{e^{-i\frac{\lambda}{2}} - e^{i\frac{\lambda}{2}}} = i \cot\left(\frac{\lambda}{2}\right). \end{aligned}$$

We need the following results related to the spectrum $\sigma(U)$.

**Theorem 3.4.** *Given the unitary operator $U$, the set*

$$S = \{e^{ix} \mid G(x) = \infty\}$$

*is a dense $G_\delta$ in $\sigma(U)$.*

**Theorem 3.5.** *$\{\lambda \mid U_\lambda$ does not have eigenvalues in $\sigma(U)\}$ is a dense $G_\delta$ set in $[0, 2\pi)$.*

### 3.1. Proof of Theorem 3.4

The set $S = \{e^{ix} \mid G(x) = \infty\}$ is dense in $\sigma(U)$. To prove this, we first recall that $G(x)$ is given by

$$G(x) = \int_0^{2\pi} \mathrm{d}\mu(t) \left(\sin^2\left(\frac{x - t}{2}\right)\right)^{-1},$$

and that

$$\Re(F(re^{ix})) = \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(x - t)} \mathrm{d}\mu(t).$$

Now, suppose that $G(x) < \infty$ over an interval $(a, b)$. Consequently,

$$\lim_{r \to 1} \Re(F(re^{ix})) = 0.$$

Then, using the fact that $\lim_{r \to 1} F(re^{ix})$ exists for Lebesgue a.e. $t \in [0, 2\pi)$ and

$$\mathrm{d}\mu(t) = f(t)\frac{\mathrm{d}t}{2\pi} + \mathrm{d}\mu_{\mathrm{s}}(t),$$

where $f(x) = \lim_{r\to 1} \Re(F(re^{ix}))$ and $d\mu_s$ is supported on

$$\{x \mid \lim_{r\to 1} \Re(F(re^{ix})) = \infty\},$$

we obtain that $\mu(a,b) = 0$; that is, $(a,b) \cap \mathrm{supp}(d\mu) = \emptyset$. Therefore,

$$S = \{e^{ix} \mid G(x) = \infty\}$$

is dense in $\sigma(U)$.

Next, we show that $S = \{e^{ix} \mid G(x) = \infty\}$ is a $G_\delta$. To do this, introduce

$$G^m(x) = \int_0^{2\pi} d\mu(t)\left(\frac{1}{m^2} + \sin^2\left(\frac{x-t}{2}\right)\right)^{-1},$$

which is a $C^\infty$ function and $G(x) = \sup_m G^m(x)$. Then,

$$\begin{aligned}\{e^{ix} \mid G(x) = \infty\} &= \{e^{ix} \mid \text{for every } n, \text{ there exists } m \text{ such that } G^m(x) > n\} \\ &= \bigcap_n \bigcup_m \{e^{ix} \mid G^m(x) > n\}\end{aligned}$$

is a $G_\delta$.

**Remark 3.6.** According to Proposition 3.2, only values of $e^{ix}$ with $G(x) < \infty$ can serve as eigenvalues of $U_\lambda$. Note that if the spectrum $\sigma(U)$ is a perfect set without isolated points, Theorem 3.5 states that the points $e^{iy}$ with $G(y) = \infty$ are locally nonenumerable within $\sigma(U)$.

On the other hand, it is evident that $\{e^{ix} \mid G(x) = \infty\} \subset \sigma(U)$. And Theorem 3.5 reveals that $\{e^{ix} \mid G(x) < \infty\}$ has an empty interior within $\sigma(U)$. Moreover, this interior is also empty within the circle $S^1 = \{e^{it} \mid 0 \le t < 2\pi\}$. In fact, the theorem provides a stronger result, suggesting that $\sigma(U_\lambda)$ could have an empty interior in $S^1$. Furthermore, if $G(x) < \infty$, it implies that the integral

$$F(x) = \int_0^{2\pi} \frac{(1-r^2) + 2ri\sin(x-t)}{1 + r^2 - 2r\cos(x-t)} d\mu(t)$$

converges absolutely, and $F(x)$ is purely imaginary as $r \to 1$.

**Lemma 3.7.** *Let $B$ be a subset of $\mathbb{R}$ that is nowhere dense, and let $H : B \to \mathbb{R}$ be a function satisfying, for $x < y$,*

$$\alpha(y-x) < H(y) - H(x) < \beta(y-x) \tag{3.2}$$

*with fixed $\alpha, \beta > 0$. Then, the image of $H$ is a set that is nowhere dense.*

*Proof.* See [7, Lemma 3.2]. ∎

**Theorem 3.8.** *The set $S = \{F(x) \mid G(x) < \infty$ and $x \in \mathrm{supp}(\mu)\}$ is a countable union of sets that are nowhere dense in $[0, 2\pi)$ ($\mathrm{int}(\overline{S}) = \emptyset$).*

*Proof.* See [7, Lemma 3.1]. ∎

### 3.2. Proof of Theorem 3.5

Let $M : (0, 2\pi) \to I$ be the function (which is a homeomorphism), where $I = \{a + ib \mid a = 0\}$ (the imaginary axis), defined by $M(\lambda) = i\cot(\frac{\lambda}{2})$. Then, by Theorem 3.8, we have that the set

$$\left\{\lambda \mid \text{there exists } x \text{ with } G(x) < \infty,\ e^{ix} \in \sigma(U),\ M(x) = i\cot\left(\frac{\lambda}{2}\right)\right\}$$

is a countable union of nowhere dense subsets. Therefore, its complement is a dense set by the Baire category theorem. But, by Proposition 3.2, this complement is exactly

$$\{\lambda \mid U_\lambda \text{ has no eigenvalues in } \sigma(U)\},$$

which is dense. Furthermore, by [5, 8], this set is also a $G_\delta$.

As a consequence of these results, we have the following corollary.

**Corollary 3.9.** *If $\{e^{it} \mid a < t < b\} \subset \sigma(U)$ and $U$ does not have absolutely continuous spectrum, then, for a generic set of $\lambda$ in $[0, 2\pi)$, $U_\lambda$ has purely singular continuous spectrum in $\{e^{it} \mid a < t < b\}$.*

*Proof.* Combine Theorem 3.5 and the multiplicative version of Birman–Krein theorem (since $(e^{i\lambda} - 1)P_\phi$ in (3.1) is trace class). ∎

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