J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1431

Published by EMS Press

Jean-François Delmas · Dylan Dronnier · Pierre-André Zitt

The effective reproduction number: Convexity, concavity and invariance

Received 10 November 2021; revised 25 November 2022

Abstract. Motivated by the question of optimal vaccine allocation strategies in heterogeneous population for epidemic models, we study various properties of the *effective reproduction number*. In the simplest case, given a fixed non-negative matrix K, this corresponds mathematically to the study of the spectral radius $R_e(\eta)$ of the matrix product $\mathrm{Diag}(\eta)K$, as a function of $\eta \in \mathbb{R}_+^n$. The matrix K and the vector η can be interpreted as a next-generation operator and a vaccination strategy. This can be generalized in an infinite-dimensional case where the matrix K is replaced by a positive integral compact operator, which is composed with a multiplication by a non-negative function η. We give sufficient conditions for the function R_e to be convex or a concave. Eventually, we provide equivalence properties on models which ensure that the function R_e is unchanged.

Keywords: integral operator, vaccination strategy, effective reproduction number.

1. Introduction

1.1. The mathematical question

For $p \in [1, +\infty]$, we consider the Lebesgue space L^p, with its usual norm $\|\cdot\|_p$, on a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$. We denote by $\|\cdot\|_{L^p}$ the operator norm on the Banach space of bounded operators from L^p to L^p. For a bounded operator T on L^p, we denote by $\rho(T) = \lim_{n\to\infty} \|T^n\|_{L^p}^{1/n}$ its spectral radius. We recall that an operator T on L^p is positive if $T(L^p_+) \subset L^p_+$, where L^p_+ denotes the set of non-negative functions in L^p. For $h \in L^\infty_+$, let $M_h\colon f \mapsto hf$ denote the bounded operator on L^p.

Jean-François Delmas: CERMICS, École des Ponts ParisTech, 6 et 8 avenue Blaise Pascal, Cité Descartes – Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France; jean-francois.delmas@enpc.fr

Dylan Dronnier: CERMICS, École des Ponts ParisTech, 6 et 8 avenue Blaise Pascal, Cité Descartes – Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France; dylan.dronnier@enpc.fr

Pierre-André Zitt: LAMA, Université Gustave Eiffel, 5 Boulevard Descartes, Cité Descartes – Champs-sur-Marne, 77454 Marne-la-Vallée Cedex 2, France; pierre-andre.zitt@univ-eiffel.fr

Mathematics Subject Classification 2020: 92D30 (primary); 47B34, 47A25, 58E17, 34D20 (secondary).

According to the Krein–Rutman theorem, if T is a positive compact operator on L^p such that $\rho(T)$ is positive, then $\rho(T)$ is also an eigenvalue. For such an operator, we define the map $R_e[T]$ on L^∞_+ by

$$R_e[T](h) = \rho(TM_h). \tag{1}$$

By homogeneity of the spectral radius, for the study of the map $R_e[T]$, it is enough to consider this map only on the subset $\Delta \subset L^\infty_+$ of non-negative measurable functions bounded by 1. Our aim is to provide sufficient conditions on T for the map $R_e[T]$ to be convex or concave on Δ. We briefly explain in the next section how this question is related to the optimal vaccination problem in epidemic models.

1.2. The epidemic motivation

In finite metapopulation models, the population is divided into $N \geq 2$ different sub-populations; this amounts to considering the discrete state space $\Omega_\mathrm{d} = \{1, \ldots, N\}$. Following [22], the entry K_{ij} of the so-called next-generation matrix K is equal to the expected number of secondary infections for people in subgroup i resulting from a single randomly selected non-vaccinated infectious person in subgroup j. The matrix K has non-negative entries and represents the compact positive operator T. Let $\eta \in \Delta = [0, 1]^N$ represent a vaccination strategy, that is, η_i is the fraction of non-vaccinated individuals in the i-th sub-population; thus $\eta_i = 0$ when the i-th sub-population is fully vaccinated, and 1 when it is not vaccinated at all – this seemingly unnatural convention is in particular motivated by the simple form of equation (1). So, the strategy $\mathbb{1} \in \Delta$, with all its entries equal to 1, corresponds to an entirely non-vaccinated population.

The *effective reproduction number* $R_e[K](\eta)$ associated to the vaccination strategy η is then the spectral radius of the matrix $K \cdot \mathrm{Diag}(\eta)$:

$$R_e[K](\eta) = \rho(K \cdot \mathrm{Diag}(\eta)), \tag{2}$$

where $\mathrm{Diag}(\eta)$ is the diagonal matrix with diagonal entries η. It may be interpreted as the mean number of infections coming from a typical case in the SIS model (where S and I stand for susceptible and infected). In particular, we denote by $R_0 = R_e[K](\mathbb{1})$ the so-called *basic reproduction number* associated to the metapopulation epidemiological model, see Lajmanovich and Yorke [26]. Let us mention that in this model if $R_0 \leq 1$, then there is no endemic equilibrium (i.e., the epidemic vanishes asymptotically), whereas if $R_0 > 1$, there exists at least one non-trivial endemic equilibrium (which means that the epidemic is persistent). With the interpretation of the function R_e in mind, it is then very natural to minimize it under a constraint on the cost of the vaccination strategies η. This constrained optimization problem appears in most of the literature for designing efficient vaccination strategies for multiple epidemic situation (SIS/SIR/SEIR) [7,10,15,16,22,29,31,38]. Note that in some of these references, the effective reproduction number is defined as the spectral radius of the matrix $\mathrm{Diag}(\eta) \cdot K$. Since the eigenvalues of $\mathrm{Diag}(\eta) \cdot K$ are exactly the eigenvalues of the matrix $K \cdot \mathrm{Diag}(\eta)$, this actually defines the same function $R_e[K]$.

Given the importance of convexity to solve optimization problems efficiently, it is natural to look for conditions on the matrix K that imply convexity or concavity for the map $R_e[K]$ defined by (2). Those properties can be useful to design vaccination strategies in the best possible way; see the companion paper [12].

1.3. *The finite-dimensional case*

In their investigation of the behavior of the map $R_e[K]$ defined in (2), Hill and Longini conjectured in [22] sufficient spectral conditions to get either concavity or convexity. More precisely, guided by explicit examples, they state that $R_e[K]$ should be convex if all the eigenvalues of K are non-negative real numbers, and that it should be concave if all eigenvalues are real, with only one positive eigenvalue.

Our first series of results show that, while this conjecture cannot hold in full generality (see Section 4.1), it is true under an additional symmetry hypothesis. Recall that a matrix K is called diagonally symmetrizable if there exist positive numbers $(d_1, \ldots, d_N)$ such that for all i, j, $d_i K_{ij} = d_j K_{ji}$. Such a matrix is diagonalizable with real eigenvalues according to the spectral theorem for symmetric matrices. The following result, which appears below in the text as Theorem 4.1, settles the conjecture for diagonally symmetrizable matrices. Let us mention that the eigenvalue λ_1 in the theorem below is non-negative and is equal to the spectral radius of K, that is, $\lambda_1 = R_e[K](\mathbb{1}) = R_0$, thanks to the Perron–Frobenius theory. We consider the function $R_e = R_e[K]$ defined on $[0, 1]^N$.

Theorem 1.1. *Let K be an $N \times N$ matrix with non-negative entries. Suppose that K is diagonally symmetrizable with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.*

(i) *If $\lambda_N \geq 0$, then the function R_e is convex.*

(ii) *If $\lambda_2 \leq 0$, then the function R_e is concave.*

Note that case (i) appears already in Cairns [7]; see Section 4.1 below for a detailed comparison with existing results. This completes results on log-convexity of the map $R_e[K]$ given in [17, 19]. Notice also that if K and K' are diagonally similar up to transposition, they define the same function R_e; see [13] for more results in this direction. Eventually, the concavity of the map $R_e[K]$ implies that K has a unique irreducible component in its Perron–Frobenius diagonalization as shown in Lemma 5.10 below.

Let us stress that diagonally symmetrizable next-generation matrices appear in a wide variety of models used in the literature. Typically, the next-generation matrix is expressed as a product of a diagonal matrix giving the activity level per group and a mixing matrix that satisfies the conditions of Busenberg and Castillo-Chavez [5]. With this form, the next-generation matrix is indeed diagonally symmetrizable.

1.4. *The general case*

We now give our main result in the setting of Section 1.1. We give in Definition 4.2 an extension to the notion of "diagonally symmetrizable" for compact operators. For example, according to Proposition 4.9, if T' is a self-adjoint compact operator on L^2 and f, g

are two non-negative measurable functions defined on Ω bounded and bounded away from 0, then the operator $T = M_f T' M_g$ is a compact diagonally symmetrizable on L^2. In particular, Corollary 4.8 states that diagonally symmetrizable compact operators on L^p, with $p \in [1, +\infty)$, have a real spectrum.

For a compact operator T, let $\mathrm{p}(T)$ (resp. $\mathrm{n}(T)$) denote the number of eigenvalues with positive (resp. negative) real part taking into account their (algebraic) multiplicity. Then, we obtain the following result given in Theorem 4.10 below.

Theorem 1.2 (Convexity/concavity of R_e). *Let T be a positive compact diagonally symmetrizable operator on L^p with $p \in [1, +\infty)$. We consider the function $R_e = R_e[T]$ defined on Δ.*

(i) *If* $\mathrm{n}(T) = 0$, *then the function R_e is convex.*

(ii) *If* $\mathrm{p}(T) = 1$, *then the function R_e is concave.*

The proof of the concavity property relies on the explicit expression of the second derivative of $R_e[T]$ when T is self-adjoint and on the Sylvester's inertia theorem.

The concavity property of $R_e[T]$ implies a strong structural property on the operator T. In order to establish this result, we present in Section 5 an atomic decomposition of the space Ω related to the operator T following [34]. In particular, we extend the notion of quasi-irreducible operator to the non-self-adjoint case and say an operator is monatomic if it has only one non-trivial irreducible component; see Definition 5.5 in Section 5.2. If T is a positive compact operator on L^p for some $p \in [1, +\infty)$ with $R_0 = R_e[T](\mathbb{1}) > 0$, where $\mathbb{1} \in \Delta$ is the constant function equal to 1, then we have the following properties:

(i) If $R_e[T]$ is concave, then T is monatomic according to Lemma 5.10.

(ii) If $\mathrm{p}(T) = 1$, then R_0 is simple and the only eigenvalue in $\mathbb{R}_+^*$, and thus T is monatomic according to Lemma 5.9.

(iii) More generally, using the decomposition from Lemma 5.3, we get that if $\mathrm{Spec}(T) \subset \mathbb{R}_- \cup \{R_0\}$ and T is a diagonally symmetrizable operator, then the function R_e is the maximum of m concave functions which are non-zero on m pairwise disjoint subsets of Δ, where m is the (algebraic) multiplicity of R_0.

