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A property of ideals of jets of functions vanishing on a set

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Abstract. For a set $E \subset \mathbb{R}^n$ that contains the origin, we consider $I^m(E)$ – the set of all m^{th} degree Taylor approximations (at the origin) of C^m functions on $\mathbb{R}^n$ that vanish on E. This set is a proper ideal in $\mathcal{P}^m(\mathbb{R}^n)$ – the ring of all m^{th} degree Taylor approximations of C^m functions on $\mathbb{R}^n$. Which ideals in $\mathcal{P}^m(\mathbb{R}^n)$ arise as $I^m(E)$ for some E? In this paper we introduce the notion of a *closed* ideal in $\mathcal{P}^m(\mathbb{R}^n)$, and prove that any ideal of the form $I^m(E)$ is closed. We do not know whether in general any closed proper ideal is of the form $I^m(E)$ for some E, however we prove in a subsequent paper that all closed proper ideals in $\mathcal{P}^m(\mathbb{R}^n)$ arise as $I^m(E)$ when $m + n \leq 5$.

Contents

1. Introduction

In this article we work in the ring $\mathcal{P}_0^m(\mathbb{R}^n)$ of m-jets (at the origin) of C^m functions on $\mathbb{R}^n$ that vanish at the origin. We write $J^m(F)$ to denote the m-jet at $\vec{0}$ of a function $F \in C^m(\mathbb{R}^n)$, i.e., its m^{th} degree Taylor approximation about the origin. Given a subset $E \subset \mathbb{R}^n$ that contains the origin, we define an ideal $I^m(E)$ in $\mathcal{P}_0^m(\mathbb{R}^n)$ by

$$I^m(E) := \{J^m(F) \mid F \in C^m(\mathbb{R}^n) \text{ and } F = 0 \text{ on } E\}.$$

Our main goal is to understand which ideals I in $\mathcal{P}_0^m(\mathbb{R}^n)$ arise as $I^m(E)$ for some E.

We introduce the notion that a given ideal I *implies* a particular jet p. The set of all the jets implied by a given ideal I is an ideal containing I, which we call the *closure* of I. We say that I is *closed* if it is equal to its closure.

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Main results in a nutshell

We will prove that $I^m(E)$ is always closed (for any m and any E); see Theorem 2.18. We know of no example of an ideal I that is closed but does not arise as $I^m(E)$ for some E. Indeed, in the rings $\mathcal{P}_0^1(\mathbb{R}^n)$, $\mathcal{P}_0^m(\mathbb{R}^1)$, $\mathcal{P}_0^2(\mathbb{R}^2)$, $\mathcal{P}_0^3(\mathbb{R}^2)$ and $\mathcal{P}_0^2(\mathbb{R}^3)$ we will classify in our subsequent paper [10] all possible closed ideals up to a natural equivalence relation, and for each ideal I on these lists we will exhibit a set E such that $I = I^m(E)$. The natural equivalence relation arises from the fact that every C^m-diffeomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ that fixes the origin induces an automorphism of $\mathcal{P}_0^m(\mathbb{R}^n)$ defined by $p \mapsto J^m(p \circ \phi)$. We say that two ideals I and I' in $\mathcal{P}_0^m(\mathbb{R}^n)$ are *equivalent* if I' is the image of I under such an automorphism. Since this equivalence relation preserves the property of an ideal being closed, in [10] we in particular prove that each closed ideal in the rings $\mathcal{P}_0^1(\mathbb{R}^n)$, $\mathcal{P}_0^m(\mathbb{R}^1)$, $\mathcal{P}_0^2(\mathbb{R}^2)$, $\mathcal{P}_0^3(\mathbb{R}^2)$ and $\mathcal{P}_0^2(\mathbb{R}^3)$ arises as $I^m(E)$ for some closed E that contains the origin.

Nullstellensatz-type point of view

Our problem is loosely analogous to the setting of the Hilbert Nullstellensatz, where one asks which polynomial ideals arise as $\mathcal{I}(V)$, the set of polynomials vanishing on an algebraic subset V of $\mathbb{C}^n$. In that setting, we may say that an ideal I *implies* a polynomial p if p belongs to the radical of I. Recall that the Nullstellensatz tells us that an ideal $I \lhd \mathbb{C}[x_1, \ldots, x_n]$ arises as $\mathcal{I}(V)$ for some algebraic subset V if and only if it is equal to its radical. When either $m = 1$, $n = 1$ or $m + n \leq 5$, we will prove in [10] that an ideal $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ arises as $I^m(E)$ for some closed subset E that contains the origin if and only if it is equal to its closure. We have not yet studied enough examples to know whether to believe that for arbitrary m and n every closed ideal in $\mathcal{P}_0^m(\mathbb{R}^n)$ arises as $I^m(E)$ for some E. Perhaps, in addition to being closed, $I^m(E)$ has further properties of which we are so far unaware.

We stress that unlike the setting of Hilbert Nullstellensatz, we are not aware of a natural map that assigns to each ideal $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ a closed subset E that contains the origin such that $I = I^m(E)$. We present numerous concrete examples in [10], and in particular it will become evident that it does not make sense to look at the zero locus of a given ideal – we will see that in $\mathcal{P}_0^2(\mathbb{R}^2)$ (where (x, y) is a standard coordinate system), $I^2(\{x = 0\})$ is the principal ideal generated by x, while $I^2(\{x(x^2 - y^3) = 0\} \cap \{x \geq 0\})$ is the principal ideal generated by x^2. The latter example also shows that $I^m(E)$ may be not radical (and in particular not real-radical).

A conjecture by N. Zobin

Recall that semi-algebraic sets and maps are those that can be described by finitely many polynomial equations and inequalities and boolean operations (for detailed exposition on semi-algebraic geometry see, for instance, [2]). In [10], we will see in particular that each closed ideal in the rings $\mathcal{P}_0^1(\mathbb{R}^n)$, $\mathcal{P}_0^m(\mathbb{R}^1)$, $\mathcal{P}_0^2(\mathbb{R}^2)$, $\mathcal{P}_0^3(\mathbb{R}^2)$ and $\mathcal{P}_0^2(\mathbb{R}^3)$ arises as $I^m(E)$ for a *semi-algebraic* set E that contains the origin. A conjecture of N. Zobin (see Problem 5 in [5]) asserts that every ideal $I^m(E)$ in $\mathcal{P}_0^m(\mathbb{R}^n)$ already arises as the ideal $I^m(V)$ for a semi-algebraic set V that contains the origin. Thus, we prove that the conjecture is true for the cases in which either $m = 1$, $n = 1$ or $m + n \leq 5$. We do not know whether the conjecture holds for general m and n, though it seems plausible to us.

Related problems

All the questions this paper addresses may be asked with C^m functions replaced by different classes of functions. For instance, fix $m, n \in \mathbb{N}$ and $m \leq k \in \mathbb{N} \cup \{\infty\}$. Given a subset $E \subset \mathbb{R}^n$ that contains the origin, define an ideal $I_k^m(E)$ in $\mathcal{P}_0^m(\mathbb{R}^n)$ by

$$I_k^m(E) := \{J^m(F) \mid F \in C^k(\mathbb{R}^n) \text{ and } F = 0 \text{ on } E\}.$$

What can we say about the ideal $I_k^m(E)$? Which ideals in $\mathcal{P}_0^m(\mathbb{R}^n)$ arise as $I_k^m(E)$ for some E? Does every ideal of the form $I_k^m(E)$ already arises as the ideal $I_k^m(V)$ for a semi-algebraic set V? We would like to study these variants, however in this paper we only deal with the case $m = k$.

The relation to extension problems

The ideal $I^m(E)$ arose naturally in connection with *Whitney's extension problem*: given a function $f : E \to \mathbb{R}$ (with $\vec{0} \in E \subset \mathbb{R}^n$), we want to decide whether f extends from E to a C^m function F on the whole $\mathbb{R}^n$. If such an extension F exists, then $I^m(E)$ expresses the ambiguity in the jet of F at the origin. Moreover, similarly to the definition of $I^m(E)$, we can define the ideal $I_{\vec{x}}^m(E)$ of m-jets of C^m functions that vanish identically on E about an arbitrary point $\vec{x} \in E$. In this terminology, the ideal $I^m(E)$ is $I_{\vec{0}}^m(E)$. Analyzing the ideals $I_{\vec{x}}^m(E)$ is often a key step for solving such Whitney type extrapolation problems (see, for instance, [4]).

We stress here that for a given set E, one can in principle compute $I^m(E)$; see [4]. In this paper we are interested is the converse question: given I, can we find an E?

Intuition on implied jets

We prepare to motivate and explain the notion of an ideal I implying a jet p without going into technicalities. We sacrifice accuracy to meet this goal. Let us set up some ad-hoc notation and definitions for this purpose (these notation and definitions will be only used in this introduction; the main text will include a rigorous exposition and in particular a detailed construction of all the examples below).

We identify m-jets with m^{th} degree Taylor polynomials. Thus, $\mathcal{P}_0^m(\mathbb{R}^n)$ consists of all at most m^{th} degree polynomials on $\mathbb{R}^n$ with a zero constant term, $J^m(F)$ is the m^{th} degree Taylor polynomial of F at the origin, and the multiplication in $\mathcal{P}_0^m(\mathbb{R}^n)$ is given by $PQ := J^m(PQ)$.

Let $\Gamma \subset \mathbb{R}^n$ be an open set whose closure contains the origin. A *C^m-flat* function on Γ is a function $F \in C^m(\Gamma)$ such that $\partial^\alpha F(\vec{x}) = o(|\vec{x}|^{m-|\alpha|})$ as $\vec{x}$ tends to the origin in Γ, for any $|\alpha| \leq m$. A *C^m-tame* function on Γ is a function $S \in C^m(\Gamma)$ such that $\partial^\alpha S(\vec{x}) = O(|\vec{x}|^{-|\alpha|})$ as $\vec{x}$ tends to the origin in Γ, for any $|\alpha| \leq m$.

Often, Γ will be a cone with vertex at the origin. If $\Gamma = \mathbb{R}^n \setminus \{\vec{0}\}$, then the C^m-flat functions are simply the C^m functions F with $J^m(F) = 0$, restricted to $\mathbb{R}^n \setminus \{\vec{0}\}$. The following simple observation will be important.

Remark 1.1. Let F be C^m-flat on Γ, and let S be C^m-tame on Γ. Then, $S \cdot F$ is C^m-flat on Γ.

We start by presenting two examples to show that the ideal $I(E)$ unsurprisingly restricts the geometry of E.

Example 1.2. In $\mathcal{P}_0^2(\mathbb{R}^2)$, suppose $x^2 + y^2 \in I^2(E)$. Then, by definition there exists $F \in C^2(\mathbb{R}^2)$ with jet $x^2 + y^2$ that vanishes on E. We have $F(x, y) - (x^2 + y^2) = o(x^2 + y^2)$ as $(x, y) \to (0, 0)$, hence $F(x, y)$ is nonzero on a punctured neighborhood of the origin. Consequently, that punctured neighborhood contains no points of E. This in turn easily implies that $I^2(E)$ consists of all jets that vanish at the origin, i.e., $I(E) = \mathcal{P}_0^2(\mathbb{R}^2)$. We thus proved that

$$x^2 + y^2 \in I^2(E) \implies (0,0) \text{ is an isolated point of } E \text{ and } I^2(E) = \mathcal{P}_0^2(\mathbb{R}^2).$$

Example 1.3. In $\mathcal{P}_0^2(\mathbb{R}^2)$, suppose $xy \in I^2(E)$. Let Γ be an open sector with vertex at the origin, whose closure in $\mathbb{R}^2 \setminus \{\vec{0}\}$ meets neither the x-axis nor the y-axis. Then, by the argument of Example 1.2, $E \cap \Gamma$ cannot contain points arbitrarily close to the origin. This tells us that the only possible "tangent directions to E at the origin" are the x-axis and the y-axis.

We formalize the lesson of Examples 1.2 and 1.3 in the following definition:

Let I be an ideal in $\mathcal{P}_0^m(\mathbb{R}^n)$ and let $\omega \in S^{n-1}$ be a unit vector (a "direction"). We say that ω is *forbidden* for I if there exist jets $Q_1, \dots, Q_L \in I$, a cone

$$\Gamma = \{\vec{x} \in \mathbb{R}^n : 0 < |\vec{x}| < r, |\vec{x}/|\vec{x}| - \omega| < \delta\},$$

(for some $r, \delta > 0$) and a constant $c > 0$, such that $\left|Q_1(\vec{x})\right| + \dots + \left|Q_L(\vec{x})\right| > c|\vec{x}|^m$ for any $\vec{x} \in \Gamma$. If ω is not forbidden, we say that ω is *allowed*. We write $\mathrm{Forb}(I)$ and $\mathrm{Allow}(I)$, respectively, to denote the sets of forbidden and allowed directions in the unit sphere.

For an ideal $I = I^m(E)$, the argument of Example 1.2 easily shows that only allowed directions may be tangent to E at the origin. Thus, as promised, I *constrains the geometry of any set E for which we hope that $I = I^m(E)$.*

Note that Example 1.2 also tells us that if an ideal is of the form $I^m(E)$ for some E in the ring $\mathcal{P}_0^2(\mathbb{R}^2)$, and it contains the jet $x^2 + y^2$, then it contains many more jets. Let us see more examples that illustrate how the existence of some jets in $I^m(E)$ can force the existence of another jet in $I^m(E)$.

Example 1.4. In $\mathcal{P}_0^3(\mathbb{R}^2)$, suppose $x(x^2 + y^2) \in I^3(E)$. Then, $x^3, x^2y, xy^2 \in I^3(E)$ as well.

Indeed, since $x(x^2 + y^2) \in I^3(E)$, there exists $F \in C^3(\mathbb{R}^2)$, with jet $x(x^2 + y^2)$, such that $F = 0$ on E. Equivalently, there exists a C^3-flat function F_1 on $\mathbb{R}^2 \setminus \{\vec{0}\}$ such that

$$x(x^2 + y^2) + F_1(x, y) = 0 \quad \text{on } E. \tag{1.1}$$

Now set $S_1 = \frac{x^2}{x^2+y^2}$, and note that S_1 is C^3-tame on $\mathbb{R}^2 \setminus \{\vec{0}\}$. Multiplying (1.1) by S_1, we have

$$x^3 + S_1 \cdot F_1 = 0 \quad \text{on } E. \tag{1.2}$$

Moreover, $S_1 \cdot F_1$ is C^3-flat on $\mathbb{R}^2 \setminus \{\vec{0}\}$, by Remark 1.1. Therefore, (1.2) shows that $x^3 \in I^3(E)$. Similar arguments using $S_2 = \frac{xy}{x^2+y^2}$ and $S_3 = \frac{y^2}{x^2+y^2}$ in place of S_1 show that also $x^2 \cdot y$ and xy^2 belong to $I^3(E)$.

The argument of Example 1.2 can be modified easily to show that any *proper* closed ideal in $\mathcal{P}_0^m(\mathbb{R}^n)$ has a non-empty set of allowed directions. Example 1.4 shows that *not* every ideal in $\mathcal{P}_0^m(\mathbb{R}^n)$ with allowed directions is closed: the principal ideal generated by $x(x^2+y^2)$ in $\mathcal{P}_0^3(\mathbb{R}^2)$ is not closed, though $\{(0, \pm 1)\}$ are allowed directions of this ideal.

Our final example in this introduction brings into play the set of allowed directions.

Example 1.5. In $\mathcal{P}_0^2(\mathbb{R}^3)$, suppose $I = I^2(E)$ contains x^2 and $y^2 - xz$. Then, I contains xy.

