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Tensorization of quasi-Hilbertian Sobolev spaces

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Abstract. The tensorization problem for Sobolev spaces asks for a characterization of how the Sobolev space on a product metric measure space $X \times Y$ can be determined from its factors. We show that two natural descriptions of the Sobolev space from the literature coincide, $W^{1,2}(X \times Y) = J^{1,2}(X, Y)$, thus settling the tensorization problem for Sobolev spaces in the case $p = 2$, when X and Y are *infinitesimally quasi-Hilbertian*, i.e., the Sobolev space $W^{1,2}$ admits an equivalent renorming by a Dirichlet form. This class includes in particular metric measure spaces X, Y of finite Hausdorff dimension as well as infinitesimally Hilbertian spaces.

More generally, for $p \in (1, \infty)$ we obtain the norm-one inclusion $\| f \|_{J^{1,p}(X,Y)} \leq \| f \|_{W^{1,p}(X \times Y)}$ and show that the norms agree on the algebraic tensor product

$$W^{1,p}(X) \otimes W^{1,p}(Y) \subset W^{1,p}(X \times Y).$$

When $p = 2$ and X and Y are infinitesimally quasi-Hilbertian, standard Dirichlet forms theory yields the density of $W^{1,2}(X) \otimes W^{1,2}(Y)$ in $J^{1,2}(X, Y)$, thus implying the equality of the spaces. Our approach raises the question of the density of $W^{1,p}(X) \otimes W^{1,p}(Y)$ in $J^{1,p}(X, Y)$ in the general case.

1. Introduction

Over the last three decades, Sobolev spaces over *metric spaces* have become a prominent feature in a plethora of geometric problems ranging from Plateau-type problems [15, 23, 24] to quasiconformal uniformization questions [25, 27] and structural problems of spaces with Ricci curvature bounds [4, 5, 17, 19]. During that time, their theory has been studied intensively and significant developments include the unification of different definitions of Sobolev spaces, the density (in energy) of Lipschitz functions in, and the reflexivity of Sobolev spaces over metric spaces in a very general setting – see, e.g., [1, 9, 14, 18, 28].

The *tensorization problem for Sobolev spaces*, first considered in [5], asks whether Sobolev regularity of a function of two variables can be deduced from the existence and integrability of directional derivatives. More precisely, let $X = (X, d_X, \mu)$ and $Y =$

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(Y, d_Y, ν) be two metric measure spaces, let $p \in [1, \infty)$ and let $\big(X \times Y, \sqrt{d_X^2 + d_Y^2}, \mu \times \nu\big)$ be their (Euclidean) product. Given $p \geq 1$, the tensorization problem asks whether the Sobolev space $W^{1,p}(X \times Y)$ coincides with the *Beppo-Levi space* $J^{1,p}(X, Y)$ consisting of functions $f \in L^p(X \times Y)$ for which $f(x, \cdot) \in W^{1,p}(Y)$ for μ-almost every $x \in X$, $f(\cdot, y) \in W^{1,p}(X)$ for ν-almost every $y \in Y$, and

$$(x, y) \mapsto \sqrt{|Df(\cdot, y)|^2(x) + |Df(x, \cdot)|^2(y)} \in L^p(X \times Y). \tag{1.1}$$

In addition, tensorization of Sobolev spaces requires that the minimal p-weak upper gradient of any $f \in J^{1,p}(X, Y)$ is given by (1.1). For the definition of $W^{1,p}(X)$ used in this paper, see Section 1.3.

While immediate in Euclidean spaces, a positive answer to the tensorization problem is non-trivial in the non-smooth setting, and needed, e.g., in the splitting theorem for RCD-spaces [17]. Further, it is of crucial importance in a variety of settings where partial derivatives can be bounded, and one wishes to obtain a bound on the full derivative, see e.g. [6, 12]. Surprisingly, the problem has remained open, even though tensorization of many other properties such as the doubling property, Poincaré inequalities and curvature lower bounds are well known. Previous partial results for $p = 2$ include the work of Ambrosio–Gigli–Savaré [4] for RCD-spaces, of Gigli–Han [20] settling the case where one factor is a closed interval in $\mathbb{R}$, and of Ambrosio–Pinamonti–Speight [6] for PI-spaces. Working in the general case $p \geq 1$ (with a finite dimensionality assumption on the factors), the authors of the present manuscript proved tensorization of Sobolev spaces assuming that *one* of the factors is a PI-space [13] (see also the independent work [16], where warped products are considered). The present work strengthens all of these results in the $p = 2$ case, and proves more general results for $p > 1$.

1.1. Tensorization in infinitesimally quasi-Hilbertian spaces

In this paper, we establish tensorization of Sobolev spaces in the important special case $p = 2$ when the factors are *infinitesimally quasi-Hilbertian*.

Definition 1.1. A metric measure space X is infinitesimally quasi-Hilbertian if there exists a closed Dirichlet form $\mathcal{E}$ with domain $W^{1,2}(X)$ such that $\sqrt{\|u\|^2_{L^2(X)} + \mathcal{E}(u, u)}$ is an equivalent norm on $W^{1,2}(X)$.

See Section 3 for the definition of Dirichlet forms. We remark that infinitesimally Hilbertian spaces, as well as spaces admitting a 2-weak differentiable structure (in particular spaces with finite Hausdorff dimension), are infinitesimally quasi-Hilbertian, cf. Proposition 3.6.

Theorem 1.2. *Suppose that X and Y are infinitesimally quasi-Hilbertian. Then we have that $W^{1,2}(X \times Y) = J^{1,2}(X, Y)$ and, for each $f \in J^{1,2}(X, Y)$, we have that*

$$|Df|(x, y)^2 = |Df(\cdot, y)|(x)^2 + |Df(x, \cdot)|(y)^2$$

for $\mu \times \nu$-almost every $(x, y) \in X \times Y$.

Remark 1.3. If the space $X \times Y$ is equipped with a product metric $\|(d_X, d_Y)\|$ induced by some norm $\|\cdot\|$ on $\mathbb{R}^2$, we obtain that $|Df| = \|(|Df(\cdot, y)|(x), |Df(x, \cdot)|(y))\|'$, where $\|\cdot\|'$ is a form of dual norm, cf. Theorem 3.3.

In particular, we have the following corollary.

Corollary 1.4. *If each of the factors X and Y is either infinitesimally Hilbertian or has finite Hausdorff dimension, then $W^{1,2}(X \times Y) = J^{1,2}(X, Y)$ with equal norms.*

Theorem 1.2 follows by combining three ingredients: 1) standard theory of Dirichlet forms and the elementary inclusion $W^{1,2}(X \times Y) \subset J^{1,2}(X, Y)$, 2) the non-trivial fact that the inclusion $W^{1,2}(X \times Y) \subset J^{1,2}(X, Y)$ has norm one, and 3) the equality of the norms on the algebraic tensor product $W^{1,2}(X) \otimes W^{1,2}(Y)$. We establish the last two results in the more general setting when $p > 1$ and the product space $X \times Y$ is equipped with a product metric given by a possibly non-Euclidean planar norm.