Considering a general positive compact operator T on L^p for some $p \in [1, +\infty)$, by [34], we provide in Corollary 5.4 the decomposition $R_e[T]$ on the irreducible atoms,

$$R_e[T] = \max_{i \in I} R_e[T_i],$$

where $T_i(\cdot) = \mathbb{1}_{\Omega_i} T(\mathbb{1}_{\Omega_i} \cdot)$ with $(\Omega_i, i \in I)$ the at most countable collection of irreducible atoms in Ω associated to T.

1.5. Structure of the paper

After recalling the mathematical framework in Section 2, we discuss invariance properties of R_e in Section 3. The convexity properties of R_e and the related conjecture of Hill and Longini are discussed in Section 4. Finally, the case of reducible operators is treated in Section 5, using the Frobenius decomposition from [34].

2. Setting, notations and previous results

2.1. Spaces, operators, spectra

All metric spaces (S, d) are endowed with their Borel σ-field denoted by $\mathcal{B}(S)$. The set $\mathcal{K}$ of compact subsets of $\mathbb{C}$ endowed with the Hausdorff distance d_{H} is a metric space, and the function rad from $\mathcal{K}$ to $\mathbb{R}_+$ defined by $\mathrm{rad}(K) = \max\{|\lambda|, \lambda \in K\}$ is Lipschitz continuous from $(\mathcal{K}, d_{\mathrm{H}})$ to $\mathbb{R}$ endowed with its usual Euclidean distance.

Let $(\Omega, \mathcal{F}, \mu)$ be a measured space, with a σ-finite (positive and non-zero) measure μ. For real-valued functions f and g defined on Ω, we may write $\langle f, g\rangle$ or $\int_\Omega fg \,\mathrm{d}\mu$ for $\int_\Omega f(x)g(x)\mu(\mathrm{d}x)$ whenever the latter is meaningful. For $p \in [1, +\infty]$, we denote by $L^p = L^p(\mu) = L^p(\Omega, \mu)$ the space of real-valued measurable functions g defined on Ω such that $\|g\|_p = (\int |g|^p \,\mathrm{d}\mu)^{1/p}$ (with the convention that $\|g\|_\infty$ is the μ-essential supremum of $|g|$) is finite, where functions which agree μ-a.e. are identified. We denote by L^p_+ the subset of L^p of non-negative functions. We define Δ as the subset of L^∞ of $[0, 1]$-valued measurable functions defined on Ω. We denote by $\mathbb{1}$ (resp. $\mathbb{0}$) the constant function on Ω equal to 1 (resp. 0).

Let $(E, \|\cdot\|)$ be a complex Banach space. We denote by $\|\cdot\|_E$ the operator norm on $\mathcal{L}(E)$, the Banach algebra of bounded operators. The spectrum $\mathrm{Spec}(T)$ of $T \in \mathcal{L}(E)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda\,\mathrm{Id}$ does not have a bounded inverse operator, where Id is the identity operator on E. Recall that $\mathrm{Spec}(T)$ is a compact subset of $\mathbb{C}$, and that the spectral radius of T is given by

$$\rho(T) = \mathrm{rad}(\mathrm{Spec}(T)) = \lim_{n\to\infty} \|T^n\|_E^{1/n}.$$

The element $\lambda \in \mathrm{Spec}(T)$ is an eigenvalue if there exists $x \in E$ such that $Tx = \lambda x$ and $x \neq 0$. Following [25], we define the (algebraic) multiplicity of $\lambda \in \mathbb{C}$ by

$$\mathrm{m}(\lambda, T) = \dim\Big(\bigcup_{k\in\mathbb{N}^*} \ker(T - \lambda\,\mathrm{Id})^k\Big),$$

so that λ is an eigenvalue if $\mathrm{m}(\lambda, T) \geq 1$. We say the eigenvalue λ of T is *simple* if $\mathrm{m}(\lambda, T) = 1$.

If E is also an algebra of functions, for $g \in E$, we denote by M_g the multiplication operator (possibly unbounded) defined by $M_g(h) = gh$ for all $h \in E$; if furthermore g is the indicator function of a set A, we simply write M_A for $M_{\mathbb{1}_A}$.

2.2. Invariance and continuity of the spectrum for compact operators

We collect some known results on the spectrum and multiplicity of eigenvalues related to compact operators. Let $(E, \|\cdot\|)$ be a complex Banach space. Let $A \in \mathcal{L}(E)$. We denote by $A^\top$ the adjoint of A. A sequence $(A_n, n \in \mathbb{N})$ of elements of $\mathcal{L}(E)$ converges strongly to $A \in \mathcal{L}(E)$ if $\lim_{n\to\infty} \|A_n x - Ax\| = 0$ for all $x \in E$. Following [1], a set of operators $\mathcal{A} \subset \mathcal{L}(E)$ is *collectively compact* if the set $\{Ax : A \in \mathcal{A}, \|x\| \leq 1\}$ is relatively compact. Recall that the spectrum of a compact operator is finite or countable and has at most one accumulation point, which is 0. Furthermore, 0 belongs to the spectrum of compact operators in infinite dimension.

We refer to [33] for an introduction to Banach lattices and positive operators; we shall only consider the real Banach lattices $L^p = L^p(\Omega, \mu)$ for $p \in [1, +\infty]$ on a measured space $(\Omega, \mathcal{F}, \mu)$ with a σ-finite positive non-zero measure, as well as their complex extension. (Recall that the norm of an operator on L^p or its natural complex extension is the same, see [18, Corollary 1.3].) A bounded operator A on L^p is positive if $A(L^p_+) \subset L^p_+$.

We say that two complex Banach spaces $(E, \|\cdot\|)$ and $(E', \|\cdot\|')$ are compatible if $(E \cap E', \|\cdot\| + \|\cdot\|')$ is a Banach space, and $E \cap E'$ is dense in E and in E'. Given two compatible spaces E and E', two operators $A \in \mathcal{L}(E)$ and $A' \in \mathcal{L}(E')$ are said to be consistent if, with $E'' = E \cap E'$, $A(E'') \subset E''$, $A'(E'') \subset E''$ and $Ax = A'x$ for all $x \in E''$.

Lemma 2.1 (Spectral properties). *Let A, B be elements of $\mathcal{L}(E)$.*

(i) *If E is a Banach lattice, and if A, B and $A - B$ are positive operators, then we have*
$$\rho(A) \geq \rho(B).$$

(ii) *If A is compact, then $A^\top$, AB and BA are compact and we have*
$$\mathrm{Spec}(A) = \mathrm{Spec}(A^\top) \quad \textit{and} \quad \mathrm{m}(\lambda, A) = \mathrm{m}(\lambda, A^\top) \quad \textit{for } \lambda \in \mathbb{C}^*, \tag{3}$$
$$\mathrm{Spec}(AB) = \mathrm{Spec}(BA) \quad \textit{and} \quad \mathrm{m}(\lambda, AB) = \mathrm{m}(\lambda, BA) \quad \textit{for } \lambda \in \mathbb{C}^*, \tag{4}$$

and in particular,
$$\rho(AB) = \rho(BA). \tag{5}$$

(iii) *Let $(E', \|\cdot\|')$ be a complex Banach space and $A' \in \mathcal{L}(E')$ such that $(E, \|\cdot\|)$ and $(E', \|\cdot\|')$ are compatible, and A and A' are consistent. If A and A' are compact, then we have*
$$\mathrm{Spec}(A) = \mathrm{Spec}(A') \quad \textit{and} \quad \mathrm{m}(\lambda, A) = \mathrm{m}(\lambda, A') \quad \textit{for } \lambda \in \mathbb{C}^*.$$

(iv) *Let $(A_n, n \in \mathbb{N})$ be a collectively compact sequence which converges strongly to A. Then, we have $\lim_{n\to\infty} \mathrm{Spec}(A_n) = \mathrm{Spec}(A)$ in $(\mathcal{K}, d_{\mathrm{H}})$, $\lim_{n\to} \rho(A_n) = \rho(A)$ and for $\lambda \in \mathrm{Spec}(A) \cap \mathbb{C}^*$, $r > 0$ such that $\lambda' \in \mathrm{Spec}(A)$ and $|\lambda - \lambda'| \leq r$ implies $\lambda = \lambda'$, and all n large enough:*
$$\mathrm{m}(\lambda, A) = \sum_{\lambda' \in \mathrm{Spec}(A_n),\, |\lambda - \lambda'| \leq r} \mathrm{m}(\lambda', A_n). \tag{6}$$

Proof. Property (i) can be found in [28, Theorem 4.2]. Property (iii) is in [8, Theorem 4.2.15].

Equation (3) from property (ii) can be deduced from [25, p. 20, Theorem]. Using [25, p. 25, Proposition], we get the second part of (4) and $\mathrm{Spec}(AB) \cap \mathbb{C}^* = \mathrm{Spec}(BA) \cap \mathbb{C}^*$, and thus (5) holds. To get the first part of (4), we only need to consider if 0 belongs to the spectrum or not. We first consider the infinite-dimensional case: as A is compact, we get that AB and BA are compact, thus 0 belongs to their spectrum. We then consider the finite-dimensional case: as $\det(AB) = \det(A)\det(B) = \det(BA)$, where A and B denote also the matrix of the corresponding operator in a given base, we get that 0 belongs to the spectrum of AB if and only if it belongs to the spectrum of BA.

We eventually check property (iv). We deduce from [1, Theorems 4.8 and 4.16] (see also (d), (g) [take care that $d(\lambda, K)$ therein is the algebraic multiplicity of λ for the compact operator K and not the geometric multiplicity] and (e) in [2, Section 3]) that $\lim_{n\to\infty} \mathrm{Spec}(A_n) = \mathrm{Spec}(A)$ and (6). Then use that the function rad is continuous to deduce the convergence of the spectral radius from the convergence of the spectra. ∎

We complete this section with an example of compatible Banach spaces. According to [8, p. 49, Problem 2.2.9], the spaces $L^p(\mu)$ are compatible for all $p \in [1, +\infty)$. We shall use the following slightly more general result. We recall that two σ-finite measures on $(\Omega, \mathcal{F})$, say μ and ν, are mutually absolutely continuous if for $A \in \mathcal{F}$, we have $\mu(A) = 0 \Leftrightarrow \nu(A) = 0$. Due to the Radon–Nikodym theorem, the σ-finite measures μ and ν are mutually absolutely continuous if and only if there exists a positive finite measurable function h such that $\mathrm{d}\nu = h\,\mathrm{d}\mu$.

Lemma 2.2 (Compatibility of L^p spaces). *Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$ which are mutually absolutely continuous, and let $p, r \in [1, +\infty)$. Then, the spaces $L^p(\mu)$ and $L^r(\nu)$ are compatible.*

Proof. First note that a property is true μ-a.e. if and only if it is true ν-a.e. since μ and ν are mutually absolutely continuous. Hence, we shall simply write that the property is true a.e. in this case.

