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# On the Moebius deformable hypersurfaces

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**Abstract.** Li, Ma and Wang [Adv. Math. 256 (2014), 156–205] have investigated the interesting class of Moebius deformable hypersurfaces, that is, the umbilic-free Euclidean hypersurfaces $f\colon M^n \to \mathbb{R}^{n+1}$ that admit non-trivial deformations preserving the Moebius metric. The classification of Moebius deformable hypersurfaces of dimension $n \geq 4$ stated in the aforementioned article, however, misses a large class of examples. In this article, we complete that classification for $n \geq 5$.

## 1. Introduction

Let $f\colon M^n \to \mathbb{R}^m$ be an isometric immersion of a Riemannian manifold $(M^n, g)$ into Euclidean space, and let $\alpha \in \Gamma(\mathrm{Hom}(TM, TM; N_f M))$ be its normal bundle-valued second fundamental form. Let $\|\alpha\|^2 \in C^\infty(M)$ be given at any point $x \in M^n$ by

$$\|\alpha(x)\|^2 = \sum_{i,j=1}^{n} \|\alpha(x)(X_i, X_j)\|^2,$$

where $\{X_i\}_{1\leq i\leq n}$ is an orthonormal basis of $T_x M$. Define $\phi \in C^\infty(M)$ by

$$\phi^2 = \frac{n}{n-1}\,(\|\alpha\|^2 - n\|\mathcal{H}\|^2), \tag{1.1}$$

where $\mathcal{H}$ is the mean curvature vector field of $f$. Notice that $\phi$ vanishes precisely at the umbilical points of $f$. The metric

$$g^* = \phi^2 g,$$

defined on the open subset of non-umbilical points of $f$, is called the *Moebius metric* determined by $f$. The metric $g^*$ is invariant under Moebius transformations of the ambient space, that is, if two immersions differ by a Moebius transformation of $\mathbb{R}^m$, then their corresponding Moebius metrics coincide.

It was shown in [9] that a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is uniquely determined, up to Moebius transformations of the ambient space, by its Moebius metric and its *Moebius shape operator* $S = \phi^{-1}(A - HI)$, where $A$ is the shape operator of $f$ with respect to a

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unit normal vector field $N$ and $H$ is the corresponding mean curvature function. A similar result holds for submanifolds of arbitrary codimension (see [9] and Section 9.8 of [4]).

Li, Ma and Wang investigated in [8] the natural and interesting problem of looking for the hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ that are not determined, up to Moebius transformations of $\mathbb{R}^{n+1}$, only by their Moebius metrics. This fits into the fundamental problem in submanifold theory of looking for data that are sufficient to determine a submanifold up to some group of transformations of the ambient space.

More precisely, an umbilic-free hypersurface $f: M^n \to \mathbb{R}^{n+1}$ is said to be *Moebius deformable* if there exists an immersion $\tilde{f}: M^n \to \mathbb{R}^{n+1}$ that shares with $f$ the same Moebius metric and is not Moebius congruent to $f$ on any open subset of $M^n$. The first result in [8] is that a Moebius deformable hypersurface with dimension $n \geq 4$ must carry a principal curvature with multiplicity at least $n-2$. As pointed out in [8], for $n \geq 5$ this is already a consequence of Cartan's classification in [1] (see also [3] and Chapter 17 of [4]) of the more general class of *conformally deformable* hypersurfaces. These are the hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ that admit a non-trivial *conformal deformation* $\tilde{f}: M^n \to \mathbb{R}^{n+1}$, that is, an immersion such that $f$ and $\tilde{f}$ induce conformal metrics on $M^n$ and do not differ by a Moebius transformation of $\mathbb{R}^{n+1}$ on any open subset of $M^n$.

According to Cartan's classification, besides the conformally flat hypersurfaces, which have a principal curvature with multiplicity greater than or equal to $n-1$ and are highly conformally deformable, the remaining ones fall into one of the following classes:

(i) *conformally surface-like hypersurfaces*, that is, those that differ by a Moebius transformation of $\mathbb{R}^{n+1}$ from cylinders and rotation hypersurfaces over surfaces in $\mathbb{R}^3$, or from cylinders over three-dimensional hypersurfaces of $\mathbb{R}^4$ that are cones over surfaces in $\mathbb{S}^3$;

(ii) *conformally ruled hypersurfaces*, i.e., hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ for which $M^n$ carries an integrable $(n-1)$-dimensional distribution whose leaves are mapped by $f$ into umbilical submanifolds of $\mathbb{R}^{n+1}$;

(iii) hypersurfaces that admit a non-trivial *conformal variation* $F: (-\varepsilon, \varepsilon) \times M^n \to \mathbb{R}^{n+1}$, that is, a smooth map defined on the product of an open interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ with $M^n$ such that, for any $t \in (-\varepsilon, \varepsilon)$, the map $f_t = F(t; \cdot)$, with $f_0 = f$, is a non-trivial *conformal deformation* of $f$;

(iv) hypersurfaces that admit a single non-trivial conformal deformation.

It was shown in [8] that, among the conformally surface-like hypersurfaces, the ones that are Moebius deformable are those that are determined by a Bonnet surface $h: L^2 \to \mathbb{Q}^3_\varepsilon$ admitting isometric deformations preserving the mean curvature function. Here $\mathbb{Q}^3_\varepsilon$ stands for a space form of constant sectional curvature $\varepsilon \in \{-1, 0, 1\}$. It was also shown in [8] that an umbilic-free conformally flat hypersurface $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 4$ (hence with a principal curvature of constant multiplicity $n-1$), admits non-trivial deformations preserving the Moebius metric if and only if it has constant Moebius curvature, that is, its Moebius metric has constant sectional curvature. Such hypersurfaces were classified in [5], and an alternative proof of the classification was given in [8]. They were shown to be, up to Moebius transformations of $\mathbb{R}^{n+1}$, either cylinders or rotation hypersurfaces over the so-called *curvature spirals* in $\mathbb{R}^2$ or $\mathbb{R}^2_+$, respectively, the latter endowed with the hyperbolic metric, or cylinders over surfaces that are cones over curvature spirals in $\mathbb{S}^2$.

It is claimed in [8] that there exists only one further example of a Moebius deformable hypersurface, which belongs to the third of the above classes in Cartan's classification of the conformally deformable hypersurfaces. Namely, the hypersurface given by

$$f = \Phi \circ (\mathrm{id} \times f_1) : M^n := \mathbb{H}^{n-3}_{-m} \times N^3 \to \mathbb{R}^{n+1}, \quad m = \sqrt{\frac{n-1}{n}}, \tag{1.2}$$

where id is the identity map of $\mathbb{H}^{n-3}_{-m}$, $f_1 \colon N^3 \to \mathbb{S}^4_m$ is Cartan's minimal isoparametric hypersurface, which is a tube over the Veronese embedding of $\mathbb{RP}^2$ into $\mathbb{S}^4_m$, and $\Phi \colon \mathbb{H}^{n-3}_{-m} \times \mathbb{S}^4_m \subset \mathbb{L}^{n-2} \times \mathbb{R}^5 \to \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-4}$ is the conformal diffeomorphism given by

$$\Phi(x, y) = \frac{1}{x_0}(x_1, \ldots, x_{n-4}, y)$$

for all $x = x_0 e_0 + x_1 e_1 + \cdots + x_{n-3} e_{n-3} \in \mathbb{L}^{n-2}$ and $y = (y_1, \ldots, y_5) \in \mathbb{S}^4 \subset \mathbb{R}^5$. Here $\{e_0, \ldots, e_{n-3}\}$ denotes a pseudo-orthonormal basis of the Lorentzian space $\mathbb{L}^{n-2}$ with $\langle e_0, e_0\rangle = 0 = \langle e_{n-3}, e_{n-3}\rangle$ and $\langle e_0, e_{n-3}\rangle = -1/2$. The deformations of $f$ preserving the Moebius metric have been shown to be actually compositions $f_t = f \circ \phi_t$ of $f$ with the elements of a one-parameter family of isometries $\phi_t \colon M^n \to M^n$ with respect to the Moebius metric; hence all of them have the same image as $f$.

The initial goal of this article was to investigate the larger class of *infinitesimally* Moebius bendable hypersurfaces, that is, umbilic-free hypersurfaces $f \colon M^n \to \mathbb{R}^{n+1}$ for which there exists a one-parameter family of immersions $f_t \colon M^n \to \mathbb{R}^{n+1}$, with $t \in (-\varepsilon, \varepsilon)$ and $f_0 = f$, such that the Moebius metrics determined by $f_t$ coincide up to the first order, in the sense that $\frac{\partial}{\partial t}|_{t=0} g^*_t = 0$. This is carried out for $n \geq 5$ in the forthcoming paper [7].

In the course of our investigation, however, we realized that the infinitesimally Moebius bendable hypersurfaces of dimension $n \geq 5$ in our classification that are not conformally surface-like are actually also Moebius deformable. Nevertheless, except for the example in the preceding paragraph, they do not appear in the classification of such hypersurfaces as stated in [8]. This has led us to revisit that classification under a different approach from that in [8].