We remark that the earlier work by the authors [13] gave stronger conclusions on tensorization when $p \neq 2$, while also employing stronger assumptions. Indeed, the isometric embedding $N^{1,p}(X \times Y) \subset J^{1,p}(X, Y)$ (a stronger conclusion than Theorems 1.5 and 1.6) is obtained under the additional assumption that the factors admit a p-weak differentiable structure for all $p \geq 1$. By focusing on the Newtonian space, and using the stronger assumptions, the authors were able to analyse the borderline case $p = 1$ (see also Remark 1.8). However, in [13], full tensorization (that is, $N^{1,p}(X \times Y) = J^{1,p}(X, Y)$ with equal norms) is only obtained under the rather restrictive assumption that one of the factors supports an appropriate Poincaré inequality. In contrast, Theorems 1.5 and 1.6 are true for *any* metric measure spaces, while full tensorization for $p = 2$ (Theorem 1.2) is obtained when the factors admit p-weak differentiable structures.

1.2. Norm inequalities and equalities in the inclusion $W^{1,p}(X \times Y) \subset J^{1,p}(X, Y)$

Let $(X \times Y, d, \mu \times \nu)$ be the product of two metric measure spaces $X = (X, d_X, \mu)$ and $Y = (Y, d_Y, \nu)$, where the product metric is given by $d = \|(d_X, d_Y)\|$ for a given planar norm $\|\cdot\|$, and let $p > 1$.

For $f \in J^{1,p}(X, Y)$, we denote by $|D_X f|$ and $|D_Y f|$ the $L^p(X \times Y)$-functions $(x, y) \mapsto |Df(\cdot, y)|(x)$ and $(x, y) \mapsto |Df(x, \cdot)|(y)$, respectively, and replace (1.1) with the comparable quantity

$$\|(|D_X f|, |D_Y f|)\|' \in L^p(X \times Y). \tag{1.2}$$

Here $\|(a, b)\|' := \sup\{at + bs :\ s, t \geq 0,\ \|(s, t)\| \leq 1\}$ is the *partial dual norm* of $\|\cdot\|$. Notice that the Euclidean norm is its own partial dual, and thus (1.1) and (1.2) coincide for $d = \sqrt{d_X^2 + d_Y^2}$. The first of the two results states that the minimal p-weak upper gradient always dominates (1.2).

Theorem 1.5. *Let $p \in (1, \infty)$. If $f \in W^{1,p}(X \times Y)$, then $f \in J^{1,p}(X, Y)$ and*

$$\|(|D_X f|, |D_Y f|)\|' \leq |Df| \tag{1.3}$$

$\mu \times \nu$-almost everywhere.

In particular, for the Euclidean product metric $d = \sqrt{d_X^2 + d_Y^2}$, Theorem 1.5 yields the inequality

$$\sqrt{|D_X f|^2 + |D_Y f|^2} \leq |Df|, \quad f \in W^{1,p}(X \times Y).$$

Although it is straightforward to obtain the estimate $\|(|D_X f|, |D_Y f|)\|' \leq C|Df|$ for some $C > 0$ independent of f from the definitions, Theorem 1.5 is new and was previously only known for general spaces when $p = 2$ and $\|\cdot\|$ is the Euclidean norm, by work of Ambrosio–Gigli–Savaré [5] via different techniques (see also [6]). Our approach uses a density in energy argument [3, 11] to reduce the proof of Theorem 1.5 to a simple, yet novel, inequality for Lipschitz functions (see Proposition 2.1 below).

Our next result establishes *equality* in (1.3) in the *algebraic tensor product* $W^{1,p}(X) \otimes W^{1,p}(Y) \subset W^{1,p}(X \times Y)$ consisting of finite sums of simple tensor products, i.e., functions of the form

$$f(x, y) = \sum_{j}^{N} \varphi_j(x)\psi_j(y), \quad \varphi_j \in W^{1,p}(X),\ \psi_j \in W^{1,p}(Y), \quad j = 1, \dots, N.$$

Theorem 1.6. *Let $p \in (1, \infty)$. If $f \in W^{1,p}(X) \otimes W^{1,p}(Y)$, then*

$$\|(|D_X f|, |D_Y f|)\|' = |Df|$$

$\mu \times \nu$-almost everywhere.

We show, using *canonical minimal upper gradients* introduced in [14], that (1.2) is a p-weak upper gradient of $f \in W^{1,p}(X) \otimes W^{1,p}(Y)$. Together with Theorem 1.5, this suffices to demonstrate Theorem 1.6. Note that having constant one in (1.3) is important for the validity of this argument.

The crux of Theorem 1.2 is that, for infinitesimally quasi-Hilbertian spaces, the algebraic tensor product is dense in the Beppo-Levi space (with $p = 2$). Indeed, this is a standard result for domains of Dirichlet forms (see Proposition 3.2), and follows easily for Sobolev spaces under the infinitesimal quasi-Hilbertianity assumption. This completes the proof of Theorem 1.2 and also raises the natural question: when is $W^{1,p}(X) \otimes W^{1,p}(Y)$ dense in $J^{1,p}(X, Y)$? We expect that some separability assumption might be necessary, and formulate the question accordingly below.

Question 1.7. *Let $p \in [1, \infty)$. If $W^{1,p}(X)$ and $W^{1,p}(Y)$ are separable, is $W^{1,p}(X) \otimes W^{1,p}(Y)$ dense in $J^{1,p}(X, Y)$?*

An affirmative answer to Question 1.7 under the stronger assumption that X and Y admit p-weak differentiable structures would already be interesting, since it covers all spaces with finite Hausdorff dimension.

Remark 1.8. The above Theorems 1.5 and 1.6 are stated for exponents $p > 1$. In the proofs we use the equality of the Newton–Sobolev space $N^{1,p}(X)$ defined by Shanmugalingam and Cheeger [9, 28], and the plan-Sobolev space $W^{1,p}(X)$ from Ambrosio, Gigli and Savaré [4]. The equality of these spaces is not yet available in the literature in the case $p = 1$. Once proven, such equality would imply Theorems 1.5 and 1.6 also in the case $p = 1$.

1.3. Notation and conventions

A map $u\colon X \to \mathbb{R}$ from a metric space X is Lipschitz if

$$\operatorname{Lip}(u) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty.$$

The space of bounded Lipschitz functions with bounded support, is denoted $\mathrm{LIP_b}(X)$. The *asymptotic Lipschitz constant* of u is defined as

$$\operatorname{Lip_a} u(x) := \limsup_{r \to 0} \operatorname{Lip}(u|_{B(x,r)}), \quad x \in X.$$

Throughout the paper, $X = (X, d_X, \mu)$ and $Y = (Y, d_Y, \nu)$ are metric measure spaces, by which we mean complete separable metric spaces equipped with measures that are finite on bounded sets. Given $p > 1$, we denote by $N^{1,p}(X)$ and $W^{1,p}(X)$ the *Newton–Sobolev space*, and the Sobolev space via test plans, respectively. Both of these spaces are defined using the *upper gradient inequality*. A function $f \in L^p(X)$ is in $N^{1,p}(X)$ if there exists a function $g \in L^p(X)$ so that

$$|u(\gamma_1) - u(\gamma_0)| \leq \int_0^1 g(\gamma_t)\, |\gamma_t'| \,\mathrm{d}t \tag{1.4}$$

holds for Mod_p-a.e. curve. On the other hand, $f \in W^{1,p}(X)$ if (1.4) holds for $\boldsymbol{\eta}$-a.e. γ for every *q-test plan* $\boldsymbol{\eta}$. Modulus is an outer measure on curve families, and test plans are a family of measures on curve families. See [4] for a definition of test plans. For the properties the modulus of a curve family, Mod_p, see [21].

