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Dynamics of a one-dimensional nonlinear poroelastic system weakly damped

Manoel Dos Santos, Mirelson Freitas, and Anderson Ramos

Abstract. In this paper, we study the long-time behavior of a nonlinear porous elasticity system. The system is subject to a viscoporous damping and a nonlinear source term which is locally Lipschitz and depends only on the volume fraction. The dynamical system associated with the solutions of the model is gradient, and under the hypothesis of equal speeds of propagation for the waves, we prove that it is also quasi-stable, which allows us to show the existence of a global attractor for the system, which is the main result of the paper.

1. Introduction

Cowin and Nunziato in [11, 12, 27] proposed a model for the mechanical behavior of elastic materials with distributed voids such that the bulk density is the product of two scalar fields, the matrix material density and the volume fraction field, based on the continuum theory for granular materials with void interstices; they established the following constitutive equations (for the one-dimensional homogeneous and isotropic porous elastic solid model with the extrinsic equilibrated body force and body force field vanishing):

$$\begin{aligned} \rho u_{tt} &= T_x, \\ J\phi_{tt} &= H_x + G, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$ is the displacement from the reference configuration, $\phi = \phi(x,t)$ is the change in volume fraction from the reference volume fraction, T is the stress, H is the equilibrated stress vector, G is the intrinsic equilibrated body force, ρ and J are positive constants. Since then, several works have been carried out considering the behavior of system solutions (1.1) as the main object of study.

In [30], Quintanilla considered the system (1.1) with the following constitutive equations:

$$T = \mu u_x + b\phi, \quad H = \delta\phi_x, \quad G = -bu_x - \xi\phi - \tau\phi_t, \tag{1.2}$$

where $\rho, \mu, J, \delta, \xi, \tau$ are positive constants and $b \neq 0$ is a constant satisfying $\mu\xi - b^2 \geq 0$. Moreover, it is assumed that the body occupies the interval $[0, L]$, with $L > 0$. Thus, the

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following system was obtained:

$$
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0,\\
J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t &= 0, \quad (x,t) \in \,]0,L[\times]0,\infty[
\end{aligned} \tag{1.3}
$$

as well as initial conditions defined in appropriate spaces (energy spaces). The term $\tau\phi_t$ in $(1.3)_2$ (which is the rate of change of the volume fraction over time) acts as a damping term for the system (porous dissipation, physically represents viscoporosity). It is worth mentioning as shown in [12] that (in the situation of an isotropic material) b is a constant that depends on the reference volume fraction, and according to (1.2), the sign of b will influence the stress tensor T and intrinsic equilibrated body force G. Note that in (1.3) there is no elastic dissipation. An interesting question is whether the term $\tau\phi_t$ is strong enough to stabilize the full system and if so at which rate? The author used the Routh–Hurwitz theorem to prove that if $\chi \neq 0$, where

$$
\chi := \frac{\rho}{\mu} - \frac{J}{\delta}, \tag{1.4}
$$

then the solutions of (1.3) decay slowly (there is no exponential decay) for boundary conditions like

$$
u(x,t) = \phi_x(x,t) = 0 \tag{1.5}
$$

and

$$
u_x(x,t) = \phi(x,t) = 0 \tag{1.6}
$$

for $x = 0, L$ and $t \in]0,\infty[$. This result means that the porous viscosity is not strong enough to guarantee the exponential stability for the system.

Santos et al. in [32] studied the system (1.3) and by using the semigroup theory of linear operators on Hilbert spaces, due to Gearhart–Herbst–Prüss–Huang (see [18, 20, 21, 29]), proved that the solutions, subject to Dirichlet–Neumann boundary conditions (1.5), decay exponentially if and only if $\chi = 0$; otherwise $\chi \neq 0$. Using a result due to Borichev and Tomilov in [5], the authors showed that the solutions decay polynomially with optimal rate $1/\sqrt{t}$.

Apalara in [2, 3] considered (1.3) subject to the boundary conditions (for $L = 1$)

$$
\begin{aligned}
u_x(0,t) = u(L,t) = \phi(0,t) = \phi_x(L,t) = 0,&\\
u(0,t) = u_x(L,t) = \phi(0,t) = \phi_x(L,t) = 0,&\quad t > 0,
\end{aligned}
$$

and Neumann–Dirichlet boundary conditions (1.6). Using the multiplicative method, the author showed, in all these cases, that the solutions decay exponentially since $\chi = 0$.

If we consider only the effect of viscoelasticity acting on the poroelastic model, we obtain the following system:

$$
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x - \gamma u_{txx} &= 0,\\
J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi &= 0, \quad (x,t) \in \,]0,L[\times]0,\infty[.
\end{aligned} \tag{1.7}
$$

Muñoz-Rivera and Quintanilla in [26] studied the model (1.7) with boundary conditions given by (1.5). The authors proved that the solutions of the system decay polynomially. (In [31], Santos et al. obtained the optimal rate $1/\sqrt{t}$ for the polynomial decay of (1.7).) It is worth mentioning that in this same work the authors studied the one-dimensional poroelastic system subject to a single dissipation mechanism, which is heat conduction. In this case, dissipation is not strong enough to produce exponential decay, but the system decays polynomially.

Magaña and Quintanilla in [23] considered, in addition to the viscoelasticity effect, the viscoporosity effect (when we insert the term $\tau\phi_t$ in the equation $(1.7)_2$). They studied the system

$$\begin{aligned} \rho u_{tt} - \mu u_{xx} - b\phi_x - \gamma u_{txx} &= 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t &= 0, \quad (x,t) \in]0,L[\times]0,\infty[, \end{aligned}$$

with boundary conditions given by (1.5). They showed that the solutions of such a system decay exponentially.

When it is assumed that the microvoid motion is quasi-static (physically, $J \to 0$ in (1.3)), there is a considerable change in the structure of the system, since the coupling of a hyperbolic equation with a parabolic equation; this is,

$$\begin{aligned} \rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \\ \tau\phi_t - \delta\phi_{xx} + bu_x + \xi\phi &= 0, \quad (x,t) \in]0,L[\times]0,\infty[. \end{aligned} \tag{1.8}$$

Magaña and Quintanilla in [24] studied the system (1.8) with boundary conditions given by (1.5). The authors proved the slow decay for the system solutions. (However, Muñoz-Rivera and Quintanilla in [26] proved that the solutions of the system decay polynomially.) Furthermore, they established that, unlike the poroelastic case with viscoelastic effect (1.7), whose solutions decay polynomially, the solutions of the quasi-static microvoid model with viscoelastic effect (when we insert the viscoelasticity term $-\gamma u_{txx}$ into equation $(1.8)_1$) decay exponentially.

Pamplona et al. in [28] studied a poroelastic (one-dimensional) model with thermal effect when viscoelasticity is present; more precisely, they addressed the following system:

$$\begin{aligned} \rho u_{tt} - \mu u_{xx} - b\phi_x + \beta\theta_x - \gamma u_{txx} &= 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi - m\theta &= 0, \\ c\theta_t - k\theta_{xx} + \beta u_{xt} + m\phi_t &= 0, \quad (x,t) \in]0,L[\times]0,\infty[, \end{aligned}$$

where θ represents the temperature of a particle $x \in [0, L]$ at instant t. The boundary conditions of the problem were given by

$$u(0,t) = u(L,t) = \phi_x(0,t) = \phi_x(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0.$$

The authors proved that the solutions of the system do not decay exponentially but decay polynomially.

Recently in [1], Al-Mahdi et al. studied the issue of general decay for the following thermoelastic porous model with memory effect:

$$
\begin{aligned}
&\rho u_{tt} - \mu u_{xx} - b\phi_x + \beta\theta_x = 0,\\
&J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi - m\theta + \int_0^t g(t-s)\phi_{xx}(x,s)ds = 0,\\
&c\theta_t - k\theta_{xx} + \beta u_{xt} + m\phi_t = 0, \qquad (x,t) \in]0,1[\times]0,\infty[,
\end{aligned}
$$

with the following (Neumann–Dirichlet–Dirichlet) boundary conditions:

$$
u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad t \in]0,\infty[.
$$

The authors proved that there exist positive constants t_1, k_1, and k_2 such that

$$
E(t) \le k_2 G_1^{-1}\left(k_1 \int_{t_1}^t \gamma(s)ds\right), \quad \forall t \ge t_1,
$$

where $E(t)$ is the energy of a solution of the system and $G_1(t)$ and $\gamma(t)$ are given functions satisfying

$$
g'(t) \le -\gamma(t)G(g(t)) \quad \text{and} \quad G_1(t) = \int_t^{\varepsilon} \frac{1}{sG'(s)}ds.
$$

Compared to linear ones, there are few works related to the asymptotic behavior of nonlinear poroelastic systems. In this case, it may happen that the objective of the work is to study the decay of the solutions of the system as in [4,22,25] or to study the existence of global attractors (Definition A.2) for the dynamical system associated with the solutions of the system; we will focus on the second. It is noteworthy that the global attractor, when it exists, is unique. Roughly speaking, a global attractor is a set to which a dynamical system evolves after a long period of time. Next, we highlight some works on the existence of global attractors for nonlinear poroelastic systems.

