



Hongjie Dong · YanYan Li · Zhuolun Yang

Optimal gradient estimates of solutions to the insulated conductivity problem in dimension greater than two

Received 10 November 2021; revised 28 April 2022

Abstract. We study the insulated conductivity problem with inclusions embedded in a bounded domain in $\mathbb{R}^n$. The gradient of solutions may blow up as ε , the distance between inclusions, approaches to 0. It was known that the optimal blow-up rate in dimension $n = 2$ is of order $\varepsilon^{-1/2}$. It has recently been proved that in dimensions $n \geq 3$, an upper bound of the gradient is of order $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$. On the other hand, optimal values of β have not been identified. In this paper, we prove that when the inclusions are balls, the optimal value of β is $[-(n-1) + \sqrt{(n-1)^2 + 4(n-2)}]/4 \in (0, 1/2)$ in dimensions $n \geq 3$.

Keywords: optimal gradient estimates, high contrast coefficients, insulated conductivity problem, degenerate elliptic equation, maximum principle.

1. Introduction and main results

First we describe the nature of the domain. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\gamma}$ ($0 < \gamma < 1$) boundary containing two $C^{2,\gamma}$ ($0 < \gamma < 1$) relatively strictly convex open sets D_1 and D_2 with $\text{dist}(D_1 \cup D_2, \partial\Omega) > c > 0$. Let

$$\varepsilon := \text{dist}(D_1, D_2)$$

and $\tilde{\Omega} := \Omega \setminus \overline{(D_1 \cup D_2)}$. The conductivity problem can be modeled by the following elliptic equation:

$$\begin{cases} \text{div}(a_k(x)\nabla u_k) = 0 & \text{in } \Omega, \\ u_k = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

Hongjie Dong: Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA; hongjie_dong@brown.edu

YanYan Li: Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA; yli@math.rutgers.edu

Zhuolun Yang: Institute for Computational and Experimental Research in Mathematics, Brown University, 121 South Main Street, Providence, RI 02903, USA; zhuolun_yang@brown.edu

Mathematics Subject Classification 2020: 35J15 (primary); 35Q74, 74E30, 74G70, 78A48 (secondary).

where $\varphi \in C^2(\partial\Omega)$ is given, and

$$a_k(x) = \begin{cases} k \in (0, \infty) & \text{in } D_1 \cup D_2, \\ 1 & \text{in } \tilde{\Omega}. \end{cases}$$

In the context of electric conduction, the elliptic coefficients a_k refer to conductivities, and the solutions u_k represent voltage potential. From an engineering point of view, it is significant to estimate the magnitude of the electric fields in the narrow region between the inclusions, which is given by $|\nabla u_k|$. This problem is analogous to the Lamé system studied by Babuška, Andersson, Smith, and Levin [5], where they analyzed numerically that when the ellipticity constants are bounded away from 0 and infinity, the gradient of solutions remains bounded independent of ε , the distance between inclusions. Later, Bonnetier and Vogelius [12] proved that when $\varepsilon = 0$, $|\nabla u_k|$ is bounded for a fixed k and circular inclusions D_1 and D_2 in dimension $n = 2$. This result was extended by Li and Vogelius [28] to general second order elliptic equations of divergence form with piecewise Hölder coefficients and general shape of inclusions D_1 and D_2 in any dimension. Furthermore, they established a stronger piecewise $C^{1,\alpha}$ control of u_k , which is independent of ε . Li and Nirenberg [27] extended this global Lipschitz and piecewise $C^{1,\alpha}$ result to general second order elliptic systems of divergence form, including the linear system of elasticity. Some higher order derivative estimates in dimension $n = 2$ were obtained in [15, 16, 18].

When k is equal to ∞ (inclusions are perfect conductors) or 0 (insulators), it was shown in [13, 22, 31] that the gradient of solutions generally becomes unbounded as $\varepsilon \rightarrow 0$. Ammari et al. in [3, 4] considered the perfect and insulated conductivity problems with circular inclusions in $\mathbb{R}^2$, and established optimal blow-up rates $\varepsilon^{-1/2}$ in both cases. Yun extended in [33, 34] the results above allowing D_1 and D_2 to be any bounded strictly convex smooth domains.

The above gradient estimates in dimension $n = 2$ were localized and extended to higher dimensions by Bao, Li, and Yin in [6, 7]. For the perfect conductor case, they proved in [6] that

$$\begin{cases} \|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq C \varepsilon^{-1/2} \|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 2, \\ \|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq C |\varepsilon \ln \varepsilon|^{-1} \|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 3, \\ \|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq C \varepsilon^{-1} \|\varphi\|_{C^2(\partial\Omega)} & \text{when } n \geq 4. \end{cases}$$

These bounds were shown to be optimal in the paper and they are independent of the shape of inclusions, as long as the inclusions are relatively strictly convex. Moreover, many works have been done in characterizing the singular behaviors of ∇u , which are significant in practical applications. For further works on the perfect conductivity problem and closely related ones, see, e.g., [1, 2, 8–11, 14–17, 20, 21, 23–26, 30] and the references therein.

For the insulated conductivity problem, it was proved in [7] that

$$\|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq C \varepsilon^{-1/2} \|\varphi\|_{C^2(\partial\Omega)} \quad \text{when } n \geq 2. \quad (1.1)$$

The upper bound is optimal for $n = 2$ as mentioned above.

Yun [35] studied the following free space insulated conductivity problem in $\mathbb{R}^3$: Let H be a harmonic function in $\mathbb{R}^3$, $D_1 = B_1(0, 0, 1 + \varepsilon/2)$, and $D_2 = B_1(0, 0, -1 - \varepsilon/2)$,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{(D_1 \cup D_2)}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, i = 1, 2, \\ u(x) - H(x) = O(|x|^{-2}) & \text{as } |x| \rightarrow \infty. \end{cases}$$

He proved that for some positive constant C independent of ε ,

$$\max_{|x_3| \leq \varepsilon/2} |\nabla u(0, 0, x_3)| \leq C \varepsilon^{(\sqrt{2}-2)/2}. \quad (1.2)$$

He also showed that this upper bound of $|\nabla u|$ on the ε -segment connecting D_1 and D_2 is optimal for $H(x) \equiv x_1$. Although this result does not provide an upper bound of $|\nabla u|$ in the complement of the ε -segment, it has added support to a long-time suspicion that the upper bound $\varepsilon^{-1/2}$ obtained for dimension $n = 3$ in [7] is not optimal.

The upper bound (1.1) was recently improved by Li and Yang [29] to

$$\|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq C \varepsilon^{-1/2+\beta} \|\varphi\|_{C^2(\partial\Omega)} \quad \text{when } n \geq 3$$

for some $\beta > 0$. When insulators are unit balls, a more explicit constant $\beta(n)$ was given by Weinkove in [32] for $n \geq 4$ by a different method. The constant $\beta(n)$ obtained in [32] presumably improves that in [29]. In particular, it was proved in [32] that $\beta(n)$ approaches $1/2$ from below as $n \rightarrow \infty$. However, the optimal blow-up rate in dimensions $n \geq 3$ remained unknown. We draw reader's attention to a recent survey paper [19] by Kang, where in the conclusions section, the three-dimensional case is described as an outstanding problem.

In this paper, we focus on the following insulated conductivity problem in dimensions $n \geq 3$, and give an optimal gradient estimate for a certain class of inclusions including two balls of any size:

$$\begin{cases} -\Delta u = 0 & \text{in } \tilde{\Omega}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, i = 1, 2, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where $\varphi \in C^2(\partial\Omega)$ is given, and $\nu = (\nu_1, \dots, \nu_n)$ denotes the inner normal vector on $\partial D_1 \cup \partial D_2$.

We use the notation $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$. After choosing a coordinate system properly, we can assume that near the origin, the part of ∂D_1 and ∂D_2 , denoted by Γ_+ and Γ_- , are respectively the graphs of two $C^{2,\gamma}$ ($0 < \gamma < 1$) functions in terms of x' . That is, for some $R_0 > 0$,

$$\begin{aligned} \Gamma_+ &= \left\{ x_n = \frac{\varepsilon}{2} + f(x'), |x'| < R_0 \right\}, \\ \Gamma_- &= \left\{ x_n = -\frac{\varepsilon}{2} + g(x'), |x'| < R_0 \right\}, \end{aligned}$$

where f and g are $C^{2,\gamma}$ functions satisfying

$$\begin{aligned} f(x') &> g(x') \quad \text{for } 0 < |x'| < R_0, \\ f(0') &= g(0') = 0, \quad \nabla_{x'} f(0') = \nabla_{x'} g(0') = 0, \end{aligned} \quad (1.4)$$

$$f(x') - g(x') = a|x'|^2 + O(|x'|^{2+\gamma}) \quad \text{for } 0 < |x'| < R_0 \quad (1.5)$$

with $a > 0$. Here and throughout the paper, we use the notation $O(A)$ to denote a quantity that can be bounded by CA , where C is some positive constant independent of ε . For $0 < r \leq R_0$, we denote

$$\Omega_r := \left\{ (x', x_n) \in \tilde{\Omega} \mid -\frac{\varepsilon}{2} + g(x') < x_n < \frac{\varepsilon}{2} + f(x'), |x'| < r \right\}. \quad (1.6)$$

By standard elliptic estimates, the solution $u \in H^1(\tilde{\Omega})$ of (1.3) satisfies

$$\|u\|_{C^1(\tilde{\Omega} \setminus \Omega_{R_0/2})} \leq C. \quad (1.7)$$

We will focus on the following problem near the origin:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega_{R_0}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_+ \cup \Gamma_-, \\ \|u\|_{L^\infty(\Omega_{R_0})} \leq 1. \end{cases} \quad (1.8)$$

It was proved in [7] that for $u \in H^1(\Omega_{R_0})$ satisfying (1.8),

$$|\nabla u(x)| \leq C(\varepsilon + |x'|^2)^{-1/2} \quad \forall x \in \Omega_{R_0}, \quad (1.9)$$

where C is a positive constant depending only on n , R_0 , a , $\|f\|_{C^2}$, and $\|g\|_{C^2}$, and is in particular independent of ε . The above-mentioned improvement on (1.1) in [29, 32] also applies to (1.9).

