Vanishing first cohomology and strong 1-boundedness for von Neumann algebras

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Abstract. We obtain a new proof of Shlyakhtenko's result which states that if G is a sofic, finitely presented group with vanishing first ℓ^2-Betti number, then $L(G)$ is strongly 1-bounded. Our proof of this result adapts and simplifies Jung's technical arguments which showed strong 1-boundedness under certain conditions on the Fuglede–Kadison determinant of the matrix capturing the relations. Our proof also features a key idea due to Jung which involves an iterative estimate for the covering numbers of microstate spaces. We also use the works of Shlyakhtenko and Shalom to give a short proof that the von Neumann algebras of sofic groups with Property (T) are strongly 1 bounded, which is a special case of another result by the authors.

1. Introduction

A *tracial von Neumann algebra* is a pair (M, τ) of a finite von Neumann algebra and a faithful normal tracial state. For every group G, there is an associated tracial von Neumann algebra, the von Neumann algebra $L(G)$ generated by the left regular representation of G on $\ell^2(G)$ with the trace given by $\langle \delta_e, (\cdot)\delta_e \rangle$, and a major theme of operator algebraic research has been how the properties of a group (algebraic, analytic, geometric, etc.) are reflected by its von Neumann algebra.

In particular, one may consider finitary approximations of the group in several senses: A group is *sofic* if the group trace can be approximated by almost representations in permutation groups; on the other hand, $L(G)$ is *Connes embeddable* if the same holds for representations in unitary groups, or if the group can be approximated by $*$-representations in matrices. Voiculescu's free entropy dimension was introduced to quantify the *amount* of approximations by matrices for a given tuple x in a von Neumann algebra [28, 29]. The standard generators for a free group $\mathbb{F}_n$ for $n \geq 2$ have many approximations, and Voiculescu used this fact to deduce that the von Neumann algebra has no Cartan subalgebras [29]. The free entropy approach has had several other applications to free group von Neumann algebras (and more generally free products) [10, 13, 15, 18, 25]. A related notion of *strong 1-boundedness* was introduced by Jung (see [20]); this is a strengthening of the condition of having free entropy dimension 1, with the useful property that it is independent of the choice of generating set. The first author reformulated strong 1-boundedness

through a numerical invariant h, called the 1-bounded entropy, which is finite if and only if M is strongly 1-bounded (see [15]).

The work of Jung [19] and Shlyakhtenko [25] investigated strong 1-boundedness in the context of polynomial relations in a $*$-algebra. In particular, Shlyakhtenko explicitly connected this to ℓ^2-Betti numbers of groups. In this paper, we present an alternative proof of Shlyakhtenko's result [25] that finitely presented sofic groups with vanishing first ℓ^2-Betti number are strongly 1-bounded (results in this direction are in Jung's paper but under somewhat restrictive hypotheses). Shlyakhtenko's result generalized Jung's earlier work [19] but with a different proof strategy using non-microstates free entropy rather than microstates free entropy. We give a purely microstates proof that streamlines Jung's original ideas and clarifies the essential ingredients and limitations of this approach. The statement of the theorem is as follows.

Theorem 1.1 ([25]). *If G is a sofic finitely presented group with vanishing first ℓ^2-Betti number, then $L(G)$ is strongly 1-bounded.*

Shlyakhtenko proved Theorem 1.1 by obtaining a key technical free probabilistic fact involving non-microstates theory and Fisher's information. Using this in combination with an inequality between the microstates and non-microstates free entropy dimensions, he obtains, as a corollary, the following generalization of Jung's result [19, Theorem 6.9].

Theorem 1.2. *Let (M, τ) be a tracial W^*-algebra generated by some $x \in M_{\mathrm{sa}}^d$. Suppose that $\|x\|_\infty < R$. Let $m \in \mathbb{N} \cup \{\infty\}$. Let $f(t_1, \ldots, t_d) \in \mathbb{C}\langle t_1, \ldots, t_d \rangle^{\oplus m}$ be a tuple of non-commutative polynomials such that $f(x) = 0$. Let*

$$D_f(x) = \begin{pmatrix} x_1 \otimes 1 - 1 \otimes x_1 & \cdots & x_d \otimes 1 - 1 \otimes x_d \\ \partial_{x_1} f(x) & \cdots & \partial_{x_d} f(x) \end{pmatrix} \in M_{m,d}(M \otimes M^{\mathrm{op}}),$$

and let $\mu_{|D_f|}$ be the spectral measure of $|D_f| = (D_f^* D_f)^{1/2}$ with respect to $\tau \otimes \tau$. If

$$\int_{[0,\infty)} |\log t| \, d\mu_{|D_f|}(t) < \infty, \tag{1.1}$$

(with the convention that $\log(0) = -\infty$), then M is strongly 1-bounded.

Strictly speaking, both Jung and Shlyakhtenko's results are about α-boundedness for general $\alpha \geq 1$, whereas the above theorem just covers $\alpha = 1$. However, the case of $\alpha = 1$ is of the most interest in applications, and the case of $\alpha > 1$ will not be relevant in our paper.

One deduces Theorem 1.1 from Theorem 1.2 through the well-known relationship between group cocycles and derivations on the group algebra. One then parameterizes the derivations in terms of their action on a self-adjoint generating set, hence obtaining a bijection between derivations and vectors z in the kernel of $\partial f(x)$. Looking at cocycles that are orthogonal to the inner cocycles results in the additional condition of $\sum_{j=1}^d [x_j, z_j] = 0$, or that z is in the kernel of the commutator operator in the first row of the matrix $D_f(x)$.

Hence, the first ℓ^2-Betti number with respect to τ of the $*$-algebra generated by x is the Murray–von Neumann dimension of the kernel of D_f.

The condition (1.1) is needed for the microstates argument to go through in the case of [19], or the more general non-microstates estimate used in [25] (the arguments are substantially different). This hypothesis is non-trivial to check in the group case, and this is where one uses the assumption of soficity. The bound (1.1) expresses positivity of a certain Fuglede–Kadison determinant, which is known for sofic groups [11]. We remark that Shlyakhtenko's results about vanishing L^2-Betti numbers have been generalized to $*$-algebras that are not group algebras [5], but this still requires some way of controlling the Fuglede–Kadison determinant.

Our proof of Theorem 1.2 is longer than Shlyakhtenko's argument, but it is more self-contained. Indeed, Shlyakhtenko's argument used the external fact that $\chi \leq \chi^*$ from [4] and the result about strong 1-boundedness and non-amenability sets from [15, Proposition A.16]. In this paper, we generalize and streamline Jung's strategy from [19], which uses iteration to bound covering numbers for smaller and smaller ε with errors controlled by the integral (1.1). Much of the technical challenge in Jung's work had to do with converting between covering numbers with respect to different non-commutative L^p-norms on the von Neumann algebra (and in fact L^p quasinorms for $p \in (0, 1)$). Our argument works mostly with L^2 norms but requires conversion between L^1 and L^2 norms at one point, and this is the main time we use a significant external ingredient, Szarek's estimates for the covering numbers of Grassmannians [26].

Another notable feature of the proof is the way in which the condition $\sum_j [x_j, z_j] = 0$ (which corresponded in cohomology to looking at cocycles orthogonal to inner cocycles) arises naturally in the microstate setting by considering the element in a unitary orbit closest to a given point x.

We also remark that polynomials in Theorem 1.2 can be replaced more generally by power series and even non-commutative trace C^2 functions in the sense of [17]; see Remark 3.10.

2. Background

2.1. Tracial von Neumann algebras and non-commutative laws

A tracial von Neumann algebra is a pair (M, τ) where M is a von Neumann algebra and $\tau: M \to \mathbb{C}$ is a faithful, normal, tracial state. The classical example is $\mathbb{M}_n(\mathbb{C})$ as a tracial von Neumann algebra with the tracial state tr_n given by

$$\mathrm{tr}_n(A) = \frac{1}{n} \sum_{i=1}^{n} A_{ii}.$$

We will primarily be interested in cases where M is *diffuse*, i.e., it has no non-zero minimal projections. The above algebra is finite-dimensional, and is thus not diffuse. One

interesting class of *diffuse* tracial von Neumann algebras are the *group von Neumann algebras*. For a discrete group, we define the *left regular representation* $\lambda\colon G \to \mathcal{U}(\ell^2(G))$ by

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h) \quad \text{for all } g, h \in G.$$

The *group von Neumann algebra of* G is then

$$L(G) = \overline{\operatorname{span}\{\lambda(g) : g \in G\}}^{\mathrm{SOT}}.$$

The linear functional $\tau\colon L(G) \to \mathbb{C}$ given by $\tau(x) = \langle x\delta_1, \delta_1 \rangle$ is a faithful, normal, tracial state (see, e.g., [21, Remark 6.7.3]). So, $(L(G), \tau)$ is a tracial von Neumann algebra. Moreover, it can be shown that $L(G)$ is diffuse if and only if G is infinite.

For a von Neumann algebra M, we use M_{sa} for the self-adjoint elements of M and $\mathcal{U}(M)$ for the unitary elements of M.

Abelian tracial von Neumann algebras correspond exactly to probability spaces, and so we may think of tracial von Neumann algebras as an instance of *non-commutative probability spaces*. Mimicking the Abelian case, for a tracial von Neumann algebra (M, τ) and $1 \le p \le \infty$, we define $\|\cdot\|_p$ on M by

$$\|x\|_p = \tau(|x|^p)^{1/p}, \quad \text{where } |x| = (x^*x)^{1/2}.$$

It can be shown [9] that this is indeed a norm on M. We use the notation $\|x\|_\infty$ for the operator norm. More generally, for $x \in M^d$, we set

$$\|(x_1, \ldots, x_d)\|_p = \begin{cases} \left(\sum_{j=1}^d \tau(|x_j|^p) \right)^{1/p}, & p \in [1, \infty), \\ \max_{j=1,\ldots,d} \|x_j\|, & p = \infty. \end{cases}$$

If (M, τ) is viewed as a non-commutative probability space, then its elements may be viewed as non-commutative random variables. In fact, a d-tuple $x = (x_1, \ldots, x_d) \in M_{\mathrm{sa}}^d$ is the non-commutative analog of an $\mathbb{R}^d$-valued random variable. In the commutative setting, an $\mathbb{R}^d$-valued random variable naturally gives rise to a *probability distribution* as a classical measure on $\mathbb{R}^d$. It is not possible to define such a measure in the non-commutative setting. However, as probability measures of compactly supported measures may be uniquely characterized by their moments, we define an analog of the notion of probability distribution as a linear functional on non-commutative polynomials.

