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Finite distortion curves: Continuity, differentiability and Lusin's (N) property

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Abstract. We define finite distortion ω-curves and we show that for some forms ω and when the distortion function is sufficiently exponentially integrable, the map is continuous, differentiable almost everywhere and has Lusin's (N) property. This is achieved through some higher integrability results about finite distortion ω-curves. It is also shown that this result is sharp both for continuity and for Lusin's (N) property. We also show that if we assume weak monotonicity for the coordinates of a finite distortion ω-curve, we obtain continuity.

1. Introduction

The study of mappings of *finite distortion* is a central theme in geometric function theory. Their importance stems in part from the various connections with other fields like holomorphic dynamics, PDEs and non-linear elasticity, to name a few. A mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, where Ω is a domain in $\mathbb{R}^n$, is said to be of finite distortion if $J_f \in L^1_{\mathrm{loc}}$ and there exists a measurable function $K_f(x) \geq 1$, which is finite almost everywhere, and is such that the following inequality is satisfied:

$$|Df(x)|^n \leq K_f(x)\, J_f(x) \quad \text{a.e.,} \tag{1.1}$$

where $|Df|$ denotes the sup norm of the differential Df, and J_f the Jacobian determinant. When $K_f(x) \leq K < \infty$ almost everywhere, where K constant, such a map is called *quasiregular* (also known as maps of bounded distortion).

Quasiregular maps are the correct higher dimensional generalization of holomophic maps in the complex plane. Starting with Reshetnyak, they have been studied extensively since the sixties, and by now their theory is well developed. We refer to the books [19, 20] for more details on quasiregular maps. In many applications, the distortion must be allowed to blow up in a controlled manner. This has led to systematic study of the more general class of mappings with finite distortion. By now, there is a quite extensive literature for these maps; we refer to the books [8, 12] for their general theory.

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One of the basic facts about maps of finite distortion concerns their continuity. In [21], Vodopyanov and Goldshtein showed that a mapping of finite distortion $f\colon \Omega \to \mathbb{R}^n$ that belongs in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has a continuous representative in the same Lebesgue class. But finite distortion mappings do not necessarily belong in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. However, in Theorem 7.1 of [10], it was shown that if we assume that $e^{\lambda K_f(x)} \in L^1_{\mathrm{loc}}$ for some large enough $\lambda > 0$, then we have that f is in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, and as a result, that it has a continuous representative. In fact, in [11] it was shown that $e^{\lambda K_f(x)} \in L^1_{\mathrm{loc}}$, for any $\lambda > 0$, is enough in order to obtain continuity. Moreover, mappings of finite distortion with $e^{\lambda K_f(x)} \in L^1_{\mathrm{loc}}$, for some $\lambda > 0$, are almost everywhere differentiable and have Lusin's (N) property, i.e., they map sets of zero (Lebesgue) measure to sets of zero measure, see [8] and also [14, 15].

Recently, there has been a lot of interest in generalizing quasiregular mappings in the setting where the range and the domain of definition do not have the same dimension, see [6, 7, 17, 18]. Pankka in [18] called such maps *quasiregular curves*, in accordance with their holomorphic counterparts, which are called holomorphic curves. In this paper, we study under what conditions do *curves of finite distortion*, which are a generalization of quasiregular curves, have the desirable properties we mentioned above. By $\Omega^n(\mathbb{R}^m)$ we denote the space of smooth differential n-forms in $\mathbb{R}^m$, where $n \le m$. In the following definition, $\omega \in \Omega^n(\mathbb{R}^m)$ is a smooth, non-vanishing and closed differential n-form. We call such forms *n-volume forms*.

Definition 1.1 (Finite distortion ω-curve). A mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, where Ω is a domain in $\mathbb{R}^n$ and $n \le m$, is called a finite distortion ω-curve if $\star f^*\omega \in L^1_{\mathrm{loc}}$ and there exists a measurable function $K_f(x) \ge 1$, which is finite almost everywhere, and is such that the following inequality is satisfied almost everywhere on Ω:

$$(\|\omega\| \circ f)|Df(x)|^n \le K_f(x)(\star f^*\omega), \tag{1.2}$$

Here, $\star f^*\omega$ is the Hodge star of the n-volume form $f^*\omega$ (the pullback of ω), that is, the function that satisfies $(\star f^*\omega)\, dx_1 \wedge \dots \wedge dx_n = f^*\omega$. In other words, if $\omega = \sum_I \phi_I(x)\, dx_I$, where the sum is over all multi-indices $I = (i_1, \dots, i_n)$, and if $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_n}$, then

$$\star f^*\omega = \sum_I \phi(f(x))\, J_I,$$

where J_I denotes the Jacobian of the map $(f_{i_1}, \dots, f_{i_n})$. The function $\|\omega\|\colon \mathbb{R}^m \to [0, \infty)$ is the pointwise comass norm of ω, defined as

$$\|\omega\|(p) = \sup\{|\omega_p(v_1, \dots, v_n)| : \ v_1, \dots, v_2 \in \mathbb{R}^m,\ |v_i| \le 1\}$$

for each $p \in \mathbb{R}^m$.

When $K_f(x) \le K < \infty$, where K is some constant, we get the class of quasiregular ω-curves. We note here that in [18], quasiregular curves are assumed to be continuous by definition. However, in [17], Pankka and Onninen showed that the continuity assumption can be dropped from the definition when the form ω has constant coefficients.

Our first result shows that, unlike the case of mappings of finite distortion, the condition $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}$ for any $\lambda > 0$ is not enough to guarantee the existence of a continuous representative even for the most simple forms. In fact, it shows that even $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is not enough (compare this with Theorem 1.4 in [11]).

Theorem 1.2. *For every $\lambda < 2$, there exists a finite distortion ω-curve $f : \mathbb{R}^2 \to \mathbb{R}^3$ for $\omega = dx \wedge dy$, such that $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ and $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}$, but f does not have a continuous representative.*

The situation is similar for Lusin's (N) condition as well. The condition $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}$, for any $\lambda > 0$, is not enough.

Theorem 1.3. *For every $\lambda < 2$, there exists a finite distortion ω-curve $f : \mathbb{R}^2 \to \mathbb{R}^4$ for $\omega = dx \wedge dy$, such that $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^4)$ and $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}$, but f does not have Lusin's* (N) *property.*

To explain why finite distortion curves differ from finite distortion maps in terms of their continuity properties, we need to introduce the important concept of weak monotonicity. A real valued function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is called weakly monotone if the following holds: for all balls B compactly contained in Ω and all constants $m, M \in \mathbb{R}$, $m \leq M$, such that

$$|M - u| - |u - m| + 2u - m - M \in W^{1,1}_0(B),$$

we have that

$$m \leq u(x) \leq M,$$

for almost every $x \in B$. Here $W^{1,1}_0(B)$ stands for the closure of $C^\infty_0(B)$, the compactly supported smooth functions, with respect to the Sobolev space norm. For continuous functions, this is equivalent to the more well known notion of monotonicity, namely

$$\operatorname*{osc}_B u \leq \operatorname*{osc}_{\partial B} u,$$

where $\operatorname{osc}_B u = \sup_{x,y \in B} |u(x) - u(y)|$.

The reason that weak monotonicity is important when discussing continuity is that the coordinate functions of a finite distortion map in a suitable Orlicz–Sobolev space are weakly monotone, see for example Theorem 7.3.1 in [12]. It is also well known that weakly monotone functions in that same Orlicz–Sobolev space have continuous representatives (for more precise formulations, see Lemma 2.1 in Section 2). Thus, roughly speaking, to show the existence of a continuous representative for a mapping of finite distortion it is enough to show that the map has sufficient regularity and its coordinates are weakly monotone.

However, the situation is different for curves of finite distortion. Their coordinate functions are not necessarily weakly monotone, as our example in Theorem 1.2 shows. On the other hand, if we assume the weak monotonicity of the coordinate maps and that $e^{\lambda K_f(x)} \in L^1_{\mathrm{loc}}$, for some $\lambda > 0$, then by adapting the arguments in Theorem 2.4 of [8] and Theorem 7.5.2 of [12], we obtain the existence of a continuous representative.

Theorem 1.4. *Let $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$ be a finite distortion ω-curve, for a bounded n-volume form ω in $\mathbb{R}^m$, such that the coordinate functions $f_1, \ldots, f_m$ are weakly monotone. Suppose that there exists $\lambda > 0$ such that $e^{\lambda K_f(x)} \in L^1_{\mathrm{loc}}$. Then f has a continuous representative in Ω.*

Our next result gives some sufficient conditions in order for a finite distortion ω-curve to have a continuous representative which is also almost everywhere differentiable and has Lusin's (N) property in the simple case of constant coefficient forms. We will call an n-volume form in $\mathbb{R}^m$, $\omega = \sum_I \phi_I(x)\, dx_I$, where the sum is over all multi-indices $I = (i_1, \ldots, i_n)$, $1 \le i_1 < \cdots < i_n \le m$, a *constant coefficient form* when the functions ϕ_I are constant. Here dx_I denotes the n-covector $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$.

Theorem 1.5. *Let $f : \Omega \to \mathbb{R}^m$ be a finite distortion ω-curve, for a constant coefficient n-volume form ω in $\mathbb{R}^m$. There exists $c_2 = c_2(n) > 0$ such that if $\exp(\lambda K_f(x)) \in L^1_{\text{loc}}$, for some $\lambda > c_2$ and $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^m)$, then f has a continuous and almost everywhere differentiable representative. Moreover, this representative has Lusin's* (N) *property, that is, it maps sets of zero n-dimensional Lebesgue measure to sets of zero n-dimensional Lebesgue measure.*

Our results in Theorems 1.2 and 1.3 essentially show that the integrability condition on the distortion cannot be relaxed.