Freitas et al. in [17] considered the following nonlinear poroelastic system:

$$
\begin{aligned}
\rho u_{tt} - \mu u_{xx} - b\phi_x + g_1(u_t) &= f_1(u,\phi),\\
J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + g_2(\phi_t) &= f_2(u,\phi), \quad (x,t) \in]0,L[\times]0,\infty[,
\end{aligned} \tag{1.9}
$$

with boundary conditions given by

$$
u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, \quad t \in]0,\infty[. \tag{1.10}
$$

In this model, the functions g_1 and g_2 represent damping terms, while f_1 and f_2 represent source terms. The authors proved, depending on the hypotheses on terms of damping and source, the existence of blowup of the solutions and the existence of a global attractor.

Freitas in [16] considered the system (1.9)–(1.10) with external sources h_1, h_2 depending on x and on t and obtained a non-autonomous dynamical system associated with the solutions of the system. In this case, the author proved the existence of minimal pullback attractors as well as the upper semicontinuity of the attractor with respect to non-autonomous perturbations.

Dos Santos et al. in [13] studied a nonlinear poroelastic system with delayed visco-porosity; more precisely, they considered the system

$$\begin{aligned} &\rho u_{tt} - \mu u_{xx} - b\phi_x + u_t + f_1(u,\phi) = h_1(x), \\ &\quad J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t \\ &\qquad +\mu_2\phi_t(x,t-\tau) + f_2(u,\phi) = h_2(x), \quad (x,t) \in]0,L[\times]0,\infty[, \end{aligned}$$

with boundary conditions given by (1.10). It was proved in the paper that if $\mu_2 < \mu_1$, then the dynamical system associated with the solutions of the system possesses global and exponential attractors. Furthermore, the fractal dimension of the attractor is finite.

Feng et al. in [15] consider the following nonlinear poroelastic system with infinite memory in the equation of volume fraction and nonlinear friction damping:

$$\begin{aligned} &\rho u_{tt} - \mu u_{xx} - b\phi_x + g(u_t) + f_1(u,\phi) = h_1(x), \\ &J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi \\ &\quad + \int_0^\infty k(s)\phi_{xx}(t-s)ds + f_2(u,\phi) = h_2(x), \quad (x,t) \in]0,L[\times]0,\infty[. \end{aligned}$$

The authors showed that the dynamical system associated with the model possesses a global attractor with finite fractal dimension.

In this paper, we consider the poroelastic system (1.1) with constitutive equations given by

$$\begin{aligned} T &= \mu u_x + b\phi, \\ H &= \delta\phi_x, \\ G &= -bu_x - \xi\phi - \tau\phi_t - f(\phi), \end{aligned} \tag{1.11}$$

with μ, δ, ξ, τ positive constants, f is a nonlinear forcing term, and $b \neq 0$ a constant satisfying

$$b^2 \leq \mu\xi. \tag{1.12}$$

Assuming that the region occupied by the material is the interval [0, 1], from (1.1) and (1.11), we obtain the following system:

$$\begin{aligned} &\rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ &J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t + f(\phi) = 0, \quad (x,t) \in]0,1[\times]0,\infty[. \end{aligned} \tag{1.13}$$

The system (1.13) is subject to the boundary conditions

$$u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, \quad t \in]0,\infty[, \tag{1.14}$$

and initial conditions

$$\begin{aligned} &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\ &\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad x \in [0,1]. \end{aligned} \tag{1.15}$$

The paper is divided as follows: in Section 2, we state the assumptions for the nonlinear source term f as well as establish the notations for the functional spaces used throughout the paper. In Section 3, we study the well-posedness of the system (1.13)–(1.15) from the point of view of semigroup theory of linear operators on Hilbert spaces (let us rewrite the system (1.13)–(1.15) as a first-order initial value problem in a given Hilbert space) and define three types of energy (in addition to the resulting energy from the norm in phase space, we define the modified energy and second energy) for the solutions of the system and conclude the section with the result of existence and uniqueness of global solutions (in time) and continuous dependence on initial data. In Section 4, assuming that $\chi = 0$ in (1.4), we obtain the main result of the work, which is the existence of a global attractor; this result will be obtained by showing that the dynamical system associated with the solutions of the system (1.13)–(1.15) is quasi-stable (quasi-stability is a method introduced by Chueshov and Lasiecka [7–9] originally developed for second-order (in time) nonlinear evolutionary models with nonlinear damping) on bounded positively invariant sets and is gradient.

2. Notations and assumptions

In this paper, we denote by $L^p(0,1)$, $1 \leq p \leq \infty$, the (Lebesgue) space of measurable functions on [0, 1], whose p-th power is Lebesgue integrable endowed with the norm $\|\cdot\|_p$. For $p = 2$, we have the product inner and norm given, respectively, by

$$(u,v)_2 := \int_0^1 u(x)v(x)dx \quad \text{and} \quad \|u\|_2^2 = (u,u)_2, \quad u,v \in L^2(0,1).$$

For $s = 1, 2$, we denote by $H^s(0,1)$ the L_2-based Sobolev space of the order s and by $H_0^1(0,1)$ the closure of $C_0^\infty(0,1)$ in $H^1(0,1)$. We consider in $H_0^1(0,1)$ the Poincaré's inequality

$$\|u\|_2 \leq \|u_x\|_2, \quad \forall u \in H_0^1(0,1).$$

Regarding the source term f in (1.13), we make the following assumptions.

(A1) $f \in C^1(\mathbb{R})$, and there exist constants $p \geq 1$, $C_f > 0$, and $C_F > 0$ such that

$$|f'(s)| \leq C_f(|s|^{p-1} + 1), \qquad \forall s \in \mathbb{R}, \tag{2.1}$$

$$F(s) := \int_0^s f(\sigma)d\sigma \geq -\frac{\beta}{2}s^2 - C_F, \quad \forall s \in \mathbb{R}, \tag{2.2}$$

for some positive constants β, satisfying

$$\beta < \frac{\delta}{2}.$$

(A2) In addition, we assume that

$$f(s)s \geq -\beta s^2 - C_f, \quad \forall s \in \mathbb{R}. \tag{2.3}$$

It follows from (2.1) that f is locally Lipschitz.

3. Well-posedness

The main goal of this section is to study the well-posedness of (1.13)–(1.15). First, we consider the Hilbert space $\mathscr{H} = [H_0^1(0,1)]^2 \times [L^2(0,1)]^2$ endowed with inner product

$$\big((u,\phi,v,\psi),(\tilde{u},\widetilde{\phi},\tilde{v},\widetilde{\psi})\big)_{\mathscr{H}} = \rho(v,\tilde{v})_2 + J(\psi,\widetilde{\psi})_2 + \delta(\phi_x,\widetilde{\phi}_x)_2 + \mu(u_x,\tilde{u}_x)_2 + b(u_x,\widetilde{\phi})_2 + b(\tilde{u}_x,\phi)_2 + \xi(\phi,\widetilde{\phi})_2$$

and norm (induced by inner product above)

$$\begin{aligned}\|(u,\phi,v,\psi)\|_{\mathscr{H}}^2 &= \rho\|v\|_2^2 + J\|\psi\|_2^2 + \delta\|\phi_x\|_2^2 + \mu\|u_x\|_2^2 + 2b(u_x,\phi)_2 + \xi\|\phi\|_2^2 \\ &= \rho\|v\|_2^2 + J\|\psi\|_2^2 + \delta\|\phi_x\|_2^2 + \left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2 \\ &\quad + \left(\mu - \frac{b^2}{\xi}\right)\|u_x\|_2^2. \end{aligned} \tag{3.1}$$

Considering

$$\varphi = \varphi(t) = (u(t),\phi(t),u_t(t),\phi_t(t)) = (u,\phi,u_t,\phi_t) \quad \text{and} \quad \varphi_0 = (u_0,\phi_0,u_1,\phi_1),$$

we rewrite (1.13)–(1.15) in the form of the following initial value problem in $\mathcal{H}$; that is,

$$\begin{cases} \varphi_t(t) = \mathcal{A}\varphi(t) + \mathcal{F}(\varphi(t)), & t > 0, \\ \varphi(0) = \varphi_0, \end{cases} \tag{3.2}$$

where $\varphi_t = \frac{d\varphi}{dt}$, $\mathcal{A} : D(\mathcal{A}) \subset \mathscr{H} \to \mathscr{H}$, and $\mathcal{F} : \mathscr{H} \to \mathscr{H}$ are given by

$$\begin{aligned} \mathcal{A}\begin{pmatrix} u \\ \phi \\ v \\ \psi \end{pmatrix} &= \begin{pmatrix} v \\ \psi \\ -\rho^{-1}(-\mu u_{xx} - b\phi_x) \\ -J^{-1}(-\delta\phi_{xx} + bu_x + \xi\phi + \tau\psi) \end{pmatrix}, \\ \mathcal{F}\begin{pmatrix} u \\ \phi \\ v \\ \psi \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ -J^{-1}f(\phi) \end{pmatrix} \end{aligned} \tag{3.3}$$

with

$$D(\mathcal{A}) = \left\{(u,\phi,v,\psi) \in \mathscr{H};\ u,\phi \in H^2(0,1),\ v,\psi \in H_0^1(0,1)\right\}.$$

The following result can be found in [3, 30, 32].