Let us prove that $L^p(\mu) \cap L^r(\nu)$ is dense in $L^p(\mu)$. Let $f \in L^r(\nu)$ such that $f > 0$ a.e. For any $g \in L^p_+(\mu)$, note that the non-decreasing sequence $(\min(g, nf), n \in \mathbb{N})$ of elements of $L^p(\mu) \cap L^r(\nu)$ converges towards g a.e.; and so, it converges in $L^p(\mu)$ according to the dominated convergence theorem. This gives $L^p(\mu) \cap L^r(\nu)$ is dense in $L^p(\mu)$ and in $L^r(\nu)$ by symmetry.

To prove that $L^p(\mu) \cap L^r(\nu)$ is complete (with respect to the norm given by the sum of the norms in $L^p(\mu)$ and $L^r(\nu)$), it is enough to check that if a sequence $(h_n, n \in \mathbb{N})$ converges to g in $L^p(\mu)$ and to f in $L^r(\nu)$, then $g = f$ a.e. This is immediate: for such a sequence, one can extract a sub-sequence which converges to g a.e. and to f a.e. ∎

2.3. The effective reproduction number R_e

For $p \in [1, +\infty)$ and $\eta \in \Delta$, the multiplication operator M_η is bounded, and if T is a compact operator on L^p, then so is TM_η. Following [10], where only integral operators were considered, and keeping similar notations, we define the *reproduction number,* associated to the positive compact operator T (on L^p for some $p \in [1, +\infty)$) as its spectral radius,

$$R_0[T] = \rho(T),$$

the *effective spectrum* function $\mathrm{Spec}[T]$ from Δ to $\mathcal{K}$ by

$$\mathrm{Spec}[T](\eta) = \mathrm{Spec}(TM_\eta),$$

and the *effective reproduction number* function $R_e[T] = \mathrm{rad} \circ \mathrm{Spec}[T]$ from Δ to $\mathbb{R}_+$ by

$$R_e[T](\eta) = \mathrm{rad}(\mathrm{Spec}(TM_\eta)) = \rho(TM_\eta).$$

Take care that

$$\mathrm{Spec}(T) = \mathrm{Spec}[T](\mathbb{1}) \quad \text{and} \quad R_0[T] = R_e[T](\mathbb{1}).$$

When there is no risk of confusion on the positive compact operator T, then we simply write R_e and R_0 for the function $R_e[T]$ and the number $R_0[T]$. We have the following immediate properties for the function $R_e[T]$ (use Lemma 2.1 (i) for the third property).

Proposition 2.3 (Elementary properties of R_e). *The function $R_e = R_e[T]$, where T is a positive compact operator on L^p with $p \in [1, +\infty)$ satisfies the following properties:*

(i) $R_e(\eta_1) = R_e(\eta_2)$ *if* $\eta_1 = \eta_2$, μ *a.s., and* $\eta_1, \eta_2 \in \Delta$,

(ii) $R_e(\mathbb{0}) = 0$ *and* $R_e(\mathbb{1}) = R_0$,

(iii) $R_e(\eta_1) \leq R_e(\eta_2)$ *for all* $\eta_1, \eta_2 \in \Delta$ *such that* $\eta_1 \leq \eta_2$,

(iv) $R_e(\lambda\eta) = \lambda R_e(\eta)$, *for all* $\eta \in \Delta$ *and* $\lambda \in [0, 1]$.

We shall use the following continuity property of the spectrum; see also [10, Proposition 3.6] for stronger results when considering integral operators and the weak topology on Δ.

Lemma 2.4 (Continuity of the spectrum). *Let T be a compact operator on L^p with $p \in [1, +\infty)$. Let $(v_n, n \in \mathbb{N})$ and $(w_n, n \in \mathbb{N})$ be two bounded sequences in L^∞ which converge to v_∞ and w_∞, respectively, and let $T_n = M_{v_n} T M_{w_n}$. Then for any $\eta \in \Delta$, as n goes to infinity, we have that*

(i) $\mathrm{Spec}[T_n](\eta)$ *converges to* $\mathrm{Spec}[T_\infty](\eta)$ *in* $\mathcal{K}$,

(ii) $R_e[T_n](\eta)$ *converges to* $R_e[T_\infty](\eta)$ *in* $\mathbb{R}$,

(iii) *for any* $\lambda \in \mathrm{Spec}(T_\infty M_\eta) \cap \mathbb{C}^*$ *and any* $r > 0$ *such that* $\lambda' \in \mathrm{Spec}(T_\infty M_\eta)$ *and* $|\lambda - \lambda'| \leq r$ *implies* $\lambda = \lambda'$, *then for all n large enough,*

$$\mathrm{m}(\lambda, T_\infty M_\eta) = \sum_{\lambda' \in \mathrm{Spec}(T_n M_\eta),\, |\lambda - \lambda'| \leq r} \mathrm{m}(\lambda', T_n M_\eta).$$

Proof. Set $T'_n = TM_{\eta v_n w_n}$ for $n \in \bar{\mathbb{N}}$, where $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. Using Lemma 2.1 (ii) for the second equality, we have that for $n \in \bar{\mathbb{N}}$,

$$\mathrm{Spec}[T_n](\eta) = \mathrm{Spec}(M_{v_n} T M_{\eta w_n}) = \mathrm{Spec}(TM_{\eta v_n w_n}) = \mathrm{Spec}(T'_n),$$

and similarly for the multiplicity. Notice the set of functions $\Delta' = \{\eta v_n w_n : \eta \in \Delta$ and $n \in \mathbb{N}\}$ is bounded in L^∞, and thus the set of multiplication operators $\{M_h : h \in \Delta'\}$ is bounded in $\mathcal{L}(L^p)$. We deduce from [1, Proposition 4.2] that the set $\{TM_h : h \in \Delta'\}$ is collectively compact. In particular, the sequence $(T'_n, n \in \mathbb{N})$ is collectively compact.

Let $h \in L^p$, we have $\|T'_\infty h - T'_n h\|_p \leq \|T\|_{L^p} \|(v_\infty w_\infty - v_n w_n)h\|_p$. Then, use dominated convergence to get that $\lim_{n\to\infty} \|(v_\infty w_\infty - v_n w_n)h\|_p = 0$. This implies that the sequence $(T'_n, n \in \mathbb{N})$ converges strongly to T'_∞. Then use Lemma 2.1 (iv) to conclude. ■

Remark 2.5 (On integral operators). Consider the positive integral operator defined by

$$T_{\mathrm{k}}(g)(x) = \int_\Omega \mathrm{k}(x, y) g(y) \mu(\mathrm{d}y), \tag{7}$$

where k is a kernel on Ω, that is, a non-negative measurable function defined on $\Omega \times \Omega$. Under the hypothesis that k has a finite double norm in L^p for some $p \in [1, +\infty)$, that is,

$$\|\mathrm{k}\|_{p,q}^p = \int_\Omega \Big(\int_\Omega |\mathrm{k}(x, y)|^q \mu(\mathrm{d}y) \Big)^{p/q} \mu(\mathrm{d}x)$$

is finite with $q = p/(p - 1)$, the operator T_{k} is compact if $p > 1$, and T_{k}^2 is compact if $p = 1$; see [20, p. 293]. When $p > 1$, one gets stronger results on the continuity of the function $R_e[T_{\mathrm{k}}]$; see [10, Theorem 3.5 and Proposition 3.6] (where $R_e[T_{\mathrm{k}}]$ is denoted by $R_e[\mathrm{k}]$ therein).

We conclude this section with a remark on the definition of the operator $M_f T M_g$ when T is a positive operator and f and g are non-negative measurable functions.

Remark 2.6 (On $M_f T M_g$). Let T be a positive compact operator on $L^p(\mu)$ for some $p \in [1, +\infty)$ and f, g be non-negative measurable functions defined on Ω. If the functions f, g are bounded, then the operator $M_f T M_g$ is a positive compact operator on $L^p(\mu)$. Motivated by Example 4.5, we shall however be interested in considering possibly unbounded functions f and g. In this case, the operator TM_g is a positive compact operator from $E = L^p((1 + g)^p \,\mathrm{d}\mu)$ to $L^p(\mu)$, and thus $M_f T M_g$ is a positive compact operator from E to $E' = L^p((1 + f)^{-1} \mathrm{d}\mu)$. Let $r \in [1, +\infty)$ and let ν be a σ-finite measure mutually absolutely continuous with μ. Taking $F = E$ or E', and using the compatibility between F and $L^r(\nu)$ given by Lemma 2.2, we deduce that there exists at most a unique continuous extension of $M_f T M_g$ as a bounded operator on $L^r(\nu)$, which we shall still denote by $M_f T M_g$. By construction, this extension, when it exists, is also positive. However, let us stress that it is not compact *a priori*.

3. Spectrum-preserving transformations

In this section, we consider a measured space $(\Omega, \mathcal{F}, \mu)$ with μ a non-zero σ-finite measure, and we discuss two operations on the positive compact operator T which leave invariant the functions $\mathrm{Spec}[T]$ and $R_e[T]$ defined on Δ. Recall the discussion on the operator $M_f T M_g$ from Remark 2.6.

Lemma 3.1. *Let T be a positive compact operator on L^p for some $p \in [1, +\infty)$ and h be a measurable non-negative function defined on Ω.*

(i) *If $M_h T$ and TM_h are positive compact operators (on L^r and L^s, respectively, with $r, s \in [1, +\infty)$ possibly distinct), then we have*

$$\mathrm{Spec}[M_h T] = \mathrm{Spec}[M_h T M_{\{h>0\}}] = \mathrm{Spec}[M_{\{h>0\}} T M_h] = \mathrm{Spec}[T M_h],$$
$$R_e[M_h T] = R_e[M_h T M_{\{h>0\}}] = R_e[M_{\{h>0\}} T M_h] = R_e[T M_h].$$

(ii) *If h is positive and if $M_h T M_{1/h}$ is a positive compact operator (on some L^r with $p, r \in [1, +\infty)$ possibly distinct), then we have*

$$\mathrm{Spec}[T] = \mathrm{Spec}[M_h T M_{1/h}] \quad \textit{and} \quad R_e[T] = R_e[M_h T M_{1/h}].$$

(iii) *The adjoint operator $T^\top$ is a positive compact operator on L^q with $q = p/(p-1)$ and we have*

$$\mathrm{Spec}[T] = \mathrm{Spec}[T^\top] \quad \textit{and} \quad R_e[T] = R_e[T^\top].$$

Let us stress that the compactness hypothesis of T can be removed in the statement of (i). Even if (ii) is a consequence of (i), we state it separately since (ii) and (iii) describe two modifications of T that leave the functions R_e and Spec invariant. See Remark 5.2 and Lemma 4.7 for other transformations on the operators which leave the functions R_e and Spec invariant. See also [13] for further results in the finite-dimensional case.