For $d \in \mathbb{N}$, we let $\mathbb{C}\langle t_1, \ldots, t_d \rangle$ be the algebra of non-commutative polynomials in d formal variables $t_1, \ldots, t_d$, i.e., the free $\mathbb{C}$-algebra with d-generators. We give $\mathbb{C}\langle t_1, \ldots, t_d \rangle$ the unique $*$-algebra structure which makes the t_j self-adjoint. By universality, if A is any $*$-algebra and $x = (x_1, \ldots, x_d) \in A^d$ is a self-adjoint tuple, then there is a unique $*$-homomorphism $\mathbb{C}\langle t_1, \ldots, t_d \rangle \to A$ which sends t_j to x_j. For $p \in \mathbb{C}\langle t_1, \ldots, t_d \rangle$. We use $p(x)$ for the image of p under this $*$-homomorphism. Given a tracial von Neumann algebra (M, τ) and $x \in M_{\mathrm{sa}}^d$, we define the *law of* x, denoted by ℓ_x, to be the linear functional $\ell_x\colon \mathbb{C}\langle t_1, \ldots, t_d \rangle \to \mathbb{C}$ given by

$$\ell_x(f) = \tau(f(x)).$$

Non-commutative laws can be characterized as follows.

Proposition 2.1 (see [1, Proposition 5.2.14]). *The following are equivalent.*

(i) *There exists a tracial von Neumann algebra (M, τ) and $x \in M_{\mathrm{sa}}^d$ such that $\ell = \ell_x$ and $\|x\|_\infty \leq R$.*

(ii) *ℓ satisfies the following conditions:*

- *$\ell(1) = 1$,*
- *$\ell(f^* f) \geq 0$ for $f \in \mathbb{C}\langle t_1, \ldots, t_d \rangle$,*
- *$\ell(fg) = \ell(gf)$ for $f, g \in \mathbb{C}\langle t_1, \ldots, t_d \rangle$,*
- *$|\ell(t_{i_1} \cdots t_{i_k})| \leq R^k$ for all $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{1, \ldots, d\}$.*

For $R > 0$, $d \in \mathbb{N}$, we let $\Sigma_{d,R}$ be the space of laws, satisfying either of the above equivalent conditions (for this specific) R. We also denote

$$\Sigma_d = \bigcup_{R > 0} \Sigma_{d,R}.$$

Since Σ_d is a space of linear functionals on $\mathbb{C}\langle t_1, \ldots, t_d \rangle$, we can give it the weak*-topology.

Remark. The proof of ((ii) implies (i)) uses the GNS construction (see [1, Proposition 5.2.14]). Namely, let $H = L^2(\ell)$ be separation-completion of $\mathbb{C}\langle t_1, \ldots, t_d \rangle$ with respect to the semi-inner product $\langle f, g \rangle_\ell = \ell(f^* g)$. It can be shown that multiplication by t_j is bounded with respect to this semi-inner product and induces a well-defined bounded, self-adjoint operator x_j on $L^2(\ell)$. Let $M = W^*(x_1, \ldots, x_d)$ be the algebra *generated* by $x_1, \ldots, x_d$, and we define $\tau: M \to \mathbb{C}$ by $\tau(x) = \langle x1, 1 \rangle$, where $1 \in \mathbb{C}\langle t_1, \ldots, t_d \rangle$ is viewed as a vector in $L^2(\ell)$. We will denote $M = W^*(\ell)$, and $\pi_l: \mathbb{C}\langle t_1, \ldots, t_d \rangle \to W^*(\ell)$ the unique $*$-homomorphism, satisfying $\pi_\ell(t_j) = x_j$.

2.2. Microstate spaces and 1-bounded entropy

Let (M, τ) be a diffuse tracial von Neumann algebra, and $x \in M_{\mathrm{sa}}^d$ for some $d \in \mathbb{N}$ with $W^*(x) = M$. Suppose that $\|x\|_\infty \leq R$. Following [28], for each open set $\mathcal{O}$ of $\Sigma_{d,R}$ and $N \in \mathbb{N}$, we define

$$\Gamma_R^{(n)}(\mathcal{O}) = \{X \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d : \ell_X \in \mathcal{O}\}.$$

When $\mathcal{O}$ is a neighborhood of ℓ_x, we call $\Gamma_R^{(n)}(\mathcal{O})$ a *microstate space* for x.

Given $d, n \in \mathbb{N}$, $p \in [1, \infty]$, $\varepsilon > 0$, and $\Omega, \Xi \subseteq \mathbb{M}_n(\mathbb{C})^d$, then Ξ is said to be $(\varepsilon, \|\cdot\|_p)$-*cover* Ω if, for every $A \in \Omega$, there is a $B \in \Xi$ with $\|A - B\|_p < \varepsilon$. We define the *covering number* of $\Omega \subseteq \mathbb{M}_n(\mathbb{C})^d$, denoted by $K_\varepsilon(\Omega, \|\cdot\|_p)$, to be the minimal cardinality of a set that $(\varepsilon, \|\cdot\|_p)$-covers Ω. For subsets of $\mathbb{M}_n(\mathbb{C})^d$ which are invariant under the conjugation action of $\mathcal{U}(n)$ on $\mathbb{M}_n(\mathbb{C})^d$, it is natural to take the orbital numbers modulo unitary conjugation. Given $n \in \mathbb{N}$, $\varepsilon > 0$ and $\Omega, \Xi \subseteq \mathbb{M}_n(\mathbb{C})^d$, we say that Ξ *orbitally* $(\varepsilon, \|\cdot\|_p)$-*covers* Ω if, for every $A \in \Omega$, there is a $B \in \Xi$ and an $n \times n$ unitary matrix V so that

$$\|A - VBV^*\|_2 < \varepsilon.$$

We define the *orbital covering number* $K_\varepsilon^{\mathrm{orb}}(\Omega, \|\cdot\|_p)$ as the minimal cardinality of a set of Ω_0 that orbitally $(\varepsilon, \|\cdot\|_p)$-covers Ω. Since we will usually be concerned with $\|\cdot\|_2$-norms, we will frequently drop $\|\cdot\|_2$ from the notation and use $K_\varepsilon^{\mathrm{orb}}(\Omega)$ instead of $K_\varepsilon^{\mathrm{orb}}(\Omega, \|\cdot\|_2)$. Let $R \in [0, \infty)$ be such that $\|x\|_\infty < R$.

For a weak*-neighborhood $\mathcal{O}$ of ℓ_x, we define

$$h_{R,\varepsilon}(\mathcal{O}) := \limsup_{n \to \infty} \frac{1}{n^2} \log K_\varepsilon^{\mathrm{orb}}(\Gamma_R^{(n)}(\mathcal{O})),$$

$$h_{R,\varepsilon}(x) := \inf_{\mathcal{O} \ni \ell_x} h_{R,\varepsilon}(\mathcal{O}),$$

where the infimum is over all weak*-neighborhoods $\mathcal{O}$ of ℓ_x. We then define

$$h_R(x) := \sup_{\varepsilon > 0} h_{R,\varepsilon}(x).$$

By [15], it follows that $h_R(x)$ is independent of R as soon as $\|x\|_\infty \leq R$, so we use $h(x)$ instead of $h_R(x)$ as soon as $\|x\|_\infty \leq R$. Moreover, if x, y are two self-adjoint tuples in M with $W^*(x) = M = W^*(y)$, then $h(x) = h(y)$. So, we may define $h(M) = h(x)$ if $W^*(x) = M$. If M is not a factor, then the 1-bounded entropy depends upon the choice of M. We will use $h(M, \tau)$ if we wish to emphasize the dependence of the 1-bounded entropy of τ. Usually, the choice of τ will be clear from the context and use $h(M)$. In [15], it is shown how to extend this definition to infinitely many variables, but we will not need this. The 1-bounded entropy characterizes strong 1-boundedness by the following result.

Theorem 2.2 (see [15, Proposition A.16]). *A tracial von Neumann algebra M is strongly 1-bounded in the sense of Jung [18] if and only if $h(M) < \infty$.*

Because of this, we will not use Jung's original definition of strong 1-boundedness [18] and will prove that algebras are strongly 1-bounded by showing that they have finite 1-bounded entropy.

3. Proof of Theorem 1.2

3.1. Sketch of the proof

In order to prove strong 1-boundedness, or equivalently that $h_R(x) < \infty$, we will estimate $h_{R,\varepsilon}(x)$ iteratively for smaller and smaller values of ε in a similar manner to Jung [18]. In particular, if $\eta \leq \varepsilon$, then we want to estimate $h_{R,\eta}(x)$ in terms of $h_{R,\varepsilon}(x)$ by covering a $(\varepsilon, \|\cdot\|_2)$-ball in the microstate space by an $(\eta, \|\cdot\|_2)$-balls.

Consider the $(\varepsilon, \|\cdot\|_2)$-ball centered at some microstate $X \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d$ with $\|X\|_\infty \leq R$. Let $D_f(X)$ denote the matrix of tensors as in the theorem statement with x replaced by X. If Y is a microstate in the $(\varepsilon, \|\cdot\|_2)$-ball of X, then, by Taylor expansion $f(Y) - f(X)$ is approximately $\partial f(X)\#(Y - X)$. By taking a high degree of approximation for our microstate space, we can make $f(Y) - f(X)$ arbitrarily small, and thus arrange that

$Y - X$ is in the approximate kernel of $\partial f(X)$. Furthermore, because we are only considering microstates up to unitary orbits, we can assume without loss of generality that Y is the closest point in its unitary orbit to X, which implies that $\sum_{j=1}^{m}[X_j, Y_j] = 0$ (see Lemma 3.3). Hence, $Y - X$ is in the approximate kernel of $D_f(X)$. Because $\int |\log t| \, d\mu_{|D_f(x)|}(t) < \infty$ and $|D_f(X)|$ converges in distribution to $|D_f(x)|$, the dimension of the kernel of $D_f(X)$ vanishes in comparison to n^2, and we can use standard estimates on covering numbers of approximate kernels to get a bound on the η-covering number.

3.2. Background on non-commutative derivatives and Taylor expansion

First, we recall Voiculescu's free difference quotient [28,30]. Consider the d-variable non-commutative polynomial algebra $\mathbb{C}\langle t_1, \ldots, t_d \rangle$. Let $\partial_j : \mathbb{C}\langle t_1, \ldots, t_d \rangle \to \mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle$ be the unique linear map, satisfying

$$\partial_j [t_{i_1} \cdots t_{i_k}] = \sum_{\alpha=1}^{k} \delta_{i_\alpha = j} t_{i_1} \cdots t_{i_{\alpha-1}} \otimes t_{i_{\alpha+1}} \cdots t_{i_k}.$$

The map ∂_j can also be characterized as the unique derivation

$$\mathbb{C}\langle t_1, \ldots, t_d \rangle \to \mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle,$$

satisfying

$$\partial_j (t_i) = \delta_{i=j} (1 \otimes 1).$$

Here, when we describe ∂_j as a "derivation", we are viewing $\mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle$ as a bimodule over $\mathbb{C}\langle t_1, \ldots, t_d \rangle$ using the multiplication operations

$$p(f \otimes g) = pf \otimes g, \quad (f \otimes g)p = f \otimes gp.$$

If $f = (f_1, \ldots, f_m) \in \mathbb{C}\langle t_1, \ldots, t_d \rangle^m$, then

$$\partial f \in \mathbb{M}_{m,d}(\mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle)$$

will denote the matrix whose (i, j) entry is $\partial_j f_i$. This matrix plays a similar role to the derivative of a function $\mathbb{R}^d \to \mathbb{C}^m$, in that it furnishes the first-order term in a non-commutative Taylor expansion for the evaluation of f on elements of a tracial von Neumann algebra.