Lemma 3.1. *The operator $\mathcal{A}$ is dissipative, i.e., $(\mathcal{A}\varphi_0,\varphi_0)_{\mathscr{H}} \le 0$ for any $\varphi_0 \in D(\mathcal{A})$ and $R(I-\mathcal{A}) = \mathscr{H}$, where $R(I-\mathcal{A})$ is the range of operator $I-\mathcal{A}$. Therefore, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$ on $\mathscr{H}$, with $\overline{D(\mathcal{A})} = \mathscr{H}$, where $\overline{D(\mathcal{A})}$ is the closure of $D(\mathcal{A})$.*

Lemma 3.2. *Suppose that f in (1.13) is locally Lipschitz continuous. Then, the operator $\mathcal{F}$ in (3.3) is locally Lipschitz continuous.*

Proof. Let $K > 0$. Then, for any $\varphi_0 = (u_0, \phi_0, v_0, \psi_0)$, $\varphi_1 = (u_1, \phi_1, v_1, \psi_1)$ in $\mathcal{H}$ such that $\|\varphi_0\|_{\mathcal{H}}, \|\varphi_1\|_{\mathcal{H}} \leq K$, from (3.1), we have

$$\begin{aligned} &\delta \|\phi_{0x}\|_2^2 \leq \|\varphi_0\|_{\mathcal{H}}^2 \leq K^2 \text{ and } \delta \|\phi_{1x}\|_2^2 \leq \|\varphi_1\|_{\mathcal{H}}^2 \leq K^2 \\ &\Longrightarrow \|\phi_{0x}\|_2 \leq \delta^{-1/2} K \text{ and } \|\phi_{1x}\|_2 \leq \delta^{-1/2} K. \end{aligned} \tag{3.4}$$

Since $\phi_0, \phi_1 \in H_0^1(0,1)$, it follows from (3.4) and from the continuous embedding

$$H_0^1(0,1) \hookrightarrow C([0,1])$$

(with embedding constant $C_0 > 0$) that

$$\begin{aligned} |\phi_0(x)| &\leq \sup_{x \in [0,1]} |\phi_0(x)| \leq C_0 \|\phi_{0x}\|_2 \leq C_0 \delta^{-1/2} K, \\ |\phi_1(x)| &\leq \sup_{x \in [0,1]} |\phi_1(x)| \leq C_0 \|\phi_{1x}\|_2 \leq C_0 \delta^{-1/2} K, \end{aligned} \tag{3.5}$$

a.e. in $[0,1]$. Let us denote by M_K the Lipschitz constant of f on interval $[-C_0\delta^{-1/2}K, C_0\delta^{-1/2}K]$ (note that M_K depends on K). Therefore, it follows from the local Lipschitz continuity of f and (3.5) that

$$|f(\phi_0(x)) - f(\phi_1(x))| \leq M_K |\phi_0(x) - \phi_1(x)| \quad \text{a.e. in } [0,1]. \tag{3.6}$$

From (3.3), (3.6) and by using Poincaré's inequality, we obtain

$$\begin{aligned} \|\mathcal{F}(\varphi_0) - \mathcal{F}(\varphi_1)\|_{\mathcal{H}}^2 &= J^{-1} \int_0^1 |f(\phi_0) - f(\phi_1)|^2 dx \leq J^{-1} M_K^2 \|\phi_{0x} - \phi_{1x}\|_2^2 \\ &\leq J^{-1} M_K^2 \|\varphi_0 - \varphi_1\|_{\mathcal{H}}^2, \end{aligned}$$

which concludes the proof of the lemma. ∎

Definition 3.3. A *strong solution* to (3.2) on $[0,T)$ is a continuous function

$$\varphi : [0,T) \to \mathcal{H},$$

continuously differentiable on $(0,T)$, $\varphi(t) \in D(\mathcal{A})$ for any $t \in (0,T)$ and (3.2) holds on $[0,T)$.

Definition 3.4. A *mild solution* to (3.2) on $[0,T]$ is a continuous function $\varphi : [0,T] \to \mathcal{H}$ satisfying the following integral equation:

$$\varphi(t) = T(t)\varphi_0 + \int_0^t T(t-s)\mathcal{F}(\varphi(s))ds, \quad 0 \leq t \leq T, \tag{3.7}$$

where $T(t)$ is the semigroup generated by $\mathcal{A}$. Note that a strong solution is a mild solution.

The (natural) energy $E(t)$ and the modified energy (functional) $\mathcal{E}(t)$ of a (mild) solution $\varphi(t) = (u, \phi, u_t, \phi_t)$ to (3.2) are given, respectively, by

$$E(t) = \frac{1}{2}\|\varphi(t)\|_{\mathcal{H}}^2$$

and

$$\mathcal{E}(t) = E(t) + \int_0^1 F(\phi)dx. \tag{3.8}$$

Also, consider the second energy (functional) $\mathcal{E}(t)$ of $U(t)$, defined by

$$\mathcal{E}(t) = \frac{1}{2}\left[\rho\|u_t\|_2^2 + J\|\phi_t\|_2^2 + \delta\|\phi_x\|_2^2 + \left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2\right].$$

It is immediately seen that

$$\mathcal{E}(t) \le E(t).$$

On the other hand, by using Poincaré's inequality, we have

$$\left\|\frac{b}{\sqrt{\xi}}u_x\right\|_2 \le \left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2 + \sqrt{\xi}\|\phi\|_2 \le \left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2 + \sqrt{\xi}\|\phi_x\|_2.$$

Therefore,

$$\|u_x\|_2^2 \le \frac{2\xi}{b^2}\left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2 + \frac{2\xi^2}{b^2}\|\phi_x\|_2^2. \tag{3.9}$$

This means that there exists a positive constant α (independent of $\varphi(t)$) such that

$$\mathcal{E}(t) \le E(t) \le \alpha\mathcal{E}(t). \tag{3.10}$$

Lemma 3.5. *Suppose that (A1)–(A2) hold. Then, for any (strong or mild) solution* $\varphi(t) = (u(t), \phi(t), u_t(t), \phi_t(t))$ *to (3.2) on* $[0, T)$*, its modified energy* $\mathcal{E}(t)$ *satisfies*

$$\frac{1}{4}\|\varphi(t)\|_{\mathcal{H}}^2 - C_f \le \mathcal{E}(t) \le M_0(\|\varphi(t)\|_{\mathcal{H}}^{p+1} + 1), \quad t \in [0, T), \tag{3.11}$$

where $M_0 > 0$ *is independent of* φ*. As a consequence, we have*

$$\lim_{t\to T^-}\|\varphi(t)\|_{\mathcal{H}} < \infty. \tag{3.12}$$

In addition, if φ *is a strong solution, then* $\mathcal{E}(t)$ *is nonincreasing.*

Proof. By integrating twice (2.1), we obtain a (generic) positive constant C_p depending on p and C_f; then using the embedding $H_0^1(0,1) \hookrightarrow L^{p+1}(0,1)$, we obtain

$$\begin{aligned}\int_0^1 F(\phi(t))dx &\le C_p(\|\phi(t)\|_{p+1}^{p+1} + 1) \le C_p(\|\phi_x(t)\|_2^{p+1} + 1)\\ &\le \widetilde{M}_0(\|\varphi(t)\|_{\mathcal{H}}^{p+1} + 1),\end{aligned} \tag{3.13}$$

where $\widetilde{M}_0$ depends on p, on embedding $H_0^1(0,1) \hookrightarrow L^{p+1}(0,1)$, and on δ. On the other hand, from (2.2) and Poincaré's inequality, we have

$$\int_0^1 F(\phi)dx \geq -\frac{\beta}{2}\|\phi\|_2^2 - C_f \geq -\frac{\beta}{4}\|\phi_x\|_2^2 - C_f. \tag{3.14}$$

From (3.13)–(3.14) and (3.8) and taking into account

$$E(t) \leq \|\varphi(t)\|_{\mathcal{H}}^{p+1} + 1,$$

we obtain (3.11).