To see this, first note that outside any conic neighborhood of the z-axis, we have $|x^2| + |y^2 - xz| > c(x^2 + y^2 + z^2)$ for some constant $c > 0$ that depends on the conic neighborhood. Therefore, $\text{Allow}(I)$ is contained in $\{(0, 0, \pm 1)\}$ and so E is tangent to the z-axis, i.e.,

$$E \subset \{(x, y, z) : |(x, y)| \leq g(|z|) \cdot |z|\}, \tag{1.3}$$

for some function $g\colon [0, \infty) \to \mathbb{R}$ that is strictly positive away from 0 and that satisfies

$$g(t) \to 0 \quad \text{as } t \to 0. \tag{1.4}$$

Let us see how, by possibly replacing g by a function that goes to zero more slowly, we may assume without loss of generality that $g \in C^2(0, \infty)$ and

$$\left(\frac{d}{dt}\right)^k g(t) = O(t^{-k} g(t)) \quad \text{for all } k \in \{0, 1, 2\}. \tag{1.5}$$

Note that replacing E with its intersection with a small ball about the origin does not affect $I^2(E)$, so we may first replace g by $\min\{g(t), 1\}$. We thus may assume without loss of generality that $0 < g(t) \leq 1$ for all $t \in (0, \infty)$. Second, we set $\tilde{g}(t) := \sup\{\frac{2t}{t+s} g(s) : s \in (0, \infty)\}$, for all $t \in (0, \infty)$. Note that this sup is finite since $g(s) \leq 1$ for all $s \in (0, \infty)$. By taking $s = t$ in the above sup, we see that

$$\tilde{g}(t) \geq g(t) \quad \text{for all } t \in (0, \infty). \tag{1.6}$$

Fix $\varepsilon > 0$. By (1.4), there exists $\delta(\varepsilon) > 0$ such that $g(s) < \varepsilon$ for all $0 < s < \delta(\varepsilon)$. So we have $(\frac{2t}{t+s}) g(s) < 2\varepsilon$ for all $0 < s < \delta(\varepsilon)$. For $s \geq \delta(\varepsilon)$ we have $(\frac{2t}{t+s}) g(s) \leq \frac{2t}{t+s} \leq \frac{2t}{\delta(\varepsilon)} < 2\varepsilon$ for $t < \tilde{\delta}(\varepsilon) := \varepsilon \cdot \delta(\varepsilon)$. These two facts together tell us that $\tilde{g}(t) \leq 2\varepsilon$ for all $t < \tilde{\delta}(\varepsilon)$, and so

$$\tilde{g}(t) \to 0 \quad \text{as } t \to 0. \tag{1.7}$$

One readily sees that if $t_1/t_2 \in [1/2, 2]$, then $\frac{1}{4} \cdot \frac{2t_1}{t_1+s} \leq \frac{2t_2}{t_2+s} \leq 4 \cdot \frac{2t_1}{t_1+s}$ for all $s \in (0, \infty)$, and so

$$\frac{1}{4} \cdot \tilde{g}(t_1) \leq \tilde{g}(t_2) \leq 4 \cdot \tilde{g}(t_1) \quad \text{if } \frac{t_1}{t_2} \in \left[\frac{1}{2}, 2\right]. \tag{1.8}$$

Let $\varphi \in C^\infty((0,\infty))$ be a non-negative function, supported in $[1/2,2]$, and not the zero function, and set

$$g^*(t) := \int_0^\infty \varphi\Big(\frac{t}{s}\Big) \cdot \tilde{g}(s)\,\frac{ds}{s}\cdot$$

Then $g^* \in C^\infty((0,\infty))$. Since $t/s \in [1/2,2]$ in the support of the integrand, we have from (1.8) that

$$c \cdot \tilde{g}(t) \le g^*(t) \le C \cdot \tilde{g}(t) \quad \text{for all } t \in (0,\infty), \tag{1.9}$$

where c and C above are positive constants depending only on φ. Moreover, for any $t \in (0,\infty)$ and $k \in \{0,1,2\}$ we have

$$\begin{aligned} \Big|\Big(\frac{d}{dt}\Big)^k g^*(t)\Big| &= \Big|\int_0^\infty s^{-k}\varphi^{(k)}\Big(\frac{t}{s}\Big)\cdot \tilde{g}(s)\,\frac{ds}{s}\Big| \\ &\le \int_0^\infty s^{-k}\Big|\varphi^{(k)}\Big(\frac{t}{s}\Big)\Big|\,\frac{ds}{s}\cdot 4\tilde{g}(t) = C' t^{-k}\tilde{g}(t), \end{aligned} \tag{1.10}$$

where C' is a positive constant depending only on φ, and again we exploit estimate (1.8) in the support of the integrand. Now, (1.9) and (1.10) imply that

$$\Big(\frac{d}{dt}\Big)^k g^*(t) = O(t^{-k} g^*(t)) \quad \text{for all } k \in \{0,1,2\}. \tag{1.11}$$

From (1.6), (1.7) and (1.9) we see that

$$g^*(t) \to 0 \quad \text{as } t \to 0 \tag{1.12}$$

and

$$C'' g^*(t) \ge g(t) \quad \text{for all } t \in (0,\infty), \tag{1.13}$$

where C'' is a positive constant depending only on φ. Finally, thanks to (1.11), (1.12) and (1.13), we may replace $g^*(t)$ by $g^+(t) := C'' g^*(t)$ and conclude that (1.3), (1.4) and (1.5) all hold.

We thus established that we may assume without loss of generality that (1.3), (1.4) and (1.5) all hold. We proceed by replacing E with its intersection with a small ball about the origin (this does not affect $I^2(E)$), and so we may also assume that $|(x,y)| \le |z|$ for any $(x,y,z) \in E$.

Let $\theta(t)$ be a C^2 (cutoff) function on $[0,\infty)$, supported in $[0,2]$ and equal to 1 on $[0,1]$. We define functions on $\mathbb{R}^3 \setminus \{\vec{0}\}$ by

$$F_1(x,y,z) = \frac{y^3}{z}\cdot \theta\Big(\frac{|(x,y)|}{g(z)|z|}\Big) \quad \text{and} \quad S(x,y,z) = \frac{y}{z}\cdot\theta\Big(\frac{|(x,y)|}{|z|}\Big).$$

One can readily verify that (1.4) and (1.5) imply that F_1 is C^2-flat on $\mathbb{R}^3 \setminus \{\vec{0}\}$ and S is C^2-tame on $\mathbb{R}^3 \setminus \{\vec{0}\}$. Recall that $y^2 - xz \in I^2(E)$. Thus, there exists a C^2-flat function on $\mathbb{R}^3 \setminus \{\vec{0}\}$, which we denote by F_2, such that

$$y^2 - xz + F_2(x,y,z) = 0 \quad \text{on } E \setminus \{\vec{0}\}. \tag{1.14}$$

Multiplying (1.14) by $-y/z$, we find that

$$xy - \frac{y^3}{z} - \frac{y}{z} F_2(x,y,z) = 0 \quad \text{on } E \setminus \{\vec{0}\}. \tag{1.15}$$

By our assumption that $|(x,y)| \le |z|$ for any $(x,y,z) \in E$ and (1.3), we have that

$$\theta\Big(\frac{|(x,y)|}{g(z)|z|}\Big) = \theta\Big(\frac{|(x,y)|}{|z|}\Big) = 1 \quad \text{on } E,$$

and so

$$F_1 = \frac{y^3}{z} \quad \text{and} \quad S = \frac{y}{z} \quad \text{on } E.$$

Consequently, (1.15) implies that

$$xy - [F_1 + SF_2] = 0 \quad \text{on } E \setminus \{\vec{0}\}. \tag{1.16}$$

Recall that F_1 and F_2 are C^2-flat on $\mathbb{R}^3 \setminus \{\vec{0}\}$, and that S is C^2-tame on $\mathbb{R}^3 \setminus \{\vec{0}\}$. Thanks to Remark 1.1, we see that $F_2 + SF_2$ is C^2-tame on $\mathbb{R}^3 \setminus \{\vec{0}\}$, and so (1.16) tells us that $xy \in I^2(E)$. This completes our analysis of Example 1.5.

We stress that we got crucial help from our knowledge of $\mathrm{Allow}(I)$, which told us where E lives and permitted us to make use of the cutoff functions $\theta(\frac{|(x,y)|}{g(z)|z|})$ and $\theta(\frac{|(x,y)|}{|z|})$, thus avoiding the singularities of the functions y^3/z and y/z.

We are now ready to define the notion of an implied jet. Unfortunately, it is not so simple.

Let I be an ideal in $\mathcal{P}_0^m(\mathbb{R}^n)$, and let $\Omega \subset S^{n-1}$ be the set of allowed directions for I. We say that I *implies* a given jet p if there exist a constant $A > 0$ and jets $Q_1, \dots, Q_L \in I$ for which the following holds:

Given $\varepsilon > 0$, there exist $\delta, r > 0$ such that for any $0 < \rho \le r$, there exist functions $F, S_1, \dots, S_L$ satisfying

(1.17) $|\partial^\alpha F(\vec{x})| \le \varepsilon \rho^{m-|\alpha|}$ for all $\rho/4 < |\vec{x}| < 4\rho$ and all $|\alpha| \le m$;

(1.18) $|\partial^\alpha S_l(\vec{x})| \le A\rho^{-|\alpha|}$ for all $\rho/4 < |\vec{x}| < 4\rho$, all $|\alpha| \le m$ and all $1 \le l \le L$;

(1.19) if $\Omega \neq \emptyset$, then $p(\vec{x}) = F(\vec{x}) + S_1(\vec{x})Q_1(\vec{x}) + S_2(\vec{x})Q_2(\vec{x}) + \cdots + S_L(\vec{x})Q_L(\vec{x})$ for all $\rho/2 < |\vec{x}| < 2\rho$ such that $\mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta$.

The point of this definition is as follows: suppose that I implies p, and suppose that $I = I^m(E)$ for some E. The functions $S_1, \dots, S_L, F$ in (1.17)–(1.19) are allowed to depend on the length scale ρ, but by patching together the $S_1, \dots, S_L, F$ arising from all small length scales, we can find functions $S_1^\#, \dots S_L^\#$ that are C^m-tame on $\mathbb{R}^n \setminus \{\vec{0}\}$ and a function $F^\#$ that is C^m-flat on $\mathbb{R}^n \setminus \{\vec{0}\}$ such that, in some punctured neighborhood of the origin, we have

$$p(\vec{x}) = F^\#(\vec{x}) + S_1^\#(\vec{x})\,Q_1(\vec{x}) + S_2^\#(\vec{x})\,Q_2(\vec{x}) + \cdots + S_L^\#(\vec{x})\,Q_L(\vec{x}) \quad \text{on } E.$$

An easy argument using Remark 1.1 then shows that $p \in I = I^m(E)$. Thus, if $I = I^m(E)$ then I contains any jet p implied by I, i.e., $I^m(E)$ is always closed (Theorem 2.18 below).

Let us see how the above definition applies to Example 1.5 by verifying that I implies $p = xy$. Recall that $\Omega = \text{Allow}(I)$ is contained in $\{(0,0,\pm 1)\}$. We take Q_1 to be $y^2 - xz$, and we take (for instance) $A = 10^9$. Given $\varepsilon > 0$, we take (for instance) $\delta = r = \varepsilon/10^9$.

Now, given $0 < \rho \le r$, we must produce functions F and S_1 satisfying (1.17), (1.18), and (1.19) above. We will take

$$F = \frac{y^3}{z} \cdot \theta\Big(\frac{|(x,y)|}{\delta \cdot \rho}\Big) \quad \text{and} \quad S_1 = -\frac{y}{z} \cdot \theta\Big(\frac{|(x,y)|}{\rho}\Big),$$

where θ is a C^3-smooth one variable cutoff function defined on $[0,\infty]$, equal to 1 on $[0,4]$, supported on $[0,8]$, and such that θ and its derivatives up to order 3 have absolute value at most 100.

The above F and S_1 satisfy (1.17),(1.18), and (1.19). In fact, for $\rho/4 < |\vec{x}| < 4\rho$, (1.18) is easily verified, and (1.17) holds since $|\partial^\alpha F(\vec{x})| = O(\delta^{3-|\alpha|} \cdot \rho^{2-|\alpha|})$ for all $|\alpha| \le 2$. To check (1.19), we note that $\theta(\frac{|(x,y)|}{\delta\rho}) = \theta(\frac{|(x,y)|}{\rho}) = 1$ for $\rho/2 < |(x,y,z)| < 2\rho$ such that $\text{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta$. Consequently, (1.19) reduces to the equation $xy = (-y/z)(y^2 - xz) + y^3/z$. Thus, as promised, I implies xy in Example 1.5. The cutoff function $\theta(\frac{|(x,y)|}{\delta \cdot \rho})$ here plays the role of the cutoff $\theta(\frac{|(x,y)|}{g(|z|)\cdot|z|})$ in Example 1.5.

Calculating implied jets

So we have defined the notion that an ideal I implies a jet p. How can we show in practice that a *given* ideal I implies a *given* jet p? In Section 3 we develop tools that answer this question, by introducing the notions of *strong implication* and *strong directional implication*. These notions use the definition of *negligible functions*, that is quite technical, but relatively easy to work with. Then, we show that if a given jet p is strongly implied by a given ideal I in any allowed direction of I, then the jet p is implied by I (Corollary 3.18). We also provide an easy algorithm to calculate the set of allowed directions of a given ideal (Corollary 3.3) – if we are given a basis of an ideal I, this algorithm always produces a set that contains $\text{Allow}(I)$, and sometimes it calculates $\text{Allow}(I)$ exactly. We routinely use these tools in [10] when we exhibit many examples of closed ideals.

Calculating the closure of an ideal

Recall once more that the closure of I consists of all the jets implied by I, and that I is closed if and only if it is equal to its closure. How can we calculate the closure of a given ideal I, i.e., how can we calculate a basis for the space of all implied jets of a given ideal? We answer this question in Section 4. In this section we show how to realize the conditions defining implied jets as conditions about the existence of sections of some semi-algebraic bundles (bundles in the sense of Fefferman–Luli; see [7]). Consequently, we also explain how, in principle, we can calculate the closure of a given ideal using well known results regarding the spaces of sections of such bundles. This algorithm is based on results from [8, 9].

We stress that the tools in Section 3 provide useful information in many particular cases in which we want to show that a given ideal implies a given jet, but (as far as we know) are not guaranteed to work in general. On the other hand, the algorithm in Section 4 is guaranteed to compute the closure of any given ideal of m-jets, but is unfortunately far too labor-intensive to use in practice.

2. Closed ideals and a necessary condition

As said, we start a rigorous construction of our theory below, and do not rely on anything defined in the introduction. Let us start by fixing notation.

2.1. Notation

We work in $\mathbb{R}^n$ with Euclidean metric, and most of the notation we use is standard.

Functions. For an open subset $U \subset \mathbb{R}^n$ and $m \in \mathbb{N} \cup \{0\}$, we denote by $C^m(U)$ the space of real valued m-times continuously differentiable functions. We use multi-index notation for derivatives: for a multi-index $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ we set $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\alpha! := \alpha_1!\alpha_2! \cdots \alpha_n!$. For $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we set $\vec{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. If $|\alpha| \le m$ and $f \in C^m(U)$, we write

$$f^{(\alpha)} := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

when $|\alpha| \neq 0$, and $f^{(\alpha)} := f$ when $|\alpha| = 0$. We sometimes write $\partial^\alpha f$ or $\partial_\alpha f$ instead of $f^{(\alpha)}$. When it is clear from the context, we will sometimes write f_{xy} or $\partial^2_{xy} f$ when $\alpha = (1, 1, 0, 0, \dots, 0)$ and $(x, y, z, \dots)$ is a coordinate system on $\mathbb{R}^n$, and other such similar conventional notation.

Asymptotic behaviors. For $f, g \in C^m(U)$ and $\vec{x}_0 \in \bar{U}$, we write

$$f(\vec{x}) = o(g(\vec{x})) \quad \text{as } \vec{x} \to \vec{x}_0 \quad \text{if} \quad \frac{f(\vec{x})}{g(\vec{x})} \to 0 \quad \text{as } \vec{x} \to \vec{x}_0.$$

If $\vec{x}_0$ is not specified, then $\vec{x}_0 = \vec{0}$, unless otherwise is clear from the context. We write

$$f(\vec{x}) = O(g(\vec{x})) \quad \text{as } \vec{x} \to \vec{x}_0$$

if

$$\left|\frac{f(\vec{x})}{g(\vec{x})}\right| \text{ is bounded in some punctured neighborhood of } \vec{x}_0.$$

Again, if $\vec{x}_0$ is not specified, then $\vec{x}_0 = \vec{0}$, unless otherwise is clear from the context.

Balls, cones, annuli and other geometric objects. For $r > 0$, we set $B(r) := \{\vec{x} \in \mathbb{R}^n : |\vec{x}| < r\}$ and set $B^\times(r) := B(r) \setminus \{\vec{0}\}$, where n should be clear from the context. For two sets $X, Y \subset \mathbb{R}^n$, we set

$$\begin{aligned} &\operatorname{dist}(X, Y) := \inf\{|\vec{x} - \vec{y}| : \vec{x} \in X, \vec{y} \in Y\} \quad \text{if } X \neq \emptyset \text{ and } Y \neq \emptyset, \\ &\operatorname{dist}(X, Y) = +\infty \quad \text{otherwise,} \end{aligned}$$

and if one of them is a singleton, we write $\operatorname{dist}(\vec{x}, Y)$ instead of $\operatorname{dist}(\{\vec{x}\}, Y)$, and similarly if the other is a singleton. We denote as usual $\mathbb{R}^n \supset S^{n-1} := \{|\vec{x}| = 1\}$, and refer to points

points in S^{n-1} as *directions*. For $\Omega \subset S^{n-1}$ and $\delta > 0$, we denote (the dome around Ω of opening δ)

$$D(\Omega, \delta) := \{\omega \in S^{n-1} : \operatorname{dist}(\omega, \Omega) < \delta\}.$$

Note that in particular if $\Omega = \emptyset$, then $D(\Omega, \delta) = \emptyset$. Given (radius) $r > 0$, we set

$$\Gamma(\Omega, \delta, r) := \bigcup_{r' \in (0,r)} r' D(\Omega, \delta) = \{\vec{x} \in \mathbb{R}^n : 0 < |\vec{x}| < r, \operatorname{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta\}.$$

Note that in particular if $\Omega = \emptyset$, then $\Gamma(\Omega, \delta, r) = \emptyset$.

For a singleton $\omega \in S^{n-1}$, we write $D(\omega, \delta)$ and $\Gamma(\omega, \delta, r)$ instead of $D(\{\omega\}, \delta)$ and $\Gamma(\{\omega\}, \delta, r)$, respectively. We call a set of the form $\Gamma(\omega, \delta, r)$ (respectively, $\Gamma(\Omega, \delta, r)$) a cone in the direction ω (respectively, around the set of directions $\Omega \subset S^{n-1}$), or a conic neighborhood of ω (respectively, of Ω). Note that some cones are non-convex. Also note that for any $\Omega \subset S^{n-1}$, $\delta > 0$ and $r > 0$, we have that $\Gamma(\Omega, \delta, \varepsilon)$ is open in $\mathbb{R}^n$, and $D(\Omega, \delta)$ is open in S^{n-1} (in the restricted topology). Finally (when n is clear from the context), for $\mathbb{R} \ni K \geq 1$ and $\mathbb{R} \ni r > 0$ we define the annulus

$$\operatorname{Ann}_K(r) := \{\vec{x} \in \mathbb{R}^n : r/K < |\vec{x}| < Kr\}.$$

2.2. Basic definitions

Definition 2.1 (Jet spaces). Let $m, n \in \mathbb{N}$. For a real valued function $f \in C^m(\mathbb{R}^n)$, we define the *m^{th} degree jet of f at the origin* (or m-jet), denoted by $J^m(f)$, to be the m^{th} degree Taylor polynomial (that has degree at most m) of f at the origin. The jet $J^m(f)$ is an element of the space of (at most) m^{th} degree (Taylor) polynomials in n variables, which we call the m^{th} degree jet space at the origin, and denote by $\mathcal{P}^m(\mathbb{R}^n)$. This is naturally a (commutative unital) ring, with multiplication given by $PQ := J^m(PQ)$, however it is not a integral domain, e.g., in $\mathcal{P}^m(\mathbb{R}^1)$ we always have $x^m \cdot x = 0$. We stress that by PQ (and $P \cdot Q$) we always mean jet product, and not the produce in the ring of polynomials, unless we say otherwise.