For each $u \in N^{1,p}(X)$ and $u \in W^{1,p}(X)$, there exists a minimal $|Du| \in L^p(X)$ so that

$$|u(\gamma_1) - u(\gamma_0)| \leq \int_0^1 |Du|(\gamma_t)\, |\gamma_t'| \,\mathrm{d}t \tag{1.5}$$

holds for "almost all" absolutely continuous curves $\gamma\colon [0,1] \to X$. For $u \in N^{1,p}(X)$, the inequality (1.5) is required to hold for Mod_p-almost every curve γ, whereas for $u \in W^{1,p}(X)$, (1.5) holds for $\boldsymbol{\eta}$-a.e. γ for every *q-test plan* $\boldsymbol{\eta}$. The minimal objects $|Du|$ associated to each case agree μ-almost everywhere (in this notation, we suppress its dependence on p and on the metric) and we have that $W^{1,p}(X) = N^{1,p}(X)$ for $p > 1$ with equal norms[1] [2,3]. Here the Sobolev space $W^{1,p}(X)$ is equipped with norm

$$\|u\|_{W^{1,p}(X)} = \big(\|u\|_{L^p(X)}^p + \||Du|\|_{L^p(X)}^p\big)^{1/p}.$$

For functions $u\colon X \times Y \to \mathbb{R}$, we will define the sliced functions, for $x \in X, y \in Y$, by

$$u_x := u(x,\cdot) : Y \to \mathbb{R} \quad \text{and} \quad u^y := u(\cdot, y) : X \to \mathbb{R}.$$

[1] The equality holds up to the subtle issue of choosing appropriate representatives: $N^{1,p}(X) \subset W^{1,p}(X)$, but for every $f \in N^{1,p}(X)$ there exists a function $\tilde{f} \in W^{1,p}(X)$ with $\tilde{f} = f$ almost everywhere. A further difference is that functions in $N^{1,p}(X)$ are defined up to capacity-a.e. equivalence, whereas functions in $W^{1,p}(X)$ are defined up to an almost everywhere equivalence. The proof is contained in Theorem 10.7 of [2].

When $u \in J^{1,p}(X,Y)$, we denote by $|D_X u|, |D_Y u| \in L^p(X \times Y)$ the functions such that

$$|D_X u|(\cdot, y) = |Du^y| \quad \text{for } \nu\text{-a.e. } y \in Y, \quad |D_Y u|(x\cdot,) = |Du_x| \quad \text{for } \mu\text{-a.e. } x \in X.$$

We equip $J^{1,p}(X,Y)$ with the norm

$$\|f\|_{J^{1,p}(X,Y)} = \Big(\int_{X\times Y} \big(|f|^p + (\|(|D_X f|, |D_Y f|)\|')^p\big)\, \mathrm{d}(\mu\times\nu)\Big)^{1/p}.$$

Remark 1.9. It is straightforward to check that $|D_X u|$ is given as the minimal p-weak upper gradient of u when $X \times Y$ is equipped with the metric $d = d_X + \sqrt{d_Y}$, and similarly for $|D_Y u|$. In particular, $|D_X u|$ and $|D_Y u|$ can be chosen Borel measurable.

2. The inclusion $W^{1,p}(X \times Y) \subset J^{1,p}(X,Y)$

In this section, we prove Theorems 1.5 and 1.6 for a general $p > 1$. In the proof of Proposition 2.1, below we will shorten the notation by using the evaluation map

$$e_X : C([0,1]; X) \times [0,1] \to X : (\gamma, t) \mapsto \gamma_t.$$

We start with a new and seemingly elementary inequality, which however has hitherto not appeared. The authors in [6] and [5] used a substantially different approach employing Hopf-lax equations, heat flows and further results. The following result is the key to our proof of the isometric inclusion, and perhaps gives a more transparent and geometric argument.

Proposition 2.1. *Let $p \in (1,\infty)$ and let $f \in \mathrm{LIP}_\mathrm{b}(X \times Y)$. Then*

$$\|(|D_X f|, |D_Y f|)\|' \le \mathrm{Lip}_\mathrm{a} f$$

$\mu \times \nu$-almost everywhere.

Proof of Proposition 2.1. The argument will proceed by finding, for a.e. point (x, y), a curve in the X- and Y-directions along which the function f has maximal derivative given by the minimal p-weak upper gradients. We do this by employing a result from [14], but we also outline in Remark 2.2 another argument inspired by one from Cheeger and Kleiner [10] after the proof, which some readers may find helpful.

Since $f \in \mathrm{LIP}_\mathrm{b}(X \times Y)$, we have that $f \in W^{1,p}(X \times Y)$ when $X \times Y$ is equipped with the distance $d_X + \sqrt{d_Y}$. By Remark 1.9 and Theorem 1.1 in [14], there exists a test plan $\boldsymbol{\eta}$ so that the disintegration $\{\boldsymbol{\pi}_{(x,y)}\}$ of the measure $\mathrm{d}\boldsymbol{\pi} := |\gamma_t'|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}$ with respect to the evaluation map $e_{X\times Y} : C([0,1]; X \times Y) \times [0,1] \to X \times Y$ satisfies

$$|D_X f|(x,y) = |Df^y|(x) = \left\| \frac{(f^y \circ \gamma)_t'}{|\gamma_t'|} \right\|_{L^\infty(\boldsymbol{\pi}_{(x,y)})} \tag{2.1}$$

for $\mu \times \nu$-almost every $(x,y) \in \{|D_X f| > 0\}$. (Notice that every rectifiable curve in $(X \times Y, d_X + \sqrt{d_Y})$ is of the form (α, y), where $y \in Y$ is a constant curve and α is a rectifiable curve in X. One could obtain (2.1) for $p > 1$ alternatively via the existence of the master test plans introduced in [26] and by using Fubini's theorem.) By applying the same argument with metric $\sqrt{d_X + d_Y}$, we similarly obtain measures $\{\tilde{\boldsymbol{\pi}}_{(x,y)}\}$ for almost every $(x, y) \in \{|D_Y f| > 0\}$ so that

$$|D_Y f|(x,y) = |Df_x|(y) = \Big\| \frac{(f_x \circ \gamma)'_t}{|\gamma'_t|} \Big\|_{L^\infty(\tilde{\boldsymbol{\pi}}_{(x,y)})}. \tag{2.2}$$