Next we consider more general classes of forms. We will call an n-volume form in $\mathbb{R}^m$, $\omega = \sum_I \phi_I(x)\, dx_I$, *bounded* if all the non-zero, real valued functions $\phi_I(x)$ are bounded in $\mathbb{R}^m$, and there is a constant $c > 0$ such that, for all $x \in \mathbb{R}^m$, there is a multi-index I_x such that $|\phi_{I_x}(x)| > c$. Notice that for a bounded form ω, there is a constant $c > 0$ so that $\|\omega\|(x) > c$, for all $x \in \mathbb{R}^m$. We will use the symbols $|\omega|_{\ell_1}$ and $|\omega|_{\inf}$ to denote

$$\sup_{x \in \mathbb{R}^m} \sum_I |\phi_I(x)| \quad \text{and} \quad \inf\{\|\omega\|(x) : x \in \mathbb{R}^m\},$$

respectively. When the functions ϕ_I are constant, we say that ω is a constant coefficient form. Notice that for bounded forms, we have that $|\omega|_{\ell_1} < \infty$ and $|\omega|_{\inf} > 0$. For an n-volume form $\omega = \sum_I \phi_I(x)\, dx_I$, we will also denote by H_ω the set of multi-indices I such that $\phi_I(x)$ is not identically zero.

We define E_f to be the following set:

$$\begin{aligned} E_f = \{x \in \Omega :\ & \text{there exists a sequence } x_n \to x \text{ such that} \\ & \star f^* dx_I(x_n) \to \infty, \text{ for some multi-index } I \in H_\omega\}. \end{aligned}$$

Notice that $\star f^* dx_I$ is a real valued function for all I. For a bounded n-form ω, we say that the pair (f, ω) satisfies *condition (D)* if the set E_f can be covered by a countable number of open balls $\{B_i\}_{i=1}^{\infty}$, and if on each ball B_i, there exists a multi-index I_i such that $\star f^* dx_{I_i} \ge \max_I \{\star f^* dx_I\}$. So roughly speaking, condition (D) says that on balls where $\star f^* \omega$ gets large, one of the terms $\star f^* dx_I$ dominates the others.

Theorem 1.6. *Let $f : \Omega \to \mathbb{R}^m$ be a finite distortion ω-curve, for a bounded n-volume form ω in $\mathbb{R}^m$, such that the pair (f, ω) satisfies condition* (D). *Suppose that $c = c(n, \omega) = \frac{n}{c_1} |\omega|_{\ell_1} / |\omega|_{\inf}$, where $c_1 = c_1(n) > 0$ constant. If $\exp(\lambda K_f(x)) \in L^1_{\text{loc}}$, for some $\lambda > c$, then there is a continuous and almost everywhere differentiable representative of f. Moreover, this representative has Lusin's* (N) *property, i.e., it maps sets of zero n-dimensional Lebesgue measure to sets of zero n-dimensional Lebesgue measure.*

Remark 1.7. The constant c_1 in Theorem 1.6 comes from a well-known result about the higher integrability if the Jacobian of mappings of finite distortion proven in [5], see

Lemma 2.3 in Section 2 for the precise statement. It is known that in two dimensions, $c_1 = 1$ (see Theorem 20.4.12 in [2]), but the exact value of this constant remains unknown in higher dimensions. Notice that when $n = 2$ and the form $\omega = dx \wedge dy$, we have that the constant above is $c = 2$. Thus Theorems 1.2 and 1.3 imply that Theorem 1.6 is sharp in this case.

An immediate corollary of the above theorems, which is worth pointing out, is the area formula (see [16]). Here, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure in $\mathbb{R}^m$.

Corollary 1.8 (Area formula). *Let f be as in Theorem 1.6 or 1.5. Then*

$$\int_E h(x)\sqrt{\det(Df^T Df)}\,dx = \int_{f(E)} \sum_{x\in f^{-1}(y)} h(x)\,d\mathcal{H}^n(y)$$

holds for all measurable functions $h\colon \mathbb{R}^n \to [0,\infty]$ and all measurable sets $E \subset \mathbb{R}^n$.

Theorems 1.6 and 1.5 follow from the fact that curves of finite distortion satisfying the assumptions of the theorems enjoy a higher than natural amount of regularity. We state the precise results, since they are of independent interest.

Theorem 1.9. *Let $f\colon \Omega \to \mathbb{R}^m$ be a finite distortion ω-curve, for a bounded n-volume form ω in $\mathbb{R}^m$, such that the pair (f,ω) satisfies condition* (D). *If* $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}(\Omega)$, *for some $\lambda > 0$, then there exists a constant $c_1 = c_1(n) > 0$ such that*

$$\star f^*\omega \log^a(e + \star f^*\omega) \in L^1_{\mathrm{loc}}(\Omega) \quad \textit{and} \quad |Df(x)|^n \log^{a-1}(e + |Df(x)|) \in L^1_{\mathrm{loc}}(\Omega),$$

for all $a < c_1\lambda\, |\omega|_{\inf}/|\omega|_{\ell_1}$.

Theorem 1.10. *Let $f\colon \Omega \to \mathbb{R}^m$ be a finite distortion ω-curve, where ω is an n-volume form with constant coefficients. If* $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}(\Omega)$, *for some $\lambda > 0$, and if $f \in W^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$, then there is a $c_3 = c_3(n) > 0$ such that*

$$\star f^*\omega \log^a(e + \star f^*\omega) \in L^1_{\mathrm{loc}}(\Omega) \quad \textit{and} \quad |Df(x)|^n \log^{a-1}(e + |Df(x)|) \in L^1_{\mathrm{loc}}(\Omega),$$

for all $a < c_3\lambda$.

It is also interesting to point out that the higher integrability implies also modulus of continuity estimates. This can be achieved through an embedding result for Orlicz–Sobolev spaces of Donaldson and Trudinger (Theorem 3.6 in [4]; see also Theorem 8.40 in [1]).

Theorem 1.11. *Let $f\colon \Omega \to \mathbb{R}^m$ be a finite distortion ω-curve, where ω is*

(i) *either an n-volume form with constant coefficients and $f \in W^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$,*

(ii) *or a bounded n-volume form such that the pair (f,ω) satisfies condition* (D).

Then there exists a constant $q = q(n,\omega) > 0$ such that if $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}(\Omega)$, *for some $\lambda > q$, then for every compact subdomain $F \subset \Omega$ and every $x, y \in F$, we have*

$$|f(x) - f(y)| \le Q\|f\|_{W^{1,P}} \int_{|x-y|^{-n}}^{\infty} \frac{P^{-1}(t)}{t^{(n+1)/n}}\,dt, \tag{1.3}$$

where $P(t) = t^n \log^a(e+t)$, with $a > n$, $Q = Q(n,F) > 0$ is a constant, and $\|f\|_{W^{1,P}} = \|f\|_P + \|Df\|_P$.

The rest of the paper is organized as follows. In Section 2, we introduce the necessary notation and terminology, and we recall results from the theory of finite distortion maps that we are going to need. In Section 3, we prove the higher integrability results of Theorems 1.9 and 1.10, while in Section 4 we give the proofs of Theorems 1.6 and 1.5. In Section 5, we prove Theorem 1.4. Finally, in Section 6, we construct the functions of Theorems 1.2 and 1.3.

2. Preliminaries on Orlicz–Sobolev spaces and finite distortion maps

Here we collect results and terminology from Orlicz–Sobolev spaces and finite distortion maps that we are going to need for the proofs of our results, we refer to [1, 12] for more details.

First we need to introduce the *Orlicz spaces* and the *Orlicz–Sobolev spaces*. An Orlicz function is a continuous and increasing function $P\colon [0,\infty) \to [0,\infty)$ with $P(0) = 0$ and $\lim_{t\to\infty} P(t) = \infty$. We will also assume that the function P is convex. The Orlicz space $L^P(\Omega)$ consists of all measurable functions $u\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\|u\|_P := \int_\Omega P(\lambda|u|) < \infty, \quad \text{for some } \lambda = \lambda(u) > 0.$$

The functional $\|\cdot\|_P$, which is usually called the Luxemburg functional, is a norm in L^P, and in fact L^P is a Banach space with this norm.

The Orlicz space L^P for $P(t) = t^p \log^a(e+t)$, $1 \le p < \infty$ and $a > 1-n$, will be denoted as $L^p \log^a L(\Omega)$.

For an Orlicz function P, the Orlicz–Sobolev space $W^{1,P}(\Omega)$ is simply the set of functions that belong in $W^{1,1}_{\mathrm{loc}}(\Omega)$ and their weak partial derivatives are also in $L^P(\Omega)$. It is easy to see that the vector valued versions of the above spaces are simply the maps whose coordinate functions are in the corresponding space of real valued functions.

For the proof of Theorem 1.4, we shall require the following result, see Theorem 7.5.1 in [12].

Lemma 2.1. *Let $u \in W^{1,P}(\Omega)$, where $P(t) = t^n/\log(e+t)$. If u is a weakly monotone function, then u has a continuous representative.*

For the proof of Theorem 1.6, we are going to need the following lemma, which is an amalgamation of Theorems B and C in [13].

Lemma 2.2. *Let $g\colon \Omega \to \mathbb{R}^m$ be a function in the Orlicz–Sobolev space $W^{1,P}(\Omega,\mathbb{R}^m)$, where $P(t) = t^n \log^a(e+t)$, with $a > n-1$. Then g has a continuous and almost everywhere differentiable representative. Moreover, this representative has Lusin's* (N) *property.*

We note that the aforementioned theorems in [13] are stated about functions in the Lorentz space. However, there is an equivalent way of defining Lorentz spaces using Orlicz spaces which we have used in the above lemma. We refer to the discussion preceding Theorems B and C in [13] for more details.