Proof. Since $R_e = \mathrm{rad} \circ \mathrm{Spec}$, we only need to prove the lemma for the function Spec. We give a detailed proof of (ii) and (iii). The proof of (i), which is very similar, is left to the reader. We first assume that T is a positive compact operator on L^p, and h and $1/h$ are bounded. The operators TM_η and $M_h T M_{\eta/h}$ and the multiplication operators M_h and $M_{1/h}$ are bounded operators on L^p. We have, using that $TM_{\eta/h}$ is compact and (4) for the second equality,

$$\mathrm{Spec}(TM_\eta) = \mathrm{Spec}(TM_{\eta/h} M_h) = \mathrm{Spec}(M_h T M_{\eta/h}).$$

Since $\eta \in \Delta$ is arbitrary, this gives that

$$\mathrm{Spec}[T] = \mathrm{Spec}[M_h T M_{1/h}].$$

In the general case, we use an approximation scheme. Let h be a positive function and assume that $T' = M_h T M_{1/h}$ is compact on L^r with $r \in [1, +\infty)$. For $n \in \mathbb{N}^*$, set

$$v_n = \mathbb{1}_{\{n \geq h \geq 1/n\}} \quad \text{and} \quad h_n = n^{-1} \vee (h \wedge n).$$

Notice that $T_n = M_{v_n} T M_{v_n}$ and $T_n'' = M_{v_n h_n} T M_{v_n/h_n}$ are positive compact operators on L^p. Let $\eta \in \Delta$. From the first part of the proof, we get

$$\mathrm{Spec}(T_n M_\eta) = \mathrm{Spec}(T_n'' M_\eta).$$

Consider also the positive compact operator on L^r defined by $T'_n = M_{v_n} T' M_{v_n}$. Since the sequence $(v_n, n \in \mathbb{N}^*)$ converges in L^∞ to $\mathbb{1}$, we deduce from Lemma 2.4 that

$$\lim_{n\to\infty} \mathrm{Spec}(T_n M_\eta) = \mathrm{Spec}(T M_\eta) \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Spec}(T'_n M_\eta) = \mathrm{Spec}(T' M_\eta).$$

Since the compact operators T'_n and T''_n are consistent and L^p and L^r are compatible according to Lemma 2.2, we deduce from Lemma 2.1 (iii) that

$$\mathrm{Spec}(T''_n M_\eta) = \mathrm{Spec}(T'_n M_\eta).$$

In conclusion, we obtain that $\mathrm{Spec}(TM_\eta) = \mathrm{Spec}(T'M_\eta)$ and thus $\mathrm{Spec}[T] = \mathrm{Spec}[T']$.

We now prove (iii). Notice that TM_η and $(TM_\eta)^\top$ are compact operators. We have

$$\mathrm{Spec}(T^\top M_\eta) = \mathrm{Spec}(M_\eta T^\top) = \mathrm{Spec}((TM_\eta)^\top) = \mathrm{Spec}(TM_\eta),$$

where we used (4) for the first equality, the self-adjointness of M_η for the second, and (3) for the third. Since this is true for any $\eta \in \Delta$, this gives $\mathrm{Spec}[T^\top] = \mathrm{Spec}[T]$. ■

Remark 3.2 (Multiplicity of the eigenvalues). Following closely the proof of item (ii) of Lemma 3.1, we also get under the assumption of Lemma 3.1 (ii) that

$$\mathrm{m}(\lambda, T) = \mathrm{m}(\lambda, M_h T M_{1/h}) \quad \text{for all } \lambda \in \mathbb{C}^*.$$

4. Sufficient conditions for convexity or concavity of R_e

4.1. A conjecture by Hill and Longini

Recall that, in the metapopulation framework with N groups, the effective reproduction number is equal to the spectral radius of the matrix $K \cdot \mathrm{Diag}(\eta)$, where the next-generation matrix K is an $N \times N$ matrix with non-negative entries and $\eta \in \Delta = [0, 1]^N$ is the vaccination strategy giving the proportion of non-vaccinated people in each group. The Hill–Longini conjecture [22] conditions on the spectrum of the next-generation matrix that should imply convexity or concavity of the effective reproduction number. The conjecture states that the function $R_e[K]$ is

(i) convex when $\mathrm{Spec}(K) \subset \mathbb{R}_+$,

(ii) concave when $\mathrm{Spec}(K) \setminus \{R_0\} \subset \mathbb{R}_-$.

It turns out that the conjecture cannot be true without additional assumptions on the matrix K. Indeed, consider the following next-generation matrix:

$$K = \begin{pmatrix} 16 & 12 & 11 \\ 1 & 12 & 12 \\ 8 & 1 & 1 \end{pmatrix}. \tag{8}$$

Its eigenvalues are approximately equal to 24.8, 2.9 and 1.3. Since R_e is homogeneous, the function is entirely determined by the value it takes on the plane $\{\eta : \eta_1 + \eta_2 +$

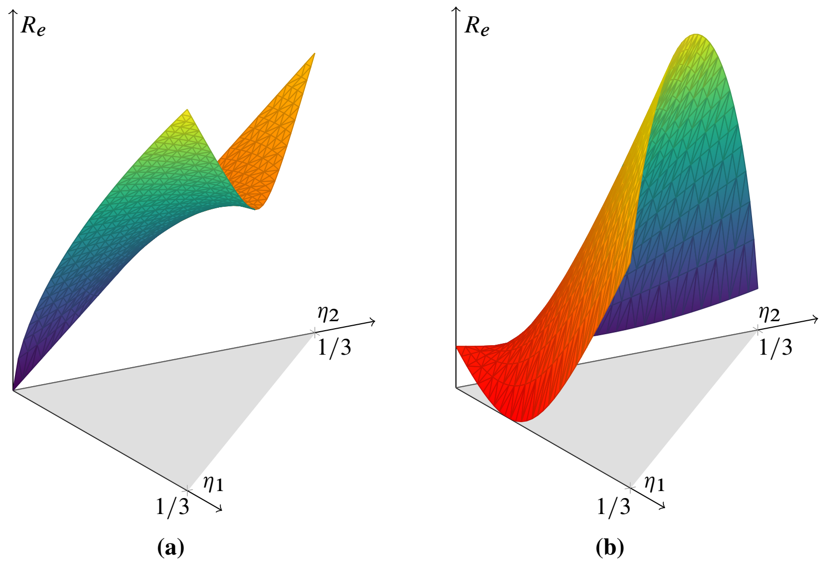

Fig. 1. Counter-example of the Hill–Longini conjecture. (a) Convex case: K given by (8). (b) Concave case: K given by (9). The plan of strategies $P = \{\eta : \eta_1 + \eta_2 + \eta_3 = 1/3\}$ is represented as a gray surface. The triangulated surface corresponds to the graph of $\eta \mapsto R_e[K](\eta)$ restricted to P.

$\eta_3 = 1/3\}$. The graph of the function R_e restricted to this set has been represented in Figure 1 (a). The view clearly shows the saddle nature of the surface. Hence, the Hill–Longini conjecture (i) is contradicted in its original formulation.

In the same manner, the eigenvalues of the next-generation matrix

$$K = \begin{pmatrix} 9 & 13 & 14 \\ 18 & 6 & 5 \\ 1 & 6 & 6 \end{pmatrix} \tag{9}$$

are approximately equal to 26.3, -1.4 and -3.9. Thus, K satisfies the condition that should imply the concavity of the effective reproduction number in the Hill–Longini conjecture (ii). However, as we can see in Figure 1 (b), the function R_e is neither convex nor concave.

Despite these counter-examples, the Hill–Longini conjecture is indeed true when making further assumption on the next-generation matrix. Let M be a square real matrix. The matrix M is *diagonally similar* to a matrix M' if there exists a non-singular real diagonal matrix D such that $M = D \cdot M' \cdot D^{-1}$. The matrix M is said to be *diagonally symmetrizable* or simply *symmetrizable* if it is diagonally similar to a symmetric matrix, or, equivalently, if M admits a decomposition $M = D \cdot S$ (or $M = S \cdot D$), where D is a diagonal matrix with positive diagonal entries and S is a symmetric matrix. If a matrix M is diagonally symmetrizable, then its eigenvalues are real since similar matrices share the same spectrum. We obtain the following result when the next-generation matrix is symmetrizable as a particular case of Theorem 4.10 below.

Theorem 4.1. *Let K be a diagonally symmetrizable $N \times N$ matrix with non-negative entries, and consider the function $R_e = R_e[K]$ defined on $\Delta = [0,1]^N$.*

(i) *If* $\operatorname{Spec}(K) \subset \mathbb{R}_+$, *then the function R_e is convex.*

(ii) *If $R_0 = R_0[K]$ is a simple eigenvalue of K and* $\operatorname{Spec}(K) \subset \mathbb{R}_- \cup \{R_0\}$, *then the function R_e is concave.*

The first point (i) has been proved by Cairns in [7]. In [19], Friedland obtained that if the next-generation matrix K is not singular and its inverse is an M-matrix (i.e., its non-diagonal coefficients are non-positive), then $R_e[K]$ is convex. Friedland's condition does not imply that K is symmetrizable nor that $\operatorname{Spec}(K) \subset \mathbb{R}_+$. On the other hand, the following matrix is symmetric definite positive (and thus R_e is convex) but its inverse is not an M-matrix:

$$K = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix} \quad \text{with inverse } K^{-1} = \begin{pmatrix} 1.4 & -1.6 & 0.4 \\ -1.6 & 2.4 & -0.6 \\ 0.4 & -0.6 & 0.4 \end{pmatrix}.$$

Thus Friedland's condition and property (i) in Theorem 4.1 are not comparable. Note that if K is diagonally symmetrizable and its inverse is an M-matrix, then the eigenvalues of K are actually non-negative thanks to [3, Chapter 6, Theorem 2.3], and one can apply Theorem 4.1 (i) to recover Friedland's result in this case.

4.2. Generalization to compact operators

In this section, we give the analog of Theorem 4.1 for positive compact operators instead of matrices. First, we proceed with some definitions. By analogy with the matrix case, we introduce the notion of diagonally symmetrizable operators.

Recall that $(\Omega, \mathcal{F}, \mu)$ is a measured space with μ a σ-finite non-zero measure. Recall the definition of consistent operators given in Section 2.2 before Lemma 2.1.

Definition 4.2 (Diagonally symmetrizable operator). A compact operator T on $L^p(\mu)$, with $p \in [1, +\infty)$, is called *diagonally symmetrizable* if there exists a σ-finite measure μ' mutually absolutely continuous with respect to μ, and a compact self-adjoint operator T' on $L^2(\mu')$ such that T and T' are consistent.

Remark 4.3 (Diagonally symmetrizable operator in finite dimension). Let us check that Definition 4.2 coincides with the definition of diagonally symmetrizable matrices in finite dimension. Let Ω be a finite set, say $\{1, \dots, n\}$, and without loss of generality assume that the measure μ, as well as μ', which can be seen as vectors of $\mathbb{R}^n$, have positive entries. The sets $L^p(\mu)$ and $L^2(\mu')$ are all equal to $\mathbb{R}^n$, and $T = T'$ can be represented by a matrix, say M, in the canonical base of $\mathbb{R}^n$. Let D be the diagonal matrix with diagonal entries μ'. Then T' being self-adjoint in $L^2(\mu')$ is equivalent to DM being symmetric, and thus the matrix M is diagonally symmetrizable (in the sense of the previous section).

