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Discrete quantum structures II: Examples

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Abstract. Part I of this paper established the basic properties of quantum predicate logic as the internal logic of discrete quantum structures. We now show that a majority of the established quantum generalizations of discrete structures are naturally axiomatizable within this internal logic. In particular, we axiomatize the quantum graphs of Duan, Severini, and Winter, the quantum metric spaces of Kuperberg and Weaver, the quantum isomorphisms of Atserias, Mančinska, Roberson, Šámal, Severini, and Varvitsiotis, and the quantum groups of Woronowicz. In each instance, we consider only those structures that are discrete in the sense that the underlying von Neumann algebra is hereditarily atomic.

1. Introduction

1.1. The internal logic of quantum sets

The category of quantum locally compact Hausdorff spaces and their maps is formally defined to be the opposite of the category of C^*-algebras and Woronowicz morphisms [11, 50]. Among these quantum locally compact Hausdorff spaces, the discrete quantum spaces are those that correspond to C^*-algebras that are c_0-direct sums of full matrix algebras. The category of discrete quantum spaces and the maps between them is thus a quantum generalization of the category of sets and functions, and discrete quantum spaces may also be called quantum sets [16]. Part I of this paper established that the quantum predicate logic of Weaver [46] is a robust internal logic for quantum sets [17]. We now recall this internal logic briefly and informally.

For technical simplicity, we work with von Neumann algebras rather than C^*-algebras. Up to an equivalence of categories, a quantum set is a von Neumann algebra that is an ℓ^∞-direct sum of full matrix algebras. Such von Neumann algebras may be called hereditarily atomic because they are characterized by the property that every von Neumann subalgebra is atomic [16, Prop. 5.4]. The Cartesian product of quantum sets is then the spacial tensor product of hereditarily atomic von Neumann algebras, which is also their categorical tensor product [12, Prop. 8.6]. If $\mathcal{X}$ and $\mathcal{Y}$ are quantum sets, then we write $\ell^\infty(\mathcal{X})$ and $\ell^\infty(\mathcal{Y})$ for the corresponding von Neumann algebras, and we may express this definition of the Cartesian product as $\ell^\infty(\mathcal{X} \times \mathcal{Y}) \cong \ell^\infty(\mathcal{X}) \mathbin{\overline{\otimes}} \ell^\infty(\mathcal{Y})$.

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In this account, a relation of arity $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ is just a projection in the von Neumann algebra $\ell^\infty(\mathcal{X}_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{X}_n)$, and the internal logic of quantum sets interprets each formula whose free variables $x_1, \ldots, x_n$ have sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$ as such a projection. Symbolically, if $\phi(x_1, \ldots, x_n)$ is such a formula, then $[\![\phi(x_1, \ldots, x_n)]\!]$ is a projection in $\ell^\infty(\mathcal{X}_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{X}_n)$. The projections of any von Neumann algebra form a complete orthomodular lattice, and this structure is used to interpret the Boolean connectives $\wedge$, $\vee$, and $\neg$ in the usual way [4, 39]. Additionally, the Sasaki projection & and the Sasaki arrow $\to$ play an important role; they are defined by $p \,\&\, q = (p \vee \neg q) \wedge q$ and $p \to q = \neg p \vee (p \wedge q)$ [13, 41].

The standard convention is that the variables $x_1, \ldots, x_n$ do not need to actually appear in the formula $\phi(x_1, \ldots, x_n)$ to be counted among its free variables. The notation $\phi(x_1, \ldots, x_n)$ only indicates that the variables that do appear freely in ϕ are among $x_1, \ldots, x_n$. Formally, the interpretation of a formula ϕ is defined relative to a finite sequence of distinct variables called a context, as in [14, Secs. D1.1–2]. Thus, $[\![(x_1, \ldots, x_n) \mid \phi(x_1, \ldots, x_n)]\!]$ is a projection in $\ell^\infty(\mathcal{X}_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{X}_n)$. The relevant bookkeeping is treated carefully in Part I [17]. If x_n does not appear freely in $\phi(x_1, \ldots, x_n)$, then $[\![(x_1, \ldots, x_n) \mid \phi(x_1, \ldots, x_n)]\!] = [\![(x_1, \ldots, x_{n-1}) \mid \phi(x_1, \ldots, x_n)]\!] \otimes 1$. If x_n does appear freely in $\phi(x_1, \ldots, x_n)$, then the expression $[\![(x_1, \ldots, x_{n-1}) \mid \phi(x_1, \ldots, x_n)]\!]$ is undefined.

Following Weaver [46, Sec. 2.6], we interpret the quantifier $\forall$ by

$$[\![(x_1, \ldots, x_{n-1}) \mid \forall x_n\, \phi(x_1, \ldots, x_n)]\!] = \sup\{p \mid p \otimes 1 \leq [\![(x_1, \ldots, x_n) \mid \phi(x_1, \ldots, x_n)]\!]\},$$

where p varies over the projections in $\ell^\infty(\mathcal{X}_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{X}_{n-1})$. We interpret the quantifier $\exists$ likewise. Each quantum set $\mathcal{X}$ has a dual quantum set $\mathcal{X}^*$ that may be defined by

$$\ell^\infty(\mathcal{X}^*) \cong \ell^\infty(\mathcal{X})^{op}.$$

If x is a variable of sort $\mathcal{X}$ and x_* is a variable of sort $\mathcal{X}^*$, then we define $[\![(x, x_*) \mid x = x_*]\!]$ to be the largest projection in $\ell^\infty(\mathcal{X} \times \mathcal{X}^*) \cong \ell^\infty(\mathcal{X}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{X})^{op}$ that is orthogonal to $p \otimes (1 - p)$ for all projections $p \in \ell^\infty(\mathcal{X})$. The ubiquitous notation

$$\forall (x = x_*)\, \psi(x, x_*, x_1, \ldots, x_n)$$

is an abbreviation for $\forall x\, \forall x_*\, (x = x_* \to \psi(x, x_*, x_1, \ldots, x_n))$.

In the parlance of symbolic logic, a term is an expression that is built up from variables using function symbols of various finite arities. The internal logic of quantum sets interprets each term $t(x_1, \ldots, x_n)$ that has sort $\mathcal{Y}$ and whose free variables $x_1, \ldots, x_n$ have sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$ as a unital normal $*$-homomorphism $\ell^\infty(\mathcal{Y}) \to \ell^\infty(\mathcal{X}_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{X}_n)$. We notate this map as $[\![(x_1, \ldots, x_n) \mid t(x_1, \ldots, x_n)]\!]$. This interpretation of terms is compositional, and in particular,

$$[\![(x_1, \ldots, x_n) \mid \phi(t(x_1, \ldots, x_n))]\!] = [\![(x_1, \ldots, x_n) \mid t(x_1, \ldots, x_n)]\!]([\![(y) \mid \phi(y)]\!]),$$

where y is a variable of sort $\mathcal{Y}$, $\phi(y)$ is a formula, and $\phi(t(x_1, \dots, x_n))$ is the result of the substitution of t for y in ϕ. This equality expresses the standard understanding of unital normal $*$-homomorphisms as a quantum generalization of functions.

It is convenient to view both the projections that interpret formulas and the homomorphisms that interpret terms as morphisms of a single category that admits a graphical calculus [37]. Up to equivalence, this category is the monoidal category of hereditarily atomic von Neumann algebras and quantum relations in the sense of Weaver [47]. Each projection p in a hereditarily atomic von Neumann algebra $M \subseteq L(H)$ is identified with the quantum relation V from M to $\mathbb{C}$ that is defined by

$$V = \{v \in L(H, \mathbb{C}) \mid vp = v\}.$$

Each unital normal $*$-homomorphism π from $M \subseteq L(H)$ to $N \subseteq L(K)$ is identified with the quantum relation W from N to M that is defined by $W = \{w \in L(K, H) \mid aw = w\pi(a)\}$ [15]. Formally, we work in an equivalent monoidal category, which we call the category of quantum sets and binary relations; it is described in detail in Part I [17] and in greater detail in [16].

1.2. Examples

Many established classes of discrete quantum structures can be naturally axiomatized within the internal logic of quantum sets. In this paper, we treat quantum graphs, quantum metric spaces, quantum posets, quantum graph homomorphisms, quantum graph isomorphisms, quantum permutations, and quantum groups that are all discrete in the sense that the underlying von Neumann algebra is hereditarily atomic.

The quantum graphs and quantum metric spaces that are considered here originate in the study of quantum error correction. Quantum graphs were introduced in [7] as the confusability graphs of quantum channels, and they were generalized to arbitrary von Neumann algebras in [47]. These quantum graphs are not closely related to metric graphs equipped with Schrödinger operators on each edge, which are also called quantum graphs [10, 33]. Quantum metric spaces in the sense of von Neumann algebras [21] generalize quantum graphs in a way that quantifies error. That research led to the notion of a quantum relation [47], which in turn led to the research in the present paper. There are other quantum generalizations of metric structure [24, 40, 45], which we do not consider here.

There are multiple natural notions of a quantum poset, even in the finite-dimensional case. A quantum partial order on $M_n(\mathbb{C})$ may be defined to be an antisymmetric subalgebra of $M_n(\mathbb{C})$ [47], a hereditarily antisymmetric subalgebra of $M_n(\mathbb{C})$ [49] or a nilpotent subalgebra of $M_n(\mathbb{C})$ [49]. We axiomatize both the first and the last of these notions with almost identical sets of nonduplicating formulas. The difference between them illustrates two natural generalizations of conjunction to the quantum setting [4, 41]. Discrete quantum posets of the first kind form a well-behaved category [19] that may be used to model recursion in the quantum setting [18].

Quantum graph homomorphisms and quantum graph isomorphisms originate in the study of quantum nonlocality. These terms refer to quantum analogs of relationships between simple graphs, rather than to relationships between quantum graphs. The notion of quantum graph homomorphism was defined in [26]. In [28], quantum graph homomorphisms were identified with the morphisms of a 2-category, and in [16], quantum graph homomorphisms were interpreted as quantum families of graph homomorphisms within the framework of noncommutative mathematics. The notion of quantum graph isomorphism was defined in [2], and quantum graph isomorphisms were identified with certain morphisms in [28]. Both notions also have analogs in the quantum commuting framework [2,32,34,35], which coincides with the usual tensor product framework when all the measurement operators are taken from a hereditarily atomic von Neumann algebra; see [12, Prop. 8.6] and [16, Prop. 5.4].

