

2-LC triangulated manifolds are exponentially many

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Abstract. We introduce “ t -LC triangulated manifolds” as those triangulations obtainable from a tree of d -simplices by recursively identifying two boundary $(d - 1)$ -faces whose intersection has dimension at least $d - t - 1$. The t -LC notion interpolates between the class of LC manifolds introduced by Durhuus and Jonsson (corresponding to the case $t = 1$), and the class of all manifolds (case $t = d$). Benedetti and Ziegler proved that there are at most $2^{d^2 N}$ triangulated 1-LC d -manifolds with N facets. Here we prove that there are at most $2^{\frac{d^3}{2} N}$ triangulated 2-LC d -manifolds with N facets. This extends an intuition by Mogami for $d = 3$ to all dimensions. We also introduce “ t -constructible complexes”, interpolating between constructible complexes (the case $t = 1$) and all complexes (case $t = d$). We show that all t -constructible pseudomanifolds are t -LC, and that all t -constructible complexes have (homotopical) depth larger than $d - t$. This extends the famous result by Hochster that constructible complexes are (homotopy) Cohen–Macaulay.

1. Introduction

Since the 1960s, Regge [19–21] and many other physicists and mathematicians, cf., e.g., [1, 2, 17], have worked to develop a discrete version of quantum gravity. In Weingarten’s dynamical triangulations (or “DT”) setup [25], smooth manifolds are approximated by equilateral triangulations. This allows to translate all metric aspects, such as curvature and volume, into simpler combinatorial calculations; for example, the partition function for gravity, which is a path integral over all possible metrics, becomes an infinite sum over all triangulations. The downside of this powerful simplification method is a convergence issue. For example, the partition function diverges to infinity, unless one restricts the sum to triangulations into a certain class, and such class happens to have exponential size. In fact, for any fixed $d \geq 2$, there are more than exponentially many triangulated d -manifolds with N facets. Here two triangulations

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are considered equal if they are “combinatorially isomorphic”: That is, if up to relabeling the vertices they have the same face poset.

In an important step for this program, Durhuus and Jonsson [11] defined “locally constructible” (LC) manifolds as those triangulated manifolds obtainable from a tree of d -simplices by recursively identifying two boundary facets whose intersection has *codimension one*. They proved that LC 3-manifolds are exponentially many [11]; and also in higher dimensions, LC d -manifolds are less than $2^{d^2 N}$ [7]. Since all polytope boundaries are LC, this idea lead to a first proof that polytopes with N facets, in fixed dimension, are exponentially many [7].

Here we define *2-LC manifolds* as those obtainable from a tree of d -simplices by recursively identifying two boundary facets that intersect in *codimension one or two*. We prove that this broader class has also exponential size.

Theorem I (Theorem 3.6). *For fixed $d \geq 3$, the number of combinatorially distinct 2-LC d -manifolds with N facets is smaller than $2^{\frac{d^3}{2} N}$.*

The bound can be extended also to 2-LC *quasimanifolds*, which are pseudo-manifolds with particularly nice face links (Theorem 3.9), but not to arbitrary 2-LC pseudomanifolds, which are more than exponentially many (Remark 3.7). Theorem I gives a precise mathematical formulation and extends to all dimensions an intuition by Mogami [18], who worked on 2-LC 3-spheres. The crucial ingredient for this novel exponential upper bound is the planarity of the links of all $(d - 3)$ -faces. In general, the link of a k -dimensional face in a triangulated d -dimensional manifold (without boundary) is a homology sphere of dimension $(d - k - 1)$. However, since homology spheres that are not spheres exist only in dimension 3 and higher, when $k = d - 3$ all links of $(d - 3)$ -faces are indeed homeomorphic to S^2 .

This brings topology into the picture. Durhuus and Jonsson conjectured in 1995 that all 3-spheres and 3-balls are LC [11]. The conjecture was disproved in 2011 by the first author and Ziegler [7]. The weaker conjecture by Mogami [18] that all 3-balls are 2-LC was also recently disproved by the first author [4]. Thus there is little hope that these combinatorial cutoffs may encompass entire topologies. But there are two other reasons why the LC notion is of mathematical importance, beyond the enumerative aspect mentioned above:

- (a) All LC-triangulable manifolds are simply connected, and conversely, all simply connected “piecewise linear” (PL) manifolds of dimension $\neq 4$ admit an LC subdivision [3].
- (b) All shellable and all constructible manifolds are LC [7].

Both results above are still valid if one replaces “LC” with “2-LC”. This triggers a natural curiosity, namely, whether for the result (b) above, for the 2-LC case, one could say more. Perhaps the “constructible” assumption can be weakened?

To answer this curiosity, we define more generally “ t -LC triangulated manifolds” as those obtainable from a tree of d -simplices by recursively identifying two boundary $(d - 1)$ -faces whose intersection has dimension at least $d - t - 1$. This notion interpolates between LC manifolds (which are the same as 1-LC) and all manifolds (the same as d -LC); the case $t = d - 1$ was also previously studied [4]. In parallel, we introduce “ t -constructible complexes” as a generalization of constructible complexes, which correspond to the $t = 1$ case. Intuitively, t -constructible d -complexes are defined recursively as those obtained by gluing two t -constructible d -complexes at a codimension-one subcomplex whose $(d - t)$ -skeleton is constructible.

With these two new properties, we will prove the following generalization of the well-known result by Hochster [15] that all constructible d -complexes are Cohen–Macaulay.

Theorem II (Propositions 2.6 and 2.10). *All t -constructible d -complexes have homotopical depth larger than $d - t$. Moreover, all t -constructible pseudomanifolds are t -LC.*

The converse of Theorem II is false, even if we restrict ourselves to 3-manifolds. In fact, in [5] there are two explicit examples (with 13 and 16 vertices, respectively) of two 3-spheres containing a non-trivial knot that is realized by just three edges in their 1-skeleton; the knots are the trefoil and the square knot, respectively. These examples have homotopical depth 3 because they are spheres, and are 1-LC by computation [5], but they are not 1-constructible because of the knot [13].