Due to the above lemma, to prove Theorem 1.6 it is enough to show that the coordinate functions f_i of our finite distortion ω-curve f are in the right Orlicz–Sobolev space.

To prove that, we require another important result about the higher integrability of finite distortion maps proven in Theorem 1.1 of [5], compare this with Theorems 1.9 and 1.10.

Lemma 2.3. *Let $g : \Omega \to \mathbb{R}^n$, $n \geq 2$, be a mapping of finite distortion. Assume that the distortion function $K_g(x)$ satisfies $\exp(\beta K_g) \in L^1_{\text{loc}}$, for some $\beta > 0$. Then there is a constant $c_1 = c_1(n)$, $0 < c_1 < 1$, such that*

$$J_g(x) \log^a(e + J_g(x)) \in L^1_{\text{loc}}(\Omega) \quad \textit{and} \quad |Dg(x)|^n \log^{a-1}(e + |Dg(x)|) \in L^1_{\text{loc}}(\Omega)$$

for all $a < c_1 \beta$.

We are going to need the following important inequality. Its proof can be found in the Appendix of [5], or in Lemma 6.2 of [8].

Lemma 2.4. *Let $x, y \geq 1$. Then, for any $a > -1$ and $b > 0$, we have*

$$xy \log^a(C(n)(xy)^{1/n}) \leq \frac{C(n)}{b} x \log^{a+1}(x^{1/n}) + C(a, b, n) \exp(by),$$

We also require a slight variant of the above inequality. We include a proof here for completeness.

Lemma 2.5. *Let $x, y \geq 1$. Then, for any $a > -1$ and $b > 0$, we have*

$$xy \log^a(e + (xy)^{1/n}) \leq \frac{C_1(a, b, n)}{b} x \log^{a+1}(e + x^{1/n}) + C_2(a, b, n) \exp(by),$$

where $C_1(a, b, n)$ and $C_2(a, b, n)$ are constants.

Proof. If $x < e^{by/2}$, then since there is a $C_2 = C_2(a, b, n)$ such that

$$y \log^a(e + (xy)^{1/n}) \leq C_2 \, e^{by/2},$$

we obtain that

$$xy \log^a(e + (xy)^{1/n}) \leq C_2 \, e^{by}.$$

If on the other hand, $x \geq e^{by/2}$, then $y \leq \frac{2}{b} \log x$. Hence we have that

$$\begin{aligned} y \log^a(e + (xy)^{1/n}) &\leq \frac{2}{b} \log x \log^a \left(e + \left(x \frac{2}{b} \log x\right)^{1/n}\right) \\ &\leq \frac{2}{b} \log(e + x) \log^a \left(e + (x^{2/b+1})^{1/n}\right) \\ &\leq \left(\frac{b+2}{b}\right)^a \frac{2n}{b} \log(e + x)^{1/n} \log^a \left(e + x^{1/n}\right) \leq C_1(a, b, n) \log^{a+1}(e + x^{1/n}), \end{aligned}$$

where

$$C_1(a, b, n) = \left(\frac{b+2}{b}\right)^a \frac{2n}{b}.$$

Thus in that case we have

$$xy \log^a(e + (xy)^{1/n}) \leq C_1(a, b, n) \, x \log^{a+1}(e + x^{1/n}),$$

and we are done. ∎

3. Higher integrability of finite distortion curves

In this section, we give the proofs of Theorems 1.9 and 1.10, which combined with Lemma 2.2, will give us Theorems 1.6 and 1.5.

Proof of Theorem 1.9. Let K be a compact subset of Ω and let $P(t) = t^n \log^a(e+t)$. We want to find the values of a for which $\int_K P(\star f^*\omega)\,dx < \infty$. To that end, write

$$\int_K P(\star f^*\omega)\,dx = I_1 + I_2,$$

where

$$I_1 = \int_{K\setminus(\bigcup_i V_i)} P(\star f^*\omega)\,dx \quad \text{and} \quad I_2 = \int_{(\bigcup_i V_i)\cap K} P(\star f^*\omega)\,dx.$$

Here, the sets V_i, $i = 1, 2, \ldots$, are open subsets covering the set E_f from condition (D) such that $\overline{V_i} \subset B_i$, where B_i are the balls of condition (D).

We now want to estimate the integrals I_1 and I_2. We start with I_1. Notice that if $\omega_y = \sum_I \phi_I(y)\,dx_I$, for $y \in \mathbb{R}^m$, then $\star f^*\omega = \sum_I \phi(f(x)) \star f^*dx_I(x)$, with $x \in \mathbb{R}^n$. Moreover, since the ω form is bounded, we have that $(\|\omega\| \circ f)(x) \geq |\omega|_{\inf}$ and that $\sum_I \phi(f(x)) \leq |\omega|_{\ell_1}$. By condition (D), we know that on the set $K \setminus (\bigcup_i V_i)$ the functions $\star f^*dx_I$, for I any multi-index, are uniformly bounded by a constant $M_2 > 0$ almost everywhere, since on this set there is no sequence x_n such that $\star f^*dx_I(x_n) \to \infty$ for any multi-index I. Hence

$$I_1 \leq \int_{K\setminus(\bigcup_i V_i)} P(C)\,dx,$$

where $C = C(\omega, f) > 0$ is a constant. The integral on the right-hand side is finite since K is compact.

To estimate I_2, notice first that

$$I_2 \leq \sum_i^{\infty} \int_{V_i\cap K} P(\star f^*\omega)\,dx. \tag{3.1}$$

On each of the sets V_i, there is a multi-index I_i such that $\star f^*dx_{I_i} \geq \max_I\{\star f^*dx_I\}$. Since there are finitely many multi-indices, we enumerate them as I_j, with $j = 1, \ldots, \binom{m}{n}$. We can now consider the finitely many sets $A_j = (\bigcup_k V_k) \cap K$, where k runs over all the sets V_i where $\star f^*dx_{I_j} \geq \max_I\{\star f^*dx_I\}$. We can rewrite (3.1) as

$$I_2 \leq \sum_j^{\binom{m}{n}} \int_{A_j} P(\star f^*\omega)\,dx.$$

We fix a j and notice that $A_j \subset W_j := (\bigcup_k B_k) \cap K_\varepsilon$, where k runs over the balls B_i for which $\star f^*dx_{I_j} \geq \max_I\{\star f^*dx_I\}$ and K_ε is a ε-neighbourhood of K. On W_j, the distortion inequality (1.2) becomes

$$(\|\omega\| \circ f)(x)|Df(x)| \leq K_f(x) \star f^*dx_{I_j} \sum_I \phi_I(f(x)). \tag{3.2}$$

Notice that if we set $f_{I_j} = (f_{j_1}, \dots, f_{j_n})$, where $I_j = (j_1, \dots, j_n)$, then by (3.2) and using the fact that the ω form is bounded and $|Df_{I_j}(x)| \le |Df(x)|$, we obtain

$$|Df_{I_j}(x)| \le \frac{|\omega|_{\ell_1}}{|\omega|_{\inf}} K_f(x) \star f^* dx_{I_j}, \quad \text{for almost every } x \in W_j. \tag{3.3}$$

Notice that $|\omega|_{\ell_1}/|\omega|_{\inf} > 1$. Hence, (3.3) implies that f_{I_j} is a mapping of finite distortion on W_j with distortion function $K_{f_{I_j}} = |\omega|_{\ell_1}/|\omega|_{\inf} K_f(x)$. By assumption, we know that $\exp(\lambda K_f) \in L^1_{\mathrm{loc}}(\Omega)$, or equivalently,

$$\exp\Big(\lambda \frac{|\omega|_{\inf}}{|\omega|_{\ell_1}} K_{f_{I_j}}\Big) \in L^1_{\mathrm{loc}}(\Omega), \quad \text{for some } \lambda > 0.$$

Therefore, by Lemma 2.3 we obtain

$$\star f^* dx_{I_j} \log^a(e + \star f^* dx_{I_j}) \in L^1_{\mathrm{loc}}(W_j),$$

for all $a < c_1 \lambda\, |\omega|_{\inf}/|\omega|_{\ell_1}$. Hence $I_2 < \infty$ for those a, as we wanted.

Next, we want to show that $\int_K \hat{P}(|Df(x)|)\, dx < \infty$, where $\hat{P}(t) = t^n \log^{a-1}(e+t)$, for all $a < c_1 \lambda\, |\omega|_{\inf}/|\omega|_{\ell_1}$. By the distortion inequality, we obtain that

$$\int_K \hat{P}(|Df(x)|)\, dx \le \int_K \hat{P}\Big(\frac{|\omega|_{\ell_1}}{|\omega|_{\inf}} K_f(x) \star f^*\omega\Big).$$

By Lemma 2.5, we have that for all $b > 0$,

$$\begin{aligned} &\hat{P}\Big(\frac{|\omega|_{\ell_1}}{|\omega|_{\inf}} K_f(x) \star f^*\omega\Big) \\ &\qquad \le \frac{C_1}{b} \star f^*\omega \log^a(e + \star f^*\omega^{1/n}) + C_2 \exp\Big(b \frac{|\omega|_{\ell_1}}{|\omega|_{\inf}} K_f(x)\Big). \end{aligned} \tag{3.4}$$

By what we have proven so far, we have that the first term of the right-hand side is integrable on K when $a < c_1 \lambda\, |\omega|_{\inf}/|\omega|_{\ell_1}$. Moreover, for suitable $b > 0$, we have that $C_2 \exp\big(b \frac{|\omega|_{\ell_1}}{|\omega|_{\inf}} K_f(x)\big) \in L^1(K)$. Hence, (3.4) gives us that $\int_K \hat{P}(|Df(x)|)\, dx < \infty$, for $a < c_1 \lambda\, |\omega|_{\inf}/|\omega|_{\ell_1}$, as we wanted. ■

The classical way to prove higher integrability results about quasiregular maps is through the so called weak reverse Hölder inequalities and Gehring's lemma, see for example [3] and [9]. The same method was adapted to the setting of quasiregular curves by Pankka and Onninen in [17]. The proof of higher integrability for finite distortion maps in [5] was inspired by the same methods.