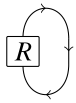
Quantum permutations are just quantum automorphisms of edgeless graphs, but they predate quantum isomorphisms considerably [44]. A quantum permutation of a finite set is also called a magic unitary [3, 28]. A subclass of quantum permutations, the quantum Latin squares, has been used for the construction of unitary error bases [30]. In the language of quantum information theory, Wang's quantum permutation group [44] may be regarded as a universal quantum permutation of the given finite set, and in the language of noncommutative mathematics, it may be regarded as the compact quantum group of all permutations of the given finite set.

As in the case of our other examples, the quantum groups that we consider are the discrete members of a larger class; discrete quantum groups are discrete locally compact quantum groups [22,23]. Infinite discrete quantum groups first occurred as the Pontryagin duals of compact matrix quantum groups [38,51]. The class of all discrete quantum groups was then implicitly defined along with the class of all compact quantum groups [52]. The first explicit definition of discrete quantum groups appears to have been given by Effros and Ruan [8], but it falls slightly outside the standard approach of starting with an operator algebra of complex-valued functions on a putative quantum space. We work with the definition of Van Daele [43]. The proof given here establishing discrete quantum groups as an example is essentially due to Vaes [42]; any flaws are the fault of the author.

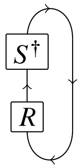
Not all established classes of discrete quantum structures are claimed to be naturally definable in quantum predicate logic. Most notably, unital normal complete positive maps, which formalize quantum channels [20], are not treated in this paper. Additionally, some structures, such as metric spaces and posets, have more than one proposed quantum generalization, not all of which have been axiomatized.

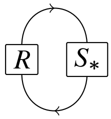
1.3. Orthogonality and trace

The category **qRel** of quantum sets and binary relations is dagger compact, i.e., strongly compact: it is a symmetric monoidal category with dual objects and a compatible involution [16, Sec. 3]. Consequently, it admits a convenient graphical calculus [1]. We briefly recall the trace of a binary relation R on a quantum set $\mathcal{X}$ [19, App. C], which is most naturally defined within this graphical calculus:

$$\mathrm{Tr}_{\mathcal{X}}(R) :=$$  .

It will be convenient for us that two binary relations R and S from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ are orthogonal to each other [16, Def. 3.8 (5)] if and only if $\mathrm{Tr}_{\mathcal{X}}(S^\dagger \circ R) = \perp$ [19, Prop. C.2]. We may express this equivalence graphically as follows:

$$R \perp S \quad \Longleftrightarrow$$ 

$$= \perp \quad \Longleftrightarrow$$ 

$$= \perp.$$

1.4. Conventions

Part II follows the conventions established in Part I [17, Sec. 1.8] with the following exceptions. We write $\forall x$ and $\exists x$ in place of $(\forall x \in \mathcal{X})$ and $(\exists x \in \mathcal{X})$, respectively, when x is a variable of sort $\mathcal{X}$. We write $\forall(x = x_*)$ and $\exists(x = x_*)$ in place of $(\forall(x = x_*) \in \mathcal{X} \times \mathcal{X}^*)$ and $(\exists(x = x_*) \in \mathcal{X} \times \mathcal{X}^*)$, respectively, when x is a variable of sort $\mathcal{X}$ and x_* is a variable of sort $\mathcal{X}^*$. We write $s = t$ in place of $E_{\mathcal{X}}(s,t)$ when s is a term of sort $\mathcal{X}$ and t is a term of sort $\mathcal{X}^*$. This notation takes advantage of the circumstance that we are now working with concrete formulas, which implicitly determine the sorts of their constituent terms.

2. Quantum graphs

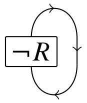
Quantum graphs are a quantum generalization of simple graphs. Quantum graphs were first defined in the context of zero-error communication to be operator systems on a finite-dimensional Hilbert space [7, Sec. II]. More generally, a quantum graph structure on an arbitrary von Neumann algebra $M \subseteq L(H)$ is an ultraweakly closed operator system V such that $m'v \in V$ and $vm' \in V$ for all $m' \in M'$ and all $v \in V$; see [47, Def. 2.6 (d)] and [48]. By definition, an operator system contains the scalar operators, and this feature reflects the convention that each vertex of a simple graph is adjacent to itself, which is natural to the context of zero-error communication. Hence, in Section 2, a simple graph is a set A that is equipped with a binary relation $\sim_A$ that is both reflexive and symmetric.

Proposition 2.1. *Let $\mathcal{X}$ be a quantum set, and let R be a binary relation on $\mathcal{X}$. Then, $I_{\mathcal{X}} \leq R$ if and only if*

$$[\![\forall(x = x_*)\, \hat{R}(x, x_*)]\!] = \top.$$

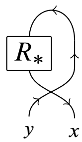
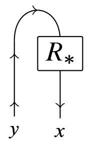
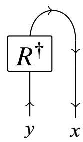
Proof. We reason that

$$[\![\forall(x = x_*)\, \hat{R}(x, x_*)]\!] = \top \iff [\![\exists(x = x_*)\, \neg\hat{R}(x, x_*)]\!] = \bot$$

$\iff$  $= \bot \iff \neg R \perp I_{\mathcal{X}} \iff I_{\mathcal{X}} \leq R.$ ∎

Lemma 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets. Let R be a binary relation from $\mathcal{X}$ to $\mathcal{Y}$, and let $S = R^\dagger$. Then, $[\![(y, x_*) \mid \hat{R}_*(x_*, y)]\!] = \hat{S}$.*

Proof. We calculate that

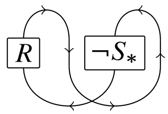
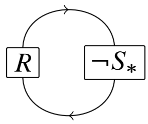
$[\![(y, x_*) \in \mathcal{Y} \times \mathcal{X}^* \mid \hat{R}_*(x_*, y)]\!] =$  $=$  $=$  $= \hat{S}.$ ∎

Lemma 2.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets. Let R and S be binary relations from $\mathcal{X}$ to $\mathcal{Y}$. Then, $R \leq S$ if and only if*

$$[\![\forall(x = x_*)\, \forall(y = y_*)\, (\hat{R}(x, y_*) \to \hat{S}_*(x_*, y))]\!] = \top. \tag{‡}$$

Proof. Taking advantage of the fact that the formulas $\hat{R}(x, y_*)$ and $\hat{S}_*(x_*, y)$ have no variables in common, we find that equation (‡) is equivalent to each of the following conditions:

$$[\![\exists(x = x_*)\, \exists(y = y_*)\, \neg(\hat{R}(x, y_*) \to \hat{S}_*(x_*, y))]\!] = \bot$$

$\iff$  $= \bot \iff$  $= \bot \iff R \perp \neg S$

$\iff R \leq S.$ ∎

Proposition 2.4. *Let $\mathcal{X}$ be a quantum set, and let R be a binary relation on $\mathcal{X}$. Then, the following are equivalent:*

(1) $R \leq R^\dagger$;

(2) $[\![\forall x_1\, \forall x_{2*}\, (\hat{R}(x_1, x_{2*}) \to \hat{R}_*(x_{2*}, x_1))]\!] = \top$;

(3) $[\![\forall(x_1 = x_{1*})\, \forall(x_2 = x_{2*})\, (\hat{R}(x_1, x_{2*}) \to \hat{R}(x_2, x_{1*}))]\!] = \top$.

Proof. Let $S = R^\dagger$. Applying Lemma 2.2 twice, we find that

$$[\![(x_1, x_{2*}) \in \mathcal{X} \times \mathcal{X}^* \mid \hat{R}_*(x_{2*}, x_1)]\!] = \hat{S}$$

and

$$[\![(x_2, x_{1*}) \in \mathcal{X} \times \mathcal{X}^* \mid \hat{S}_*(x_{1*}, x_2)]\!] = \hat{R}.$$

Combining the former equality with Proposition 3.2.2 of Part I [17], we conclude that condition (2) is equivalent to condition (1). Combining the latter equality with Lemma 2.3, we conclude that condition (3) is equivalent to condition (1). Indeed, we have that

$$[\![(x_2, x_{1*}) \in \mathcal{X} \times \mathcal{X}^* \mid \hat{S}_*(x_{1*}, x_2)]\!] = [\![(x_2, x_{1*}) \in \mathcal{X} \times \mathcal{X}^* \mid \hat{R}(x_2, x_{1*})]\!]. \qquad \blacksquare$$

For each atom X of a quantum set $\mathcal{X}$, let $\mathrm{inc}_X \in L(X, \bigoplus \mathrm{At}(\mathcal{X}))$ be the inclusion isometry of X into the ℓ^2-direct sum of the atoms of $\mathcal{X}$.

Theorem 2.5. *Let $\mathcal{X}$ be a quantum set. Then, there is a one-to-one correspondence between quantum graph structures V on $\ell^\infty(\mathcal{X})$ in the sense of* [47, Def. 2.6 (d)] *and binary relations R on $\mathcal{X}$ such that*

(1) $[\![\forall(x = x_*)\, \hat{R}(x, x_*)]\!] = \top$;

(2) $[\![\forall(x_1 = x_{1*})\, \forall(x_2 = x_{2*})\, (\hat{R}(x_1, x_{2*}) \to \hat{R}(x_2, x_{1*}))]\!] = \top$.

The correspondence is given by $R(X_1, X_2) = \mathrm{inc}_{X_2}^\dagger \cdot V \cdot \mathrm{inc}_{X_1}$, for $X_1, X_2 \in \mathrm{At}(\mathcal{X})$.