2. t -Constructible versus t -LC

As in [7], to which we refer for all definitions, we shall work with *simplicial regular CW-complexes*: These are finite regular CW-complexes where for every proper face F , the interval $[0, F]$ in the face poset of the complex is Boolean. The facets (i.e., the inclusion-maximal faces) of any simplicial regular CW-complex K are therefore simplices; K is *pure* if all facets have the same dimension. Let σ be a face of K . The *star* of σ in K is the subcomplex $\text{St}(\sigma, K) = \{s \in K \mid \exists \tau \in K \text{ s. t. } \sigma \subset \tau \text{ and } s \subset \tau\}$. The *link* of σ in K is the subcomplex $\text{link}(\sigma, K) = \{\tau \in \text{St}(\sigma, K) \mid \tau \cap \sigma = \emptyset\}$. The *boundary* of K is the subcomplex $\partial K = \{s \in K \mid \exists! \tau \in K \text{ s. t. } s \subsetneq \tau\}$. The faces of K that do not belong to ∂K are called *interior*. If K is a simplicial complex, link and boundary commute, in the sense that $\text{link}(\sigma, \partial K) = \partial \text{link}(\sigma, K)$ for all σ . Moreover, if the dimension of σ is k , for a $(d - 1)$ -face $\tau \in \text{St}(\sigma, K)$ and the corresponding $(d - k - 2)$ -face $\tau' \in \text{link}(\sigma, K)$, we know τ' is a boundary face of the $\text{link}(\sigma, K)$ if and only if τ is a boundary face of K .

By a d -pseudomanifold, we mean a finite regular CW-complex P that is pure d -dimensional, simplicial, and such that each $(d - 1)$ -dimensional cell belongs to at most two d -cells. The boundary of the pseudomanifold P , denoted by ∂P , is the smallest subcomplex of P containing all the $(d - 1)$ -cells of P that belong to exactly one d -cell of P . According to this convention, adopted in [7], a pseudomanifold needs not be a simplicial complex; it might be disconnected; and its boundary might not be a pseudomanifold, as shown by a cone over disjoint segments. A *tree of d -simplices* is a triangulation of the d -dimensional ball whose dual graph is a tree. From now on, we use the word “faces” as synonymous of “cells”. We also adopt the convention that the empty set is a face of dimension -1 .

Definition 2.1 (t -LC). Let $d > 1$ be an integer. Let $t \in \{1, \dots, d\}$. We call t -LC pseudomanifolds the d -dimensional pseudomanifolds obtainable from a tree of d -simplices by recursively identifying two boundary $(d - 1)$ -faces whose intersection has dimension at least $d - 1 - t$.

A t -LC gluing in the boundary of a pseudomanifold is the identification of two boundary facets Δ and Δ' whose intersection is at least $(d - t - 1)$ -dimensional. The glued facets become interior, so they are not available for further gluings. A t -local construction for a pseudomanifold M is a sequence of t -LC gluings that obtains M from some tree of d -simplices T . From Definition 2.1, it is clear that all t -LC pseudomanifolds are also $(t + 1)$ -LC. Three values of t have already been studied in the literature:

- For $t = 1$, the “1-LC” notion is the same as the “LC” notion in [7, 11].
- For $t = d - 1$, “ $(d - 1)$ -LC pseudomanifolds” are the same as the “Mogami pseudomanifolds” introduced in [4] and named after [18].
- For $t = d$, all d -dimensional strongly connected pseudomanifolds are d -LC.

Note that there is a “big jump” from $t = d - 1$ to $t = d$: Any non-simply connected manifold is an example of a d -LC pseudomanifold that is not $(d - 1)$ -LC.

Recall that *constructible complexes* are defined inductively in the following way:

- every simplex, and every 0-complex, is constructible;
- a d -dimensional pure simplicial complex C that is not a simplex is constructible if and only if it can be written as $C = C_1 \cup C_2$, where C_1 and C_2 are constructible d -complexes, and $C_1 \cap C_2$ is a (pure) constructible $(d - 1)$ -complex.

Note that when $d = 1$, “constructible” is synonymous with “connected”.

Definition 2.2 (t -constructible). Let $t \leq d$ be positive integers. We define t -constructibility for d -dimensional simplicial complexes recursively, as follows:

- every simplex is t -constructible;

- a 1-dimensional complex is t -constructible if and only if it is connected;
- a d -dimensional pure simplicial complex C that is not a simplex is t -constructible if and only if it can be written as $C = C_1 \cup C_2$, where C_1 and C_2 are t -constructible d -complexes, and $C_1 \cap C_2$ is a pure $(d - 1)$ -complex whose $(d - t)$ -skeleton is constructible.

Note that when $t = 1$, “1-constructible” is synonymous with “constructible”.

Example 2.3. Inside the boundary of some shellable 3-ball B , pick two triangles that belong to different tetrahedra in B and that share exactly one vertex v . Let X be the subcomplex of ∂B formed by these two triangles. Glue together two identical copies B', B'' of B by identifying the corresponding subcomplexes $X' \equiv X''$. Let P be the resulting 3-dimensional pseudomanifold. The link of v in P is topologically an annulus, so it cannot be constructible: in fact, it is easy to see from the recursive definition that the only constructible 2-manifolds with non-empty boundary are triangulated 2-disks. Hence, P is not constructible, because constructibility is closed under taking links, cf. [8, Section 11.2]. Yet P is 2-constructible, because by taking $C_1 = B'$ and $C_2 = B''$ in the definition above, their intersection is a pure 2-complex X whose 1-skeleton is a connected graph.

Remark 2.4. It is an open question whether all k -skeleta of constructible complexes are themselves constructible. Should the answer to this problem be positive, then it is easy to see, by induction, that if $t < d$ all t -constructible d -complexes are also $(t + 1)$ -constructible.

Recall that a (pure) d -dimensional complex C is called

- *homotopy Cohen–Macaulay* if for any face F , for all $i < \dim \text{link}(F, C)$,

$$\pi_i(\text{link}(F, C)) = 0;$$

- *Cohen–Macaulay*, if for any face F , for all $i < \dim \text{link}(F, C)$,

$$H_i(\text{link}(F, C)) = 0.$$

By Hurewicz theorem, the first notion is stronger. By choosing $F = \emptyset$, one sees immediately that all the homotopy groups from the 0-th to the $(d - 1)$ -st of any homotopy Cohen–Macaulay complex must be trivial. (And analogously, all homology groups from the 0-th to the $(d - 1)$ -st of Cohen–Macaulay complexes are trivial.) Thus any homology sphere that is not simply connected, like the 16-vertex triangulation in [9], is an example of a Cohen–Macaulay complex that is not homotopy Cohen–Macaulay. Recall also that the *homotopical depth* of a simplicial complex C is defined as the maximum k such that the k -skeleton of C is homotopic Cohen–Macaulay [16, Chapter 3.6.1]. The (*homological*) *depth* of C is defined algebraically,

but it can be characterized as the maximum k such that the k -skeleton of C is Cohen–Macaulay [23, Theorem 4.8].