Note that $\mathcal{P}^m(\mathbb{R}^n)$ is a finite dimensional vector space, and that for any $f, g \in C^m(\mathbb{R}^n)$, we have $J^m(f \cdot g) = J^m(f) \cdot J^m(g)$. In order to avoid confusion, we will use the following notation.

Notation 2.2. Let $\mathcal{F} \subset \mathcal{P}^m(\mathbb{R}^n)$ be a family of jets. We denote by $\langle \mathcal{F} \rangle_m$ the ideal in $\mathcal{P}^m(\mathbb{R}^n)$ generated by $\mathcal{F}$. For instance, in the single variable case, we have $\langle x \rangle_m = \operatorname{span}_{\mathbb{R}}\{x, x^2, \ldots, x^m\}$.

Definition 2.3 (Order of vanishing of functions). Let $m, n \in \mathbb{N}$, and let $f \in C^m(\mathbb{R}^n)$. *The order of vanishing* (*at the origin*) of f is said to be

- the minimal $1 \leq m' \leq m$ such that $J^{m'}(f) \neq 0$, if such m' exists and $f(\vec{0}) = 0$;
- more than m, if $J^m(f) = 0$;
- 0, if $f(\vec{0}) \neq 0$.

The order of vanishing of a jet $p \in \mathcal{P}^m(\mathbb{R}^n)$ is the order of vanishing of the polynomial p.

Note that every C^m-diffeomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ that fixes the origin induces an automorphism of $\mathcal{P}^m(\mathbb{R}^n)$ defined by $p \mapsto J^m(p \circ \phi)$. This automorphism does *not* preserve in general the degree (as a polynomial) of a jet, however it always preserves the order of vanishing.

Definition 2.4. We set

$$\mathcal{P}_0^m(\mathbb{R}^n) := \{p \in \mathcal{P}^m(\mathbb{R}^n) \mid \text{the order of vanishing of } p \text{ is not } 0\}.$$

Note that $\mathcal{P}_0^m(\mathbb{R}^n)$ is a (commutative) ring, but it is not unital. It is also a finite dimensional vector space. Equivalently, one can define $\mathcal{P}_0^m(\mathbb{R}^n)$ as the unique maximal ideal in $\mathcal{P}^m(\mathbb{R}^n)$.

Definition 2.5. Let $E \subset \mathbb{R}^n$ be a closed set containing the origin. We define

$$I^m(E) := \{p \in \mathcal{P}^m(\mathbb{R}^n) \mid \exists f \in C^m(\mathbb{R}^n),\, f|_E = 0,\, J^m(f) = p\}.$$

2.2.1. Simple observations. Let $E, E' \subset \mathbb{R}^n$ be two closed sets, both containing the origin.

(i) $I^m(E)$ only depends on the local behavior of E around the origin: if there exists $r > 0$ such that $E \cap B(r) = E' \cap B(r)$, then $I^m(E) = I^m(E')$.

(ii) $I^m(E)$ is an ideal in $\mathcal{P}^m(\mathbb{R}^n)$: indeed, let $p \in I^m(E)$ and $p' \in \mathcal{P}^m(\mathbb{R}^n)$. Then, there exists $f \in C^m(\mathbb{R}^n)$ such that $J^m(f) = p$ and $f|_E = 0$. Note that $f \cdot p'$ is a C^m function on $\mathbb{R}^n$ that vanishes on E, and $J^m(f \cdot p') = p \cdot p'$. We conclude that $p \cdot p' \in I^m(E)$, i.e., $I^m(E) \lhd \mathcal{P}^m(\mathbb{R}^n)$.

(iii) $I^m(E) \subset \mathcal{P}_0^m(\mathbb{R}^n)$, in fact it is an ideal: $I^m(E) \lhd \mathcal{P}_0^m(\mathbb{R}^n)$.

(iv) $I^m(E \cup E') \subset I^m(E) \cap I^m(E')$.

(v) $I^m(E) \subset I^m(E \cap E')$.

2.3. Allowed and forbidden directions of an ideal and tangent directions of a set

Definition 2.6 (Allowed and forbidden directions of jets and ideals).

• Let $p_1, p_2, \dots, p_L \in \mathcal{P}_0^m(\mathbb{R}^n)$ be jets, and let $\omega \in S^{n-1}$ be a direction. We say that ω is a *forbidden direction* of $p_1, \dots, p_L$ if the following holds:

(2.1) There exist $c, \delta, r > 0$ such that

$$|p_1(\vec{x})| + |p_2(\vec{x})| + \cdots + |p_L(\vec{x})| > c \cdot |\vec{x}|^m \quad \text{for all } \vec{x} \in \Gamma(\omega, \delta, r).$$

Otherwise, we say that ω is an *allowed direction* of $p_1, \dots, p_L$.

• Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal. A direction $\omega \in S^{n-1}$ is said to be a *forbidden direction* of I if there exist $p_1, p_2, \dots, p_L \in I$ such that ω is a forbidden direction of $p_1, p_2, \dots, p_L$. Otherwise, we say that ω is an *allowed direction* of I.

We denote the sets of forbidden and allowed directions of I by $\mathrm{Forb}(I)$ and $\mathrm{Allow}(I)$, respectively. Note that the set $\mathrm{Allow}(I) \subset S^{n-1}$ is always closed.

Definition 2.7 (Tangent, forbidden and allowed directions of a set). Let $E \subset \mathbb{R}^n$ be a closed subset containing the origin and let $\omega \in S^{n-1}$. We say that E *is tangent to the direction* ω if

$$E \cap \Gamma(\omega, \delta, r) \neq \emptyset \quad \text{for all } \delta > 0 \text{ and all } r > 0.$$

We denote by $T(E) \subset S^{n-1}$ the set of all directions to which E is tangent. Finally, for a fixed $m \in \mathbb{N}$, we say that ω *is a forbidden (respectively, allowed) direction of* E, if ω is a forbidden (respectively, allowed) direction of $I^m(E)$. We denote the sets of forbidden and allowed directions of E by $\mathrm{Forb}(E)$ and $\mathrm{Allow}(E)$, respectively (here m should be implicitly understood from the context).

Lemma 2.8. *Let* $\vec{0} \in E \subset \mathbb{R}^n$ *be a closed subset and* $m \in \mathbb{N}$. *Then,* $T(E) \subset \mathrm{Allow}(E)$.

Proof. Let $\omega \in T(E)$. Assume towards a contradiction that $\omega \in \mathrm{Forb}(E) = \mathrm{Forb}(I^m(E))$. By Definition 2.6 and (2.1), there exist $p_1, \ldots, p_L \in I^m(E)$ such that the following holds:

(2.2) There exist $c, \delta, r > 0$ such that

$$|p_1(\vec{x})| + |p_2(\vec{x})| + \cdots + |p_L(\vec{x})| > c \cdot |\vec{x}|^m \quad \text{for all } \vec{x} \in \Gamma(\omega, \delta, r).$$

By Definition 2.5, for $1 \leq l \leq L$ there exists $f_l \in C^m(\mathbb{R}^n)$ such that $f_l(\vec{x}) = p_l(\vec{x}) + r_l(\vec{x})$, where

$$r_l(\vec{x}) = o(|\vec{x}|^m) \tag{2.3}$$

and $f_l|_E = 0$. As $\omega \in T(E)$, there exists a sequence of points

$$\{\vec{x}_i\}_{i=1}^{\infty} \subset \Gamma(\omega, \delta, r) \cap E \text{ converging to the origin.} \tag{2.4}$$

On any of these points we have

$$0 = |f_1(\vec{x}_i)| + \cdots + |f_L(\vec{x}_i)| = |p_1(\vec{x}_i) + r_1(\vec{x}_i)| + \cdots + |p_L(\vec{x}_i) + r_L(\vec{x}_i)|, \tag{2.5}$$

and dividing (2.5) by $|x_i|^m \neq 0$, we get

$$\begin{aligned} 0 &= \frac{|p_1(\vec{x}_i) + r_1(\vec{x}_i)| + \cdots + |p_L(\vec{x}_i) + r_L(\vec{x}_i)|}{|\vec{x}_i|^m} \\ &\geq \frac{|p_1(\vec{x}_i)| + \cdots + |p_L(\vec{x}_i)|}{|\vec{x}_i|^m} - \frac{|r_1(\vec{x}_i)| + \cdots + |r_L(\vec{x}_i)|}{|\vec{x}_i|^m}. \end{aligned} \tag{2.6}$$

Now (2.6) is a contradiction, as (2.2) and (2.4) imply that $(|p_1(\vec{x}_i)| + \cdots + |p_L(\vec{x}_i)|)/|\vec{x}_i|^m$ is bounded from below by c, while (2.3) and (2.4) imply that, as $\vec{x}_i$ approaches the origin, $(|r_1(\vec{x}_i)| + \cdots + |r_L(\vec{x}_i)|)/|\vec{x}_i|^m$ goes to zero. ∎

Lemma 2.9. *Let* $\vec{0} \in E \subset \mathbb{R}^n$ *be a closed subset, let* $m \in \mathbb{N}$ *and denote* $\Omega = \mathrm{Allow}(I^m(E))$. *Then, given* $\delta > 0$, *there exists* $\bar{r} > 0$ *such that*

$$\begin{aligned} &E \cap B^{\times}(\bar{r}) \subset \{\vec{x} \in B^{\times}(\bar{r}) : \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta\} \quad \textit{if } \Omega \neq \emptyset, \textit{ and} \\ &E \cap B^{\times}(\bar{r}) = \emptyset \quad \textit{if } \Omega = \emptyset. \end{aligned} \tag{2.7}$$

Proof. Assume $\Omega \neq \emptyset$ and that the lemma does not hold. Then there exist $\delta > 0$ and a sequence of points $\{\vec{x}_i\}_{i=1}^{\infty} \subset E \setminus \{\vec{0}\}$ converging to the origin such that $\mathrm{dist}(\vec{x}_i/|\vec{x}_i|, \Omega) \geq \delta$ for any $i \in \mathbb{N}$. Since S^{n-1} is compact, by possibly diluting the sequence $\{\vec{x}_i\}_{i=1}^{\infty}$ we may assume that $\vec{x}_i/|\vec{x}_i| \to \omega^* \in S^{n-1}$ as $i \to \infty$. In particular, $\mathrm{dist}(\omega^*, \Omega) \geq \delta$ and so $\omega^* \in \mathrm{Forb}(I^m(E))$. By Lemma 2.8, $\omega^* \notin T(E)$, so there exist $\delta^*, r^* > 0$ such that

$$E \cap \Gamma(\omega^*, \delta^*, r^*) = \emptyset.$$

Since $\vec{x}_i$ is converging to the origin, we may assume $0 < |\vec{x}_i| < r^*$ for large enough i, and since $\vec{x}_i/|\vec{x}_i| \to \omega^*$, we may assume $|\vec{x}_i/|\vec{x}_i| - \omega^*| < \delta^*$ for large enough i. Combining these two we have that, for large enough i,

$$\vec{x}_i \in \Gamma(\omega^*, \delta^*, r^*).$$

But $\vec{x}_i \in E$, so we have

$$E \cap \Gamma(\omega^*, \delta^*, r^*) \neq \emptyset,$$

which is a contradiction. The proof of the case $\Omega = \emptyset$ is almost identical but slightly simpler (essentially repeat the proof but omit the condition "$\mathrm{dist}(\vec{x}_i/|\vec{x}_i|, \Omega) \geq \delta$, for any $i \in \mathbb{N}$" in the very beginning), thus we leave it to the reader to verify the details. ∎

2.4. Closed ideals – a fundamental property of ideals of the form $I^m(E)$

Recall that (when n is clear from the context) for $K \geq 1$ and $r > 0$, we set

$$\mathrm{Ann}_K(r) := \{\vec{x} \in \mathbb{R}^n : r/K < |\vec{x}| < Kr\}.$$

Definition 2.10 (Implied jets). Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal, denote $\Omega := \mathrm{Allow}(I)$, and let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial. We say that I *implies* p (or that p is implied by I) if there exist a constant $A > 0$ and $Q_1, Q_2, \ldots, Q_L \in I$ such that the following holds:

For any $\varepsilon > 0$, there exist $\delta, r > 0$ such that for any $0 < \rho \leq r$, there exist functions $F, S_1, S_2, \ldots, S_L \in C^m(\mathrm{Ann}_4(\rho))$ satisfying

$$|\partial^\alpha F(\vec{x})| \leq \varepsilon \rho^{m-|\alpha|} \quad \text{for all } \vec{x} \in \mathrm{Ann}_4(\rho) \text{ and all } |\alpha| \leq m; \tag{2.8}$$

$$|\partial^\alpha S_l(\vec{x})| \leq A \rho^{-|\alpha|} \quad \text{for all } \vec{x} \in \mathrm{Ann}_4(\rho), \text{ all } |\alpha| \leq m \text{ and all } 1 \leq l \leq L; \tag{2.9}$$

$$p(\vec{x}) = F(\vec{x}) + S_1(\vec{x})Q_1(\vec{x}) + S_2(\vec{x})Q_2(\vec{x}) + \cdots + S_L(\vec{x})Q_L(\vec{x}) \quad \text{for all } \vec{x} \in \mathrm{Ann}_2(\rho) \text{ such that } \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta. \tag{2.10}$$

Remark 2.11. Definition 2.10 asserts the existence of functions $S_1, \ldots, S_L$ for some $L \in \mathbb{N}$. One can define "1-implied jets" by only allowing $L = 1$, so for instance (2.10) becomes "$p(\vec{x}) = F(\vec{x}) + S_1(\vec{x})Q_1(\vec{x})$ for all $\vec{x} \in \mathrm{Ann}_2(\rho)$ such that $\mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta$". Clearly, if a jet is 1-implied by an ideal, it is also implied by the same ideal. We do not know whether the converse also holds; in particular, we are not able to construct an ideal I and a jet $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ such that p is implied by I, but p is not 1-implied by I. Similar remarks can be made regarding Definition 3.11 and Definition 3.13.

Definition 2.12 (The closure of an ideal). Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal. We define its *implication closure* (or simply *closure*) $\mathrm{cl}(I)$ by

$$\mathrm{cl}(I) := \{p \in \mathcal{P}_0^m(\mathbb{R}^n) \mid p \text{ is implied by } I\}.$$

We say that I is *closed* if $I = \mathrm{cl}(I)$.

Remark 2.13. Let $p \in I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$. Then, we can take $A = 1, l = 1, Q_1 = p, S_1 = 1$ and $F = 0$, and for any $\varepsilon > 0$ take any $\delta, r > 0$, to prove $p \in \mathrm{cl}(I)$. Thus, we always have $I \subset \mathrm{cl}(I)$. Moreover, the closure of any ideal is an ideal as well.