Let $\varphi(t) = (u(t), \phi(t), u_t(t), \phi_t(t))$ be a strong solution of (3.2). Multiplying $(1.13)_1$ by u_t and $(1.13)_2$ by ϕ_t, integrating by parts over $[0,1]$ and adding the results, we obtain

$$\frac{d\mathcal{E}}{dt}(t) = -\tau\|\phi_t\|_2^2. \tag{3.15}$$

For any $t_1 \leq t_2$ in $[0,T)$, by integrating the above equality from t_1 to t_2 with respect to t, we have

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) = -\int_{t_1}^{t_2} \tau\|\phi_t(t)\|_2^2 dt \leq 0. \tag{3.16}$$

Therefore, $\mathcal{E}(t)$ is nonincreasing; as a consequence of the equality above, one obtains (for strong solution) that

$$\mathcal{E}(t) \leq \mathcal{E}(0), \quad t \in [0,T). \tag{3.17}$$

Therefore, the proof of lemma is complete. ∎

Theorem 3.6 (Existence of global solution). *Suppose that (A1) holds. Then,*

(a) *if $\phi_0 \in \mathcal{H}$, then (3.2) has a unique global mild solution;*

(b) *if $\phi_0 \in D(\mathcal{A})$, then the mild solution obtained in (a) is strong solution;*

(c) *if $\varphi^1(t)$ and $\varphi^2(t)$ are mild solutions to (3.2) with initial data φ_0^1 and φ_0^2, respectively, then there exists a positive constant $c_0 = c_0(\varphi_0^1, \varphi_0^2)$ depending on φ_0^1 and on φ_0^2 such that, for every $T > 0$, we have*

$$\|\varphi^1(t) - \varphi^2(t)\|_{\mathcal{H}} \leq e^{c_0 t}\|\varphi_0^1 - \varphi_0^2\|_{\mathcal{H}}, \quad 0 \leq t \leq T. \tag{3.18}$$

Proof. (a)–(b) Since $\mathcal{A}$ is infinitesimal generator of a C_0-semigroup and $\mathcal{F}$ is locally Lipschitz (because f satisfies (2.1)), it follows from [6, Proposition 4.3.3 and Proposition 4.3.9] that there exists $T_{\max} > 0$ such that if $\phi_0 \in \mathcal{H}$, then (3.2) has a unique mild solution on $[0, T_{\max})$, and if $\phi_0 \in D(\mathcal{A})$, then (3.2) has a unique strong solution on $[0, T_{\max})$. Because (3.12) holds, it follows from [6, Proposition 4.3.3], we have $T_{\max} = \infty$.

(c) Let us assume that $\varphi^1(t)$ and $\varphi^2(t)$ are strong solutions. By (3.11) and (3.17), there exists a positive constant $M = M(\varphi_0^1, \varphi_0^2)$ such that

$$\|\varphi^1(t) - \varphi^2(t)\|_{\mathcal{H}} \leq \|\varphi^1(t)\|_{\mathcal{H}} + \|\varphi^2(t)\|_{\mathcal{H}} \leq M, \quad \forall t \geq 0,$$

and let $c_0 > 0$ be the Lipschitz constant of $\mathcal{F}$ corresponding to closed ball centered at origin and radius M. Since $T(t)$ is a semigroup of contractions, it follows from (3.7) that

$$\begin{aligned}\|\varphi^1(t) - \varphi^2(t)\|_{\mathcal{H}} &\leq \|T(t)[\varphi_0^1 - \varphi_0^2]\|_{\mathcal{H}} + \int_0^t \|T(t-s)[\mathcal{F}(\varphi^1(t)) - \mathcal{F}(\varphi^2(t))]\|_{\mathcal{H}} ds \\ &\leq \|\varphi_0^1 - \varphi_0^2\|_{\mathcal{H}} + \int_0^t \|\mathcal{F}(\varphi^1(t)) - \mathcal{F}(\varphi^2(t))\|_{\mathcal{H}} ds \\ &\leq \|\varphi_0^1 - \varphi_0^2\|_{\mathcal{H}} + c_0 \int_0^t \|\varphi^1(t) - \varphi^2(t)\|_{\mathcal{H}} ds.\end{aligned}$$

Thus, applying the Grönwall's lemma (see [6, Lemma 4.2.1]) to the above inequality, we obtain (3.18). ∎

Remark 3.7. Let $\varphi(t)$ be a (global) mild solution to (3.2) with initial data $\varphi_0 \in \mathcal{H}$ obtained by Theorem 3.6. Since $D(\mathcal{A})$ is dense in $\mathcal{H}$, we can obtain a sequence $\{\varphi_0^n\} \subset D(\mathcal{A})$, with $\varphi_0^n \to \varphi_0$ in $\mathcal{H}$. For each n, by Theorem 3.6, let $\varphi^n(t)$ be the (global) strong solution to (3.2) corresponding to initial data φ_0^n. It follows from [6, Proposition 4.3.7 (ii)] that for any $T > 0$ we have

$$\varphi^n \to \varphi \quad \text{in } C([0,T];\mathcal{H}).$$

In other words, every (global) mild solution to (3.2) is uniform limit on $[0, T]$ of strong solution for any $T > 0$. Therefore, (3.16) and (3.18) hold for mild solution.

Remark 3.8. A mild solution $\varphi(t)$ to (3.2) produces a nonlinear semigroup $S(t)$ on $\mathcal{H}$ given by

$$S(t)\varphi_0 := \varphi(t) = (u(t), \phi(t), u_t(t), \phi_t(t)), \quad \varphi_0 = (u_0, \phi_0, u_1, \phi_1) \in \mathcal{H}. \tag{3.19}$$

Thanks to the regularity of $\varphi(t)$, we have $S(t)$ a C_0-semigroup; thus, $(\mathcal{H}, S(t))$ is a (nonlinear) dynamical system.

Remark 3.9. Considering $X = H_0^1(0,1)^2$, $Y = L^2(0,1)^2$, and $v(t) = (u(t), \phi(t))$, we obtain that a semigroup (3.19) satisfies the Assumption A.5.

4. Global attractor

We denote by $\mathcal{N}$ the set of stationary points (Definition A.1 (c)) of $(\mathcal{H}, S(t))$; in this case, it follows from [6, Lemma 10.2.1] and (3.3) that

$$\begin{aligned}\mathcal{N} &= \{\varphi_0 \in D(\mathcal{A}) |\ \mathcal{A}\varphi_0 + \mathcal{F}(\varphi_0) = 0\} \\ &= \big\{(u_0, \phi_0, 0, 0) \in D(\mathcal{A})\ |\ -\mu u_{xx} - b\phi_x = 0, \\ &\qquad\qquad -\delta\phi_{xx} + bu_x + \xi\phi + f(\phi) = 0\big\}.\end{aligned}$$

Lemma 4.1. *Suppose that (A1)–(A2). Then, the set $\mathcal{N}$ of the stationary points of $(\mathcal{H}, S(t))$ is bounded in $\mathcal{H}$.*

Proof. Let $\varphi_0 = (u_0, \phi_0, 0, 0) \in \mathcal{N}$ be arbitrary. Then, (u_0, ϕ_0) satisfies

$$\begin{cases} -\mu u_{0xx} - b\phi_{0x} = 0, \\ -\delta\phi_{0xx} + bu_{0x} + \xi\phi_0 + f(\phi_0) = 0. \end{cases} \tag{4.1}$$

Multiplying the first equation in (4.1) by u_0 and the second by ψ_0, respectively, taking the sum and integrating over [0, 1], we get

$$\delta\|\phi_{0x}\|_2^2 + \mu\|u_{0x}\|_2^2 + 2b(u_{0x}, \phi_0)_2 + \xi\|\phi_0\|_2^2 = -\int_0^1 f(\phi_0)\phi_0 dx. \tag{4.2}$$

From (2.3) and by using Poincaré's inequality, we obtain

$$-\int_0^1 f(\phi_0)\phi_0 dx \leq \beta\|\phi_0\|_2^2 + C_f \leq \frac{\delta}{2}\|\phi_{0x}\|_2^2 + C_f. \tag{4.3}$$