We give an example of diagonally symmetrizable integral operator motivated by the epidemiological framework of Example 4.5 below. Recall from Remark 2.5 that a kernel k on Ω is a non-negative measurable function defined on $\Omega \times \Omega$.

Proposition 4.4 (Diagonally symmetrizable integral operators). *Let $p \in (1, +\infty)$ and let q be its conjugate, k be a symmetric kernel on Ω^2, and f, g be two positive measurable functions on Ω such that*

$$\int_\Omega f(x)^p \Big(\int_\Omega k(x,y)^q g(y)^q \mu(\mathrm{d}y) \Big)^{p/q} d\mu(x) < +\infty,$$

$$\int_{\Omega^2} f(x) g(x) k^2(x,y) f(y) g(y) \mu(\mathrm{d}x) \mu(\mathrm{d}y) < +\infty.$$

Then, the integral operator $T\colon u \mapsto (x \mapsto \int_\Omega f(x) k(x,y) g(y) u(y) \mu(\mathrm{d}y))$ on $L^p(\mu)$ is compact positive and diagonally symmetrizable.

Proof. The measure $\mathrm{d}\mu' = (g/f)\mathrm{d}\mu$ is σ-finite and mutually absolutely continuous with respect to μ. Consider the integral operator

$$T'\colon u \mapsto \Big(x \mapsto \int_\Omega f(x) k(x,y) f(y) u(y) \mu'(\mathrm{d}y) \Big).$$

The integrability assumptions ensure that T is compact on $L^p(\mu)$ and T' is compact (and in fact, Hilbert–Schmidt) on $L^2(\mu')$; see Remark 2.5. According to Lemma 2.2, the Banach spaces $L^p(\mu)$ and $L^2(\mu')$ are compatible. Since the operators T and T' are defined by the same kernel formula on their respective space, they are consistent. Finally, the compact operator T' is clearly self-adjoint on $L^2(\mu')$. This implies that T is diagonally symmetrizable. ■

Example 4.5 (Epidemics on graphon). Consider the SIS model on graphon introduced in [11, Example 1.3]. In this example, the next-generation operator is an integral operator, as defined in Remark 2.5, associated to the kernel k given by

$$\mathrm{k}(x,y) = \frac{\beta(x) W(x,y) \theta(y)}{\gamma(y)},$$

where $\beta(x)$ represents the susceptibility, $\theta(x)$ the infectiousness and $\gamma(x)$ the recovery rate of the individuals with trait x, and W corresponds to the graph of the contacts within the population. More precisely, for $x, y \in \Omega$, the quantity $W(x,y) \in [0,1]$ represents the density of contacts between individuals with traits x and y and is equal to $W(y,x)$ by construction. We deduce from Proposition 4.4 that if $\beta \in L^p(\mu)$ and $\theta/\gamma \in L^q(\mu)$ with $p \in (1, +\infty)$ and q its conjugate, then the integral operator T_k with kernel k defined by (7) is diagonally symmetrizable.

Remark 4.6 (Related notions). An operator T on a Hilbert space is classically called symmetrizable if there exists a positive bounded self-adjoint operator H such that HT is self-adjoint; this notion is discussed, for example, in [21, 32, 36]. Our definition is closer in spirit to [27], where symmetrizability is discussed for operators on Banach spaces with respect to a scalar product. In the matrix case, our setting is a bit more restrictive than general symmetrizability since we symmetrize by a diagonal matrix with positive terms. In the general case, the conditions are not comparable, since we do not impose any upper nor lower bound assumption on the density $\mathrm{d}\nu/\mathrm{d}\mu$.

We complete Section 3 with another example of operators having the same effective spectrum.

Lemma 4.7. *Let T be a diagonally symmetrizable compact operator on $L^p(\mu)$, with $p \in [1, +\infty)$, and let T' be the associated self-adjoint operator from Definition 4.2. Then, we have that on Δ,*

$$\mathrm{Spec}[T] = \mathrm{Spec}[T'], \quad R_e[T] = R_e[T'] \quad \textit{and} \quad \mathrm{m}(\lambda, T) = \mathrm{m}(\lambda, T') \quad \textit{for } \lambda \in \mathbb{C}^*.$$

Proof. Let μ' be the measure from Definition 4.2. Recall that the Banach spaces $L^p(\mu)$ and $L^2(\mu')$ are compatible thanks to Lemma 2.2. Let $\eta \in \Delta$. Since M_η is bounded (both on $L^p(\mu)$ and $L^2(\mu')$), the operators TM_η and $T'M_\eta$, acting on $L^p(\mu)$ and $L^2(\mu')$, respectively, are both compact. Since T and T' are consistent, the operators TM_η and $T'M_\eta$ are also consistent. Then use Lemma 2.1 (iii) to conclude. ■

The next corollary is immediate as the spectrum of a self-adjoint operator, say T', is real and its spectral radius is zero if and only if $T' = 0$.

Corollary 4.8. *Let T be a compact operator on $L^p(\mu)$, with $p \in [1, +\infty)$. If T is diagonally symmetrizable, then its spectrum is real, and T cannot be quasi-nilpotent: $R_0(T) = 0$ if and only if $T = 0$.*

For a compact operator T, let $\mathrm{p}(T)$ and $\mathrm{n}(T)$ denote the number of its eigenvalues with positive and negative real part, respectively, taking into account their (algebraic) multiplicity

$$\mathrm{p}(T) = \sum_{\mathrm{Re}(\lambda)>0} \mathrm{m}(\lambda, T) \quad \text{and} \quad \mathrm{n}(T) = \sum_{\mathrm{Re}(\lambda)<0} \mathrm{m}(\lambda, T).$$

We now give a consequence of Sylvester's inertia theorem [6, Theorem 6.1].

Proposition 4.9 (Sylvester). *Let T be a compact diagonally symmetrizable operator on $L^p(\mu)$, with $p \in [1, +\infty)$. Let f, g be positive bounded measurable functions defined on Ω which are also bounded away from 0. Then the compact operator $M_f T M_g$ on $L^p(\mu)$ is diagonally symmetrizable with the same inertia as T,*

$$\mathrm{p}(T) = \mathrm{p}(M_f T M_g) \quad \textit{and} \quad \mathrm{n}(T) = \mathrm{n}(M_f T M_g).$$

Proof. First note that if h is a positive bounded and bounded away from 0, then for any $r \in [1, +\infty)$ and σ-finite non-zero measure ν, the multiplication operator M_h is bounded with bounded inverse on $L^r(\nu)$. In particular, the operator $\widetilde{T} = M_f T M_g$ is a compact operator on $L^p(\mu)$ as T is compact.

Let T' be the compact self-adjoint operator on $L^2(\mu')$ associated to T from Definition 4.2. The measure $\mathrm{d}\widetilde{\mu}' = (g/f)\,\mathrm{d}\mu'$ is σ-finite and mutually absolutely continuous with respect to both μ' and μ. The mapping $\Phi = M_{\sqrt{f/g}}$ is an isometry between the Hilbert spaces $L^2(\widetilde{\mu}')$ and $L^2(\mu')$.

We now define the operator $\widetilde{T}'$ on $L^2(\widetilde{\mu}')$ by

$$\widetilde{T}' = \Phi \circ (M_{\sqrt{fg}} T' M_{\sqrt{fg}}) \circ \Phi^{-1}.$$

Since T' is compact, the operator $\widetilde{T}'$ is also compact. Since f and g are bounded and bounded away from 0, the sets $L^p(\mu) \cap L^2(\mu')$ and $L^p(\mu) \cap L^2(\widetilde{\mu}')$ are equal. Since T and T' coincide on this set, so do $M_f T M_g = \widetilde{T}$ and $\widetilde{T}'$. The operator $M_{\sqrt{fg}} T' M_{\sqrt{fg}}$ is bounded and symmetric on $L^2(\mu')$, and therefore self-adjoint. Since Φ is an isometry, we deduce that $\widetilde{T}'$ is self-adjoint on $L^2(\widetilde{\mu}')$. Therefore, the operator $\widetilde{T} = M_f T M_g$ on $L^p(\mu)$ is diagonally symmetrizable.

We now establish the following string of equalities:

$$\mathrm{p}(T) = \mathrm{p}(T') = \mathrm{p}(M_{\sqrt{fg}} T' M_{\sqrt{fg}}) = \mathrm{p}(\widetilde{T}') = \mathrm{p}(\widetilde{T}) = \mathrm{p}(M_f T M_g). \tag{10}$$

By Lemma 4.7, $\mathrm{p}(T) = \mathrm{p}(T')$ and $\mathrm{p}(\widetilde{T}') = \mathrm{p}(\widetilde{T})$. Since $M_{\sqrt{fg}}$ is invertible in $L^2(\mu')$ and T' is self-adjoint (thus with real eigenvalues), we get, using the generalization of Sylvester's inertia theorem [6, Theorem 6.1] (the definition of inertia in that paper being consistent with the definition of $\mathrm{p}(\cdot)$ and $\mathrm{n}(\cdot)$, which can be checked using [6, Theorem 4.5 (ii)]) that

$$\mathrm{p}(T') = \mathrm{p}(M_{\sqrt{fg}} T' M_{\sqrt{fg}}).$$

Finally, since Φ is an isometry, $\mathrm{p}(M_{\sqrt{fg}} T' M_{\sqrt{fg}}) = \mathrm{p}(\widetilde{T}')$, and (10) is justified.

The equalities are similar for the number of negative eigenvalues $\mathrm{n}(\cdot)$. ∎

The following result is the analog of Theorem 4.1 for positive compact operators. Note that if T is a positive compact operator with $R_0[T] > 0$, then $R_0[T]$ is an eigenvalue of T thanks to the Krein–Rutman theorem, see [34, Corollary 9], and thus $\mathrm{p}(T) \geq 1$.

Theorem 4.10 (Convexity/concavity of R_e). *Let T be a positive compact diagonally symmetrizable operator on $L^p(\mu)$, with $p \in [1, +\infty)$. We consider the function $R_e = R_e[T]$ defined on Δ.*

(i) *If $\mathrm{n}(T) = 0$, then the function R_e is convex.*

(ii) *If $\mathrm{p}(T) = 1$, then the function R_e is concave.*

The proof for a positive self-adjoint operator T is given in Section 4.3 for the convex case and in Section 4.4 for the concave case when T is compact. The extension to diagonally symmetrizable positive compact operators follows directly from Lemma 4.7.