Proof. This is an immediate consequence of Propositions 2.1 and 2.4 and the equivalence between binary relations and quantum relations described in Appendix A.2 of Part I [17]. ■

Corollary 2.6. *Furthermore, assume that $\mathrm{At}(\mathcal{X}) = \{H\}$ for some nonzero finite-dimensional Hilbert space H. Then, there is a one-to-one correspondence between operator systems $V \subseteq L(H)$ and binary relations R on $\mathcal{X}$ satisfying conditions* (1) *and* (2) *in the statement of Theorem* 2.5. *The correspondence is given by $R(H, H) = V$.*

Proof. In the special case $\mathrm{At}(\mathcal{X}) = \{H\}$, we have that $\ell^\infty(\mathcal{X}) = L(H)$ and that $\ell^\infty(\mathcal{X})' = \mathbb{C}1_H$, so every operator system on H is also a quantum graph structure on $L(H)$. The isometry inc_H is simply the identity on H. ■

3. Quantum preordered sets

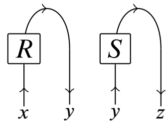
A quantum preorder on a von Neumann algebra $M \subseteq L(H)$ is an ultraweakly closed algebra $N \subseteq L(H)$ that contains M' [47, Def. 2.6 (b)]. Quantum preorders have not attracted much research interest except as a stepping stone to quantum partial orders [47, Def. 2.6 (c)]; this is the role that they play here too.

Before considering quantum preorders, we tersely recall Corollary 3.3.3 of Part I [17].

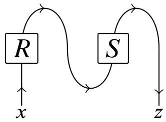
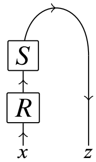
Lemma 3.1. *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be quantum sets. Let R be a binary relation from $\mathcal{X}$ to $\mathcal{Y}$, let S be a binary relation from $\mathcal{Y}$ to $\mathcal{Z}$, and let T be the binary relation from $\mathcal{X}$ to $\mathcal{Z}$ defined by $T = S \circ R$. Then,*

$$[\![(x, z_*) \in \mathcal{X} \times \mathcal{Z}^* \mid \exists(y = y_*)\, (\hat{R}(x, y_*) \wedge \hat{S}(y, z_*))]\!] = \hat{T}.$$

Proof. We calculate that

$$[\![(x, y_*, y, z_*) \in \mathcal{X} \times \mathcal{Y}^* \times \mathcal{Y} \times \mathcal{Z}^* \mid \hat{R}(x, y_*) \wedge \hat{S}(y, z_*)]\!] =$$  ,

$$[\![(x, z_*) \in \mathcal{X} \times \mathcal{Z}^* \mid \exists(y = y_*)\,(\hat{R}(x, y_*) \wedge \hat{S}(y, z_*))]\!]$$

$=$  $=$  $= \hat{T}.$ ∎

Lemma 3.2. *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be quantum sets. Let R be a binary relation from $\mathcal{X}$ to $\mathcal{Y}$, let S be a binary relation from $\mathcal{Y}$ to $\mathcal{Z}$, and let T be a binary relation from $\mathcal{X}$ to $\mathcal{Z}$. Then, the following are equivalent:*

(1) $S \circ R \leq T$;

(2) $[\![\forall x\, \forall z_*\, (\exists(y = y_*)\,(\hat{R}(x, y_*) \wedge \hat{S}(y, z_*)) \to \hat{T}(x, z_*))]\!] = \top$;

(3) $[\![\forall(x = x_*)\, \forall(y = y_*)\, \forall(z = z_*)\, ((\hat{R}(x, y_*) \wedge \hat{S}(y, z_*)) \to \hat{T}_*(x_*, z))]\!] = \top$.

Proof. Let $T_0 = S \circ R$. By Lemma 3.1,

$$\begin{aligned}&[\![(x, z_*) \in \mathcal{X} \times \mathcal{Z}^* \mid \exists(y = y_*)\,(\hat{R}(x, y_*) \wedge \hat{S}(y, z_*))]\!]\\ &\quad = \hat{T}_0 = [\![(x, z_*) \in \mathcal{X} \times \mathcal{Z}^* \mid \hat{T}_0(x, z_*)]\!],\end{aligned}$$

so by Proposition 3.2.2 of Part I [17], condition (2) is equivalent to $T_0 \leq T$, i.e., to condition (1).

No two of the formulas $\hat{R}(x, y_*)$, $\hat{S}(y, z_*)$, and $\hat{T}_*(x_*, z)$ have variables in common, which implies that the formula $\neg((\hat{R}(x, y_*) \wedge \hat{S}(y, z_*)) \to \hat{T}_*(x_*, z))$ has the same interpretation as the formula $\hat{R}(x, y_*) \wedge \hat{S}(y, z_*) \wedge \neg\hat{T}_*(x_*, z)$. Similarly, the formula

$$\neg(\exists(y = y_*)\,(\hat{R}(x, y_*) \wedge \hat{S}(y, z_*)) \to \hat{T}_*(x_*, z))$$

has the same interpretation as the formula $\exists(y = y_*)\,(\hat{R}(x, y_*) \wedge \hat{S}(y, z_*)) \wedge \neg\hat{T}_*(x_*, z)$. It follows by Theorem 3.3.2 of Part I [17] and the duality between the quantifier expressions $\forall(y = y_*)$ and $\exists(y = y_*)$ that the formula

$$\forall(y = y_*)\,((\hat{R}(x, y_*) \wedge \hat{S}(y, z_*)) \to \hat{T}_*(x_*, z))$$

has the same interpretation as the formula

$$\exists(y = y_*)\,(\hat{R}(x, y_*) \wedge \hat{S}(y, z_*)) \to \hat{T}_*(x_*, z).$$

We conclude that condition (3) is equivalent to condition (2) by Proposition 2.4. ∎

Theorem 3.3. *Let $\mathcal{X}$ be a quantum set. Then, there is a one-to-one correspondence between quantum preorders V on $\ell^\infty(\mathcal{X})$ in the sense of* [47, Def. 2.6 (b)] *and binary relations R on $\mathcal{X}$ such that*

(1) $[\![\forall(x = x_*)\, \hat{R}(x, x_*)]\!] = \top$;

(2) $[\![\forall(x_1 = x_{1*})\, \forall(x_2 = x_{2*})\, \forall(x_3 = x_{3*})\, ((\hat{R}(x_1, x_{2*}) \wedge \hat{R}(x_2, x_{3*})) \to \hat{R}_*(x_{1*}, x_3))]\!] = \top$.

The correspondence is given by $R(X_1, X_2) = \mathrm{inc}_{X_2}^\dagger \cdot V \cdot \mathrm{inc}_{X_1}$, *for* $X_1, X_2 \in \mathrm{At}(\mathcal{X})$.

Proof. This is an immediate consequence of Proposition 2.1 and Lemma 3.2 and the equivalence between binary relations and quantum relations described in Appendix A.2 of Part I [17]. ∎

Corollary 3.4. *Let $\mathcal{X}$ be a quantum set. Then, there is a one-to-one correspondence between von Neumann algebras $M \subseteq \ell^\infty(\mathcal{X})$ and binary relations R on $\mathcal{X}$ such that*

(1) $[\![\forall(x = x_*)\, \hat{R}(x, x_*)]\!] = \top$;

(2) $[\![\forall(x_1 = x_{1*})\, \forall(x_2 = x_{2*})\, (\hat{R}(x_1, x_{2*}) \to \hat{R}(x_2, x_{1*}))]\!] = \top$;

(3) $[\![\forall(x_1 = x_{1*})\, \forall(x_2 = x_{2*})\, \forall(x_3 = x_{3*})\, ((\hat{R}(x_1, x_{2*}) \wedge \hat{R}(x_2, x_{3*})) \to \hat{R}_*(x_{1*}, x_3))]\!] = \top$.

The correspondence is given by $R(X_1, X_2) = \mathrm{inc}_{X_2}^\dagger \cdot M' \cdot \mathrm{inc}_{X_1}$, *for* $X_1, X_2 \in \mathrm{At}(\mathcal{X})$.

Proof. As an immediate consequence of Theorems 2.5 and 3.3, we have a one-to-one correspondence between quantum equivalence relations V on $\ell^\infty(\mathcal{X})$ in the sense of [47, Def. 2.6 (a)] and binary relations R on $\mathcal{X}$ satisfying conditions (1), (2), and (3). A quantum equivalence relation on $\ell^\infty(\mathcal{X})$ is just a von Neumann algebra that contains $\ell^\infty(\mathcal{X})'$, and such von Neumann algebras are in one-to-one correspondence with the von Neumann algebras contained in $\ell^\infty(\mathcal{X})$ via the commutant operation. ∎

The significance of Corollary 3.4 is that, according to the noncommutative dictionary, the unital ultraweakly closed $*$-subalgebras of $\ell^\infty(\mathcal{X})$ correspond to the quotients of $\mathcal{X}$.

4. Quantum posets

The example of quantum posets is quite similar to the example of quantum preordered sets, but it is nevertheless significant, both because quantum posets are of inherent interest [18, 49] and because this example touches on a basic feature of quantum logic. This basic feature is that the notion of inconsistency between two predicates has multiple natural generalizations to the quantum setting. More generally, the notion of the conjunction of two predicates has multiple natural generalizations to the quantum setting.

The phenomenon is familiar: a pair of predicates P and Q may be inconsistent in the sense that $P \wedge Q = \bot$, or they may be inconsistent in the stronger sense that $P \perp Q$. Physically, a pair of predicates P and Q that are inconsistent in the former sense but not in the latter sense correspond to Boolean observables with the property that no state provides the certainty of both P and Q, but there are states that provide the certainty of P and the possibility of Q, and vice versa. This distinction in turn involves the distinction between the meet connective $\wedge$ and the Sasaki projection connective &. Both connectives have a natural role in quantum predicate logic; see Lemma A.6.1 of Part I [17] and [9].

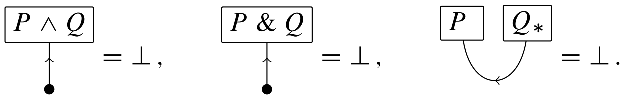
Proposition 4.1. *Let $\mathcal{X}$ be a quantum set, and let P and Q be predicates on $\mathcal{X}$. Then,*

(1) $P \wedge Q = \bot_{\mathcal{X}}$ *if and only if* $[\![\forall x \, \neg(P(x) \wedge Q(x))]\!] = \top$;

(2) $P \perp Q$ *if and only if* $[\![\forall x \, \neg(P(x) \,\&\, Q(x))]\!] = \top$;

(3) $P \perp Q$ *if and only if* $[\![\forall (x = x_*) \, \neg(P(x) \wedge Q_*(x_*))]\!] = \top$.