Recall that if (M, τ) is a tracial von Neumann algebra and $f \in \mathbb{C}\langle t_1, \ldots, t_d \rangle$ and $x = (x_1, \ldots, x_d) \in M_{\mathrm{sa}}^d$, then the evaluation of $f(x)$ is the image of f under the unique unital $*$-homomorphism $\mathbb{C}\langle t_1, \ldots, t_d \rangle \to M$ given by $t_j \mapsto x_j$. The evaluation of $f = (f_1, \ldots, f_m)$ on $x = (x_1, \ldots, x_d)$ is defined by $(f_1(x), \ldots, f_m(x))$. Moreover, $f, g \in \mathbb{C}\langle t_1, \ldots, t_d \rangle$; we set

$$(f \otimes g)(x) = f(x) \otimes g(x)^{\mathrm{op}} \in M \otimes M^{\mathrm{op}},$$

where M^{op} denotes the opposite algebra of M[1], and $\otimes$ is the algebraic tensor product. By extending this operation linearly, we can define

$$F(x) \in \mathbb{M}_{m,d}(M \otimes M^{\mathrm{op}}) \quad \text{for } F \in \mathbb{M}_{m,d}(\mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle).$$

For $a, b, x \in M$, we define

$$(a \otimes b^{\mathrm{op}})\#x = axb.$$

This extends to a bilinear map $(M \otimes M^{\mathrm{op}}) \times M \to M$. If $A \in \mathbb{M}_{m,d}(M \otimes M^{\mathrm{op}})$ and $x \in M_{\mathrm{sa}}^d$, we define $A\#x \in M^m$ as the vector with entries

$$(A\#x)_i = \sum_{j=1}^{d} A_{i,j}\#x_j.$$

The first-order Taylor approximation is as follows. Note that in contrast with the classical Taylor approximation where the error estimates are typically given in the Euclidean norm or 2-norm on $\mathbb{R}^m$, we have to mix different non-commutative p-norms of $y - x$ in the estimates.

Lemma 3.1. *Let $f \in \mathbb{C}\langle t_1, \ldots, t_d \rangle^m$ and let $R > 0$. Then, there exist constants A_f, B_f, C_f depending only on f and R such that for every tracial von Neumann algebra (M, τ) and $x, y \in M_{\mathrm{sa}}^d$ with $\|x\|_\infty, \|y\|_\infty \leq R$, we have*

$$\|f(x)\|_\infty \leq A_f, \tag{3.1}$$

$$\|f(y) - f(x)\|_2 \leq B_f\|y - x\|_2, \tag{3.2}$$

$$\|f(y) - f(x) - \partial f(x)\#(y - x)\|_1 \leq C_f\|y - x\|_2^2. \tag{3.3}$$

Proof. The case of general m will follow from applying the $m = 1$ case componentwise. For the $m = 1$ case, to verify the claims for every non-commutative polynomial f, it suffices to check them for $f(t_1, \ldots, t_d) = t_j$ and show that they are preserved under linear combinations and products.

(1) For $f(t_1, \ldots, t_d) = t_j$, the claims hold with $A_f = R$, $B_f = 1$, $C_f = 0$ since $\partial_i f = \delta_{i=j}(1 \otimes 1)$.

(2) If f and g satisfy the claims and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ satisfies the claims with $A_{\alpha f + \beta g} = |\alpha|A_f + |\beta|A_g$ and the same for the B's and C's.

(3) Suppose that $f, g \in \mathbb{C}\langle t_1, \ldots, t_d \rangle$ satisfy the conclusions of the lemma. Then, fg satisfies (3.1) with $A_{fg} = A_f A_g$. Moreover, by writing

$$(fg)(y) - (fg)(y) = (f(y) - f(x))g(y) + f(x)(g(y) - g(x))$$

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[1] M^{op} is an algebra with the same addition and $*$-operation but the order of multiplication is reversed; note that M^{op} is a tracial von Neumann algebra.

and using the L^2-L^∞-Hölder inequality, fg satisfies (3.2) with $B_{fg} = B_f A_g + A_f B_g$. Similarly, using algebraic manipulations and the fact that ∂_j is a derivation,

$$
\begin{aligned}
(fg)(y) - (fg)(x) - \partial(fg)(x)\#(y-x) &= [f(y)-f(x)-\partial f(x)\#(y-x)]g(x) \\
&\quad + f(x)[g(y)-g(x)-\partial g(x)\#(y-x)] \\
&\quad + (f(y)-f(x))(g(y)-g(x)).
\end{aligned}
$$

We estimate the first two terms by the L^1-L^∞ Hölder inequality and the third term by the L^2-L^2 Hölder inequality and thus obtain that fg satisfies (3.3) with $C_{fg} = C_f A_g + C_g A_f + B_f B_g$. ∎

The following lemma will be needed to show that the spectral measures of certain operators on $\mathbb{M}_n(\mathbb{C})^d$ associated with matricial microstates for $x \in M_{\mathrm{sa}}^d$ converge as $n \to \infty$ to the spectral measures of corresponding operators from a tracial von Neumann algebra. In the following, for a tracial von Neumann algebra M, we denote by $M \overline{\otimes} M^{\mathrm{op}}$ the tracial von Neumann algebraic tensor product of M, equipped with the trace $\tau_M \otimes \tau_{M^{\mathrm{op}}}$. If $M \overline{\otimes} M^{\mathrm{op}}$ is represented on the Hilbert space H, then $\mathbb{M}_{m,d}(M \overline{\otimes} M^{\mathrm{op}})$ are represented as operators $H^d \to H^m$. Also, $\mathbb{M}_d(M \overline{\otimes} M^{\mathrm{op}})$ is a tracial von Neumann algebra and can be equipped with the normalized trace $\mathrm{tr}_d \otimes \tau_M \otimes \tau_{M^{\mathrm{op}}}$, where tr_d is the normalized trace on $\mathbb{M}_d(\mathbb{C})$. Moreover, $\mathcal{P}(\mathbb{R})$ denotes the space of probability measures on $\mathbb{R}$ equipped with the weak* topology as linear functionals on $C_0(\mathbb{R})$.

Lemma 3.2. *Let $d, m \in \mathbb{N}$, $f \in \mathbb{C}\langle t_1, \ldots, t_d \rangle^m$, and $R > 0$. For $\ell \in \Sigma_{d,R}$, let*

$$
\pi_\ell: \mathbb{C}\langle t_1, \ldots, t_d \rangle \to W^*(\ell)
$$

be the GNS construction corresponding to ℓ as in Remark after Proposition 2.1. Let

$$
F \in \mathbb{M}_{m,d}(\mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle),
$$

consider $F(\pi_\ell(t_1), \ldots, t_d)) \in \mathbb{M}_{m,d}(W^*(\ell) \otimes W^*(\ell)^{\mathrm{op}})$, and let $\mu_{|F(\pi_\ell(t_1, \ldots, t_d))|}$ be the spectral measure of

$$
|F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))| = (F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))^* F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d)))^{1/2}
$$

as an element of $\mathbb{M}_d(W^*(\ell) \overline{\otimes} W^*(\ell)^{\mathrm{op}})$. Then, the map

$$
\Sigma_{d,R} \to \mathcal{P}(\mathbb{R}) : \ell \mapsto \mu_{|F(\pi_\ell(t_1, \ldots, t_d))|}
$$

is weak*-weak* continuous.

Proof. Because $F(x)$ is a linear combination of simple tensors of polynomials, there is some universal constant C depending on F and R such that $\|F(x)\|_{\mathbb{M}_{m,d}(M \overline{\otimes} M^{\mathrm{op}})} \leq K$ for every tuple of operators with $\|x\|_\infty \leq R$. In particular, the spectral measure of $|F(x)|$ is supported on $[0, K]$. Hence, it suffices to show that, for every $\phi \in C([0, K])$, the map

$$
\ell \mapsto (\mathrm{tr}_d \otimes \tau_{W^*(\ell)} \otimes \tau_{W^*(\ell)}^{\mathrm{op}})(\phi(|F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))|))
$$

is continuous. By the Stone–Weierstrass theorem, it suffices to consider the case when $\phi(s) = \psi(s^2)$ where ψ is a polynomial. In this case,

$$\phi(|F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))|) = \psi(F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))^* F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d)))$$

The right-hand side is just an element of $\mathbb{M}_d(\mathbb{C}\langle t_1, \ldots, t_d\rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d\rangle)$ applied to the operators $\pi_\ell(t_1), \ldots, \pi_\ell(t_d)$. Hence,

$$(\mathrm{tr}_d \otimes \tau_{\mathrm{W}^*(\ell)} \otimes \tau^{\mathrm{op}}_{\mathrm{W}^*(\ell)})(\phi(|F(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))|))$$
$$= (\tau_{\mathrm{W}^*(\ell)} \otimes \tau^{\mathrm{op}}_{\mathrm{W}^*(\ell)})(G(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))),$$

where $G \in \mathbb{C}\langle t_1, \ldots, t_d\rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d\rangle$ is $1/d$ times the sum of the diagonal entries of this matrix of tensors of polynomials. Since G is a linear combination of simple tensors, it suffices to show the continuity of the map

$$\ell \mapsto (\tau_{\mathrm{W}^*(\ell)} \otimes \tau^{\mathrm{op}}_{\mathrm{W}^*(\ell)})((f \otimes g)(\pi_\ell(t_1), \ldots, \pi_\ell(t_d))),$$

where $f, g \in \mathbb{C}\langle t_1, \ldots, t_d\rangle$. But the right-hand side is equal to $\ell(f)\ell(g)$, and $\ell \mapsto \ell(f)\ell(g)$ is continuous by definition of the weak* topology. ∎

3.3. Covering the microstate space

We now give the details of the argument sketched in Section 3.1. We begin with the orbital optimization trick. This lemma also appears in [12, Lemma 1.14], where it is related to non-commutative optimal transport theory.