Hence, the next proposition generalizes Hochster’s result that all constructible complexes are (homotopy) Cohen–Macaulay [15].

Definition 2.5 (*t*-homotopy-CM, *t*-CM). Let $t \leq d$ be positive integers. A (pure) d -complex C is called

- *t*-homotopy-CM if it has homotopical depth $> d - t$;
- *t*-CM if it has depth $> d - t$.

Note that “1-homotopy-CM” and “1-CM” are synonymous with “homotopy Cohen–Macaulay” and “Cohen–Macaulay”, respectively.

Proposition 2.6. *Let $t \leq d$ be positive integers. For d -dimensional simplicial complexes, we have the following hierarchy:*

$$\{t\text{-constructible}\} \subset \{t\text{-homotopy-CM}\} \subset \{t\text{-CM}\}.$$

Proof. The claim for $t = 1$ (and arbitrary d) is a well-known result by Hochster [15], so we shall focus on the case $2 \leq t \leq d$. Also, the second inclusion is straightforward, so we shall focus on the first one. Let C be a t -constructible d -complex with N facets. If C is a simplex, then it has homotopical depth d , and $d \geq d - t + 1$. If C is not a simplex, then $C = C_1 \cup C_2$, where C_1 and C_2 are t -constructible d -complexes, and $C_1 \cap C_2$ is a pure $(d - 1)$ -dimensional complex whose $(d - t)$ -skeleton is constructible. By the inductive assumption with respect to N , the $(d - t + 1)$ -skeleta of C_1 and of C_2 are homotopy Cohen–Macaulay. Since we are in the case $t \geq 2$, it is easy to see that their intersection is

$$U \stackrel{\text{def}}{=} (d - t + 1)\text{-skel}(C_1 \cap C_2)$$

(see, e.g., Jonsson [16, Lemma 3.32] for the homotopical statement and Hibi [14, p. 98] for the homological one) that the union of the $(d - t + 1)$ -skeleta of C_1 and C_2 is homotopy Cohen–Macaulay. In other words, both the homotopical depth and the homological depth of $C_1 \cup C_2$ are at least $d - t + 1$. ■

Lemma 2.7. *Let $t \leq d$ be positive integers.*

- (i) *If a d -complex is t -constructible, for all $0 \leq k \leq d$ its k -skeleton is strongly connected.*
- (ii) *If a d -pseudomanifold is t -LC, for all $0 \leq k \leq d$ its k -skeleton is strongly connected.*

Proof. (i) We proceed by induction on the number of d -faces. If C is a simplex, the k -skeleton of C is even shellable [10]. If not, then $C = C_1 \cup C_2$, where C_1 and C_2

are t -constructible d -complexes, and $C_1 \cap C_2$ is a pure $(d - 1)$ -dimensional complex whose $(d - t)$ -skeleton is constructible. By inductive assumption, the k -skeleta of C_1 and of C_2 both have connected dual graphs. Moreover, any $(k - 1)$ -face σ of $C_1 \cap C_2$ is contained in some d -face of C_i , and in particular in some k -faces of C_i for $i = 1, 2$. Therefore, there is an edge “across σ ” in the dual graph of C that connects the dual graphs of the k -skeleta of C_1 and of C_2 .

(ii) The k -skeleton of a tree of simplices is strongly connected. Any subsequent boundary gluing does not destroy this property. ■

Lemma 2.8. *Let $t \leq d$ be positive integers. Let C be a d -dimensional pseudomanifold. If C can be split in the form $C = C_1 \cup C_2$, where C_1 and C_2 are t -LC d -pseudomanifolds and $C_1 \cap C_2$ is a pure $(d - 1)$ -complex with strongly connected $(d - t)$ -skeleton, then C is t -LC.*

Proof. The proof is a direct generalization of that of [7, Lemma 2.23]. Fix a t -local construction for C_1 and C_2 , and call T_i the tree along which C_i is constructed. Pick a $(d - 1)$ -face σ in $C_1 \cap C_2$, which thus specifies a $(d - 1)$ -face in the boundary of C_1 and of C_2 . Let C' be the pseudomanifold obtained by identifying the two copies of σ . Clearly, C' has a t -local construction along the tree obtained by joining T_1 and T_2 by an edge across σ : Just redo the same t -LC gluings of the C_i 's. Now if $C_1 \cap C_2$ consists of only one simplex, then $C = C'$ and we are done. Otherwise, by the assumption, we can label the facets of $C_1 \cap C_2$ by $\sigma_0, \dots, \sigma_m$, so that $\sigma_0 = \sigma$ and for each $k \geq 1$ the facet σ_k intersects the union $\sigma_0 \cup \dots \cup \sigma_{k-1}$ in a subcomplex of $\partial\sigma_k$ of dimension $\geq d - t - 1$. Now for each i , identify the two copies of the facet σ_i inside C' . All these gluings are t -LC, and eventually yield C . ■

Corollary 2.9. *Let C be a d -pseudomanifold. If $C = C_1 \cup C_2$, where C_1 and C_2 are t -LC d -pseudomanifolds and $C_1 \cap C_2$ is a pure $(d - 1)$ -complex of depth $\geq d - t$, then C is t -LC.*

Proof. The $(d - t)$ -skeleton of $C_1 \cap C_2$ is Cohen–Macaulay, hence strongly connected. ■

Proposition 2.10. *Let $t \leq d$ be positive integers. For d -dimensional pseudomanifolds, we have the following hierarchy:*

$$\{t\text{-constructible}\} \subset \{t\text{-LC}\} \subset \{\text{all}\},$$

and for $d \geq t + 2$ all inclusions are strict.