Lemma 2.14. *Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal, denote $\Omega := \mathrm{Allow}(I)$, and let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial that is implied by I. Then, given $K \geq 4$, there exist a constant $\tilde{A} > 0$ and $Q_1, Q_2, \ldots, Q_L \in I$ such that the following holds:*

For any $\varepsilon > 0$, there exist $\delta, r > 0$ such that for any $0 < \rho \leq r$, there exist functions $F, S_1, S_2, \ldots, S_L \in C^m(\mathbb{R}^n)$ satisfying

$$|\partial^\alpha F(\vec{x})| \leq \varepsilon \rho^{m-|\alpha|} \text{ for all } \vec{x} \in \mathbb{R}^n \text{ and all } |\alpha| \leq m; \tag{2.11}$$

$$|\partial^\alpha S_l(\vec{x})| \leq \tilde{A} \rho^{-|\alpha|} \text{ for all } \vec{x} \in \mathbb{R}^n \text{, all } |\alpha| \leq m \text{ and all } 1 \leq l \leq L; \tag{2.12}$$

$$p(\vec{x}) = F(\vec{x}) + S_1(\vec{x})\, Q_1(\vec{x}) + S_2(\vec{x})\, Q_2(\vec{x}) + \cdots + S_L(\vec{x})\, Q_L(\vec{x}) \quad \text{for all } \vec{x} \in \mathrm{Ann}_K(\rho) \text{ such that } \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta. \tag{2.13}$$

Proof. Let $A > 0$ and $Q_1, Q_2, \ldots, Q_L \in I$ be as in Definition 2.10, and let $K \geq 4$. Fix $\varepsilon > 0$ and set $\tilde{\varepsilon} := \frac{\varepsilon}{2C''}$, where C'' is a constant depending only on m, n, K and A, to be determined below. Corresponding to $\tilde{\varepsilon}$, let $\delta, r > 0$ be as in Definition 2.10, and set $\tilde{r} = \frac{r}{1000K}$. Suppose $0 < \rho \leq \frac{r}{1000K}$. We introduce a partition of unity $\{\theta_\nu\}_{\nu=1,\ldots,\nu_{\max}}$, satisfying:

- for all $\nu = 1, \ldots, \nu_{\max}$, $\theta_\nu \in C^\infty(\mathbb{R}^n)$ and $\mathrm{supp}\,\theta_\nu \subset \mathrm{Ann}_2(\rho_\nu)$ for some $\frac{\rho}{1000K} \leq \rho_\nu \leq 1000K\rho$.
- $\nu_{\max} \leq C$;
- $1 = \sum_{\nu=1}^{\nu_{\max}} \theta_\nu(\vec{x})$ for any $\vec{x} \in \mathrm{Ann}_{40K}(\rho)$;
- $|\partial^\alpha \theta_\nu(\vec{x})| \leq C' \rho_\nu^{-|\alpha|} \leq (1000K)^m C' \rho^{-|\alpha|}$ for any $\vec{x} \in \mathbb{R}^n$, $\nu = 1, \ldots, \nu_{\max}$ and $|\alpha| \leq m$,

where above C and C' are positive constants depending only on m, n, K and A. By Definition 2.10, for each $\nu = 1, \ldots, \nu_{\max}$ there exist functions $F^\nu, S_1^\nu, S_2^\nu, \ldots, S_L^\nu \in C^m(\mathrm{Ann}_4(\rho_\nu))$ satisfying

$$|\partial^\alpha F^\nu(\vec{x})| \leq \tilde{\varepsilon} \rho_\nu^{m-|\alpha|} \leq (1000K)^m \tilde{\varepsilon} \rho^{m-|\alpha|} \quad \text{for all } \vec{x} \in \mathrm{Ann}_4(\rho_\nu) \text{ and all } |\alpha| \leq m; \tag{2.14}$$

$$|\partial^\alpha S_l^\nu(\vec{x})| \leq A \rho_\nu^{-|\alpha|} \leq (1000K)^m A \rho^{-|\alpha|} \quad \text{for all } \vec{x} \in \mathrm{Ann}_4(\rho),\ |\alpha| \leq m \text{ and all } 1 \leq l \leq L; \tag{2.15}$$

$$p(\vec{x}) = F^\nu(\vec{x}) + S_1^\nu(\vec{x})\, Q_1(\vec{x}) + S_2^\nu(\vec{x})\, Q_2(\vec{x}) + \cdots + S_L^\nu(\vec{x})\, Q_L(\vec{x}) \quad \text{for all } \vec{x} \in \mathrm{Ann}_2(\rho_\nu) \text{ such that } \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta. \tag{2.16}$$

Define

$$F = \sum_{\nu=1}^{\nu_{\max}} \theta_\nu F^\nu \in C^m(\mathbb{R}^n),$$

and for $l = 1, \ldots, L$ define

$$S_l = \sum_{\nu=1}^{\nu_{\max}} \theta_\nu S_l^\nu \in C^m(\mathbb{R}^n).$$

Then, (2.14)–(2.16), together with the properties of $\{\theta_\nu\}_{\nu=1,\ldots,\nu_{\max}}$, imply that

(2.17) $|\partial^\alpha F(\vec{x})| \le C'' \tilde{\varepsilon} \rho^{m-|\alpha|} = \frac{\varepsilon}{2} \rho^{m-|\alpha|} < \varepsilon \rho^{m-|\alpha|}$ for all $\vec{x} \in \mathbb{R}^n$ and all $|\alpha| \le m$;

(2.18) $|\partial^\alpha S_l(\vec{x})| \le \tilde{A} \rho_\nu^{-|\alpha|}$ for all $\vec{x} \in \mathbb{R}^n$, $|\alpha| \le m$ and all $1 \le l \le L$;

(2.19)
$$\begin{aligned} p(\vec{x}) &= \sum_{\nu=1}^{\nu_{\max}} \theta_\nu(\vec{x}) p(\vec{x}) \\ &= \sum_{\nu=1}^{\nu_{\max}} \theta_\nu(\vec{x}) [F^\nu(\vec{x}) + S_1^\nu(\vec{x}) Q_1(\vec{x}) + S_2^\nu(\vec{x}) Q_2(\vec{x}) + \cdots + S_L^\nu(\vec{x}) Q_L(\vec{x})] \\ &= F(\vec{x}) + S_1(\vec{x}) Q_1(\vec{x}) + S_2(\vec{x}) Q_2(\vec{x}) + \cdots + S_L(\vec{x}) Q_L(\vec{x}) \\ &\qquad \text{for all } \vec{x} \in \mathrm{Ann}_K(\rho) \text{ such that } \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta, \end{aligned}$$

where above C'' and $\tilde{A}$ are positive constants depending only on m, n, K and A.

Finally, for a fixed $\varepsilon > 0$, we let $\delta, \tilde{r} > 0$ be as above. So for a given $0 < \rho \le \tilde{r}$ we have produced functions $F, S_1, S_2, \ldots, S_L \in C^m(\mathbb{R}^n)$ satisfying (2.17)–(2.19), which are equivalent to (2.11)–(2.13), and so the lemma holds. ∎

Lemma 2.15. *Definition 2.10 is invariant with respect to C^m coordinate changes around the origin, namely: let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal, let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial, and assume I implies p. Let $\phi: \mathbb{R}^n \to \mathbb{R}^n$ be a C^m-diffeomorphism that fixes the origin, recall it induces an automorphism of $\mathcal{P}^m(\mathbb{R}^n)$ defined by $p \mapsto J^m(p \circ \phi)$, and denote this automorphism by ϕ^*. Then, $\phi^*(I)$ implies $\phi^*(p)$. In particular, the property of an ideal being closed is invariant with respect to C^m coordinate changes around the origin.*

Proof. The image of any cone under any C^m coordinate change around the origin contains a cone. Thus, the notion of allowed directions of an ideal is invariant under C^m coordinate changes around the origin, up to a linear coordinate change (that arises from the Jacobian of the C^m coordinate change). The image of any annulus centered at the origin under any C^m coordinate change around the origin is contained in an annulus centered at the origin. These two facts together with Lemma 2.14 prove the lemma. We leave it to the reader to verify the details. ∎

Remark 2.16. Another invariant of a given ideal I with respect to C^m coordinate changes is $\dim \mathrm{span}_{\mathbb{R}} \mathrm{Allow}(I)$ – we will use this fact often in [10].

Corollary 2.17 (Sets with no allowed directions). *The only closed ideal $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ such that* $\mathrm{Allow}(I) = \emptyset$ *is* $I = \mathcal{P}_0^m(\mathbb{R}^n)$. *Moreover, if $E \subset \mathbb{R}^n$ is a closed subset containing the origin such that* $\mathrm{Allow}(I^m(E)) = \emptyset$, *then the origin is an isolated point of E and $I^m(E) = \mathcal{P}_0^m(\mathbb{R}^n)$.*

Proof. If I is closed and $\mathrm{Allow}(I) = \emptyset$, then any jet in $\mathcal{P}_0^m(\mathbb{R}^n)$ is implied by I as (2.10) vacuously holds (to satisfy (2.8) and (2.9), one may take $A = 1, l = 1, Q_1 = 0, S_1 = 0$ and $F = 0$). This proves that the only closed ideal $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ such that $\mathrm{Allow}(I) = \emptyset$ is $I = \mathcal{P}_0^m(\mathbb{R}^n)$. The "moreover" part follows from Lemma 2.9, the observation (i) in Subsection 2.2.1, and the fact that $I^m(\{\vec{0}\}) = \mathcal{P}_0^m(\mathbb{R}^n)$. ∎

Theorem 2.18. *Fix $m, n \in \mathbb{N}$ and let $E \subset \mathbb{R}^n$ be a closed subset containing the origin. Then, $I^m(E) \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ is a closed ideal.*

Proof. Denote $\Omega = \mathrm{Allow}(I^m(E))$. If $\Omega = \emptyset$, then the theorem follows from Corollary 2.17. So we assume $\Omega \neq \emptyset$. Let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be implied by $I^m(E)$, and let $A > 0$ and $Q_1, \dots, Q_L \in I^m(E)$ be such that the upshot of Definition 2.10 holds. For $k \in \mathbb{N}$, set $\varepsilon_k = 2^{-k}$ and let δ_k, r_k correspond to ε_k as in Definition 2.10. Given $0 < \rho \le r_k$, there exist functions $F, S_1, S_2, \dots, S_L \in C^m(\mathrm{Ann}_4(\rho))$ such that

$$|\partial^\alpha F(\vec{x})| \le \varepsilon_k \rho^{m-|\alpha|} \text{ for all } \vec{x} \in \mathrm{Ann}_4(\rho) \text{ and all } |\alpha| \le m; \tag{2.20}$$

$$|\partial^\alpha S_l(\vec{x})| \le A\rho^{-|\alpha|} \text{ for all } \vec{x} \in \mathrm{Ann}_4(\rho), \text{ all } |\alpha| \le m \text{ and all } 1 \le l \le L; \tag{2.21}$$

$$\begin{aligned} p(\vec{x}) = F(\vec{x}) + S_1(\vec{x})\,Q_1(\vec{x}) + S_2(\vec{x})\,Q_2(\vec{x}) + \dots + S_L(\vec{x})\,Q_L(\vec{x}) \\ \text{for all } \vec{x} \in \mathrm{Ann}_2(\rho) \text{ such that } \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta_k. \end{aligned} \tag{2.22}$$

By Lemma 2.9, there exists $\bar{r}_k > 0$ such that

$$E \cap B^\times(4\bar{r}_k) \subset \{\vec{x} \in B^\times(4\bar{r}_k) : \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta_k\}. \tag{2.23}$$

Replacing $\bar{r}_k$ by a smaller number preserves (2.23), so without loss of generality we may assume that, for any $k \in \mathbb{N}$, $\bar{r}_k$ is a negative integer power of 2, $\bar{r}_k < r_k$, and $\bar{r}_{k+1} \le 2^{-10}\bar{r}_k$. In particular, $\bar{r}_k \to 0$ as $k \to \infty$.

Set $\nu_{\min}$ to be the unique integer such that $\bar{r}_1 = 2^{-\nu_{\min}}$. For $\mathbb{N} \ni \nu \ge \nu_{\min}$, set $k(\nu)$ to be the unique integer such that $\bar{r}_{k(\nu)+1} < 2^{-\nu} \le \bar{r}_{k(\nu)}$. Then, for any $\nu \ge \nu_{\min}$ we in particular have $2^{-\nu} \le \bar{r}_{k(\nu)} < r_{k(\nu)}$, so setting $\rho := 2^{-\nu}$ in (2.20)–(2.22) above, we obtain functions $F^\nu, S_1^\nu, S_2^\nu, \dots, S_L^\nu \in C^m(\mathrm{Ann}_4(2^{-\nu}))$ such that

$$|\partial^\alpha F^\nu(\vec{x})| \le \varepsilon_{k(\nu)} (2^{-\nu})^{m-|\alpha|} \text{ for all } \vec{x} \in \mathrm{Ann}_4(2^{-\nu}) \text{ and all } |\alpha| \le m; \tag{2.24}$$

$$|\partial^\alpha S_l^\nu(\vec{x})| \le A(2^{-\nu})^{-|\alpha|} \text{ for all } \vec{x} \in \mathrm{Ann}_4(2^{-\nu}), \text{ all } |\alpha| \le m \text{ and all } 1 \le l \le L; \tag{2.25}$$

$$\begin{aligned} p(\vec{x}) = F^\nu(\vec{x}) + S_1^\nu(\vec{x})\,Q_1(\vec{x}) + S_2^\nu(\vec{x})\,Q_2(\vec{x}) + \dots + S_L^\nu(\vec{x})\,Q_L(\vec{x}) \\ \text{for all } \vec{x} \in \mathrm{Ann}_2(2^{-\nu}) \text{ such that } \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta_{k(\nu)}. \end{aligned} \tag{2.26}$$

Since $2 \cdot 2^{-\nu} < 4\bar{r}_{k(\nu)}$, we have that (2.23) and (2.26) imply that

$$\begin{aligned} p(\vec{x}) = F^\nu(\vec{x}) + S_1^\nu(\vec{x})Q_1(\vec{x}) + S_2^\nu(\vec{x})Q_2(\vec{x}) + \dots + S_L^\nu(\vec{x})Q_L(\vec{x}) \\ \text{for all } \vec{x} \in E \cap \mathrm{Ann}_2(2^{-\nu}). \end{aligned} \tag{2.27}$$

Note that $k(\nu) \to \infty$ as $\nu \to \infty$, and so

$$\varepsilon_{k(\nu)} = 2^{-k(\nu)} \to 0 \quad \text{as } \nu \to \infty. \tag{2.28}$$

Let $\{\theta_\nu \in C^m(\mathbb{R}^n)\}_{\nu \geq \nu_{\min}}$ be such that, for some constant $C_1 > 0$ (that depends only on m and n), we have

$$0 \leq \theta_\nu(\vec{x}) \leq 1 \text{ for all } \vec{x} \in \mathbb{R}^n \text{ and } \operatorname{supp}(\theta_\nu) \subset \operatorname{Ann}_2(2^{-\nu}) \text{ for all } \nu \geq \nu_{\min}; \tag{2.29}$$

$$|\partial^\alpha \theta_\nu(\vec{x})| \leq C_1 (2^{-\nu})^{-|\alpha|} \text{ for all } \vec{x} \in \mathbb{R}^n, \text{ all } |\alpha| \leq m \text{ and all } \nu \geq \nu_{\min}; \tag{2.30}$$

$$\sum_{\nu \geq \nu_{\min}} \theta_\nu(\vec{x}) = 1 \text{ for all } 0 < |\vec{x}| \leq 2^{-10} \cdot 2^{-\nu_{\min}}. \tag{2.31}$$

We now define $F, S_1, S_2, \ldots, S_L \in C^m(\mathbb{R}^n \setminus \{\vec{0}\})$ by

$$F(\vec{x}) := \sum_{\nu \geq \nu_{\min}} \theta_\nu(\vec{x}) F^\nu(\vec{x}) \quad \text{and} \quad S_l(\vec{x}) := \sum_{\nu \geq \nu_{\min}} \theta_\nu(\vec{x}) S_l^\nu(\vec{x}).$$

Fix an integer μ and suppose $2^{-(\mu+1)} < |\vec{x}| \leq 2^{-\mu}$. Then, $\vec{x} \in \operatorname{supp}(\theta_\nu) = \operatorname{Ann}_2(2^{-\nu})$ (recall (2.29)) only for ν satisfying $|\nu - \mu| \leq 2$, since $2^{-(\mu+1)} < |\vec{x}| < 2 \cdot 2^{-\nu}$ and $\frac{1}{2} 2^{-\nu} < |\vec{x}| < 2^{-\mu}$. Consequently, (2.24) and (2.30) yield that for some constants $C_2, C_3 > 0$ (that depend only on m and n), we have

$$|\partial^\alpha F(\vec{x})| \leq \sum_{|\nu-\mu| \leq 2; \nu \geq \nu_0} C_2\, \varepsilon_{k(\nu)} (2^{-\nu})^{m-|\alpha|} \leq C_3 \Big[\sum_{|\nu-\mu| \leq 2; \nu \geq \nu_0} \varepsilon_{k(\nu)} \Big] (2^{-\mu})^{m-|\alpha|}$$
$$\text{for all } 2^{-(\mu+1)} < |\vec{x}| \leq 2^{-\mu}, |\alpha| \leq m \text{ and all } \mu \in \mathbb{Z}. \tag{2.32}$$

For a constant $C_4 > 0$ (that depends only on m and n), we have also

$$|\partial^\alpha S_l(\vec{x})| \leq C_4\, A (2^{-\nu})^{-|\alpha|} \quad \text{for } 2^{-\nu-1} < |\vec{x}| \leq 2^{-\nu}, \text{ for all } |\alpha| \leq m \text{ and all } \nu \geq \nu_{\min}. \tag{2.33}$$

Recall that $\bar{r}_1 = 2^{-\nu_{\min}}$, and (2.28), (2.32) and (2.33) imply that for some constant $C_5 > 0$ (that depends only on m and n), we have

$$|\partial^\alpha F(\vec{x})| = o(|\vec{x}|^{m-\alpha}) \quad \text{as } |\vec{x}| \to 0 \text{ for all } |\alpha| \leq m, \tag{2.34}$$

$$|\partial^\alpha S_l(\vec{x})| \cdot |\vec{x}|^{|\alpha|} < C_5 A \quad \text{for all } \vec{x} \in B^\times(2^{-10} \cdot \bar{r}_1) \text{ and all } |\alpha| \leq m. \tag{2.35}$$

Now, (2.27), (2.29) and (2.31) imply that

$$p(\vec{x}) = F(\vec{x}) + S_1(\vec{x})\, Q_1(\vec{x}) + S_2(\vec{x})\, Q_2(\vec{x}) + \cdots + S_L(\vec{x})\, Q_L(\vec{x})$$
$$\text{for all } \vec{x} \in E \cap B^\times(2^{-10} \cdot \bar{r}_1). \tag{2.36}$$

Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that χ identically equals 1 on $B^\times(2^{-11} \cdot \bar{r}_1)$ and $\operatorname{supp}(\chi) \subset (2^{-10} \cdot \bar{r}_1)$. Then, (2.36) implies that, for all $\vec{x} \in E \setminus \{\vec{0}\}$,

$$p(\vec{x}) = [(1 - \chi(\vec{x})) \cdot p(\vec{x}) + \chi(\vec{x}) \cdot F(\vec{x})]$$
$$+ \chi(\vec{x}) \cdot [S_1(\vec{x})\, Q_1(\vec{x}) + S_2(\vec{x})\, Q_2(\vec{x}) + \cdots + S_L(\vec{x})\, Q_L(\vec{x})]. \tag{2.37}$$