Proof. Each of the three equivalences may be established via the corresponding condition depicted graphically below:



■

Corollary 4.2. *The one-to-one correspondence of Theorem 3.3 restricts to a one-to-one correspondence between quantum partial orders V on $\ell^\infty(\mathcal{X})$ in the sense of* [47, Def. 2.6 (c)] *and binary relations R on $\mathcal{X}$ satisfying conditions* (1) *and* (2) *in the statement of Theorem 3.3 as well as*

(3) $[\![\forall x_1 \, \forall x_{2*} \, ((\hat{R}(x_1, x_{2*}) \wedge \hat{R}_*(x_{2*}, x_1)) \to x_1 = x_{2*})]\!] = \top$.

Proof. Let $S = R^\dagger$. By Lemma 2.2, we have that

$$[\![(x_1, x_{2*}) \in \mathcal{X} \times \mathcal{X}^* \mid \hat{R}_*(x_{2*}, x_1)]\!] = \hat{S}.$$

By Proposition 3.2.2 of Part I [17], we conclude that condition (3) is equivalent to $\hat{R} \wedge \hat{S} \leq E_{\mathcal{X}}$, i.e., to $R \wedge R^\dagger \leq I_{\mathcal{X}}$, i.e., to $V \cap V^\dagger \subseteq \ell^\infty(\mathcal{X})'$. ■

Lemma 4.3. *Let $\mathcal{X}$ be a quantum set. There is a one-to-one correspondence between binary relations R on $\mathcal{X}$ such that $I_{\mathcal{X}} \leq R$, $R \circ R \leq R$, and $R \,\&\, R^\dagger \leq I_{\mathcal{X}}$ and binary relations S on $\mathcal{X}$ such that $S \perp I_{\mathcal{X}}$ and $S \circ S \leq S$. The correspondence is given by $R \mapsto R \wedge \neg I_{\mathcal{X}}$ and by $S \mapsto S \vee I_{\mathcal{X}}$.*

Proof. We gather a couple of basic facts. First, for each binary relation R on $\mathcal{X}$ such that $I_{\mathcal{X}} \leq R$, we have that

$$\begin{aligned} R \,\&\, R^\dagger \leq I_{\mathcal{X}} \quad &\Longleftrightarrow \quad R \leq R^\dagger \to I_{\mathcal{X}} \quad \Longleftrightarrow \quad R \leq \neg R^\dagger \vee I_{\mathcal{X}} \\ &\Longleftrightarrow \quad R \wedge \neg I_{\mathcal{X}} \leq \neg R^\dagger \vee I_{\mathcal{X}} \quad \Longleftrightarrow \quad (R \wedge \neg I_{\mathcal{X}}) \perp (R \wedge \neg I_{\mathcal{X}})^\dagger, \end{aligned}$$

with the first equivalence following via the Sasaki adjunction and the third equivalence following by orthomodularity. Second, for each binary relation S on $\mathcal{X}$, we have that $S \perp S^\dagger \Leftrightarrow \mathrm{Tr}_{\mathcal{X}}(S \circ S) = \bot \Leftrightarrow S \circ S \perp I_{\mathcal{X}}$ [19, App. C].

Let R be a binary relation on $\mathcal{X}$ such that $I_{\mathcal{X}} \leq R$, $R \circ R \leq R$ and $R \,\&\, R^\dagger \leq I_{\mathcal{X}}$, and let $S = R \wedge \neg I_{\mathcal{X}}$. Clearly, $S \perp I_{\mathcal{X}}$. Furthermore, $S \perp S^\dagger$, which implies that $S \circ S \leq \neg I_{\mathcal{X}}$. We now calculate that $S \circ S \leq (R \circ R) \wedge \neg I_{\mathcal{X}} \leq R \wedge \neg I_{\mathcal{X}} = S$. Therefore, S satisfies both $S \perp I_{\mathcal{X}}$ and $S \circ S \leq S$.

Let S be a binary relation on $\mathcal{X}$ such that $S \perp I_{\mathcal{X}}$ and $S \circ S \leq S$, and let $R = S \vee I_{\mathcal{X}}$. Clearly, $I_{\mathcal{X}} \leq R$. Furthermore,

$$R \circ R = (S \circ S) \vee (S \circ I_{\mathcal{X}}) \vee (I_{\mathcal{X}} \circ S) \vee (I_{\mathcal{X}} \circ I_{\mathcal{X}}) \leq S \vee I_{\mathcal{X}} = R.$$

Finally, we observe that $S = R \wedge \neg I_{\mathcal{X}}$ by orthomodularity, and we reason that $S \perp I_{\mathcal{X}}$ implies $S \circ S \perp I_{\mathcal{X}}$, which implies $S \perp S^\dagger$, leading to $R \,\&\, R^\dagger \leq I_{\mathcal{X}}$. Therefore, R satisfies all three inequalities $I_{\mathcal{X}} \leq R$, $R \circ R \leq R$, and $R \,\&\, R^\dagger \leq I_{\mathcal{X}}$.

We have shown that the construction $R \mapsto R \wedge \neg I_{\mathcal{X}}$ takes binary relations of the first kind to binary relations of the second kind, and that the construction $S \mapsto S \vee I_{\mathcal{X}}$ takes binary relations of the second kind to binary relations of the first kind. The two constructions invert each other by orthomodularity. ■

Theorem 4.4. *Let H be a nonzero finite-dimensional Hilbert space, and let $\mathcal{X}$ be the quantum set defined by* $\mathrm{At}(\mathcal{X}) = \{H\}$. *Then, there is a one-to-one correspondence between nilpotent algebras $A \subseteq L(H)$ in the sense of* [49, Sec. 6] *and binary relations R on $\mathcal{X}$ satisfying conditions* (1) *and* (2) *in the statement of Theorem* 3.3 *as well as*

(3) $[\![\forall x_1\, \forall x_{2*}\, ((\hat{R}(x_1, x_{2*}) \,\&\, \hat{R}_*(x_{2*}, x_1)) \rightarrow x_1 = x_{2*})]\!] = \top.$

This correspondence is given by $R(H, H) = A + \mathbb{C}1_H$.

Proof. Condition (1) is equivalent to the inequality $I_{\mathcal{X}} \leq R$ by Proposition 2.1. Condition (2) is equivalent to the inequality $R \circ R \leq R$ by Lemma 3.2. Condition (3) is equivalent to the inequality $R \,\&\, R^\dagger \leq I_{\mathcal{X}}$, as in the proof of Corollary 4.2. Thus, by Lemma 4.3, the binary relations R on $\mathcal{X}$ satisfying conditions (1), (2), and (3) are in one-to-one correspondence with the binary relations S on $\mathcal{X}$ satisfying $S \perp I_{\mathcal{X}}$ and $S \circ S \leq S$, which is given by

$$R = S \vee I_{\mathcal{X}},$$

i.e., by $R(H, H) = S(H, H) + \mathbb{C}1_H$.

We also have a one-to-one correspondence between the binary relations S on $\mathcal{X}$ and subspaces A of $L(H)$, which is given by $A = S(H, H)$. The binary relations S that satisfy $S \perp I_{\mathcal{X}}$ correspond to subspaces of trace-zero operators, and the binary relations S that satisfy $S \circ S \leq S$ correspond to subalgebras. Thus, the binary relations S on $\mathcal{X}$ that satisfy both inequalities correspond to subalgebras of trace-zero operators. These are exactly the nilpotent subalgebras because an operator $a \in L(H)$ is nilpotent if and only if $\mathrm{Tr}_H(a^n) = 0$ for each positive integer n.

Combining the one-to-one correspondences, we find that binary relations R on $\mathcal{X}$ satisfying conditions (1), (2), and (3) correspond to nilpotent subalgebras A of $L(H)$ via the equation $R(H, H) = A + \mathbb{C}1_H$. ■

5. Functions between quantum sets

We have characterized functions between quantum sets as binary relations satisfying a pair of intelligible formulas, which certainly characterize ordinary functions within the dagger compact category of ordinary sets and ordinary binary relations; see Definition 2.6.1 and Theorem 3.4.2 of Part I [17]. Now, we give streamlined characterizations of functions, injective functions and surjective functions between quantum sets, emphasizing our recovery of the corresponding concepts from noncommutative geometry.

Proposition 5.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets. Then, there is a one-to-one correspondence between unital normal $*$-homomorphisms ϕ from $\ell^\infty(\mathcal{Y})$ to $\ell^\infty(\mathcal{X})$ and binary relations F on $\mathcal{X}$ such that*

(1) $[\![\forall x\, \exists y_*\, \hat{F}(x, y_*)]\!] = \top$;

(2) $[\![\forall(x = x_*)\, \forall(y_1 = y_{1*})\, \forall(y_2 = y_{2*})\, ((\hat{F}(x, y_{1*}) \wedge \hat{F}_*(x_*, y_2)) \to y_1 = y_{2*})]\!] = \top$.

These are exactly the functions F from $\mathcal{X}$ to $\mathcal{Y}$ [16, Def. 4.1]. *This correspondence is given by $\phi = F^\star$* [16, Thm. 7.4].

Proof. This proposition combines Theorem 3.4.2 of Part I [17] with Lemmas 2.2 and 3.2. In Lemma 3.2, we set $R = F^\dagger$, $S = F$, and $T = E_\mathcal{Y}$ to infer that

$$[\![\forall y_1\, \forall y_{2*}\, (\exists(x = x_*)\, (\hat{R}(y_1, x_*) \wedge \hat{F}(x, y_{2*})) \to y_1 = y_{2*})]\!] = \top$$

is equivalent to

$$[\![\forall(y_1 = y_{1*})\, \forall(x = x_*)\, \forall(y_2 = y_{2*})\, ((\hat{R}(y_1, x_*) \wedge \hat{F}(x, y_{2*})) \to y_{1*} = y_2)]\!] = \top. \quad (\S)$$

By Lemma 2.2,

$$[\![(y_1, x_*) \in \mathcal{Y} \times \mathcal{X}^* \mid \hat{F}_*(x_*, y_1)]\!] = \hat{R} = [\![(y_1, x_*) \in \mathcal{Y} \times \mathcal{X}^* \mid \hat{R}(y_1, x_*)]\!],$$

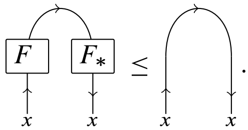
so we can replace $\hat{R}(y_1, x_*)$ by $\hat{F}_*(x_*, y_1)$ in both formulas. We thus recover the definition of a function graph (Definition 2.6.1 of Part I [17]). Similarly, appealing to the fact that $I_\mathcal{X}$ is self-adjoint, we can replace $y_{1*} = y_2$ by $y_2 = y_{1*}$ in formula (§). Exchanging the variables y_1 and y_2 and permuting the quantifiers, we recover condition (2). ∎

Proposition 5.2. *The one-to-one correspondence in Proposition 5.1 restricts to a one-to-one correspondence between surjective unital normal $*$-homomorphisms ϕ from $\ell^\infty(\mathcal{Y})$ to $\ell^\infty(\mathcal{X})$ and binary relations F from $\mathcal{X}$ to $\mathcal{Y}$ satisfying conditions* (1) *and* (2) *in the statement of that proposition, together with*

(3) $[\![\forall x\, \forall x_*\, (F(x) = F_*(x_*) \to x = x_*)]\!] = \top$.