Lemma 3.3. *Let* $X, Y \in \mathbb{M}_n(\mathbb{C})^d_{\mathrm{sa}}$. *There exists a unitary matrix* U *that minimizes* $\|X - UYU^*\|_2$, *and any such unitary satisfies*

$$\sum_{j=1}^d [X_j, UY_jU^*] = 0.$$

Proof. A minimizer exists because the unitary group is compact and the function $U \mapsto \|X - UYU^*\|_2$ is continuous. Suppose that U is a minimizer and let $A \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}$. Then

$$0 \leq \|X - e^{itA}UYU^*e^{-itA}\|_2^2 - \|X - UYU\|_2^2$$
$$= 2\langle X, e^{itA}UYU^*e^{-itA} - UYU^*\rangle.$$

Differentiating at $t = 0$, we get

$$0 = \sum_{j=1}^d \mathrm{tr}_n(X_j i[A, UY_jU^*]) = \sum_{j=1}^d \mathrm{tr}_n(i[UY_jU^*, X_j]A).$$

Because A was arbitrary, we have

$$\sum_{j=1}^d [UY_jU^*, X_j] = 0. \qquad \blacksquare$$

Next, we will give an initial form of the iterative estimate in terms of an auxiliary quantity $\Psi_{R,\eta,\delta,\varepsilon}(x, f)$ measuring the size of approximate kernels of $D_f(X)$. For a neighborhood $\mathcal{O}$ of ℓ_x in $\Sigma_{d,R}$, define

$$\Psi_{R,\eta,\delta,\varepsilon}(\mathcal{O}, f)$$
$$= \limsup_{n \to \infty} \frac{1}{n^2} \sup_{X \in \Gamma_R^{(n)}(\mathcal{O})} \log K_\eta\big(\{Z : \|Z\| \leq 2R, \|Z\|_2 < \delta, \|D_f(X)\#Z\|_1 < \varepsilon\}, \|\cdot\|_2\big).$$

Note that $\Psi_{R,\eta,\delta,\varepsilon}(\mathcal{O}, f)$ is monotone in $\mathcal{O}$. We define

$$\Psi_{R,\eta,\delta,\varepsilon}(x, f) = \inf_{\mathcal{O}} \Psi_{R,\eta,\delta,\varepsilon}(\mathcal{O}, f).$$

At this point, the reader may be wondering why we use $\|D_f(X)\#Z\|_1 < \varepsilon$ instead of $\|D_f(X)\#Z\|_2 < \varepsilon$. The reason is that the error estimate in the non-commutative Taylor expansion requires the 1-norm rather than the 2-norm; that is,

$$\|f(Y) - f(X) - \partial f(X)\#(Y - X)\|_1 \leq C\|Y - X\|_2^2.$$

Later, we will work to estimate this in terms of the approximate kernel with the error measured in 2-norm.

Lemma 3.4. *With the setup of Theorem* 1.2*, there is a constant $C > 0$ (depending only upon f and R) so that for all $\varepsilon, \eta > 0$ we have that*

$$h_{R,\eta}(x) \leq h_{R,\varepsilon}(x) + \Psi_{2R,\eta/2,2\varepsilon,C\varepsilon^2}(x, f).$$

Proof. Fix the neighborhood

$$\mathcal{U} = \left\{\ell : \sum_{j=1}^m \ell(f_j^* f_j)^{1/2} < \varepsilon^2\right\} \subseteq \Sigma_{d,R}.$$

In order to estimate $h_{R,\eta}(x, \|\cdot\|_2)$, pick a neighborhood $\mathcal{O}$ of ℓ_x, and then we will cover the microstate space $\Gamma^{(n)}(\mathcal{O} \cap \mathcal{U})$ by orbital $(\eta, \|\cdot\|_2)$-balls. Recall that if a set can be covered by a certain number of ε-balls with centers not necessarily in that set, then it can be covered with the same number of 2ε-balls with centers in the set. Hence, there exists a set $\Omega \subseteq \Gamma_R^{(n)}(\mathcal{O} \cap \mathcal{U})$ of cardinality at most $K_\varepsilon(\Gamma_R^{(n)}(\mathcal{O} \cap \mathcal{U}), \|\cdot\|_2)$ such that the $(2\varepsilon, \|\cdot\|_2)$-balls centered at X in Ω cover $\Gamma_R^{(n)}(\mathcal{O} \cap \mathcal{U})$.

We want to cover each of the orbital $(\varepsilon, \|\cdot\|_2)$-balls by orbital $(\eta, \|\cdot\|_2)$-balls. If Y is in the orbital $(2\varepsilon, \|\cdot\|_2)$-ball around X, then because we only need to cover Y up to unitary equivalence, we can assume without loss of generality that Y is the element of its orbit that is closest to X in $\|\cdot\|_2$, and thus $\sum_j [X_j, Y_j] = 0$ by Lemma 3.3. Recall by Lemma 3.1,

$$f(Y) - f(X) = \partial f(X)\#(Y - X) + \Delta_f(X, Y),$$

where the error term $\Delta_f(X, Y)$ satisfies

$$\|\Delta_f(X, Y)\|_1 \leq C_f \|Y - X\|_2^2 \leq 4 C_f \varepsilon^2$$

for a constant C_f depending only on f and R. By our choice of $\mathcal{U}$, we have

$$\|f(X)\|_1 \leq \sum_{j=1}^{\infty} \mathrm{tr}_n (f_j(X)^* f_j(X))^{1/2} < \varepsilon^2,$$

and similarly $\|f(Y)\|_1 < \varepsilon^2$. It follows that

$$\|\partial f(X) \# (Y - X)\|_1 < (2 + 4 C_f) \varepsilon^2.$$

Let $Z = Y - X$. Note that

$$\sum_{j=1}^{d} [X_j, Z_j] = \sum_{j=1}^{d} [X_j, Y_j] = 0.$$

Since

$$\sum_{j=1}^{d} [X_j, Z_j] = 0,$$

we have

$$\|D_f(X) \# (Y - X)\|_1 = \|(\partial f)(X) \# (Y - X)\|_1 < (2 + 4 C_f) \varepsilon^2.$$

Also, $\|Z\|_\infty \leq 2R$. Of course, the number of $(\eta, \|\cdot\|_2)$-balls needed to cover the set of Z's obtained in this way is at most

$$\sup_{X \in \Gamma_R^{(n)}(\mathcal{O})} K_\eta \big(\{ Z : \|Z\| \leq 2R, \|Z\|_2 < 2\varepsilon, \|D_f(X) \# Z\|_1 < (2 + 4 C_f) \varepsilon^2 \}, \|\cdot\|_2 \big).$$

It follows that

$$K_\eta^{\mathrm{orb}} \big(\Gamma_R^{(n)}(\mathcal{O} \cap \mathcal{U}), \|\cdot\|_2 \big)$$

$$\leq \frac{1}{N^2} K_\varepsilon^{\mathrm{orb}} \big(\Gamma_R^{(n)}(\mathcal{O} \cap \mathcal{U}), \|\cdot\|_2 \big)$$

$$\times \sup_{X \in \Gamma_R^{(n)}(\mathcal{O})} K_{\eta/2} \bigg(\bigg\{ Z : \|Z\| \leq 2R, \sum_{j=1}^{d} [Z_j, X_j] = 0, \|Z\|_2 < 2\varepsilon,$$

$$\|\partial f(X) \# Z\|_1 < (2 + C_f) \varepsilon^2 \bigg\}, \|\cdot\|_2 \bigg).$$

Apply $\limsup_{n \to \infty} (1/n^2) \log$ to obtain

$$h_{R,\eta}(\mathcal{O} \cap \mathcal{U}, \|\cdot\|_2) \leq h_{R,\varepsilon}(\mathcal{O} \cap \mathcal{U}) + \Psi_{2R, \eta/2, 2\varepsilon, (2 + C_f)\varepsilon^2}(\mathcal{O} \cap \mathcal{U}, f).$$

Because all the covering numbers are monotone in the "$\mathcal{O}$" variable, taking the infimum over all $\mathcal{O}$ yields the same result whether or not we intersect with $\mathcal{U}$ first. Thus, upon taking the infimum with respect to $\mathcal{O}$, we obtain the asserted result. ∎

3.4. Covering the approximate kernel

In order to convert our estimate with the $\|\cdot\|_1$-approximate kernel to an estimate with the $\|\cdot\|_2$-approximate kernel, we will estimate in Lemma 3.6 the $\|\cdot\|_2$-covering number of the intersection of a $\|\cdot\|_1$-ball and a $\|\cdot\|_\infty$ ball. We employ Szarek's covering estimate [26] in a similar way to [16]. For convenience of the reader, we state the lemma explicitly here.

Lemma 3.5 ([16]). *There exists a universal constant C such that for $t \geq 0$,*

$$K_\varepsilon\big(\{P \in \mathbb{M}_n(\mathbb{C}) \text{ projection}, \operatorname{tr}_n(P) \leq t\}, \|\cdot\|_\infty\big)$$
$$= K_\varepsilon\big(\{P \in \mathbb{M}_n(\mathbb{C}) \text{ projection}, \operatorname{tr}_n(P) \geq 1 - t\}, \|\cdot\|_\infty\big)$$
$$\leq (1 + nt)\left(\frac{C}{\varepsilon}\right)^{2n^2 t}.$$

Lemma 3.6. *There is a universal constant C such that for $t > 0$ and $\varepsilon \leq 3R$,*

$$\limsup_{n \to \infty} \frac{1}{n^2} \log K_\varepsilon(B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, t\varepsilon), \|\cdot\|_\infty) \leq 12t \log \frac{CR}{\varepsilon}.$$

Proof. By Lemma 3.5, there exists a set Ξ of projections of rank at least $n(1 - 3t)$ such that every projection P of rank at least $n(1 - 3t)$ satisfies $\|P - Q\|_\infty < \varepsilon/3R$ for some $Q \in \Xi$ and such that

$$|\Xi| \leq (1 + nt)\left(\frac{6C_1 R}{\varepsilon}\right)^{6n^2 t}.$$

Next, for each $Q \in \Xi$, observe that $(1 - Q)\mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}$ is a Hilbert space of real dimension at most $6n^2 t$, and hence for some constant C_2,

$$K_{\varepsilon/3}((1 - Q)B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, R), \|\cdot\|_\infty) \leq \left(\frac{3C_2 R}{\varepsilon}\right)^{6n^2 t}.$$

Therefore, we may choose a set Ω_Q with

$$|\Omega_Q| \leq \left(\frac{3C_2 R}{\varepsilon}\right)^{2n^2 t}$$

that $(\varepsilon/3, \|\cdot\|_\infty)$-covers $(1 - Q)B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, R)$.

