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A note on closedness of convex hull of sets

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Abstract. The closedness of certain sets plays an important role in mathematics, especially in convex and functional analysis. In this short note, we give some connections between closedness of Minkowski sum of sets, closedness of the convex hull of sets, and other properties of the topological vector space.

1. Introduction

Let X be a topological vector space over the field of real numbers. For $A, B \subset X$, we define a *Minkowski sum* of these sets as

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and *scalar multiplication* as

$$\lambda A = \{\lambda a : a \in A\}.$$

Since for $\lambda \neq 0$ the function $f_\lambda : X \to X$, $f_\lambda(x) = \lambda x$ is a linear homeomorphism, therefore if A is closed, compact, open, dense, convex, then so is the set λA for $\lambda \neq 0$. If $\lambda = 0$, then obviously $\lambda A = \{0\}$.

Similarly, if A, B are compact, open, dense, then so is the set $A + B$.

But the situation when the sets A and B are closed is more complicated; namely, the sum of two closed sets need not to be closed, what can be seen by taking

$$A = \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}, \quad B = \mathbb{Z} \subset \mathbb{R},$$

where $\mathbb{Z}$ denotes the set of integers.

Generally, the following can be shown.

Let X be a real topological vector space, and let $A, B \subset X$. If A is closed and B is compact, then the set $A + B$ is closed.

But the connections of the closedness of the Minkowski sum of sets with other properties of the topological vector space as reflexivity, separation, etc. may be really interesting (see [5]).

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For a subset A of topological vector space X, by $\mathrm{conv}(A)$ we denote the *convex hull* of the set A, i.e., the smallest (with respect to inclusion) convex set that contains a set A. It is easy to see that

$$\mathrm{conv}(A) = \bigcup_{n=1}^{\infty} \big\{s_1 x_1 + \dots + s_n x_n : s, \dots, s_n \geq 0,\, s_1 + \dots + s_n = 1,\, x_1, \dots, x_n \in A\big\}.$$

By $\overline{\mathrm{conv}}(A)$ we denote the *closed convex hull* of the set A, i.e., the closure of the set $\mathrm{conv}(A)$.

Notice that the convex hull of the closed set need not to be closed even if the set A is compact.

If the sets A and B are convex, then

$$\mathrm{conv}(A \cup B) = \big\{sa + tb : s, t \geq 0,\, s + t = 1,\, a \in A,\, b \in B\big\}.$$

In this short note, we give some connections between closedness of Minkowski sum of sets, closedness of the convex hull of sets, and other properties of the topological vector space X.

Main results

We start with the following theorem.

Theorem 1. *Let X be a topological vector space, and let $A, B \subset X$ be a convex subsets. Consider the sets $A_1 = \{0\} \times A$, $B_1 = \{1\} \times B \subset \mathbb{R} \times X$. If the set $\mathrm{conv}(A_1 \cup B_1)$ is closed, then the set $A + B$ is closed.*

Proof. Denote by τ the linear topology given on X. Consider a topological vector space $X_1 = \mathbb{R} \times X$ with the product topology.

Let

$$A_0 = \{0\} \times A, \quad B_1 = \{1\} \times B.$$

Assume that $\mathrm{conv}(A_1 \cup B_1)$ is closed. Denote by $\pi : X_1 \to \mathbb{R}$ the canonical projection. Since π is continuous and

$$\left\{\frac{1}{2}\right\} \times \left(\frac{1}{2}A + \frac{1}{2}B\right) = \mathrm{conv}(A_1 \cup B_1) \cap \pi^{-1}\left(\left\{\frac{1}{2}\right\}\right),$$

the set $\{\frac{1}{2}\} \times (\frac{1}{2}A + \frac{1}{2}B)$ is closed and it implies the closedness of the set $\frac{1}{2}A + \frac{1}{2}B$. Now, from the equality

$$A + B = 2\left(\frac{1}{2}A + \frac{1}{2}B\right),$$

we obtain that the set $A + B$ is closed. ∎

Remark 1. The inverse theorem is not true since for sets

$$A = \{0\} \subset \mathbb{R}, \quad B = [0, \infty) \subset \mathbb{R}$$

we have

$$A + B = [0, \infty)$$

and

$$\operatorname{conv}(A_1 \cup B_1) = ((0, 1] \times (0, \infty)) \cup ([0, 1] \times \{0\}),$$

so the set $A + B$ is closed, but $\operatorname{conv}(A \cup B)$ is not closed.