We now define $\tilde{F} \in C^m(\mathbb{R}^n)$ (thanks to (2.34)) and $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_L \in C^m(\mathbb{R}^n \setminus \{\vec{0}\})$ by

$$\tilde{F} := \begin{cases} (1-\chi(\vec{x}))\cdot p(\vec{x}) + \chi(\vec{x})\cdot F(\vec{x}) & \text{if } \vec{x} \neq \vec{0}, \\ 0 & \text{if } \vec{x} = \vec{0}, \end{cases} \quad \text{and} \quad \tilde{S}_l(\vec{x}) := \chi(\vec{x})\cdot S_l(\vec{x}),$$

and thanks to (2.37), (2.34) and (2.35), respectively, we get

$$p(\vec{x}) = \tilde{F}(\vec{x}) + \tilde{S}_1(\vec{x})\,Q_1(\vec{x}) + \tilde{S}_2(\vec{x})\,Q_2(\vec{x}) + \cdots + \tilde{S}_L(\vec{x})\,Q_L(\vec{x}) \quad \text{for all } \vec{x} \in E \setminus \{\vec{0}\}; \tag{2.38}$$

$$J^m(\tilde{F}) = 0; \tag{2.39}$$

$$|\partial^\alpha \tilde{S}_l(\vec{x})| = O(|\vec{x}|^{-\alpha}) \text{ for all } |\alpha| \leq m \text{ and all } 1 \leq l \leq L. \tag{2.40}$$

We recall that for any $1 \leq l \leq L$, $Q_l \in I^m(E)$, so there exists $F_l^\# \in C^m(\mathbb{R}^n)$ with $J^m(F_l^\#) = 0$ such that $Q_l + F_l^\# = 0$ on E. So now (2.38) implies that

$$\begin{aligned} p(\vec{x}) &= \left[\tilde{F}(\vec{x}) - \sum_{l=1}^{L} \tilde{S}_l(\vec{x}) F_l^\#(\vec{x})\right] + \sum_{l=1}^{L} \tilde{S}_l(\vec{x})\,[Q_l(\vec{x}) + F_l^\#(\vec{x})] \\ &= \left[\tilde{F}(\vec{x}) - \sum_{l=1}^{L} \tilde{S}_l(\vec{x}) F_l^\#(\vec{x})\right] \quad \text{for all } \vec{x} \in E \setminus \{\vec{0}\}. \end{aligned} \tag{2.41}$$

Finally, we define

$$F^\#(\vec{x}) := \begin{cases} \sum_{l=1}^{L} \tilde{S}_l(\vec{x}) F_l^\#(\vec{x}) - \tilde{F}(\vec{x}) & \text{if } \vec{x} \neq \vec{0}, \\ 0 & \text{if } \vec{x} = \vec{0}. \end{cases}$$

Since $\tilde{F}, F_1^\#, F_2^\#, \ldots, F_L^\# \in C^m(\mathbb{R}^n)$ and $J^m(\tilde{F}) = J^m(F_1^\#) = J^m(F_2^\#) = \cdots = J^m(F_L^\#) = 0$, we get from (2.40) that $F^\# \in C^m(\mathbb{R}^n)$ and $J^m(F^\#) = 0$. Moreover, from (2.41) we get that $p + F^\#$ vanishes identically on E. We conclude that $p \in I^m(E)$, i.e., $I^m(E)$ is indeed closed. ■

Question 2.19 (Being closed is sufficient). *Fix $m, n \in \mathbb{N}$. Is it true that for any closed ideal $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ there exists a closed subset $E \subset \mathbb{R}^n$ containing the origin such that $I = I^m(E)$?*

Question 2.20 (Semi-algebraic sets suffice). *Fix $m, n \in \mathbb{N}$ and let $E \subset \mathbb{R}^n$ be a closed subset containing the origin. Is it true that there always exists a semi-algebraic subset $E' \subset \mathbb{R}^n$ containing the origin such that $I^m(E) = I^m(E')$?*

3. Calculating implied jets

3.1. Calculating the allowed directions of an ideal

Definition 3.1 (Lowest degree homogenous part). Let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be a non-zero jet. We may always uniquely write $p = p_k + q$, with p_k a homogenous polynomial of degree k, where k is the order of vanishing of p (as p is not the zero jet, we have $1 \leq k \leq m$, see

Definition 2.3), and q is a (possibly zero) polynomial of order of vanishing more than k. We call p_k *the lowest degree homogenous part of* p.

The proof of the following lemma is straightforward, and so left to the reader.

Lemma 3.2. *Let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be a non-zero jet and let $\omega \in S^{n-1}$ be a direction. Let p_k be the lowest degree homogenous part of p. Then, the following are equivalent:*

(3.1) *there exist a cone $\Gamma(\omega, \delta, r)$ and $c > 0$ such that*

$$|p(\vec{x})| > c \cdot |\vec{x}|^k \text{ for all } \vec{x} \in \Gamma(\omega, \delta, r);$$

(3.2) $p_k(\omega) \neq 0$.

Corollary 3.3. *Let $I = \langle p_1, p_2, \dots, p_t \rangle_m \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal. Then,*

$$\mathrm{Allow}(I) \subset \bigcap_{i=1}^{t} \{\textit{the zero set in } S^{n-1} \textit{ of the lowest degree homogenous part of } p_i\}.$$

Moreover, if $p_1, p_2, \dots, p_t$ are all homogenous, then

$$\mathrm{Allow}(I) = \bigcap_{i=1}^{t} \{\textit{the zero set in } S^{n-1} \textit{ of } p_i\}.$$

Proof. The first part follows immediately from Lemma 3.2 and the fact that (3.1) obviously implies (2.1). It is left to the reader to verify the "moreover" part, as it now follows easily from Definition 2.6. ∎

3.2. Negligible functions

Definition 3.4 (Negligible functions). Let $U \subset \mathbb{R}^n$ be open and let $\Omega \subset S^{n-1}$. A function $F \in C^m(U)$ is *$(m$-$)$negligible for* Ω if for all $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ such that the following hold:

(3.3) $\Gamma(\Omega, \delta, r) \subset U$;

(3.4) $|\partial^\alpha F(\vec{x})| \leq \varepsilon |\vec{x}|^{m-|\alpha|}$ for all $\vec{x} \in \Gamma(\Omega, \delta, r)$ and all $|\alpha| \leq m$;

(3.5) $$\left|\partial^\alpha F(\vec{x}) - \sum_{|\gamma| \leq m-|\alpha|} \frac{1}{\gamma!} \partial^{\alpha+\gamma} F(\vec{y}) \cdot (\vec{x} - \vec{y})^\gamma\right| \leq \varepsilon |\vec{x} - \vec{y}|^{m-|\alpha|}$$

for all $\vec{x}, \vec{y} \in \Gamma(\Omega, \delta, r)$ distinct, and all $|\alpha| \leq m$.

Example 3.5 (Follows easily from Taylor's theorem). If there exists $r > 0$ such that $B(r) \subset U$ and $J^m(F) = 0$, then F is negligible for Ω, for any $\Omega \subset S^{n-1}$.

Definition 3.6 (Whitney-negligible functions). Let $U \subset \mathbb{R}^n$ be open and let $\Omega \subset S^{n-1}$. A function $F \in C^m(U)$ is *Whitney-$(m$-$)$negligible for* Ω if for all $\varepsilon > 0$ there exist $\delta > 0$, $r > 0$ and $F_\varepsilon \in C^m(\mathbb{R}^n \setminus \{\vec{0}\})$ such that the following hold:

(3.6) $\Gamma(\Omega, \delta, r) \subset U$;

(3.7) $|\partial^\alpha F_\varepsilon(\vec{x})| \leq \varepsilon |\vec{x}|^{m-|\alpha|}$ for all $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$ and all $|\alpha| \leq m$;

(3.8) $F_\varepsilon(\vec{x}) = F(\vec{x})$ for all $\vec{x} \in \Gamma(\Omega, \delta, r)$.

We will show that a function is negligible if and only if it is Whitney-negligible (Lemma 3.8). Before we do that, we state and prove a preliminary lemma that follows from Whitney's extension theorem.

Lemma 3.7. *Fix $m, n \in \mathbb{N}$. There exists a constant $C(m,n) > 0$ depending only on m and n such that the following holds*:

Let $\emptyset \neq \Omega \subset S^{n-1}$, $\delta > 0$, and $F \in C^m(U_\delta)$, where $U_\delta = \{\vec{x} \in \mathbb{R}^n : 1/10 < |\vec{x}| < 1, \operatorname{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta\}$. Also set $\tilde{U}_\delta = \{\vec{x} \in \mathbb{R}^n : 1/2 \leq |\vec{x}| \leq 2/3, \operatorname{dist}(\vec{x}/|\vec{x}|, \Omega) \leq \delta/2\}$. Let $M > 0$ be such that

$$|\partial^\alpha F(\vec{x})| \leq M \ \text{ for all } \vec{x} \in U_\delta \text{ and all } |\alpha| \leq m; \tag{3.9}$$

$$\left|\partial^\alpha F(\vec{x}) - \sum_{|\gamma| \leq m-|\alpha|} \frac{1}{\gamma!} \partial^{\alpha+\gamma} F(\vec{y}) \cdot (\vec{x}-\vec{y})^\gamma \right| \leq M|\vec{x}-\vec{y}|^{m-|\alpha|} \tag{3.10}$$
$$\text{for all } \vec{x}, \vec{y} \in U_\delta \text{ distinct, and all } |\alpha| \leq m.$$

Then, there exists $\tilde{F} \in C^m(\mathbb{R}^n)$ such that

$$|\partial^\alpha \tilde{F}(\vec{x})| \leq C(m,n) \cdot M \ \text{ for all } \vec{x} \in \mathbb{R}^n \text{ and all } |\alpha| \leq m; \tag{3.11}$$

$$\tilde{F}(\vec{x}) = F(\vec{x}) \ \text{ for all } \vec{x} \in \tilde{U}_\delta. \tag{3.12}$$

Proof. We start by checking that the following holds:

$$\left|\partial^\alpha F(\vec{x}) - \sum_{|\gamma| \leq m-|\alpha|} \frac{1}{\gamma!} \partial^{\alpha+\gamma} F(\vec{y}) \cdot (\vec{x}-\vec{y})^\gamma \right| = o(|\vec{x}-\vec{y}|^{m-|\alpha|}) \tag{3.13}$$
$$\text{as } |\vec{x}-\vec{y}| \to 0, \text{ subject to } \vec{x}, \vec{y} \in \tilde{U}_\delta \text{ distinct, and all } |\alpha| \leq m.$$

Indeed, suppose (3.13) does not hold. Then, there exist a multi-index α_0 with $|\alpha_0| \leq m$, a number $\eta > 0$ and for any $\nu \in \mathbb{N}$ distinct points $\vec{x}_\nu, \vec{y}_\nu \in \tilde{U}_\delta$ such that $|\vec{x}_\nu - \vec{y}_\nu| \to 0$ as $\nu \to \infty$ and, for all $\nu \in \mathbb{N}$,

$$\left|\partial^{\alpha_0} F(\vec{x}_\nu) - \sum_{|\gamma| \leq m-|\alpha_o|} \frac{1}{\gamma!} \partial^{\alpha_0+\gamma} F(\vec{y}_\nu) \cdot (\vec{x}_\nu-\vec{y}_\nu)^\gamma \right| \geq \eta \cdot |\vec{x}_\nu-\vec{y}_\nu|^{m-|\alpha_0|}. \tag{3.14}$$

Since $\tilde{U}_\delta$ is a compact that is contained in the open set U_δ, by passing to a subsequence we may assume $\vec{x}_\nu, \vec{y}_\nu \to \vec{z} \in U_\delta$ as $\nu \to \infty$, and that moreover there exists a closed ball B centered at $\vec{z}$ such that $B \subset U_\delta$ and $\vec{x}_\nu, \vec{y}_\nu \in B$ for any $\nu \in \mathbb{N}$. Note that F, and all of its derivatives up to order m, are uniformly continuous on the closed ball B. Let $\omega(\cdot)$ be the modulus of continuity of the m^{th} derivatives of F on B. Then,

$$\omega(t) \to 0 \quad \text{as } t \to 0, \tag{3.15}$$

and by Taylor's theorem there exists a constant $\tilde{C}(m,n) > 0$ depending only on m and n such that

$$\left|\partial^\alpha F(\vec{x}) - \sum_{|\gamma| \leq m-|\alpha|} \frac{1}{\gamma!} \partial^{\alpha+\gamma} F(\vec{y}) \cdot (\vec{x}-\vec{y})^\gamma \right| \leq \tilde{C}(m,n) \cdot \omega(|\vec{x}-\vec{y}|) \cdot |\vec{x}-\vec{y}|^{m-|\alpha|} \tag{3.16}$$
$$\text{for all } \vec{x}, \vec{y} \in B \text{ distinct, and all } |\alpha| \leq m.$$

Applying (3.15) and (3.16) with $\vec{x} = \vec{x}_\nu$ and $\vec{y} = \vec{y}_\nu$ we get

(3.17)
$$\left|\partial^{\alpha_0} F(\vec{x}_\nu) - \sum_{|\gamma| \le m - |\alpha_o|} \frac{1}{\gamma!} \partial^{\alpha_0+\gamma} F(\vec{y}_\nu) \cdot (\vec{x}_\nu - \vec{y}_\nu)^\gamma\right| = o(\left|\vec{x}_\nu - \vec{y}_\nu\right|^{m-|\alpha_0|}) \quad \text{as } \nu \to \infty.$$

Now, (3.17) clearly contradicts (3.14), and so we proved that (3.13) holds.

Set $P^{\vec{x}} = J_{\vec{x}} F$ (the m^{th}-jet of F about $\vec{x}$) for any $\vec{x} \in \tilde{U}_\delta$. Now (3.9), (3.10) and (3.13) tell us that

$$\text{(3.18)} \qquad |(\partial^\alpha P^{\vec{x}})(\vec{x})| \le M \quad \text{for all } \vec{x} \in \tilde{U}_\delta \text{ and all } |\alpha| \le m;$$

$$\text{(3.19)} \qquad \left|(\partial^\alpha P^{\vec{x}})(\vec{x}) - \sum_{|\gamma| \le m-|\alpha|} \frac{1}{\gamma!} [(\partial^{\alpha+\gamma} P^{\vec{y}})(\vec{y})] \cdot (\vec{x} - \vec{y})^\gamma\right| \le M|\vec{x} - \vec{y}|^{m-|\alpha|}$$
$$\text{for all } \vec{x}, \vec{y} \in \tilde{U}_\delta \text{ distinct, and for all } |\alpha| \le m;$$

and

$$\text{(3.20)} \quad \left|(\partial^\alpha P^{\vec{x}})(\vec{x}) - \sum_{|\gamma| \le m-|\alpha|} \frac{1}{\gamma!} [(\partial^{\alpha+\gamma} P^{\vec{y}})(\vec{y})] \cdot (\vec{x} - \vec{y})^\gamma\right| = o(|\vec{x} - \vec{y}|^{m-|\alpha|})$$
$$\text{as } |\vec{x} - \vec{y}| \to 0, \text{ subject to } \vec{x}, \vec{y} \in \tilde{U}_\delta \text{ distinct, for all } |\alpha| \le m.$$

The above (3.18)–(3.20) are the hypothesis of the classical Whitney extension theorem (see [11] and Theorem 2.3 in [6]), and so there exists a constant $C(m,n) > 0$ depending only on m and n and $\tilde{F} \in C^m(\mathbb{R}^n)$ such that

$$\text{(3.21)} \qquad \left|\partial^\alpha \tilde{F}(\vec{x})\right| \le C(m,n) \cdot M \quad \text{for all } \vec{x} \in \mathbb{R}^n \text{ and all } |\alpha| \le m;$$

and

$$\text{(3.22)} \qquad J_{\vec{x}}(\tilde{F}) = P^{\vec{x}} (= J_{\vec{x}} F) \quad \text{for all } \vec{x} \in \tilde{U}_\delta.$$

In particular, (3.22) implies that

$$\text{(3.23)} \qquad \tilde{F}(\vec{x}) = F(\vec{x}) \quad \text{for all } \vec{x} \in \tilde{U}_\delta,$$

which together with (3.21) implies that (3.11) and (3.12) indeed hold. ■

Lemma 3.8. *Definitions 3.4 and 3.6 coincide. That is, let $U \subset \mathbb{R}^n$ be open, let $\Omega \subset S^{n-1}$ and let $F \in C^m(U)$. Then, F is negligible for Ω if and only if F is Whitney-negligible for Ω.*

Proof. If $\Omega = \emptyset$, then one readily sees that any $F \in C^m(U)$ is both negligible for Ω and Whitney-negligible for Ω, so we only need to show the equivalence of Definitions 3.4 and 3.6 for $\Omega \neq \emptyset$. Let F be Whitney-negligible for $\Omega \neq \emptyset$ and fix $\varepsilon > 0$. Let δ, r and F_ε be such that (3.7) and (3.8) hold. We immediately have from (3.7) and (3.8) that

$$\text{(3.24)} \qquad |\partial^\alpha F(\vec{x})| \le \varepsilon|\vec{x}|^{m-|\alpha|} \quad \text{for all } \vec{x} \in \Gamma(\Omega, \delta, r) \text{ and all } |\alpha| \le m.$$

Moreover, (3.7) and (3.8) together with Taylor's theorem imply that for some constant $C_1 > 0$ (that depends only on m and n) we have

$$\left|\partial^\alpha F(\vec{x}) - \sum_{|\gamma|\le m-|\alpha|} \frac{1}{\gamma!}\partial^{\alpha+\gamma}F(\vec{y})\cdot(\vec{x}-\vec{y})^\gamma\right| \le C_1\varepsilon|\vec{x}-\vec{y}|^{m-|\alpha|} \tag{3.25}$$

$$\text{for all } \vec{x},\vec{y}\in\Gamma(\Omega,\delta,r) \text{ distinct, and all } |\alpha|\le m.$$

We conclude that F is negligible for U.