Combining (4.2) and (4.3), we arrive at

$$\|\phi_0\|_{\mathcal{H}}^2 \leq 2C_f,$$

and the proof is complete. ■

4.1. Gradient system

Lemma 4.2. *Suppose that (A1) holds. Then,*

(a) *$(\mathcal{H}, S(t))$ is gradient (Definition A.1 (d)) with Lyapunov function $\Phi : \mathcal{H} \to \mathbb{R}$ given by*

$$\Phi(u_0, \phi_0, u_1, \phi_1) = \frac{1}{2}\|(u_0, \phi_0, u_1, \phi_1)\|_{\mathcal{H}}^2 + \int_0^1 F(\phi_0)dx; \tag{4.4}$$

(b) *the Lyapunov function (4.4) is bounded from above on every bounded subset of $\mathcal{H}$ and the set $\Phi_R := \{\varphi_0 \in \mathcal{H} | \Phi(\varphi_0) \leq R\}$ is bounded for any $R > 0$.*

Proof. (a) Let $\varphi_0 \in \mathcal{H}$ and $S(t)\varphi_0 = (u(t), \phi(t), u_t(t), \phi_t(t))$. By a density argument, we can assume that $S(t)\varphi_0$ is a strong solution to (3.2). Note that if $\mathcal{E}(t)$ is the modified energy of $S(t)\varphi_0$, then

$$\Phi(S(t)\varphi_0) := \mathcal{E}(t) = \frac{1}{2}\|(u(t), \phi(t), u_t(t), \phi_t(t))\|_{\mathcal{H}}^2 + \int_0^1 F(\phi(t))dx.$$

Then, it follows from Lemma 3.5 that $t \mapsto \Phi(S(t)\varphi_0)$ is nonincreasing. In addition, if $\Phi(S(t)\varphi_0) = \Phi(\varphi_0)$ for all $t \geq 0$ and some $\varphi_0 = (u_0, \phi_0, u_1, \phi_1) \in D(\mathcal{A})$ (in this case, $S(t)$ is strong solution), from (3.15), we have

$$-\tau\|\phi_t\|_2^2 = \frac{d}{dt}\Phi(S(t)\varphi_0) = \frac{d}{dt}\Phi(\varphi_0) = 0, \quad \forall t \geq 0.$$

Thus, $\phi(x,t) = \phi(x)$ for all $t \geq 0$. Then, by $(1.13)_2$, we find

$$bu_x(x,t) = \delta\phi_{xx}(x) - \xi\phi(x) - f(\phi(x)) =: g(x).$$

Integrating the above equality over $[0, x]$, we obtain a function $h = h(t)$ such that

$$u(x,t) = \frac{1}{b}\int_0^x g(s)ds + h(t), \quad \forall t \geq 0.$$

By using the boundary condition $u(0,t) = 0$ for all $t \geq 0$ yields $h(t) = 0$ for all $t \geq 0$. Therefore, u does not depend on t; thus,

$$u(x,t) = u(x) = \frac{1}{b}\int_0^x g(s)ds \Longrightarrow u_t(x,t) = 0, \quad \forall t \geq 0.$$

Then, $S(t)\varphi_0 = \varphi_0 = (u_0, \phi_0, 0, 0)$ is a stationary point of $(\mathcal{H}, S(t))$. This proves that Φ is a strict Lyapunov function for $(\mathcal{H}, S(t))$. Finally, (b) follows promptly from estimates (3.11). The proof is complete. ∎

4.2. Quasi-stability

Theorem 4.3. *If $\chi = 0$ in (1.4), then the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable (Definition A.6) on every bounded positively invariant set $\mathcal{O} \subset \mathcal{H}$. Therefore, it follows from Theorem A.7 that $(\mathcal{H}, S(t))$ is asymptotically smooth.*

Proof. For $i = 1, 2$, let $\varphi^i(t) = S(t)\varphi_0^i = (u^i, \phi^i, u_t^i, \phi_t^i)$ be strong solution to (3.2), with $\varphi_0^i \in \mathcal{O}$, where $\mathcal{O}$ is a bounded positively invariant set of $\mathcal{H}$; then,

$$\varphi(t) = \varphi^1(t) - \varphi^2(t) = (u(t), \phi(t), u_t(t), \phi_t(t)),$$

where $u = u^1 - u^2$ and $\phi = \phi^1 - \phi^2$ is solution to

$$\begin{aligned} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, &\quad (x,t) \in]0,1[\times]0,\infty[, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \tau\phi_t + G(\phi) = 0, &\quad (x,t) \in]0,1[\times]0,\infty[, \end{aligned} \tag{4.5}$$

with boundary conditions

$$u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0$$

and initial conditions

$$\varphi(0) = \varphi^1(0) - \varphi^2(0)$$

and

$$G(\phi) = f(\phi^1) - f(\phi^2).$$

We defined the energy $E(t)$ of $\varphi(t)$ by

$$E(t) = \frac{1}{2}\|\varphi(t)\|_{\mathcal{H}} = \frac{1}{2}\|S(t)\varphi_0^1 - S(t)\varphi_0^2\|_{\mathcal{H}}.$$

The rest of the proof will be divided into the following 8 steps.

Step 1. Multiplying $(4.5)_1$ by u_t, $(4.5)_2$ by ϕ_t, integrating over [0, 1], and adding the results, we obtain

$$\frac{dE}{dt}(t) = -\tau\|\phi_t\|_2^2 - (G(\phi), \phi_t)_2. \tag{4.6}$$

Moreover, since $\mathcal{O}$ is bounded and positively invariant and f is locally Lipschitz, by the same argument as in the proof of Lemma 3.2, there exists a positive constant $M_{\mathcal{O}}$ that depends only on $\mathcal{O}$ but does not depend on $\varphi(t)^i$ such that

$$|f(\phi^1(x,t)) - f(\phi^2(x,t))| \leq M_{\mathcal{O}}|\phi^1(x,t) - \phi^2(x,t)|, \quad (x,t) \in [0,1] \times [0,\infty[. \tag{4.7}$$

Therefore, by using (4.7) and Young's inequality, we have

$$\begin{aligned} -(G(\phi), \phi_t)_2 &\leq M_{\mathcal{O}} \int_0^1 |\phi^1 - \phi^2|\,|\phi_t| dx \leq \frac{M_{\mathcal{O}}^2}{2\tau}\|\phi^1 - \phi^2\|_2^2 + \frac{\tau}{2}\|\phi_t\|_2^2 \\ &\leq \frac{M_{\mathcal{O}}^2}{2\tau}(\|u\|_2^2 + \|\phi\|_2^2) + \frac{\tau}{2}\|\phi_t\|_2^2. \end{aligned} \tag{4.8}$$

It follows from (4.6) and (4.8) that

$$\frac{dE}{dt}(t) \leq -\frac{\tau}{2}\|\phi_t\|_2^2 + C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2),$$

where $C_{\mathcal{O}}$ is a generic constant depending on $\mathcal{O}$.

Step 2. We consider the following functional:

$$I_1(t) := -\rho(u_t, u)_2 - J(\phi_t, \phi)_2 - \frac{\tau}{2}\|\phi\|_2^2.$$

Taking the derivative of $I(t)$ and by using (4.5), we obtain

$$\begin{aligned} \frac{dI_1}{dt}(t) = &-\rho\|u_t\|_2^2 - J\|\phi_t\|_2^2 + \delta\|\phi_x\|_2^2 + \mu\|u_x\|_2^2 + 2b(u_x, \phi)_2 + \xi\|\phi\|_2^2 \\ &+ (G(\phi), \phi)_2. \end{aligned} \tag{4.9}$$

By using a similar argument as in (4.8) and Poincaré's inequality, we have

$$(G(\phi), \phi)_2 \leq C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2) + \|\phi_x\|_2^2. \tag{4.10}$$

Now, applying Young's and Poincaré's inequalities and (3.9), we have

$$\begin{aligned} 2b(u_x, \phi)_2 &\leq \|u_x\|_2^2 + b^2\|\phi_x\|_2^2 \\ &\leq \frac{2\xi}{b^2}\left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2 + \left(\frac{2\xi^2}{b^2} + b^2\right)\|\phi_x\|_2^2. \end{aligned} \tag{4.11}$$

From (4.9)–(4.11), we obtain

$$\begin{aligned} \frac{dI_1}{dt}(t) \leq &-\rho\|u_t\|_2^2 - J\|\phi_t\|_2^2 + C_{1,1}\|\phi_x\|_2^2 + C_{1,2}\left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2 \\ &+ C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2), \end{aligned}$$

where $C_{1,1}$ and $C_{1,2}$ are positive constants independent of $\varphi(t)^i$ and $\mathcal{O}$.