To prove Theorem 1.10, we will adapt the arguments in [5] to our setting. To this end, we need the Hardy–Littlewood maximal function, which is defined for any locally integrable mapping $g : \mathbb{R}^n \to \mathbb{R}$ by

$$M(g)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y)|\, dy.$$

We start with a generalized version of Gehring's lemma which is implicitly contained in the proof of Theorem 1.1 in [5].

Lemma 3.1 (Generalized Gehring's lemma). *Let $g, K\colon \Omega \to \mathbb{R}$ be functions such that $g \in L^1_{\mathrm{loc}}(\Omega)$, $K(x)$ is measurable and $1 \le K(x) < \infty$ a.e., and $\exp(\beta K(x)) \in L^1_{\mathrm{loc}}$ for some $\beta > 0$. Suppose that these functions satisfy the generalized weak reverse Hölder inequality*

$$\fint_{\frac{1}{2}B} |g(x)|\,dx \le C_1(n)\Big(\fint_B (K(x)|g(x)|)^s\,dx\Big)^{1/s}, \tag{3.5}$$

for some $s < 1$ and all balls $B \subset\subset \Omega$. Then there is a constant $c_1 = c_1(n) > 0$ such that

$$\begin{aligned}\fint_{\frac{1}{2}B} |g(x)| \log^a\Big(e + \frac{|g(x)|}{\fint_B |g(y)|\,dy}\Big)\,dx \\ \le C_2(n,\beta,s)\fint_B \exp(\beta K(x))\,dx \fint_B |g(x)|\,dx,\end{aligned} \tag{3.6}$$

for all $a < c_1(n,s)\beta$.

Proof. Fix a ball $B_0(x_0, r_0) \subset\subset \Omega$. Since both (3.5) and (3.6) are homogeneous with respect to g, we can assume without loss of generality that $\fint_{B_0} |g(x)|\,dx = 1$. Let $d(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus B_0)$, and define the functions $h_1, h_2\colon \mathbb{R}^n \to \mathbb{R}$ by

$$h_1(x) = d(x)^n |g(x)| \quad \text{and} \quad h_2(x) = \chi_{B_0}(x),$$

for all $x \in \Omega$ and 0 outside of Ω. We are going to prove that for all balls $B \subset \mathbb{R}^n$, either

$$\Big(\fint_B h_1(x)\,dx\Big)^{1/n} \le C_1(n)\Big(\fint_{2B} (K(x)h_1(x))^s\,dx\Big)^{1/sn}, \tag{3.7}$$

in case $3B \subset B_0$, or

$$\Big(\fint_B h_1(x)\,dx\Big)^{1/n} \le C_3(n)\Big(\fint_{2B} h_2(x)\,dx\Big)^{1/n}, \tag{3.8}$$

when $3B$ is not contained in B_0. We can assume that $B \cap B_0 \ne \emptyset$, since otherwise the left-hand side is zero and thus the inequalities hold trivially. So suppose that $3B \subset B_0$. Then a simple geometric argument shows that

$$\max_B d(x) \le 4 \min_{2B} d(x).$$

Hence, by (3.5), we have

$$\begin{aligned}\Big(\fint_B h_1(x)\,dx\Big)^{1/n} &\le \max_B d(x)\Big(\fint_B |g(x)|\,dx\Big)^{1/n} \\ &\le C_1(n)\min_{2B} d(x)\Big(\fint_{2B} (K(x)|g(x)|)^s dx\Big)^{1/sn} \le C_1(n)\Big(\fint_{2B} (K(x)h_1(x))^s dx\Big)^{1/sn}.\end{aligned}$$

On the other hand, suppose that $3B$ is not contained in B_0. Since B_0 intersects B, again a simple geometric argument shows that

$$\max_B d(x) \le \max_{2B} d(x) \le C_4(n))|2B \cap B_0|^{1/n}.$$

Hence, using the fact that $\int_{B_0} |g(x)|\,dx = 1$, we have that

$$\Big(\frac{1}{|B|}\int_B h_1(x)\,dx\Big)^{1/n} \le \max_B d(x)\Big(\frac{1}{|B|}\int_{B\cap B_0} |g(x)|\,dx\Big)^{1/n}$$
$$\le C_4(n)\Big(\frac{|2B\cap B_0|}{|B|}\int_{B_0} |g(x)|\,dx\Big)^{1/n} \le C_3(n)\Big(\fint_{2B} h_2(x)\,dx\Big)^{1/n}.$$

By (3.7), we have that when the sup is achieved for a ball B with $3B \subset B_0$, then

$$M(h_1)(y)^{1/n} \le C_1(n)\, M((Kh_1)^s)(y)^{1/sn};$$

otherwise, by (3.8), we have that

$$M(h_1)(y)^{1/n} \le C_4(n)\, M(h_2)^{1/n}.$$

Notice that $M(h_2)(y) \le 1$, and thus there exists $\lambda_1(n)$ such that, for every $\lambda > \lambda_1$, the set $\{y \in \mathbb{R}^n : C_4(n)\, M(h_2)(y) \ge \lambda^n\}$ is empty. Hence, when λ large, we have

$$\{y \in \mathbb{R}^n : M(h_1)(y) \ge \lambda^n\} \subset \{y \in \mathbb{R}^n : C_1(n)\, M((Kh_1)^s)(y) \ge \lambda^{sn}\},$$

which implies that

$$|\{y \in \mathbb{R}^n : M(h_1)(y) \ge \lambda^n\}| \le |\{y \in \mathbb{R}^n : C_1(n)\, M((Kh_1)^s)(y) \ge \lambda^{sn}\}|.$$

We want to apply Proposition 2.1 in [5] (twice) to conclude that

$$\int_{\{h_1>\lambda^n\}} h_1(x)\,dx \le C_5(n)\,\lambda^{n-sn}\int_{\{C_6(n,s)Kh_1>\lambda^n\}} (K(x)h_1(x))^s\,dx. \tag{3.9}$$

To do that, we need to show that h_1 and $(Kh_1)^s$ are in $L^1(\mathbb{R}^n)$. This is easy to see for h_1, since $g \in L^1_{\mathrm{loc}}$. For $(Kh_1)^s$, we can argue using the inequality

$$ab \le \exp(\kappa a) + \frac{2b}{\kappa}\log(e + b/\kappa)$$

and the fact that $K(x)$ is exponentially integrable and h_1 is in L^1. We refer to the proof of Theorem 1.4 for a more detailed argument.

Let $a > 0$ be a constant, and set

$$\Psi(\lambda) = \frac{n-sn}{a}\log^a(\lambda) + \log^{a-1}\lambda.$$

Notice that

$$\Phi(\lambda) := \frac{d}{d\lambda}\Psi(\lambda) = \frac{n-sn}{\lambda}\log^{a-1}(\lambda) + \frac{a-1}{\lambda}\log^{a-2}\lambda,$$

and that $\Phi(\lambda) > 0$ for all $\lambda > \lambda_2(n,s) = e^{1/(n-sn)}$. Moreover,

$$\lambda^{n-sn}\,\Phi(\lambda) = \frac{d}{d\lambda}(\lambda^{n-sn}\log^{a-1}\lambda).$$

We multiply both sides of (3.9) by $\Phi(\lambda)$ and we integrate with respect to λ over the interval (λ_0, j), where $\lambda_0 = \max\{\lambda_1, \lambda_2\}$ and for $j > \lambda_0$ large. Note that the functions $\Phi(\lambda)h_1(x)$ and $(K(x)h_1(x))^s \lambda^{s-sn}\Phi(\lambda)$ are non-negative on the sets we are integrating over. Hence, by the Fubini–Tonelli theorem, we can change the order of integration to obtain

$$\begin{aligned}\int_{A_1} h_1(x)\int_{\lambda_0}^{h_1^{1/n}} \Phi(\lambda)\,d\lambda\,dx + \int_{A_1'} h_1(x)\int_{\lambda_0}^{j} \Phi(\lambda)\,d\lambda\,dx \\ \le C_5(n)\int_{A_2}(K(x)h_1(x))^s\int_{\lambda_0}^{(C_6Kh_1)^{1/n}} \lambda^{n-sn}\,\Phi(\lambda)\,d\lambda\,dx \\ + C_5(n)\int_{A_2'}(K(x)h_1(x))^s\int_{\lambda_0}^{j}\lambda^{n-sn}\,\Phi(\lambda)\,d\lambda\,dx,\end{aligned}$$

where

$$\begin{aligned}A_1 &= \{\lambda_0 < h_1^{1/n} < j\}, \quad A_1' = \{h_1^{1/n} \ge j\},\\ A_2 &= \{\lambda_0 < (C_6Kh_1)^{1/n} < j\}, \quad A_2' = \{(C_6Kh_1)^{1/n} \ge j\}.\end{aligned}$$