Our main results of this paper are as follows.

Theorem 1.1. *For $n \geq 3$, $\varepsilon \in (0, 1/4)$, let $u \in H^1(\Omega_{R_0})$ be a solution of (1.8) with f , g satisfying (1.4) and (1.5). Then there exists a positive constant C depending only on n , R_0 , γ , a positive lower bound of a , and an upper bound of $\|f\|_{C^{2,\gamma}}$ and $\|g\|_{C^{2,\gamma}}$, such that*

$$|\nabla u(x)| \leq C \|u\|_{L^\infty(\Omega_{R_0})} (\varepsilon + |x'|^2)^{(\alpha-1)/2} \quad \forall x \in \Omega_{R_0/2}, \quad (1.10)$$

where

$$\alpha = \alpha(n) := \frac{-(n-1) + \sqrt{(n-1)^2 + 4(n-2)}}{2} \in (0, 1). \quad (1.11)$$

Note that $\alpha(n)$ is monotonically increasing in n , and

$$\alpha(n) = 1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right) \quad \text{as } n \rightarrow \infty.$$

For $n = 3$, the exponent $(\alpha - 1)/2 = (\sqrt{2} - 2)/2$ is the same as the exponent in (1.2). For $n \geq 4$, the exponent $(\alpha - 1)/2$ is strictly greater than the one obtained in [32].

A consequence of Theorem 1.1 is, in view of (1.7), as follows.

Corollary 1.2. For $n \geq 3$, $\varepsilon \in (0, 1/4)$, let D_1, D_2 be two balls of radii r_1, r_2 , centered at $(0', \varepsilon/2 + r_1)$ and $(0', -\varepsilon/2 - r_2)$, respectively. Let $u \in H^1(\tilde{\Omega})$ be the solution of (1.3). Then there exists a positive constant C depending only on n, r_1, r_2 , and $\|\partial\Omega\|_{C^2}$ such that

$$\|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq C \|\varphi\|_{C^2(\partial\Omega)} \varepsilon^{(\alpha-1)/2}, \quad (1.12)$$

where α is given by (1.11).

Estimate (1.12) is optimal as shown in the following theorem.

Theorem 1.3. For $n \geq 3$, $\varepsilon \in (0, 1/4)$, let $\Omega = B_5$, and D_1, D_2 be the unit balls center at $(0', 1 + \varepsilon/2)$ and $(0', -1 - \varepsilon/2)$, respectively. Let $\varphi = x_1$ and $u \in H^1(\tilde{\Omega})$ be the solution of (1.3). Then there exists a positive constant C depending only on n , such that

$$\|\nabla u\|_{L^\infty(\tilde{\Omega} \cap B_{2\sqrt{\varepsilon}})} \geq \frac{1}{C} \varepsilon^{(\alpha-1)/2}, \quad (1.13)$$

where α is given by (1.11).

Remark 1.4. Estimate (1.13) holds for all C^2 domains Ω and C^4 relatively strictly convex open sets D_1, D_2 that are axially symmetric with respect to x_n -axis. A modification of the proof of the theorem yields the result.

Let us give a brief description of the proof of Theorem 1.3. Consider

$$\bar{u}(x') = \oint_{-\varepsilon/2+g(x') < x_n < \varepsilon/2+f(x')} u(x', x_n) dx_n, \quad |x'| < 1,$$

where $f(x') = -\sqrt{1 - |x'|^2} + 1$ and $g(x') = \sqrt{1 - |x'|^2} - 1$. In the polar coordinates, $\bar{u}(x') = \bar{u}(r, \xi)$, where $x' = (r, \xi)$, $0 < r < 1$, and $\xi \in \mathbb{S}^{n-2}$. Since the boundary value φ depends only on x_1 and is odd in x_1 , the projection of $\bar{u}(r, \cdot)$ to the span of the spherical harmonics is $U_{1,1}(r)Y_{1,1}(\xi)$, where $Y_{1,1}$ is $x_1|_{\mathbb{S}^{n-2}}$ modulo a harmless positive normalization constant,

$$U_{1,1}(r) = \oint_{-\varepsilon/2+g(x') < x_n < \varepsilon/2+f(x')} \hat{u}(r, x_n) dx_n,$$

and

$$\hat{u}(r, x_n) = \int_{\mathbb{S}^{n-2}} u(r, \xi, x_n) Y_{1,1}(\xi) d\xi.$$

We analyze the equations satisfied by $U_{1,1}(r)$ and $\hat{u}(r, x_n)$ and establish a lower bound

$$U_{1,1}(r) \geq \frac{1}{C} r^\beta (\varepsilon + r^2)^{(\alpha-\beta)/2}, \quad 0 < r < 1,$$

where $\beta = (2\alpha^2 + \alpha(n-1))/(n-3+\alpha)$ and C is a positive constant independent of ε . It follows that

$$\|\bar{u}(\sqrt{\varepsilon}, \cdot)\|_{L^2(\mathbb{S}^{n-2})} \geq |U_{1,1}(\sqrt{\varepsilon})| \geq \frac{1}{C} \varepsilon^{\alpha/2},$$

and, consequently, there exist $\xi_0 \in \mathbb{S}^{n-2}$, $x_n \in (-\varepsilon/2 + g(x'), \varepsilon/2 + f(x'))$ such that

$$|u(\sqrt{\varepsilon}, \xi_0, x_n)| \geq \frac{1}{C} \varepsilon^{\alpha/2}.$$

Estimate (1.13) follows since $u(0) = 0$ by the oddness of u in x_1 .

Theorems 1.1 and 1.3 will be proved in Sections 2 and 3, respectively.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Without loss of generality, we may assume $a = 1$. Namely, we consider

$$f(x') - g(x') = |x'|^2 + O(|x'|^{2+\gamma}) \quad \text{for } 0 < |x'| < R_0.$$

We perform a change of variables by setting

$$\begin{cases} y' = x', \\ y_n = 2\varepsilon \left(\frac{x_n - g(x') + \varepsilon/2}{\varepsilon + f(x') - g(x')} - \frac{1}{2} \right), \end{cases} \quad \forall (x', x_n) \in \Omega_{R_0}. \quad (2.1)$$

This change of variables maps the domain Ω_{R_0} to a cylinder of height 2ε , denoted by $Q_{R_0, \varepsilon}$, where

$$Q_{s,t} := \{y = (y', y_n) \in \mathbb{R}^n \mid |y'| < s, |y_n| < t\}$$

for $s, t > 0$. Moreover, $\det(\partial_x y) = 2\varepsilon(\varepsilon + f(x') - g(x'))^{-1}$. Let $u(x) \in H^1(\Omega_{R_0})$ be a solution of (1.8) and let $v(y) = u(x)$. Then v satisfies

$$\begin{cases} -\partial_i(a^{ij}(y)\partial_j v(y)) = 0 & \text{in } Q_{R_0, \varepsilon}, \\ a^{nj}(y)\partial_j v(y) = 0 & \text{on } \{y_n = -\varepsilon\} \cup \{y_n = \varepsilon\} \end{cases} \quad (2.2)$$

with $\|v\|_{L^\infty(Q_{R_0, \varepsilon})} \leq 1$, where the coefficient matrix $(a^{ij}(y))$ is given by

$$(a^{ij}(y)) = \frac{2\varepsilon(\partial_x y)(\partial_x y)^t}{\det(\partial_x y)} = \begin{pmatrix} \varepsilon + |y'|^2 & 0 & \cdots & 0 & a^{1n} \\ 0 & \varepsilon + |y'|^2 & \cdots & 0 & a^{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \varepsilon + |y'|^2 & a^{n-1,n} \\ a^{n1} & a^{n2} & \cdots & a^{n,n-1} & \frac{4\varepsilon^2 + \sum_{i=1}^{n-1} |a^{in}|^2}{\varepsilon + f(y') - g(y')} \end{pmatrix} + \begin{pmatrix} e^1 & 0 & \cdots & 0 & 0 \\ 0 & e^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} a^{ni} &= a^{in} = -2\varepsilon \partial_i g(y') - (y_n + \varepsilon) \partial_i (f(y') - g(y')), \\ e^i &= f(y') - g(y') - |y'|^2 \end{aligned}$$

for $i = 1, \dots, n-1$. By (1.4), we know that for $i = 1, \dots, n-1$,

$$|a^{ni}(y)| = |a^{in}(y)| \leq C\varepsilon|y'| \quad \text{and} \quad |e^i(y')| \leq C|y'|^{2+\gamma}. \quad (2.3)$$

Note that $e^1, \dots, e^{n-1}$ depend only on y' and are independent of y_n . We define

$$\bar{v}(y') := \int_{-\varepsilon}^{\varepsilon} v(y', y_n) dy_n. \quad (2.4)$$