To state our result, we need to recall some terminology. Let $f \colon M^n \to \mathbb{R}^{n+1}$ be an oriented hypersurface with respect to a unit normal vector field $N$. Then the family of hyperspheres $x \in M^n \mapsto S(h(x), r(x))$ with radius $r(x)$ and center $h(x) = f(x) + r(x)N(x)$ is enveloped by $f$. If, in particular, $1/r$ is the mean curvature of $f$, such family of hyperspheres is called the *central sphere congruence* of $f$.

Let $\mathbb{V}^{n+2}$ denote the light cone in the Lorentz space $\mathbb{L}^{n+3}$ and let $\Psi = \Psi_{v,w,C} \colon \mathbb{R}^{n+1} \to \mathbb{L}^{n+3}$ be the isometric embedding onto

$$\mathbb{E}^{n+1} = \mathbb{E}^{n+1}_w = \{u \in \mathbb{V}^{n+2} : \langle u, w\rangle = 1\} \subset \mathbb{L}^{n+3}$$

given by

$$\Psi(x) = v + Cx - \frac{1}{2}\|x\|^2 w, \tag{1.3}$$

in terms of $w \in \mathbb{V}^{n+2}$, $v \in \mathbb{E}^{n+1}$ and a linear isometry $C \colon \mathbb{R}^{n+1} \to \{v, w\}^\perp$. Then we have that the congruence of hyperspheres $x \in M^n \mapsto S(h(x), r(x))$ is determined by the map $S\colon M^n \to \mathbb{S}^{n+2}_{1,1}$ that takes values in the Lorentzian sphere

$$\mathbb{S}^{n+2}_{1,1} = \{x \in \mathbb{L}^{n+3} \colon \langle x, x\rangle = 1\}$$

and is defined by

$$S(x) = \frac{1}{r(x)}\,\Psi(h(x)) + \frac{r(x)}{2}\,w,$$

in the sense that $\Psi(S(h(x), r(x))) = \mathbb{E}^{n+1} \cap S(x)^{\perp}$ for all $x \in M^n$. The map $S$ has rank $0 < k < n$, that is, it corresponds to a $k$-parameter congruence of hyperespheres, if and only if $\lambda = 1/r$ is a principal curvature of $f$ with constant multiplicity $n-k$ (see Section 9.3 of [4] for details). In this case, $S$ gives rise to a map $s\colon L^k \to \mathbb{S}^{n+2}_{1,1}$ such that $S \circ \pi = s$, where $\pi\colon M^n \to L^k$ is the canonical projection onto the quotient space of leaves of $\ker(A - \lambda I)$.

**Theorem 1.1.** *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a Moebius deformable hypersurface that is not conformally surface-like on any open subset and has a principal curvature of constant multiplicity $n-2$. Then the central sphere congruence of $f$ is determined by a minimal space-like surface $s\colon L^2 \to \mathbb{S}^{n+2}_{1,1}$.*

*Conversely, any simply connected hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, whose central sphere congruence is determined by a minimal space-like surface $s\colon L^2 \to \mathbb{S}^{n+2}_{1,1}$ is Moebius deformable. In fact, $f$ is Moebius bendable: it admits precisely a one-parameter family of conformal deformations, all of which share with $f$ the same Moebius metric.*

**Remarks 1.2.** (1) Particular examples of Moebius deformable hypersurfaces $f\colon M^n \to \mathbb{R}^{n+1}$ that are not conformally surface-like on any open subset and have a principal curvature of constant multiplicity $n-2$ are the minimal hypersurfaces of rank two. These are well-known to admit a one-parameter associated family of isometric deformations, all of which are also minimal of rank two. The elements of the associated family, sharing with $f$ the same induced metric, all have the same scalar curvature and, being minimal, also share with $f$ the same Moebius metric. These examples are not comprised in the statement of Proposition 9.2 in [8] and, since the elements of the associated family of a minimal hypersurface of rank two do not have in general the same image, neither in the statement of Theorem 1.5 therein.

(2) More general examples are the compositions $f = P \circ h$ of minimal hypersurfaces $h\colon M^n \to \mathbb{Q}^{n+1}_c$ of rank two with a "stereographic projection" $P$ of $\mathbb{Q}^{n+1}_c$ (minus one point if $c > 0$) onto $\mathbb{R}^{n+1}$. The latter are precisely the hypersurfaces $f\colon M^n \to \mathbb{R}^{n+1}$ with a principal curvature of constant multiplicity $n-2$ whose central sphere congruences are determined by minimal space-like surfaces $s\colon L^2 \to \mathbb{S}^{n+2}_{1,1} \subset \mathbb{L}^{n+3}$ such that $s(L)$ is contained in a hyperplane of $\mathbb{L}^{n+3}$ orthogonal to a vector $T \in \mathbb{L}^{n+3}$ satisfying $-\langle T, T\rangle = c$ (see, e.g., Corollary 3.4.6 in [6]).

(3) The central sphere congruence of the hypersurface given by (1.2) is a Veronese surface in a sphere $\mathbb{S}^4 \subset \mathbb{S}^{n+2}_{1,1}$.

(4) The proof of Theorem 1.1 makes use of some arguments in the classification of the conformally deformable hypersurfaces of dimension $n \geq 5$ given in Chapter 17 of [4].

## 2. Preliminaries

In this short section, we recall some basic definitions and state Wang's fundamental theorem for hypersurfaces in Moebius geometry.

Let $f: M \to \mathbb{R}^{n+1}$ be an umbilic-free immersion with Moebius metric $g^* = \langle \cdot, \cdot \rangle^*$ and Moebius shape operator $S$. The *Blaschke tensor* $\psi$ of $f$ is the endomorphism defined by

$$\langle \psi X, Y \rangle^* = \frac{H}{\phi} \langle SX, Y \rangle^* + \frac{1}{2\phi^2} \big( \|\text{grad}^* \phi\|_*^2 + H^2 \big) \langle X, Y \rangle^* - \frac{1}{\rho} \text{Hess}^* \phi(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$, where grad$^*$ and Hess$^*$ stand for the gradient and Hessian, respectively, with respect to $g^*$. The *Moebius form* $\omega \in \Gamma(T^*M)$ of $f$ is defined by

$$\omega(X) = -\frac{1}{\phi} \langle \text{grad}^* H + S \text{grad}^* \phi, X \rangle^*.$$

The Moebius shape operator, the Blaschke tensor and the Moebius form of $f$ are Moebius invariant tensors that satisfy the conformal Gauss and Codazzi equations:

$$R^*(X, Y) = SX \wedge^* SY + \psi X \wedge^* Y + X \wedge^* \psi Y, \tag{2.1}$$

$$(\nabla^*_X S)Y - (\nabla^*_Y S)X = \omega(X)Y - \omega(Y)X, \tag{2.2}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\nabla^*$ denotes the Levi-Civita connection, $R^*$ the curvature tensor and $\wedge^*$ the wedge product with respect to $g^*$. We also point out for later use that the Moebius shape operator $S = \phi^{-1}(A - HI)$, besides being traceless, has constant norm $\sqrt{(n-1)/n}$.

The following fundamental result was proved by Wang (see Theorem 3.1 in [9]).

**Proposition 2.1.** *Two umbilic-free hypersurfaces $f_1, f_2: M^n \to \mathbb{R}^{n+1}$ are conformally* (*Moebius*) *congruent if and only if they share the same Moebius metric and the same Moebius second fundamental form* (*up to sign*).

## 3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In the first subsection, we use the theory of flat bilinear forms to give an alternative proof of a key proposition proved in [8] on the structure of the Moebius shape operators of Moebius deformable hypersurfaces. The proof of Theorem 1.1 is provided in the subsequent subsection.

### 3.1. Moebius shape operators of Moebius deformable hypersurfaces

The starting point for the proof of Theorem 1.1 is Proposition 3.3 below, which gives the structure of the Moebius shape operator of a Moebius deformable hypersurface of dimension $n \geq 5$ that carries a principal curvature of multiplicity $(n-2)$ and is not conformally surface-like on any open subset.

First we provide, for the sake of completeness, an alternative proof for $n \geq 5$, based on the theory of flat bilinear forms, of a result first proved for $n \geq 4$ by Li, Ma and Wang in [8] (see Theorem 6.1 therein) on the structure of the Moebius shape operators of any pair of Euclidean hypersurfaces of dimension $n \geq 5$ that are Moebius deformations of each other (see Proposition 3.2 below).

Recall that if $W^{p,q}$ is a vector space of dimension $p+q$ endowed with an inner product $\langle\langle\ ,\ \rangle\rangle$ of signature $(p,q)$, and $V$, $U$ are finite dimensional vector spaces, then a bilinear form $\beta\colon V \times U \to W^{p,q}$ is said to be *flat* with respect to $\langle\langle\ ,\ \rangle\rangle$ if

$$\langle\langle \beta(X,Y), \beta(Z,T)\rangle\rangle - \langle\langle \beta(X,T), \beta(Z,Y)\rangle\rangle = 0$$

for all $X, Z \in V$ and $Y, T \in U$. It is called *null* if

$$\langle\langle \beta(X,Y), \beta(Z,T)\rangle\rangle = 0$$

for all $X, Z \in V$ and $Y, T \in U$. Thus a null bilinear form is necessarily flat.