Remark 4.11 (Rank-one operator). The so-called *configuration model* occurs in finite dimension when the next-generation matrix has rank one. This corresponds to a classical mixing structure called the *proportionate mixing* introduced by [30] and used in many different epidemiological models. Motivated by the finite-dimensional case, we consider a *configuration* kernel k defined by

$$\mathrm{k} = f \otimes g, \quad \text{where } (f \otimes g)(x, y) = f(x)g(y),$$

with $f \in L^p$ and $g \in L^q$ for some $p \in (1, +\infty)$ and $q = p/(p-1)$. We also suppose that $\mu(fg > 0) > 0$. Let T_k denote the integral operator with kernel k, see Remark 2.5. According to Proposition 4.4, with $k = \mathbb{1}_{\{f>0\}} \otimes \mathbb{1}_{\{g>0\}}$ and $h \in \{f, g\}$ replaced by $h + h'\mathbb{1}_{\{h=0\}}$ for some positive function $h' \in L^p(\mu) \cap L^q(\mu)$, we deduce that the integral operator T_k on $L^p(\mu)$ is compact positive and diagonally symmetrizable. Since T_k is of rank one, we deduce from Theorem 4.10 that $R_e[T_\mathrm{k}]$ is convex and concave and thus linear. This can be checked directly as it is immediate to notice that

$$R_e[T_\mathrm{k}](\eta) = \int_\Omega fg\eta \, \mathrm{d}\mu.$$

We shall provide in a forthcoming work a deeper study of configuration kernels in the context of epidemiology.

4.3. The convex case

The proof of property (i) in Theorem 4.10 relies on an idea from [19] (see therein just before Theorem 4.3). Suppose that T is a self-adjoint operator on $L^2 = L^2(\mu)$ such that $\mathrm{Spec}(T) \subset \mathbb{R}_+$. As $R_0[T] = 0$ implies $T = 0$ and thus $R_e[T] = 0$, we shall only consider the case $R_0[T] > 0$. Since T is a self-adjoint positive semi-definite operator on L^2, there exists a self-adjoint positive semi-definite operator Q on L^2 such that $Q^2 = T$. Thanks to (5), we have for $\eta \in \Delta$,

$$R_e[T](\eta) = \rho(TM_\eta) = \rho(Q^2 M_\eta) = \rho(QM_\eta Q).$$

Since the self-adjoint operator $QM_\eta Q$ on L^2 is also positive semi-definite, we deduce from the Courant–Fischer–Weyl min-max principle that

$$R_e[T](\eta) = \rho(QM_\eta Q) = \sup_{u \in L^2(\mu)\setminus\{0\}} \frac{\langle u, QM_\eta Qu\rangle}{\langle u, u\rangle}.$$

Since the map $\eta \mapsto \langle u, QM_\eta Qu\rangle$ defined on Δ is linear, we deduce that $\eta \mapsto R_e[T](\eta)$ is convex as a supremum of linear functions.

4.4. The concave case

The proof of property (ii) in Theorem 4.10 relies on a computation of the second derivative of the function R_e. Let T be a positive compact self-adjoint operator on $L^2(\mu)$ such that $\mathrm{p}(T) = 1$. Let Δ^* be the subset of Δ of the functions which are bounded away from 0. The set Δ^* is a dense convex subset of Δ (for the $L^2(\mu)$-convergence or simple convergence). The function $R_e = R_e[T]$ is continuous on Δ, see Lemma 2.4 (indeed, with the notations therein, take $v_n = 1$, $w_n \in \Delta$ and notice that $R_e[T_n](\mathbb{1}) = R_e[T](w_n)$ converges to $R_e[T_\infty](\mathbb{1}) = R_e[T](w_\infty)$). So it suffices to prove that $R_e = R_e[T]$ is concave on Δ^*. Let η_0, η_1 be elements of Δ^*, and set $\eta_\alpha = (1-\alpha)\eta_0 + \alpha\eta_1$ for $\alpha \in [0,1]$ (which is also an element of Δ^*). We write $T_\alpha = TM_{\eta_\alpha}$, so that $T_\alpha = T_0 + \alpha TM$, where $M = M_{\eta_1-\eta_0}$ is the multiplication by $(\eta_1 - \eta_0)$ operator, and, with $R(\alpha) = R_e(\eta_\alpha)$,

$$R(\alpha) = \rho(T_\alpha) = \rho(T_0 + \alpha TM).$$

So, to prove that R_e is concave on Δ^* (and thus on Δ), it is enough to prove that $\alpha \mapsto R(\alpha)$ is concave on $(0,1)$. Thanks to Sylvester's inertia theorem stated in Proposition 4.9 (with $f = 1$ and $g = \eta_\alpha$), we also get that $\mathrm{p}(T_\alpha) = \mathrm{p}(T) = 1$. This implies that $R(\alpha)$ is positive and a simple eigenvalue.

We consider the following scalar product on $L^2(\mu)$ defined by $\langle u, v\rangle_\alpha = \langle u, \eta_\alpha v\rangle$. The operator T_α is self-adjoint and compact on $L^2(\eta_\alpha \mathrm{d}\mu)$ with spectrum $\mathrm{Spec}(T_\alpha)$ thanks to Lemmas 2.1 (iii) and 2.2. Let $(\lambda_n, n \in I = [\![0, N[\![)$, with $N \in \mathbb{N} \cup \{\infty\}$, be an enumeration of the non-zero eigenvalues of T_α with their multiplicity so that $\lambda_0 = R(\alpha) > 0$, and thus $\lambda_n < 0$ for $n \in I^* = I \setminus \{0\}$, and denote by $(u_n, n \in I)$ a corresponding sequence of orthogonal eigenvectors (in $L^2(\eta_\alpha \mathrm{d}\mu)$). The functions $v_\alpha = u_0$ and $\phi_\alpha = \eta_\alpha u_0$ are the right and left eigenvectors for T_α (seen as an operator on $L^2(\mu)$) associated to $R(\alpha)$.

We now follow [24] to get that $\alpha \mapsto R(\alpha) = \rho(T_0 + \alpha TM)$ is analytic and compute its second derivative. Let π_α be the projection on the $(\langle\cdot,\cdot\rangle_\alpha)$-orthogonal of v_α, and define

$$S_\alpha = (T_\alpha - R(\alpha))^{-1}\pi_\alpha.$$

In other words, S_α maps u_0 to 0 and u_i to $(\lambda_i - R(\alpha))^{-1}u_i$. Let $\alpha \in (0,1)$ and let ε be small enough so that $\alpha + \varepsilon \in [0,1]$. We have

$$T_{\alpha+\varepsilon} = T_\alpha + \varepsilon TM,$$

and thus $\|T_{\alpha+\varepsilon} - T_\alpha\|_{L^2(\eta_\alpha \mathrm{d}\mu)} = O(\varepsilon)$. Using [24, Theorem 2.6] on the Banach space $L^2(\eta_\alpha\, \mathrm{d}\mu)$, we get that

$$R(\alpha+\varepsilon) = R(\alpha) + \varepsilon\langle v_\alpha, TMv_\alpha\rangle_\alpha - \varepsilon^2\langle v_\alpha, TMS_\alpha TMv_\alpha\rangle_\alpha + O(\varepsilon^3).$$

Let $N_\alpha = M_{1/\eta_\alpha}M = MM_{1/\eta_\alpha}$ be the multiplication by $(\eta_1 - \eta_0)/\eta_\alpha$ bounded operator. Since $\alpha \mapsto R(\alpha)$ is analytic and T is self-adjoint (with respect to $\langle\cdot,\cdot\rangle$), we get that

$$\begin{aligned} R''(\alpha) &= -2\langle v_\alpha, TMS_\alpha TMv_\alpha\rangle_\alpha \\ &= -2\langle MT_\alpha v_\alpha, S_\alpha TMv_\alpha\rangle \\ &= -2R(\alpha)\langle Mv_\alpha, S_\alpha TMv_\alpha\rangle \\ &= -2R(\alpha)\langle N_\alpha v_\alpha, S_\alpha T_\alpha N_\alpha v_\alpha\rangle_\alpha. \end{aligned}$$

Since the kernel and the image of T_α are orthogonal (in $L^2(\eta_\alpha \mathrm{d}\mu)$), and the latter is generated by $(u_n, n \in I)$, we have the decomposition $N_\alpha v_\alpha = g + \sum_{n\in I} a_n u_n$ with $g \in \mathrm{Ker}(T_\alpha)$ and $a_n = \langle N_\alpha v_\alpha, u_n\rangle_\alpha / \langle u_n, u_n\rangle_\alpha$. This gives, with $I^* = I \setminus \{0\}$,

$$R''(\alpha) = 2R(\alpha)\sum_{n\in I^*} \frac{\lambda_n}{R(\alpha) - \lambda_n}\, a_n^2 \langle u_n, u_n\rangle_\alpha.$$

Since $\lambda_n < 0$ for all $n \in I^*$, we deduce that $R''(\alpha) \leq 0$, and thus $\alpha \mapsto R(\alpha)$ is concave on $[0, 1]$. This implies that $R_e[T]$ is concave.

Remark 4.12. The same proof with obvious changes gives that if T is a positive quasi-irreducible compact self-adjoint operator (see Section 5.2 for the precise definition of quasi-irreducible operator) such that $\mathrm{n}(T) = 0$, then $R_e[T]$ is convex on Δ. Then, using the decomposition of a compact operator on its irreducible atoms (see Section 5.1 and more precisely Lemma 5.3) and the fact that the maximum of convex functions is convex (used in (14)), we can recover Theorem 4.10 (i).

5. The reproduction number and reducible positive compact operators

Following [34], we present in Section 5.1 the atomic decomposition of a positive compact operator T on L^p, where $p \in [1, +\infty)$ and state a formula which "reduces" the effective reproduction function of T on the whole space to the ones of the restriction of T to each atoms (or irreducible components); see Corollary 5.4 below. Then, we consider the notion of quasi-irreducible and monatomic operators in Section 5.2, and provide some properties of monatomic operators and prove that if the effective reproduction number is concave then the operator is monatomic.

5.1. Atomic decomposition

Our presentation is a direct application of the Frobenius decomposition [23, 34, 35] or the "super diagonal" form [14, Part II.2]. For convenience, we follow [34] for positive compact operators on $L^p(\mu)$ for some $p \in [1, +\infty)$; see also [4, Lemma 5.17] in the case of integral operators with symmetric kernel. We stress that the results in [34] are stated under the hypothesis that μ is a finite measure, but it is elementary to check that the main results (Theorems 7 and 8 therein) also hold if the measure μ is σ-finite.

For $A, B \in \mathcal{F}$, we write $A \subset B$ a.e. if $\mu(B^c \cap A) = 0$ and $A = B$ a.e. if $A \subset B$ a.e. and $B \subset A$ a.e. Let T be a positive compact operator on L^p for some $p \in [1, +\infty)$. Let $f_0 \in L^p$ and $g_0 \in L^q$ be positive functions and consider the operator $T_0 = M_{g_0} T M_{f_0}$ from L^∞ to L^1. We define the function k_T on $\mathcal{F}^2$ for $A, B \in \mathcal{F}$ as

$$\mathrm{k}_T(B, A) = \int_B (T_0 \mathbb{1}_A)(x) \mu(\mathrm{d}x) = \langle \mathbb{1}_B, T_0 \mathbb{1}_A \rangle. \tag{11}$$

It is clear from (11) that the family of sets (B, A) such that $\mathrm{k}_T(B, A) = 0$ does not depend on the choice of the positive functions $f_0 \in L^p$ and $g_0 \in L^q$. If the measure μ is finite, then one can take $f_0 = g_0 = \mathbb{1}$ and thus $T_0 = T$.