Proof. By Lemma 2.7, the $(d - t)$ -skeleton of any t -constructible $(d - 1)$ -complex is strongly connected. So Lemma 2.8 (or Corollary 2.9, together with Proposition 2.6) immediately implies by induction that all t -constructible complexes are t -LC. To show

the strictness of this inclusion, take $(d - t - 2)$ consecutive cones over the example constructed in Proposition 2.11, part (ii), and apply Proposition 2.11, part (i), and Proposition 2.12 below. Finally, the second inclusion is obvious, and its strictness (already for $d \geq t + 1$) will be shown in Corollary 2.14 below. ■

Proposition 2.11. *Let t be any positive integer.*

- (i) *For any $d > t$, the link of any vertex in a t -constructible d -complex is t -constructible.*
- (ii) *There exists a $(t + 2)$ -dimensional pseudomanifold M that is t -LC, but has a vertex link that is neither t -LC nor t -constructible.*

Proof. (i) We proceed by induction on the number of d -faces. Let C be a t -constructible d -complex and let v be a vertex of C . If C is a simplex, the claim is clear. If not, then $C = C_1 \cup C_2$, where C_1 and C_2 are t -constructible d -complexes, $C_1 \cap C_2$ is a pure $(d - 1)$ -complex, and $S := (d - t)$ -skel($C_1 \cap C_2$) is constructible. If v is not in C_1 , then $\text{link}(v, C) = \text{link}(v, C_2)$, so the claim follows by inductive assumption, since C_2 has fewer facets. The case $v \notin C_2$ is symmetric. So without loss of generality, we may assume $v \in C_1 \cap C_2$. Set $L := \text{link}(v, C)$ and $L_i := \text{link}(v, C_i)$ ($i = 1, 2$). Clearly, $L = L_1 \cup L_2$. By induction assumption, both L_1 and L_2 are t -constructible. It is easy to see that $L_1 \cap L_2 = \text{link}(v, C_1 \cap C_2)$ is pure $(d - 2)$ -dimensional. Thus to conclude that L is t -constructible, we need to show that the $(d - 1 - t)$ -skeleton of $L_1 \cap L_2$ is constructible. Since constructibility is closed under taking links, and since S is constructible, it suffices to show that

$$(d - 1 - t)\text{-skel}(L_1 \cap L_2) = \text{link}(v, S). \tag{2.1}$$

Let us prove relation (2.1). If σ is in the left-hand side, then there is a $(d - 2)$ -face F of $L_1 \cap L_2$ containing σ . The $(d - 1)$ -face $v * F$ is thus a facet of $C_1 \cap C_2$. Since $v * F$ contains $v * \sigma$, it follows that $v * \sigma$ is in S , so σ is in $\text{link}(v, S)$. Conversely, if σ is in $\text{link}(v, S)$, then $v * \sigma$ is contained in some $(d - 1)$ -face F of $C_1 \cap C_2$. But then the $(d - 2)$ -face $F' = F \setminus \{v\}$ contains σ and belongs to $L_1 \cap L_2$. So σ is contained in the $(d - 1 - t)$ -skeleton of $L_1 \cap L_2$.

(ii) Let H_i be the $(t + 1)$ -simplex of consecutive vertices $[i, i + 1, \dots, i + t + 1]$. Let $m \geq t + 3$. Consider the “tree of m d -simplices”

$$H_1, H_2, \dots, H_m$$

and let P be the simplicial complex obtained from it by gluing together the first and the last vertex, i.e., vertices 1 and $m + t + 1$. Clearly, P is strongly connected, but the link of 1 in P consists of two disjoint t -simplices. Now choose any shellable $(t + 2)$ -ball B that contains in its boundary a copy of P . Glue together two copies

of B by identifying the corresponding copies of P , and call M the resulting $(t + 2)$ -dimensional pseudomanifold. By Lemma 2.8, M is t -LC, because the t -skeleton of P is strongly connected. However, the link of 1 inside M consists of two $(t + 1)$ -balls glued together at a disjoint union of two t -simplices: In other words, the link of 1 inside M is homeomorphic to $\mathbb{S}^1 \times \mathbb{I}^t$, and thus homotopy equivalent to a 1-sphere. Since it is not simply connected, the link of 1 inside M is not t -LC, and in particular not t -constructible. In fact, by Proposition 2.6, any triangulation of the link of 1, being homeomorphic to $\mathbb{S}^1 \times \mathbb{I}^t$, is not t -constructible. In particular, M (and any subdivision of it) cannot be t -constructible by part (i) above. ■

Proposition 2.12. *Let $t \leq d$ be positive integers. Let C be a d -pseudomanifold. Let v be a new point. Then C is t -LC if and only if $v * C$ is t -LC.*

Proof. The “only if” direction is easy: Suppose C is obtained from a tree of d -simplices T with a sequence of t -LC gluings, where the i -th gluing identifies faces F_i and F'_i with intersection of dimension $\geq d - 1 - t$. Then $v * C$ is obtained from the tree $v * T$ of $(d + 1)$ -simplices with the sequence of gluings that at the step i glues together $v * F_i$ and $v * F'_i$; and the intersection

$$(v * F_i) \cap (v * F'_i) = v * (F_i \cap F'_i)$$

has dimension $\geq d - t$, so all these steps are legitimate t -LC gluings.

The “if” direction is perhaps more surprising because, as we saw in Proposition 2.11, the t -LC property is not maintained by links. Yet a similar argument of [7, Proposition 3.25] works. Suppose $v * C$ is t -LC and let $T_i \rightarrow T_{i+1}$ be any step in some t -local construction of $v * C$. This step glues two d -faces F and G of ∂T_i sharing a $(d - t)$ -face σ . Since F and G will end up in the interior of $v * C$, both contain a copy of v , since $C \subset \partial(v * C)$. If σ contains v' , a copy of v , then by gluing $F = v' * F'$ with $G = v' * G'$, we glue $(d - 1)$ -faces F' and G' sharing the $(d - t - 1)$ -face τ , where $\sigma = v' * \tau$. But if σ does not contain a copy a v , then $\sigma \in C$. Gluing F and G corresponds to possibly many gluings of $(d - 1)$ -faces F' and G' , where F', G' share σ and do not contain any copy of v . Hence performing these gluings one by one, following the t -local construction of $v * C$, we eventually obtain a t -local construction for C . ■

Corollary 2.13. *Let $t \leq d$ be positive integers. If a d -pseudomanifold is t -LC, its suspension is t -LC.*

Proof. Let A be a t -LC d -pseudomanifold. Let v_1, v_2 be two new vertices. Let $C_i = v_i * A$. By Proposition 2.12, each C_i is t -LC. Glue C_1 and C_2 together by identifying the two copies of A . By Lemma 2.7, A has strongly connected $(d + 1 - t)$ -skeleton. By Lemma 2.8, $C_1 \cup C_2$, which is the suspension of A , is t -LC. ■

Corollary 2.14. *For any $d \geq 2$, for any $1 \leq t \leq d - 1$, there exists a d -pseudomanifold that is $(t + 1)$ -LC, but not t -LC.*