**Proposition 3.1.** *Let $f_1, f_2\colon M^n \to \mathbb{R}^{n+1}$ be umbilic-free immersions that share the same Moebius metric $\langle\ ,\ \rangle^*$. Let $S_i$ and $\psi_i$, $i = 1,2$, denote their corresponding Moebius shape operators and Blaschke tensors. Then, for each $x \in M^n$, the bilinear form $\Theta\colon T_xM \times T_xM \to \mathbb{R}^{2,2}$ defined by*

$$\Theta(X,Y) = \Big(\langle S_1X, Y\rangle^*, \frac{1}{\sqrt{2}}\langle \Psi_+X, Y\rangle^*, \langle S_2X, Y\rangle^*, \frac{1}{\sqrt{2}}\langle \Psi_-X, Y\rangle^*\Big),$$

*where $\Psi_\pm = I \pm (\psi_1 - \psi_2)$, is flat with respect to the (indefinite) inner product $\langle\langle \cdot, \cdot \rangle\rangle$ in $\mathbb{R}^{2,2}$. Moreover, $\Theta$ is null for all $x \in M^n$ if and only if $f_1$ and $f_2$ are Moebius congruent.*

*Proof.* Using (2.1) for $f_1$ and $f_2$, we obtain

$$\begin{aligned}\langle\langle \Theta(X,Y), &\Theta(Z,W)\rangle\rangle - \langle\langle \Theta(X,W), \Theta(Z,Y)\rangle\rangle \\ &= \langle (S_1Z \wedge^* S_1X)Y, W\rangle^* - \langle (S_2Z \wedge^* S_2X)Y, W\rangle^* \\ &\quad + \langle ((\psi_1 - \psi_2)Z \wedge^* X)Y, W\rangle^* + \langle (Z \wedge^* (\psi_1 - \psi_2)X)Y, W\rangle^* \\ &= 0\end{aligned}$$

for all $x \in M^n$ and $X, Y, Z, W \in T_xM$, which proves the first assertion.

Assume now that $\Theta$ is null for all $x \in M^n$. Then

$$\begin{aligned}0 = \langle\langle \Theta(X,Y), \Theta(Z,W)\rangle\rangle &= \langle S_1X, Y\rangle^* \langle S_1Z, W\rangle^* - \langle S_2X, Y\rangle^* \langle S_2Z, W\rangle^* \\ &+ \frac{1}{2}\langle (I + (\psi_1 - \psi_2))X, Y)\rangle^* \langle (I + (\psi_1 - \psi_2))Z, W)\rangle^* \\ &- \frac{1}{2}\langle (I - (\psi_1 - \psi_2))X, Y)\rangle^* \langle (I - (\psi_1 - \psi_2))Z, W)\rangle^*\end{aligned}$$

for all $x \in M^n$ and $X, Y, Z, W \in T_xM$. This is equivalent to

$$\begin{aligned}&\langle S_1X, Y\rangle^* S_1 - \langle S_2X, Y\rangle^* S_2 + \frac{1}{2}\langle (I + (\psi_1 - \psi_2))X, Y)\rangle^* (I + (\psi_1 - \psi_2)) \\ &\quad - \frac{1}{2}\langle (I - (\psi_1 - \psi_2))X, Y)\rangle^* (I - (\psi_1 - \psi_2)) \\ &= \langle S_1X, Y\rangle^* S_1 - \langle S_2X, Y\rangle^* S_2 + \langle X, Y\rangle^* (\psi_1 - \psi_2) + \langle (\psi_1 - \psi_2)X, Y\rangle^* I \\ &= 0\end{aligned} \tag{3.1}$$

for all $x \in M^n$ and $X, Y \in T_xM$. Now we use that

$$(3.2) \qquad (n-2)\langle \psi_i X, Y\rangle^* = \mathrm{Ric}^*(X,Y) + \langle S_i^2 X, Y\rangle^* - \frac{n^2 s^* + 1}{2n}\langle X, Y\rangle^*$$

for all $X, Y \in T_xM$, where $\mathrm{Ric}^*$ and $s^*$ are the Ricci and scalar curvatures of the Moebius metric (see, e.g., Proposition 9.20 in [4]), which implies that

$$\mathrm{tr}\,\psi_1 = \frac{n^2 s^* + 1}{2n} = \mathrm{tr}\,\psi_2.$$

Therefore, taking traces in (3.1) yields

$$\langle (\psi_1 - \psi_2)X, Y\rangle^* = 0$$

for all $x \in M^n$ and $X, Y \in T_xM$. Thus $\psi_1 = \psi_2$, and hence $\langle S_1 X, Y\rangle^* S_1 = \langle S_2 X, Y\rangle^* S_2$. In particular, $S_1$ and $S_2$ commute. Let $\lambda_i$ and $\rho_i$, $1 \le i \le n$, denote their respective eigenvalues. Then $\lambda_i\lambda_j = \rho_i\rho_j$ for all $1 \le i, j \le n$ and, in particular, $\lambda_i^2 = \rho_i^2$ for any $1 \le i \le n$. If $\lambda_1 = \rho_1 \neq 0$, then $\lambda_j = \rho_j$ for any $j$, and then $S_1 = S_2$. Similarly, if $\lambda_1 = -\rho_1 \neq 0$, then $S_1 = -S_2$. Therefore, in any case, $f_1$ and $f_2$ are Moebius congruent by Proposition 2.1. ■

**Proposition 3.2.** *Let $f_1, f_2 \colon M^n \to \mathbb{R}^{n+1}$, $n \ge 5$, be umbilic-free immersions that are Moebius deformations of each other. Then there exists a distribution $\Delta$ of rank $(n-2)$ on an open and dense subset $\mathcal{U} \subset M^n$ such that, for each $x \in \mathcal{U}$, $\Delta(x)$ is contained in eigenspaces of the Moebius shape operators of both $f_1$ and $f_2$ at $x$ correspondent to a common eigenvalue (up to sign).*

*Proof.* First notice that, for each $x \in M^n$, the kernel

$$\mathcal{N}(\Theta) := \{Y \in T_xM \ : \ \Theta(X,Y) = 0 \text{ for all } X \in T_xM\}$$

of the flat bilinear form $\Theta \colon T_xM \times T_xM \to \mathbb{R}^{2,2}$ given by Proposition 3.1 is trivial, for if $Y \in T_xM$ belongs to $\mathcal{N}(\Theta)$, then $\langle \Psi_+ Y, Y\rangle^* = 0 = \langle \Psi_- Y, Y\rangle^*$, which implies that $\langle Y, Y\rangle = 0$, and hence $Y = 0$.

Now, by Proposition 2.1 and the last assertion in Proposition 3.1, the flat bilinear form $\Theta$ is not null on any open subset of $M^n$, for $f_1$ and $f_2$ are not Moebius congruent on any open subset of $M^n$. Let $\mathcal{U} \subset M^n$ be the open and dense subset where $\Theta$ is not null. Since $n \ge 5$, it follows from Lemma 4.22 in [4] that, at any $x \in \mathcal{U}$, there exists an orthogonal decomposition $\mathbb{R}^{2,2} = W_1^{1,1} \oplus W_2^{1,1}$ according to which $\Theta$ decomposes as $\Theta = \Theta_1 + \Theta_2$, where $\Theta_1$ is null and $\Theta_2$ is flat with $\dim \mathcal{N}(\Theta_2) \ge n-2$.

We claim that $\Delta = \mathcal{N}(\Theta_2)$ is contained in eigenspaces of both $S_1$ and $S_2$ at any $x \in \mathcal{U}$. In order to prove this, take any $T \in \Gamma(\Delta)$, so that $\Theta(X,T) = \Theta_1(X,T)$ for any $X \in T_xM$, and hence $\langle\langle \Theta(X,T), \Theta(Z,Y)\rangle\rangle = 0$ for all $X, Y, Z \in T_xM$. Equivalently,

$$(3.3) \quad \langle S_1 X, T\rangle^* S_1 - \langle S_2 X, T\rangle^* S_2 + \langle (\psi_1 - \psi_2)X, T\rangle^* I + \langle X, T\rangle^* (\psi_1 - \psi_2) = 0$$

for any $X \in T_xM$. In particular, for $X$ orthogonal to $T$,

$$\langle S_1 X, T\rangle^* S_1 - \langle S_2 X, T\rangle^* S_2 + \langle (\psi_1 - \psi_2)X, T\rangle I = 0.$$