Notice that

$$\int_{A_2}(K(x)h_1(x))^s\int_{\lambda_0}^{j}\lambda^{n-sn}\,\Phi(\lambda)\,d\lambda\,dx \le \int_{A_2} K(x)h_1(x)\log^{a-1}(C_6K(x)h_1(x))^{1/n}\,dx,$$

and thus we obtain

$$\begin{aligned}\int_{A_1} h_1(x)(\Psi(h_1^{1/n}) - \Psi(\lambda_0))\,dx + \int_{A_1'} h_1(x)(\Psi(j) - \Psi(\lambda_0))\,dx \\ \le C_5(n)\Big(\int_{A_2} C_6(n)K(x)h_1(x)\log^{a-1}(C_6K(x)h_1(x))^{1/n}\,dx \\ + \int_{A_2'}(K(x)h_1(x))^s j^{n-sn}\log^{a-1} j\,dx\Big),\end{aligned}$$

which implies

$$\begin{aligned}\frac{n-sn}{a}\Big(\int_{A_1} h_1(x)\log^a(h_1^{1/n})\,dx + \int_{A_1'} h_1(x)\log^a j\,dx\Big) \\ \le C_7(n)I_1 + C_5(n)I_2 + C_8(n,s,a)\Big(\int_{A_1} h_1(x)\,dx + \int_{A_1'} h_1(x)\,dx\Big),\end{aligned} \tag{3.10}$$

where

$$I_1 = \int_{A_2} K(x)h_1(x)\log^{a-1}(C_6K(x)h_1(x))^{1/n}\,dx$$

and

$$I_2 = \int_{A_2'}(K(x)h_1(x))^s j^{n-sn}\log^{a-1} j\,dx.$$

Notice that since $\int_{B_0} |g(x)|\,dx = 1$ and $d(x) \le r_0$, we have that

$$\int_{A_1} h_1(x)\,dx + \int_{A_1'} h_1(x)\,dx \le 2\int_{B_0} d(x)^n\,|g(x)|\,dx \le C_9(n)\,|B_0|.$$

Hence, (3.10) becomes

$$\begin{aligned}(3.11)\qquad \frac{1}{a}\Big(\int_{A_1} h_1(x)\log^a(h_1^{1/n})\,dx + \int_{A_1'} h_1(x)\log^a j\,dx\Big)\\ \le C_{10}(n,s)\,I_1 + C_{11}(n,s)\,I_2 + C_{12}(n,s,a)\,|B_0|.\end{aligned}$$

Notice that $\{h_1 > \lambda_0^n\} \subset \{C_6(n)K(x)h_1(x) > \lambda_0^n\}$, since its not hard to see that $C_6(n) > 1$. Hence we can write

$$\begin{aligned} I_1 = &\int_{E_1} K(x)h_1(x)\log^a(C_6K(x)h_1(x))^{1/n}\,dx\\ &+ \int_{E_2} K(x)h_1(x)\log^a(C_6K(x)h_1(x))^{1/n}\,dx,\end{aligned}$$

where

$$E_1 = A_2 \cap \{h_1^{1/n} \le \lambda_0\} \quad \text{and} \quad E_2 = \{h_1^{1/n} > \lambda_0\} \cap \{(C_6Kh_1)^{1/n} < j\} \subset A_1.$$

The first integral on the right-hand side is bounded by $C_{13}(n,s)\int_{B_0} e^{\beta K(x)}$. We will call the second integral P_0.

Moreover, we can do the same for the integral I_2 and write it as a sum of integrals over the sets A_1', $A_2' \setminus A_1' \cap \{h_1^{1/n} \le \lambda_0\}$ and $A_2' \setminus A_1' \cap \{h_1^{1/n} > \lambda_0\} \subset A_1$, which we call P_1, P_2 and P_3, respectively. Again, the integral over the set $A_2' \setminus A_1' \cap \{h_1^{1/n} \le \lambda_0\}$ is bounded by $C_{14}(n,s)\int_{B_0} e^{\beta K(x)}$. Furthermore, notice that

$$P_3 \le \int_{A_1} C_6^{n-sn}\,K(x)h_1(x)\log^{a-1}(C_6K(x)h_1(x))^{1/n} =: P_4.$$

Hence (3.11) gives us

$$\begin{aligned}&\frac{1}{a}\Big(\int_{A_1} h_1(x)\log^a(h_1^{1/n})\,dx + \int_{A_1'} h_1(x)\log^a j\,dx\Big)\\ (3.12)\quad &\le C_{10}(n,s)P_0 + C_{11}(n,s)(P_1+P_4) + C_{15}(n,s)\int_{B_0} e^{\beta K(x)} + C_{12}(n,s,a)\,|B_0|.\end{aligned}$$

We apply Lemma 2.4 and obtain

$$\begin{aligned} P_0 &\le C(n,a)\int_{A_1} e^{\beta K(x)}\,dx + \frac{C_6^{1/n}}{\beta}\int_{A_1} h_1(x)\log^a(h_1(x))^{1/n}\,dx,\\ P_4 &\le C'(n,a)\int_{A_1} e^{\beta K(x)}\,dx + \frac{C_6^{n-sn+1/n}}{\beta}\int_{A_1} h_1(x)\log^a(h_1(x))^{1/n}\,dx,\end{aligned}$$

and

$$P_1 \le C''(n,a) \int_{A_1'} e^{\delta K(x)^s}\, dx + \frac{1}{\delta} \int_{A_1'} h_1^s(x)\, j^{n-sn} \log^{a-1} j \log\big(h_1^{s/n} j^{1-s} \log^{(a-1)/s} j\big)\, dx,$$

for some $\delta > 0$ which will be fixed later. We call P_5 the second integral on the right-hand side. Using the above estimates and the fact that $A_1, A_1' \subset B_0$ and $e^{\delta K(x)^s} < c e^{\beta K(x)}$ for some constant $c = c(\delta) > 0$, (3.12) gives

$$\begin{aligned} &\frac{1}{a}\Big(\int_{A_1} h_1(x) \log^a (h_1^{1/n})\, dx + \int_{A_1'} h_1(x) \log^a j\, dx \Big) \\ &\qquad \le \frac{C_{16}(n,s)}{\beta} \int_{A_1} h_1(x) \log^a (h_1(x))^{1/n}\, dx + \frac{C_{11}(n,s)}{\delta} P_5 \\ &\qquad\quad + C_{17}(n,s,a,\beta) \int_{B_0} e^{\beta K(x)}\, dx + C_{12}(n,s,a)\, |B_0|, \end{aligned} \tag{3.13}$$

Put $a = \beta/(2C_{16}(n,s))$. It is not hard to see that for all j and for large enough $\delta > 0$, we have that

$$\frac{C_{11}}{\delta} P_5 \le \frac{1}{a} \int_{A_1'} h_1(x) \log^a j\, dx.$$

Hence, (3.13) implies

$$\int_{A_1} h_1(x) \log^a (h_1^{1/n})\, dx \le C_{17}(n,s,a,\beta) \int_{B_0} e^{\beta K(x)}\, dx + C_{12}(n,s,a)\, |B_0|.$$

It is easy to see that $|B_0| < \int_{B_0} e^{\beta K(x)} dx$. Hence, if we take limits as $j \to \infty$, then by the monotone convergence theorem we obtain

$$\int_{\{h_1 > \lambda_0^n\}} h_1(x) \log^a (h_1^{1/n})\, dx \le C_{18}(n,s,\beta) \int_{B_0} e^{\beta K(x)} dx. \tag{3.14}$$

We are now ready to prove (3.6). First notice that since $\lambda_0^n > 1$, there is a constant $k > 0$ such that $x \log^a(e + x) \le kx \log^a(x)$, for all $x > \lambda_0^n$. Now notice that

$$\begin{aligned} \int_{B_0} h_1(x) \log^a (e + h_1)\, dx = &\int_{\{h_1 > \lambda_0^n\}} h_1(x) \log^a (e + h_1)\, dx \\ &+ \int_{\{h_1 \le \lambda_0^n\} \cap B_0} h_1(x) \log^a (e + h_1)\, dx. \end{aligned}$$

The second integral on the right-hand side is bounded by $C_{19}(n,s)|B_0|$. Thus

$$\int_{B_0} h_1(x) \log^a (e + h_1)\, dx \le k \int_{\{h_1 > \lambda_0^n\}} h_1(x) \log^a (h_1)\, dx + C_{19}(n,s)\, |B_0|.$$

Using (3.14), we obtain

$$\int_{B_0} h_1(x) \log^a (e + h_1)\, dx \le C_{20}(n,s,\beta) \int_{B_0} e^{\beta K(x)} dx.$$

Notice that in the ball $\frac{1}{2}B_0$ we have that $d(x)^n \geq r_0^n/2^n \geq C_{23}(n)|B_0|$. Hence the above inequality implies

$$C_{21}(n)|B_0| \int_{\frac{1}{2}B_0} |g(x)| \log^a(e + C_{21}|B_0||g(x)|)\, dx \leq C_{22}(n,s,\beta) \int_{B_0} e^{\beta K(x)} dx.$$

Using the normalization $\fint_{B_0} |g(x)|\, dx = 1$ and the inequality $c \log^a(e+x) < \log^a(e+cx)$, for $c < 1$, in the case that $C_{21}(n) < 1$ we obtain

$$\fint_{\frac{1}{2}B_0} |g(x)| \log^a\Big(e + \frac{|g(x)|}{\fint_{B_0} |g(y)|\, dy}\Big)\, dx \leq C_2(n,\beta,s) \fint_{B_0} \exp(\beta K(x))\, dx \fint_{B_0} |g(x)|\, dx,$$

as we wanted. ■

Next, we have the analogues of the weak reverse Hölder inequalities for finite distortion curves.