Then $\bar{v}$ satisfies

$$\operatorname{div}((\varepsilon + |y'|^2)\nabla \bar{v}) = - \sum_{i=1}^{n-1} \partial_i (\overline{a^{in} \partial_n v}) - \sum_{i=1}^{n-1} \partial_i (e^i \partial_i \bar{v}) \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1} \quad (2.5)$$

with $\|\bar{v}\|_{L^\infty(B_{R_0})} \leq 1$, where $\overline{a^{in} \partial_n v}$ is the average of $a^{in} \partial_n v$ with respect to y_n in $(-\varepsilon, \varepsilon)$. Since $\partial u / \partial \nu = 0$ on Γ_+ and Γ_- , we have by (1.4) and (1.9) that

$$|\partial_n u(x)| \leq C \left| \sum_{i=1}^{n-1} x_i \partial_i u \right| \leq C \quad \forall x \in \Gamma_+ \cup \Gamma_-.$$

By the harmonicity of $\partial_n u$, estimate (1.7), and the maximum principle,

$$|\partial_n u| \leq C \quad \text{in } \Omega_{R_0}, \quad (2.6)$$

and consequently,

$$|\partial_n v| \leq \frac{C(\varepsilon + |y'|^2)}{\varepsilon} \quad \text{in } Q_{R_0, \varepsilon}. \quad (2.7)$$

Therefore, equation (2.5) can be rewritten as

$$\operatorname{div}((\varepsilon + |y'|^2)\nabla \bar{v}) = \sum_{i=1}^{n-1} \partial_i F_i \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1}, \quad (2.8)$$

where $F_i := -\overline{a^{in} \partial_n v} - e^i \partial_i \bar{v}$ satisfies, using (1.9) and (2.3),

$$|F_i| \leq C(|y'|(\varepsilon + |y'|^2) + |y'|^{2+\gamma}(\varepsilon + |y'|^2)^{-1/2}).$$

For $\gamma, s \in \mathbb{R}$, we introduce the following norm:

$$\|F\|_{\varepsilon, \gamma, s, B_{R_0}} := \sup_{y' \in B_{R_0}} \frac{|F(y')|}{|y'|^\gamma (\varepsilon + |y'|^2)^{1-s}}.$$

Proposition 2.1. *For $n \geq 3$, $s \geq 0$, $1 + \gamma - 2s > 0$, $1 + \gamma - 2s \neq \alpha$, $\varepsilon > 0$, and $R_0 > 0$, let $\bar{v} \in H^1(B_{R_0})$ be a solution of*

$$\operatorname{div}((\varepsilon + |y'|^2)\nabla \bar{v}) = \operatorname{div} F + G \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1},$$

where $F, G \in L^\infty(B_{R_0})$ satisfy

$$\|F\|_{\varepsilon, \gamma, s, B_{R_0}} < \infty, \quad \|G\|_{\varepsilon, \gamma-1, s, B_{R_0}} < \infty. \quad (2.9)$$

Then for any $R \in (0, R_0/2)$, we have

$$\begin{aligned} & \left(\int_{\partial B_R} |\bar{v} - \bar{v}(0)|^2 d\sigma \right)^{1/2} \\ & \leq C(\|F\|_{\varepsilon, \gamma, s, B_{R_0}} + \|G\|_{\varepsilon, \gamma-1, s, B_{R_0}} + \|\bar{v} - \bar{v}(0)\|_{L^2(\partial B_{R_0})}) R^{\tilde{\alpha}}, \end{aligned} \quad (2.10)$$

where $\tilde{\alpha} := \min\{\alpha, 1 + \gamma - 2s\}$, α is given in (1.11), and C is some positive constant depending only on n, γ, s , and R_0 , and is independent of ε .

For the proof, we use an iteration argument based on the following two lemmas.

Lemma 2.2. For $n \geq 3$, $\varepsilon > 0$, and $R_0 > 0$, let $v_1 \in H^1(B_{R_0})$ satisfy

$$\operatorname{div}((\varepsilon + |y'|^2)\nabla v_1) = 0 \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1}. \quad (2.11)$$

Then for any $0 < \rho < R \leq R_0$, we have

$$\left(\int_{\partial B_\rho} |v_1(y') - v_1(0)|^2 d\sigma \right)^{1/2} \leq \left(\frac{\rho}{R} \right)^\alpha \left(\int_{\partial B_R} |v_1(y') - v_1(0)|^2 d\sigma \right)^{1/2},$$

where α is given in (1.11).

Proof. By the elliptic theory, $v_1 \in C^\infty(B_{R_0})$. Without loss of generality, we assume that $v_1(0) = 0$. By scaling, it suffices to prove the lemma for $R = 1$. Denote

$$y' = (r, \xi) \in (0, 1) \times \mathbb{S}^{n-2}.$$

We can rewrite (2.11) as

$$\partial_{rr} v_1 + \left(\frac{n-2}{r} + \frac{2r}{\varepsilon + r^2} \right) \partial_r v_1 + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-2}} v_1 = 0 \quad \text{in } B_1 \setminus \{0\}.$$

Take the decomposition

$$v_1(y') = \sum_{k=1}^{\infty} \sum_{i=1}^{N(k)} V_{k,i}(r) Y_{k,i}(\xi), \quad y' \in B_1 \setminus \{0\}, \quad (2.12)$$

where $Y_{k,i}$ is a k -th degree spherical harmonics, that is,

$$-\Delta_{\mathbb{S}^{n-2}} Y_{k,i} = k(k+n-3) Y_{k,i}$$

and $\{Y_{k,i}\}_{k,i}$ forms an orthonormal basis of $L^2(\mathbb{S}^{n-2})$. Here we used the fact that $V_{0,1} = 0$ because $v_1(0) = 0$. Then $V_{k,i}(r) \in C^2(0, 1)$ is given by

$$V_{k,i}(r) = \int_{\mathbb{S}^{n-2}} v_1(y') Y_{k,i}(\xi) d\xi$$

and satisfies

$$L_k V_{k,i} := V_{k,i}''(r) + \left(\frac{n-2}{r} + \frac{2r}{\varepsilon + r^2} \right) V_{k,i}'(r) - \frac{k(k+n-3)}{r^2} V_{k,i}(r) = 0 \quad \text{in } (0, 1)$$

for each $k \in \mathbb{N}$, $i = 1, 2, \dots, N(k)$. For any $k \in \mathbb{N}$, let

$$\alpha_k := \frac{-(n-1) + \sqrt{(n-1)^2 + 4k(k+n-3)}}{2}.$$

For any $c \in \mathbb{R}$, we have, by a direct computation,

$$L_k r^c = r^{c-2} \left(c^2 + \left(n-3 + \frac{2r^2}{\varepsilon + r^2} \right) c - k(k+n-3) \right) \quad \text{in } (0, 1).$$

Thus for $c > 0$ sufficiently small, we have

$$L_k r^{-c} \leq 0 \quad \text{and} \quad L_k r^{\alpha_k} \leq 0 \quad \text{in } (0, 1).$$

Therefore, for any $\gamma > 0$,

$$L_k(\pm V_{k,i}(r) - \gamma r^{-c} - |V_{k,i}(1)|r^{\alpha_k}) \geq 0 \quad \text{in } (0, 1).$$

Since $v_1 \in L^\infty(B_1)$, we know that $V_{k,i}(r)$ is bounded in $(0, 1)$, so we have

$$\pm V_{k,i}(r) - \gamma r^{-c} - |V_{k,i}(1)|r^{\alpha_k} < 0 \quad \text{as } r \searrow 0.$$

Clearly,

$$\pm V_{k,i}(r) - \gamma r^{-c} - |V_{k,i}(1)|r^{\alpha_k} < 0 \quad \text{when } r = 1.$$

By the maximum principle,

$$|V_{k,i}(r)| \leq \gamma r^{-c} + r^{\alpha_k} |V_{k,i}(1)| \quad \text{for } 0 < r < 1.$$

Sending $\gamma \rightarrow 0$, we have

$$|V_{k,i}(r)| \leq r^{\alpha_k} |V_{k,i}(1)| \quad \text{for } 0 < r < 1. \quad (2.13)$$

It follows from (2.12) and (2.13) that

$$\begin{aligned} \oint_{\partial B_\rho} |v_1(y')|^2 d\sigma &= \sum_{k=1}^{\infty} \sum_{i=1}^{N(k)} |V_{k,i}(\rho)|^2 \\ &\leq \rho^{2\alpha} \sum_{k=1}^{\infty} \sum_{i=1}^{N(k)} |V_{k,i}(1)|^2 = \rho^{2\alpha} \oint_{\partial B_1} |v_1(y')|^2 d\sigma. \quad \blacksquare \end{aligned}$$

Lemma 2.3. For $n \geq 3$, $s \geq 0$, $1 + \gamma - 2s > 0$, and $\varepsilon > 0$, suppose that $F, G \in L^\infty(B_1)$ satisfy (2.9) with $R_0 = 1$, and $v_2 \in H_0^1(B_1)$ satisfies

$$\operatorname{div}((\varepsilon + |y'|^2)\nabla v_2) = \operatorname{div} F + G \quad \text{in } B_1 \subset \mathbb{R}^{n-1}. \quad (2.14)$$

Then we have

$$\|v_2\|_{L^\infty(B_1)} \leq C(\|F\|_{\varepsilon, \gamma, s, B_1} + \|G\|_{\varepsilon, \gamma-1, s, B_1}),$$

where $C > 0$ depends only on n , γ , and s , and is in particular independent of ε .