Assume that $T$ is not an eigenvector of $S_1$. Then there exists $X$ orthogonal to $T$ such that $\langle S_1X, T\rangle^* \neq 0$. Since $f_1$ is umbilic-free, we must have $\langle S_2X, T\rangle^* \neq 0$. Thus $S_1$ and $S_2$ are mutually diagonalizable. Let $X_1, \dots, X_n$ be an orthonormal diagonalizing basis of both $S_1$ and $S_2$ with respective eigenvalues $\lambda_i$ and $\rho_i$, $1 \leq i \leq n$. Since $T$ is not an eigenvector, there are at least two distinct eigenvalues, say, $0 \neq \lambda_1 \neq \lambda_2$, with corresponding eigenvectors $X_1$ and $X_2$, such that $\langle X_1, T\rangle^* \neq 0 \neq \langle X_2, T\rangle^*$. Thus (3.3) yields

$$\lambda_1\langle X_1, T\rangle^* S_1 - \rho_1\langle X_1, T\rangle^* S_2 + \langle(\psi_1 - \psi_2)X_1, T\rangle^* I + \langle X_1, T\rangle^*(\psi_1 - \psi_2) = 0,$$
$$\lambda_2\langle X_2, T\rangle^* S_1 - \rho_2\langle X_2, T\rangle^* S_2 + \langle(\psi_1 - \psi_2)X_2, T\rangle^* I + \langle X_2, T\rangle^*(\psi_1 - \psi_2) = 0.$$

It follows from (3.2) that $(n-2)(\psi_1 - \psi_2) = S_1^2 - S_2^2$. Hence

$$\lambda_1 S_1 - \rho_1 S_2 + \frac{1}{n-2}(\lambda_1^2 - \rho_1^2) I + (\psi_1 - \psi_2) = 0,$$
$$\lambda_2 S_1 - \rho_2 S_2 + \frac{1}{n-2}(\lambda_2^2 - \rho_2^2) I + (\psi_1 - \psi_2) = 0.$$

Taking traces in the above expressions, we obtain

$$\lambda_1^2 - \rho_1^2 = 0 = \lambda_2^2 - \rho_2^2.$$

On the other hand, the above relations also yield

$$\lambda_1\lambda_i - \rho_1\rho_i + \frac{1}{n-2}(\lambda_i^2 - \rho_i^2) = 0,$$
$$\lambda_2\lambda_i - \rho_2\rho_i + \frac{1}{n-2}(\lambda_i^2 - \rho_i^2) = 0,$$

for any $1 \leq i \leq n$. Assume first that $\lambda_1 = \rho_1$, and hence $\lambda_2 = \rho_2$. Then the preceding expressions become

$$(\lambda_i - \rho_i)\Big(\lambda_j + \frac{1}{n-2}(\lambda_i + \rho_i)\Big) = 0$$

for $j = 1, 2$ and $1 \leq i \leq n$. Since $S_1 \neq S_2$ and both tensors have vanishing trace, there must exist at least two directions for which $\lambda_i - \rho_i \neq 0$. For such a fixed direction, say $k$, we have

$$\lambda_j + \frac{1}{n-2}(\lambda_k + \rho_k) = 0,$$

with $j = 1, 2$. Thus $\lambda_1 = \lambda_2$, which is a contradiction.

Similarly, if we assume $\lambda_1 = -\rho_1$, we obtain that $\lambda_2 = -\rho_2$, and then

$$(\lambda_i + \rho_i)\Big(\lambda_j + \frac{1}{n-2}(\lambda_i - \rho_i)\Big) = 0$$

for $j = 1, 2$ and $1 \leq i \leq j$. By the same argument as above, we see that $\lambda_1 = \lambda_2$, reaching again a contradiction. Therefore $T$ must be an eigenvector of $S_1$. Since $S_2$ is not a multiple of the identity, taking $X$ orthogonal to $T$ we see from (3.3) that $T$ must also be an eigenvector of $S_2$. Given that $T \in \Gamma(\Delta)$ was chosen arbitrarily, we conclude that $\Delta$ is contained in eigenspaces of both $S_1$ and $S_2$.

Let $\mu_1$ and $\mu_2$ be such that $S_1|_\Delta = \mu_1 I$ and $S_2|_\Delta = \mu_2 I$. By (3.3), we have

$$\mu_1^2 - \mu_2^2 + \frac{2}{n-2}(\mu_1^2 - \mu_2^2) = 0.$$

Thus $\mu_1^2 - \mu_2^2 = 0$, and hence $\mu_1 = \pm\mu_2$.

It remains to argue that $\dim \Delta = n-2$. After changing the normal vector of either $f_1$ or $f_2$, if necessary, one can assume that $\mu_1 = \mu_2 := \mu$. Since $S_1|_\Delta = \mu I = S_2|_\Delta$, if $\dim \Delta = n-1$ then the condition $\operatorname{tr}(S_1) = 0 = \operatorname{tr}(S_2)$ would imply that $S_1 = S_2$, a contradiction. ■

Now we make the extra assumptions that $f$ is not conformally surface-like on any open subset of $M^n$ and has a principal curvature with constant multiplicity $n-2$.

**Proposition 3.3.** *Let $f_1\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a Moebius deformable hypersurface with a principal curvature $\lambda$ of constant multiplicity $n-2$. Assume that $f_1$ is not conformally surface-like on any open subset of $M^n$. If $f_2\colon M^n \to \mathbb{R}^{n+1}$ is a Moebius deformation of $f_1$, then the Moebius shape operators $S_1$ and $S_2$ of $f_1$ and $f_2$, respectively, have constant eigenvalues $\pm\sqrt{(n-1)/2n}$ and 0, and the eigenspace $\Delta$ correspondent to $\lambda$ as a common kernel. In particular, $\lambda$ and the corresponding principal curvature of $f_2$ coincide with the mean curvatures of $f_1$ and $f_2$, respectively. Moreover, the Moebius forms of $f_1$ and $f_2$ vanish on $\Delta$.*

For the proof of Proposition 3.3, we will make use of Lemma 3.4 below (see Theorem 1 in [2] or Corollary 9.33 in [4]), which characterizes conformally surface-like hypersurfaces among hypersurfaces of dimension $n$ that carry a principal curvature with constant multiplicity $n-2$ in terms of the splitting tensor of the corresponding eigenbundle. Recall that, given a distribution $\Delta$ on a Riemannian manifold $M^n$, its *splitting tensor* $C\colon \Gamma(\Delta) \to \Gamma(\operatorname{End}(\Delta^\perp))$ is defined by

$$C_T X = -\nabla_X^h T$$

for all $T \in \Gamma(\Delta)$ and $X \in \Gamma(\Delta^\perp)$, where $\nabla_X^h T = (\nabla_X T)_{\Delta^\perp}$.

**Lemma 3.4.** *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface with a principal curvature of multiplicity $n-2$ and let $\Delta$ denote its eigenbundle. Then $f$ is conformally surface-like if and only if the splitting tensor of $\Delta$ satisfies $C(\Gamma(\Delta)) \subset \operatorname{span}\{I\}$.*

*Proof of Proposition* 3.3. Since $f_1$ has a principal curvature $\lambda$ of constant multiplicity $n-2$, it follows from Proposition 3.2 that, after changing the normal vector field of either $f_1$ or $f_2$, if necessary, we can assume that the Moebius shape operators $S_1$ and $S_2$ of $f_1$ and $f_2$ have a common eigenvalue $\mu$ with the same eigenbundle $\Delta$ of rank $n-2$.

Let $\lambda_i$, $i = 1, 2$, be the eigenvalues of $S_1|_{\Delta^\perp}$. In particular, $\lambda_1 \neq \mu \neq \lambda_2$. The conditions $\operatorname{tr}(S_1) = 0 = \operatorname{tr}(S_2)$ and $\|S_1\|^2 = (n-1)/n = \|S_2\|^2$ imply that $S_1$ and $S_2$ have the same eigenvalues. Then we must also have $\lambda_1 \neq \lambda_2$, for otherwise $S_1$ and $S_2$ would coincide.

Let $X, Y \in \Gamma(\Delta^\perp)$ be an orthonormal frame of eigenvectors of $S_1|_{\Delta^\perp}$ with respect to $g^*$. Then $S_1 X = \lambda_1 X$, $S_1 Y = \lambda_2 Y$, $S_2 X = b_1 X + cY$ and $S_2 Y = cX + b_2 Y$ for some smooth functions $b_1$, $b_2$ and $c$. Since $\operatorname{tr}(S_1) = 0 = \operatorname{tr}(S_2)$ and $\|S_1\|^{*2} = (n-1)/n = \|S_2\|^{*2}$, we have

$$\lambda_1 + \lambda_2 + (n-2)\mu = 0, \tag{3.4}$$

$$\lambda_1^2 + \lambda_2^2 + (n-2)\mu^2 = \frac{n-1}{n}, \tag{3.5}$$

$$b_1 + b_2 + (n-2)\mu = 0, \tag{3.6}$$

$$b_1^2 + b_2^2 + 2c^2 + (n-2)\mu^2 = \frac{n-1}{n}. \tag{3.7}$$

Thus the first assertion in the statement will be proved once we show that $\mu$ vanishes identically. The last assertion will then be an immediate consequence of (2.2).