Step 3. Now, we consider

$$\underbrace{I_2(t) := \rho(u_t, w)_2 + J(\phi_t, \phi)_2 + \frac{\tau}{2}\|\phi\|_2^2,}_{1} \quad \text{where } \underbrace{-w_x = \frac{b}{\mu}\phi,\ w(0) = w(1) = 0\,.}_{2} \tag{4.12}$$

Then,

$$\frac{dI_2}{dt}(t) = -\delta\|\phi_x\|_2^2 - \xi\|\phi\|_2^2 - (G(\phi), \phi)_2 + J\|\phi_t\|_2^2 + \mu\|w_x\|_2^2 + \rho(u_t, w_t)_2. \tag{4.13}$$

It follows from (4.12), Poincaré's inequality, and (4.8) that

$$\mu\|w_x\|_2^2 = \frac{b^2}{\mu}\|\phi\|_2^2, \quad \|w_t\|_2^2 \le \|w_{tx}\|_2^2 = \frac{b^2}{\mu^2}\|\phi_t\|_2^2 \tag{4.14}$$

and

$$(G(\phi), \phi)_2 \le C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2) + \frac{\delta}{2}\|\phi\|_2^2 \le C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2) + \frac{\delta}{2}\|\phi_x\|_2^2. \tag{4.15}$$

Combining (4.13), (4.14), and (4.15), we arrive at

$$\frac{dI_2}{dt}(t) \le -\frac{\delta}{2}\|\phi_x\|_2^2 + \left(C_{2,1} + \frac{C_{2,2}}{\varepsilon_1}\right)\|\phi_t\|_2^2 + \varepsilon_1\|u_t\|_2^2 + C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2)$$

for any $\varepsilon_1 > 0$, where $C_{2,1}$ and $C_{2,2}$ are positive constants independent of $\varphi^i(t)$ and $\mathcal{O}$.

Step 4. Now, we consider the following functional:

$$I_3(t) := J\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi, \phi_t\right)_2 + \frac{bJ}{\sqrt{\xi}}(\phi_x, u_t)_2.$$

If $\frac{\rho}{\mu} = \frac{J}{\delta}$ (i.e., $\chi = 0$), then

$$\begin{aligned}\frac{dI_3}{dt}(t) = {} & \frac{\delta b}{\sqrt{\xi}}[\phi_x u_x]_{x=0}^{x=1} + \left(\frac{b^2 J}{\rho\sqrt{\xi}} - \sqrt{\xi}\delta\right)\|\phi_x\|_2^2 - \sqrt{\xi}\left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2 \\ & - \tau\left(\phi_t, \frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_2 - \left(G(\phi), \frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_2 + J\sqrt{\xi}\|\phi_t\|_2^2.\end{aligned} \tag{4.16}$$

By using Hölder's and Young's inequalities, we obtain

$$-\tau\left(\phi_t, \frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_2 \le \frac{\tau^2}{\sqrt{\xi}}\|\phi_t\|_2^2 + \frac{\sqrt{\xi}}{4}\left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2, \tag{4.17}$$

and by (4.8), we obtain

$$-\left(G(\phi), \frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_2 \le C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2) + \frac{\sqrt{\xi}}{4}\left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2. \tag{4.18}$$

Since $\frac{b^2 J}{\rho\sqrt{\xi}} - \sqrt{\xi}\delta \leq 0$, it follows from (4.16), (4.17), and (4.18) that

$$\frac{dI_3}{dt}(t) \leq \frac{\delta b}{\sqrt{\xi}}[\phi_x u_x]_{x=0}^{x=1} - \frac{\sqrt{\xi}}{2}\left\| \frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi \right\|_2^2 + C_{3,1}\|\phi_t\|_2^2 + C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2),$$

where $C_{3,1}$ is a positive constant independent of $\varphi^i(t)$ and $\mathcal{O}$.

Step 5. Now, we consider the following functional:

$$I_4(t) := J\delta(q\phi_t, \phi_x)_2,$$

where $q : [0,1] \to \mathbb{R}$ is a function given by $q(x) = 2 - 4x$. Then,

$$\frac{dI_4}{dt}(t) = -\delta^2[\phi_x^2(1) + \phi_x^2(0)] + 2\delta^2\|\phi_x\|_2^2 - \delta\sqrt{\xi}\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi, q\phi_x\right)_2$$
$$- \delta\tau(q\phi_t, \phi_x)_2 - \delta(qG(\phi), \phi_x)_2 + 2J\delta\|\phi_t\|_2^2.$$

By using Hölder's and Young's inequalities and (4.8), we obtain

$$\frac{dI_4}{dt}(t) \leq -\delta^2[\phi_x^2(1) + \phi_x^2(2)] + \left(C_{4,1} + \frac{C_{4,2}}{\varepsilon_2}\right)\|\phi_x\|_2^2 + C_{4,3}\|\phi_t\|_2^2$$
$$+ \varepsilon_2\left\| \frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi \right\|_2^2 + C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2) \tag{4.19}$$

for any $\varepsilon_2 > 0$, where $C_{4,1}$, $C_{4,2}$, and $C_{4,3}$ are positive constants independent of $\varphi^i(t)$ and $\mathcal{O}$.

Step 6. Now, we consider the following functional:

$$I_5(t) := \rho(qu_t, u_x)_2.$$

Then, considering $\frac{\rho}{\mu} = \frac{J}{\delta}$, we obtain

$$\frac{dI_5}{dt}(t) = -\frac{b^2}{\xi}[u_x^2(1) + u_x^2(0)] + 2\rho\|u_t\|_2^2 + 2\mu\|u_x\|_2^2 + b(q\phi_x, u_x)_2.$$

Applying Hölder's and Young's inequalities, (3.9), and (1.12), we get

$$\frac{dI_5}{dt}(t) \leq -\frac{b^2}{\xi}[u_x^2(1) + u_x^2(0)] + 2\rho\|u_t\|_2^2 + C_{5,1}\|\phi_x\|_2^2$$
$$+ C_{5,2}\left\| \frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi \right\|_2^2, \tag{4.20}$$

where $C_{5,1}$ and $C_{5,2}$ are positive constants independent of $\varphi^i(t)$ and $\mathcal{O}$.

Step 7. From Hölder's and Young's inequalities, for any $\varepsilon > 0$, we have

$$\frac{\delta b}{\sqrt{\xi}}[\phi_x u_x]_{x=0}^{x=1} \leq \frac{\varepsilon b^2}{\xi}[u_x^2(1) + u_x^2(0)] + \frac{\delta^2}{4\varepsilon}[\phi_x^2(1) + \phi_x^2(0)]. \tag{4.21}$$

From (4.19), (4.20), and (4.21), we obtain

$$\begin{aligned}\frac{\delta b}{\sqrt{\xi}}[\phi_x u_x]_{x=0}^{x=1} \leq &-\varepsilon\frac{dI_5}{dt}(t) - \frac{1}{4\varepsilon}\frac{dI_4}{dt}(t) + 2\varepsilon\rho\|u_t\|_2^2 \\ &+ \left[\varepsilon C_{5,1} + \frac{1}{4\varepsilon}\left(C_{4,1} + \frac{C_{4,2}}{\varepsilon_2}\right)\right]\|\phi_x\|_2^2 + \frac{C_{4,3}}{4\varepsilon}\|\phi_t\|_2^2 \\ &+ \left(\varepsilon C_{5,2} + \frac{\varepsilon_2}{4\varepsilon}\right)\left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2 + \frac{1}{4\varepsilon}C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2).\end{aligned}$$

Step 8. Consider the following Lyapunov functional $\mathcal{L}(t)$, given by

$$\mathcal{L}(t) := NE(t) + N_1 I_1(t) + N_2 I_2(t) + I_3(t) + \frac{1}{4\varepsilon}I_4(t) + \varepsilon I_5(t). \tag{4.22}$$

Note that by using (3.9) we can to obtain $\gamma > 0$ such that

$$|\mathcal{L}(t) - NE(t)| \leq \gamma E(t).$$

Therefore, for sufficiently large N, there exist $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t). \tag{4.23}$$

From (4.22), we have

$$\begin{aligned}\frac{d\mathcal{L}}{dt}(t) \leq &-\left[\frac{N\tau}{2} + N_1 J - N_2\left(C_{2,1} + \frac{C_{2,2}}{\varepsilon_1}\right) - C_{3,1} - \frac{C_{4,3}}{4\varepsilon}\right]\|\phi_t\|_2^2 \\ &- \left[N_1\rho - N_2\varepsilon_1 - 2\varepsilon\rho\right]\|u_t\|_2^2 \\ &- \left[\frac{N_2\delta}{2} - N_1 C_{1,1} - \varepsilon C_{5,1} - \frac{1}{4\varepsilon}\left(C_{4,1} + \frac{C_{4,2}}{\varepsilon_2}\right)\right]\|\phi_x\|_2^2 \\ &- \left[\frac{\sqrt{\xi}}{2} - N_1 C_{1,2} - \varepsilon C_{5,2} - \frac{\varepsilon_2}{4\varepsilon}\right]\left\|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right\|_2^2 \\ &+ \left[N + N_1 + N_2 + 1 + \frac{1}{4\varepsilon}\right]C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2).\end{aligned}$$