Proof. Without loss of generality, we assume $\|F\|_{\varepsilon, \gamma, s, B_1} + \|G\|_{\varepsilon, \gamma-1, s, B_1} = 1$. Denote $r = |y'|$. We can rewrite (2.14) as

$$\begin{aligned} \Delta v_2 + \frac{2r}{\varepsilon + r^2} \partial_r v_2 &= \partial_i (F_i (\varepsilon + r^2)^{-1}) + 2F_i y_i (\varepsilon + r^2)^{-2} \\ &\quad + G (\varepsilon + r^2)^{-1} \quad \text{in } B_1. \end{aligned} \quad (2.15)$$

We use Moser's iteration argument. By the definitions,

$$\begin{aligned} |F_i (\varepsilon + r^2)^{-1}| &\leq r^{\gamma-2s} \|F\|_{\varepsilon, \gamma, s, B_1}, \\ |F_i y_i (\varepsilon + r^2)^{-2}| &\leq r^{\gamma-2s-1} \|F\|_{\varepsilon, \gamma, s, B_1}, \\ |G (\varepsilon + r^2)^{-1}| &\leq r^{\gamma-2s-1} \|G\|_{\varepsilon, \gamma-1, s, B_1}. \end{aligned}$$

For $p \geq 2$, we multiply equation (2.15) by $-|v_2|^{p-2} v_2$ and integrate by parts to obtain

$$\begin{aligned} (p-1) \int_{B_1} |\nabla v_2|^2 |v_2|^{p-2} dy' &- \int_{B_1} \frac{2r}{\varepsilon + r^2} \partial_r v_2 (|v_2|^{p-2} v_2) dy' \\ &\leq C(p-1) \int_{B_1} |\nabla v_2| |v_2|^{p-2} r^{\gamma-2s} dy' + C \int_{B_1} |v_2|^{p-1} r^{\gamma-2s-1} dy'. \end{aligned}$$

The second term on the left-hand side is equal to

$$-\frac{1}{p} \int_{\mathbb{S}^{n-2}} \int_0^1 \frac{2r^{n-1}}{\varepsilon + r^2} \partial_r |v_2|^p dr d\theta = \frac{1}{p} \int_{\mathbb{S}^{n-2}} \int_0^1 \partial_r \left(\frac{2r^{n-1}}{\varepsilon + r^2} \right) |v_2|^p dr d\theta \geq 0.$$

Therefore, by Hölder's inequality and using $1 + \gamma - 2s > 0$,

$$\begin{aligned} (p-1) \int_{B_1} |\nabla v_2|^2 |v_2|^{p-2} dy' &\leq C(p-1) \| |\nabla v_2| |v_2|^{(p-2)/2} \|_{L^2(B_1)} \|v_2^{p-2}\|_{L^{n/(n-2)}(B_1)}^{1/2} \|r^{2(\gamma-2s)}\|_{L^{n/2}(B_1)}^{1/2} \\ &\quad + C \| |v_2|^{p-1} \|_{L^{n/(n-2)}(B_1)} \|r^{\gamma-2s-1}\|_{L^{n/2}(B_1)} \\ &\leq C(p-1) \| |\nabla v_2| |v_2|^{(p-2)/2} \|_{L^2(B_1)} \|v_2^{p-2}\|_{L^{n/(n-2)}(B_1)}^{1/2} \\ &\quad + C \| |v_2|^{p-1} \|_{L^{n/(n-2)}(B_1)}. \end{aligned}$$

By Young's inequality,

$$\begin{aligned} \frac{p-1}{2} \int_{B_1} |\nabla v_2|^2 |v_2|^{p-2} dy' &\leq C(p-1) \| |v_2|^{p-2} \|_{L^{n/(n-2)}(B_1)} \\ &\quad + C \| |v_2|^{p-1} \|_{L^{n/(n-2)}(B_1)}. \end{aligned} \quad (2.16)$$

Taking $p = 2$ in the above, we have, by Hölder's inequality,

$$\begin{aligned} \int_{B_1} |\nabla v_2|^2 dy' &\leq C + C \|v_2\|_{L^{n/(n-2)}(B_1)} \\ &\leq C + C \|v_2\|_{L^{2n/(n-2)}(B_1)}. \end{aligned}$$

Applying the Sobolev–Poincaré inequality on the left-hand side, we have

$$\|v_2\|_{L^{2n/(n-2)}(B_1)}^2 \leq C \int_{B_1} |\nabla v_2|^2 dy' \leq C + C \|v_2\|_{L^{2n/(n-2)}(B_1)},$$

which implies

$$\|v_2\|_{L^{2n/(n-2)}(B_1)} \leq C. \quad (2.17)$$

From (2.16), by Hölder's inequality,

$$\begin{aligned} \frac{4(p-1)}{p^2} \int_{B_1} |\nabla|v_2|^{p/2}|^2 dy' &= (p-1) \int_{B_1} |\nabla v_2|^2 |v_2|^{p-2} dy' \\ &\leq Cp \|v_2\|_{L^{np/(n-2)}(B_1)}^{p-2} + C \|v_2\|_{L^{np/(n-2)}(B_1)}^{p-1}, \end{aligned}$$

which implies that

$$\|\nabla|v_2|^{p/2}\|_{L^2(B_1)}^{2/p} \leq \max_{i \in \{1,2\}} (Cp^i)^{1/p} \|v_2\|_{L^{np/(n-2)}(B_1)}^{(p-i)/p}.$$

Then by the Sobolev inequality and Young's inequality, we have

$$\begin{aligned} \|v_2\|_{L^{\frac{(n-1/2)p}{n-5/2}}(B_1)} &\leq \max_{i \in \{1,2\}} (Cp^i)^{1/p} \left(\frac{p-i}{p} \|v_2\|_{L^{np/(n-2)}(B_1)} + \frac{i}{p} \right) \\ &\leq (Cp^2)^{1/p} \left(\|v_2\|_{L^{np/(n-2)}(B_1)} + \frac{2}{p} \right). \end{aligned}$$

For $k \geq 0$, let

$$p_k = 2 \left(\frac{(n-1/2)(n-2)}{(n-5/2)n} \right)^k \frac{n}{n-2}.$$

Iterating the relations above, we have, by (2.17),

$$\begin{aligned} \|v_2\|_{L^{p_k}} &\leq \prod_{i=0}^{k-1} (Cp_i^2)^{2/p_i} \|v_2\|_{L^{p_0}(B_1)} + \sum_{i=0}^{k-1} \prod_{j=i}^{k-1} (Cp_j^2)^{2/p_j} \frac{4}{p_i} \\ &\leq C \|v_2\|_{L^{2n/(n-2)}(B_1)} + C \leq C, \end{aligned} \quad (2.18)$$

where C is a positive constant depending on n , γ , and s , and is in particular independent of k . The lemma is concluded by taking $k \rightarrow \infty$ in (2.18). ■

Proof of Proposition 2.1. Without loss of generality, we assume that $\bar{v}(0) = 0$ and

$$\|F\|_{\varepsilon, \gamma, s, B_{R_0}} + \|G\|_{\varepsilon, \gamma-1, s, B_{R_0}} + \|\bar{v}\|_{L^2(\partial B_{R_0})} = 1.$$

Consider

$$\omega(\rho) := \left(\int_{\partial B_\rho} |\bar{v}|^2 d\sigma \right)^{1/2}.$$

For $0 < \rho \leq R/2 \leq R_0/2$, we write $\bar{v} = v_1 + v_2$ in B_R , where v_2 satisfies

$$\operatorname{div}((\varepsilon + |y'|^2)\nabla v_2) = \operatorname{div} F + G \quad \text{in } B_R$$

and $v_2 = 0$ on ∂B_R . Thus v_1 satisfies

$$\operatorname{div}((\varepsilon + |y'|^2)\nabla v_1) = 0 \quad \text{in } B_R$$

and $v_1 = \bar{v}$ on ∂B_R . By Lemma 2.2,

$$\left(\int_{\partial B_\rho} |v_1(y') - v_1(0)|^2 d\sigma \right)^{1/2} \leq \left(\frac{\rho}{R} \right)^\alpha \left(\int_{\partial B_R} |v_1(y') - v_1(0)|^2 d\sigma \right)^{1/2}. \quad (2.19)$$