These are exactly the injective functions F from $\mathcal{X}$ to $\mathcal{Y}$ [16, Def. 4.3 (1)].

Proof. Appealing to Proposition 3.2.2 and Lemma 3.5.2 of Part I [17], we find that condition (3) is equivalent to the following inequality:



Straightening the wire, we conclude that condition (3) is equivalent to the inequality $F^\dagger \circ F \le I_{\mathcal{X}}$, i.e., to F being injective. A function F is injective if and only if $F^\star$ is surjective [16, Prop. 8.4]. ■

Proposition 5.3. *The one-to-one correspondence in Proposition 5.1 restricts to a one-to-one correspondence between injective unital normal $*$-homomorphisms ϕ from $\ell^\infty(\mathcal{Y})$ to $\ell^\infty(\mathcal{X})$ and binary relations from $\mathcal{X}$ to $\mathcal{Y}$ satisfying conditions* (1) *and* (2) *in the statement of that proposition, together with*

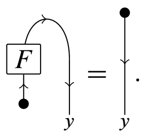
(4) $[\![\forall y_* \, \exists x \; F(x) = y_*]\!] = \top$.

These are exactly the surjective functions F from $\mathcal{X}$ to $\mathcal{Y}$ [16, Def. 4.3 (2)].

Proof. We glean from the proof of Theorem 3.4.2 of Part I [17] that $\top_{\mathcal{Y}} \circ F = \top_{\mathcal{X}}$, simply because F is a function. Similarly, condition (4) is equivalent to the inequality

$$\top_{\mathcal{X}} \circ F^\dagger = \top_{\mathcal{Y}},$$

as we infer from its depiction below:



By the definition of a function from $\mathcal{X}$ to $\mathcal{Y}$, the function F satisfies the inequality $F \circ F^\dagger \le I_{\mathcal{Y}}$. It is surjective if and only if $F \circ F^\dagger = I_{\mathcal{Y}}$. If $F \circ F^\dagger = I_{\mathcal{Y}}$, then

$$\top_{\mathcal{X}} \circ F^\dagger \ge \top_{\mathcal{Y}} \circ F \circ F^\dagger = \top_{\mathcal{Y}} \circ I_{\mathcal{X}},$$

and therefore, $\top_{\mathcal{X}} \circ F^\dagger = \top_{\mathcal{Y}}$. Inversely, if $F \circ F^\dagger \ne I_{\mathcal{Y}}$, then the inequality $F \circ F^\dagger \le I_{\mathcal{Y}}$ implies that $(F \circ F^\dagger)(W, Y) = 0$ for some atom $W \in \mathrm{At}(\mathcal{Y})$ and all atoms $Y \in \mathrm{At}(\mathcal{Y})$. Thus, $\mathcal{Y}$ has a nonempty subset $\mathcal{W}$ whose inclusion function J satisfies $F \circ F^\dagger \circ J = \bot_{\mathcal{W}}^{\mathcal{Y}}$, where $\bot_{\mathcal{W}}^{\mathcal{Y}}$ denotes the minimum binary relation from $\mathcal{W}$ to $\mathcal{Y}$. If we suppose that $\top_{\mathcal{X}} \circ F^\dagger = \top_{\mathcal{Y}}$, then we may calculate that $\bot_{\mathcal{W}} = \top_{\mathcal{Y}} \circ F \circ F^\dagger \circ J = \top_{\mathcal{X}} \circ F^\dagger \circ J = \top_{\mathcal{Y}} \circ J = \top_{\mathcal{W}}$, contradicting that $\mathcal{W}$ is nonempty. Therefore, if $F \circ F^\dagger \ne I_{\mathcal{Y}}$, then $\top_{\mathcal{X}} \circ F^\dagger \ne \top_{\mathcal{Y}}$.

We conclude that F satisfies condition (4) if and only if it is surjective. A function F is surjective if and only if $F^\star$ is injective [16, Prop. 8.1]. ■

Noting the similarity between the proof of Proposition 5.3 and the proof of Theorem 3.4.2 of Part I [17], the reader may well ask whether a binary relation R from $\mathcal{X}$ to $\mathcal{Y}$ is surjective in the sense of [16, Def. 4.3 (2)] if and only if $F \circ \top_{\mathcal{X}}^{\dagger} = \top_{Y}^{\dagger}$ or equivalently $[\![\forall y_* \, \exists x \, R(x, y_*)]\!] = \top$. The answer is no, by the same simple example that appeared at the end of [16, Sec. 4], which we now revisit. Thus, the notion of surjectivity for binary relations has multiple natural generalizations to the quantum setting.

Let $\mathcal{X}$ be the quantum set whose only atom is $\mathbb{C}^2$, let $a \in L(\mathbb{C}^2, \mathbb{C}^2)$ be an invertible matrix that is not a scalar multiple of a unitary matrix, and let R be the binary relation on $\mathcal{X}$ defined by $R(\mathbb{C}^2, \mathbb{C}^2) = \mathbb{C}a$. By our choice of a, the binary relation R does not satisfy the inequality $R \circ R^{\dagger} \geq I_{\mathcal{X}}$. However, by our choice of a, the binary relation R is clearly invertible, so it does satisfy the inequality $R \circ \top_{\mathcal{X}}^{\dagger} \geq \top_{\mathcal{Y}}^{\dagger}$, as a simple consequence of $\top_{\mathcal{X}}^{\dagger} \geq R^{-1} \circ \top_{\mathcal{Y}}^{\dagger}$. We conclude that R does not satisfy $R \circ R^{\dagger} \geq I_{\mathcal{X}}$ and that it does satisfy $R \circ \top_{\mathcal{X}}^{\dagger} = \top_{\mathcal{Y}}^{\dagger}$.

6. Quantum metric spaces

A quantum metric on a von Neumann algebra $M \subseteq L(H)$ is a family of ultraweakly closed subspaces $(V_\alpha \subseteq L(H) \mid \alpha \in [0, \infty))$ such that $V_\alpha V_\beta \subseteq V_{\alpha+\beta}$ for all $\alpha, \beta \in [0, \infty)$, $V_\alpha = \bigcap_{\beta > \alpha} V_\beta$ for all $\alpha \in [0, \infty)$ and $V_0 = M'$ [21, Defs. 2.1 (a) and 2.3]. Intuitively, each subspace V_α consists of those operators which transform the configuration of the system to a configuration that is at most distance $\alpha \in [0, \infty)$ away. In the motivating example of quantum Hamming distance, $M = M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})$ and V_α is the span of operators $a_1 \otimes \cdots \otimes a_n$, with $a_i \neq 1$ for at most α indices $i \in \{1, \ldots, n\}$. Thus, V_α is spanned by operators that corrupt at most α qubits. Note that these quantum metrics generalize metrics for which the distance between two points may be infinite [21, Sec. 2.1, Prop. 2.5].

Lemma 6.1. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be quantum sets, and let $A_1, \ldots, A_m$ be ordinary sets. Let $t_1, \ldots, t_m$ be terms of sorts ‘$A_1, \ldots,$ ‘A_m, whose free variables are among $x_1, \ldots, x_n$ of sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $x_{1*}, \ldots, x_{n*}$ of sorts $\mathcal{X}_1^*, \ldots, \mathcal{X}_n^*$. Let r be an ordinary relation of arity $(A_1, \ldots, A_m)$. If ‘$r(t_1, \ldots, t_n)$ is nonduplicating, then the following are equivalent:*

(1) $[\![\forall(x_1 = x_{1*}) \cdots \forall(x_n = x_{n*})\, ‘r(t_1, \ldots, t_m)]\!] = \top$*;*

(2) *for all* $(a_1, \ldots, a_m) \in A_1 \times \cdots \times A_m \setminus r$*,*

$$[\![\exists(x_1 = x_{1*}) \cdots \exists(x_n = x_{n*})\, (t_1 = ‘a_1 \wedge \cdots \wedge t_m = ‘a_m)]\!] = \bot.$$

The following proof includes formulas that may be infinite conjunctions or infinite disjunctions of other formulas. Formally, such formulas are not part of our language, but they pose no special difficulty. Indeed, Definition 2.3.1 of Part I [17] excludes infinitary conjunction exclusively for the sake of the exposition. In light of Lemma A.6.1 of Part I [17], we may regard each infinitary conjunction $\bigwedge_{(a_1,\ldots,a_n)\in A^n} \phi(‘a_1, \ldots, ‘a_m, y_1, \ldots, y_n)$ as abbreviating

$$\forall x_1 \cdots \forall x_n \, \phi(x_1, \ldots, x_n, y_1, \ldots, y_n),$$

and we may do similarly for each infinitary conjunction

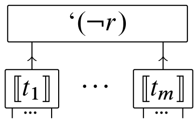
$$\bigwedge_{(a_1,\dots,a_n)\in A^n} \phi(‘a_{1*},\dots,‘a_{m*},y_1,\dots,y_n).$$

Proof of Lemma 6.1. The relation $\neg r = A_1 \times \cdots \times A_m \setminus r$ may be regarded as a morphism in the dagger compact category of ordinary sets and binary relations. As a binary relation from $A_1 \times \cdots \times A_m$ to the singleton set $\{*\}$, it is defined by

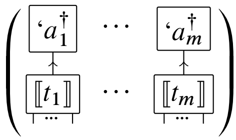
$$\neg r = \sup\{a_1^\dagger \times \cdots \times a_m^\dagger \mid (a_1,\dots,a_m) \in A_1 \times \cdots \times A_m \setminus r\}.$$

For brevity, we will write $\bar{a} = (a_1,\dots,a_m)$:

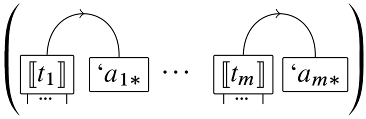
$$[\![‘(\neg r)(t_1,\dots,t_n)]\!] =$$



$$= \bigvee_{(a_1,\dots,a_m)\in\neg r}$$



$$= \bigvee_{(a_1,\dots,a_m)\in\neg r}$$



$$= [\![\textstyle\bigvee_{\bar{a}\in\neg r}(t_1 = ‘a_{1*} \wedge \cdots \wedge t_m = ‘a_{m*})]\!],$$

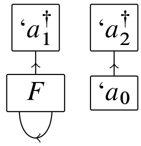
$$\begin{aligned}
&[\![\forall(x_1 = x_{1*})\cdots\forall(x_n = x_{n*})\,‘r(t_1,\dots,t_m)]\!]\\
&\quad= [\![\neg\exists(x_1 = x_{1*})\cdots\exists(x_n = x_{n*})\,\neg‘r(t_1,\dots,t_m)]\!]\\
&\quad= [\![\neg\exists(x_1 = x_{1*})\cdots\exists(x_n = x_{n*})\,(\neg‘r)(t_1,\dots,t_m)]\!]\\
&\quad= [\![\neg\exists(x_1 = x_{1*})\cdots\exists(x_n = x_{n*})\,‘(\neg r)(t_1,\dots,t_m)]\!]\\
&\quad= [\![\neg\exists(x_1 = x_{1*})\cdots\exists(x_n = x_{n*})\,\textstyle\bigvee_{\bar{a}\in\neg r}(t_1 = ‘a_{1*} \wedge \cdots \wedge t_m = ‘a_{m*})]\!]\\
&\quad= [\![\neg\textstyle\bigvee_{\bar{a}\in\neg r}\exists(x_1 = x_{1*})\cdots\exists(x_n = x_{n*})\,(t_1 = ‘a_{1*} \wedge \cdots \wedge t_m = ‘a_{m*})]\!]\\
&\quad= \bigwedge_{\bar{a}\in\neg r}[\![\neg\exists(x_1 = x_{1*})\cdots\exists(x_n = x_{n*})\,(t_1 = ‘a_{1*} \wedge \cdots \wedge t_m = ‘a_{m*})]\!].
\end{aligned}$$

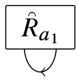
For the second equality, we appeal to Lemma 3.5.2 of Part I [17]; [16, Thm. B.8] implies that for each function from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ and each binary relation R from $\mathcal{Y}$ to $\mathbf{1}$, we have that $\neg(R \circ F) = (\neg R) \circ F$. For the penultimate equality, we appeal to Theorem 3.3.2 of Part I [17] and either to Proposition 3.2.3 of Part I [17] or to the fact that the composition of binary relations between quantum sets respects arbitrary joins in both arguments, according to our gloss of the infinitary disjunction symbol. The conjunction of binary relations is equal to $\top$ if and only if each conjunct is equal to $\top$, so the statement of the theorem follows. ∎

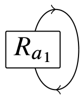
Proposition 6.2. *Let $\mathcal{X}$ be a quantum set, let A be an ordinary set, and let $F\colon \mathcal{X} \times \mathcal{X}^* \to ‘A$ be a function. For each $a \in A$, let R_a be the binary relation on $\mathcal{X}$ defined by $\hat{R}_a = ‘a^\dagger \circ F$. Let $a_0 \in A$. Then, $R_{a_0} \geq I_{‘A}$ if and only if $[\![\forall(x_1 = x_{1*})\,‘(=_A)(F(x_1,x_{1*}),‘a_0)]\!] = \top$.*

Proof. Applying Lemma 6.1, we find that the following are equivalent:

$$[\![\forall(x_1 = x_{1*})\, `(=_A)(F(x_1, x_{1*}), `a_0)]\!] = \top$$

$\iff$ for all distinct $a_1, a_2 \in A$,  $= \bot$

$\iff$ for all $a_1 \in A$ distinct from a_0,  $= \bot$

$\iff$ for all $a_1 \in A$ distinct from a_0,  $= \bot$

$\iff$ for all $a_1 \in A$ distinct from a_0, $R_{a_1} \perp I_{`A}$

$\iff$ $I_{`A} \leq R_{a_0}$.

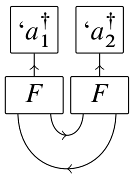
The second equivalence follows from the fact that $`a_2^\dagger \circ `a_0 = \top$ or $`a_2^\dagger \circ `a_0 = \bot$ according to whether $a_2 = a_0$ or $a_2 \neq a_0$. The last equivalence follows from the fact that $\{`a^\dagger \circ F \mid a \in A\}$ consists of pairwise-orthogonal relations of arity $(\mathcal{X}, \mathcal{X}^*)$ whose join is the maximum relation of arity $(\mathcal{X}, \mathcal{X}^*)$, and thus, $\{R_a \mid a \in A\}$ consists of binary relations on $\mathcal{X}$ whose join is the maximum binary relation on $\mathcal{X}$. ■

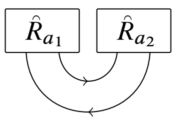
Proposition 6.3. *Let $\mathcal{X}$ be a quantum set, let A be an ordinary set, and let $F\colon \mathcal{X} \times \mathcal{X}^* \to `A$ be a function. For each $a \in A$, let $R_a$ be the binary relation on $\mathcal{X}$ defined by $\hat{R}_a = `a^\dagger \circ F$. Then, $R_a^\dagger = R_a$ for all $a \in A$ if and only if*

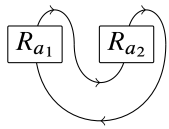
$$[\![\forall(x_1 = x_{1*})\, \forall(x_2 = x_{2*})\, `(=_A)(F(x_1, x_{2*}), F(x_2, x_{1*}))]\!] = \top.$$

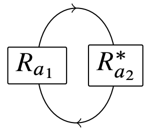
Proof. Applying Lemma 6.1, we find that the following are equivalent:

$$[\![\forall(x_1 = x_{1*})\, \forall(x_2 = x_{2*})\, `(=_A)(F(x_1, x_{2*}), F(x_2, x_{1*}))]\!] = \top$$

$\iff$ for all distinct $a_1, a_2 \in A$,  $= \bot$

$\iff$ for all distinct $a_1, a_2 \in A$,  $= \bot$

$\iff$ for all distinct $a_1, a_2 \in A$,  $= \bot$

$\iff$ for all distinct $a_1, a_2 \in A$,  $= \bot$

$\iff$ for all distinct $a_1, a_2 \in A$, $R_{a_1} \perp R^\dagger_{a_2}$

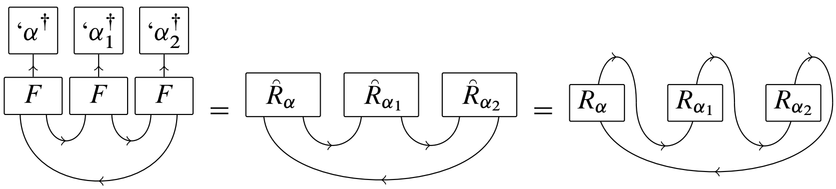
$\iff$ for all $a \in A$, $R_a \leq R^\dagger_a$.

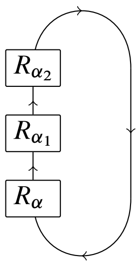
The second-to-last equivalence holds because $R^*_{a_2} = (R^\dagger_{a_2})_*$. The last equivalence holds because $\{R_a \mid a \in A\}$ consists of pairwise-orthogonal relations on $\mathcal{X}$ whose join is the maximum relation on $\mathcal{X}$. The inequality $R_a \leq R^\dagger_a$ for all $a \in A$ is equivalent to the equality $R_a = R^\dagger_a$ for all $a \in A$ because the adjoint operation is an order isomorphism. ∎

Proposition 6.4. *Let $\mathcal{X}$ be a quantum set, and let $F\colon \mathcal{X} \times \mathcal{X}^* \to$ ‘$[0,\infty]$ be a function. For each $\alpha \in [0,\infty]$, let R_α be the binary relation on $\mathcal{X}$ defined by $\hat{R}_\alpha =$ ‘$\alpha^\dagger \circ F$. Then, $R_{\alpha_2} \circ R_{\alpha_1} \leq \bigvee_{\alpha \leq \alpha_1 + \alpha_2} R^\dagger_\alpha$ for all $\alpha_1, \alpha_2 \in [0,\infty]$ if and only if*

$$[\![\forall(x_1 = x_{1*})\,\forall(x_2 = x_{2*})\,\forall(x_3 = x_{3*})\,‘(\hat{\leq})(F(x_1, x_{2*}), ‘(+)(F(x_2, x_{3^*}), F(x_3, x_{1*})))]\!] = \top.$$

Proof. Refer to [19, App. C] for the basic properties of the trace on binary relations. Let $\alpha, \alpha_1, \alpha_2 \in [0,\infty]$. We compute that



$=$  $= \mathrm{Tr}_{\mathcal{X}}(R_{\alpha_2} \circ R_{\alpha_1} \circ R_\alpha).$

Let r be the ordinary relation of arity $([0,\infty], [0,\infty], [0,\infty])$ defined by

$$r = (\hat{\leq}) \circ (\mathrm{id}_{[0,\infty]} \times (+)) = \{(\alpha, \alpha_1, \alpha_2) \in [0,\infty]^3 \mid \alpha \leq \alpha_1 + \alpha_2\}.$$