Lemma 3.2 (Generalized weak reverse Hölder inequality). *Let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a finite distortion ω-curve, where ω is an n-volume form with constant coefficients, and let $B \subset\subset \Omega$. Then*

$$\fint_{\frac{1}{2}B} \star f^*\omega\, dx \leq C(n,m)\Big(\fint_B (K_f(x) \star f^*\omega)^{n/(n+1)}\, dx\Big)^{(n+1)/n}.$$

Proof. By following the argument in the proof of Lemma 6.1 in [17], where the authors prove a Cacciopoli-type inequality for quasiregular ω-curves, we can show that for all balls $B(y,r) \subset\subset \Omega$ and all functions $\phi \in C_0^\infty(B)$, we have

$$\int_B \phi(x)(\star f^*\omega)(x)\, dx \leq \|\omega\| \int_B |\nabla\phi(x)||f(x) - f_B||Df(x)|^{n-1}\, dx,$$

where $f_B = \fint_B f(x)\, dx$ (integration is meant coordinate-wise). Indeed, since ω is exact, there exists an $(n-1)$-form τ such that $d\tau = \omega$, $\tau_{f_B} = 0$. Since ω is a constant coefficient form, it is easy to see that τ is $\|\omega\|$-Lipschitz, meaning that $\|\tau\|(y) \leq \|\omega\||y - f_B|$ for all $y \in \mathbb{R}^m$. Hence

$$\begin{aligned}\int_B \phi(\star f^*\omega) &= \int_B \phi f^* d\tau = \int_B \phi\, df^*\tau = \int_B d(\phi f^*\tau) - \int_B d\phi \wedge f^* d\tau \\ &= -\int_B d\phi \wedge f^* d\tau \leq \int_B |\nabla\phi|(\|\tau\| \circ f)|Df|^{n-1} \leq \|\omega\| \int_B |\nabla\phi||f - f_B||Df|^{n-1},\end{aligned}$$

as we wanted. It is important to note that this argument uses the fact that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. We choose now ϕ so that $\phi(x) = 1$ when $x \in \frac{1}{2}B$, $0 \leq \phi \leq 1$, and $|\nabla\phi| \leq 2/r$. With this choice for ϕ and by applying Hölder's inequality, we obtain

$$\begin{aligned}&\int_{\frac{1}{2}B} (\star f^*\omega)(x)\, dx \\ &\qquad \leq \frac{2\|\omega\|}{r}\Big(\int_B |f(x) - f_B|^{n^2}\, dx\Big)^{1/n^2} \Big(\int_B |Df(x)|^{n^2/(n+1)}\, dx\Big)^{n^2-1/n^2}.\end{aligned} \tag{3.15}$$

By applying the Poincaré–Sobolev inequality (see for example Theorem A.18 in [8]) coordinate-wise, it is easy to see that

$$\Big(\int_B |f(x)-f_B|^{n^2}\,dx\Big)^{1/n^2} \le C_0(n,m)\Big(\int_B |Df(x)|^{n^2/(n+1)}\,dx\Big)^{(n+1)/n^2}.$$

Hence, (3.15) becomes

$$\fint_{\frac{1}{2}B} (\star f^*\omega)(x)\,dx \le \|\omega\| C(n,m)\Big(\fint_B |Df(x)|^{n^2/(n+1)}\,dx\Big)^{(n+1)/n}.$$

Applying the distortion inequality (1.2) in the above inequality gives

$$\fint_{\frac{1}{2}B} (\star f^*\omega)(x)\,dx \le C(n,m)\Big(\fint_B (K_f(x)\star f^*\omega)^{n/(n+1)}\,dx\Big)^{(n+1)/n},$$

which is what we wanted. ∎

The proof of Theorem 1.10 readily follows.

Proof of Theorem 1.10. The fact that

$$\star f^*\omega \log^a(e+\star f^*\omega) \in L^1_{\mathrm{loc}}(\Omega) \tag{3.16}$$

follows immediately from Lemmas 3.1 and 3.2. On the other hand,

$$|Df(x)|^n \log^{a-1}(e+|Df(x)|) \in L^1_{\mathrm{loc}}(\Omega)$$

follows by using (3.16) and the inequality in Lemma 2.4. ∎

4. Proofs of Theorems 1.6, 1.5 and 1.11

Proof of Theorem 1.6. For $k \ge 1$, let

$$U_k = \{x \in \Omega\colon \operatorname{dist}(x,\partial\Omega) > 1/k\} \cap B(0,k),$$

where $B(0,k)$ is the open ball centred at 0 and of radius k. If we take

$$c = c(n,\omega) = \frac{n}{c_1}\frac{|\omega|_{\ell_1}}{|\omega|_{\inf}},$$

then we can find a p so that $n-1<p$ and $p+1<\lambda c_1|\omega|_{\inf}/|\omega|_{\ell_1}$, since $\lambda > c$. By Theorem 1.9, now we have that f is in $W^{1,P}(U_k)$ for $P(t)=t^n\log^p(e+t)$ and some $p>n-1$. Hence, Lemma 2.2 implies that f has a continuous and almost everywhere differentiable representative in U_k, which also has Lusin's (N) property, for each large enough k. Since $U_k \subset U_{k+1}$ and $\bigcup_{k=1}^{\infty} U_k = \Omega$, this immediately implies that f has a representative in Ω with all the above properties. ∎

Proof of Theorem 1.5. The proof follows in the same way as that of Theorem 1.6, except that instead of Theorem 1.9, we use Theorem 1.10. ∎

Proof of Theorem 1.11. First we assume that F is a ball compactly contained in Ω. We want to apply Theorem 3.6 in [4] to the coordinate functions of f in F. To do that, we need to know that they are in $W^{1,P}_{\text{loc}}(\Omega)$, with $P(t)$ such that

$$\int_1^\infty \frac{P^{-1}(t)}{t^{(n+1)/n}}\, dt < \infty.$$

It is not hard to see that this last condition is satisfied when $P(t) = t^n \log^a(e+t)$, with $a > n$. By Theorems 1.9 and 1.10, there is a constant q such that if $\exp(\lambda K_f(x)) \in L^1_{\text{loc}}(\Omega)$ for some $\lambda > q$, then we have that $f \in W^{1,P}_{\text{loc}}(\Omega)$. Thus, by applying Theorem 3.6 in [4] to the coordinate functions, it is easy to see that (1.3) holds and we are done. For a general compact set F, the result follows routinely. We sketch the proof for completeness. Cover the set F by a finite number of balls $B_i \subset \Omega$, $i = 1, \dots, k$, consider the segment connecting x and y (if the segment is not in F, just consider a polygonal path), and find points x_i, $i = 0, \dots, k' < k$, such that $x_0 = x$, $x_{k'} = y$ and $x_i \in B_i \cap B_{i+1}$, $i = 1, \dots, k'-1$ (by renaming the balls if necessary). By applying the special case of the theorem for balls in B_i, we can conclude that

$$|f(x_i) - f(x_{i+1})| \leq Q_i \|f\|_{W^{1,P}} \int_{|x_i - x_{i+1}|^{-n}}^\infty \frac{P^{-1}(t)}{t^{(n+1)/n}},$$

for $i = 1, \dots, k' < k$. The triangle inequality and the fact that $|x_i - x_{i+1}|^{-n} > |x-y|^{-n}$ give us the required inequality. ∎

5. Monotonicity implies continuity

Here we prove Theorem 1.4. The proof is essentially an adaptation of that of Lemma 2.8 in [8].

Proof of Theorem 1.4. From Lemma 2.1, it is enough to show that the coordinate functions are in $W^{1,P}(\Omega)$, where $P(t) = t^n/\log(e+t)$. For that it is enough to show that $|Df| \in L^n \log^{-1} L(\Omega)$.

First notice that, because the function $P(t)$, $t > 0$, is increasing, and because ω is a bounded form, we have that

$$\begin{aligned}\frac{|Df|^n}{\log(e+|Df|)} &\leq \frac{c_0 K_f(x) \star f^*\omega}{\log(e + (c_0 K_f(x) \star f^*\omega)^{1/n})} \\ &\leq \frac{c_0 K_f(x) \star f^*\omega}{\log(e + (c_0 \star f^*\omega)^{1/n})} \leq n \frac{c_0 K_f(x) \star f^*\omega}{\log(e + c_0 \star f^*\omega)},\end{aligned}$$

where $c_0 > 0$ is a constant that depends on ω. Hence, for any compact set $K \subset \Omega$, we have that

$$\int_K \frac{|Df|^n}{\log(e+|Df|)}\, dx \leq n \int_K \frac{c_0 K_f(x) \star f^*\omega}{\log(e + c_0 \star f^*\omega)}\, dx. \tag{5.1}$$

By using (5.1) and the inequality (see Lemma 2.7 in [8] for a proof)

$$ab \le \exp(\kappa a) + \frac{2b}{\kappa}\log(e + b/\kappa), \quad \text{for all } a \ge 1, b \ge 0, \kappa > 0,$$

with

$$a = K_f(x), \quad b = \frac{c_0 \star f^*\omega}{\log(e + c_0 \star f^*\omega)} \quad \text{and} \quad \kappa = \lambda,$$

we obtain

$$\int_K \frac{|Df|^n}{\log(e+|Df|)}dx \le n\int_K \exp(\lambda K_f(x))\,dx + \frac{2n}{\lambda}\int_K \frac{c_0 \star f^*\omega}{\log(e+c_0\star f^*\omega)} \log\left(e + \frac{c_0 \star f^*\omega}{\lambda \log(e+c_0\star f^*\omega)}\right)dx.$$

The first integral is finite by assumption. We can split the second integral into two integrals, over the sets

$$A_1 = \{x \in K : \lambda \log(e + c_0 \star f^*\omega) \le 1\} \quad \text{and} \quad A_2 = K \setminus A_1.$$

On A_2, the integrand is dominated by $c_0 \star f^*\omega$, which is integrable, while on A_1, notice that $\star f^*\omega \le e^{1/\lambda - 1}/c_0$, and since the function $t/\log(e+t)$ is increasing, we obtain that the integrand is bounded in A_1, and we are done. ■