The umbilicity of $\Delta$, together with (2.2) evaluated in orthonormal sections $T$ and $S$ of $\Delta$ with respect to $g^*$, imply that $\omega_1(T) = T(\mu) = \omega_2(T)$, where $\omega_i$ is the Moebius form of $f_i$, $1 \le i \le 2$. Taking the derivative of (3.4) and (3.5) with respect to $T \in \Gamma(\Delta)$, we obtain

$$T(\lambda_1) = \frac{(n-2)(\mu - \lambda_2)}{\lambda_2 - \lambda_1} T(\mu) \quad \text{and} \quad T(\lambda_2) = \frac{(n-2)(\lambda_1 - \mu)}{\lambda_2 - \lambda_1} T(\mu).$$

The $X$ and $Y$ components of (2.2) for $S_1$ evaluated in $X$ and $T \in \Gamma(\Delta)$ give, respectively,

$$(\mu - \lambda_1)\langle \nabla^*_X T, X\rangle^* = T(\lambda_1) - T(\mu) = -\frac{n\lambda_2}{\lambda_2 - \lambda_1} T(\mu) \tag{3.8}$$

and

$$(\mu - \lambda_2)\langle \nabla^*_X T, Y\rangle^* = (\lambda_1 - \lambda_2)\langle \nabla^*_T X, Y\rangle^*. \tag{3.9}$$

Similarly, the $X$ and $Y$ components of (2.2) for $S_1$ evaluated in $Y$ and $T$ give, respectively,

$$(\mu - \lambda_1)\langle \nabla^*_Y T, X\rangle^* = (\lambda_2 - \lambda_1)\langle \nabla^*_T Y, X\rangle^* \tag{3.10}$$

and

$$(\mu - \lambda_2)\langle \nabla^*_Y T, Y\rangle^* = T(\lambda_2) - T(\mu) = \frac{n\lambda_1}{\lambda_2 - \lambda_1} T(\mu). \tag{3.11}$$

We claim that $S_1$ and $S_2$ do not commute, that is, that $c \neq 0$. Assume otherwise. Then equations (3.4) to (3.7) imply that $S_2 X = \lambda_2 X$ and $S_2 Y = \lambda_1 Y$. Hence, the $X$ and $Y$ components of (2.2) for $S_2$ evaluated in $X$ and $T \in \Gamma(\Delta)$ give, respectively,

$$(\mu - \lambda_2)\langle \nabla^*_X T, X\rangle^* = T(\lambda_2) - T(\mu) \tag{3.12}$$

and

$$(\mu - \lambda_1)\langle \nabla^*_X T, Y\rangle^* = (\lambda_2 - \lambda_1)\langle \nabla^*_T X, Y\rangle^*. \tag{3.13}$$

Similarly, the $X$ and $Y$ components of (2.2) for $S_2$ evaluated in $Y$ and $T$ give, respectively,

$$(\mu - \lambda_2)\langle \nabla^*_Y T, X\rangle^* = (\lambda_1 - \lambda_2)\langle \nabla^*_T Y, X\rangle^* \tag{3.14}$$

and

$$(\mu - \lambda_1)\langle \nabla^*_Y T, Y\rangle^* = T(\lambda_1) - T(\mu). \tag{3.15}$$

Adding (3.9) and (3.13) yields

$$(2\mu - \lambda_1 - \lambda_2)\langle \nabla^*_X T, Y\rangle^* = 0.$$

Similarly, equation s(3.10) and (3.14) give

$$(2\mu - \lambda_1 - \lambda_2)\langle \nabla^*_Y T, X\rangle^* = 0.$$

If $(2\mu - \lambda_1 - \lambda_2)$ does not vanish identically, there exists an open subset $U \subset M^n$ where $\langle \nabla^*_X T, Y\rangle^* = 0 = \langle \nabla^*_Y T, X\rangle^*$. Now, from (3.8) and (3.12) we obtain

$$(\lambda_2 - \lambda_1)\langle \nabla^*_X T, X\rangle^* = T(\lambda_1 - \lambda_2).$$

Similarly, using (3.11) and (3.15) we have

$$(\lambda_1 - \lambda_2)\langle \nabla^*_Y T, Y\rangle^* = T(\lambda_2 - \lambda_1).$$

The preceding equations imply that the splitting tensor $C^*$ of $\Delta$ with respect to the Moebius metric satisfies $C^*_T \in \operatorname{span}\{I\}$ for any $T \in \Gamma(\Delta|_U)$. From the relation between the Levi-Civita connections of conformal metrics, we obtain

$$C^*_T = C_T - T(\log\phi)\, I, \tag{3.16}$$

where $\phi$ is the conformal factor of $g^*$ with respect to the metric induced by $f_1$ and $C$ is the splitting tensor of $\Delta$ corresponding to the latter metric. Therefore, we also have $C_T \in \operatorname{span}\{I\}$ for any $T \in \Gamma(\Delta|_U)$, and hence $f_1|U$ is conformally surface-like by Lemma 3.4, a contradiction. Thus $(2\mu - \lambda_1 - \lambda_2)$ must vanish everywhere, which, together with (3.4), implies that also $\mu$ is everywhere vanishing. Hence $\lambda_1 = -\lambda_2$, and therefore $S_1 = -S_2$, which is again a contradiction, and proves the claim.

Now we compute

$$\begin{aligned}\langle (\nabla^*_T S_2)X, X\rangle^* &= \langle \nabla^*_T(b_1 X + cY), X\rangle^* - \langle S_2\nabla^*_T X, X\rangle^*\\ &= T(b_1) + c\langle \nabla^*_T Y, X\rangle^* - c\langle \nabla^*_T X, Y\rangle^* = T(b_1) + 2c\langle \nabla^*_T Y, X\rangle^*.\end{aligned}$$

In a similar way,

$$\langle (\nabla^*_T S_2)Y, Y\rangle^* = T(b_2) + 2c\langle \nabla^*_T X, Y\rangle^*.$$

Adding the preceding equations and using (3.6) yield

$$\langle (\nabla^*_T S_2)X, X\rangle^* + \langle (\nabla^*_T S_2)Y, Y\rangle^* = (2-n)T(\mu). \tag{3.17}$$

From (2.2), we obtain

$$\begin{aligned}\langle (\nabla^*_T S_2)X, X\rangle^* &= \langle (\nabla^*_X S_2)T, X\rangle^* + T(\mu) = \mu\langle \nabla^*_X T, X\rangle^* - \langle \nabla^*_X T, S_2 X\rangle^* + T(\mu)\\ &= (\mu - b_1)\langle \nabla^*_X T, X\rangle^* - c\langle \nabla^*_X T, Y\rangle^* + T(\mu),\end{aligned}$$

and similarly,

$$\langle (\nabla^*_T S_2)Y, Y\rangle^* = (\mu - b_2)\langle \nabla^*_Y T, Y\rangle^* - c\langle \nabla^*_Y T, X\rangle^* + T(\mu).$$

Substituting the preceding expressions in (3.17) gives

$$nT(\mu)+(\mu-b_1)\langle\nabla^*_X T, X\rangle^*+(\mu-b_2)\langle\nabla^*_Y T, Y\rangle^* = c\langle\nabla^*_X T, Y\rangle^*+c\langle\nabla^*_Y T, X\rangle^*. \tag{3.18}$$

Let us first focus on the terms on the left-hand side of the above equation. Using (3.8) and (3.11), we obtain

$$\begin{aligned} nT(\mu) &+ (\mu-b_1)\langle\nabla^*_X T, X\rangle^* + (\mu-b_2)\langle\nabla^*_Y T, Y\rangle^* \\ &= nT(\mu) - \frac{n\lambda_2(\mu-b_1)}{(\mu-\lambda_1)(\lambda_2-\lambda_1)}\,T(\mu) + \frac{n\lambda_1(\mu-b_2)}{(\mu-\lambda_2)(\lambda_2-\lambda_1)}\,T(\mu) \\ &= \frac{(n-1)(\lambda_1-b_1)}{(\mu-\lambda_2)(\lambda_2-\lambda_1)}\,T(\mu). \end{aligned}$$

For the right-hand side of (3.18), using (3.9) and (3.10) we have

$$\begin{aligned} c(\langle\nabla^*_X T, Y\rangle^* + \langle\nabla^*_Y T, X\rangle^*) &= c\Big(\frac{\lambda_1-\lambda_2}{\mu-\lambda_2}\,\langle\nabla^*_T X, Y\rangle^* + \frac{\lambda_2-\lambda_1}{\mu-\lambda_1}\,\langle\nabla^*_T Y, X\rangle^*\Big) \\ = c\,\frac{(\lambda_1-\lambda_2)(\mu-\lambda_1+\mu-\lambda_2)}{(\mu-\lambda_1)(\mu-\lambda_2)}\,\langle\nabla^*_T X, T\rangle^* &= c\,\frac{n\mu(\lambda_1-\lambda_2)}{(\mu-\lambda_1)(\mu-\lambda_2)}\,\langle\nabla^*_T X, Y\rangle^*. \end{aligned}$$

Therefore (3.17) becomes

$$(n-1)(b_1-\lambda_1)\,T(\mu) = nc\mu(\lambda_1-\lambda_2)^2\langle\nabla^*_T X, T\rangle^*. \tag{3.19}$$

Now evaluate (2.2) for $S_2$ in $X$ and $T$. More specifically, the $Y$ component of that equation is

$$T(c) = (\mu-b_2)\,\langle\nabla^*_X T, Y\rangle^* - c\,\langle\nabla^*_X T, X\rangle + (b_2-b_1)\,\langle\nabla^*_T X, Y\rangle^*.$$