Let F be negligible for Ω and fix $\varepsilon > 0$. Let δ and r be such that (3.3), (3.4) and (3.5) hold, and without loss of generality assume $r < 1$. We recall that

$$\Gamma(\Omega,\delta,r) = \{\vec{x}\in\mathbb{R}^n : 0<|\vec{x}|<r, \operatorname{dist}(\vec{x}/|\vec{x}|,\Omega)<\delta\},$$

and for any $0<\rho\le r$, we set

$$E(\rho) := \{\vec{x}\in\mathbb{R}^n : \rho/10<|\vec{x}|<\rho, \operatorname{dist}(\vec{x}/|\vec{x}|,\Omega)<\delta\}.$$

We then have from (3.4) and (3.5) that for some constant $C_2>0$, that depends only on m and n,

$$|\partial^\alpha F(\vec{x})| \le C_2\varepsilon\rho^{m-|\alpha|} \quad \text{for all } \vec{x}\in E(\rho) \text{ and all } |\alpha|\le m; \tag{3.26}$$

$$\left|\partial^\alpha F(\vec{x}) - \sum_{|\gamma|\le m-|\alpha|} \frac{1}{\gamma!}\partial^{\alpha+\gamma}F(\vec{y})\cdot(\vec{x}-\vec{y})^\gamma\right| \le C_2\varepsilon|\vec{x}-\vec{y}|^{m-|\alpha|} \tag{3.27}$$

$$\text{for all } \vec{x},\vec{y}\in E(\rho) \text{ distinct, and all } |\alpha|\le m.$$

Define a function $G\in C^m(\rho^{-1}E(\rho))$ by $G(\vec{x}) := F(\rho\vec{x})$. From (3.26) and (3.27) we get that

$$\left|\partial^\alpha G(\vec{x})\right| \le C_2\varepsilon\rho^m \quad \text{for all } \vec{x}\in\rho^{-1}E(\rho) \text{ and all } |\alpha|\le m; \tag{3.28}$$

$$\left|\partial^\alpha G(\vec{x}) - \sum_{|\gamma|\le m-|\alpha|} \frac{1}{\gamma!}\partial^{\alpha+\gamma}G(\vec{y})\cdot(\vec{x}-\vec{y})^\gamma\right| \le C_2\varepsilon\rho^m|\vec{x}-\vec{y}|^{m-|\alpha|} \tag{3.29}$$

$$\text{for all } \vec{x},\vec{y}\in\rho^{-1}E(\rho) \text{ distinct, and all } |\alpha|\le m.$$

Define

$$\hat{E}(\rho) := \{\vec{x}\in\mathbb{R}^n : \rho/2<|\vec{x}|<2\rho/3,\ \operatorname{dist}(\vec{x}/|\vec{x}|,\Omega)<\delta/2\}.$$

Applying Lemma 3.7 we find that there exists $G_\rho\in C^m(\mathbb{R}^n)$ such that

$$|\partial^\alpha G_\rho(\vec{x})| \le C(m,n)\cdot C_2\varepsilon\rho^m \quad \text{for all } \vec{x}\in\mathbb{R}^n \text{ and all } |\alpha|\le m; \tag{3.30}$$

$$G_\rho(\vec{x}) = G(\vec{x}) \quad \text{for all } \vec{x}\in\rho^{-1}\hat{E}(\rho). \tag{3.31}$$

Define a function $F_\rho\in C^m(\mathbb{R}^n)$ by $F_\rho(\vec{x}) := G_\rho(\rho^{-1}\vec{x})$. From (3.30) and (3.31) we get that

$$|\partial^\alpha F_\rho(\vec{x})| \le C(m,n)\cdot C_2\varepsilon\rho^{m-|\alpha|} \quad \text{for all } \vec{x}\in\mathbb{R}^n \text{ and all } |\alpha|\le m; \tag{3.32}$$

$$F_\rho(\vec{x}) = F(\vec{x}) \quad \text{for all } \vec{x}\in\hat{E}(\rho). \tag{3.33}$$

For $k \in \mathbb{N} \cup \{0\}$ set $\rho_k := (\frac{9}{10})^k r$ and let $\theta_k \in C^m(\mathbb{R}^n)$ be such that the following hold:

$$\sum_{k=0}^{\infty} \theta_k(\vec{x}) = 1 \quad \text{for all } 0 < |\vec{x}| \le r/10; \tag{3.34}$$

$$\operatorname{supp} \theta_k \subset \{\rho_k/2 \le |\vec{x}| \le 2\rho_k/3\}; \tag{3.35}$$

$$|\partial^\alpha \theta_k(\vec{x})| \le C_3 \cdot \rho_k^{-|\alpha|} \quad \text{for all } \vec{x} \in \mathbb{R}^n \text{ and all } |\alpha| \le m, \tag{3.36}$$

where C_3 is some constant depending on m and n only. Define $\tilde{F} \in C^m(\mathbb{R}^n \setminus \{\vec{0}\})$ by

$$\tilde{F}(\vec{x}) := \sum_{k=0}^{\infty} \theta_k(\vec{x}) \cdot F_{\rho_k}(\vec{x}).$$

We have from (3.32), (3.35) and (3.36) that for some constant $C_4 > 0$ that depends only on m and n,

$$|\partial^\alpha \tilde{F}(\vec{x})| \le C_4 \varepsilon |\vec{x}|^{m-|\alpha|} \quad \text{for all } \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\} \text{ and all } |\alpha| \le m, \tag{3.37}$$

and from (3.33), (3.34) and (3.35) we have

$$\tilde{F}(\vec{x}) = F(\vec{x}) \quad \text{for all } \vec{x} \in \Gamma(\Omega, \delta/2, r/10). \tag{3.38}$$

We conclude that from (3.37) and (3.38) that F is Whitney-negligible for U. ∎

Example 3.9. Set $n = 3$, $m = 2$ and (x, y, z) a standard coordinate system on $\mathbb{R}^3$. Let $U = \mathbb{R}^3 \setminus \{z = 0\}$, $\Omega = \{(0, 0, \pm 1)\} \subset S^2$ and let $F(x, y, z) = y^3/z \in C^2(U)$. Then, F is negligible for $\Omega \cap D(\omega, 10^{-3})$, for any $\omega \in \Omega$.

Indeed, fixing $\omega \in \Omega$, we have $\Omega \cap D(\omega, 10^{-3}) = \{\omega\}$. Given $\varepsilon > 0$ (and we are allowed to assume $\varepsilon < 1$), one has to find $\delta, r > 0$ such that (3.3), (3.4) and (3.5) hold, with Ω being replaced by $\{\omega\}$. We claim that taking $\delta = 10^{-100} \cdot \varepsilon^{10}$ and $r = 1$, we satisfy (3.3), (3.4) and (3.5): indeed, (3.3) holds trivially. In order to see (3.4) we note that on $\Gamma(\omega, 10^{-100} \cdot \varepsilon^{10}, 1)$, we have $|y| < |\frac{\varepsilon}{10} z|$. So on this cone we have:

$$|F^{(0,0,0)}(x, y, z)| = \left|\frac{y^3}{z}\right| < \left|\frac{\varepsilon^3 z^2}{1000}\right| < \varepsilon(x^2 + y^2 + z^2) = \varepsilon|\vec{x}|^2;$$

$$|F^{(0,1,0)}(x, y, z)| = \left|\frac{3y^2}{z}\right| < \left|\frac{3\varepsilon^2 z}{100}\right| < \varepsilon(x^2 + y^2 + z^2)^{1/2} = \varepsilon|\vec{x}|;$$

$$|F^{(0,0,1)}(x, y, z)| = \left|\frac{-y^3}{z^2}\right| < \left|\frac{\varepsilon^3 z}{1000}\right| < \varepsilon(x^2 + y^2 + z^2)^{1/2} = \varepsilon|\vec{x}|;$$

$$|F^{(0,2,0)}(x, y, z)| = \left|\frac{6y}{z}\right| < \left|\frac{6}{10}\varepsilon\right| < \varepsilon; \quad |F^{(0,0,2)}(x, y, z)| = \left|\frac{2y^3}{z^3}\right| < \left|\frac{2\varepsilon^3}{1000}\right| < \varepsilon;$$

$$|F^{(0,1,1)}(x, y, z)| = \left|\frac{-3y^2}{z^2}\right| < \left|\frac{3\varepsilon^2}{100}\right| < \varepsilon.$$

All the other partial derivatives of F are identically zero, so we showed that indeed (3.4) holds. Finally, as $\Gamma(\omega, 10^{-100} \cdot \varepsilon^{10}, 1)$ is convex, Taylor's theorem together with (3.4) implies (3.5).

Lemma 3.10 (Patching negligible functions). *Let $U \subset \mathbb{R}^n$ be open and let $\Omega \subset S^{n-1}$. Let $\delta_1 \dots, \delta_K > 0$, let $\omega_1, \dots, \omega_K \in S^{n-1}$ and define $\Omega_k := \Omega \cap D(\omega_k, \delta_k)$ for any $1 \le k \le K$. Suppose that for any $1 \le k \le K$ we are given $F_k \in C^m(U)$ such that F_k is Whitney-negligible (or equivalently, negligible) for Ω_k. Suppose that moreover we are given $\theta_1, \dots, \theta_K \in C^m(\mathbb{R}^n \setminus \{\vec{0}\})$ and a constant $\hat{C} > 0$ such that for any $1 \le k \le K$ we have*

$$\operatorname{supp}\theta_k \subset \left\{\vec{x} \in \mathbb{R}^n : \left|\frac{\vec{x}}{|\vec{x}|} - \omega_k\right| \le \frac{2}{3}\,\delta_k\right\}; \quad \textit{and} \tag{3.39}$$

$$|\partial^\alpha \theta_k(\vec{x})| \le \hat{C}\,|\vec{x}|^{-|\alpha|} \quad \textit{for all } \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\} \textit{ and all } |\alpha| \le m. \tag{3.40}$$

Then, $F(\vec{x}) = \sum_{k=1}^K \theta_k(\vec{x}) \cdot F_k(\vec{x})$ is Whitney-negligible (or equivalently, negligible) for Ω.

Proof. Fix $\varepsilon > 0$ and for each $1 \le k \le K$ let $r_k, \tilde{\delta}_k > 0$ and $\tilde{F}_{\varepsilon,k} \in C^m(\mathbb{R}^n \setminus \{\vec{0}\})$ be such that (3.6), (3.7) and (3.8) hold for Ω_k. So we have

$$|\partial^\alpha \tilde{F}_{\varepsilon,k}(\vec{x})| \le \varepsilon|\vec{x}|^{m-|\alpha|} \ \text{ for all } \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\} \text{ and all } |\alpha| \le m; \tag{3.41}$$

$$\tilde{F}_{\varepsilon,k}(\vec{x}) = F_k(\vec{x}) \ \text{ for all } \vec{x} \in \Gamma(\Omega_k, \tilde{\delta}_k, r_k). \tag{3.42}$$

Since (3.6), (3.8) and (3.42) are preserved under replacing δ and $\tilde{\delta}_k$ by smaller numbers, we may assume without loss of generality that $\tilde{\delta}_k \le \delta_k$ for each $1 \le k \le K$. Set

$$\tilde{F}_\varepsilon(\vec{x}) := \sum_{k=1}^K \theta_k(\vec{x}) \cdot \tilde{F}_{\varepsilon,k}(\vec{x}).$$

By (3.40) and (3.41) for some constant $C_1 > 0$ (that may be dependent on K as well, in addition to m and n) we have

$$|\partial^\alpha \tilde{F}_\varepsilon(\vec{x})| \le C_1 \hat{C}\,\varepsilon|\vec{x}|^{m-|\alpha|} \quad \text{for all } \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\} \text{ and all } |\alpha| \le m. \tag{3.43}$$

Now set $r := 10^{-10}\min\{r_1, \dots, r_K\}$, $\delta := 10^{-9}\min\{\tilde{\delta}_1, \dots, \tilde{\delta}_K\}$, and let $\vec{x} \in \Gamma(\Omega, \delta, r)$. In particular, we have $0 < |\vec{x}| < r \le r_k$ for any $1 \le k \le K$. Suppose that moreover $\vec{x} \in \operatorname{supp}\theta_k$ for some $1 \le k \le K$. We then have by (3.39) that

$$\left|\frac{\vec{x}}{|\vec{x}|} - \omega_k\right| \le \frac{2}{3}\,\delta_k,$$

and moreover, since $\vec{x} \in \Gamma(\Omega, \delta, r)$, we also have

$$\left|\frac{\vec{x}}{|\vec{x}|} - \omega''\right| \le 10^{-9}\,\tilde{\delta}_k \le 10^{-9}\,\delta_k \quad \text{for some } \omega'' \in \Omega.$$

We conclude that

$$|\omega'' - \omega_k| \le \frac{2}{3}\,\delta_k + 10^{-9}\,\delta_k < \delta_k \quad \text{for some } \omega'' \in \Omega,$$

hence $\omega'' \in \Omega_k$, and so $\operatorname{dist}(\vec{x}/|\vec{x}|, \Omega_k) < \tilde{\delta}_k$. Recall that also $0 < |\vec{x}| \le r_k$, and therefore $\vec{x} \in \Gamma(\Omega_k, \tilde{\delta}_k, r_k)$. We conclude from (3.42) that

$$\tilde{F}_{\varepsilon,k}(\vec{x}) = F_k(\vec{x}) \quad \text{for all } \vec{x} \in \Gamma(\Omega, \delta, r) \cap \operatorname{supp}\theta_k. \tag{3.44}$$

Consequently, we have

$$\tilde{F}_\varepsilon(\vec{x}) = \sum_{k=1}^{K} \theta_k(\vec{x}) \cdot \tilde{F}_{\varepsilon,k}(\vec{x}) = \sum_{k=1}^{K} \theta_k(\vec{x}) \cdot F_k(\vec{x}) = F(\vec{x}) \quad \text{for all } \vec{x} \in \Gamma(\Omega, \delta, r),$$

which together with (3.43) proves that F is Whitney-negligible for Ω. ∎

3.3. Strong directional implication and strong implication

Definition 3.11 (Strong directional implication). Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal and let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial. We say that I *strongly implies* p *in the direction* $\omega \in S^{n-1}$ if there exist $\delta_\omega > 0$, $r_\omega > 0$, polynomials $Q_1, \dots, Q_L \in I$, functions $S_1, \dots, S_L \in C^m(\Gamma(\omega, \delta_\omega, r_\omega))$, positive constants $C_1, \dots, C_L > 0$ and a function $F \in C^m(\Gamma(\omega, \delta_\omega, r_\omega))$ such that the following hold:

(3.45) F is negligible for $\mathrm{Allow}(I) \cap D(\omega, \delta_\omega)$;

(3.46) $|\partial^\alpha S_l(\vec{x})| \le C_l\, |\vec{x}|^{-|\alpha|}$ for all $|\alpha| \le m$, all $1 \le l \le L$ and all $\vec{x} \in \Gamma(\omega, \delta_\omega, r_\omega)$;

(3.47) $p(\vec{x}) = \sum_{l=1}^{L} S_l(\vec{x}) \cdot Q_l(\vec{x}) + F(\vec{x})$ for all $\vec{x} \in \Gamma(\omega, \delta_\omega, r_\omega)$.

Remark 3.12. If $\omega \notin \mathrm{Allow}(I)$, then I always strongly implies p in the direction ω, for any $p \in \mathcal{P}_0^m(\mathbb{R}^n)$. Indeed, recall that $\mathrm{Allow}(I)$ is closed (see Definition 2.6) and fix some $\delta_\omega > 0$ such that $\mathrm{Allow}(I) \cap D(\omega, \delta_\omega) = \emptyset$. Now (3.45)–(3.47) hold with $F = p$, $L = C_1 = r_\omega = 1$, $Q_1 = S_1 = 0$.

Definition 3.13 (Strong implication). Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal and let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial. We say that I *strongly implies* p if there exist $r_0 > 0$, polynomials $Q_1, \dots, Q_L \in I$, functions $S_1, \dots, S_L \in C^m(B^\times(r_0))$, positive constants $C_1, \dots, C_L > 0$ and a function $F \in C^m(B^\times(r_0))$ such that the following hold:

(3.48) F is negligible for $\mathrm{Allow}(I)$;

(3.49) $|\partial^\alpha S_l(\vec{x})| \le C_l\, |\vec{x}|^{-|\alpha|}$ for all $|\alpha| \le m$, all $1 \le l \le L$ and all $\vec{x} \in B^\times(r_0)$;

(3.50) $p(\vec{x}) = \sum_{l=1}^{L} S_l(\vec{x}) \cdot Q_l(\vec{x}) + F(\vec{x})$ for all $\vec{x} \in B^\times(r_0)$.

Remark 3.14. Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal and let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be a jet. Clearly, if I strongly implies p, then I strongly implies p in the direction ω for any $\omega \in \mathrm{Allow}(I)$. The following Lemma 3.15 shows that the converse also holds.