A set $A \in \mathcal{F}$ is T-*invariant*, or simply *invariant*, when there is no ambiguity on the operator T if $\mathrm{k}_T(A^c, A) = 0$. We recall that $\mathcal{J}$ is called a closed ideal of L^p (with $p \in [1, +\infty)$) if and only if it is equal to $\mathcal{J}_B = \{f \in L^p : f \mathbb{1}_B = 0\}$ for some measurable set $B \in \mathcal{F}$; see [33, Section III.1, Example 2] or [37, Section III.2]. Notice that a set $A \in \mathcal{F}$ is invariant if and only if the ideal $\mathcal{J}_A$ is invariant for T, that is, $T(\mathcal{J}_A) \subset \mathcal{J}_A$.

A positive compact operator T on L^p is (ideal)-*irreducible* if the only closed invariant ideals are $\{0\}$ and L^p. Thus, the positive compact operator T is irreducible if and only if any T-invariant set A is such that either $\mu(A) = 0$ or $\mu(A^c) = 0$. According to [9, Theorem 3] (see also [37, Section III.3] for an elementary presentation in L^p), if T is an irreducible positive compact operator on L^p, then either $R_0[T] > 0$, or the situation is degenerate in the sense that Ω is an atom of μ (that is, for all $A \in \mathcal{F}$, we have either $\mu(A) = 0$ or $\mu(A^c) = 0$) and $T = 0$.

Let $\mathcal{A}$ be the set of T-invariant sets, and notice that $\mathcal{A}$ is stable by countable unions and countable intersections. Let $\mathcal{F}_{\text{inv}} = \sigma(\mathcal{A})$ be the σ-field generated by $\mathcal{A}$. Then, thanks to [34, Theorem 8], the operator T restricted to an atom of μ in $\mathcal{F}_{\text{inv}}$ is irreducible. We shall only consider non-degenerate atoms, and say that the atom (of μ in $\mathcal{F}_{\text{inv}}$) is non-zero if the restriction of the operator T to this atom has a positive spectral radius. We denote by $(\Omega_i, i \in I)$ the at most countable (but possibly empty) collection of non-zero atoms of μ in $\mathcal{F}_{\text{inv}}$. The atoms are defined up to an a.e. equivalence and can be chosen to be pairwise disjoint. For $i \in I$, we set

$$T_i = M_{\Omega_i} T M_{\Omega_i},$$

which is a positive compact operator on L^p. Note that

$$T \geq T', \quad \text{where } T' = \sum_{i \in I} T_i. \tag{12}$$

We now give some properties of the Frobenius decomposition.

Remark 5.1 (Properties of the Frobenius decomposition). We have, with $i \in I$,

(i) By definition of the non-zero atoms, $\mu(\Omega_i) > 0$ and T restricted to Ω_i is irreducible with positive spectral radius, that is, $R_0[T_i] > 0$.

(ii) According to [34, Theorem 8], the spectral radius of T_i is a simple eigenvalue of T_i such that $\mathrm{m}(R_0(T_i), T_i) = 1$.

(iii) According to [34, Theorem 7], for all $\lambda \in \mathbb{C}^*$, we have

$$\mathrm{m}(\lambda, T) = \sum_{j \in I} \mathrm{m}(\lambda, T_j). \tag{13}$$

(iv) Consider the complement of the non-zero atoms, say $\Omega_0 = (\bigcup_{j \in I} \Omega_j)^c$ (with the convention that 0 does not belong to the set of indices I). Then, the restriction of T to Ω_0 is quasi-nilpotent, that is $R_e[T](\mathbb{1}_{\Omega_0}) = 0$.

From those properties, we deduce the following elementary results:

(v) The cardinal of the set of indices $i \in I$ such that $R_0[T_i] = R_0[T]$ is exactly equal to the multiplicity of R_0[k] for T, that is, $\mathrm{m}(R_0[T], T)$.

(vi) There exists at least one non-zero atom ($\sharp I \geq 1$) if and only if $R_0[T] > 0$.

(vii) The operator T is quasi-nilpotent if and only if there is no non-zero atom ($\sharp I = 0$).

(viii) If $A \in \mathcal{F}$ invariant implies A^c invariant (which is in particular the case if T is self-adjoint and $p = 2$), then we have $T = \sum_{i \in I} T_i$, and thus the restriction of T to Ω_0 is zero (intuitively T is block diagonal).

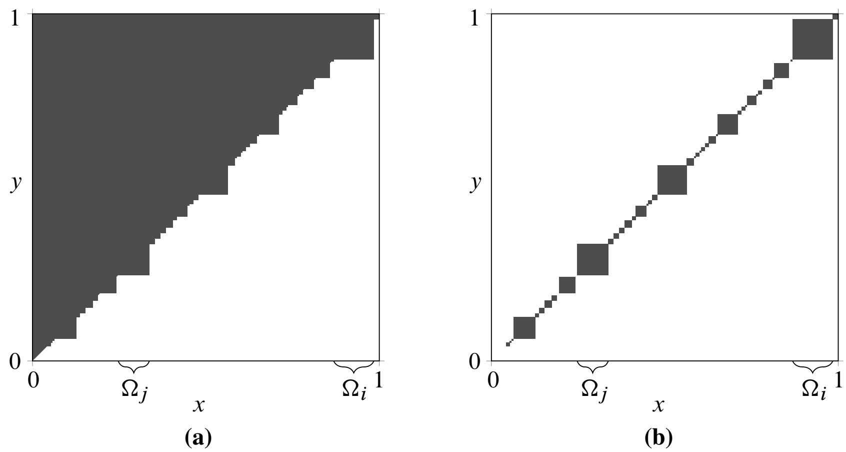

Fig. 2. Example of a kernel k on $\Omega = [0, 1]$ and the kernel $\mathrm{k}' = \sum_{i \in I} \mathrm{k}_i$, with $\mathrm{k}_i(x, y) = \mathbb{1}_{\Omega_i}(x)\mathrm{k}(x, y)\mathbb{1}_{\Omega_i}(y)$ and $(\Omega_i, i \in I)$ the non-zero atoms. (a) A representation of the kernel k with the white zone included into $\{\mathrm{k} = 0\}$. (b) A representation of the kernel $\mathrm{k}' = \sum_{i \in I} \mathrm{k}_i$ with the white zone included into $\{\mathrm{k}' = 0\}$. We have $\mathrm{Spec}[T_\mathrm{k}] = \mathrm{Spec}[T_{\mathrm{k}'}]$ and thus $R_e[T_\mathrm{k}] = R_e[T_{\mathrm{k}'}]$.

Remark 5.2. Assume that $T = T_\mathrm{k}$ is an integral operator with kernel k on $\Omega = [0, 1]$; see Remark 2.5. Then, the operators T_i are integral operators with respective kernels k_i given by $\mathrm{k}_i(x, y) = \mathbb{1}_{\Omega_i}(x)\mathrm{k}(x, y)\mathbb{1}_{\Omega_i}(y)$, and the operator $T' = T_{\mathrm{k}'}$ is also an integral operator with kernel $\mathrm{k}' = \sum_{i \in I} \mathrm{k}_i$. We represent in Figure 2 (a) an example of a kernel k with its atomic decomposition using a "nice" order on Ω (see [14,23,35] on the existence of such an order relation; intuitively, the kernel is upper block triangular: the population on the "left" of an atom does not infect the population on the "right" of this atom) and in Figure 2 (b) the corresponding kernel k′. Notice that $\mathrm{k}(\Omega_i, \Omega_j) = 0$ for j "smaller" than i, where $\mathrm{k}(A, B) = \int_{\Omega^2} \mathbb{1}_A(x)\mathrm{k}(x, y)\mathbb{1}_B(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y)$ is a consistent notation with (11).

For $i \in I$ and $\eta \in \Delta$, we set $\eta_i = \eta \mathbb{1}_{\Omega_i}$ and recall that $T_i = M_{\Omega_i} T M_{\Omega_i}$. We now give the decomposition of $R_e[T]$ according to the irreducible components $(\Omega_i, i \in I)$ of T.

Lemma 5.3. *Let T be a positive compact operator on L^p for some $p \in [1, +\infty)$. With the convention that the maximum of an empty space is 0, we have for $\eta \in \Delta$,*

$$R_e[T](\eta) = \max_{i \in I} R_e[T_i](\eta_i) = \max_{i \in I} R_e[T_i](\eta) = \max_{i \in I} R_e[T](\eta \mathbb{1}_{\Omega_i}), \tag{14}$$

and more generally,

$$\mathrm{m}(\lambda, TM_\eta) = \sum_{i \in I} \mathrm{m}(\lambda, T_i M_\eta) \quad \textit{for all } \lambda \in \mathbb{C}^*. \tag{15}$$

Before proving the lemma, we first state a direct consequence of (15), in the spirit of Section 3 on a spectrum-preserving transformation. Recall that $T' = \sum_{i \in I} T_i$ by formula (12).