Since $\tilde{v}_2(y') := v_2(Ry')$ satisfies

$$\operatorname{div}((R^{-2}\varepsilon + |y'|^2)\nabla \tilde{v}_2) = \operatorname{div} \tilde{F} + \tilde{G} \quad \text{in } B_1,$$

where $\tilde{F}(y') := R^{-1}F(Ry')$ and $\tilde{G}(y') := G(Ry')$ satisfy

$$\begin{aligned} \|\tilde{F}\|_{R^{-2}\varepsilon, \gamma, s, B_1} &= R^{1+\gamma-2s} \|F\|_{\varepsilon, \gamma, s, B_R}, \\ \|\tilde{G}\|_{R^{-2}\varepsilon, \gamma-1, s, B_1} &= R^{1+\gamma-2s} \|G\|_{\varepsilon, \gamma-1, s, B_R}, \end{aligned}$$

we apply Lemma 2.3 to $\tilde{v}_2$ with ε replaced by $R^{-2}\varepsilon$ to obtain

$$\|v_2\|_{L^\infty(B_R)} \leq CR^{1+\gamma-2s}. \quad (2.20)$$

Since $\bar{v}(0) = v_1(0) + v_2(0) = 0$, we have $|v_1(0)| = |v_2(0)|$. Combining (2.19) and (2.20) yields, using $\bar{v} = v_1 + v_2$, and $\bar{v} = v_1$ on ∂B_R ,

$$\begin{aligned} \omega(\rho) &\leq \left(\int_{\partial B_\rho} |v_1(y') - v_1(0)|^2 d\sigma \right)^{1/2} + \left(\int_{\partial B_\rho} |v_2(y') - v_2(0)|^2 d\sigma \right)^{1/2} \\ &\leq \left(\frac{\rho}{R} \right)^\alpha \left(\int_{\partial B_R} |v_1(y')|^2 d\sigma \right)^{1/2} + \left(\frac{\rho}{R} \right)^\alpha |v_1(0)| + 2\|v_2\|_{L^\infty(B_R)} \\ &\leq \left(\frac{\rho}{R} \right)^\alpha \omega(R) + CR^{1+\gamma-2s}. \end{aligned} \quad (2.21)$$

For a positive integer k , we take $\rho = 2^{-i-1}R_0$ and $R = 2^{-i}R_0$ in (2.21) and iterate from $i = 0$ to $k - 1$. We have, using $1 + \gamma - 2s \neq \alpha$,

$$\begin{aligned} \omega(2^{-k}R_0) &\leq 2^{-k\alpha}\omega(R_0) + C \sum_{i=1}^k 2^{-(k-i)\alpha} (2^{1-i}R_0)^{1+\gamma-2s} \\ &\leq 2^{-k\alpha}\omega(R_0) + C 2^{-k\alpha} R_0^{1+\gamma-2s} \frac{1 - 2^{k(\alpha-1-\gamma+2s)}}{1 - 2^{\alpha-1-\gamma+2s}}. \end{aligned}$$

It follows that

$$\omega(2^{-k}R_0) \leq 2^{-k\tilde{\alpha}}(\omega(R_0) + CR_0^{1+\gamma-2s}).$$

For any $\rho \in (0, R_0/2)$, let k be the integer such that $2^{-k-1}R_0 < \rho \leq 2^{-k}R_0$. Then

$$\omega(\rho) \leq C\rho^{\tilde{\alpha}} \quad \forall \rho \in \left(0, \frac{R_0}{2}\right).$$

Therefore, (2.10) is proved. ■

Proof of Theorem 1.1. Without loss of generality, we assume that $a = 1$, $u(0) = 0$ and $\|u\|_{L^\infty(\Omega_{R_0})} = 1$. We make the change of variables (2.1), and let $v(y) = u(x)$. Then v satisfies (2.2). Let $\bar{v}$ be defined as in (2.4). By (1.9),

$$\|\nabla \bar{v}(y')\|_{\varepsilon, 0, s_0+1, B_{R_0}} < \infty,$$

where $s_0 = 1/2$. Then $\bar{v}$ satisfies equation (2.8) with F satisfying

$$\|F\|_{\varepsilon, \gamma-2s_0, 0, B_{R_0}} < \infty.$$

By (2.7),

$$|v(y', y_n) - \bar{v}(y')| \leq 2\varepsilon \max_{y_n \in (-\varepsilon, \varepsilon)} |\partial_n v(y', y_n)| \leq C(\varepsilon + |y'|^2) \quad \text{in } Q_{R_0, \varepsilon}. \quad (2.22)$$

By decreasing γ if necessary, we may assume that $1 + \gamma - 2s_0 = \gamma < \alpha$. By Proposition 2.1 and (2.22), we have

$$\begin{aligned} \int_{Q_{2\varepsilon^{1/2}, \varepsilon}} |v - \bar{v}(0)|^2 dy &\leq C \int_{Q_{2\varepsilon^{1/2}, \varepsilon}} |v - \bar{v}|^2 dy + C \int_{Q_{2\varepsilon^{1/2}, \varepsilon}} |\bar{v} - \bar{v}(0)|^2 dy \\ &\leq C\varepsilon^{\tilde{\alpha}}, \end{aligned} \quad (2.23)$$

where $\tilde{\alpha} = \min\{\alpha, 1 + \gamma - 2s_0\}$. Let $\tilde{a}^{ij}(y) = a^{ij}(\varepsilon^{1/2}y)$ and $\tilde{v}(y) = v(\varepsilon^{1/2}y) - \bar{v}(0)$. Then $\tilde{v}$ satisfies

$$\begin{cases} -\partial_i(\tilde{a}^{ij}(y)\partial_j\tilde{v}(y)) = 0 & \text{in } Q_{2, \varepsilon^{1/2}}, \\ \tilde{a}^{nj}(y)\partial_j\tilde{v}(y) = 0 & \text{on } \{y_n = -\varepsilon^{1/2}\} \cup \{y_n = \varepsilon^{1/2}\}. \end{cases}$$

It is straightforward to verify that

$$\frac{I}{C} \leq \tilde{a} \leq CI \quad \text{and} \quad \|\nabla \tilde{a}\|_{L^\infty(Q_{2, \varepsilon^{1/2}})} \leq C.$$

Now we define

$$S_l := \{y \in \mathbb{R}^n \mid |y'| < 2, (2l-1)\varepsilon^{1/2} < y_n < (2l+1)\varepsilon^{1/2}\}$$

for any integer l , and

$$S := \{y \in \mathbb{R}^n \mid |y'| < 2, |y_n| < 2\}.$$

Note that $Q_{2, \varepsilon^{1/2}} = S_0$. We take the even extension of $\tilde{v}$ with respect to $y_n = \varepsilon^{1/2}$ and then take the periodic extension (so that the period is equal to $4\varepsilon^{1/2}$). More precisely, we define, for any $l \in \mathbb{Z}$, a new function $\hat{v}$ by setting

$$\hat{v}(y) := \tilde{v}(y', (-1)^l(y_n - 2l\varepsilon^{1/2})) \quad \forall y \in S_l.$$

We also define the corresponding coefficients, for $k = 1, 2, \dots, n-1$,

$$\hat{a}^{nk}(y) = \hat{a}^{kn}(y) := (-1)^l \tilde{a}^{kn}(y', (-1)^l(y_n - 2l\varepsilon^{1/2})) \quad \forall y \in S_l,$$

and for other indices,

$$\hat{a}^{ij}(y) := \tilde{a}^{ij}(y', (-1)^l(y_n - 2l\varepsilon^{1/2})) \quad \forall y \in S_l.$$

Then $\hat{v}$ and $\hat{a}^{ij}$ are defined in the infinite cylinder $Q_{2,\infty}$. In particular, by using the conormal boundary conditions, it is easily seen that $\hat{v}$ satisfies the equation

$$\partial_i(\hat{a}^{ij}\partial_j\hat{v}) = 0 \quad \text{in } S.$$

By [27, Proposition 4.1], [29, Lemma 2.1] and (2.23),

$$\|\nabla\hat{v}\|_{L^\infty(\frac{1}{2}S)} \leq C\|\hat{v}\|_{L^2(S)} \leq C\varepsilon^{\tilde{\alpha}/2},$$

which implies, after reversing the changes of variables,

$$\|\nabla u\|_{L^\infty(\Omega_{\varepsilon^{1/2}})} \leq C\varepsilon^{(\tilde{\alpha}-1)/2}.$$

For any $R \in (\varepsilon^{1/2}, R_0/4)$, by Proposition 2.1 and (2.22), we have

$$\int_{Q_{4R,\varepsilon} \setminus Q_{R/2,\varepsilon}} |v - \bar{v}(0)|^2 dy \leq CR^{2\tilde{\alpha}}.$$

This implies

$$\int_{\Omega_{4R} \setminus \Omega_{R/2}} |u - \bar{v}(0)|^2 dx \leq CR^{2\tilde{\alpha}}.$$

We make a change of variables by setting

$$\begin{cases} z' = x', \\ z_n = 2R^2 \left(\frac{x_n - g(x') + \varepsilon/2}{\varepsilon + f(x') - g(x')} - \frac{1}{2} \right), \end{cases} \quad \forall (x', x_n) \in \Omega_{4R} \setminus \Omega_{R/2}.$$

This change of variables maps the domain $\Omega_{4R} \setminus \Omega_{R/2}$ to $Q_{4R,R^2} \setminus Q_{R/2,R^2}$. Let $w(z) = u(x) - \bar{v}(0)$, so that $w(z)$ satisfies

$$\begin{cases} -\partial_i(b^{ij}(z)\partial_j w(z)) = 0 & \text{in } Q_{4R,R^2} \setminus Q_{R/2,R^2}, \\ b^{nj}(z)\partial_j w(z) = 0 & \text{on } \{z_n = -R^2\} \cup \{z_n = R^2\}, \end{cases}$$

where

$$(b^{ij}(z)) = \frac{(\partial_x z)(\partial_x z)^t}{\det(\partial_x z)}.$$

It is straightforward to verify that

$$\frac{I}{C} \leq b(z) \leq CI \quad \text{and} \quad \|\nabla b\|_{L^\infty(Q_{4R,R^2} \setminus Q_{R/2,R^2})} \leq CR^{-1}.$$

We can apply the “flipping argument” as above to get

$$\|\nabla w\|_{L^\infty(Q_{2R,R^2} \setminus Q_{R,R^2})} \leq CR^{\tilde{\alpha}-1},$$

which implies

$$\|\nabla u\|_{L^\infty(\Omega_{2R} \setminus \Omega_R)} \leq CR^{\tilde{\alpha}-1}$$

for any $R \in (\varepsilon^{1/2}, R_0/4)$. Therefore, we have improved the upper bound $|\nabla u(x)| \leq C(\varepsilon + |x'|^2)^{-s_0}$ to $|\nabla u(x)| \leq C(\varepsilon + |x'|^2)^{(\tilde{\alpha}-1)/2}$, where

$$\frac{\tilde{\alpha} - 1}{2} = \min \left\{ \frac{\alpha - 1}{2}, -s_0 + \frac{\gamma}{2} \right\}.$$

If $-s_0 + \gamma/2 < (\alpha - 1)/2$, we take $s_1 = s_0 - \gamma/2$ and repeat the argument above. We may decrease γ if necessary so that

$$\frac{\alpha - 1}{2} \neq -s_0 + k \frac{\gamma}{2} \quad \text{for any } k = 1, 2, \dots$$

After repeating the argument finite number of times, we obtain estimate (1.10). $\blacksquare$

3. Optimality

In this section, we prove Theorem 1.3. We will make use of the following lemma.