6. Counterexamples to continuity and Lusin's condition

Proof of Theorem 1.2. Let $z = (x, y)$ and $f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y))$, where we take

$$f_1(z) = \begin{cases} \frac{x}{|z|}\,[e + \log(1/|z|)]^{-1} & \text{for } |z| \le 1,\\ x/e & \text{for} |z| > 1,\end{cases}$$

$$f_2(z) = \begin{cases} \frac{y}{|z|}\,[e + \log(1/|z|)]^{-1} & \text{for } |z| \le 1,\\ y/e & \text{for} |z| > 1,\end{cases}$$

$$f_3(z) = \log\log(e + |\log|z||).$$

Let $F(z) = (f_1(z), f_2(z))$. This map is then a mapping of finite distortion, meaning that $F \in W^{1,1}_{\text{loc}}$ and that $|DF|^2 \le K'(z)\,J_F$ for some function $K'(z)$ which is finite almost everywhere. Moreover, it is true that $e^{\lambda K'(z)} \in L^1_{\text{loc}}(\mathbb{R}^2)$, for all $\lambda < 2$. In fact, we have that

$$|DF(z)| = \begin{cases} \frac{1}{|z|(e-\log|z|)} & \text{for } |z| \le 1,\\ 1/e & \text{for } |z| > 1,\end{cases} \qquad J_F = \begin{cases} \frac{1}{|z|^2\,(e-\log|z|)^3} & \text{for } |z| \le 1,\\ 1/e^3 & \text{for } |z| > 1,\end{cases}$$

and

$$K'(z) = \begin{cases} (e - \log|z|) & \text{for } |z| \le 1,\\ e & \text{for } |z| > 1.\end{cases}$$

Moreover,

$$|Df_3(z)| \le \frac{1}{|z|(e-\log|z|)\log(e-\log|z|)}, \quad \text{for } |z| \le 1,$$
$$|Df_3(z)| \le \frac{1}{|z|(e+\log|z|)\log(e+\log|z|)}, \quad \text{for } |z| > 1.$$

It is also easy to see that $\star f^*\omega = J_F$, for $\|\omega\| = 1$, and that (f, ω) satisfies condition (D). Putting everything together, we obtain

$$|Df|^2 \le (|DF| + |Df_3|)^2 \le \Big(\sqrt{K'(z)J_F} + \frac{\sqrt{K'(z)\,J_F}}{\log(e+|\log|z||)}\Big)^2$$
$$= \Big(\frac{1}{\log(e+|\log|z||)} + 1\Big)^2 K'(z)\,J_F.$$

So that (1.1) is true with

$$K(z) = \Big(\frac{1}{\log(e+|\log|z||)} + 1\Big)^2 K'(z).$$

Notice that for any $\varepsilon > 0$, there is an $r(\varepsilon) = r > 0$ such that $\frac{1}{\log(e+|\log|z||)} < \varepsilon$, for $|z| \le r$. Also, notice that $K(z)$ is bounded in $\mathbb{R}^2 \setminus B(0,r)$. Hence,

$$\int_{\mathbb{R}^2} \exp(\lambda K(z))\,dz = \int_{\mathbb{R}^2\setminus B(0,r)} \exp(\lambda K(z))\,dz + \int_{B(0,r)} \exp(\lambda K(z))\,dz,$$

and the first integral is finite for all $\lambda > 0$, while

$$\int_{B(0,r)} \exp(\lambda K(z))\,dz \le \int_{B(0,r)} \exp(\lambda(1+\varepsilon)^2 K'(z))\,dz,$$

which is finite for all $\lambda < 2/(1+\varepsilon)^2$. Since $\varepsilon > 0$ can be arbitrary, we have $e^{\lambda K(z)} \in L^1_{\mathrm{loc}}$, for any given $\lambda < 2$. However, the map f does not have a continuous representative since its third coordinate f_3 does not (f_3 has a logarithmic singularity at 0). ∎

For the construction in Theorem 1.3, we will need a special case of Lemma 5.1 in [13].

Lemma 6.1. *There exists a function $u \in W^{1,n}_{\mathrm{loc}}(B(0,1))$ so that u is non-negative, radial, continuous outside the origin, tends to ∞ when $x \to 0$, and satisfies*

$$\int_{B(0,1)} \Phi(|\nabla u(x)|)\,dx < \infty, \tag{6.1}$$

where $\Phi(t) = t^n \log^{n-1}(e+t)\log^{n-1}(\log(e+t))$ and

$$|\nabla u(x)| \le \frac{1}{|x|(1-\log|x|)\log|\log|x||}, \quad \text{when } |x| \text{ is small enough.} \tag{6.2}$$

Proof. Let $\phi(t) = t^{-n}\log^{-n}(e+t)\log^{-n}(\log(e+t))$. We define the real functions

$$h_k(t) := \inf\{s > 0 : \phi(2s) \le (2^k t)^n\}, \quad t \in (0,\infty).$$

Notice that

$$h_k(t) \le \frac{1}{t(1-\log t)\log|\log t|}, \tag{6.3}$$

for all $k \ge 1$ and all t small enough. Indeed, it is enough to show that

$$\phi\Big(\frac{2}{t(1-\log t)\log|\log t|}\Big) \le (2^k t)^n,$$

which is equivalent to

$$\frac{(1-\log t)\log|\log t|}{\log\big(e+\frac{2}{t(1-\log t)\log|\log t|}\big)\log\big(\log\big(e+\frac{2}{t(1-\log t)\log|\log t|}\big)\big)} \le 2^{k-1}.$$

Notice that the left-hand side goes to 1 as $t \to 0$, so the above inequality is true for all $k \ge 1$ and t small enough. Next we wish to show that $\int_0^1 h_k(t)dt = \infty$. To that end, notice that we can find a σ_k such that $\phi(2\sigma_k) < 2^{kn}$, which implies that

$$\{(t,s) : s > \sigma_k, 0 < t < 2^{-k}\phi^{1/n}(2s)\} \subset \{(t,s) : 0 < t < 1, 0 < s \le h_k(t)\}.$$

Hence, by Fubini's theorem, we have that

$$\begin{aligned}\int_0^1 h_k(t)\,dt &= \int_{\{(t,s):0<t<1,\,0<s\le h_k(t)\}} dt\,ds \ge \int_{\{(t,s):s>\sigma_k,\,0<t<2^{-k}\phi^{1/n}(2s)\}} dt\,ds\\ &= 2^{-k}\int_{\sigma_k}^{\infty}\phi^{1/n}(2s)\,ds.\end{aligned}$$

It is not hard to see that the last integral is ∞. Hence, we can define a decreasing sequence $\{a_k\}$ of positive real number such that $a_1 = 1$ and

$$\int_{a_{k+1}}^{a_k} h_k(x)\,dx = 1, \quad \text{for all } k \in \mathbb{N}.$$

The above implies that $h_k(a_{k+1}) \ge 1$ and thus $\phi(2) \ge 2^{nk}a_{k+1}^n$. Hence $a_k \to 0$. We set

$$u(x) = k + \int_{|x|}^{a_k} h_k, \quad \text{when } a_{k+1} \le |x| \le a_k,$$

which is continuous outside the origin, radial, non-negative and $\lim_{x\to 0} u(x) = \infty$. Moreover, (6.3) implies

$$|\nabla u(x)| = |h_k(|x|)| \le \frac{1}{|x|(1-\log|x|)\log|\log|x||},$$

which proves (6.2). Finally, since $\phi(h_k(t)) \le (2^k t)^n$, we have that

$$h_k(t)^n \log^{n-1}(e + h_k(t)) \log^{n-1}(\log(e + h_k(t))) \le (2^k t)^{1-n} h_k(t),$$

and thus

$$\begin{aligned}\int_{B(0,1)} \Phi(|\nabla u(x)|)\, dx &\le \sum_k \int_{\{a_{k+1}\le |x| \le a_k\}} (2^k t)^{1-n} h_k(|x|)\, dx \\ &= \sum_k 2\pi \int_{a_{k+1}}^{a_k} r^{n-1} (2^k r)^{1-n} h_k(r)\, dx = 2\pi \sum_k 2^{k(1-n)} < \infty.\end{aligned}$$

Hence (6.1) is proven, and since $t^n \le \Phi(t)$ for all $t > 0$, we have that

$$|\nabla u(x)|^n \le \Phi(|\nabla \phi(x)|),$$

which implies that $u \in W^{1,n}_{\mathrm{loc}}(B(0,1))$. ■

Proof of Theorem 1.3. We define $f = (f_1, f_2, f_3, f_4)$, where $F(z) = (f_1(z), f_2(z))$, with $z = (x, y)$, will be a mapping of finite distortion. The function $G = (f_3, f_4)$ will be constructed by adapting the construction in Theorem 5.2 of [13].

We start by constructing F first.

Construction of F. The function F will be obtained as the limit of a sequence of finite distortion mappings. First we set $z = (x, y)$ and

$$g(z) = \begin{cases} \frac{z}{|z|}[1 + \log(1/|z|)]^{-1} & \text{for } |z| \le 1, \\ z & \text{for } |z| > 1. \end{cases}$$

Similarly as in the proof of Theorem 1.2, it is easy to calculate that

$$|Dg(z)| = \begin{cases} \frac{1}{|z|(1-\log|z|)} & \text{for } |z| \le 1, \\ 1 & \text{for } |z| > 1, \end{cases} \quad J_g = \begin{cases} \frac{1}{|z|^2(1-\log|z|)^3} & \text{for } |z| \le 1, \\ 1 & \text{for } |z| > 1, \end{cases}$$

and

$$K_g(z) = \begin{cases} (1 - \log|z|) & \text{for } |z| \le 1, \\ 1 & \text{for } |z| > 1. \end{cases}$$

Take four disjoint balls $B_{i,0}(c_{i,0}, R_0)$, $i = 1, 2, 3, 4$, with centres $c_{i,0}$ in the segment $[-1, 1] \times \{0\}$ and radii to be determined later. Let also $A_{i,0}(c_{i,0}, R_0, r_0)$ denote the annuli of the same centres, outer radius R_0 and inner radius r_0, which is also to be defined later. Define the finite distortion map

$$f_1(z) = \begin{cases} R_0(1 - \log(R_0))\, g(z - c_{i,0}) & \text{for } |z| \in A_{i,0}(c_{i,0}, R_0, r_0), \\ \frac{R_0(1-\log R_0)}{r_0(1-\log r_0)}\, z & \text{for } |z| \in B(c_{i,0}, r_0), \\ z & \text{otherwise.} \end{cases}$$