We claim $\Omega = \bigcup_{Q \in \Xi} \Omega_Q$ is an $(\varepsilon, \|\cdot\|_\infty)$-covering of

$$B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, \delta\varepsilon/3).$$

Let $A \in B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, t\varepsilon)$, and let $\mu_{|A|}$ be the spectral measure of $|A|$, which is supported on $[0, R]$. Let $P = 1_{[0, \varepsilon/3)}(|A|)$. Note that

$$\mu_{|A|}([\varepsilon/3, \infty)) \leq \frac{3}{\varepsilon} \int_\varepsilon^\infty x \, d\mu_{|A|}(x) \leq \frac{3\|A\|_1}{\varepsilon} \leq 3t.$$

Therefore,

$$\operatorname{rank}(P) = n\mu_{|A|}([0, \varepsilon/3)) \geq n(1 - 3t).$$

Choose $Q \in \Xi$ such that $\|P - Q\|_\infty < \varepsilon/3R$. There is some $B \in \Omega_Q$ such that $\|B - QA\|_\infty < \varepsilon/3$. Observe that

$$\begin{aligned}
\|A - B\| &\leq \|PA\|_\infty + \|(P - Q)A\|_\infty + \|QA - B\|_\infty \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3R}R + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

It follows that

$$\begin{aligned}
&K_\varepsilon(B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, t\varepsilon), \|\cdot\|_\infty) \\
&\qquad \leq (1 + 3nt)\left(\frac{6C_1 R}{\varepsilon}\right)^{6n^2 t}\left(\frac{3C_2 R}{\varepsilon}\right)^{6n^2 t}.
\end{aligned}$$

Let $C = \max(6C_1, 3C_2)$. Then

$$\begin{aligned}
&\frac{1}{n^2}\log K_\varepsilon(B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, t\varepsilon), \|\cdot\|_\infty) \\
&\qquad \leq \frac{1}{n^2}\log(1 + 3nt) + 12t\log\frac{CR}{\varepsilon}.
\end{aligned}$$

Taking $n \to \infty$, we obtain the desired estimate. ∎

The second ingredient for estimating $\Psi_{R, \eta, \delta, \varepsilon}(x, f)$ is the following standard estimate for covering numbers of approximate kernels of operators on a Hilbert space. Of course, we will apply this lemma to the operator $D_f(X)\#$ from the Hilbert space $\mathbb{M}_n(\mathbb{C})^d$ with the normalized Hilbert–Schmidt norm $\|\cdot\|_2$ to the Hilbert space $\mathbb{M}_n(\mathbb{C})^m$ with $\|\cdot\|_2$. We remark that $B(\mathbb{M}_n(\mathbb{C})^d)$ is isomorphic to $\mathbb{M}_d(\mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})^{\mathrm{op}})$ acting on $\mathbb{M}_n(\mathbb{C})^d$ with the # operation, and the normalized trace on $B(\mathbb{M}_n(\mathbb{C})^d)$ corresponds to

$$\operatorname{tr}_d \otimes\tau_{\mathbb{M}_n(\mathbb{C})} \otimes \tau_{\mathbb{M}(\mathbb{C})^{\mathrm{op}}}.$$

Lemma 3.7. *There is a universal constant $C > 0$ with the following property. Let $\mathcal{H}$, $\mathcal{K}$ be (complex) Hilbert spaces with $\mathcal{H}$ finite-dimensional, and let $T \in B(\mathcal{H}, \mathcal{K})$. Fix $R > 0$. For any $\delta, \varepsilon, \eta > 0$ with $\eta < 3$, we have that*

$$K_\eta(\{\xi \in \mathcal{H} : \|\xi\| < \delta, \|T\xi\| < \varepsilon\}) \leq \left(\frac{C\delta}{\eta}\right)^{2\dim(\mathcal{H})\mu_{|T|}([0, \frac{2\varepsilon}{\eta}])},$$

where $\mu_{|T|}$ is the spectral measure of $|T|$ with respect to the normalized trace on $B(\mathcal{H})$.

Proof. Let $P = 1_{[0, \frac{2\varepsilon}{\eta}]}(|T|)$. Suppose that $\xi \in \mathcal{H}$ and $\|T\xi\| < \varepsilon$. Then, by functional calculus,

$$\|\xi - P\xi\| = \|1_{(\frac{2\varepsilon}{\eta}, \infty)}(|T|)\xi\| \leq \frac{\eta}{2\varepsilon}\|T\xi\| < \frac{\eta}{2}.$$

Thus, $\{\xi \in \mathcal{H} : \|\xi\| < \delta, \|T\xi\| < \varepsilon\}$ is contained in the $\eta/2$-neighborhood of $B_{P(\mathcal{H})}(0, \delta)$. Thus,

$$K_\eta(\{\xi \in \mathcal{H} : \|\xi\| < \delta, \|T\xi\| < \varepsilon\}) \leq K_{\eta/2}(B_{P(\mathcal{H})}(0, \delta)) \leq \left(\frac{C\delta}{\eta}\right)^{2\dim(P\mathcal{H})},$$

since the real dimension of $P\mathcal{H}$ is twice the complex dimension. Then, note that

$$\dim(P\mathcal{H}) = \operatorname{tr}_{\dim(\mathcal{H})}(P)\dim(\mathcal{H}) = \mu_{|T|}([0, 2\varepsilon/\eta])\dim(\mathcal{H}). \qquad \blacksquare$$

Lemma 3.8. *Let* $t \in (0, 1/3]$ *and suppose that* $Rt \geq \varepsilon$. *Then for some constants* C_2 *and* C_3 *depending on* f, *we have*

$$\Psi_{R,\eta,\delta,\varepsilon}(x, f) \leq \mu_{|D_f(x)|}([0, 2\varepsilon/t\eta])\log\frac{C_2\delta}{\eta} + 12mt\log\frac{C_3Rmt}{\varepsilon}.$$

Proof. Let $\mathcal{O}$ be a neighborhood of ℓ_x and $X \in \Gamma^{(n)}(\mathcal{O})$. We want to estimate the $(\eta, \|\cdot\|_2)$ covering number of

$$B_{\mathbb{M}_n(\mathbb{C})^d_{\mathrm{sa}}, \|\cdot\|_\infty}(0, R) \cap B_{\mathbb{M}_n(\mathbb{C})^d_{\mathrm{sa}}, \|\cdot\|_2}(0, \delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_1}(0, \varepsilon)).$$

There exists a constant C_1 depending on f such that

$$\|D_f(X)\#Z\|_\infty \leq C_1\|Z\|_\infty,$$

and in particular, this is bounded by $2C_1R$ when $\|Z\|_\infty \leq 2R$. Hence, it suffices to estimate the $(\eta, \|\cdot\|_2)$-covering number of

$$B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, \delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_\infty}(0, 2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_1}(0, \varepsilon)),$$

where we use $D_f(X)$ to denote the linear transformation $D_f(X)\# : \mathbb{M}_n(\mathbb{C})^d \to \mathbb{M}_n(\mathbb{C})^m$. Fix a set $\Omega \subseteq \mathbb{M}_n(\mathbb{C})^m$ that $(\frac{\varepsilon}{2t}, \|\cdot\|_2)$-covers

$$B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_\infty}(0, 2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_1}(0, \varepsilon)$$

and satisfies

$$\begin{aligned}
|\Omega| &\leq K_{\varepsilon/2t}(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_\infty}(0, 2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_1}(0, \varepsilon), \|\cdot\|_2) \\
&\leq K_{\varepsilon/2mt}(B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, 2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, \varepsilon), \|\cdot\|_\infty)^m,
\end{aligned}$$

where for the last several steps we used that $\|\cdot\|_2 \leq m\|\cdot\|_\infty$ on $\mathbb{M}_n(\mathbb{C})^m_{\mathrm{sa}}$ and that

$$B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_\infty}(0, 2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_1}(0, \varepsilon)$$

is contained in the product of m copies of $B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, 2C_1R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, \varepsilon)$. Then

$$\begin{aligned}
&B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, \delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_\infty}(0, C_1R) \cap B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_1}(0, \varepsilon)) \\
&\subseteq \bigcup_{Y \in \Omega} B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, \delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_2}(Y, \tfrac{\varepsilon}{2t})).
\end{aligned}$$

For each $Y \in \Omega$, if $B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, \delta) \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_2}(Y, \varepsilon/2t))$ is non-empty, then pick some Z_Y in this set so that

$$
B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, \delta) \cap D_f(X)^{-1}\left(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_2}\left(Y, \frac{\varepsilon}{2t}\right)\right)
$$
$$
\subseteq B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, \delta) \cap D_f(X)^{-1}\left(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_2}\left(D_f(X)\#Z_Y, \frac{\varepsilon}{2t}\right)\right)
$$
$$
\subseteq Z_Y + \left(B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, 2\delta) \cap D_f(X)^{-1}\left(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_2}\left(0, \frac{\varepsilon}{t}\right)\right)\right).
$$

By Lemma 3.7,

$$
K_\eta\left(B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, 2\delta) \cap D_f(X)^{-1}\left(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_2}\left(0, \frac{\varepsilon}{t}\right)\right), \|\cdot\|_2\right)
$$
$$
\leq \left(\frac{C_2 \delta}{\eta}\right)^{2dn^2 \mu_{|D_f(X)|}([0, 2\varepsilon/t\eta])}.
$$

In particular,

$$
\frac{1}{n^2} \log K_\eta\left(B_{\mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d, \|\cdot\|_\infty}(0, R) \cap B_{\mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d, \|\cdot\|_2}(0, \delta)\right.
$$
$$
\left. \cap D_f(X)^{-1}(B_{\mathbb{M}_n(\mathbb{C})^m, \|\cdot\|_1}(0, \varepsilon)), \|\cdot\|_2\right)
$$
$$
\leq 2d\left(\sup_{X \in \Gamma^{(n)}(\mathcal{U})} \mu_{|D_f(X)|}([0, 2\varepsilon/t\eta])\right) \log \frac{C_2 \delta}{\eta}
$$
$$
+ \frac{m}{n^2} \log K_{\varepsilon/2mt}(B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, 2C_1 R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, \varepsilon), \|\cdot\|_\infty).
$$

By Lemma 3.6,

$$
\limsup_{n \to \infty} \frac{1}{n^2} \log K_{\varepsilon/2mt}(B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_\infty}(0, 2C_1 R) \cap B_{\mathbb{M}_n(\mathbb{C}), \|\cdot\|_1}(0, \varepsilon), \|\cdot\|_\infty)
$$
$$
\leq 12t \log \frac{C_3 Rmt}{\varepsilon}.
$$

Now observe that as $\mathcal{O}$ shrinks to $\{\ell_x\}$, the measures $\mu_{|D_f(X)|}$ for $X \in \Gamma^{(n)}(\mathcal{O})$ converge uniformly in distribution to $\mu_{|D_f(x)|}$ using Lemma 3.2. Thus, we have

$$
\limsup_{n \to \infty} \sup_{X \in \Gamma^{(n)}(\mathcal{U})} \mu_{|D_f(X)|}([0, 2\varepsilon/t\eta]) \leq \mu_{D_f(x)}([0, 2\varepsilon/t\eta]).
$$

Thus, when we take the lim sup as $n \to \infty$, we obtain the assertion of the theorem. ∎

3.5. Iteration of the estimates

By combining Lemmas 3.4 and 3.8, we obtain the following bounds.