Remark 2. The sets A_1, B_1 in Theorem 1 cannot be replaced by the sets A, B even if the sets A, B are closed bounded and convex since in any nonreflexive Banach space X there exists closed bounded and convex sets $A, B \subset X$ such that $A \subset B$ and $A + B$ is not closed. To see this, let $f : X \to \mathbb{R}$ be a linear functional with norm 1 which does not attain its norm on the unit ball

$$B(0, 1) = \{x \in X : \|x\| \leq 1\}.$$

Let $K_n = \{x \in X : f(x) \geq 1 \text{ and } \|x\| \leq n\}$. It can be shown that $K_n + B(0, 1)$ is not closed set for every $n \in \mathbb{N}$ (see [5]). But for sufficiently large n there exists $x_0 \in X$ such that $x_0 + B(0, 1) \subset K_n$; obviously, the sets $A = x_0 + B(0, 1)$, $B = K_n$ satisfy the conditions that $\operatorname{conv}(A \cup B) = K_n$ is closed bounded and convex set, but $A + B$ is not closed.

Remark 3. It is known that Minkowski sum of compact set and closed set is a closed set. The sets A, B defined in Remark 1 show that convex hull of compact convex set and closed convex set need not to be closed. So, the closedness of sum of sets does not imply the closedness of convex hull of sets even if the sets are both closed and convex and one of them is compact.

Now, we show the following theorem.

Theorem 2. *Let X be a real topological vector space and A a convex and compact subset of X. If B is closed convex and bounded subset of X, then the set* $\operatorname{conv}(A \cup B)$ *is closed.*

Proof. Let (z_t), $z_t \in \operatorname{conv}(A \cup B)$ be a net convergent to some z_0. Then,

$$z_t = \alpha_t a_t + \beta_t b_t,$$

where $\alpha_t, \beta_t \geq 0$, $\alpha_t + \beta_t = 1$, $a_t \in A$, $b_t \in B$.

By assumptions, there exist a subnet (z_σ) and $\alpha_0 \in [0, 1]$, $a_0 \in A$ such that $z_\sigma \to z_0$, $a_\sigma \to a_0$, $\alpha_\sigma \to \alpha_0$.

We have to consider two cases.

(1) If $\alpha_0 = 1$, then because B is bounded therefore $\beta_\sigma b_\sigma \to 0$, and hence,

$$z_0 = a_0 \in A \subset \operatorname{conv}(A \cup B).$$

(2) If $\alpha_0 \neq 1$, then by the equality

$$b_\sigma = \frac{z_\sigma - \alpha_\sigma a_\sigma}{\beta_\sigma}$$

the subnet b_σ is convergent to some $b_0 \in B$, and hence,

$$z_0 = \alpha_0 a_0 + \beta_0 b_0 \in \mathrm{conv}(A \cup B),$$

which proves that the set $\mathrm{conv}(A \cup B)$ is closed. ■

Corollary 1. *Let X be a reflexive Banach space, and let $A, B \subset X$ be closed convex and bounded sets; then, the set* $\mathrm{conv}(A \cup B)$ *is closed.*

Proof. If a Banach space is reflexive, then any closed bounded and convex set is compact in the weak topology (see [1]), and by Theorem 2, the convex hull of any two such sets is closed. ■

Remark 4. If X is a Banach space and for any two closed bounded and convex sets $A, B \subset X$ the closedness of the set $\mathrm{conv}(A \cup B)$ implies the closedness of $A + B$, then X must be a reflexive Banach space. It can be easily seen by taking sets $A = K_n$ and $B = x_0 + B(0, 1)$ constructed in Remark 2. We still do not know whether closedness of convex hull of any two closed bounded and convex subsets of Banach space X implies the reflexivity of X.

Remark 5. Denote by $\mathcal{B}(X)$ the family of all nonempty, closed, convex, and bounded subsets of topological vector space X. If X is a reflexive Banach space, then the structure $(\mathcal{B}(X), +, \cdot, \subset)$ is a partially ordered abstract convex cone. For $A, B \in \mathcal{B}(X)$, we have

$$\sup\{A, B\} = \mathrm{conv}(A \cup B).$$

These properties play an important role in embedding theorems (see [2–4, 6–8]).

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