Lemma 3.1. *For $\varepsilon > 0$, there exists a unique solution $h \in L^\infty((0, 1)) \cap C^\infty((0, 1])$ of*

$$Lh := h''(r) + \left(\frac{n-2}{r} + \frac{2r}{\varepsilon + r^2} \right) h'(r) - \frac{n-2}{r^2} h(r) = 0, \quad 0 < r < 1, \quad (3.1)$$

satisfying $h(1) = 1$. Moreover, $h \in C([0, 1])$, $h(0) = 0$, and for

$$\beta \geq \frac{2\alpha^2 + \alpha(n-1)}{n-3+2\alpha},$$

there exist positive constants $C(\varepsilon)$ and $C(\beta)$ such that

$$r < h(r) < \min\{C(\varepsilon)r, r^\alpha\} \quad \text{and} \quad h(r) \geq \frac{1}{C(\beta)} r^\beta (\varepsilon + r^2)^{(\alpha-\beta)/2} \quad \text{for } 0 < r < 1, \quad (3.2)$$

where α is given by (1.11) and h is strictly increasing in $[0, 1]$.

Proof. For $0 < a < 1$, let $h_a \in C^2([a, 1])$ be the solution of $Lh_a = 0$ in $(a, 1)$ satisfying $h_a(a) = a$ and $h_a(1) = 1$. Since $Lr > 0$ and $Lr^\alpha < 0$ in $(0, 1)$, by the maximum principle and the strong maximum principle,

$$r < h_a(r) < r^\alpha, \quad a < r < 1.$$

Sending $a \rightarrow 0$ along a subsequence, $h_a \rightarrow h$ in $C_{\text{loc}}^2((0, 1])$ for some $h \in C([0, 1]) \cap C^\infty((0, 1])$ satisfying $r \leq h(r) \leq r^\alpha$, $Lh = 0$ in $(0, 1)$, and $h(0) = 0$. By the strong maximum principle,

$$r < h(r) < r^\alpha, \quad 0 < r < 1.$$

Let $v = r(1 - r^{1/2}/2)$. By a direct computation,

$$Lv = -\frac{1}{4}\left(n - \frac{1}{2}\right)r^{-1/2} + \frac{1}{\varepsilon}O(r) \quad \text{as } r \rightarrow 0.$$

Hence $Lv < 0$ in $(0, r_0(\varepsilon))$, for some small $r_0(\varepsilon)$. Recall that $Lh = 0$ and $h < r^\alpha$ in $(0, r_0(\varepsilon))$, $h(0) = v(0) = 0$. By the maximum principle, we have

$$h \leq \frac{h(r_0(\varepsilon))}{v(r_0(\varepsilon))}v \leq C(\varepsilon)r \quad \text{in } (0, r_0(\varepsilon)),$$

where $C(\varepsilon) = r_0^\alpha(\varepsilon)/v(r_0(\varepsilon))$.

For $\beta \in \mathbb{R}$, let $U(r) = r^\beta(\varepsilon + r^2)^{(\alpha-\beta)/2}$. By a direct computation,

$$\begin{aligned} LU = r^{\beta-2}(\varepsilon + r^2)^{(\alpha-\beta)/2} & \left\{ (\beta - \alpha)^2 \left(\frac{r^2}{\varepsilon + r^2} \right)^2 + [(2\beta + n - 1)(\alpha - \beta) + 2\beta] \right. \\ & \left. \times \left(\frac{r^2}{\varepsilon + r^2} \right) + (n - 2 + \beta)(\beta - 1) \right\}, \quad 0 < r < 1. \end{aligned}$$

Consider the second order polynomial

$$p(x) := (\beta - \alpha)^2 x^2 + [(2\beta + n - 1)(\alpha - \beta) + 2\beta]x + (n - 2 + \beta)(\beta - 1), \quad x \in [0, 1].$$

Since $p(1) = 0$, a sufficient condition for $LU \geq 0$ in $(0, 1)$ is

$$\begin{aligned} p'(x) &= 2(\beta - \alpha)^2 x + (2\beta + n - 1)(\alpha - \beta) + 2\beta \\ &\leq 2(\beta - \alpha)^2 + (2\beta + n - 1)(\alpha - \beta) + 2\beta \leq 0. \end{aligned}$$

This is equivalent to $\beta \geq (2\alpha^2 + \alpha(n - 1))/(n - 3 + 2\alpha)$. Therefore, U is a subsolution of (3.1) when $\beta \geq (2\alpha^2 + \alpha(n - 1))/(n - 3 + 2\alpha)$. Estimate (3.2) follows.

Next we show that h is strictly increasing in $(0, 1)$. If not, there exists an $r_0 \in (0, 1)$ such that $h'(r_0) = 0$ and $h''(r_0) \leq 0$. Since $h(r_0) > 0$, we have $Lh(r_0) < 0$, a contradiction.

Finally, we show the uniqueness of h . Let $h_2 \in L^\infty((0, 1)) \cap C^\infty((0, 1])$ be a solution of (3.1) in $(0, 1)$ satisfying $h_2(1) = 1$. Then $w(r) := h_2(r)/h(r)$ satisfies

$$(Gw')' = 0, \quad 0 < r < 1,$$

where $G = h^2 r^{n-2}(\varepsilon + r^2)$. Therefore, for some constants C_0 and C_1 , we have

$$h_2(r) = h(r)w(r) = h(r) \int_r^1 \frac{C_0}{h^2(s)s^{n-2}(\varepsilon + s^2)} ds + C_1 h(r), \quad 0 < r < 1.$$

By the first inequality in (3.2),

$$h(r) \int_r^1 \frac{1}{h^2(s)s^{n-2}(\varepsilon + s^2)} ds \rightarrow +\infty$$

as $r \rightarrow 0$. Therefore, since h_2 and h are bounded, $C_0 = 0$, $C_1 = 1$, and $h_2 = h$. ■

Proof of Theorem 1.3. Step 1. By the elliptic theory, the fact that $\tilde{\Omega}$ is symmetric in x_1 , and the fact that φ is odd in x_1 , we know that u is smooth and u is odd in x_1 . In $\{(x', x_n) \in \mathbb{R}^n \mid |x'| < 1\}$, f and g can be written as

$$f(x') = \frac{|x'|^2}{2} + O(|x'|^4) \quad \text{and} \quad g(x') = -f(x') = -\frac{|x'|^2}{2} + O(|x'|^4),$$

respectively. In Ω_1 , where Ω_r is defined as in (1.6), we define

$$\bar{u}(x') = \int_{-\varepsilon/2+g(x')}^{\varepsilon/2+f(x')} u \, dx_n.$$

We perform the change of variables (2.1) with $R_0 = 1$, and let $v(y) = u(x)$. Note that $\bar{v}(y') = \bar{u}(x')$, where we define $\bar{v}$ as (2.4). By the same argument as in Section 2, we know that $\bar{v}$ satisfies equation (2.5) with $R_0 = 1$, $a^{in}(y) = 2y_n \partial_i g(y')$, and $e^i(y') = f(y') - g(y') - |y'|^2$. Therefore, by reversing the change of variables (2.1), we have

$$\int_{-\varepsilon}^{\varepsilon} y_n \partial_n v \, dy_n = \int_{-\varepsilon/2+g(x')}^{\varepsilon/2+f(x')} x_n \partial_n u \, dx_n.$$

Hence $\bar{u}$ satisfies

$$\operatorname{div}((\varepsilon + |x'|^2) \nabla \bar{u}) = \operatorname{div} F \quad \text{in } B_1 \subset \mathbb{R}^{n-1},$$

where

$$\begin{aligned} F_i &= -2\partial_i g(x') \overline{\partial_n u x_n} - e^i(x') \partial_i \bar{u} \\ &= -2(x_i + O(|x'|^3)) \overline{\partial_n u x_n} + O(|x'|^4) \partial_i \bar{u}, \end{aligned}$$