Let us fix $(x, y) \in X \times Y$ where both (2.1) and (2.2) hold. For any $\varepsilon > 0$, there exist $(\alpha, t_0) \in e_X^{-1}(x)$ and $(\beta, s_0) \in e_Y^{-1}(y)$ such that

$$(1-\varepsilon)|D_X f|(x,y) \le \frac{(f^y \circ \alpha)'_{t_0}}{|\alpha'_{t_0}|} \quad \text{and} \quad (1-\varepsilon)|D_Y f|(x,y) \le \frac{(f_x \circ \beta)'_{s_0}}{|\beta'_{s_0}|}, \tag{2.3}$$

and the limits in all the relevant quantities exist. Let $a, b \ge 0$ and define the curves

$$\tilde{\alpha}(t) = \alpha \le \Big(t_0 + \frac{a}{|\alpha'_{t_0}|}\, t\Big) \quad \text{and} \quad \tilde{\beta}(s) = \beta\Big(s_0 - \frac{b}{|\beta'_{s_0}|}\, s\Big)$$

in a small neighbourhood of the origin. Then

$$\begin{aligned} a\,\frac{(f^y \circ \alpha)'_{t_0}}{|\alpha'_{t_0}|} + b\,\frac{(f_x \circ \beta)'_{s_0}}{|\beta'_{s_0}|} &= (f^y \circ \tilde{\alpha})'_0 - (f_x \circ \tilde{\beta})'_0 \\ &= \lim_{h \to 0^+} \frac{[f(\tilde{\alpha}(h), y) - f(x,y)] - [f(x, \tilde{\beta}(h)) - f(x,y)]}{h} \\ &= \lim_{h \to 0^+} \frac{f(\tilde{\alpha}(h), y) - f(x, \tilde{\beta}(h))}{h} \\ &\le \mathrm{Lip}_{\mathrm{a}} f(x,y) \limsup_{h \to 0^+} \frac{d((\tilde{\alpha}(h), y), (x, \tilde{\beta}(h)))}{h}. \end{aligned}$$

Note however that

$$\frac{d((\tilde{\alpha}(h), y), (x, \tilde{\beta}(h)))}{h} = \Big\| \Big(\frac{d_X(\tilde{\alpha}(h), x)}{h}, \frac{d_Y(\tilde{\beta}(h), y)}{h} \Big) \Big\| \xrightarrow{h \to 0^+} \|(a,b)\|.$$

Using (2.3), we arrive at

$$(1-\varepsilon)\,[a\,|D_X f|(x,y) + b\,|D_Y f|(x,y)] \le \|(a,b)\|\, \mathrm{Lip}_{\mathrm{a}} f(x,y). \tag{2.4}$$

Taking supremum over all $a, b \ge 0$ with $\|(a,b)\| = 1$ in (2.4) yields

$$(1-\varepsilon)\|(|D_X f|(x,y), |D_Y f|(x,y))\|' \le \mathrm{Lip}_{\mathrm{a}} f(x,y) \quad \mu \times \nu\text{-a.e. } (x,y) \in X \times Y.$$

Since $\varepsilon > 0$ is arbitrary, the claim now follows. ∎

Remark 2.2. In the previous proof, the use of [14] is convenient, but the existence of curves α and β as in (2.3) with nearly maximal derivative is actually a much weaker conclusion. In fact, in the category of doubling spaces satisfying a Poincaré inequality, their existence follows from the work of Cheeger and Kleiner, see Theorem 4.2 in [10]. They gave a characterization of the minimal p-weak upper gradient of a Lipschitz function as a maximal directional derivative. In fact, the first part of the proof, which does not use the doubling or Poincaré assumptions, shows that a function $\hat{g}$ defined using the maximal directional derivatives is an upper gradient. A minimal p-weak upper gradient is a.e. less than this upper gradient, and from this the existence of α and β can be deduced. This idea played a central role in later developments, such as the seminal work of Bate [7] characterizing Lipschitz differentiability spaces.

Proposition 2.1 now implies Theorem 1.5.

Proof of Theorem 1.5. Let $f \in W^{1,p}(X)$. By the density in energy (cf. [3] or [11] for an alternate proof), there exists a sequence $(f_j) \subset \mathrm{LIP_b}(X)$ such that $f_j \to f$ and $\mathrm{Lip_a} f_j \to |Df|$ in $L^p(X)$ as $j \to \infty$. We also have, for $a, b \geq 0$ and any non-negative $\varphi \in C_b(X)$, that

$$\int_{X\times Y} \varphi(a|D_X f|+b|D_Y f|)\,\mathrm{d}(\mu\times\nu) \leq \liminf_{j\to\infty} \int_{X\times Y} \varphi(a|D_X f_j|+b|D_Y f_j|)\,\mathrm{d}(\mu\times\nu),$$

(cf. Remark 1.9 and the lower semicontinuity of the Cheeger energy [3, 9]). By Proposition 2.1 (and the definition of the partial dual norm $\|\cdot\|'$), this implies that

$$\begin{aligned}\int_{X\times Y} \varphi(a|D_X f|+b|D_Y f|)\,\mathrm{d}(\mu\times\nu) &\leq \|(a,b)\| \liminf_{j\to\infty} \int_{X\times Y} \varphi \,\mathrm{Lip_a} f_j \,\mathrm{d}(\mu\times\nu) \\ &= \|(a,b)\| \int_{X\times Y} \varphi |Df|\,\mathrm{d}(\mu\times\nu)\end{aligned}$$

for arbitrary a, b and φ. Thus $a|D_X f| + b|D_Y f| \leq \|(a,b)\||Df|$ $\mu\times\nu$-a.e. for every $a, b \geq 0$. Then, by choosing a countable dense set of real numbers $a, b \geq 0$, we obtain the pointwise inequality

$$\|(|D_X f|, |D_Y f|)\|' = \sup_{(a,b)} \frac{a|D_X f| + b|D_Y f|}{\|(a,b)\|} \leq |Df| \quad \mu\times\nu\text{-a.e.} \qquad ■$$

Next we prove Theorem 1.6. In the proof, we identify $\mathbb{R}^N$ with $(\mathbb{R}^N)^*$ in the standard way by identifying $\bar{a} \in \mathbb{R}^N$ with the functional $x \mapsto \bar{a}\cdot x \in (\mathbb{R}^N)^*$.