Corollary 5.4. *Let T be a positive compact operator on L^p for some $p \in [1, +\infty)$. We have*

$$\mathrm{Spec}[T] = \mathrm{Spec}[T'] = \bigcup_{i\in I} \mathrm{Spec}[T_i] \quad \textit{and} \quad R_e[T] = R_e[T'] = \max_{i\in I} R_e[T_i].$$

Proof of Lemma 5.3. Let T' be a positive compact operator on L^p. Recall the kernel $\mathrm{k}_{T'}$ defined in (11). For $A \in \mathcal{F}$, let $\mathrm{m}(\lambda, T', A)$ denote the multiplicity (possibly equal to 0) of the eigenvalue $\lambda \in \mathbb{C}^*$ for the operator $T' M_A$. A direct application of [34, Lemma 11] (which holds also if μ is a σ-finite measure) gives that for $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$ a.e. and $\mathrm{k}_{T'}(B, A) = 0$, we have for all $\lambda \in \mathbb{C}^*$ that

$$\mathrm{m}(\lambda, T', A \cup B) = \mathrm{m}(\lambda, T', A) + \mathrm{m}(\lambda, T', B), \tag{16}$$

and thus

$$R_e[T'](\mathbb{1}_A + \mathbb{1}_B) = \max(R_e[T'](\mathbb{1}_A), R_e[T'](\mathbb{1}_B)).$$

Let $A, B \in \mathcal{F}$ be such that $A \cap B = \emptyset$ a.e. and $\mathrm{k}_T(B, A) = 0$. Let $\eta \in \Delta$. Clearly, we have $\mathrm{k}_{TM_\eta}(B, A) \le \mathrm{k}_T(B, A)$ and thus $\mathrm{k}_{TM_\eta}(B, A) = 0$. Use (16) to get that for $\eta \in \Delta$ and $\lambda \in \mathbb{C}^*$,

$$\mathrm{m}(\lambda, TM_\eta, A \cup B) = \mathrm{m}(\lambda, TM_\eta, A) + \mathrm{m}(\lambda, TM_\eta, B).$$

Then, an immediate adaptation of the proof of [34, Theorem 7] gives that for all $\lambda \in \mathbb{C}^*$,

$$\mathrm{m}(\lambda, TM_\eta, \Omega) = \sum_{i\in I} \mathrm{m}(\lambda, TM_\eta, \Omega_i). \tag{17}$$

By definition of $\mathrm{m}(\lambda, \cdot, \cdot)$, we get

$$R_e[T](\eta) = \max\{|\lambda| : \mathrm{m}(\lambda, TM_\eta, \Omega) > 0\},$$
$$R_e[TM_{\Omega_i}](\eta) = \max\{|\lambda| : \mathrm{m}(\lambda, TM_\eta, \Omega_i) > 0\}.$$

This gives

$$R_e[T](\eta) = \max_{i\in I} R_e[TM_{\Omega_i}](\eta).$$

To conclude, notice, using Lemma 3.1 (i) for the second equality, that

$$R_e[T](\eta \mathbb{1}_{\Omega_i}) = R_e[TM_{\Omega_i}](\eta) = R_e[M_{\Omega_i} TM_{\Omega_i}](\eta) = R_e[T_i](\eta) = R_e[T_i](\eta_i).$$

Similarly, we deduce (15) from (17). ■

5.2. *Monatomic operators and applications*

Recall that for a measurable subset $A \subset \Omega$, M_A stands for the multiplication operator by $\mathbb{1}_A$. Following [4, Definition 2.11], a positive compact operator T is *quasi-irreducible* if there exists a measurable set $\Omega_{\mathrm{a}} \subset \Omega$ such that $\mu(\Omega_{\mathrm{a}}) > 0$, $T = M_{\Omega_{\mathrm{a}}} TM_{\Omega_{\mathrm{a}}}$ and T restricted to Ω_{a} is irreducible with positive spectral radius. The quasi-irreducible property is natural in the setting of positive compact self-adjoint operators; in a more general setting, one would still want to consider a positive compact operator with only one irreducible component. This motivates the next definition. Recall the atomic decomposition of the previous section.

Definition 5.5 (Monatomic operator). Let T be a positive compact operator on L^p with some $p \in [1, +\infty)$. The operator is monatomic if there exists a unique non-zero atom ($\sharp I = 1$).

In a sense, the operator T is "truly reducible" when $\sharp I \geq 2$. We shall give in a forthcoming work other characterizations of monatomic operator.

Remark 5.6 (Link between (quasi-)irreducible and monatomic operators). Irreducible positive compact operators with positive spectral radius and quasi-irreducible positive compact operators are monatomic, and we have $T = T_{\mathrm{a}}$, where $T_{\mathrm{a}} = M_{\Omega_{\mathrm{a}}} T M_{\Omega_{\mathrm{a}}}$ and Ω_{a} is the non-zero atom, with $\Omega_{\mathrm{a}} = \Omega$ in the reducible case.

Remark 5.7 (Reducibility for integral operators). We consider an integral operator T_{k} with kernel k, see Remark 2.5, and we say the kernel k is irreducible, quasi-irreducible or monatomic whenever the integral operator T_{k} satisfies the corresponding property. Then, the notion of irreducibility of a kernel depends only on its support. Indeed, provided that the measure μ is finite and the kernel so that all the operators are well defined and compact, the kernel k is irreducible (resp. quasi-irreducible, resp. monatomic) if and only if the kernel $\mathbb{1}_{\{\mathrm{k}>0\}}$ is irreducible (resp. quasi-irreducible, resp. monatomic). Furthermore, the corresponding integral operators have the same atoms.

We have represented in Figure 3 (a) a monatomic kernel k on $\Omega = [0, 1]$ and in Figure 3 (b) the kernel $\mathrm{k_a}$ (with $\mathrm{k_a}(x, y) = \mathbb{1}_{\Omega_{\mathrm{a}}}(x)\mathrm{k}(x, y)\mathbb{1}_{\Omega_{\mathrm{a}}}(y)$) associated to the quasi-irreducible integral operator $T_{\mathrm{a}} = M_{\Omega_{\mathrm{a}}} T_{\mathrm{k}} M_{\Omega_{\mathrm{a}}}$; the set $\Omega = [0, 1]$ being "nicely ordered" so that the representation of the kernels are upper triangular. Using the epidemic interpretation of Remark 5.8 below, we also represented the subset Ω_{i} of the population infected by the non-zero atom Ω_{a}.

Remark 5.8 (Epidemiological interpretation). In the infinite-dimensional SIS model developed in [11], the space $(\Omega, \mathcal{F}, \mu)$ represents all the traits of the population with $\mu(\mathrm{d}y)$ the infinitesimal size of the population with trait y. The next-generation operator is given by the integral operator T_{k}; see (7), where the kernel $\mathrm{k} = k/\gamma$ is defined in terms of a transmission rate kernel k and a recovery rate function γ by the formula $\mathrm{k}(x, y) = k(x, y)/\gamma(y)$ and has a finite double norm in L^p for some $p \in (1, +\infty)$; the basic reproduction number $R_0 = R_0[T_{\mathrm{k}}]$ is then the spectral radius of T_{k}. Intuitively, $\mathrm{k}(x, y) > 0$ (resp. $= 0$) means that individuals with trait y can (resp. cannot) infect individuals with trait x.

When the integral operator T_{k} is monatomic, with non-zero atom Ω_{a}, then the population with trait in Ω_{a} can infect itself as well as the population with other distinct traits, say Ω_{i}. The population with trait Ω_{i} can only infect itself (but not Ω_{a}!), and there is no persistent epidemic outside $\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{i}}$. We shall see in a forthcoming paper that the set $\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{i}}$ is characterized as the smallest invariant set containing the atom Ω_{a}.

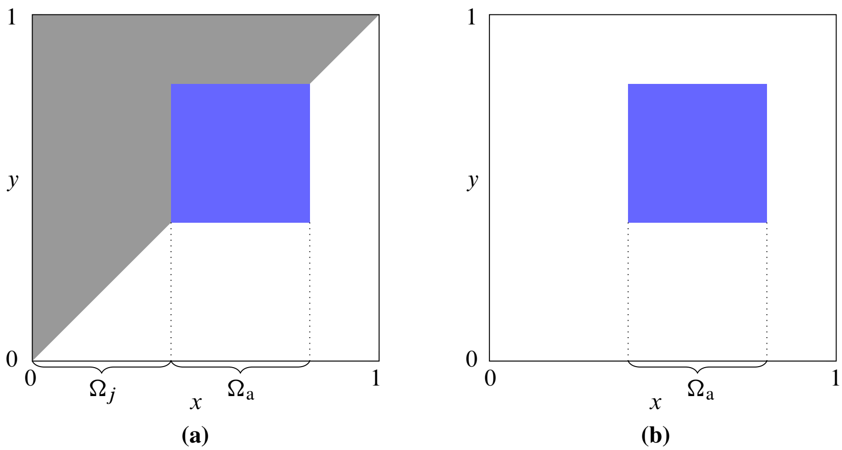

Fig. 3. Example of kernels k and k_a of a monatomic integral operator T_k and the quasi-irreducible integral operator $T_a = T_{k_a}$ on $\Omega = [0, 1]$, with non-zero atom Ω_a. (a) A representation of a monatomic kernel. (b) A representation of a quasi-irreducible kernel. The kernels are zero on the white zone and are irreducible when restricted to the zone.

From Lemma 5.3, we deduce the following two results related to monatomic operators.

Lemma 5.9. *Let T be a positive compact operator on L^p with some $p \in [1, +\infty)$, and set $R_0 = R_0[T]$. If the operator T is monatomic, then $R_0 > 0$ and R_0 is simple (i.e., $\mathrm{m}(R_0, T) = 1$). If R_0 is simple and the only eigenvalue in $(0, +\infty)$, then the operator T is monatomic.*

Proof. Let T be monatomic, so that there exists only one non-zero atom, say Ω_a. Set $T_a = M_{\Omega_a} T M_{\Omega_a}$. Since the restriction of T_a (or T) to Ω_a is irreducible and non-zero, we deduce from [9, Theorem 3] that its spectral radius is positive, and thus $R_0[T_a] > 0$. Using Lemma 5.3, this implies that $R_0[T] = R_0[T_a] > 0$. According to [34, Theorem 8], we get that $R_0[T_a]$ is simple for T_a. Since according to (13) $\mathrm{m}(\lambda, T) = \mathrm{m}(\lambda, T_a)$ for all $\lambda \in \mathbb{C}^*$, we deduce that $R_0[T]$ is simple for T.

For the second part, if T is not monatomic and $R_0[T] > 0$, we deduce that there exist at least two non-zero atoms, and thus $\sharp I \geq 2$ (if there is no non-zero atom, then T would be quasi-nilpotent and $R_0[T] = 0$). The restrictions of T to those non-zero atoms have positive spectral radius according to [9, Theorem 3] and thus at least one positive eigenvalue by the Krein–Rutman theorem. We deduce from (13) that T has at least two positive eigenvalues (counting their multiplicity if they are equal). This gives the result by contraposition. ∎

Lemma 5.10. *Let T be a positive compact operator on L^p for some $p \in [1, +\infty)$ such that $R_0[T] > 0$. If the function $R_e[T]$ is concave on Δ, then the operator T is monatomic.*

Proof. Since $R_0[T]$ is positive, we deduce that T is not quasi-nilpotent. Suppose that T is not monatomic. This means that the cardinal of the at most countable set I in decomposition (14) is at least 2. So let T_1 and T_2 be two quasi-irreducible components of T, where we assume that $\{1,2\} \subset I$. Let Ω_1 and Ω_2 denote their respective non-zero atoms. Without loss of generality, we can suppose that $R_0[T_2] \geq R_0[T_1] > 0$. Consider the strategies $\eta_1 = \mathbb{1}_{\Omega_1}$ and $\eta_2 = R_0[T_1]R_0[T_2]^{-1}\mathbb{1}_{\Omega_2}$ (which both belong to Δ). For $\theta \in [0,1]$, we deduce from (14) and the homogeneity of the spectral radius that $R_e[T](\theta\eta_1 + (1-\theta)\eta_2) = R_e[T_1]\max(\theta, 1-\theta)$. Since $\theta \mapsto \max(\theta, 1-\theta)$ is not concave, we deduce that $R_e[T]$ is not concave on Δ. ∎

Funding. This work is partially supported by Labex Bézout reference ANR-10-LABX-58.

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