$\overline{\partial_n u x_n}$ is the average of $\partial_n u x_n$ with respect to x_n in $(-\varepsilon/2 + g(x'), \varepsilon/2 + f(x'))$. We have, by (2.6), $|x_n| \leq C(\varepsilon + |x'|^2)$, and by (1.10),

$$|F(x')| \leq C(n) |x'| (\varepsilon + |x'|^2), \quad x' \in B_1. \quad (3.3)$$

Again, we denote $Y_{k,i}$ to be a k -th degree normalized spherical harmonics so that $\{Y_{k,i}\}_{k,i}$ forms an orthonormal basis of $L^2(\mathbb{S}^{n-2})$, $Y_{1,1}$ to be the one after normalizing $x_1|_{\mathbb{S}^{n-2}}$, and $x' = (r, \xi)$. Since $\bar{u}$ is odd with respect to $x_1 = 0$, and in particular $\bar{u}(0) = 0$, we have the following decomposition:

$$\bar{u}(x') = U_{1,1}(r) Y_{1,1}(\xi) + \sum_{k=2}^{\infty} \sum_{i=1}^{N(k)} U_{k,i}(r) Y_{k,i}(\xi), \quad x' \in B_1 \setminus \{0\}, \quad (3.4)$$

where $U_{k,i}(r) = \int_{\mathbb{S}^{n-2}} \bar{u}(r, \xi) Y_{k,i}(\xi) \, d\xi$ and $U_{k,i} \in C([0, 1)) \cap C^\infty((0, 1))$. Since $\bar{u}(0) = 0$ and $\varepsilon + |x'|^2$ is independent of ξ , $U_{1,1}$ satisfies $U_{1,1}(0) = 0$ and

$$LU_{1,1} := U_{1,1}''(r) + \left(\frac{n-2}{r} + \frac{2r}{\varepsilon + r^2} \right) U_{1,1}'(r) - \frac{n-2}{r^2} U_{1,1}(r) = H(r), \quad 0 < r < 1,$$

where

$$\begin{aligned} H(r) &= \int_{\mathbb{S}^{n-2}} \frac{(\operatorname{div} F)Y_{1,1}(\xi)}{\varepsilon + r^2} d\xi = \int_{\mathbb{S}^{n-2}} \frac{\partial_r F_r + (1/r)\nabla_\xi F_\xi}{\varepsilon + r^2} Y_{1,1}(\xi) d\xi \\ &= \partial_r \left(\int_{\mathbb{S}^{n-2}} \frac{F_r}{\varepsilon + r^2} Y_{1,1}(\xi) d\xi \right) + \int_{\mathbb{S}^{n-2}} \frac{2rF_r Y_{1,1}}{(\varepsilon + r^2)^2} - \frac{F_\xi \nabla_\xi Y_{1,1}}{r(\varepsilon + r^2)} d\xi \\ &=: A'(r) + B(r), \quad 0 < r < 1, \end{aligned}$$

and $A(r), B(r) \in C^1([0, 1))$ satisfy, in view of (3.3),

$$|A(r)| \leq C(n)r, \quad |B(r)| \leq C(n), \quad 0 < r < 1. \quad (3.5)$$

Step 2. We will prove, for some constant $C_1(\varepsilon)$, that

$$U_{1,1}(r) = C_1(\varepsilon)h(r) + O(r^{1+\alpha}), \quad 0 < r < 1. \quad (3.6)$$

We use the method of reduction of order to write down a bounded solution v satisfying $Lv = H$ in $(0, 1)$, and then give an estimate $v = O(r^{1+\alpha})$.

Let $h \in C([0, 1]) \cap C^\infty((0, 1])$ be the solution of $Lh = 0$ satisfying $h(0) = 0$ and $h(1) = 1$ as in Lemma 3.1. Let $v = hw$ and

$$w(r) := \int_0^r \frac{1}{h^2(s)s^{n-2}(\varepsilon + s^2)} \int_0^s h(\tau)\tau^{n-2}(\varepsilon + \tau^2)H(\tau) d\tau ds, \quad 0 < r < 1.$$

By a direct computation,

$$Lv = L(hw) = hw'' + \left[2h' + \left(\frac{n-2}{r} + \frac{2r}{\varepsilon + r^2} \right) h \right] w' = \frac{h}{G} (Gw')' = H,$$

where $G = h^2 r^{n-2}(\varepsilon + r^2)$. By (3.5) and the fact that $h' > 0$, we can estimate

$$\begin{aligned} &\int_0^s h(\tau)\tau^{n-2}(\varepsilon + \tau^2)H(\tau) d\tau \\ &= \int_0^s h(\tau)\tau^{n-2}(\varepsilon + \tau^2)A'(\tau) d\tau + O(1)h(s)s^{n-1}(\varepsilon + s^2) \\ &= - \int_0^s h' \tau^{n-2}(\varepsilon + \tau^2)A(\tau) d\tau + O(1)h(s)s^{n-1}(\varepsilon + s^2) \\ &= O(1)s^{n-1}(\varepsilon + s^2) \int_0^s h'(\tau) d\tau + O(1)h(s)s^{n-1}(\varepsilon + s^2) \\ &= O(1)h(s)s^{n-1}(\varepsilon + s^2). \end{aligned}$$

Therefore, using (3.2),

$$|v(r)| \leq Ch(r) \int_0^r \frac{s}{h(s)} ds = O(r^{1+\alpha}), \quad 0 < r < 1.$$

Since $U_{1,1} - v$ is bounded and satisfies $L(U_{1,1} - v) = 0$ in $(0, 1)$, according to Lemma 3.1, $U_{1,1} - v = C_1(\varepsilon)h$. Hence (3.6) follows.

Step 3. We will show that $C_1(\varepsilon) > 1/C$ for some positive ε -independent constant C .

Denote

$$x = (r, \xi, x_n) \in \mathbb{R}_+ \times \mathbb{S}^{n-2} \times \mathbb{R}$$

and write (1.3) under the condition of Theorem 1.3 as the following:

$$\begin{cases} u_{rr} + \frac{n-2}{r}u_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-2}}u + u_{nn} = 0 & \text{in } B_5 \setminus \overline{(D_1 \cup D_2)}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \ i = 1, 2, \\ u = x_1 & \text{on } \partial B_5. \end{cases} \quad (3.7)$$

Let

$$\hat{u}(r, x_n) = \int_{\mathbb{S}^{n-2}} u(r, \xi, x_n) Y_{1,1}(\xi) d\xi.$$

Since u is odd in x_1 , we have

$$\hat{u}(0, x_n) = 0 \quad \text{for any } x_n.$$

Multiplying (3.7) by $Y_{1,1}(\xi)$ and integrating over $\mathbb{S}^{n-2}$, we know that $\hat{u}(r, x_n)$ satisfies

$$\begin{cases} \hat{u}_{rr} + \frac{n-2}{r}\hat{u}_r - \frac{n-2}{r^2}\hat{u} + \hat{u}_{nn} = 0 & \text{in } \hat{B}_5 \setminus \overline{(\hat{D}_1 \cup \hat{D}_2)}, \\ \frac{\partial \hat{u}}{\partial \nu} = 0 & \text{on } \partial \hat{D}_i, \ i = 1, 2, \\ \hat{u} = 0 & \text{on } \{r = 0\}, \\ \hat{u} = r & \text{on } \partial \hat{B}_5, \end{cases} \quad (3.8)$$

where

$$\begin{aligned} \hat{B}_5 &:= \{(r, x_n) \in \mathbb{R}_+ \times \mathbb{R} \mid r^2 + x_n^2 < 25\}, \\ \hat{D}_i &:= \left\{ (r, x_n) \in \mathbb{R}_+ \times \mathbb{R} \mid r^2 + \left(x_n - (-1)^i \left(1 + \frac{\varepsilon}{2} \right) \right)^2 < 1 \right\}, \end{aligned}$$

and ν is the unit inner normal of $\partial \hat{D}_i$. Clearly, $\hat{v}(r) = r$ satisfies the first line of (3.8), and $\partial \hat{v} / \partial \nu < 0$ on $\partial \hat{D}_i$, $i = 1, 2$. Thus, we know that r is a subsolution of (3.8), and therefore $\hat{u} \geq r$. Then

$$U_{1,1}(r) = \oint_{-\varepsilon/2 + g(x') < x_n < \varepsilon/2 + f(x')} \hat{u}(r, x_n) dx_n \geq r.$$

By (3.6), (3.2), and the above, we have

$$r \leq U_{1,1}(r) = C_1(\varepsilon)h(r) + O(r^{1+\alpha}) \leq C_1(\varepsilon)r^\alpha + \frac{1}{2}r \quad \forall 0 < r \leq r_0,$$

where r_0 is a small constant independent of ε , which implies that

$$C_1(\varepsilon) \geq \frac{1}{2}r_0^{1-\alpha}.$$

Step 4. Completion of the proof of Theorem 1.3.