Proof of Theorem 1.6. Let

$$h(x,y) = \sum_{i=1}^{N} f_i(x)\, g_i(y) \in W^{1,p}(X) \otimes W^{1,p}(Y),$$

where $f_1, \ldots, f_N \in W^{1,p}(X)$ and $g_1, \ldots, g_N \in W^{1,p}(Y)$. Since each function in $W^{1,p}(X)$ is a.e. equal to a Newton–Sobolev function (Theorem 10.7 in [2]), we can choose Newton–Sobolev representatives for each f_j and g_j and consider the maps $\varphi = (f_1, \ldots, f_N) \in N^{1,p}(X;\mathbb{R}^N)$ and $\psi = (g_1, \ldots, g_N) \in N^{1,p}(Y;\mathbb{R}^N)$. By Lemma 4.2 in [14], we have the following:

(1) there exist maps $\Phi\colon X\times(\mathbb{R}^N)^*\to[0,\infty]$ and $\Psi\colon Y\times(\mathbb{R}^N)^*\to[0,\infty]$ so that $\Phi(\cdot,\boldsymbol{\xi})$ is the minimal p-weak upper gradient for $\boldsymbol{\xi}\circ\varphi$ and $\Psi(\cdot,\boldsymbol{\zeta})$ is the minimal p-weak upper gradient of $\boldsymbol{\zeta}\circ\psi$ for every $\boldsymbol{\xi},\boldsymbol{\zeta}\in(\mathbb{R}^N)^*$;

(2) there are families of curves Γ_X,Γ_Y with $\mathrm{Mod}_p(\Gamma_X)=\mathrm{Mod}_p(\Gamma_Y)=0$ so that, for every $\alpha\notin\Gamma_X$ and every $\boldsymbol{\xi}\in(\mathbb{R}^N)^*$, the function $\boldsymbol{\xi}\circ\varphi$ is absolutely continuous on α with upper gradient $\Phi(x,\boldsymbol{\xi})$, and for every $\beta\notin\Gamma_Y$ and every $\boldsymbol{\zeta}\in(\mathbb{R}^N)^*$, the function $\boldsymbol{\zeta}\circ\psi$ is absolutely continuous on β with upper gradient $\Psi(x,\boldsymbol{\zeta})$;

(3) and for each $\alpha\notin\Gamma_X$ and each $\beta\notin\Gamma_Y$, there exist null sets $E_\alpha\subset[0,1]$, $E_\beta\subset[0,1]$ so that for every $\boldsymbol{\xi},\boldsymbol{\zeta}\in(\mathbb{R}^N)^*$, we have
$$|(\boldsymbol{\xi}\circ\varphi\circ\alpha)'_t|\le\Phi(\alpha_t,\boldsymbol{\xi})\,|\alpha'_t|\quad\text{for every } t\in[0,1]\setminus E_\alpha$$
and
$$|(\boldsymbol{\zeta}\circ\psi\circ\beta)'_t|\le\Psi(\beta_t,\boldsymbol{\zeta})\,|\beta'_t|\quad\text{for every } t\in[0,1]\setminus E_\beta.$$

First, we show that $h\in W^{1,p}(X\times Y)$ and that $|Dh|\le\|(|D_Xh|,|D_Yh|)\|'$, which follows from showing that
$$g:=\|(|D_Xh|,|D_Yh|)\|'$$
is a p-weak upper gradient of h.

Note that $|D_Xh|(x,y)=\Phi(x,(g_i(y)))$ and $|D_Yh|(x,y)=\Psi(y,(f_i(x)))$ for $\mu\times\nu$-almost every x,y. Thus, it suffices to show that $\|(\Phi(x,(g_i(y))),\Psi(y,(f_i(x)))\|$ is a p-weak upper gradient of h. Let Γ be the collection of absolutely continuous curves $\gamma=(\alpha,\beta)$ so that $\alpha\notin\Gamma_X$ and $\beta\notin\Gamma_Y$. The complement of Γ has zero Mod_p-modulus, as follows fairly directly from the definition of modulus and the characterization of families of zero modulus, see Lemma 5.2.8 in [21]: there exists a function $g\in L^p(X\times Y)$ so that $\int_\gamma g\,\mathrm{d}s=\infty$ for each $\gamma\notin\Gamma$).

Fix $\gamma\in\Gamma$. Since $f_i\circ\alpha$ and $g_i\circ\beta$ are absolutely continuous, so is h as a product and sum of absolutely continuous functions. Further, it is differentiable a.e. and the Leibniz rule applies:
$$(h\circ\gamma)'_t=\sum_{i=1}^N(f_i\circ\alpha)'_t\,g_i(\beta_t)+\sum_{i=1}^N f_i(\alpha_t)\,(g_i\circ\beta)'_t.$$

Now,
$$\Big|\sum_{i=1}^N(f_i\circ\alpha)'_t\,g_i(\beta_t)\Big|\le\Phi(\alpha_t,(g_i(\beta_t)))\,|\alpha'_t|,\quad t\notin E_\alpha,\ \text{and}$$
$$\Big|\sum_{i=1}^N(g_i\circ\beta)'_t\,f_i(\alpha_t)\Big|\le\Psi(\beta_t,(f_i(\beta_t)))\,|\beta'_t|,\quad t\notin E_\beta.$$

Thus, for a.e. t, we have
$$\begin{aligned}|(h\circ\gamma)'_t|&\le\Phi(\alpha_t,(g_i(\beta_t)))|\alpha'_t|+\Psi(\beta_t,(f_i(\beta_t))|\beta'_t|\\&\le\|(\Phi(\alpha_t,(g_i(\beta_t))),\Psi(\beta_t,(f_i(\alpha_t))))\|'\,\|(|\alpha'_t|,|\beta'_t|)\|\\&=\|(\Phi(\alpha_t,(g_i(\beta_t))),\Psi(\beta_t,(f_i(\alpha_t))))\|'\,|\gamma'_t|)=g(\gamma_t)\,|\gamma'_t|.\end{aligned}$$

By integrating this, we obtain the upper gradient inequality (1.4). This shows that g is a p-weak upper gradient of h, whence $|Dh| \leq \|(|D_X h|, |D_Y h|)\|'$ holds $\mu \times \nu$-almost everywhere. Theorem 1.5 gives the opposite inequality, and completes the proof of the claim. ∎

3. Infinitesimally quasi-Hilbertian spaces

In this section, we complete the proof of Theorem 1.2. We begin with the definition of closed Dirichlet forms.

Definition 3.1. Let $\mathcal{D} \subset L^2(X)$ be a vector subspace. A Dirichlet form $\mathcal{E}$ (with domain $\mathcal{D}$) is a map $\mathcal{E}\colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, which satisfies

(1) $\mathcal{E}$ is bilinear,

(2) $\mathcal{E}$ is symmetric, i.e., $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ for each $u, v \in \mathcal{D}$,

(3) $\mathcal{E}$ is non-negative, i.e., $\mathcal{E}(u, u) \geq 0$ for each $u \in \mathcal{D}$, and

(4) $\mathcal{D}$ is dense in $L^2(X)$.

(5) $\mathcal{E}(f \circ u, f \circ u) \leq \mathcal{E}(u, u)$ for all $u \in L^2(X)$ and for all bounded 1-Lipschitz functions $f\colon \mathbb{R} \to \mathbb{R}$.

We say that $\mathcal{E}$ is closed if $\mathcal{D}$, when equipped with the norm

$$\|f\|_{\mathcal{E}} := \sqrt{\|f\|^2_{L^2(X)} + \mathcal{E}(f, f)},$$

is complete.