Proof. By induction on d . For $d = 2$, any surface different from the sphere is 2-LC, but not 1-LC [11]. For $d \geq 3$, if $t = d - 1$, any non-simply connected d -manifold is $(t + 1)$ -LC, but not t -LC. If instead $t \leq d - 2$, by the inductive assumption there is a $(d - 1)$ -pseudomanifold that is $(t + 1)$ -LC, but not t -LC; coning over it, by Proposition 2.12 we conclude. ■

Remark 2.15. The pseudomanifold constructed in the previous corollary is not a manifold. At the moment, we do not know an explicit example of a d -manifold that is 2-LC but not 1-LC. (See also Example 3.10 below.) A good candidate for $d = 5$ might be the double suspension S of the 16-vertex Poincaré homology sphere by Björner and Lutz [9]. This S is not 1-constructible, because it is not PL; some experiments with the random discrete Morse algorithm [6] seem to suggest that S is likely not 1-LC either. On the positive side, we do know that S is (at most) 3-LC, in view of Corollary 2.13, part (ii), applied twice.

Corollary 2.16. *For any $d \geq 3$, not all triangulated d -balls are 2-LC.*

Proof. In view of Proposition 2.12, we only need to construct a 3-ball that is not 2-LC (or equivalently, not Mogami), a task that was already carried out in [4]. ■

Remark 2.17. For any $t \leq d$, being t -LC is a property that is algorithmically recognizable, simply by trying all possible “spanning trees of d -simplices” and all possible boundary matchings. Moreover, for any $t \leq d - 1$, being t -LC implies being simply connected. Thus in view of the Poincaré conjecture (proven for $d \geq 5$ by Smale [22] and for $d = 4$ by Freedman [12]), if M is a d -manifold with the homology of a sphere, the fact that $M - \Delta$ is t -LC for some facet Δ implies that M is a d -sphere. So were all d -balls t -LC for some $t \leq d - 1$, then for any manifold M we could decide if M is a d -sphere or not just, first by checking whether M has the homology of a sphere, and then by checking whether M minus some facet is t -LC. In conclusion, Novikov’s theorem on the algorithmic unrecognizability of d -spheres for $d \geq 5$ [24] implies that for every $d \geq 5$ and every $t \leq d - 1$, there must exist d -balls that are not t -LC. It is conjectured that also 4-balls are not algorithmically recognizable, which would imply in the same non-constructive way that some 4-balls are not 3-LC.

3. An exponential bound for 2-LC manifolds

Since 1-LC d -manifolds are exponentially many, while d -LC d -manifolds (which is the same as saying “all d -manifolds”) are more than exponentially many, a natural

question is whether one can give exponential bounds for t -LC d -manifolds also for some t larger than 1. In this section, we realize a first step in this direction: For fixed $d \geq 3$, we prove that there are less than $2^{\frac{d^3}{2}N}$ combinatorially distinct simplicial 2-LC d -manifolds with N facets (Theorem 3.6).

3.1. Excluding some gluings

First we establish which 2-LC gluings can actually lead to a manifold without boundary.

Lemma 3.1. *Only the 2-LC gluings satisfying all the conditions below can lead to a triangulated manifold without boundary:*

- (i) *preserving orientability of links of $(d - 3)$ -faces (Figure 1 (a));*
- (ii) *planar with respect to the involved links of $(d - 3)$ -faces (Figure 1 (b));*
- (iii) *impacting only on the boundaries of the links of $(d - 3)$ -faces.*

Moreover, the number of ways we can glue the boundary facets to one another is completely determined by taking into account only gluings corresponding to edges of the boundaries of links of $(d - 3)$ -faces.

Proof. Let T be a tree of d -simplices. The link of a $(d - 3)$ -face σ in T is a triangulated disk, whose boundary circle is the boundary link of σ . Consider two boundary facets E_1 and E_2 sharing σ , and corresponding to edges $[a_0, a_1] \sim [b_0, b_1]$ in

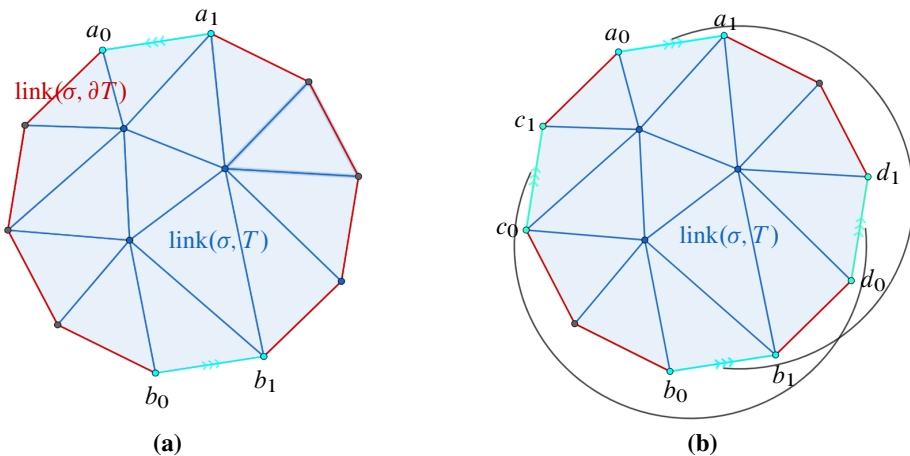


Figure 1. Ways of gluing that do not lead to a manifold without boundary. (a) This gluing makes the link non-orientable, so the link will never be homeomorphic to S^2 . (b) A non-planar gluing, by the Jordan–Schoenflies theorem, makes the final link not homeomorphic to S^2 .

link(σ, T). There are two options for such an identification: $a_0 \sim b_1$ and $a_1 \sim b_0$, or $a_0 \sim b_0$ and $a_1 \sim b_1$. As displayed in Figure 1 (a), the first option makes the link non-orientable, which cannot be fixed by any gluing. Also, if we further glue other two boundary facets E_3 and E_4 with respect to σ , it is clear that any “non-planar matching” creates a contradiction with the Jordan–Schoenflies theorem, cf. Figure 1 (b). So (i) and (ii) are clear. Now, any gluing $E_1 \sim E_2$ can affect the link of some other $(d - 3)$ -face δ in two possible ways:

(1) If δ is contained in E_i for some $i = 1, 2$, then there is a $(d - 3)$ -face δ' such that $\delta \sim \delta'$ as a consequence of $E_1 \sim E_2$. Suppose by contradiction that $\text{link}(\delta, T)$ and $\text{link}(\delta', T)$ are, or become, connected by an edge $[a_0, a_1]$ which is an interior edge of one of these links, say of $\text{link}(\delta, T)$. Then $[a_0, a_1]$ corresponds to an interior $(d - 1)$ -face F_1 of T in $\text{St}(\delta, T)$ and it also corresponds to a $(d - 1)$ -face F_2 of T in $\text{St}(\delta', T)$. After the identification, F_1 and F_2 share the $(d - 3)$ -face $\delta \sim \delta'$ and the edge $[a_0, a_1]$. In order to end up with a simplicial complex, we need to identify F_1 and F_2 at some point, which is not possible since F_1 is an interior $(d - 1)$ -face. En passant, note that $\text{link}(\delta, T)$ and $\text{link}(\delta', T)$ become connected with at least one boundary edge, since they are contained in E_i . Suppose the links are, or become, connected by a vertex v that is an interior vertex of one of the original links, say of $\text{link}(\delta, T)$. In that case, we create an S^1 just around v (containing no other vertex) in the interior of $\text{link}(\delta, T)$. In the link of $\delta \sim \delta'$ in the new complex, v appears on both sides of the S^1 , because now the two original links are connected by a boundary edge and at the same time no interior triangle or interior edge is identified. Which is a contradiction.

(2) If δ is contained in any E_i , it is not identified with another $(d - 3)$ -face. Here, it may happen that an edge or a vertex gets identified within the link itself. We can use the same reasoning as for the first case to conclude that no identifications can happen in the interior of the link.

The arguments above work not only for a tree of d -simplices T , but also for any pseudomanifold obtained from T by performing 2-LC gluings that satisfy the conditions of this lemma. After connecting links or after an identification within one link, an interior edge of the link of a $(d - 3)$ -face can only correspond to

- either one interior $(d - 1)$ -face (if it was an interior edge of the link already);
- or two boundary facets (if it is a glued boundary edge of the link).

All other options cannot lead to a manifold. So the only possible issue is an interior edge $[a_0, a_1]$ in the link of a $(d - 3)$ -face that corresponds to two distinct boundary facets. Assume E_1, E_2 are the two boundary facets in a pseudomanifold M obtained from T after some number of allowed 2-LC gluings; assume also that E_1 and E_2 are in $\text{St}(\delta, \partial M)$, where δ is a $(d - 3)$ -face of M , and that E_1, E_2 share an edge $[a_0, a_1] \in \text{link}(\delta, M)$. Now suppose we need to identify $[a_0, a_1]$ with another edge

$[b_0, b_1] \in \text{link}(\delta, M)$. The edge $[b_0, b_1]$ corresponds to (at least one) $(d - 1)$ -face of $\text{St}(\delta, \partial M)$. Now we have three $(d - 1)$ -faces sharing δ and $[a_0, a_1] \sim [b_0, b_1]$. So to get a simplicial complex at the end, we need to identify all three of them, which is not possible. In conclusion, no identification can be performed within interiors of links of $(d - 3)$ -faces. Which proves part (iii).

As for the final claim: If there is an interior edge $[a_0, a_1]$ of $\text{link}(\delta, M)$ that corresponds to two distinct boundary facets E_1 and E_2 in $\text{St}(\delta, \partial M)$ for some $(d - 3)$ -face δ of M , then E_1 and E_2 share the face δ and $[a_0, a_1]$. Hence, they have to be identified at some point, in order to get a simplicial complex at the end of the 2-local construction. So the edge $[a_0, a_1]$ does not add to the number of ways in which one can glue boundary facets. Note that E_1, E_2 can be identified with respect to δ by a 2-LC gluing. ■

The link of any $(d - 3)$ -face during a 2-local construction consists topologically of possibly punctured disks, connected by boundary edges and vertices to one another, or even to themselves.

Lemma 3.2. *The 2-LC gluings that force an identification between two distinct connected components of the boundary of the link of a $(d - 3)$ -face do not lead to a manifold.*

Proof. Denote M a pseudomanifold obtained from a tree of d -simplices by performing 2-LC gluings that satisfy the conditions of Lemma 3.1 and this lemma. Assume for a contradiction that we identify vertices v_0 and v_1 from two distinct connected components of $\partial \text{link}(\delta, M)$, where δ is a $(d - 3)$ -face of M . We then create an S^1 in the interior of the link with the connected component containing v_0 fully inside this S^1 , and the connected component containing v_1 fully outside. In the link of δ in the new complex, $v_0 \sim v_1$ appears on both sides of the S^1 , which is a contradiction using Jordan–Schoenflies theorem as we need this link to eventually become S^2 . ■

3.2. Bounding the 2-LC gluings

Lemma 3.3. *Let σ be a $(d - 3)$ -face in a tree T of N d -simplices. Let m be the number of facets of ∂T containing σ . Let $M(m)$ be the number of ways we can glue those boundary facets among each other. Then we have the inequality*

$$M(m) \leq C_m \leq 4^m, \tag{3.1}$$

where $C_m = \frac{1}{m+1} \binom{2m}{m}$ is the m -th Catalan number.

Proof. When we glue with respect to the $(d - 3)$ -face σ , we choose a boundary facet containing σ and

- (a) either we leave it unidentifed (Figure 2 (a)), which corresponds to choosing an edge of $\partial\text{link}(\sigma, T)$ and “leaving it alone”;
- (b) or we glue it with another boundary facet containing σ (Figure 2 (b)), which corresponds to choosing two edges of $\partial\text{link}(\sigma, T)$ and matching them.