First, we choose in order

$$N_1 < \frac{\sqrt{\xi}}{12C_{1,2}}, \quad \varepsilon < \min\left\{\frac{\sqrt{\xi}}{12C_{5,2}}, \frac{N_1}{8}\right\}, \quad \text{and} \quad \varepsilon_2 < \frac{\sqrt{\xi}\varepsilon}{3}.$$

Then, we have

$$\frac{\sqrt{\xi}}{2} - N_1 C_{1,2} - \varepsilon C_{5,2} - \frac{\varepsilon_2}{4\varepsilon} > \frac{\sqrt{\xi}}{4} > 0.$$

Now, we choose $N_2 > 0$ such that

$$\frac{N_2 \delta}{2} > N_1 C_{1,1} + \varepsilon C_{5,1} + \frac{1}{4\varepsilon}\left(C_{4,1} + \frac{C_{4,2}}{\varepsilon_2}\right).$$

Next, we consider

$$\varepsilon_1 < \frac{N_1 \rho}{4N_2},$$

and we obtain

$$N_1 \rho - N_2 \varepsilon_1 - 2\varepsilon\rho > \frac{\rho N_1}{2} > 0.$$

Finally, we choose N such that

$$\frac{N\tau}{2} > N_2\left(C_{2,1} + \frac{C_{2,2}}{\varepsilon_1}\right) + C_{3,1} + \frac{C_{4,3}}{4\varepsilon}.$$

Therefore, we can to obtain a positive constant α_0 such that

$$\frac{d\mathcal{L}}{dt}(t) \leq -\alpha_0 \mathcal{E}(t) + C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2).$$

From (3.10) and (4.23), we obtain a positive constant γ_0 such that

$$\frac{d\mathcal{L}}{dt}(t) \leq -\gamma_0 \mathcal{L}(t) + C_{\mathcal{O}}(\|u\|_2^2 + \|\phi\|_2^2).$$

Therefore,

$$\frac{d}{dt}\left[e^{\gamma_0 t}\mathcal{L}(t)\right] \leq C_{\mathcal{O}} e^{\gamma_0 t}(\|u\|_2^2 + \|\phi\|_2^2),$$

implying

$$\mathcal{L}(t) \leq \mathcal{L}(t)e^{-\gamma_0 t} + C_{\mathcal{O}} \int_0^t e^{-\gamma_0 (t-s)}(\|u(s)\|_2^2 + \|\phi(s)\|_2^2)ds.$$

From (4.23) again,

$$E(t) \leq \frac{\gamma_2}{\gamma_1} E(0)e^{-\gamma_0 t} + \frac{C_{\mathcal{O}}}{\gamma_1} \int_0^t e^{-\gamma_0 (t-s)} ds \sup_{0 \leq s \leq t} (\|u(s)\|_2^2 + \|\phi(s)\|_2^2). \tag{4.24}$$

Then, considering

$$\begin{aligned} a(t) &:= e^{c_0 t}, \quad b(t) := \frac{\gamma_2}{\gamma_1} e^{-\gamma_0 t}, \\ c(t) &:= \frac{2C_{\mathcal{O}}}{\gamma_1} \int_0^t e^{-\gamma_0 (t-s)} ds, \quad m_X(u,\phi) = \|u\|_2^2 + \|\phi\|_2^2, \end{aligned} \tag{4.25}$$

the assertion of the lemma follows from (4.24) and (3.18). Therefore, the proof is completed. ∎

4.3. Main result

The main result of this paper is given by the following theorem.

Theorem 4.4. *Suppose that $\chi = 0$ and (A1)–(A2) hold. Then,*

(a) *$(\mathcal{H}, S(t))$ possesses a unique compact global attractor $\mathfrak{A} \subset \mathcal{H}$ which is characterized by the unstable manifold of the set of stationary solutions $\mathcal{N}$, i.e., $\mathfrak{A} = \mathbb{M}_+(\mathcal{N})$;*

(b) *$\mathfrak{A}$ has finite fractal dimension;*

(c) *$\mathfrak{A}$ is bounded in $\mathcal{H}_0 = H^2(0,1)^2 \times H_0^1(0,1)^2$. Moreover, every full trajectory $z = \{(u(t), \phi(t), u_t(t), \phi_t) \mid t \in \mathbb{R}\}$ in $\mathfrak{A}$ has the following regularity properties:*

$$\|(u(t), \phi(t))\|^2_{H^2(0,l)^2} + \|(u_t(t), \phi_t(t))\|^2_{H_0^1(0,l)^2} + \|(u_{tt}, \phi_{tt})\|^2_{L^2(0,l)^2} \leq R_1^2$$

for some constant $R_1 > 0$ dependent of $\mathfrak{A}$.

Proof. (a) Since $(\mathcal{H}, S(t))$ is asymptotically smooth (Theorem 4.3) and gradient, with Lyapunov function Φ satisfying the property (b) of Lemma 4.2, it follows from Theorem A.8 that $(\mathcal{H}, S(t))$ has a global attractor which we will represent by $\mathfrak{A}$, given by $\mathfrak{A} = \mathbb{M}_+(\mathcal{N})$.

(b) Since $(\mathcal{H}, S(t))$ is quasi-stable on the attractor $\mathfrak{A}$ (because it is bounded and invariant), then, by using Theorem A.9, we know that $\mathfrak{A}$ has finite fractal dimension.

(c) Since the system $(\mathcal{H}, S(t))$ is quasi-stable on $\mathfrak{A}$ with $c_\infty = \sup_{t \in \mathbb{R}^+} c(t) = C_{\mathcal{A}} < \infty$, where $c(t)$ is defined in (4.25), it follows from Theorem A.11 that any bounded full trajectory $z = \{(u(t), \phi(t), u_t(t), \phi_t) \mid t \in \mathbb{R}\}$ has the following regularity properties:

$$u_t, \phi_t \in L^\infty(\mathbb{R}, H_0^1(0,1)) \cap C(\mathbb{R}, L^2(0,1)), \quad u_{tt}, \phi_{tt} \in L^\infty(\mathbb{R}, L^2(0,1)).$$

Using (1.13) and noting that the nonlinear term f is locally continuous, we have

$$\begin{aligned} \mu u_{xx} &= \rho u_{tt} - b\phi_x \in L^\infty(\mathbb{R}, L^2(0,1)), \\ \delta\phi_{xx} &= J\phi_{tt} + bu_x + \xi\phi + \tau\phi_t + f(\phi) \in L^\infty(\mathbb{R}, L^2(0,1)). \end{aligned}$$

Therefore, we conclude that $\mathfrak{A}$ is bounded in $\mathcal{H}_0$ and the proof is complete. ∎

5. Conclusion

With the same hypotheses (A1)–(A2) for f and some small adaptations of the calculations performed in this work, it is possible to prove the existence of a global attractor for the following nonlinear porous elastic with delay:

$$\begin{aligned} &\rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ &J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t \\ &\quad + \mu_2 u_t(x, t-\tau) + f(\phi) = 0, \quad (x,t) \in \,]0,1[\, \times \,]0,\infty[, \end{aligned} \tag{5.1}$$

where $\mu_1 > 0$ and μ_2 are constants and subject to the boundary conditions (1.14). The system (5.1) was recently studied by Dos Santos et al. in [14] who established that if

$$|\mu_2| < \mu_1 \quad \text{and} \quad \chi = \frac{\rho}{\mu} - \frac{J}{\delta} = 0, \tag{5.2}$$

then the semigroup associated with the solutions of the system is exponentially uniformly stable. Therefore, using the same calculations found in this work (with minor adaptations), we obtain the following result.

Theorem 5.1. *Suppose that (A1)–(A2) and* (5.2) *hold. Then, the dynamical system generated by solutions of* (5.1) *subject to* (1.14) *possesses a unique global attractor* $\mathfrak{A}_d$ *characterized by unstable manifold emanating from set of its stationary points.*

A. Some prolegomena of infinite-dimensional dynamical systems

In this section, for the convenience of the reader, we will enunciate some definitions and results regarding the theory of infinite-dimensional dynamical systems, necessary for the results contained in this work. More details about the concepts and results presented in this section can be found in [7–10, 19].

In this section, we consider H a Banach space with norm $\|\cdot\|_H$ and $\mathbf{T} = \{T(t)\}_{t\geq 0}$ a strongly continuous semigroup on H. Therefore, the pair $(H, \mathbf{T})$ represents a dynamical system on H.