Next we recall the tensor product of Dirichlet forms. Recall the notation $u^y = u(\cdot, y)$ and $u_x = u(x, \cdot)$ for $u\colon X \times Y \to \mathbb{R}$ and $(x, y) \in X \times Y$. If $(\mathcal{E}_X, \mathcal{D}_X)$ and $(\mathcal{E}_Y, \mathcal{D}_Y)$ are Dirichlet forms on X and Y, respectively, the domain of their tensor product $(\mathcal{E}, \mathcal{D})$ is defined by

$$\mathcal{D} = \big\{u \in L^2(X \times Y) :\ u^y \in \mathcal{D}_X\ \nu\text{-a.e. } y \in Y,\ u_x \in \mathcal{D}_Y\ \mu\text{-a.e. } x \in X,\ \mathcal{E}(u, u) < \infty\big\},$$

where

$$\mathcal{E}(u, u) := \int_X \mathcal{E}_Y(u_x, u_x)\, \mathrm{d}\mu(x) + \int_Y \mathcal{E}_Y(u^y, u^y)\, \mathrm{d}\nu(y).$$

The bilinear form $\mathcal{E}$ is given by polarization:

$$\mathcal{E}(u, v) = \frac{\mathcal{E}(u + v, u + v) - \mathcal{E}(u - v, u - v)}{4}.$$

We observe that the algebraic tensor product $\mathcal{D}_X \otimes \mathcal{D}_Y \subset L^2(X \times Y)$ is contained in $\mathcal{D}$. Here the algebraic tensor product is given by

$$\mathcal{D}_X \otimes \mathcal{D}_Y = \Big\{\sum_{i=1}^N a_i(x) b_i(y) : a_i \in \mathcal{D}_X, b_i \in \mathcal{D}_Y, N \in \mathbb{N}\Big\}.$$

We refer to [8], Chapter V, for the basic properties of the tensor product of Dirichlet forms, and record here the density of $\mathcal{D}_X \otimes \mathcal{D}_Y$ in $\mathcal{D}$, cf. Proposition 2.1.3 (b) in [8].

Proposition 3.2. *Let $(\mathcal{E}_X, \mathcal{D}_X)$ and $(\mathcal{E}_Y, \mathcal{D}_Y)$ be closed Dirichlet forms on X and Y, and let $\mathcal{E}$ be their tensor product. Then the algebraic tensor product $\mathcal{D}_X \otimes \mathcal{D}_Y$ is dense in $\mathcal{D}$ with respect to $\|\cdot\|_{\mathcal{E}}$.*

We will next see how Proposition 3.2 implies the density of $W^{1,2}(X) \otimes W^{1,2}(Y)$ in $J^{1,2}(X,Y)$ (and thus Theorem 1.2) for infinitesimally quasi-Hilbertian spaces. Recall that X is said to be infinitesimally quasi-Hilbertian if there exists a Dirichlet form $(\mathcal{E}, \mathcal{D})$ on X with $\mathcal{D} = W^{1,2}(X)$ and $\|\cdot\|_{\mathcal{E}}$ equivalent to $\|\cdot\|_{W^{1,2}(X)}$. Theorem 1.2 is a special case of the following theorem with $\|(a,b)\| := \sqrt{a^2+b^2}$.

Theorem 3.3. *Suppose X and Y are infinitesimally quasi-Hilbertian and equip the product space $X \times Y$ with the metric $d = \|(d_X, d_Y)\|$ for a given norm $\|\cdot\|$ on $\mathbb{R}^2$. Then $W^{1,2}(X \times Y) = J^{1,2}(X,Y)$ and*

$$\|(|D_X f|, |D_Y f|)\|' = |Df|$$

for every $f \in J^{1,2}(X,Y)$.

In the proof, we use the notation $A(u) \lesssim B(u)$ to indicate that there exists a constant $C > 0$, independent of u, such that $B(u) \leq CA(u)$, and we write $A(u) \simeq B(u)$ if $A(u) \lesssim B(u)$ and $B(u) \lesssim A(u)$.

Proof. Let $\mathcal{E}_X$ and $\mathcal{E}_Y$ be closed Dirichlet forms on X and Y with domains $W^{1,2}(X)$ and $W^{1,2}(Y)$, respectively, such that $\|f\|_{\mathcal{E}_X} \simeq \|f\|^2_{W^{1,2}(X)}$ and $\|g\|_{\mathcal{E}_Y} \simeq \|g\|^2_{W^{1,2}(Y)}$ for all $f \in W^{1,2}(X)$ and $g \in W^{1,2}(Y)$. Then

$$\begin{aligned}
\|u\|^2_{J^{1,2}(X,Y)} &= \|u\|^2_{L^2(X\times Y)} + \int_{X\times Y} (\|(|D_X u|, |D_Y u|)\|')^2 \, \mathrm{d}(\mu\times\nu) \\
&\simeq \int_{X\times Y} (|u|^2 + |D_X u|^2 + |D_Y u|^2) \, \mathrm{d}(\mu\times\nu) \\
&\simeq \int_X \|u_x\|^2_{W^{1,2}(Y)} \mathrm{d}\mu(x) + \int_Y \|u^y\|^2_{W^{1,2}(Y)} \, \mathrm{d}\nu(y) \\
&\simeq \int_X (\|u_x\|^2_{L^2(Y)} + \mathcal{E}_Y(u_x, u_x)) \, \mathrm{d}\mu(x) + \int_Y (\|u^y\|^2_{L^2(X)} + \mathcal{E}_X(u^y, u^y)) \, \mathrm{d}\nu(y) \\
&\simeq \|u\|^2_{L^2(X\times Y)} + \mathcal{E}(u,u)
\end{aligned}$$

whenever $u \in L^2(X \times Y)$ is such that $u_x \in W^{1,2}(Y)$ for μ-a.e. $x \in X$ and $u^y \in W^{1,2}(X)$ for ν-a.e. $y \in Y$. From this, it follows that $\mathcal{D} = J^{1,2}(X,Y)$ and that

$$\|u\|_{\mathcal{E}} \simeq \|u\|_{J^{1,2}(X,Y)}, \quad u \in J^{1,2}(X,Y).$$

We now prove the claim in Theorem 3.3. Let $u \in J^{1,2}(X,Y)$. By Proposition 3.2, there is a sequence $(u_j) \subset W^{1,2}(X) \otimes W^{1,2}(Y)$ such that $\|u_j - u\|_{\mathcal{E}} \to 0$ as $j \to \infty$. Theorem 1.6 implies that

$$\|u_j - u_l\|_{W^{1,2}(X\times Y)} = \|u_j - u_l\|_{J^{1,2}(X,Y)} \lesssim \|u_j - u_l\|_{\mathcal{E}}$$

for each $j, l \in \mathbb{N}$. Thus (u_j) is a Cauchy sequence in $W^{1,2}(X \times Y)$ and its limit (in $W^{1,2}(X \times Y)$) agrees almost everywhere with u since u is the L^2-limit of (u_j). It follows that $u \in W^{1,2}(X \times Y)$ and, by Theorem 1.5, that $\|(|D_X u|, |D_Y u|)\|' \le |Du|$. However, since $u_j \to u$ in $J^{1,2}(X, Y)$ as $j \to \infty$, we have that

$$|Du_j| = \|(|D_X u_j|, |D_Y u_j|)\|' \xrightarrow{j\to\infty} \|(|D_X u|, |D_Y u|)\|' \quad \text{in } L^2(X \times Y)$$

and thus $\|(|D_X u|, |D_Y u|)\|'$ is a 2-weak upper gradient of u (cf. Proposition 7.3.7 in [21]), implying that $|Du| \le \|(|D_X u|, |D_Y u|)\|'$. This completes the proof. ∎

Remark 3.4. The proof of Theorem 3.3 also yields the following statement: if X and Y are infinitesimally quasi-Hilbertian spaces, then their product $X \times Y$ with the metric $d = \|(d_X, d_Y)\|$ is also infinitesimally quasi-Hilbertian.