It follows, in view of (3.6), step 3, and (3.2), that there exists some positive constant r_0 independent of ε such that

$$U_{1,1}(r) \geq \frac{1}{C}h(r) + O(r^{1+\alpha}) \geq \frac{1}{2C}h(r), \quad 0 < r \leq r_0. \quad (3.9)$$

By (3.2),

$$h(r) \geq \frac{1}{C}r^\beta(\varepsilon + r^2)^{(\alpha-\beta)/2} \quad \text{for } \beta = \frac{2\alpha^2 + \alpha(n-1)}{n-3+\alpha}. \quad (3.10)$$

By (3.4), (3.9), and (3.10), we have

$$\left(\int_{\mathbb{S}^{n-2}} |\bar{u}(\sqrt{\varepsilon}, \xi)|^2 d\xi \right)^{1/2} \geq |U_{1,1}(\sqrt{\varepsilon})| \geq \frac{1}{C}h(\sqrt{\varepsilon}) \geq \frac{1}{C}\varepsilon^{\alpha/2}.$$

Then, there exists a $\xi_0 \in \mathbb{S}^{n-2}$ such that $|\bar{u}(\sqrt{\varepsilon}, \xi_0)| \geq \varepsilon^{\alpha/2}/C$. Since $\bar{u}$ is the average of u in the x_n direction, there exists an x_n such that

$$|u(\sqrt{\varepsilon}, \xi_0, x_n)| \geq \frac{1}{C}\varepsilon^{\alpha/2}. \quad (3.11)$$

Estimate (1.13) follows from (3.11) and $u(0) = 0$. ■

Funding. H. Dong is partially supported by Simons Fellows Award 007638, NSF Grant DMS-2055244, and the Charles Simonyi Endowment at the Institute of Advanced Study. Y. Y. Li is partially supported by NSF Grants DMS-1501004, DMS-2000261, and Simons Fellows Award 677077. Z. Yang is partially supported by NSF Grants DMS-1501004, DMS-2000261, and the Simons Bridge Postdoctoral Fellowship at Brown University.

References

- [1] Ammari, H., Ciraolo, G., Kang, H., Lee, H., Yun, K.: [Spectral analysis of the Neumann–Poincaré operator and characterization of the stress concentration in anti-plane elasticity](#). Arch. Ration. Mech. Anal. **208**, 275–304 (2013) Zbl 1320.35165 MR 3021549
- [2] Ammari, H., Davies, B., Yu, S.: [Close-to-touching acoustic subwavelength resonators: Eigenfrequency separation and gradient blow-up](#). Multiscale Model. Simul. **18**, 1299–1317 (2020) Zbl 1459.35091 MR 4128998
- [3] Ammari, H., Kang, H., Lee, H., Lee, J., Lim, M.: [Optimal estimates for the electric field in two dimensions](#). J. Math. Pures Appl. (9) **88**, 307–324 (2007) Zbl 1136.35095 MR 2384571
- [4] Ammari, H., Kang, H., Lim, M.: [Gradient estimates for solutions to the conductivity problem](#). Math. Ann. **332**, 277–286 (2005) Zbl 1129.78308 MR 2178063
- [5] Babuška, I., Andersson, B., Smith, P. J., Levin, K.: [Damage analysis of fiber composites. Part I: Statistical analysis on fiber scale](#). Comput. Methods Appl. Mech. Engrg. **172**, 27–77 (1999) Zbl 0956.74048 MR 1685902
- [6] Bao, E. S., Li, Y. Y., Yin, B.: [Gradient estimates for the perfect conductivity problem](#). Arch. Ration. Mech. Anal. **193**, 195–226 (2009) Zbl 1173.78002 MR 2506075
- [7] Bao, E. S., Li, Y. Y., Yin, B.: [Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions](#). Comm. Partial Differential Equations **35**, 1982–2006 (2010) Zbl 1218.35230 MR 2754076

- [8] Bao, J. G., Li, H. G., Li, Y. Y.: [Gradient estimates for solutions of the Lamé system with partially infinite coefficients](#). Arch. Ration. Mech. Anal. **215**, 307–351 (2015) Zbl 1309.35160 MR 3296149
- [9] Bao, J. G., Li, H. G., Li, Y. Y.: [Gradient estimates for solutions of the Lamé system with partially infinite coefficients in dimensions greater than two](#). Adv. Math. **305**, 298–338 (2017) Zbl 1353.35079 MR 3570137
- [10] Bonnetier, E., Triki, F.: [Pointwise bounds on the gradient and the spectrum of the Neumann–Poincaré operator: The case of 2 discs](#). In: Multi-scale and high-contrast PDE: from modelling, to mathematical analysis, to inversion, Contemp. Math. 577, American Mathematical Society, Providence, RI, 81–91 (2012) Zbl 1328.35126 MR 2985067
- [11] Bonnetier, E., Triki, F.: [On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2D](#). Arch. Ration. Mech. Anal. **209**, 541–567 (2013) Zbl 1280.49029 MR 3056617
- [12] Bonnetier, E., Vogelius, M.: [An elliptic regularity result for a composite medium with “touching” fibers of circular cross-section](#). SIAM J. Math. Anal. **31**, 651–677 (2000) Zbl 0947.35044 MR 1745481
- [13] Budiansky, B., Carrier, G. F.: [High shear stresses in stiff-fiber composites](#). J. Appl. Mech. **51**, 733–735 (1984) Zbl 0549.73061
- [14] Capdeboscq, Y., Yang Ong, S. C.: [Quantitative Jacobian determinant bounds for the conductivity equation in high contrast composite media](#). Discrete Contin. Dyn. Syst. Ser. B **25**, 3857–3887 (2020) Zbl 1454.35092 MR 4147367
- [15] Dong, H., Li, H.: [Optimal estimates for the conductivity problem by Green’s function method](#). Arch. Ration. Mech. Anal. **231**, 1427–1453 (2019) Zbl 1412.35082 MR 3902466
- [16] Dong, H., Zhang, H.: [On an elliptic equation arising from composite materials](#). Arch. Ration. Mech. Anal. **222**, 47–89 (2016) Zbl 1351.35041 MR 3519966
- [17] Gorb, Y.: [Singular behavior of electric field of high-contrast concentrated composites](#). Multi-scale Model. Simul. **13**, 1312–1326 (2015) Zbl 1333.35266 MR 3418221
- [18] Ji, Y.-G., Kang, H.: [Spectrum of the Neumann–Poincaré operator and optimal estimates for transmission problems in the presence of two circular inclusions](#). Int. Math. Res. Not. IMRN **2023**, 7638–7685 (2023) Zbl 1519.35087 MR 4584711
- [19] Kang, H.: [Quantitative analysis of field concentration in presence of closely located inclusions of high contrast](#). In: Proceedings of the International Congress of Mathematicians (July 6–14, 2022), Proc. Int. Cong. Math. 7, European Mathematical Society Press, Berlin, 5680–5699 (2023)
- [20] Kang, H., Lim, M., Yun, K.: [Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities](#). J. Math. Pures Appl. (9) **99**, 234–249 (2013) Zbl 1411.74047 MR 3007847
- [21] Kang, H., Lim, M., Yun, K.: [Characterization of the electric field concentration between two adjacent spherical perfect conductors](#). SIAM J. Appl. Math. **74**, 125–146 (2014) Zbl 1293.35314 MR 3162415
- [22] Keller, J. B.: [Stresses in narrow regions](#). J. Appl. Mech. **60**, 1054–1056 (1993) Zbl 0803.73042
- [23] Kim, J., Lim, M.: [Electric field concentration in the presence of an inclusion with eccentric core-shell geometry](#). Math. Ann. **373**, 517–551 (2019) Zbl 1416.78007 MR 3968879
- [24] Li, H. G.: [Asymptotics for the electric field concentration in the perfect conductivity problem](#). SIAM J. Math. Anal. **52**, 3350–3375 (2020) Zbl 1447.35140 MR 4126320
- [25] Li, H. G., Li, Y. Y., Yang, Z.: [Asymptotics of the gradient of solutions to the perfect conductivity problem](#). Multiscale Model. Simul. **17**, 899–925 (2019) Zbl 1426.78022 MR 3977105
- [26] Li, H. G., Wang, F., Xu, L.: [Characterization of electric fields between two spherical perfect conductors with general radii in 3D](#). J. Differential Equations **267**, 6644–6690 (2019) Zbl 1428.35557 MR 4001067

- [27] Li, Y. Y., Nirenberg, L.: [Estimates for elliptic systems from composite material](#). *Comm. Pure Appl. Math.* **56**, 892–925 (2003) [Zbl 1125.35339](#) [MR 1990481](#)
- [28] Li, Y. Y., Vogelius, M.: [Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients](#). *Arch. Ration. Mech. Anal.* **153**, 91–151 (2000) [Zbl 0958.35060](#) [MR 1770682](#)
- [29] Li, Y. Y., Yang, Z.: [Gradient estimates of solutions to the insulated conductivity problem in dimension greater than two](#). *Math. Ann.* **385**, 1775–1796 (2023) [Zbl 1511.35052](#) [MR 4566706](#)
- [30] Lim, M., Yun, K.: [Blow-up of electric fields between closely spaced spherical perfect conductors](#). *Comm. Partial Differential Equations* **34**, 1287–1315 (2009) [Zbl 1188.78011](#) [MR 2581974](#)
- [31] Markenscoff, X.: [Stress amplification in vanishingly small geometries](#). *Comput. Mech.* **19**, 77–83 (1996) [Zbl 0895.73004](#)
- [32] Weinkove, B.: [The insulated conductivity problem, effective gradient estimates and the maximum principle](#). *Math. Ann.* **385**, 1–16 (2023) [Zbl 1518.35279](#) [MR 4542709](#)
- [33] Yun, K.: [Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape](#). *SIAM J. Appl. Math.* **67**, 714–730 (2007) [Zbl 1189.35324](#) [MR 2300307](#)
- [34] Yun, K.: [Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections](#). *J. Math. Anal. Appl.* **350**, 306–312 (2009) [Zbl 1198.35042](#) [MR 2476915](#)
- [35] Yun, K.: [An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions](#). *J. Differential Equations* **261**, 148–188 (2016) [Zbl 1382.35095](#) [MR 3487255](#)