Definition A.1. Regarding a dynamical system $(H, \mathbf{T})$, we have the following definitions.

(a) $(H, \mathbf{T})$ is *dissipative* if it has an bounded absorbing set; i.e., there exists a bounded set $B \subset H$ such that for any bounded set $D \subset H$ there exists $t_D > 0$ satisfying

$$S(t)D \subset B, \quad \forall t \geq t_D.$$

(b) $(H, \mathbf{T})$ is said to be *asymptotically smooth* if for any bounded positively invariant set $B \subset H$, i.e., $T(t)B \subset B$, there exists a compact set $K \subset \overline{B}$, where $\overline{B}$ is the closure of B, such that

$$\lim_{t\to+\infty} \text{dist}_H(S(t)B, K) = 0,$$

where $\text{dist}_H(A, B)$, for $A, B \subset H$, denotes

$$\text{dist}_H(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_H.$$

(c) We say that $v \in H$ is a *stationary point* of $(H, \mathbf{T})$ if $T(t)v = v$ for any $t > 0$. Let $\mathcal{N}$ be the set of stationary points of $(H, \mathbf{T})$; the *unstable manifold* emanating from $\mathcal{N}$, represented by $\mathbb{M}^u_+(\mathcal{N})$, is the set of all $w \in H$ such that there is a full trajectory $\{\gamma(t);\ t \in \mathbb{R}\}$ satisfying

$$\gamma(0) = w \quad \text{and} \quad \lim_{t\to\infty} \text{dist}_H(\gamma(t), \mathcal{N}) = 0.$$

(d) $(H, \mathbf{T})$ is called *gradient* if there exists a strict Lyapunov function on H; i.e., there exists a continuous function $\Psi : H \to \mathbb{R}$ such that $t \mapsto \Psi(S(t)v)$ is nonincreasing for any $v \in H$, and if $\Psi(S(t)v) = \Phi(v)$ for all $t > 0$ and some $v \in H$, then v is a stationary point of $(H, \mathbf{T})$.

Definition A.2. A compact set $\mathfrak{A} \subset H$ is a *global attractor* for $(H, \mathbf{T})$ if it is an invariant set, i.e., $T(t)\mathfrak{A} = \mathfrak{A}$, for all $t \geq 0$ and uniformly attracts bounded sets; in other words, for every bounded set $B \subset H$, we have

$$\lim_{t \to +\infty} \operatorname{dist}_H(S(t)B, \mathfrak{A}) = 0.$$

Definition A.3. Let $n(M, \gamma)$ be the minimal number of closed balls of radius $\gamma > 0$ which covers a set $M \subset H$. The fractal dimension of a compact set $M \subset H$, represented by $\dim_f^H M$, is defined as the following limit:

$$\dim_f^H M := \lim_{\gamma \to 0} \sup \frac{\ln n(M, \gamma)}{\ln(1/\gamma)}.$$

Definition A.4. An *exponential attractor* to $(H, \mathbf{T})$ is a compact set $\mathfrak{A}_{exp} \subset H$ which is positively invariant set; i.e., $T(t)\mathfrak{A}_{exp} \subset \mathfrak{A}_{exp}$ for all $t \geq 0$, $\mathfrak{A}_{exp}$ has finite fractal dimension in H and $\mathfrak{A}_{exp}$ attracts bounded sets of H at an exponential rate; i.e., for any bounded set $D \subset H$, there exist constants t_D, C_D, and $\gamma_D > 0$ such that

$$\operatorname{dist}_H(S(t)D, \mathfrak{A}_{exp}) \leq C_D e^{-\gamma_D(t - t_D)}, \quad \forall t \geq t_D.$$

Assumption A.5. *Let us consider that the space phase H and evolution operator $T(t)$ has the following format: $H = X \times Y$, where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are reflexive Banach spaces with X compactly embedding in Y, $\|\cdot\|_H^2 = \|\cdot\|_X^2 + \|\cdot\|_Y^2$, and for any $v_0 \in H$, we have $T(t)v_0 = (v(t), v_t(t))$, with*

$$v \in C([0, \infty); X) \cap C^1([0, \infty); Y).$$

Definition A.6. We say that a dynamical system $(H, \mathbf{T})$, satisfying Assumption A.5, is *quasi-stable* on a set $B \subset H$ if there exist nonnegative functions $a, b, c : [0, \infty) \to \mathbb{R}$ and a compact seminorm $m_X(\cdot)$ on X such that $a(t)$ e $c(t)$ are locally bounded, $b(t) \in L^1(0, \infty)$, $\lim_{t \to \infty} b(t) = 0$, and for every $v_1, v_2 \in B$ and $t > 0$, we have

$$\|T(t)v_1 - T(t)v_2\|_H^2 \leq a(t)\|v_1 - v_2\|_H^2$$

and

$$\|T(t)v_1 - T(t)v_2\|_H^2 \leq b(t)\|v_1 - v_2\|_H^2 + c(t) \sup_{0 \leq s \leq t} [m_X(v^1(s) - v^2(s))]^2, \quad \text{(A.1)}$$

where $T(t)v_i = (v^i(t), v_t^i(t))$ for $i = 1, 2$.

Theorem A.7 ([9, Proposition 7.9.4]). *If* $(H, \mathbf{T})$ *satisfies Assumption* A.5 *and is quasi-stable on every bounded positively invariant subset of* H*, then* $(H, \mathbf{T})$ *is asymptotically smooth.*

Theorem A.8 ([9, Corollary 7.5.7]). *If* $(H, \mathbf{T})$ *is asymptotically smooth gradient such that its Lyapunov function* Φ *is bounded from above on every bounded subset of* H*, the set* $\{v \in H; \Phi(v) \leq R\}$ *is bounded for any* $R > 0$*, and its set of stationary points* $\mathcal{N}$ *is bounded, then* $(H, \mathbf{T})$ *has a global attractor* $\mathfrak{A} = \mathbb{M}_+^u(\mathcal{N})$.

Theorem A.9 ([9, Theorem 7.9.6]). *Suppose that* $(H, \mathbf{T})$ *satisfies Assumption* A.5*, possesses a global attractor* $\mathfrak{A}$*, and is quasi-stable on* $\mathbf{A}$*. Then, the fractal dimension of* $\mathfrak{A}$ *is finite.*

Theorem A.10 ([9, Theorem 7.9.9]). *Let* $(H, \mathbf{T})$ *satisfy Assumption* A.5*. Assume that* $(H, \mathbf{T})$ *is dissipative, quasi-stable on some bounded absorbing set* B *and there exists a extended space* $\widetilde{H} \supseteq H$ *such that*

$$\|T(t_1)z - T(t_2)z\|_{\widetilde{H}} \leq C_{B,T}|t_1 - t_2|^\gamma, \quad \forall t_1, t_2 \in [0, T],$$

where $C_{B,T} > 0$ *and* $\gamma \in (0, 1]$ *are constants. Then,* $(H, \mathbf{T})$ *has a generalized exponential attractor* $\mathfrak{A}_{exp} \subset H$ *whose fractal dimension is finite in* $\widetilde{H}$.

Theorem A.11 ([9, Theorem 7.9.8]). *Let* $(H, \mathbf{T})$ *satisfy Assumption* A.5 *and possess a compact global attractor* $\mathfrak{A}$ *such that* $(H, \mathbf{T})$ *is quasi-stable on* $\mathfrak{A}$*. Since* (A.1) *holds, if* $c_\infty = \sup_{t \in [0,\infty)} c(t) < \infty$*, then any full trajectory* $\{\gamma(t) = (v(t), v_t(t)); t \in \mathbb{R}\}$*, in the global attractor has the following regularity properties:*

$$v_t \in L^\infty(\mathbb{R}; X) \cap C(\mathbb{R}; Y), \quad v_{tt} \in L^\infty(\mathbb{R}; Y).$$

In addition, there exists $R > 0$ *depending on* c_∞*, on* μ_X*, and on embedding of* $X \hookrightarrow Y$ *such that*

$$\|v_t(t)\|_X^2 + \|v_{tt}(t)\|_Y^2 \leq R^2, \quad \forall t \in \mathbb{R}.$$

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Manoel Dos Santos
Faculty of Exact Sciences and Technology, Federal University of Pará, Manoel de Abreu Street, 68440-000 Abaetetuba, PA, Brazil; jeremias@ufpa.br

Mirelson Freitas
Faculty of Mathematics, Federal University of Pará, Raimundo Santana Cruz Street, 68721-000 Salinópolis, PA, Brazil; mirelson@ufpa.br

Anderson Ramos
Faculty of Mathematics, Federal University of Pará, Raimundo Santana Cruz Street, 68721-000 Salinópolis, PA, Brazil; ramos@ufpa.br