When an edge in the link is “left alone”, this has no impact on how the remaining edges should be matched. Therefore, the number of gluing configurations with the first gluing of the type (a) is $M(m - 1)$. Instead, when we glue a pair of edges, we divide the boundary of the link of σ into two parts (Figure 2 (b)): the first contains m_1 edges (dashed lines in the figure), the second m_2 edges (dotted lines in the figure), with $m_1 + m_2 = m - 2$. Since only planar gluings are allowed, the number of gluings with a first gluing of type (b) is thus $M(m_1) \cdot M(m_2)$. So

$$M(m) = M(m - 1) + \sum_{m_1+m_2=m-2} M(m_1)M(m_2). \tag{3.2}$$

Set $M(0) = M(1) = 1$. By definition, $M(m) \leq M(m + 1)$. Moreover,

$$\sum_{m_1+m_2=m-2} M(m_1)M(m_2) \leq \sum_{m_1+m_2=m-1} M(m_1)M(m_2) - M(m - 1). \tag{3.3}$$

Putting inequalities (3.2) and (3.3) together, we get by induction

$$M(m) \leq \sum_{m_1+m_2=m-1} M(m_1)M(m_2) \leq C_m < 4^m.$$

This completes the proof. ■

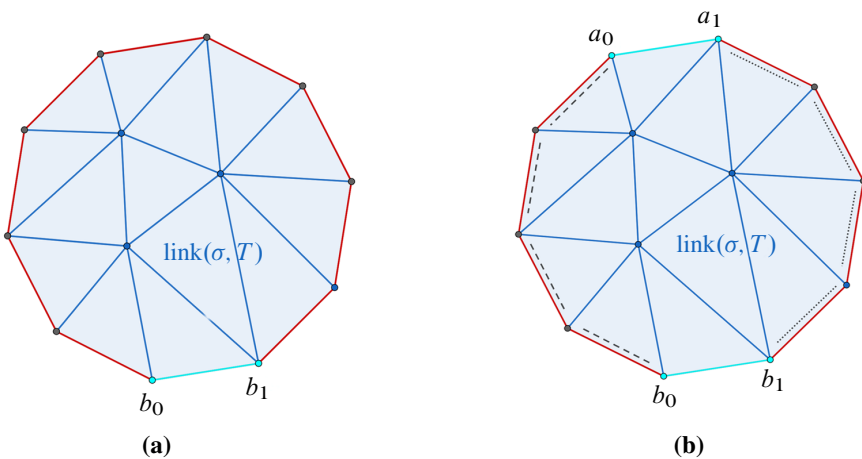


Figure 2. Two options for gluing in the very first step.

Lemma 3.4. Consider an intermediate complex K in a 2-local construction. The number $M(m)$ bounds the number of possible gluing configurations among the boundary facets containing a given $(d - 3)$ -face of K .

Proof. Let m be the number of boundary edges of $\text{link}(\sigma, K)$. By Lemma 3.2, we can partition the edges into sets S_1, S_2, \dots, S_k such that each S_i contains m_i edges that can be glued among themselves but cannot be glued to any edge outside S_i . Since $\sum m_i = m$, by (3.1)

$$M(m_1)M(m_2) \cdots M(m_k) \leq 4^{m_1+m_2+\cdots+m_k} = 4^m. \quad \blacksquare$$

3.3. Proof of the main theorem

Lemma 3.5. Let N and d be positive integers, $d \geq 3$. Let T be a tree of N d -simplices. Let N_i be the number of boundary i -faces in T . Set $D := 1 + \frac{N(d-1)}{2}$. Then

$$N_{d-1} = 2D, \quad N_{d-2} = dD, \quad N_{d-3} = \frac{d}{6}(Nd^2 + 2N - 3Nd + 3d - 3). \quad (3.4)$$

Proof. Note that all the i -faces with $i \leq d - 2$ are on the boundary of T . It is easy to see that $N_{d-1} = 2 + N(d - 1)$, or in other words $N_{d-1} = 2D$. By counting, or by [7], we get

$$N_{d-2} = \frac{d}{2}(N(d - 1) + 2) = dD.$$

Now, any d -simplex contains

$$\binom{d + 1}{d - 2} = \frac{(d + 1)d(d - 1)}{6}$$

$(d - 3)$ -faces, and any $(d - 1)$ -simplex contains

$$\binom{d}{d - 2} = \frac{d(d - 1)}{2}$$

$(d - 3)$ -faces. So

$$\begin{aligned} N_{d-3} &= N \frac{(d + 1)d(d - 1)}{6} - (N - 1) \frac{d(d - 1)}{2} \\ &= \frac{d}{6}(Nd^2 + 2N - 3Nd + 3d - 3). \quad \blacksquare \end{aligned}$$

Theorem 3.6. For any $d \geq 3$, the number of combinatorially distinct 2-LC d -manifolds with N facets, for N large, is smaller than $2^{\frac{d^3}{2}N}$.

Proof. By [7, Corollary 4.3], the number of trees with N facets is at most

$$\left(d \binom{d}{d-1}\right)^{d-1} < (de)^N. \tag{3.5}$$

So we want to obtain an exponential upper bound for all manifolds obtainable from a given, fixed tree. Any tree T of N d -simplices has $2D = 2 + N(d - 1)$ boundary facets. Hence D disjoint couples need to be glued together. Following [11], we partition any such a perfect matching into rounds. The first round consists of couples that share a $(d - 3)$ -face in the boundary of the initial tree. Recursively, the $(i + 1)$ -st round consists of all couples that get to have common $(d - 3)$ -faces only after a gluing in the i -th round. Denote by m_1 (respectively, by $m_{1,\sigma}$) the number of the boundary facets glued in the first round (respectively, glued in the first round with respect to σ). Choose a set \mathfrak{F} of n_1 $(d - 3)$ -faces, the ones with respect to which we shall perform the first round of gluings. We have

$$\sum_{\sigma \in \mathfrak{F}} m_{1,\sigma} = m_1.$$

Therefore, the number of possible gluing configurations in the initial tree, while fixing the n_1 $(d - 3)$ -faces in \mathfrak{F} , is bounded by

$$\prod_{\sigma \in \mathfrak{F}} M(m_{1,\sigma}) \leq \prod_{\sigma \in \mathfrak{F}} 4^{m_{1,\sigma}} = 4^{m_1}.$$

There are $\binom{N_{d-3}}{n_1}$ ways to choose the n_1 $(d - 3)$ -faces in \mathfrak{F} . As $n_1 \leq N_{d-3}$, the number of possibilities in the first round is therefore at most

$$\sum_{n_1=1}^{\frac{m_1}{2}} \binom{N_{d-3}}{n_1} 4^{m_1} < \sum_{n_1=0}^{N_{d-3}} \binom{N_{d-3}}{n_1} 4^{m_1} = 2^{N_{d-3}} 4^{m_1}.$$

Now for each couple in the first round, at most $\left(\frac{d(d-1)}{2} - 1\right)$ distinct $(d - 3)$ -faces become identified. So the total number of identified $(d - 3)$ -faces in the first round is at most

$$L_2 := \frac{m_1}{2} \left(\frac{d(d-1)}{2} - 1\right).$$

We select n_2 out of these L_2 $(d - 3)$ -faces for the second round. The number of possible gluing configurations in the second round of gluings, while fixing the n_2 $(d - 3)$ -faces, is bounded by

$$\prod_{\sigma \text{ is one of the } n_2 \text{ } (d-3)\text{-faces}} 4^{m_{2,\sigma}} = 4^{m_2}.$$