Inductively, now, inside each of the balls $B_{i,n-1}(c_{i,n-1}, r_{n-1})$, $i = 1, \dots, 4^n$, take four disjoint balls with centres on the segment $[-1, 1] \times \{0\}$. Thus we obtain the collection of balls $B_{i,n}(c_{i,n}, R_n)$, $i = 1, \dots 4^{n+1}$. We define the finite distortion map

$$f_{n+1}(z) = \begin{cases} R_n(1 - \log(R_n))\, g(z - c_{i,n}) & \text{for } |z| \in A_{i,n}(c_{i,n}, R_n, r_n), \\ \frac{R_n(1-\log R_n)}{r_n(1-\log r_n)}\, z & \text{for } |z| \in B(c_{i,n}, r_n), \\ z & \text{otherwise.} \end{cases}.$$

Let

$$F_n(z) = f_1 \circ \cdots \circ f_n,$$

which will be a sequence of finite distortion mappings. It is easy to see that F_n converges in the $W^{1,2}$-norm to a finite distortion function $F \in W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, for suitable choice of R_n. Indeed, a computation shows that for F to be in $W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, it is enough to have $\sum_{n=0}^{\infty} a_n < \infty$, where

$$a_n = \prod_{j=0}^{n-1} \frac{R_j^2(1 - \log R_j)^2}{r_j^2(1 - \log r_j)^2}\, R_n^2(1 - \log R_n)^2.$$

If we choose R_n small enough, we can make $a_n < 1/2^n$ so that $F \in W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. We want F to have exponentially integrable distortion. It is easy to see that for $\lambda < 2$ we have

$$\begin{aligned} \int_{\mathbb{R}^2} \exp(\lambda K_F(z))\, dz &= \sum_{n=0}^{\infty} 4^{n+1} \int_{A_{i,n}(c_{i,n}, R_n, r_n)} \exp(\lambda K_{f_{n+1}}(z))\, dz \\ &= \sum_{n=0}^{\infty} 4^{n+1} \int_0^{2\pi} \int_{r_n}^{R_n} e^{\lambda} e^{-\lambda \log r}\, r\, dr\, d\theta \\ &= \sum_{n=0}^{\infty} \frac{2\pi\, 4^{n+1}\, e^{\lambda}}{-\lambda + 2} (R_n^{-\lambda+2} - r_n^{-\lambda+2}) \le \sum_{n=0}^{\infty} \frac{2\pi\, 4^{n+1}\, e^{\lambda}}{-\lambda + 2} R_n^{-\lambda+2}. \end{aligned}$$

Thus, if we take

$$R_n \le \left(\frac{-\lambda + 2}{2\pi\, e^{\lambda}\, 8^n} \right)^{1/(-\lambda+2)},$$

then

$$\sum_{n=0}^{\infty} \frac{2\pi\, 4^{n+1}\, e^{\lambda}}{-\lambda + 2} R_n^{-\lambda+2} < \infty,$$

and F will be exponentially integrable for any $\lambda < 2$. Finally, it is easy to check that for $z \in A_{i,n}(c_{i,n}, R_n, r_n)$, we have that

$$J_F(z) = a_n \frac{1}{|z - c_{i,n}|^2(1 - \log|z - c_{i,n}|)^3} \quad \text{and} \quad K_F(z) = 1 - \log|z - c_{i,n}|.$$

Notice that so far we have no restriction placed on r_n and R_n other than R_n being small enough and $r_n < R_n$.

The function G that we are going to construct will be such that $G \in W^{1,2}_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$, and for any fixed $\varepsilon > 0$ and suitable sequences r_n and $R'_n < R_n$, it will satisfy

$$|DG(z)| \le \sqrt{a_n}\, \frac{\varepsilon}{|z - c_{i,n}|(1 - \log|z - c_{i,n}|)} = \varepsilon\sqrt{K_F(z)\,J_F(z)},$$

for all $z \in A_{i,n}(c_{i,n}, R'_n, r_n)$, and $|DG(z)| = 0$ otherwise. Moreover, we will have that $G([-1,1] \times \{0\}) = [-1,1]^2$. Assuming that we have constructed this function, we will then have that the map $f = (F, G) \in W^{1,2}_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$ will satisfy the distortion inequality

$$|Df(z)|^2 \le (|DF(z)| + |DG(z)|)^2 \le (1+\varepsilon)^2 K_F(z)\,J_F(z) = K_f(z) \star f^*\omega,$$

almost everywhere. Since K_F is exponentially integrable, so is K_f. Since ε can be arbitrarily small, K_f can be arbitrarily close to λ-exponentially integrable. Moreover, it is easy to see that $F([-1,1] \times \{0\}) = [-1,1] \times \{0\}$. Hence, $f([-1,1] \times \{0\})$ will be a surface with non zero Hausdorff 2-measure, which means f does not have Lusin's (N) property. All that remains is to construct G.

Construction of G. We are going to construct recursively a sequence of functions G_n in $W^{1,2}_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$ which will converge uniformly to the desired function G. We start by partitioning the square $[-1,1]^2$ in dyadic squares. We set

$$S_0 = \{[-1,0] \times [0,1], [-1,0] \times [-1,0], [0,1] \times [0,1], [0,1] \times [-1,0]\},$$

and more generally, S_n will be a collection of 4^{n+1} squares obtained by partitioning each of the squares of S_{n-1} into four. For $i = 1, \dots, 4^{n+1}$, we will denote by $K_{i,n}$ the elements of S_n, and by $w_{i,n}$, their centres. Also we let $w_{0,-1} = 0$.

We proceed with the definition of $\{G_n\}$. First we define $G_0 = 0$. Assuming that we have defined $G_0, \dots, G_n$, we construct G_{n+1} as follows. We use the sequences R_n and r_n from the construction of F. First, we can choose R_0 small enough so that the two dimensional version of the function u from Lemma 6.1 satisfies (6.2) when $|z| \le R_0$. Moreover, since $\int_{B(0,1)} \Phi(|\nabla u(z)|)\, dz < \infty$, we can choose R_n small enough so that

$$\int_{B(0,R_n)} \Phi(|\nabla u(z)|)\, dz < \frac{1}{2^{n+1}\, 4^{n+1}}, \tag{6.4}$$

for all $n = 0, 1, \dots$ Finally, for any fixed $\varepsilon > 0$, we first temporarily choose $r_n = R'_n$ and then take $R'_n < R_n$ small enough so that

$$\frac{1}{\log|\log|z - c_{i,n}||} \le \varepsilon\sqrt{a_n}, \tag{6.5}$$

when $|z - c_{i,n}| \le R'_n$. This is possible since a_n is constant on the annulus $A_{i,n}(c_{i,n}, R_n, r_n)$. Next, since $u(z) \to \infty$ as $z \to 0$, we can in fact reselect $r_n < R'_n$ so that

$$u(r_n, r_n) - u(R'_n, R'_n) = d_n,$$

where d_n denotes the distance between the centres of a square in S_n and one of its subdividing squares in S_{n+1}, $K_{j,n+1} \subset K_{i,n}$. Remember there is no restriction on r_n, and decreasing it makes a_n larger, so (6.5) is not affected by that. Define G_{n+1} to be G_n outside of the balls $B_{i,n}(c_{i,n}, R'_n)$. When $r_n \leq |z - c_{i,n}| \leq R'_n$, define

$$G_{n+1}(z) = w_{j,n-1} + (u(z - c_{i,n}) - u(R'_n, R'_n)) \frac{w_{i,n} - w_{j,n-1}}{|w_{i,n} - w_{j,n-1}|},$$

where j is the unique index such that $K_{i,n} \subset K_{j,n-1}$. Finally, when $|z - c_{i,n}| \leq r_n$, we define

$$G_{n+1}(z) = w_{i,n}.$$

Next, we need to show that the sequence G_n converges uniformly. To prove that, notice that G_n and G_{n+1} differ only inside the balls $B_{i,n}(c_{i,n}, R'_n)$, where G_n is constant while G_{n+1} maps the ball to a segment of length d_n. Hence, for $m > n$,

$$\|G_m - G_n\|_\infty \leq \sum_{k=n}^{m-1} \|G_{k+1} - G_k\|_\infty \leq \sum_{k=n}^{m-1} d_k.$$

Since the lengths d_k shrink at a geometric rate, we have that G_n is a uniformly Cauchy sequence, and we are done. Thus the limit map G will be continuous, and it is easy to see that (6.4) implies

$$\sum_{i=1}^{4^{n+1}} \int_{B_{i,n}(c_{i,n}, R'_n)} \Phi(|DG_{n+1}(z)|)\, dz < \frac{1}{2^{n+1}},$$

which in turn implies that $G \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$. Moreover, from (6.2) and (6.5) we obtain that

$$|DG(z)| = |\nabla u(z)| \leq \frac{\varepsilon \sqrt{a_n}}{|z - c_{i,n}|(1 - \log|z - c_{i,n}|)},$$

when $r_n \leq |z - c_{i,n}| \leq R'_n$. Finally, it is easy to see that for any $w_{i,n}$, there exists a sequence of points $z_k \in A_{i,k}(c_{i,k}, R'_k, r_k)$, converging to some point in the segment $[-1, 1] \times \{0\}$, such that $G(z_k) = w_{i,n}$. Since the set of all centres $\{w_{i,n}\}_{i,n \in \mathbb{N}}$ is dense in the square $[-1, 1]^2$ and since G is continuous, we have that

$$G([-1, 1] \times \{0\}) = [-1, 1]^2,$$

as we wanted. ∎

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