Corollary 3.9. *Let $t \in (0, 1/3]$ and $\eta \leq \varepsilon$ and $Rt \geq \varepsilon$. Then*

$$h_{R,\eta}(x) \leq h_{R,\varepsilon}(x) + \mu_{|D_f(x)|}\left(\left[0, \frac{C_1\varepsilon^2}{t\eta}\right]\right) \log \frac{C_2\varepsilon}{\eta} + 12mt \log \frac{C_3Rmt}{\varepsilon^2}. \quad (3.4)$$

In particular, if ε is sufficiently small (depending on R and f), we can take $\eta = \varepsilon^{4/3}$ and $t = \varepsilon^{1/3}$ to get

$$h_{R,\varepsilon^{4/3}}(x, \|\cdot\|_2) \leq h_{R,\varepsilon}(x, \|\cdot\|_2) + \mu_{|D_f(x)|}([0, C_1\varepsilon^{1/3}]) \log(C_2\varepsilon^{-1/3})$$
$$+ 12m\varepsilon^{1/3} \log(C_3Rm\varepsilon^{-5/3}). \quad (3.5)$$

Proof of Theorem 1.2. Fix some ε sufficiently small that we can apply (3.5). By repeated application of that estimate,

$$h_{R,\varepsilon^{4^k/3^k}}(x) \leq h_{R,\varepsilon}(x) + \sum_{j=0}^{k-1}\left(\mu_{|D_f(x)|}([0, C_1\varepsilon^{4^j/3^{j+1}}]) \log(C_2\varepsilon^{-4^j/3^{j+1}})\right.$$
$$\left. + 12m\varepsilon^{4^j/3^{j+1}} \log(C_3Rm\varepsilon^{-5\cdot4^j/3^{j+1}})\right).$$

Recall that $h_{R,\eta}(x, \|\cdot\|_2)$ decreases to $h(x)$ as $\eta \to 0$. Thus,

$$h(M) = h_R(x) \leq h_{R,\varepsilon}(x) + \sum_{j=0}^{\infty}\left(\mu_{|D_f(x)|}([0, C_1\varepsilon^{4^j/3^{j+1}}]) \log(C_2\varepsilon^{-4^j/3^{j+1}})\right.$$
$$\left. + 12m\varepsilon^{4^j/3^{j+1}} \log(C_3Rm\varepsilon^{-5\cdot4^j/3^{j+1}})\right).$$

Of course, because $\Gamma_R^{(n)}(\mathcal{O})$ is always contained in $B_{\mathbb{M}_n(\mathbb{C})^d, \|\cdot\|_2}(0, R)$, the first term $h_{R,\varepsilon}(x)$ is automatically finite. The summability in j of the term

$$12m\varepsilon^{4^j/3^{j+1}} \log(C_3Rm\varepsilon^{-5\cdot4^j/3^{j+1}})$$

in the series is straightforward: $t \log(1/t^5)$ is bounded by a constant times $t^{3/2}$; hence, we can estimate the terms by a constant times $\varepsilon^{4^j/3^j}$ which is in turn bounded by a geometric series. Thus, to complete the argument, it suffices to show the summability of the first term. We rewrite

$$\sum_{j=0}^{\infty}\mu_{|D_f(x)|}([0, C_1\varepsilon^{4^j/3^{j+1}}]) \log(C_2\varepsilon^{-4^j/3^{j+1}}) = \int_0^{\infty} \phi(t) \, d\mu_{|D_f(x)|}(t),$$

where

$$\phi(t) = \sum_{j=0}^{\infty} \log(C_2\varepsilon^{-4^j/3^{j+1}}) 1_{[0, C_1\varepsilon^{4^j/3^{j+1}}]}(t).$$

We claim that $\phi(t) \leq A + B \log(1/t)$ for some constants A and B (depending on ε and all the parameters in the theorem), and this claim is sufficient to complete the proof because $\mu_{|D_f(x)|}$ is a compactly supported probability measure, and we assumed that

$$\int_0^{\infty} \log(1/t) \, d\mu_{|D_f(x)|}(t) < \infty.$$

For every $t \in [0, C_1\varepsilon)$, there exists a unique $k \in \mathbb{N}$ such that

$$C_1\varepsilon^{4^k/3^{k+1}} < t \leq C_1\varepsilon^{4^{k-1}/3^k}.$$

Then

$$
\begin{aligned}
\phi(t) &= \sum_{j=0}^{k-1} \log(C_2\varepsilon^{-4^j/3^{j+1}}) \\
&= \sum_{j=0}^{k-1} \left(\log C_2 + \frac{4^j}{3^{j+1}} \log \frac{1}{\varepsilon} \right) \\
&\leq \left(\log C_2 + \frac{1}{3} \log \frac{1}{\varepsilon} \right) \sum_{j=0}^{k-1} \left(\frac{4}{3} \right)^j \\
&\leq 12 \left(\log C_2 + \frac{1}{3} \log \frac{1}{\varepsilon} \right) \frac{4^{k-1}}{3^k} \\
&\leq 12 \left(\log C_2 + \frac{1}{3} \log \frac{1}{\varepsilon} \right) \frac{\log(1/t) + \log C_1}{\log(1/\varepsilon)} \\
&= A + B \log \frac{1}{t},
\end{aligned}
$$

for some constants A and B.   ∎

Remark. Given the apparent freedom to choose parameters in (3.4), one might wonder whether it is possible to improve the argument to allow a weaker hypothesis on $\mu_{|D_f(x)|}$ than integrability of the logarithm. But, in fact, this hypothesis is necessary for any argument based on (3.4) to bound $h(x)$. Indeed, suppose that we choose a sequence ε_k decreasing to zero and $t_k \in (\varepsilon_k/R, 1/3)$, and suppose that

$$\sum_{k=0}^{\infty} \mu_{|D_f(x)|}\left(\left[0, \frac{C_1\varepsilon_n^2}{t_k\varepsilon_{k+1}} \right] \right) \log \frac{C_2\varepsilon_k}{\varepsilon_{k+1}} < \infty.$$

Since ε_k is decreasing and $t_k \leq 1/3$, we have $C_1\varepsilon_k^2/t_k\varepsilon_{n+1} \geq 3C_2\varepsilon_k$. Since $\varepsilon_k < 1$, we have $\log(C_2\varepsilon_k/\varepsilon_{k+1}) \geq \log(C_2/\varepsilon_{k+1})$. Hence,

$$
\begin{aligned}
\mu_{|D_f(x)|}\left(\left[0, \frac{C_1\varepsilon_k^2}{t_k\varepsilon_{k+1}} \right] \right) \log \frac{C_2\varepsilon_k}{\varepsilon_{k+1}} &\geq \mu_{|D_f(x)|}((3C_1\varepsilon_{k+1}, 3C_1\varepsilon_k]) \log(C_2/\varepsilon_{k+1}) \\
&\geq \int_{(3C_1\varepsilon_{k+1}, 3C_1\varepsilon_k]} \log(3C_1C_2/t) \, d\mu_{|D_f(x)|}(t).
\end{aligned}
$$

Hence, if the sum converges, then

$$\int_0^1 \log(1/t) \, \mu_{|D_f(x)|}(t) < \infty.$$

Remark 3.10. Although we have stated Theorem 1.2 only for polynomial f for simplicity, the same argument works for more general non-commutative functions. Indeed, it only requires that f has a Taylor expansion and error estimate as in Lemma 3.1 and that the spectral measure of $|\partial f|$ is the large-n limit of the spectral measures of corresponding operators on $M_n(\mathbb{C})^d$ as in Lemma 3.2. This holds for instance if f is given by a non-commutative power series with radius of convergence $R' > R$ as in [28, Section 3.3]. More generally, it applies to the non-commutative C^2 functions of [17] (as well as those of [7]). Roughly speaking, [17, Section 3.2] defines a space $C_{tr}^k(\mathbb{R}^{*d})$ consisting of functions f that can be evaluated on self-adjoint d-tuples $(x_1, \ldots, x_d)$ from every tracial von Neumann algebra (M, τ) such that f is a Fréchet C^k map $M_{sa}^d \to M$, and the Fréchet derivatives of order $j \le k$, viewed as multilinear maps $(M_{sa}^d)^j \to M$, satisfy

$$\|\partial^j f(x)[y_1, \ldots, y_j]\|_p \le \text{constant}(f, j, R)\|y_1\|_{p_1} \cdots \|y_j\|_{p_j}$$

whenever $1/p = 1/p_1 + \cdots + 1/p_j$ and $\|x\|_\infty \le R$ such that trace polynomials are dense. In particular, the space is cooked up so that Taylor expansions with error estimates inspired by the non-commutative Hölder's inequality, such as Lemma 3.1, will hold. Furthermore, [17, Section 4.4] describes a trace (as well as a log-determinant for invertible elements) on the algebra $C_{tr}^{k-1}(\mathbb{R}^{*d}, \mathcal{M}^1)$ in which the first derivatives $\partial_j f$ of a trace C^k function f live. Extending this trace to $d \times d$ matrices over $C_{tr}^{k-1}(\mathbb{R}^{*d}, \mathcal{M}^1)$ enables us to make sense of the spectral measure of $\partial f(x)^* \partial f(x)$. This also applies to the operator $D_f f(x)$ in Theorem 1.2 since the $t_j \otimes 1 - 1 \otimes t_j$ defines an element of $C_{tr}^{k-1}(\mathbb{R}^{*d}, \mathcal{M}^1(\mathbb{R}^{*1}))$ for each j. Furthermore, thanks to the way that the trace on $C_{tr}^{k-1}(\mathbb{R}^{*d}, \mathcal{M}^1)$ describes the asymptotic behavior of traces on matrices (see [17, Section 4.5]), Lemma 3.2 generalizes to this setting. Hence, *mutatis mutandis* Theorem 1.2 generalizes to $f \in C_{tr}^2(\mathbb{R}^{*d})^m$.

We have now completed the proof of Theorem 1.2. We refer the reader to Section 4.2 for a proof that Theorem 1.2 implies Theorem 1.1.

4. Connections to ℓ^2-invariants of sofic groups

In this section, we recall the connection between ℓ^2-cohomology and the non-commutative difference quotient (Section 4.1) exploited by Shlyakhtenko [25] as well as his argument why Theorem 1.2 implies Theorem 1.1 (Section 4.2). Then, we show how the argument for Theorem 1.1, together with Shalom's result [24], furnishes an alternative proof of strong 1-boundedness for the von Neumann algebras of sofic Property (T) groups (Section 4.3).

4.1. Cocycles, derivations, and the free difference quotient

This subsection describes how to translate from group cohomology to derivations on the group algebra to the kernel of the free difference quotient ∂f for a function f associated with a group presentation, following [6, 23, 25, 27].