Applying Lemma 6.1, we find that the following are equivalent:

$$[\![\forall(x_1 = x_{1*})\,\forall(x_2 = x_{2*})\,\forall(x_3 = x_{3*})\,`(\hat{\le})(F(x_1, x_{2*}), `(+)(F(x_2, x_{3*}), F(x_3, x_{1*})))]\!] = \top$$

$$\iff [\![\forall(x_1 = x_{1*})\,\forall(x_2 = x_{2*})\,\forall(x_3 = x_{3*})\,`r(F(x_1, x_{2*}), F(x_2, x_{3*}), F(x_3, x_{1*}))]\!] = \top$$

$$\iff \text{for all } \alpha, \alpha_1, \alpha_2 \in [0, \infty] \text{ such that } \alpha > \alpha_1 + \alpha_2, \quad \mathrm{Tr}_{\mathcal{X}}(R_{\alpha_2} \circ R_{\alpha_1} \circ R_\alpha) = \bot$$

$$\iff \text{for all } \alpha, \alpha_1, \alpha_2 \in [0, \infty] \text{ such that } \alpha > \alpha_1 + \alpha_2, \quad \mathrm{Tr}_{\mathcal{X}}((R_\alpha^\dagger)^\dagger \circ (R_{\alpha_2} \circ R_{\alpha_1})) = \bot$$

$$\iff \text{for all } \alpha, \alpha_1, \alpha_2 \in [0, \infty] \text{ such that } \alpha > \alpha_1 + \alpha_2, \quad R_{\alpha_2} \circ R_{\alpha_1} \perp R_\alpha^\dagger$$

$$\iff \text{for all } \alpha_1, \alpha_2 \in [0, \infty], \quad R_{\alpha_2} \circ R_{\alpha_1} \le \bigvee_{\alpha \le \alpha_1 + \alpha_2} R_\alpha^\dagger.$$

The first equivalence follows by the graphical calculus via Lemma 3.5.2 of Part I [17]. ∎

Let A be an ordinary set. The relations $`(=_A)$ and $E_{`A}$ are both, in some sense, equality relations on $`A$, but they are distinct because they have different arities. The relation $`(=_A)$ has arity $(`A, `A)$, and the relation $E_{`A}$ has arity $(`A, (`A)^*)$. Nevertheless, they are very closely related because $`A$ and $(`A)^*$ are naturally isomorphic via the "conjugation" function $C_A\colon `A \to (`A)^*$, defined by $C_A(\mathbb{C}_a, \mathbb{C}_a^*) = L(\mathbb{C}_a, \mathbb{C}_a^*)$ for all $a \in A$, with the other components vanishing. The function $C_A$ is intuitively the identity on $`A$, and this can be arranged to hold formally [16, App. D].

Theorem 6.5. *Let $\mathcal{X}$ be a quantum set. Let T be the binary relation on $`[0, \infty]$ defined by $T = `(\le)$. Then, there is a one-to-one correspondence between quantum pseudometrics $(V_\beta \mid \beta \in [0, \infty))$ on $\ell^\infty(\mathcal{X})$ in the sense of* [21, Def. 2.3] *and functions $F\colon \mathcal{X} \times \mathcal{X}^* \to `[0, \infty]$ such that*

(1) $[\![\forall(x_1 = x_{1*})\, F(x_1, x_{1*}) = `0_*]\!] = \top$;

(2) $[\![\forall(x_1 = x_{1*})\,\forall(x_2 = x_{2*})\, F(x_1, x_{2*}) = C_{[0,\infty]}(F(x_2, x_{1*}))]\!] = \top$;

(3) $[\![\forall(x_1 = x_{1*})\,\forall(x_2 = x_{2*})\,\forall(x_3 = x_{3*})\,\hat{T}(F(x_1, x_{2*}), `(+)_*(F_*(x_{1*}, x_3), F_*(x_{3*}, x_2)))]\!] = \top$.

This correspondence is given by $\sum_{\alpha \in [0,\beta]} R_\alpha(X_1, X_2) = \mathrm{inc}_{X_2}^\dagger \cdot V_\beta \cdot \mathrm{inc}_{X_1}$, for $\beta \in [0, \infty)$ and $X_1, X_2 \in \mathrm{At}(\mathcal{X})$, where $\hat{R}_\alpha = `\alpha^\dagger \circ F$ for each $\alpha \in [0, \infty]$.

Proof. Let $A = [0, \infty]$, and let $a_0 = 0$. The equation $\hat{R}_\alpha = `\alpha^\dagger \circ F$, for $\alpha \in A$, defines a one-to-one correspondence between functions $F\colon \mathcal{X} \times \mathcal{X}^* \to `A$ and indexed families $(R_\alpha \mid \alpha \in A)$ of pairwise disjoint binary relations on $\mathcal{X}$ whose join is $\top_{\mathcal{X}}^{\mathcal{X}}$, the maximum binary relation on $\mathcal{X}$. In the context of this correspondence, condition (1) is equivalent to the equation in Proposition 6.2 because

$$E_{`A} \circ (F \times `0_*) = E_{`A} \circ (F \times (C_A \circ `0)) = E_{`A} \circ (I_{`A} \times C_A) \circ (F \times `0) = `(=_A) \circ (F \times `0).$$

Similarly, condition (2) is equivalent to the equation in Proposition 6.3 because

$$E_{`A} \circ (F \times (C_A \circ F)) = E_{`A} \circ (I_{`A} \times C_A) \circ (F \times F) = `(=_A) \circ (F \times F).$$

Thus, F satisfies conditions (1) and (2) if and only if $R_0 \geq I_{`A}$ and $R_\alpha^\dagger = R_\alpha$ for all $\alpha \in [0, \infty]$.

Assuming conditions (1) and (2), Lemma 2.2 implies that for all $\alpha \in [0, \infty]$ we have $\hat{R}_{\alpha *} \circ B_{\mathcal{X},\mathcal{X}^*} = \hat{R}_\alpha$, i.e., $`\alpha_*^\dagger \circ F_* \circ B_{\mathcal{X},\mathcal{X}^*} = `\alpha^\dagger \circ F$, where $B_{\mathcal{X},\mathcal{X}^*}$ is the braiding of $\mathcal{X}$ and $\mathcal{X}^*$. Thus, $`\alpha_*^\dagger \circ F_* \circ B_{\mathcal{X},\mathcal{X}^*} = `\alpha_*^\dagger \circ C_A \circ F$ for all $\alpha \in A$; we conclude that $F_* \circ B_{\mathcal{X},\mathcal{X}^*} = C_A \circ F$. We now calculate that

$$\begin{aligned}
&`(\hat{\leq}) \circ (F \times (`(+) \circ (F \times F))) = `(\hat{\leq}) \circ (F \times (`(+) \circ B_{`[0,\infty],`[0,\infty]} \circ (F \times F)))\\
&= `(\hat{\leq}) \circ (F \times (`(+) \circ (F \times F) \circ B_{\mathcal{X}\times\mathcal{X}^*,\mathcal{X}\times\mathcal{X}^*}))\\
&= \hat{T} \circ (I_{`A} \times C_A) \circ (F \times (`(+) \circ (F \times F) \circ B_{\mathcal{X}\times\mathcal{X}^*,\mathcal{X}\times\mathcal{X}^*}))\\
&= \hat{T} \circ (F \times (C_A \circ `(+) \circ (F \times F) \circ B_{\mathcal{X}\times\mathcal{X}^*,\mathcal{X}\times\mathcal{X}^*}))\\
&= \hat{T} \circ (F \times (`(+)_* \circ (C_A \times C_A) \circ (F \times F) \circ B_{\mathcal{X}\times\mathcal{X}^*,\mathcal{X}\times\mathcal{X}^*}))\\
&= \hat{T} \circ (F \times (`(+)_* \circ (F_* \times F_*) \circ (B_{\mathcal{X},\mathcal{X}^*} \times B_{\mathcal{X},\mathcal{X}^*}) \circ B_{\mathcal{X}\times\mathcal{X}^*,\mathcal{X}\times\mathcal{X}^*}))\\
&= \hat{T} \circ (F \times (`(+)_* \circ (F_* \times F_*))) \circ (I_{\mathcal{X}\times\mathcal{X}^*} \times ((B_{\mathcal{X},\mathcal{X}^*} \times B_{\mathcal{X},\mathcal{X}^*}) \circ B_{\mathcal{X}\times\mathcal{X}^*,\mathcal{X}\times\mathcal{X}^*})).
\end{aligned}$$

We conclude that, modulo conditions (1) and (2), condition (3) is equivalent to the equation in Proposition 6.4.

Therefore, we have a one-to-one correspondence between functions $F\colon \mathcal{X} \times \mathcal{X}^* \to `[0, \infty]$, satisfying conditions (1), (2), and (3), and indexed families $(R_\alpha \mid \alpha \in [0, \infty])$ of pairwise orthogonal binary relations on $\mathcal{X}$, whose join is $\top_{\mathcal{X}}^{\mathcal{X}}$ and which satisfy $R_0 \geq I_{\mathcal{X}}$, $R_\alpha^\dagger = R_\alpha$ for all $\alpha \in [0, \infty]$ and $R_{\alpha_2} \circ R_{\alpha_1} \leq \bigvee_{\alpha \leq \alpha_1 + \alpha_2} R_\alpha$ for all $\alpha_1, \alpha_2 \in [0, \infty]$.

The equation $\bigvee_{\alpha \leq \beta} R_\alpha = S_\beta$ defines a one-to-one correspondence between indexed families $(R_\alpha \mid \alpha \in [0, \infty])$ such that $\bigvee_\alpha R_\alpha = \top_{\mathcal{X}}^{\mathcal{X}}$ and $R_{\alpha_1} \perp R_{\alpha_2}$ for all distinct $\alpha_1, \alpha_2 \in [0, \infty]$ and indexed families $(S_\beta \mid \beta \in [0, \infty))$ such that $S_{\beta_0} = \bigwedge_{\beta > \beta_0} S_\beta$ for all $\beta_0 \in [0, \infty)$. The existence of this correspondence becomes readily apparent by identifying the binary relations R_α and the binary relations S_β with projections in the hereditarily atomic von Neumann algebra $\ell^\infty(\mathcal{X} \times \mathcal{X}^*)$. In the context of this correspondence, $R_0 \geq I_{\mathcal{X}}$ is equivalent to $S_0 \geq I_{\mathcal{X}}$, $R_\alpha^\dagger = R_\alpha$ for all $\alpha \in [0, \infty]$ is equivalent to $S_\beta^\dagger = S_\beta$ for all $\beta \in [0, \infty)$, and $R_{\alpha_2} \circ R_{\alpha_1} \leq \bigvee_{\alpha \leq \alpha_1 + \alpha_2} R_\alpha$ for all $\alpha_1, \alpha_2 \in [0, \infty]$ is equivalent to $S_{\beta_1} \circ S_{\beta_2} \leq S_{\beta_1 + \beta_2}$ for all $\beta_1, \beta_2 \in [0, \infty)$. Families $(S_\beta \mid \beta \in [0, \infty))$ with these four properties are in obvious one-to-once correspondence with quantum pseudometrics $(V_\beta \mid \beta \in [0, \infty))$ on $\ell^\infty(\mathcal{X})$ via the expression $S_\beta(X_1, X_2) = \mathrm{inc}_{X_2}^\dagger \cdot V_\beta \cdot \mathrm{inc}_{X_1}$, as in Appendix A.2 of Part I [17].