Substituting (3.8) and (3.9) in the above equation, and using (3.4) and (3.6), we obtain

$$\begin{aligned} T(c) &= \frac{(\mu-b_2)(\lambda_1-\lambda_2)}{\mu-\lambda_2}\,\langle\nabla^*_T X, Y\rangle^* + \frac{cn\lambda_2}{(\mu-\lambda_1)(\lambda_2-\lambda_1)}\,T(\mu) \\ &\quad + (b_2-b_1)\langle\nabla^*_T X, Y\rangle^* \\ &= \frac{\mu\lambda_1-\mu\lambda_2-b_2\lambda_1+b_2\lambda_2+\mu b_2-\mu b_1-\lambda_2 b_2+b_1\lambda_2}{\mu-\lambda_2}\,\langle\nabla^*_T X, Y\rangle^* \\ &\quad + \frac{cn\lambda_2}{(\mu-\lambda_1)(\lambda_2-\lambda_1)}\,T(\mu) \\ &= \frac{n\mu(\lambda_1-b_1)}{\mu-\lambda_2}\,\langle\nabla^*_T X, Y\rangle^* + \frac{cn\lambda_2}{(\mu-\lambda_1)(\lambda_2-\lambda_1)}\,T(\mu). \end{aligned} \tag{3.20}$$

Similarly, the $X$ component of (2.2) for $S_2$ evaluated in $Y$ and $T$ gives

$$T(c) = (\mu-b_1)\,\langle\nabla^*_Y T, X\rangle^* - c\,\langle\nabla^*_Y T, Y\rangle^* + (b_2-b_1)\,\langle\nabla^*_T X, Y\rangle^*.$$

Substituting (3.10) and (3.11) in the above equation, we obtain

$$T(c) = -\frac{cn\lambda_1}{(\mu-\lambda_2)(\lambda_2-\lambda_1)}\, T(\mu) + \frac{n\mu(\lambda_1-b_1)}{\mu-\lambda_1}\,\langle\nabla^*_T X, Y\rangle^*. \tag{3.21}$$

Using (3.5), it follows from (3.20) and (3.21) that

$$(n-1)cT(\mu) = n\mu(\lambda_1-b_1)(\lambda_1-\lambda_2)^2\langle\nabla^*_T X, Y\rangle^*. \tag{3.22}$$

Comparing (3.19) and (3.22) yields

$$\mu\big((\lambda_1-b_1)^2+c^2\big)\langle\nabla^*_T X, Y\rangle^* = 0.$$

Since $(\lambda_1-b_1)^2+c^2 \neq 0$, for otherwise the immersions would be Moebius congruent, then $\mu\langle\nabla^*_T X, Y\rangle^* = 0$.

If $\mu$ does not vanish identically, then there is an open subset $U$ where $\langle\nabla^*_T X, Y\rangle^* = 0$ for any $T \in \Gamma(\Delta)$. Then (3.9) and (3.10) imply that the splitting tensor of $\Delta$ with respect to the Moebius metric satisfies $C^*_T \in \operatorname{span}\{I\}$ for any $T \in \Gamma(\Delta)$. As before, this implies that the splitting tensor of $\Delta$ with respect to the metric induced by $f_1$ also satisfies $C_T \in \operatorname{span}\{I\}$ for any $T \in \Gamma(\Delta)$, and hence $f_1|_U$ is conformally surface-like by Lemma 3.4, a contradiction. Thus $\mu$ must vanish identically. ∎

### 3.2. Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1. First we recall one further definition.

Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface that carries a principal curvature of constant multiplicity $n-2$ with corresponding eigenbundle $\Delta$. Let $C\colon \Gamma(\Delta) \to \Gamma(\operatorname{End}(\Delta^\perp))$ be the splitting tensor of $\Delta$. Then $f$ is said to be *hyperbolic* (respectively, *parabolic* or *elliptic*) if there exists $J \in \Gamma(\operatorname{End}(\Delta^\perp))$ satisfying the following conditions:

(i) $J^2 = I$ and $J \neq I$ (respectively, $J^2 = 0$, with $J \neq 0$, or $J^2 = -I$),

(ii) $\nabla^h_T J = 0$ for all $T \in \Gamma(\Delta)$,

(iii) $C(\Gamma(\Delta)) \subset \operatorname{span}\{I, J\}$, but $C(\Gamma(\Delta)) \not\subset \operatorname{span}\{I\}$.

*Proof of Theorem* 1.1. Let $f_2\colon M^n \to \mathbb{R}^{n+1}$ be a Moebius deformation of $f_1 := f$. By Proposition 3.3, the Moebius shape operators $S_1$ and $S_2$ of $f_1$ and $f_2$, respectively, share a common kernel $\Delta$ of dimension $n-2$. Let $S_i$, $i = 1, 2$, denote also the restriction $S_i|_{\Delta^\perp}$, and define $D \in \Gamma(\operatorname{End}(\Delta^\perp))$ by

$$D = S_1^{-1}S_2.$$

It follows from Proposition 3.3 that $\det D = 1$ at any point of $M^n$, while Proposition 2.1 implies that $D$ cannot be the identity endomorphism up to sign on any open subset $U \subset M^n$, for otherwise $f_1|_U$ and $f_2|_U$ would be Moebius congruent by Lemma 3.4. Therefore, we can write $D = aI + bJ$, where $b$ does not vanish on any open subset of $M^n$, and $J \in \Gamma(\operatorname{End}(\Delta^\perp))$ satisfies $J^2 = \varepsilon I$, with $\varepsilon \in \{1, 0, -1\}$, $J \neq I$ if $\varepsilon = 1$, and $J \neq 0$ if $\varepsilon = 0$.

From the symmetry of $S_2$ and the fact that $b$ does not vanish on any open subset of $M^n$, we see that $S_1 J$ must be symmetric. Moreover, given that $\operatorname{tr} S_1 = 0 = \operatorname{tr} S_2$, also $\operatorname{tr} S_1 J = 0$.

Assume first that $J^2 = 0$. Let $X, Y \in \Gamma(\Delta^\perp)$ be orthogonal vector fields, with $Y$ of unit length (with respect to the Moebius metric $g^*$), such that $JX = Y$ and $JY = 0$. Replacing $J$ by $\|X\|^* J$, if necessary, we can assume that also $X$ has unit length. Let $\alpha, \beta, \gamma \in C^\infty(M)$ be such that $S_1 X = \alpha X + \beta Y$ and $S_1 Y = \beta X + \gamma Y$, so that $S_1 JX = \beta X + \gamma Y$ and $S_1 JY = 0$. From the symmetry of $S_1 J$ and the fact that $\operatorname{tr} S_1 J = 0$, we obtain $\beta = 0 = \gamma$, and hence $\alpha = \operatorname{tr} S_1 = 0$. Thus $S_1 = 0$, which is a contradiction.

Now assume that $J^2 = I$, $J \neq I$. Let $X, Y$ be a frame of unit vector fields (with respect to $g^*$) satisfying $JX = X$ and $JY = -Y$. Write $S_1 X = \alpha X + \beta Y$ and $S_1 Y = \gamma X + \delta Y$ for some $\alpha, \beta, \gamma, \delta \in C^\infty(M)$, so that $S_1 JX = \alpha X + \beta Y$ and $S_1 JY = -\gamma X - \delta Y$. Since $\operatorname{tr} S_1 J = 0 = \operatorname{tr} S_1$, then $\alpha = 0 = \delta$. The symmetry of $S_1$ and $S_2 J$ implies that $\beta = 0 = \gamma$, which is again a contradiction.

Therefore, the only possible case is that $J^2 = -I$. Let $X, Y \in \Gamma(\Delta^\perp)$ be a frame of unit vector fields such that $JX = Y$ and $JY = -X$. Write, as before, $S_1 X = \alpha X + \beta Y$ and $S_1 Y = \gamma X + \delta Y$ for some $\alpha, \beta, \gamma, \delta \in C^\infty(M)$. Then $S_1 JX = \gamma X + \delta Y$ and $S_1 JY = -\alpha X - \beta Y$, hence $\beta = \gamma$, for $\operatorname{tr} S_1 J = 0$. From the symmetry of $S_1$, we obtain

$$\langle S_1 JX, Y\rangle = \langle JX, S_1 Y\rangle = \langle Y, S_1 Y\rangle = \gamma\langle X, Y\rangle + \delta = \beta\langle X, Y\rangle + \delta,$$

and similarly,

$$\langle S_1 JY, X\rangle = -\alpha - \beta\langle X, Y\rangle.$$

Comparing the two preceding equations, and taking into account the symmetry of $S_1 J$ and the fact that $\operatorname{tr} S_1 = 0$, we obtain that $\beta\langle X, Y\rangle = 0$. If $\beta$ is nonzero, then $X$ and $Y$ are orthogonal to each other. This is also the case if $\beta$, hence also $\gamma$, is zero, for in this case $X$ and $Y$ are eigenvectors of $S_1$. Thus, in any case, we conclude that $J$ acts as a rotation of angle $\pi/2$ on $\Delta^\perp$.