Lemma 3.15. *Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal and let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial. If I strongly implies p in the direction ω for any $\omega \in$ $\mathrm{Allow}(I)$, then I strongly implies p.*

Proof. By Remark 3.12 and our assumption, we have that I strongly implies p in any direction $\omega \in S^{n-1}$. Fix $Q_1, \dots, Q_L$ a basis of I (as a vector space). Then, for each $\omega \in S^{n-1}$, there exist $\delta_\omega > 0$, $r_\omega > 0$, functions $S_1^\omega, \dots S_L^\omega \in C^m(\Gamma(\omega, \delta_\omega, r_\omega))$, positive constants $C_1^\omega, \dots, C_L^\omega > 0$ and a function $F^\omega \in C^m(\Gamma(\omega, \delta_\omega, r_\omega))$ such that the following hold:

(3.51) F^ω is negligible for $\mathrm{Allow}(I) \cap D(\omega, \delta_\omega)$;

(3.52) $|\partial^\alpha S_l^\omega(\vec{x})| \le C_l^\omega |\vec{x}|^{-|\alpha|}$ for all $|\alpha| \le m$, all $1 \le l \le L$ and all $\vec{x} \in \Gamma(\omega, \delta_\omega, r_\omega)$;

(3.53) $$p(\vec{x}) = \sum_{l=1}^{L} S_l^\omega(\vec{x}) \cdot Q_l(\vec{x}) + F^\omega(\vec{x}) \text{ for all } \vec{x} \in \Gamma(\omega, \delta_\omega, r_\omega).$$

By compactness of S^{n-1}, there exists finitely many $\omega_1, \dots, \omega_K$ such that $S^{n-1} = \bigcup_{k=1}^K D(\omega_k, \delta_{\omega_k}/100)$. Fix $\tilde{\theta}_1, \dots, \tilde{\theta}_K \in C^\infty(S^{n-1})$ such that $\sum_{k=1}^K \tilde{\theta}_k(\omega) = 1$ for any $\omega \in S^{n-1}$ and $\operatorname{supp}\tilde{\theta}_k \subset D(\omega_k, \delta_{\omega_k}/50)$ for all $1 \le k \le K$. Defining $\theta_k(\vec{x}) := \tilde{\theta}_k(\vec{x}/|\vec{x}|)$ for all $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$ and $1 \le k \le K$, we get that for some constant $\hat{C} > 0$ (depending on m, n and $\theta_1, \dots, \theta_K$), we have

(3.54) $$\theta_1, \dots, \theta_K \in C^\infty(\mathbb{R}^n \setminus \{\vec{0}\});$$

(3.55) $$\sum_{k=1}^K \theta_k(\vec{x}) = 1 \text{ for all } \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\};$$

(3.56) $$\theta_k(\vec{x}) = 0 \text{ if } |\vec{x}/|\vec{x}| - \omega_k| > 2\delta_{\omega_k}/3 \text{ for all } 1 \le k \le K;$$

(3.57) $$|\partial^\alpha \theta_k(\vec{x})| \le \hat{C}\, |\vec{x}|^{-|\alpha|} \text{ for all } \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\} \text{ and all } |\alpha| \le m.$$

Set $\hat{r} = \min\{r_{\omega_1}, \dots, r_{\omega_k}\}$. Thanks to (3.56), we can define $F, S_1, S_2, \dots, S_L \in C^m(B^\times(\hat{r}))$ as follows: for all $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$,

(3.58) $$F(\vec{x}) := \sum_{k=1}^K \theta_k(\vec{x})\, F^{\omega_k}(\vec{x}) \quad \text{and} \quad S_l(\vec{x}) := \sum_{k=1}^K \theta_k(\vec{x})\, S_l^{\omega_k}(\vec{x}).$$

Thanks to (3.51), (3.56), (3.57) and Lemma 3.10, we get that

(3.59) F is negligible for $\mathrm{Allow}(I)$.

Thanks to (3.52) and (3.57), we get that for some constant $C' > 0$ (depending on $m, n, \hat{C}$ and $\{C_l^{\omega_k}\}_{1 \le k \le K, 1 \le l \le L}$), we have

(3.60) $$|\partial^\alpha S_l(\vec{x})| \le C'|\vec{x}|^{-|\alpha|} \quad \text{for all } |\alpha| \le m, \text{ all } 1 \le l \le L \text{ and all } \vec{x} \in B^\times(\hat{r}).$$

Finally, from (3.53), (3.56) and (3.55), we get that

(3.61) $$p(\vec{x}) = \sum_{l=1}^{L} S_l(\vec{x}) \cdot Q_l(\vec{x}) + F(\vec{x}) \quad \text{for all } \vec{x} \in B^\times(\hat{r}).$$

Combining (3.59), (3.60) and (3.61), we proved that I strongly implies p. ∎

Lemma 3.16. *Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal and let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial. If I strongly implies p (as in Definition 3.13), then I implies p (as in Definition 2.10).*

Proof. Assume that I strongly implies p and denote $\Omega := \mathrm{Allow}(I)$. Now let $r_0 > 0$, $Q_1, \ldots, Q_L \in I$, $S_1, \ldots, S_L \in C^m(B^\times(r_0))$, $C_1, \ldots, C_L > 0$ and $F \in C^m(B^\times(r_0))$ be such that (3.48), (3.49) and (3.50) hold. Setting $A := \max\{C_1, \ldots, C_L\} > 0$ we have

(3.62) F is negligible for $\mathrm{Allow}(I)$;

(3.63) $|\partial^\alpha S_l(\vec{x})| \le A|\vec{x}|^{-|\alpha|}$ for all $|\alpha| \le m$, all $1 \le l \le L$ and all $\vec{x} \in B^\times(r_0)$;

$$p(\vec{x}) = \sum_{l=1}^{L} S_l(\vec{x}) \cdot Q_l(\vec{x}) + F(\vec{x}) \text{ for all } \vec{x} \in B^\times(r_0). \tag{3.64}$$

Fix $\varepsilon > 0$. By (3.62), Definition 3.6 and Lemma 3.8, there exist $\delta, r > 0$ (and without loss of generality $r < r_0$) and $F_\varepsilon \in C^m(\mathbb{R}^n \setminus \{\vec{0}\})$ such that

$$|\partial^\alpha F_\varepsilon(\vec{x})| \le \varepsilon|\vec{x}|^{m-|\alpha|} \quad \text{for all } \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\} \text{ and all } |\alpha| \le m; \tag{3.65}$$

$$F_\varepsilon(\vec{x}) = F(\vec{x}) \quad \text{for all } \vec{x} \in \Gamma(\Omega, \delta, 4r). \tag{3.66}$$

Let $0 < \rho \le r$. From (3.63) and (3.65) we have that, for some constant $C_1 > 0$ (depending only on m and n), the following hold:

$$|\partial^\alpha S_l(\vec{x})| \le C_1 A\rho^{-|\alpha|} \text{ for all } \vec{x} \in \mathrm{Ann}_4(\rho) \text{ and all } |\alpha| \le m; \tag{3.67}$$

$$|\partial^\alpha F_\varepsilon(\vec{x})| \le C_1 \varepsilon\rho^{m-|\alpha|} \text{ for all } \vec{x} \in \mathrm{Ann}_4(\rho) \text{ and all } |\alpha| \le m. \tag{3.68}$$

Finally, from (3.64) and (3.66) we immediately get that

$$\begin{aligned} p(\vec{x}) = F_\varepsilon(\vec{x}) + S_1(\vec{x})\, Q_1(\vec{x}) + S_2(\vec{x})\, Q_2(\vec{x}) + \cdots + S_L(\vec{x})\, Q_L(\vec{x}) \\ \text{for all } \vec{x} \in \mathrm{Ann}_2(\rho) \text{ such that } \mathrm{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta. \end{aligned} \tag{3.69}$$

We thus showed that there exists a constant $A > 0$ such that, given $\varepsilon > 0$, there exist $\delta, r > 0$ such that for any $0 < \rho \le r$ there exist functions $F_\varepsilon, S_1, \ldots, S_L \in C^m(\mathrm{Ann}_4(\rho))$ such that (3.67), (3.68) and (3.69) hold. That is, we showed that I implies p. ■

Remark 3.17. We do not know whether the converse of Lemma 3.16 holds, i.e.: let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal and $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial. Assume that I implies p (as in Definition 2.10). Is it always true that I strongly implies p (as in Definition 3.13)?

Corollary 3.18. *Let $I \lhd \mathcal{P}_0^m(\mathbb{R}^n)$ be an ideal and let $p \in \mathcal{P}_0^m(\mathbb{R}^n)$ be some polynomial. If I strongly implies p in the direction ω for any $\omega \in$* Allow(I) (*as in Definition* 3.11), *then I implies p* (*as in Definition* 2.10).

Proof. It follows immediately from Lemma 3.15 and Lemma 3.16. ■

Example 3.19. Set $n = 3$, $m = 2$ and (x, y, z) a standard coordinate system on $\mathbb{R}^3$. Let $I \lhd \mathcal{P}_0^2(\mathbb{R}^3)$ be such that $x^2, y^2 - xz \in I$. Then, $xy \in \mathrm{cl}(I)$. In particular, $\langle x^2, y^2 - xz\rangle_2$ is not closed.

Indeed, by Corollary 3.3 we have $\mathrm{Allow}(I) \subset \{(0, 0, \pm 1)\}$. By Example 3.9, the function $F = y^3/z$ is negligible for $\Omega \cap D(\omega, 10^{-3})$, for any $\omega \in \mathrm{Allow}(I)$. So by Definition 3.11, setting $L = 1$, $S_1 = -y/z$ and $Q_1 = y^2 - xz$, we have that I strongly implies the jet

$$xy = S_1 \cdot Q_1 + F = (-y/z) \cdot (y^2 - xz) + y^3/z$$

in the direction ω, for any $\omega \in \mathrm{Allow}(I)$. By Corollary 3.18, I implies xy, i.e., $xy \in \mathrm{cl}(I)$.

4. An algorithm to calculate the closure of an ideal

4.1. Background on bundles

We provide the necessary background on *bundles*, introduced and studied in ([4, 7]). The notation in this section is slightly different from the rest of this paper, as we adopt here the standard notation in the literature when such bundles are used. Thus, we start by fixing notation.

Notation 4.1. Fix natural numbers $m, n, \mathcal{D} \geq 1$. We write $J_{\vec{x}}(F)$ (or $J^m_{\vec{x}} F$) to denote the m^{th} degree Taylor polynomial of a real valued function $F \in C^m(U)$, when U is an open neighborhood of $\vec{x}$. The vector space of all such polynomials is denoted by $\mathcal{P}$ (note that as a finite dimensional real vector space $\mathcal{P}$ can be identified with $\mathbb{R}^{\mathcal{D}^*}$ for some $\mathcal{D}^* \in \mathbb{N}$, so we can talk about semi-algebraic subsets of $\mathcal{P}$). The ring of m-jets (of real valued functions) at $\vec{x} \in \mathbb{R}^n$ is the vector space $\mathcal{P}$, with multiplication $P \odot_{\vec{x}} Q = J_{\vec{x}}(PQ)$. Then, for two C^m functions around $\vec{x}$, we have $J_{\vec{x}}(FG) = J_{\vec{x}}(F) \odot_{\vec{x}} J_{\vec{x}}(G)$. We write $\mathcal{R}_{\vec{x}} = (\mathcal{P}, \odot_{\vec{x}})$ to denote the ring of m-jets at $\vec{x}$.

The m-jet of a (C^m) $\mathbb{R}^{\mathcal{D}}$-valued function $\vec{F} = (F_1, \ldots, F_{\mathcal{D}})$ at $\vec{x} \in \mathbb{R}^n$ is defined to be $J^m_{\vec{x}}(\vec{F}) = (J^m_{\vec{x}} F_1, \ldots, J^m_{\vec{x}} F_{\mathcal{D}}) \in \mathcal{P}^{\mathcal{D}}$. The multiplication $Q \odot_{\vec{x}} (P_1, \ldots, P_{\mathcal{D}}) := (Q \odot_{\vec{x}} P_1, \ldots, Q \odot_{\vec{x}} P_{\mathcal{D}})$ for $Q, P_1, \ldots, P_{\mathcal{D}} \in \mathcal{P}$ turns $\mathcal{P}^{\mathcal{D}}$ into a $\mathcal{R}_{\vec{x}}$ module, which we denote by $\mathcal{R}^{\mathcal{D}}_{\vec{x}}$. We will examine $\mathcal{R}_{\vec{x}}$ submodules of $\mathcal{R}^{\mathcal{D}}_{\vec{x}}$, and we adopt the unusual convention that $\{0\}$, $\mathcal{R}^{\mathcal{D}}_{\vec{x}}$ and the empty set are all allowed as submodules of $\mathcal{R}^{\mathcal{D}}_{\vec{x}}$.

Warning. Before we proceed, we stress that this section deals with ideals in $\mathcal{P}^m_0(\mathbb{R}^n)$ and how to calculate their closures. At no point in this section do we make any assumption on an ideal $I \lhd \mathcal{P}^m_0(\mathbb{R}^n)$ being of the form $I^m(E)$ for some closed subset $E \subset \mathbb{R}^n$ containing the origin, nor do we show any ideal is of this form. The sets denoted E below play the role of the base space for our bundles, and have nothing to do with closed subsets $E \subset \mathbb{R}^n$ containing the origin that we studied in previous sections of this paper. We use the same letter to stay in line with the standard notation in the literature.

Definition 4.2 (Bundles and sections). Let $E \subset \mathbb{R}^n$ be compact. A *bundle* (over E) is a family

$$\mathcal{H} = \{H_{\vec{x}}\}_{\vec{x} \in E} \text{ such that, for any } \vec{x} \in E, \quad H_{\vec{x}} \text{ is a translate of an } \mathcal{R}_{\vec{x}} \text{ submodule of } \mathcal{R}^{\mathcal{D}}_{\vec{x}}. \tag{4.1}$$

We call $H_{\vec{x}}$ the *fiber* of $\mathcal{H}$ over $\vec{x}$. The bundle (4.1) is called *semi-algebraic* if the set

$$\{(\vec{x}, \vec{P}) \in \mathbb{R}^n \times \mathcal{P}^{\mathcal{D}} : \vec{x} \in E \text{ and } \vec{P} \in H_{\vec{x}}\} \subset \mathbb{R}^n \times \mathcal{P}^{\mathcal{D}}$$

is semi-algebraic. When we say that we *are given* (respectively, *compute*) a bundle, we mean that we are given (respectively, compute) this set[1].

A *section* of the bundle (4.1) is a (C^m) $\mathbb{R}^{\mathcal{D}}$-valued function $\vec{F} = (F_1, \ldots, F_D)$, defined on $\mathbb{R}^n$, such that $J_{\vec{x}} \vec{F} \in H_{\vec{x}}$ for any $\vec{x} \in E$. The C^m norm of a section $\vec{F}$ is defined by

$$\|\vec{F}\|_{C^m} := \sup \left\{ |\partial^\alpha F_i(\vec{x})| \right\}_{1 \leq i \leq \mathcal{D},\, |\alpha| \leq m,\, \vec{x} \in \mathbb{R}^n} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

[1] For detailed discussions of the notion of "computing a set" see Section 2.5 in [8], or more generally [1].

For any $k \in \mathbb{N}$, the *k-norm* of the bundle (4.1), denoted $\|\mathcal{H}\|_k$, is the infimum over all $\mathbb{R} \ni M > 0$ for which the following holds:

$$\text{(4.2)}\quad \begin{aligned} &\text{For any } \vec{x}_1, \dots, \vec{x}_k \in E \text{ there exist } P_1 \in H_{\vec{x}_1}, \dots, P_k \in H_{\vec{x}_k} \text{ such that} \\ &|\partial^\alpha P_i(\vec{x}_i)| \le M \text{ for all } 1 \le i \le k \text{ and all } |\alpha| \le m, \\ &\text{and, for } \vec{x}_i \ne \vec{x}_j,\ |\partial^\alpha (P_i - P_j)(\vec{x}_i)| \le M|x_i - x_j|^{m-|\alpha|} \text{ for all } |\alpha| \le m. \end{aligned}$$

If no such M exists, we set $\|\mathcal{H}\|_k = \infty$.

Given a bundle $\mathcal{H}$, we would like to know whether it has a section, and moreover, if a section exists, we want to know how small can we take its C^m norm. By our convention, $\mathcal{H}$ may have empty fibers, in which case clearly a section does not exist. To answer these questions, [4, 7] proved the existence and studied the properties of the *stable Glaeser refinement* of a bundle. We refer the reader to [4, 7] for the definition and study of these stable refinements, and list here only the properties we will use: given a bundle $\mathcal{H} = \{H_{\vec{x}}\}_{\vec{x}\in E}$, the stable Glaeser refinement of $\mathcal{H}$ is another bundle $\tilde{\mathcal{H}} = \{\tilde{H}_{\vec{x}}\}_{\vec{x}\in E}$. In particular, it satisfies the following.

Theorem 4.3. *There exist an integer constant $k^\# \ge 1$ and positive constants $c, C > 0$, all three depending only on m, n and $\mathcal{D}$, such that the following holds. Let $\mathcal{H} = \{H_{\vec{x}}\}_{\vec{x}\in E}$ be a bundle, and let $\tilde{\mathcal{H}} = \{\tilde{H}_{\vec{x}}\}_{\vec{x}\in E}$ be its stable Glaeser refinement. Then*

(4.3) *$\mathcal{H}$ has a section if and only if $\tilde{\mathcal{H}}_{\vec{x}}$ is non empty for any $\vec{x} \in E$;*

(4.4) *if $\mathcal{H}$ has a section, then $c\|\tilde{\mathcal{H}}\|_{k^\#} \le \inf\{\|\vec{F}\|_{C^m} : \vec{F}$ is a section of $\mathcal{H}\} \le C\|\tilde{\mathcal{H}}\|_{k^\#}$;*

(4.5) *if $\mathcal{H}$ is semi-algebraic, then so is $\tilde{\mathcal{H}}$, and moreover, $\tilde{\mathcal{H}}$ can be computed if $\mathcal{H}$ is given.*

The scalar case ($\mathcal{D} = 1$) of (4.3) and (4.4) are proven in [4]. The general case ($\mathcal{D} \ge 1$) of (4.3) is proven in [7] by reduction to the scalar case. The same reduction can be used to prove the general case of (4.4). The proof of (4.5) is a routine application of standard properties of semi-algebraic sets[2].

Definition 4.4. The *norm* of the bundle (4.1), denoted $\|\mathcal{H}\|$, is defined to be the $k^\#$-norm of (4.1), where $k^\#$ is the integer constant from Theorem 4.3 (formally this $k^\#$ is not unique, so we choose and fix one such $k^\#$).

Taylor's theorem easily implies that there exists a constant C', depending only on m, n and $\mathcal{D}$, such that $\|\mathcal{H}\| \le C'\|\vec{F}\|_{C^m}$ for any section $\vec{F}$ of $\mathcal{H}$.