Infinitesimally Hilbertian spaces are infinitesimally quasi-Hilbertian (recall that X is infinitesimally Hilbertian if $\|\cdot\|_{W^{1,2}(X)}$ is given by an inner product). Another important class of examples are spaces X admitting a 2-weak differentiable structure in the sense of [14] or, equivalently, spaces with finitely generated tangent module $L^2(T^*X)$ in the sense of Gigli.

The idea of the proof is that the fibers of the co-tangent bundle are associated with a natural norm. Indeed, the charts of the 2-weak differential structure correspond to Borel measurable sets U, where the co-tangent bundle can be locally trivialized as $U \times \mathbb{R}^n$ together with a norm $|\cdot|_x$ on $\mathbb{R}^n$ for each $x \in U$. This norm is *Borel measurable* in the sense that $x \mapsto |v|_x$ is Borel for each $v \in \mathbb{R}^n$. In the finite dimensional setting, by using John's ellipsoids, such a norm can be replaced by an equivalent inner product. Lemma 3.5 further guarantees a *measurable* choice of such an inner product $\langle\cdot,\cdot\rangle_x$ which is comparable to the norm $|\cdot|_x$ on the cotangent space; see [9], p. 460, for an original reference for such an argument. An inner product $\langle\cdot,\cdot\rangle_x$ is said to be Borel measurable if $x \mapsto \langle v, w\rangle_x$ is Borel for each $v, w \in \mathbb{R}^n$.

Lemma 3.5. *For every $k \in \mathbb{N}$, there exists a constant $c(k)$ so that, for every $|\cdot|$ norm on a k-dimensional vector space V, with $k \in \mathbb{N}$, there exists an inner product $\langle,\rangle$ on V so that*

$$c(k)^{-1}\sqrt{\langle v, v\rangle} \le |v| \le c(k)\sqrt{\langle v, v\rangle}.$$

Moreover, if $U \subset X$ is a Borel measurable subset, $v \mapsto \|v\|_x$ is a Borel measurable choice of norm on V, then the inner product $\langle\cdot,\cdot\rangle_x$ can be chosen to be Borel measurable.

Proof. Let V^* be the dual vector space to V equipped with the dual norm given by $|v^*| = \sup_{v\in V, |v|=1}\langle v^*, v\rangle$. Let $B_1 \subset V^*$ be the closed unit ball with respect to this norm, and let λ be the k-dimensional Hausdorff measure associated to the metric on V^* induced by the dual norm. By a classical lemma of Kirchheim, $\lambda(B_1) = \omega_k$, where ω_k is the volume of the k-dimensional unit ball; see Lemma 6 in [22].

We define the inner product via the natural map $V \to (V^*)^* \to L^\infty(B_1) \to L^2(B_1, \lambda)$, and set

$$\langle v, w\rangle = \int_{B_1} \langle v, v^*\rangle\, \overline{\langle w, v^*\rangle}\, \mathrm{d}\lambda_{v^*}. \tag{3.1}$$

The required inequality is true for $v = 0$, and thus we consider the case where $v \ne 0$. We immediately get $\langle v, v\rangle \le |v|^2\omega_k$, since $\langle v, v^*\rangle \le |v|$, for each $v^* \in B_1$. Next, take a

$w^* \in B_1$ so that $\langle w^*, v\rangle = |v|$. Let $\omega = w^*/2$, for which we have $B = B_{1/4}(\omega) \subset B_1$. Further, for every $a^* \in B$ we have $\langle a^*, v\rangle \geq \langle \omega, v\rangle - \langle \omega - a^*, v\rangle \geq |v|/2 - |v|/4 \geq |v|/4$. Thus,

$$\langle v, v\rangle \geq \frac{\lambda(B)|v|^2}{16} \geq \frac{\omega_k}{4^{k+2}}|v|^2,$$

from which the claim follows.

We are left to argue measurability. Both the unit ball B_1 in the dual space and the measure λ vary in x. Denote these by λ_x and $B_{1,x}$. Fix any inner product and the corresponding Lebesgue measure λ_0 on V^*. Then, by translation invariance, λ_x is given by

$$\lambda_x = \lambda_0(B_{1,x})^{-1}\omega_k\lambda_0.$$

We write (3.1) slightly differently as

$$\langle v, w\rangle_x = \int_{V^*}\langle v, v^*\rangle\overline{\langle w, v^*\rangle}\,\xi(v^*, x)\,\lambda_0(B_{1,x})^{-1}\,\omega_k\,\mathrm{d}\lambda_0(v^*). \tag{3.2}$$

Here $\xi(v^*, x) = 1$ when $v^* \in B_{1,x}$, and otherwise $\xi(v^*, x) = 0$. Both $x \to \xi(v^*, x)$ (for any $v^* \in V^*$) and

$$x \mapsto \lambda_0(B_{1,x}) = \int_{V^*}\xi(v^*, x)\,\mathrm{d}\lambda_0(v^*)$$

are directly seen to be Borel measurable in x, and the claim follows. ∎

Proposition 3.6. *Suppose X is a metric measure space such that $L^2(T^*X)$ is finitely generated. Then there exists a closed (and local and regular) Dirichlet form $\mathcal{E}$ with domain $W^{1,2}(X)$ and such that, for some $C > 0$,*

$$\frac{1}{C}\|Du\|^2_{L^2(X)} \leq \mathcal{E}(u,u) \leq C\|Du\|^2_{L^2(X)}, \quad u \in W^{1,2}(X).$$

In particular, X is infinitesimally quasi-Hilbertian.

Proof. By Theorem 1.11 in [14], X admits a 2-weak differential structure, that is, there are countably many disjoint 2-weak charts (U_i, φ_i), $i \in I$, such that $\mu(X \setminus \bigcup_i U_i) = 0$ and the dimensions of the Lipschitz maps $\varphi_i : U_i \to \mathbb{R}^{n_i}$ satisfy $N := \sup_i n_i < \infty$. Here, each U_i is a Borel set. Moreover, for each $x \in U_i$, $i \in I$, there is a Borel measurable norm $|\cdot|_x$ on $(\mathbb{R}^{n_i})^*$ such that

$$|Df|(x) = |\mathrm{d}_x f|_x \quad \mu\text{-a.e. } x \in U_i$$

for every $f \in W^{1,2}(X)$, where $\mathrm{d}f : U_i \to (\mathbb{R}^{n_i})^*$ is the p-weak differential of f.