Equation (2.2) and the fact that $\omega_i|_\Delta = 0$ imply that the splitting tensor of $\Delta$ with respect to the Moebius metric satisfies

$$\nabla_T^{*h} S_i = S_i C_T^*$$

for all $T \in \Gamma(\Delta)$ and $1 \le i \le 2$, where

$$(\nabla_T^{*h} S_i) X = \nabla_T^{*h} S_i X - S_i \nabla_T^{*h} X$$

for all $X \in \Gamma(\Delta^\perp)$ and $T \in \Gamma(\Delta)$. Here $\nabla_T^{*h} X = (\nabla_T^* X)_{\Delta^\perp}$. In particular,

$$S_i C_T^* = C_T^{*t} S_i, \quad 1 \le i \le 2.$$

Therefore,

$$S_1 D C_T^* = S_2 C_T^* = C_T^{*t} S_2 = C_T^{*t} S_1 D = S_1 C_T^* D,$$

and hence

$$[D, C_T^*] = 0.$$

This implies that $C_T^*$ commutes with $J$, and hence $C_T^* \in \operatorname{span}\{I, J\}$ for any $T \in \Gamma(\Delta)$. It follows from (3.16) that also the splitting tensor $C$ of $\Delta$ corresponding to the metric induced by $f$ satisfies $C_T \in \operatorname{span}\{I, J\}$ for any $T \in \Gamma(\Delta)$. Moreover, by Lemma 3.4 and the assumption that $f$ is not surface-like on any open subset, we see that $C(\Gamma(\Delta)) \not\subset \operatorname{span}\{I\}$ on any open subset. Now, since $J$ acts as a rotation of angle $\pi/2$ on $\Delta^\perp$, then $\nabla_T^h J = 0$. We conclude that $f$ is elliptic with respect to $J$.

By Proposition 3.3, the central sphere congruence $S\colon M^n \to \mathbb{S}^{n+2}_{1,1}$ of $f$ is a two-parameter congruence of hyperspheres, which therefore gives rise to a surface $s\colon L^2 \to \mathbb{S}^{n+2}_{1,1}$ such that $s = S \circ \pi$, where $\pi\colon M^n \to L^2$ is the (local) quotient map onto the space of leaves of $\Delta$. Since $\nabla^h_T J = 0 = [C_T, J]$ for any $T \in \Gamma(\Delta)$, it follows from Corollary 11.7 in [4] that $J$ is projectable with respect to $\pi$, that is, there exists $\bar{J} \in \mathrm{End}(TL)$ such that $\bar{J} \circ \pi_* = \pi_* \circ J$. In particular, the fact that $J^2 = -I$ implies that $\bar{J}^2 = -I$, where we denote also by $I$ the identity endomorphism of $TL$.

Now observe that, since $f_2$ shares with $f_1$ the same Moebius metric, its induced metric is conformal to the metric induced by $f_1$. Moreover, $f_2$ is not Moebius congruent to $f_1$ on any open subset of $M^n$ and $f_1$ has a principal curvature of constant multiplicity $(n-2)$. Thus $f_1$ is a so-called *Cartan hypersurface*. By the proof of the classification of Cartan hypersurfaces given in Chapter 17 of [4] (see Lemma 17.4 therein), the surface $s$ is *elliptic* with respect to $\bar{J}$, that is, for all $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$ we have

$$\alpha^s(\bar{J}\bar{X}, \bar{Y}) = \alpha^s(\bar{X}, \bar{J}\bar{Y}). \tag{3.23}$$

We claim that $\bar{J}$ is an orthogonal tensor, that is, it acts as a rotation of angle $\pi/2$ on each tangent space of $L^2$. The minimality of $s$ will then follow from this, the fact that $\bar{J}^2 = -I$, and (3.23).

In order to show the orthogonality of $\bar{J}$, we use the fact that the metric $\langle \cdot, \cdot \rangle'$ on $L^2$ induced by $s$ is related to the metric of $M^n$ by

$$\langle \bar{Z}, \bar{W} \rangle' = \langle (A - \lambda I)Z, (A - \lambda I)W \rangle \tag{3.24}$$

for all $\bar{Z}, \bar{W} \in \mathfrak{X}(L)$, where $A$ is the shape operator of $f$, $\lambda$ is the principal curvature of $f$ having $\Delta$ as its eigenbundle, which coincides with the mean curvature $H$ of $f$ by Proposition 3.3, and $Z$ and $W$ are the horizontal lifts of $\bar{Z}$ and $\bar{W}$, respectively. Notice that $(A - \lambda I)$ is a multiple of $S_1$. Since $S_1 J$ is symmetric, then also $(A - \lambda I)J$ is symmetric. Therefore, given any $\bar{X} \in \mathfrak{X}(L)$ and denoting by $X \in \Gamma(\Delta^\perp)$ its horizontal lift, we have

$$\begin{aligned} \langle \bar{X}, \bar{J}\bar{X} \rangle' &= \langle (A - \lambda I)X, (A - \lambda I)JX \rangle = \langle (A - \lambda I)J(A - \lambda I)X, X \rangle \\ &= \langle J(A - \lambda I)X, (A - \lambda I)X \rangle = 0, \end{aligned}$$

where in the last equality we have used that $J$ acts as a rotation of angle $\pi/2$ on $\Delta^\perp$. Using again the symmetry of $(A - \lambda I)J$, the proof of the orthogonality of $\bar{J}$ is completed by noticing that

$$\begin{aligned} \langle \bar{J}\bar{X}, \bar{J}\bar{X} \rangle' &= \langle (A - \lambda I)JX, (A - \lambda I)JX \rangle = \langle J(A - \lambda I)JX, (A - \lambda I)X \rangle \\ &= \langle JJ^t(A - \lambda I)X, (A - \lambda I)X \rangle = -\langle J^2(A - \lambda I)X, (A - \lambda I)X \rangle = \langle \bar{X}, \bar{X} \rangle'. \end{aligned}$$

Conversely, assume that the central sphere congruence of $f\colon M^n \to \mathbb{R}^{n+1}$, with $M^n$ simply connected, is determined by a space-like minimal surface $s\colon L^2 \to \mathbb{S}^{n+2}_{1,1}$. Let $\bar{J} \in \Gamma(\mathrm{End}(TL))$ represent a rotation of angle $\pi/2$ on each tangent space. Then $\bar{J}^2 = -I$ and the second fundamental form of $s$ satisfies (3.23) by the minimality of $s$. In particular, $s$ is elliptic with respect to $\bar{J}$. By Lemma 17.4 in [4], the hypersurface $f$ is elliptic with respect to the lift $J \in \Gamma(\mathrm{End}(\Delta^\perp))$ of $\bar{J}$, where $\Delta$ is the eigenbundle correspondent to the principal curvature $\lambda$ of $f$ with multiplicity $n-2$, which coincides with its mean curvature. Therefore, the splitting tensor of $\Delta$ satisfies $C_T \in \text{span}\{I, J\}$ for any $T \in \Gamma(\Delta)$. Since $(A-\lambda I)C_T$ is symmetric for any $T \in \Gamma(\Delta)$, as follows from the Codazzi equation, and $C(\Gamma(\Delta)) \not\subset \text{span}\{I\}$ on any open subset, for $f$ is not conformally surface-like on any open subset, then $(A-\lambda I)J$ is also symmetric.

By Theorem 17.5 in [4], the set of conformal deformations of $f$ is in one-to-one correspondence with the set of tensors $\bar{D} \in \Gamma(\text{End}(TL))$ with $\det \bar{D} = 1$ that satisfy the Codazzi equation

$$\big(\nabla'_{\bar{X}}\bar{D}\big)\bar{Y} - \big(\nabla'_{\bar{Y}}\bar{D}\big)\bar{X} = 0$$

for all $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$, where $\nabla'$ is the Levi-Civita connection of the metric induced by $s$. For a general elliptic hypersurface, this set either consists of a one-parameter family (continuous class) or of a single element (discrete class; see Section 11.2 and Exercise 11.3 in [4]). The surface $s$ is then said to be of the complex type of first or second species, respectively. For a minimal surface $s\colon L^2 \to \mathbb{S}^{n+2}_{1,1}$, each tensor $\bar{J}_\theta = \cos\theta I + \sin\theta \bar{J}$, $\theta \in [0, 2\pi)$, satisfies both the condition $\det \bar{J}_\theta = 1$ and the Codazzi equation, since it is a parallel tensor in $L^2$. Thus $\{\bar{J}_\theta\}_{\theta\in[0,2\pi)}$ is *the* one-parameter family of tensors in $L^2$ having determinant one and satisfying the Codazzi equation. In particular, the surface $s$ is of the complex type of first species. Therefore, the hypersurface $f$ admits a one-parameter family of conformal deformations, each of which determined by one of the tensors $\bar{J}_\theta \in \text{End}(TL)$, $\theta \in [0, 2\pi)$. The proof of Theorem 1.1 will be completed once we prove that any of such conformal deformations shares with $f$ the same Moebius metric.