For a $*$-algebra A and an A-A bimodule $\mathcal{H}$, let $\text{Der}(A, \mathcal{H})$ denote the set of derivations $\delta \colon A \to \mathcal{H}$. If (M, τ) is a tracial von Neumann algebra and $A \subseteq M$ is a weak*-dense

∗-subalgebra, then one bimodule of interest is $L^2(M) \otimes L^2(M)$, where A acts on the left by left multiplying by $a \otimes 1$ and on the right by right multiplying by $1 \otimes a$. We have a commuting action of $M \overline{\otimes} M^{\mathrm{op}}$ on $L^2(M) \otimes L^2(M)$ where $a \otimes b^{\mathrm{op}}$ acts on $c \otimes d$ by sending it to $cb \otimes ad$. We use $\#_{\mathrm{in}}$ for this action, so

$$(a \otimes b^{\mathrm{op}})\#_{\mathrm{in}}(\xi) = (1 \otimes a)\xi(b \otimes 1);$$

it is straightforward to verify that this action extends to a normal representation of $M \overline{\otimes} M^{\mathrm{op}}$ on $L^2(M) \otimes L^2(M)$. Moreover, for all $x \in M \overline{\otimes} M^{\mathrm{op}}$, all $a, b \in M$, and all $\xi \in L^2(M) \otimes L^2(M)$,

$$x\#_{\mathrm{in}}((a \otimes 1)\xi(1 \otimes b)) = (a \otimes 1)(x\#_{\mathrm{in}}\xi)(1 \otimes b).$$

This produces an action of $M \overline{\otimes} M^{\mathrm{op}}$ on $\mathrm{Der}(A, L^2(M) \otimes L^2(M))$ by

$$(x\delta)(a) = x\#_{\mathrm{in}}(\delta(a)) \quad \text{for all } x \in M \overline{\otimes} M^{\mathrm{op}}, \, a \in A.$$

So, we may regard $\mathrm{Der}(A, L^2(M) \otimes L^2(M))$ as a module over $M \overline{\otimes} M^{\mathrm{op}}$, and so it makes sense by [22] to consider

$$\dim_{M \overline{\otimes} M^{\mathrm{op}}}(\mathrm{Der}(A, L^2(M) \otimes L^2(M))).$$

We have a special class of derivatives called the *inner derivations*. We say that δ is inner if there is a $\xi \in L^2(M) \otimes L^2(M)$ with $\delta(a) = [a, \xi]$. We let $\mathrm{Inn}(A, L^2(M) \otimes L^2(M))$ be the inner derivations, and let

$$H^1(A, \tau) = \frac{\mathrm{Der}(A, L^2(M) \otimes L^2(M))}{\mathrm{Inn}(A, L^2(M) \otimes L^2(M))}.$$

We define the first ℓ^2-Betti number of A by

$$\beta^1_{(2)}(A, \tau) = \dim_{M \overline{\otimes} M^{\mathrm{op}}}(H^1(A, \tau)).$$

This definition is due to Connes–Shlyakhtenko [6].

Proposition 4.1. *Let G be a countable, discrete group, let τ be the canonical trace, and set $M = L(G)$. Then*

(i) $\beta^1_{(2)}(G) = \beta^1_{(2)}(\mathbb{C}[G], \tau)$. *In particular, if G is infinite, then*

$$\beta^1_{(2)}(G) + 1 = \dim_{M \overline{\otimes} M^{\mathrm{op}}}(\mathrm{Der}(\mathbb{C}[G], L^2(M) \otimes L^2(M))).$$

(ii)   *Suppose that G is finitely generated, and suppose that $g_1, \ldots, g_r$ is a finite generating set. Set*

$$x = (\mathrm{Re}(g_1), \mathrm{Im}(g_1), \mathrm{Re}(g_2), \mathrm{Im}(g_2), \ldots, \mathrm{Re}(g_k), \mathrm{Im}(g_k)) \in (\mathbb{C}[G]_{\mathrm{sa}})^{2r},$$

where

$$\operatorname{Re}(a) = \frac{a + a^*}{2}, \quad \operatorname{Im}(a) = \frac{a - a^*}{2i} \quad \text{for all } a \in \mathbb{C}[G].$$

Let J be the kernel of the homomorphism

$$\operatorname{ev}_x \colon \mathbb{C}\langle t_1, \ldots, t_{2r} \rangle \to \mathbb{C}[G].$$

Then, G is finitely presented if and only if J is finitely generated as a two-sided ideal.

Proof. (i) This is [6, Proposition 2.3], [23, Corollary 3.6], [27, Section 4].

(ii) Let $\mathbb{F}_r$ be the free group on letters $a_1, \ldots, a_r$. Consider the surjective homomorphism $q \colon \mathbb{F}_r \to G$ so that $q(a_j) = g_j$; we continue to use q to denote the linear extension $q \colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G]$. Let

$$y = (\operatorname{Re}(a_1), \operatorname{Im}(a_1), \operatorname{Re}(a_2), \operatorname{Im}(a_2), \ldots, \operatorname{Re}(a_k), \operatorname{Im}(a_k)) \in (\mathbb{C}[\mathbb{F}_r])_{\mathrm{sa}}^{2r},$$

so $\operatorname{ev}_x = q \circ \operatorname{ev}_y$. Let B be the ideal in $\mathbb{C}\langle t_1, \ldots, t_{2k} \rangle$ generated by

$$\{[t_{2j-1}, t_{2j}] : j = 1, \ldots, k\} \cup \{t_{2j-1}^2 + t_{2j}^2 - 1 : j = 1, \ldots, k\},$$

and let

$$\pi \colon \mathbb{C}\langle t_1, \ldots, t_k \rangle \to \mathbb{C}\langle t_1, \ldots, t_k \rangle / B$$

be the quotient map. Then, the kernel of ev_y contains π, and so ev_y descends to a map

$$\overline{\operatorname{ev}}_y \colon \mathbb{C}\langle t_1, \ldots, t_{2k} \rangle / B \to \mathbb{C}[\mathbb{F}_r]$$

with $\operatorname{ev}_y = \overline{\operatorname{ev}}_y \circ \pi$. For every $1 \leq j \leq k$, the element

$$u_j = t_{2j-1} + i t_{2j} + B \in \mathbb{C}\langle t_1, \ldots, t_{2k} \rangle / B$$

is unitary, and so there is a unique map $\phi \colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}\langle t_1, \ldots, t_{2k} \rangle / B$ which satisfies $\phi(a_j) = u_j$. Routine calculations verify that $\phi, \overline{\operatorname{ev}}_y$ are mutual inverses to each other, and so $\phi \circ \operatorname{ev}_y = \pi$.

First, suppose that G is finitely presented, and let F be a finite subset of the kernel of $q : \mathbb{F}_r \to G$ so that $\ker(q)$ is the smallest normal subgroup containing F. It is direct to verify that the kernel of $q : \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G]$ is the smallest ideal in $\mathbb{C}[\mathbb{F}_r]$ containing $\{w - 1 : w \in F\}$. For $w \in F$, let $Q_w \in \mathbb{C}\langle t_1, \ldots, t_{2k} \rangle$ be any element so that $\pi(Q_w) = \phi(w)$. We leave it as an exercise to show that J is generated as a two-sided ideal by

$$\{Q_w - 1 : w \in F\} \cup \{[t_{2j-1}, t_{2j}] : j = 1, \ldots, k\} \cup \{t_{2j-1}^2 + t_{2j}^2 - 1 : j = 1\}.$$

This shows that J is finitely generated as a two-sided ideal.

Now suppose that J is finitely generated as a two-sided ideal, say by $F_1, \ldots, F_k$. Set $N = \ker(q \colon G \to \mathbb{F}_r)$, and $Q_j = \operatorname{ev}_y(F_j)$. Then, $\pi(F_1), \ldots, \pi(F_k)$ generate $\ker(q \circ \overline{\operatorname{ev}}_y)$

as a two-sided ideal. Since $\overline{\mathrm{ev}}_y$ is an isomorphism, it follows that $Q_1, \ldots, Q_k$ generate $\ker(q\colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G])$ as a two-sided ideal. Observe that $\ker(q\colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G])$ is generated as a two-sided ideal by $\{x - 1 : x \in N\}$. So, for $j = 1, \ldots, k$, we can find a finite $F_j \subseteq N$ so that Q_j is in the two-sided ideal generated by $\{x - 1 : x \in F_j\}$. Let $F = \bigcup_{j=1}^{k} F_j$, and let I be the two-sided in $\mathbb{C}[\mathbb{F}_r]$ generated by $\{x - 1 : x \in F\}$. Then, $Q_j \in I$ for all j, and so $I = \ker(q\colon \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G])$. If $\widetilde{N}$ is the normal subgroup of G generated by R, then I is the kernel of the natural quotient map

$$\mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[\mathbb{F}_r/\widetilde{N}].$$

But $\widetilde{N} \le N$, and we saw above that I is the kernel of the natural quotient map

$$\mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[\mathbb{F}_r/N].$$

So, $N = \widetilde{N}$, and this establishes that G is finitely presented. ∎

The following may be argued exactly as in [25, Lemma 3.1].

Proposition 4.2. *Let (M, τ) be a tracial von Neumann algebra and let $x \in M_{\mathrm{sa}}^k$ be such that $W^*(x) = M$. Let A be the $*$-algebra generated by x, and let J be the kernel of $\mathrm{ev}_x\colon \mathbb{C}\langle t_1, \ldots, t_k \rangle \to A$. Suppose that $(F_j)_{j=1}^{\infty}$ is a sequence which generates J as a two-sided ideal in $\mathbb{C}\langle t_1, \ldots, t_k \rangle$. Then, the map*

$$\delta \mapsto (\delta(x_j))_{j=1}^{k}$$

is an M-M bimodular isomorphism

$$\mathrm{Der}(A, L^2(M) \otimes L^2(M)) \to \bigcap_{j=1}^{\infty} \ker((\partial F_j)(x)\#).$$

4.2. Strong 1-boundedness from vanishing ℓ^2-Betti numbers

In this section, we need the following notation. Given a group G, we view $\mathbb{C}(G) \subseteq L(G)$ by sending $\sum_g a_g g \to \sum_g a_g \lambda(g)$. This induces natural inclusion $\mathbb{M}_{m,n}(\mathbb{C}(G)) \subseteq \mathbb{M}_{m,n}(L(G))$. Given $A \in \mathbb{M}_{m,n}(\mathbb{C}(G))$, we let $\mu_{|A|}$ be the spectral measure of $(A^*A)^{1/2}$ with respect to the trace $\mathrm{Tr} \otimes \tau$, with τ defined as in Section 2. We define

$$\overset{+}{\underset{L(G)}{\det}}(A) = \exp\left(\int_{(0,\infty)} \log(t)\, d\mu_{|A|}(t) \right).$$

We have explained how to get from ℓ^2-Betti number conditions as in Theorem 1.1 to conditions on ∂f for some tuple f of non-commutative polynomials as in Theorem 1.2. The other main ingredient needed to prove Theorem 1.1 is the positivity of Fuglede–Kadison determinant. The following theorem of Elek and Szabo is the main way we know of to guarantee the positivity of Fuglede–Kadison determinants.