The number of options for the second round is therefore at most

$$\sum_{n_2=1}^{L_2} \binom{L_2}{n_2} 4^{m_2} < \sum_{n_2=0}^{L_2} \binom{L_2}{n_2} 4^{m_2} = 2^{L_2} 4^{m_2}.$$

The same way, the number of possibilities in the i -th round, $i \geq 2$, is at most

$$\sum_{n_i=1}^{L_i} \binom{L_i}{n_i} 4^{m_i} < 2^{L_i} 4^{m_i},$$

where

$$L_i := \frac{m_{i-1}}{2} \left(\frac{d(d-1)}{2} - 1 \right).$$

The following is therefore an upper bound on the total number of 2-LC d -manifolds without boundary that can be constructed from a given tree:

$$\begin{aligned} & \sum_{f=1}^D \sum_{\substack{m_1, m_2, \dots, m_f \\ \sum m_i = 2D \\ m_i \text{ even}, m_i \geq 2}} \left(\sum_{n_1=0}^{N_{d-3}} \binom{N_{d-3}}{n_1} 4^{m_1} \right) \left(\sum_{n_2=0}^{L_2} \binom{L_2}{n_2} 4^{m_2} \right) \dots \left(\sum_{n_f=0}^{L_f} \binom{L_f}{n_f} 4^{m_f} \right) \\ & \leq \sum_{f=1}^D \sum_{\substack{m_1, m_2, \dots, m_f \\ \sum m_i = 2D \\ m_i \text{ even}, m_i \geq 2}} 2^{N_{d-3}} \cdot 2^{\sum_{i=2}^f L_i} \cdot 4^{\sum_{i=1}^f m_i} \\ & \leq \sum_{f=1}^D 2^{\frac{d}{6}(Nd^2 + 2N - 3Nd + 3d - 3)} \cdot 2^{D \left(\frac{d(d-1)}{2} - 1 \right)} \cdot 4^{2D} \cdot \sum_{\substack{m_1, m_2, \dots, m_f \\ \sum m_i = 2D \\ m_i \text{ even}, m_i \geq 2}} 1 \end{aligned} \tag{3.6}$$

$$= 2^{\frac{d}{6}(Nd^2 + 2N - 3Nd + 3d - 3) + D \left(\frac{d(d-1)}{2} - 1 \right) + 4D} \cdot \sum_{f=1}^D \binom{D-1}{f-1} \tag{3.7}$$

$$= 2^{N \frac{d-1}{12} (5d^2 - 7d + 24) + d(d-1) + 3}. \tag{3.8}$$

Some explanation: Inequality (3.6) follows from relations (3.4) and from

$$\sum_{i=1}^f m_i = 2D \quad \text{and} \quad \sum_{i=2}^f L_i = \frac{1}{2} \sum_{i=2}^f m_i \left(\frac{d(d-1)}{2} - 1 \right) \leq \frac{1}{2} 2D \left(\frac{d(d-1)}{2} - 1 \right),$$

which hold because at the end, we glue all the $2D$ boundary faces. Inequality (3.7) follows from the fact that the number of compositions of $2D$ with f parts all even, is the same as the number of compositions of D with f parts, which is $\binom{D-1}{f-1}$. Equality (3.8) is by Newton’s binomial formula $\sum_{f=1}^D \binom{D-1}{f-1} = 2^{D-1}$, and the definition

of D . Consequently, via inequality (3.5), the number of 2-LC d -manifolds without boundary with N facets is at most

$$(de)^N \cdot 2^N \frac{d-1}{12} (5d^2 - 7d + 24) \cdot 2^{d(d-1)+3} < 2^N \frac{5d^3}{12} \cdot 2^{d(d-1)+3} < 2^N \frac{d^3}{2}.$$

The same proof works also for d -manifolds with boundary: we simply stop the matching process earlier. ■

Remark 3.7. The exponential bound of Theorem 3.6 cannot be extended from manifolds to pseudomanifolds. In fact, already for $d = 3$, the family of the cones $v * S$, where S is any triangulated surface and v is a new vertex, shows that 2-LC 3-pseudomanifolds are more than exponentially many. Nevertheless, it is possible to expand Theorem 3.6 by only allowing those 2-LC pseudomanifolds that are obtained with gluings that satisfy all the conditions from Lemmas 3.1 and 3.2. This can be formally rephrased as follows.

Definition 3.8. A 2-LC quasimanifold is any pseudomanifold obtainable from a tree of d -simplices by performing only 2-LC gluings that are

- orientable with respect to the involved links of $(d - 3)$ -faces;
- planar with respect to the involved links of $(d - 3)$ -faces;
- with an impact only on the boundaries of the links of $(d - 3)$ -faces;
- not forcing any identification between two distinct connected components of the boundary of the link of a $(d - 3)$ -face.

Moreover, we require any two $(d - 1)$ -faces that share an interior edge of the link of a $(d - 3)$ -face to be glued together. With the same proof of Theorem 3.6, we conclude the following assertion.

Theorem 3.9. For any $d \geq 3$, the number of combinatorially distinct 2-LC d -quasimanifolds with N facets, for N large, is smaller than $2^{\frac{d^3}{2}N}$.

Example 3.10. In the boundary of a tree of tetrahedra, choose two triangles σ_1 and σ_2 that intersect at a vertex and that do not belong to the same tetrahedron or to adjacent tetrahedra. Let P be the 3-dimensional pseudomanifold obtained by gluing σ_1 and σ_2 . Then P is a 2-LC quasimanifold. However, P cannot be LC, because it is not homeomorphic to any of the possible topologies of LC 3-pseudomanifolds, as characterized by Durhuus and Jonsson [11, Theorem 2]. This example highlights how the class of 2-LC d -quasimanifolds, bounded by Theorem 3.9, is much larger than the class of LC d -quasimanifolds, already for $d = 3$.

Remark 3.11. We conjecture that 3-LC d -manifolds are more than exponentially many for every $d \geq 3$. Note that a much stronger statement, “3-LC d -spheres are

more than exponentially many for every $d \geq 3$ ”, would be immediately implied, via suspensions, by a positive solution to Gromov’s problem of whether there are more than exponentially many 3-spheres.

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