Definition 4.5 (Parametrized bundles). Let $\hat{E} \subset \mathbb{R}^S$ be any set and let $E \subset \mathbb{R}^n$ be a compact set. We call a point $\xi \in \hat{E}$ a parameter. For each $\xi \in \hat{E}$, let $\mathcal{H}^\xi = \{H^\xi_{\vec{x}}\}_{\vec{x}\in E}$ be a bundle over E. The family $\mathcal{H} = \{H^\xi_{\vec{x}}\}^{\xi\in\hat{E}}_{\vec{x}\in E}$ is called a *parametrized bundle*. The *stable*

[2]More precisely, the stable Glaeser refinement is constructed by iterating finitely many times the process of *Glaeser refinement* (until this process stabilizes, hence the term "stable"). Each iteration is defined by formulae that are first order definable in the language of real fields with parameters from $\mathbb{R}$, and so each iteration preserves the property of the bundle being semi-algebraic. We refer the reader to [4] for further details on the iterated refinement process, and to [3] for exposition on the model theoretic notion of definability in first order languages.

Glaeser refinement of $\mathcal{H}$ is the parametrized bundle $\tilde{\mathcal{H}} = \{\tilde{H}^{\xi}_{\vec{x}}\}^{\xi\in\hat{E}}_{\vec{x}\in E}$, where for each fixed $\xi \in \hat{E}$, $\{\tilde{H}^{\xi}_{\vec{x}}\}_{\vec{x}\in E}$ is the stable Glaeser refinement of $\mathcal{H}^{\xi}$.

A parametrized bundle $\mathcal{H} = \{H^{\xi}_{\vec{x}}\}^{\xi\in\hat{E}}_{\vec{x}\in E}$ is called *semi-algebraic* if the set

$$\left\{(\xi, \vec{x}, \vec{P}) \in \mathbb{R}^S \times \mathbb{R}^n \times \mathcal{P}^{\mathcal{D}} : \xi \in \hat{E}, \vec{x} \in E \text{ and } \vec{P} \in H^{\xi}_{\vec{x}}\right\} \subset \mathbb{R}^S \times \mathbb{R}^n \times \mathcal{P}^{\mathcal{D}}$$

is semi-algebraic. When we say that we *are given* (respectively, *compute*) a parametrized bundle, we mean that we are given (respectively, compute) this set.

The following theorem is a generalization of (4.5), and is (again) a routine application of standard properties of semi-algebraic sets.

Theorem 4.6. *Let $\mathcal{H}$ be a semi-algebraic parametrized bundle, and let $\tilde{\mathcal{H}}$ be its stable Glaeser refinement. Then, $\tilde{\mathcal{H}}$ is semi-algebraic as well, and moreover $\tilde{\mathcal{H}}$ can be computed if $\mathcal{H}$ is given.*

4.2. The algorithm

For the remainder of this section, we fix the following: let $I \lhd \mathcal{P}^m_0(\mathbb{R}^n)$ be an ideal, let $Q_1, \dots, Q_L$ be a basis of I (as a vector space), denote $\Omega := \operatorname{Allow}(I)$, and let $p \in \mathcal{P}^m_0(\mathbb{R}^n)$ be some polynomial. We further assume that $\Omega \neq \emptyset$, as if $\Omega = \emptyset$ then we already know by Corollary 2.17 that any jet in $\mathcal{P}^m_0(\mathbb{R}^n)$ is implied by I, and so $\operatorname{cl}(I) = \mathcal{P}^m_0(\mathbb{R}^n)$.

Definition 4.7. We say that condition $\mathcal{C}(A, \varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$ holds if there exist functions $F, S_1, S_2, \dots, S_L \in C^m(\operatorname{Ann}_4(\rho))$ satisfying (2.8), (2.9) and (2.10).

Definition 4.7 immediately implies the following lemma.

Lemma 4.8. *I implies p if and only if there exists $A > 0$ such that for any $\varepsilon > 0$ there exist $\delta, r > 0$ for which, for all $\rho \in (0, r]$, condition $\mathcal{C}(A, \varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$ holds.*

Scaling $F, S_1, S_2, \dots, S_L \in C^m(\operatorname{Ann}_4(\rho))$ that satisfy $\mathcal{C}(A, \varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$ by setting

$$\tilde{F}(\vec{x}) := \varepsilon^{-1}\rho^{-m}F(\rho\vec{x}) \quad \text{and} \quad \tilde{S}_l(\vec{x}) := A^{-1}S_l(\rho\vec{x}) \quad \text{for any } 1 \le l \le L,$$

one easily sees that condition $\mathcal{C}(A, \varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$ is equivalent to the following condition $\mathcal{C}^*(A, \varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$:

Definition 4.9. We say that condition $\mathcal{C}^*(A, \varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$ holds if there exist functions $\tilde{F}, \tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_L \in C^m(\{1/4 < |\vec{x}| < 4\})$ such that the following hold:

$$|\partial^\alpha \tilde{F}(\vec{x})| \le 1 \quad \text{for } 1/4 < |\vec{x}| < 4 \text{ and all } |\alpha| \le m; \tag{4.6}$$

$$|\partial^\alpha \tilde{S}_l(\vec{x})| \le 1 \quad \text{for } 1/4 < |\vec{x}| < 4, \text{ all } |\alpha| \le m \text{ and all } 1 \le l \le L; \tag{4.7}$$

$$p(\rho\vec{x}) = \varepsilon\rho^m\tilde{F}(\vec{x}) + A\tilde{S}_1(\vec{x})Q_1(\rho\vec{x}) + \cdots + A\tilde{S}_L(\vec{x})Q_L(\rho\vec{x}) \quad \text{for all } \vec{x} \text{ such that } 1/2 < |\vec{x}| < 2 \text{ and } \operatorname{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta. \tag{4.8}$$

So we in fact proved the following lemma (that follows from Lemma 4.8).

Lemma 4.10. *I implies p if and only if there exists $A > 0$ such that for any $\varepsilon > 0$ there exist $\delta, r > 0$ for which, for all $\rho \in (0, r]$, condition $\mathcal{C}^*(A, \varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L)$ holds.*

We will now introduce another condition, and then see how it relates to condition $\mathcal{C}^*(A, \varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L)$.

Definition 4.11. We say that condition $\mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L; A)$ holds if there exist functions $F^*, S_1^*, S_2^*, \ldots, S_L^* \in C^m(\mathbb{R}^n)$ such that the following hold:

$$|\partial^\alpha F^*(\vec{x})| \le A \quad \text{for all } \vec{x} \in \mathbb{R}^n \text{ and all } |\alpha| \le m; \tag{4.9}$$

$$|\partial^\alpha S_l^*(\vec{x})| \le A \quad \text{for all } \vec{x} \in \mathbb{R}^n, \text{ all } |\alpha| \le m \text{ and all } 1 \le l \le L; \quad \text{and} \tag{4.10}$$

$$p(\rho\vec{x}) = \varepsilon\rho^m F^*(\vec{x}) + S_1^*(\vec{x})Q_1(\rho\vec{x}) + \cdots + S_L^*(\vec{x})Q_L(\rho\vec{x}) \quad \text{for all } \vec{x} \text{ such that } 1/2 < |\vec{x}| < 2 \text{ and } \operatorname{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta. \tag{4.11}$$

Fix $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi = 1$ on $\{1/2 < |\vec{x}| < 2\}$ and $\chi = 0$ outside $\{1/4 < |\vec{x}| < 4\}$, and denote $C_\chi = \|\chi\|_{C^m}$ (this is a constant depending only m and n). Assume that condition $\mathcal{C}^*(A, \varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L)$ holds. Then, there exists a constant $\hat{C} > 0$ depending only on C_χ and m (so only on m and n) such that condition $\mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L; \hat{C}A + \hat{C})$ holds: indeed, replacing the functions $\tilde{F}, \tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_L \in C^m(\{1/4 < |\vec{x}| < 4\})$ that satisfy $\mathcal{C}^*(A, \varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L)$ by setting

$$F^*(\vec{x}) := \chi(\vec{x}) \cdot \tilde{F}(\vec{x}) \quad \text{and} \quad S_l^*(\vec{x}) := A \cdot \chi(\vec{x}) \cdot \tilde{S}_l(\vec{x}) \quad \text{for any } 1 \le l \le L,$$

one easily sees that condition $\mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L; \hat{C}A + \hat{C})$ holds.

Vise versa, now assume that condition $\mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L; A)$ holds. Then, condition $\mathcal{C}^*(A, A\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L)$ holds: indeed, if we replace the functions $F^*, S_1^*, S_2^*, \ldots, S_L^* \in C^m(\mathbb{R}^n)$ that satisfy $\mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L; A)$ by setting

$$\tilde{F}(\vec{x}) := A^{-1} \cdot F^*(\vec{x}) \quad \text{and} \quad \tilde{S}_l := A^{-1} \cdot \tilde{S}_l(\vec{x}) \quad \text{for any } 1 \le l \le L,$$

one easily sees that condition $\mathcal{C}^*(A, A\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L)$ holds.

We found that there exists a constant $\hat{C}$ depending only on m, n such that

$$\mathcal{C}^*(A, \varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L) \implies \mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L; \hat{C}A + \hat{C})$$

and

$$\mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L; A) \implies \mathcal{C}^*(A, A\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L).$$

We conclude that the following lemma holds (and now follows easily from Lemma 4.10).

Lemma 4.12. *I implies p if and only if there exists $A > 0$ such that for any $\varepsilon > 0$ there exist $\delta, r > 0$ for which, for all $\rho \in (0, r]$, condition $\mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \ldots, Q_L; A)$ holds.*

We are now ready to introduce the relevant parametrized bundle. We define a bundle

$$\mathcal{H} = \{H_{\vec{x}}^{\xi}\}_{\vec{x} \in E(\xi)}^{\xi \in \hat{E}} \tag{4.12}$$

where

(4.13) $$\hat{E} := (\varepsilon, \delta, r, \rho, p, Q_1, \dots, Q_L) \in (0,\infty)^4 \times \mathcal{P}^{L+1};$$

(4.14) $$E := \{\vec{x} \in \mathbb{R}^n : 1/2 \le |\vec{x}| \le 2\},$$

and for each $\xi \in \hat{E}$ and $\vec{x} \in E$, we define

(4.15) $$H^{\xi}_{\vec{x}} := \big\{(P_0, P_1, \dots, P_L) \in \mathcal{P}^{L+1} : p(\rho\vec{x}) = \varepsilon \rho^m P_0(\vec{x}) + \textstyle\sum_{l=1}^{L} P_l(\vec{x}) Q_l(\rho\vec{x})\big\}$$
$$\text{if } \operatorname{dist}(\vec{x}/|\vec{x}|, \Omega) < \delta \text{ and } 1/2 < |\vec{x}| < 2,$$

and

(4.16) $$H^{\xi}_{\vec{x}} := \mathcal{P}^{L+1} \quad \text{otherwise.}$$

Remark 4.13. The parametrized bundle $\mathcal{H}$ given by (4.12)–(4.16) is semi-algebraic. That follows from the fact that this bundle is given by formulae that are first order definable in the language of real fields with parameters from $\mathbb{R}$. In order to see that the set $\Omega := \text{Allow}(I)$ is semi-algebraic, note that, by the triangle inequality, we have the following formula for this set:

$$\text{Allow}(I) = \Big\{\omega \in S^{n-1} : \text{ there do not exist } c, \delta, r > 0 \text{ such that}$$
$$\frac{|Q_1(\vec{x})| + |Q_2(\vec{x})| + \cdots + |Q_L(\vec{x})|}{|\vec{x}|^m} > c \quad \text{for any } \vec{x} \in \Gamma(\omega, \delta, r)\Big\}.$$

We leave it to the reader to verify the remaining details.

Comparing now (4.12)–(4.16) with (4.9)–(4.11), we readily see that for a fixed $\xi = (\varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$, condition $\mathcal{C}^{**}(\varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L; A)$ holds if and only if $\mathcal{H}^{\xi}$ admits a section with C^m-norm at most A.

We conclude that the following lemma holds (now follows easily from Lemma 4.12).

Lemma 4.14. *I implies p if and only if there exists $A > 0$ such that for any $\varepsilon > 0$ there exist $\delta, r > 0$ for which, for all $\rho \in (0, r]$, the bundle $\mathcal{H}^{\xi}$, with $\xi = (\varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$, admits a section with C^m-norm at most A.*

Now let $\tilde{\mathcal{H}} = \{\tilde{H}^{\xi}_{\vec{x}}\}^{\xi \in \hat{E}}_{\vec{x} \in E}$ be the stable Glaeser refinement of $\mathcal{H}$, and let $k^{\#}$ be the integer constant from Theorem 4.3 (see also remark in parenthesis in the end of Definition 4.4). Then Lemma 4.14 and Theorem 4.3 imply the following:

Proposition 4.15. *I implies p if and only if there exists $M > 0$ such that for any $\varepsilon > 0$ there exist $\delta, r > 0$ for which, for all $\rho \in (0, r]$, denoting $\xi = (\varepsilon, \delta, r, \rho; p, Q_1, \dots, Q_L)$, the following holds:*

For any $\vec{x}_1, \dots, \vec{x}_{k^{\#}} \in E$ there exist $P_1 \in \tilde{H}^{\xi}_{\vec{x}_1}, \dots, P_{k^{\#}} \in \tilde{H}^{\xi}_{\vec{x}_{k^{\#}}}$ such that

$$|\partial^{\alpha} P_i(\vec{x}_i)| \le M \text{ for all } 1 \le i \le k^{\#} \text{ and all } |\alpha| \le m, \text{ and}$$

$$|\partial^{\alpha}(P_i - P_j)(\vec{x}_i)| \le M\, |\vec{x}_i - \vec{x}_j|^{m-|\alpha|}$$
$$\text{for all } 1 \le i, j \le k^{\#} \ (\vec{x}_i, \vec{x}_j \text{ distinct}) \text{ and all } |\alpha| \le m.$$

Proposition 4.15 gives rise to an algorithm to compute the closure of a given ideal I of m-jets: given a basis $Q_1, \dots, Q_L$ for I as a vector space, we form the parametrized bundle $\mathscr{H}$ defined by (4.12)–(4.16), and then compute its stable Glaeser refinement $\tilde{\mathscr{H}}$. We know that $\tilde{\mathscr{H}}$ is semi-algebraic thanks to Remark 4.13 and (4.5). Once $\tilde{\mathscr{H}}$ is known, Proposition 4.15 expresses the condition that I implies p as a first order definable condition in the language of real fields with parameters from $\mathbb{R}$. Consequently, standard semi-algebraic technology allows us to compute the set of all p implied by I as a semi-algebraic set. Since the set of such p is a vector subspace of $\mathscr{P}_0^m(\mathbb{R}^n)$, another application of standard semi-algebraic technology computes a basis for that subspace. Thus, in principle, we have computed the closure of a given ideal I of m-jets.

We have proved the following result.

Theorem 4.16. *Let $I \lhd \mathscr{P}_0^m(\mathbb{R}^n)$ be an ideal. Then, the above procedure computes* (*in principle*) *the closure of I.*

Remark 4.17. Fix $L \in \mathbb{N}$. Then, Theorem 4.16 together with standard semi-algebraic geometry arguments show that there exists a semi-algebraic map that maps any basis of an ideal $Q_1, \dots, Q_L \in \mathscr{P}_0^m(\mathbb{R}^n)$ to a basis for the closure of the ideal, i.e., a basis for $\mathrm{cl}(\langle Q_1, \dots, Q_L\rangle_m)$.

Recall that $\mathscr{P}_0^m(\mathbb{R}^n)$ is a finite dimensional vector space, and fix $P_1, \dots, P_{\mathscr{D}^*}$, some basis (as a vector space) of $\mathscr{P}_0^m(\mathbb{R}^n)$. Then, we can naturally identify $\mathscr{P}_0^m(\mathbb{R}^n)$ with $\mathbb{R}^{\mathscr{D}^*}$. Fix an integer $1 \le L \le \mathscr{D}^*$. We can now identify each ordered set of jets $Q_1, \dots Q_L \in \mathscr{P}_0^m(\mathbb{R}^n)$, not necessarily all different jets, as a point in $\mathbb{R}^{L\cdot\mathscr{D}^*}$. Then, each point in $\mathbb{R}^{L\cdot\mathscr{D}^*}$ defines a vector space $I = \mathrm{span}_{\mathbb{R}}\{Q_1, \dots Q_L\}$ of dimension at most L. So we can think of any point in $\mathbb{R}^{L\cdot\mathscr{D}^*}$ as a vector space in $\mathscr{P}_0^m(\mathbb{R}^n)$ of dimension at most L, where not every two different points represent necessarily different vector spaces.

Note that the condition "$Q_1, \dots, Q_L$ are linearly independent" is first order definable in the language of real fields with parameters from $\mathbb{R}$. Moreover, note that the condition "$\mathrm{span}_{\mathbb{R}}\{Q_1, \dots Q_L\} = \langle Q_1, \dots Q_L\rangle_m$" is also first order definable in the language of real fields with parameters from $\mathbb{R}$. We conclude that the set of points in $\mathbb{R}^{L\cdot\mathscr{D}^*}$ that represent bases of ideals of dimension L is a semi-algebraic set.

Now Remark 4.17 implies the following.

Theorem 4.18. *Fix $1 \le L \le \dim \mathscr{P}_0^m(\mathbb{R}^n)$. Then, the set of points in $\mathbb{R}^{L\cdot\mathscr{D}^*}$ that represent bases of closed ideals of dimension L is semi-algebraic.*

We do not know whether the set of points in $\mathbb{R}^{L\cdot\mathscr{D}^*}$ that represent bases of ideals of dimension L of the form $I^m(E)$ for some closed $E \subset \mathbb{R}^n$ is semi-algebraic. We also do not know whether the set of points in $\mathbb{R}^{L\cdot\mathscr{D}^*}$ that represent bases of ideals of dimension L of the form $I^m(E)$ for some semi-algebraic closed $E \subset \mathbb{R}^n$ is semi-algebraic. These question are closely related to Question 2.19 and Question 2.20.

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