Theorem 4.3 ([11, Theorem 5]). *Let G be a countable, discrete, sofic group, and $m, n \in \mathbb{N}$. Fix $A \in \mathbb{M}_{m,n}(\mathbb{Z}(G))$. Then*

$$\det_{L(G)}^{+} (A) \geq 1.$$

Note that if G is as in the statement of the above theorem and $A \in \mathbb{M}_{m,n}(\mathbb{Q}(G))$ for some $m, n \in \mathbb{N}$, then there is a $q \in \mathbb{N}$ so that $qA \in M_{m,n}(\mathbb{Z}(G))$. Thus

$$\det_{L(G)}^{+} (A) = \frac{1}{q} \det_{L(G)}^{+} (qA) \geq \frac{1}{q} > 0.$$

Having collected the appropriate background material on derivations and L^2-Betti numbers, we now discuss why Theorem 1.2 implies Theorem 1.1.

Proof of Theorem 1.1 from Theorem 1.2. Let $G = \langle g_1, \ldots, g_s | w_1, \ldots, w_l \rangle$ be a finite presentation of G. For $1 \leq j \leq s$, set

$$x_{2j-1} = \frac{g_j + g_j^{-1}}{2}, \quad x_{2j} = \frac{g_j - g_j^{-1}}{2i},$$

and set $x = (x_1, x_2, \ldots, x_{2s}) \in (\mathbb{C}[G]_{\mathrm{sa}})^{2k}$. Let $q : \mathbb{C}[\mathbb{F}_r] \to \mathbb{C}[G]$ and $y \in \mathbb{C}[\mathbb{F}_r]_{\mathrm{sa}}^{2k}$ be as in the proof of Proposition 4.1 (ii). For $j = 1, \ldots, l + 2s$, define $f_j \in \mathbb{C}\langle t_1, \ldots, t_{2k} \rangle$ by

$$f_j = \begin{cases} w_j(t_1 + it_2, t_3 + it_4, \ldots, t_{2s-1} + it_{2s}), & \text{if } 1 \leq j \leq l \\ t_{2j-1}t_{2j} - t_{2j}t_{2j-1}, & \text{if } l + 1 \leq j \leq l + s \\ t_{2j-1}^2 + t_{2j}^2 - 1, & \text{if } l + s + 1 \leq j \leq l + 2s. \end{cases}$$

By the proof of Proposition 4.1 (ii), we see that the kernel of $\mathrm{ev}_x : \mathbb{C}\langle t_1, \ldots, t_{2s} \rangle \to \mathbb{C}[G]$ is generated (as an ideal) by

$$\{f_1, f_2, \ldots, f_{l+2s}\}.$$

Set $f = (f_1, \ldots, f_{l+2s})$. Let D_f be as in the statement of Theorem 1.2. We leave it as an exercise to verify that $D_f \in M_{l+2s+1,2s}(\mathbb{Q}(G \times G))$. By Theorem 4.3, we have that $\det_{L(G)}^{+}(D_f) > 0$, i.e.

$$\int_{(0,\infty)} \log(t) \, d\mu_{|D_f|}(t) > -\infty.$$

All that remains is to verify that D_f is injective. Recall that the $(1, j)$ entry of D_f is $x_j \otimes 1 - 1 \otimes x_j$ and the remaining rows are given by the matrix of partial derivatives ∂f discussed in Section 3.2. Suppose that $\xi \in [L^2(M) \otimes L^2(M)]^{2s}$ and $D_f \# \xi = 0$. This implies that $(\partial f)(x) \# \xi = 0$. By Proposition 4.2, we see that there is a derivation $\delta : \mathbb{C}[G] \to L^2(M) \otimes L^2(M)$ so that $\xi_j = \delta(x_j)$ for $j = 1, \ldots, 2s$. By Proposition 4.1 and the fact that $\beta_{(2)}^1(G) = 0$, we find that δ is approximately inner. Thus, we may choose a sequence $\zeta_n \in L^2(M) \otimes L^2(M)$ so that for all $j = 1, \ldots, 2s$

$$\xi_j = \lim_{n \to \infty} [x_j, \zeta_n] = \lim_{n \to \infty} (x_j \otimes 1 - 1 \otimes x_j^{\mathrm{op}}) \# \zeta_n.$$

Since $D_f \# \xi = 0$, we have that

$$(x_j \otimes 1 - 1 \otimes x_j^{\mathrm{op}}) \# \xi_j = 0 \quad \text{for all } j = 1, \dots, 2s.$$

Thus, for all $j = 1, \dots, 2s$,

$$\|\xi_j\|_2^2 = \lim_{n \to \infty} \langle \xi_j, (x_j \otimes 1 - 1 \otimes x_j^{\mathrm{op}}) \# \zeta_n \rangle = \lim_{n \to \infty} \langle (x_j \otimes 1 - 1 \otimes x_j^{\mathrm{op}}) \# \xi_j, \zeta_n \rangle = 0.$$

So, we have shown that $\xi = 0$. Thus, D_f is injective, and this completes the proof. ∎

More generally, the same proof shows that if (A, τ) is any tracial $*$-algebra and

- $\beta_{(2)}^1(A, \tau) = 0$,

- there exists a generating tuple $x \in A_{\mathrm{sa}}^d$ and $f \in \mathbb{C}\langle t_1, \dots, t_d \rangle^{\oplus m}$ so that $\{f_1, \dots, f_m\}$ generates ev_x as an ideal, and with $\det_A((\partial f)(x)) > 0$,

then $W^*(A, \tau)$ is strongly 1-bounded. This recovers the case $n = \mathrm{rank}((\partial F)(x))$ of [25, Theorem 2.5].

4.3. Strong 1-boundedness of Property (T) sofic groups from Theorem 1.1

The vanishing of first ℓ^2-Betti numbers for Property (T) groups was obtained in [3, Corollary 6]. We will need a little more than the above result to give a short proof that sofic group with Property (T) are strongly 1-bounded. Specifically, we will need the full strength of the Delorme–Guichardet Theorem [8, 14], which is about cohomology of groups with values in a unitary representation. This is because we will need not just the cohomology with values in the left regular representation of a group, but in the quasi-regular representation on ℓ^2 of a coset spaces. Let G be a countable, discrete group and $\pi \colon G \to \mathcal{U}(\mathcal{H})$ a unitary representation. A *cocycle for π* is a map $\beta \colon G \to \mathcal{H}$ which satisfies

$$\beta(gh) = \pi(g)\beta(h) + \beta(g) \quad \text{for all } g, h \in G.$$

We say that β is *inner* if there is a $\xi \in \mathcal{H}$ so that $\beta(g) = \pi(g)\xi - \xi$. The Delorme–Guichardet theorem says that G has (T) if and only if for every cocycle on G with values in a unitary representation is inner. See [2, Section 2.12] for a proof.

Lemma 4.4. *Let $\widetilde{G}, G$ be Property (T) groups and let $q \colon \widetilde{G} \to G$ be a surjective homomorphism. Let $\mathcal{H}$ be an $L(G) - L(G)$ bimodule, and view $\mathcal{H}$ as a bimodule over $\mathbb{C}[\widetilde{G}]$ via q. Then, every derivation $\delta \colon \mathbb{C}[\widetilde{G}] \to \mathcal{H}$ is inner.*

Proof. Suppose that $\delta \colon \mathbb{C}[\widetilde{G}] \to \mathcal{H}$ is a derivation. Define $\beta \colon \widetilde{G} \to \mathcal{H}$ by $\beta(x) = \delta(x)u_{q(x)}^{-1}$. The fact that δ is a derivation implies, by a direct calculation, that β is a cocycle for π. By the Delorme–Guichardet theorem and the fact that $\widetilde{G}$ has Property (T), we know that β is inner; i.e., there is a $\xi \in \mathcal{H}$ so that $\beta(x) = u_{q(x)}\xi u_{q(x)}^{-1} - \xi$ for all $x \in \widetilde{G}$. So for all $x \in \widetilde{G}$,

$$\delta(x) = \beta(x)u_{q(x)} = u_{q(x)}\xi - \xi u_{q(x)},$$

and this verifies that δ is inner. ∎

We will be primarily interested in the following special case of the above lemma.

Corollary 4.5. *Let $\widetilde{G}, G$ be infinite Property (T) groups, and let $q: \widetilde{G} \to G$ be a surjective homomorphism. Set $M = L(G)$. Then, every derivation*

$$\delta: \mathbb{C}[\widetilde{G}] \to L^2(M) \otimes L^2(M)$$

is inner.

We now show that Property (T) sofic groups are strongly 1-bounded. We argue directly from [25] using Shalom's theorem on the structure of Property (T) groups.

Corollary 4.6. *Let G be an infinite Property (T) sofic group. Then, $L(G)$ is strongly 1-bounded.*

Proof. Since G has Property (T), it is finitely generated. By a theorem of Shalom [24, Theorem 6.7], there is a finitely presented Property (T) group $\widetilde{G}$ and a surjective homomorphism $q: \widetilde{G} \to G$. It may be that $\widetilde{G}$ is not sofic. However, we will still be able to use the soficity of G to apply Shlyakhtenko's results to our setting.

Let $\widetilde{S}$ be a finite generating set of $\widetilde{G}$ and set $S = q(\widetilde{S})$. Then, there is a finite set R of words in S so that $\widetilde{G}$ has a presentation $\langle S | R \rangle$. Use S to build self-adjoint generators $x = (x_1, \ldots, x_m)$ of $\mathbb{C}[G]$ which have lifts $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_r)$ to generators of $\widetilde{G}$. Now use the relations R to produce $F_1, \ldots, F_r \in \mathbb{Q}[i]\langle t_1, \ldots, t_m \rangle$ with the property that if J is the ideal generated by $F_1, \ldots, F_r$, then the natural map $\mathbb{C}\langle t_1, \ldots, t_r \rangle \to \mathbb{C}[\widetilde{G}]$ given by $F \mapsto F(\widetilde{x})$ has kernel J. Let $F = (F_1, \ldots, F_r)$. By the proof of Proposition 4.2, we have that

$$\ker((\partial F)(x)\#) \cong \mathrm{Der}(\mathbb{C}[\widetilde{G}], L^2(M) \otimes L^2(M))$$

with $M = L(G)$. By the preceding corollary, it follows that $\ker((\partial F)(x))$ corresponds under this isomorphism to the inner derivations $\mathbb{C}[\widetilde{G}] \to L^2(M) \otimes L^2(M)$, and since M is diffuse,

$$\dim_{M \overline{\otimes} M^{\mathrm{op}}}(\ker((\partial F)(x)\#)) = \dim_{M \overline{\otimes} M^{\mathrm{op}}}(\mathrm{Inn}(\mathbb{C}[\widetilde{G}], L^2(M) \otimes L^2(M))) = 1.$$

Further, since $F_1, \ldots, F_r \in \mathbb{Q}[i]\langle t_1, \ldots, t_r \rangle$, we know from the soficity of G and Theorem 4.3 that $\det_M^+((\partial F)(x)) > 0$. Thus, Shlyakhtenko's theorem [25] implies that M is strongly 1-bounded (this also follows from our proof of Theorem 1.1 from Theorem 1.2; see the discussion at the end of the previous section). ∎

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