Let $f_\theta\colon M^n \to \mathbb{R}^{n+1}$ be the conformal deformation of $f$ determined by $\bar{J}_\theta$. Let $F_\theta\colon M^n \to \mathbb{V}^{n+2}$ be the *isometric light-cone representative* of $f_\theta$, that is, $F_\theta$ is the isometric immersion of $M^n$ into the light-cone $\mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ given by $F_\theta = \varphi_\theta^{-1}(\Psi \circ f_\theta)$, where $\varphi_\theta$ is the conformal factor of the metric $\langle\cdot,\cdot\rangle_\theta$ induced by $f_\theta$ with respect to the metric $\langle\cdot,\cdot\rangle$ of $M^n$, that is, $\langle\cdot,\cdot\rangle_\theta = \varphi_\theta^2\langle\cdot,\cdot\rangle$, and $\Psi\colon \mathbb{R}^n \to \mathbb{V}^{n+2}$ is the isometric embedding of $\mathbb{R}^n$ into $\mathbb{V}^{n+2}$ given by (1.3). As shown in the proof of Lemma 17.2 in [4], as part of the proof of the classification of Cartan hypersurfaces of dimension $n \geq 5$ given in Chapter 17 therein, the second fundamental form of $F_\theta$ is given by

$$\alpha^{F_\theta}(X,Y) = \langle AX, Y\rangle\mu - \langle (A-\lambda I)X, Y\rangle\zeta + \langle (A-\lambda I)J_\theta X, Y\rangle\bar{\zeta} \tag{3.25}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\{\mu, \zeta, \bar{\zeta}\}$ is an orthonormal frame of the normal bundle of $F_\theta$ in $\mathbb{L}^{n+3}$ with $\mu$ space-like, $\lambda = -\langle\mu, F_\theta\rangle^{-1}$ and $\zeta = \lambda F_\theta + \mu$ (hence $\langle\zeta,\zeta\rangle = -1$). Here $J_\theta$ is the horizontal lift of $\bar{J}_\theta$, which has been extended to $TM$ by setting $J_\theta|_\Delta = I$.

Let $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$ be an orthonormal frame such that $\bar{J}\bar{X} = \bar{Y}$ and $\bar{J}\bar{Y} = -\bar{X}$, and let $X, Y \in \Gamma(\Delta^\perp)$ be the respective horizontal lifts. It follows from (3.24) that $\{(A-\lambda I)X, (A-\lambda I)Y\}$ is an orthonormal frame of $\Delta^\perp$. From the symmetry of $(A-\lambda I)J$ and $(A-\lambda I)$, we have

$$\begin{aligned}\langle J(A-\lambda I)X, (A-\lambda I)X\rangle &= \langle (A-\lambda I)J(A-\lambda I)X, X\rangle \\ &= \langle (A-\lambda I)X, (A-\lambda I)JX\rangle = \langle \bar{X}, \bar{J}\bar{X}\rangle' = 0.\end{aligned}$$

In a similar way, one verifies that $\langle J(A-\lambda I)Y, (A-\lambda I)Y\rangle = 0$ and

$$\langle J(A-\lambda I)Y, (A-\lambda I)X\rangle = 1 = -\langle J(A-\lambda I)X, (A-\lambda I)Y\rangle.$$

Thus $J$ acts on $\Delta^\perp$ as a rotation of angle $\pi/2$. The symmetry of both $(A-\lambda I)J$ and $(A-\lambda I)$ implies that $\operatorname{tr}(A-\lambda I)=0=\operatorname{tr}(A-\lambda I)J$, hence

$$\operatorname{tr}(A-\lambda I)J_\theta = 0 \tag{3.26}$$

for all $\theta\in[0,2\pi)$.

Now we use the relation between the second fundamental forms of $f_\theta$ and $F_\theta$, given by equation 9.32 in [4], namely,

$$\alpha^{F_\theta}(X,Y)=\langle\varphi(A_\theta-H_\theta I)X,Y\rangle_2\tilde N-\psi(X,Y)F_\theta-\langle X,Y\rangle\zeta_2, \tag{3.27}$$

where $\langle\cdot,\cdot\rangle_\theta=\varphi_\theta^2\langle\cdot,\cdot\rangle$ is the metric induced by $f_\theta$, $A_\theta$ and $H_\theta$ are its shape operator and mean curvature, respectively, $\psi$ is a certain symmetric bilinear form, $\tilde N\in\Gamma(N_FM)$, with $\langle\tilde N,F_\theta\rangle=0$, is a unit space-like vector field, and $\zeta_2\in\Gamma(N_FM)$ satisfies $\langle\tilde N,\zeta_2\rangle=0=\langle\zeta_2,\zeta_2\rangle$ and $\langle F_\theta,\zeta_2\rangle=1$. Equations (3.25) and (3.27) give

$$\langle(A-\lambda I)J_\theta X,Y\rangle=\langle\alpha^{F_\theta}(X,Y),\bar\zeta\rangle=\varphi_\theta\langle(A_\theta-H_\theta I)X,Y\rangle\langle\tilde N,\bar\zeta\rangle-\langle X,Y\rangle\langle\zeta_2,\bar\zeta\rangle$$

for all $X,Y\in\mathfrak{X}(M)$, or equivalently,

$$(A-\lambda I)J_\theta=\varphi_\theta\langle\tilde N,\bar\zeta\rangle(A_\theta-H_\theta I)-\langle\zeta_2,\bar\zeta\rangle I. \tag{3.28}$$

Using that

$$\operatorname{tr}(A-\lambda I)J_\theta=0=\operatorname{tr}(A_\theta-H_\theta I),$$

we obtain from the preceding equation that $\langle\zeta_2,\bar\zeta\rangle=0$. Thus $\bar\zeta\in\operatorname{span}\{F_\theta,\zeta_2\}^\perp$, and hence $\bar\zeta=\pm\tilde N$. Therefore, (3.28) reduces to

$$(A-\lambda I)J_\theta=\pm\varphi(A_\theta-H_\theta I). \tag{3.29}$$

In particular, $(A_\theta-H_\theta I)|_\Delta=0$, hence also $S_\theta|_\Delta=0$, where $S_\theta=\phi_\theta^{-1}(A_\theta-H_\theta I)$ is the Moebius shape operator of $f_\theta$, with $\phi_\theta$ given by (1.1) for $f_\theta$. Since the Moebius shape operator of an umbilic-free immersion is traceless and has constant norm $\sqrt{(n-1)/n}$, then $S_\theta$ must have constant eigenvalues $\sqrt{(n-1)/2n}$, $-\sqrt{(n-1)/2n}$ and 0. The same holds for the Moebius second fundamental form $S_1$ of $f$, which has also $\Delta$ as its kernel. We conclude that the eigenvalues of $(A_\theta-H_\theta I)|_{\Delta^\perp}$ are

$$\delta_1=\phi_\theta\sqrt{(n-1)/2n}\quad\text{and}\quad\delta_2=-\phi_\theta\sqrt{(n-1)/2n}$$

and, similarly, that the eigenvalues of $(A-\lambda I)|_{\Delta^\perp}$ are

$$\lambda_1=\phi_1\sqrt{(n-1)/2n}\quad\text{and}\quad\lambda_2=-\phi_1\sqrt{(n-1)/2n},$$

where $\phi_1$ is given by (1.1) with respect to $f$. On the other hand, since

$$\det((A-\lambda I)J_\theta)=\det((A-\lambda I),$$

for $\det J_\theta=1$, and both $(A-\lambda I)$ and $(A-\lambda I)J_\theta$ are traceless (see (3.26)), it follows that $(A-\lambda I)$ and $(A-\lambda I)J_\theta$ have the same eigenvalues. This and (3.29) imply that

$$\phi_1^2=\varphi_\theta^2\,\phi_\theta^2,$$

hence the Moebius metrics of $f$ and $f_\theta$ coincide. ∎

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## References

[1] Cartan, E.: La déformation des hypersurfaces dans l'espace conforme réel à $n \geq 5$ dimensions. *Bull. Soc. Math. France* **45** (1917), 57–121.

[2] Dajczer, M., Florit, L. A. and Tojeiro, R.: On a class of submanifolds carrying an extrinsic totally umbilical foliation. *Israel J. Math.* **125** (2001), 203–220.

[3] Dajczer, M. and Tojeiro, R.: On Cartan's conformally deformable hypersurfaces. *Michigan Math. J.* **47** (2000), no. 3, 529–557.

[4] Dajczer, M. and Tojeiro, R.: *Submanifold theory. Beyond an introduction*. Universitext, Springer, New York, 2019.

[5] Guo, Z., Li, T., Lin, L., Ma, X. and Wang, C.: Classification of hypersurfaces with constant Möbius curvature in $\mathbb{S}^{m+1}$. *Math. Z.* **271** (2012), no. 1-2, 193–219.

[6] Hertrich-Jeromin, U.: *Introduction to Möbius differential geometry*. London Mathematical Society Lecture Note Series 300, Cambridge University Press, Cambridge, 2003.

[7] Jimenez, M. I. and Tojeiro, R.: Infinitesimally Moebius bendable hypersurfaces. Preprint 2023, arXiv:2307.16341.

[8] Li, T., Ma, X. and Wang, C.: Deformation of hypersurfaces preserving the Möbius metric and a reduction theorem. *Adv. Math.* **256** (2014), 156–205.

[9] Wang, C.: Moebius geometry of submanifolds in $\mathbb{S}^n$. *Manuscripta Math.* **96** (1998), no. 4, 517–534.

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