By Lemma 3.5, there exists $c(N) >$ so that for each $i \in I$ and $x \in U_i$, there exists a Borel measurable inner product $\langle\cdot,\cdot\rangle_x$ on $(\mathbb{R}^{n_i})^*$ with the property that $x \mapsto \langle \mathrm{d}_x f, \mathrm{d}_x f\rangle_x : U_i \to \mathbb{R}$ is measurable, and

$$c(N)^{-1}\sqrt{\langle \mathrm{d}_x f, \mathrm{d}_x f\rangle_x} \leq |\mathrm{d}_x f|_x \leq c(N)\sqrt{\langle \mathrm{d}_x f, \mathrm{d}_x f\rangle_x}$$

for every $f \in W^{1,2}(X)$ and $x \in U_i$.

Define a bi-linear form $\mathcal{E}\colon W^{1,2}(X) \times W^{1,2}(X) \to \mathbb{R}$ by

$$\mathcal{E}(u, v) = \sum_{i \in I} \int_{U_i} \langle \mathrm{d}_x u, \mathrm{d}_x v \rangle_x \, \mathrm{d}\mu(x).$$

Clearly $\mathcal{E}$ is a Dirichlet form with domain $W^{1,2}(X)$, and

$$c(N)^{-2} \|Du\|^2_{L^2(X)} \le \mathcal{E}(u, u) \le c(N)^2 \|Du\|^2_{L^2(X)}, \quad u \in W^{1,2}(X).$$

That $\mathcal{E}$ satisfies the Markov property, (5) in Definition 3.1, follows from the fact that $\mathrm{d}(f \circ u) = f'(u(x))\mathrm{d}u$ for C^1-functions f. See the proof of Proposition 4.10 in [14] for a similar argument. Note that, by lower-semicontinuity and closability of the differential (Corollary 4.9 in [14]), one gets that for any 1-Lipschitz f, we have $\mathrm{d}(f \circ u) = a(x)\,\mathrm{d}u$ for some $|a(x)| \le 1$.

Since $W^{1,2}(X)$ is a Banach space, it follows that $\mathcal{E}$ is closed. It is not difficult to check that $\mathcal{E}$ is local and regular, and we leave it to the interested reader (see e.g. [8], Chapter I). This completes the proof. ∎

Proof of Corollary 1.4. Infinitesimally Hilbertian spaces are trivially infinitesimally quasi-Hilbertian. Spaces of finite Hausdorff dimension admit a 2-weak differentiable structure and their L^2-cotangent module is finitely generated, cf. Theorems 1.5 and 1.11 in [14]. By Proposition 3.6, spaces of finite Hausdorff dimension are therefore infinitesimally quasi-Hilbertian. Corollary 1.4 follows immediately from this and Theorem 1.2. ∎

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References

[1] Ambrosio, L., Colombo, M. and Di Marino, S.: Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope. In *Variational methods for evolving objects*, pp. 1–58. Adv. Stud. Pure Math. 67, Math. Soc. Japan, Tokyo, 2015.

[2] Ambrosio, L., Di Marino, S. and Savaré, G.: On the duality between p-modulus and probability measures. *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 8, 1817–1853.

[3] Ambrosio, L., Gigli, N. and Savaré, G.: Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. *Rev. Mat. Iberoam.* **29** (2013), no. 3, 969–996.

[4] Ambrosio, L., Gigli, N. and Savaré, G.: Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. *Invent. Math.* **195** (2014), no. 2, 289–391.

[5] Ambrosio, L., Gigli, N. and Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded from below. *Duke Math. J.* **163** (2014), no. 7, 1405–1490.

[6] Ambrosio, L., Pinamonti, A. and Speight, G.: Tensorization of Cheeger energies, the space $H^{1,1}$ and the area formula for graphs. *Adv. Math.* **281** (2015), 1145–1177.

[7] Bate, D.: Structure of measures in Lipschitz differentiability spaces. *J. Amer. Math. Soc.* **28** (2015), no. 2, 421–482.

[8] Bouleau, N. and Hirsch, F.: *Dirichlet forms and analysis on Wiener space*. De Gruyter Studies in Mathematics 14, Walter de Gruyter & Co., Berlin, 1991.

[9] Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.* **9** (1999), no. 3, 428–517.

[10] Cheeger, J. and Kleiner, B.: Differentiability of Lipschitz maps from metric measure spaces to Banach spaces with the Radon–Nikodým property. *Geom. Funct. Anal.* **19** (2009), no. 4, 1017–1028.

[11] Eriksson-Bique, S.: Density of Lipschitz functions in energy. *Calc. Var. Partial Differential Equations* **62** (2023), no. 2, article no. 60, 23 pp.

[12] Eriksson-Bique, S., Giovannardi, G., Korte, R., Shanmugalingam, N. and Speight, G.: Regularity of solutions to the fractional Cheeger-Laplacian on domains in metric spaces of bounded geometry. *J. Differential Equations* **306** (2022), 590–632.

[13] Eriksson-Bique, S., Rajala, T. and Soultanis, E.: Tensorization of p-weak differentiable structures. Preprint 2021, arXiv: 2206.05046.

[14] Eriksson-Bique, S. and Soultanis, E.: Curvewise characterizations of minimal upper gradients and the construction of a Sobolev differential. To appear in *Anal. PDE.*

[15] Fitzi, M. and Wenger, S.: Area minimizing surfaces of bounded genus in metric spaces. *J. Reine Angew. Math.* **770** (2021), 87–112.

[16] Ghinassi, S., Giri, V. and Elisa, N.: Tensorization of Sobolev spaces. Preprint 2023, arXiv: 2305.05804.

[17] Gigli, N.: The splitting theorem in non-smooth context. Preprint 2013, arXiv: 1302.5555.

[18] Gigli, N.: On the differential structure of metric measure spaces and applications. *Mem. Amer. Math. Soc.* **236** (2015), no. 1113, vi+91.

[19] Gigli, N.: Nonsmooth differential geometry – an approach tailored for spaces with Ricci curvature bounded from below. *Mem. Amer. Math. Soc.* **251** (2018), no. 1196, v+161.

[20] Gigli, N. and Han, B.-X.: Sobolev spaces on warped products. *J. Funct. Anal.* **275** (2018), no. 8, 2059–2095.

[21] Heinonen, J., Koskela, P., Shanmugalingam, N. and Tyson, J. T.: *Sobolev spaces on metric measure spaces*. New Mathematical Monographs 27, Cambridge University Press, Cambridge, 2015.

[22] Kirchheim, B.: Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.* **121** (1994), no. 1, 113–123.

[23] Lytchak, A. and Wenger, S.: Area minimizing discs in metric spaces. *Arch. Ration. Mech. Anal.* **223** (2017), no. 3, 1123–1182.

[24] Lytchak, A. and Wenger, S.: Intrinsic structure of minimal discs in metric spaces. *Geom. Topol.* **22** (2018), no. 1, 591–644.

[25] Lytchak, A. and Wenger, S.: Canonical parameterizations of metric disks. *Duke Math. J.* **169** (2020), no. 4, 761–797.

[26] Pasqualetto, E.: Testing the Sobolev property with a single test plan. *Studia Math.* **264** (2022), no. 2, 149–179.

[27] Rajala, K.: Uniformization of two-dimensional metric surfaces. *Invent. Math.* **207** (2017), no. 3, 1301–1375.

[28] Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana* **16** (2000), no. 2, 243–279.

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