In summary, the equation $\hat{R}_\alpha = `\alpha^\dagger \circ F$, for $\alpha \in [0, \infty]$, defines a one-to-one correspondence between functions $F\colon \mathcal{X} \times \mathcal{X}^* \to `[0, \infty]$, satisfying conditions (1), (2), and (3), and families $(R_\alpha \mid \alpha \in [0, \infty])$ of binary relations on $\mathcal{X}$, satisfying

(a) $R_{\alpha_1} \perp R_{\alpha_2}$ for distinct $\alpha_1, \alpha_2 \in [0, \infty]$;

(b) $\bigvee_\alpha R_\alpha = \top_{\mathcal{X}}^{\mathcal{X}}$;

(c) $R_0 \geq I_{\mathcal{X}}$;

(d) $R_\alpha^\dagger = R_\alpha$ for $\alpha \in [0, \infty]$;

(e) $R_{\alpha_2} \circ R_{\alpha_1} \leq \bigvee_{\alpha \leq \alpha_1 + \alpha_2} R_\alpha$ for $\alpha_1, \alpha_2 \in [0, \infty]$.

Furthermore, the equation $\sum_{\alpha \leq \beta} R_\alpha(X_1, X_2) = \mathrm{inc}_{X_2}^\dagger \cdot V_\beta \cdot \mathrm{inc}_{X_1}$, for $\beta \in [0, \infty)$, defines a one-to-one correspondence between such families $(R_\alpha \mid \alpha \in [0, \infty])$ and quantum pseudometrics $(V_\beta \mid \beta \in [0, \infty))$ on $\ell^\infty(\mathcal{X})$. Thus, the theorem is proved. ∎

Corollary 6.6. *The one-to-one correspondence of Theorem* 6.5 *restricts to a one-to-one correspondence between quantum metrics* $(V_\beta \mid \beta \in [0, \infty))$ *on* $\ell^\infty(\mathcal{X})$ *in the sense of* [21, Def. 2.3] *and functions* $F\colon \mathcal{X} \times \mathcal{X}^* \to {}`[0, \infty]$ *satisfying conditions* (1)*,* (2)*, and* (3) *in the statement of that theorem, together with*

(4) $[\![\forall x_1\, \forall x_{2*}\, (F(x_1, x_{2*}) = 0_* \to x_1 = x_{2*})]\!] = \top$.

Proof. We have that $[\![(x_1, x_{2*}) \in \mathcal{X} \times \mathcal{X}^* \mid F(x_1, x_{2*}) = 0_*]\!] = {}`0^\dagger \circ F = \hat{R}_0$ by the graphical calculus, so condition (4) is equivalent to $\hat{R}_0 \leq E_{\mathcal{X}}$ by Proposition 3.2.2 of Part I [17]. We now reason that $\hat{R}_0 \leq E_{\mathcal{X}}$ is equivalent to $R_0 \leq I_{\mathcal{X}}$, which is equivalent to $V_0 \leq \ell^\infty(\mathcal{X})'$ via the correspondence between binary relations and quantum relations given in Appendix A.2 of Part I [17] because $\ell^\infty(\mathcal{X})'$ is the identity quantum relation on $\ell^\infty(\mathcal{X})$. ∎

7. Quantum families of graph isomorphisms

The graph coloring game is played by two cooperating players, traditionally named Alice and Bob, against a referee. The parameters of the game are a finite simple graph G and a finite set of colors C, and the rules of the game are such that, classically, Alice and Bob have a winning strategy if and only if G can be properly colored by C. Alice and Bob are forbidden from communicating with each other during the course of the game, and the existence of a proper graph coloring ensures that they are able to successfully coordinate their responses to the referee. However, if Alice and Bob possess entangled quantum systems, then they may have a winning strategy even if no proper graph coloring exists. In this case, one says that a quantum graph coloring exists. Section I.A of [16] contains a longer discussion of the graph coloring game and its connection to quantum sets.

The graph coloring game can be generalized to the graph homomorphism game, whose parameters are two finite simple graphs G and H. Classically, Alice and Bob have a winning strategy if and only if there exists a graph homomorphism from G to H. We show that Alice and Bob have a winning strategy utilizing finite entangled quantum systems if and only if there exists a nonempty quantum family of graph homomorphisms from G to H, suitably expressed in the quantum predicate logic of this paper. The family is quantum in the sense that it is indexed by a quantum set. This equivalence between the existence a winning strategy utilizing finite entangled quantum systems and the existence of a nonempty quantum family of graph homomorphisms is essentially already present in [16, Prop. 1.2] and even in [5, Sec. II]. Thus, the novelty of Proposition 7.4 consists primarily in the axiomatization of these quantum families of graph homomorphisms in quantum predicate logic.

Similarly, we axiomatize quantum families of graph *isomorphisms*. Naturally, Alice and Bob have a winning strategy for the graph isomorphism game [2, Sec. 1.1], possibly utilizing finite entangled quantum systems, if and only if there exists a nonempty quantum family of graph isomorphisms. The example of the quantum families of graph isomorphisms is particularly significant due to extraordinary progress in understanding the quantum isomorphism relation [25, 27–29].

Lemma 7.1. *Let $\mathcal{X}$ be a quantum set, and let A be an ordinary set. Then, there is a one-to-one correspondence between functions $F\colon \mathcal{X} \to `A$ and families $(P_a \in \mathrm{Rel}(\mathcal{X}) \mid a \in A)$ such that $P_{a_1} \perp P_{a_2}$ for all $a_1 \neq a_2$ and $\bigvee_{a\in A} P_a = \top_{`A}$. This correspondence is given by $P_a = `a^\dagger \circ F$.*

Proof. The equation $P_a = `a^\dagger \circ F$, for $a \in A$, defines a one-to-one correspondence between binary relations $F$ from $\mathcal{X}$ to $`A$ and indexed families $(P_a \in \mathrm{Rel}(\mathcal{X}) \mid a \in A)$, essentially by the definition of a binary relation between quantum sets. It only remains to show that the inequalities $F \circ F^\dagger \leq I_{`A}$ and $F^\dagger \circ F \geq I_{\mathcal{X}}$ are together equivalent to the stated conditions on $(P_a \mid a \in A)$ under this correspondence. Reasoning in terms of the trace on binary relations [19, App. C],

$$\begin{aligned} &F \circ F^\dagger \leq I_{`A} \iff F \circ F^\dagger \perp \neg I_{`A} \iff \mathrm{Tr}(\neg I_{`A}^\dagger \circ F \circ F^\dagger) = \perp \\ &\iff \mathrm{Tr}\Big(\Big(\bigvee_{a_1\neq a_2} `a_1 \circ `a_2^\dagger\Big) \circ F \circ F^\dagger\Big) = \perp \iff \bigvee_{a_1\neq a_2} \mathrm{Tr}(`a_1 \circ `a_2^\dagger \circ F \circ F^\dagger) = \perp \\ &\iff \text{for all distinct } a_1, a_2 \in A, \quad \mathrm{Tr}((`a_2^\dagger \circ F) \circ (a_1^\dagger \circ F)^\dagger) = \perp \\ &\iff \text{for all distinct } a_1, a_2 \in A, \quad P_{a_1} \perp P_{a_2}. \end{aligned}$$

Under the assumption $F \circ F^\dagger \leq I_{`A}$, the inequality $F^\dagger \circ F \geq I_{\mathcal{X}}$ is equivalent to the equality $\top_{`A} \circ F = \top_{\mathcal{X}}$ by [16, Lems. 6.4 and B.4]:

$$F \circ F^\dagger \leq I_{`A} \iff \top_{`A} \circ F = \top_{\mathcal{X}} \iff \Big(\bigvee_{a\in A} `a^\dagger\Big) \circ F = \top_{\mathcal{X}} \iff \bigvee_{a\in A} P_a = \top_{\mathcal{X}}. \quad \blacksquare$$

Lemma 7.2. *Let A and B be ordinary sets, and let $\mathcal{X}$ be a quantum set. We have a one-to-one correspondence between families $(P_{ab} \in \mathrm{Rel}(\mathcal{X}) \mid a \in A,\ b \in B)$ such that*

(1) *$P_{ab_1} \perp P_{ab_2}$ for all $a \in A$ and distinct $b_1, b_2 \in B$;*

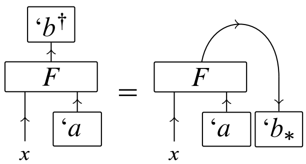
(2) *$\bigvee_{b\in B} P_{ab} = \top_{\mathcal{X}}$ for all $a \in A$,*

and functions $F\colon \mathcal{X} \times `A \to `B$. It is given by the equation $P_{ab} = `b^\dagger \circ F \circ (I_{\mathcal{X}} \times `a)$ for $a \in A$ and $b \in B$.

Proof. The quantum set $\mathcal{X} \times `A$ is a coproduct of copies of $\mathcal{X}$, with one copy for each element of $A$. The inclusions of this coproduct are exactly the functions $I_\mathcal{X} \times `a$ for $a \in A$. Thus, we have a one-to-one correspondence between families of functions $(F_a : \mathcal{X} \to `B \mid a \in A)$ and functions $F : \mathcal{X} \times `A \to `B$, which is given by $F_a = F \circ (I_\mathcal{X} \times `a)$ for $a \in A$. The statement of the lemma then follows by Lemma 7.1. ■

Proposition 7.3. *Let A and B be ordinary sets, let $\mathcal{X}$ be a quantum set, and let F be a function $\mathcal{X} \times `A \to `B$. For each $a \in A$ and each $b \in B$, let $P_{ab} = `b^\dagger \circ F \circ (I_\mathcal{X} \times `a)$. Then, $\bigvee_{a \in A} P_{ab} = \top_\mathcal{X}$ for all $b \in B$ if and only if $[\![\forall x\, \forall b_*\, \exists a\; F(x,a) = b_*]\!] = \top$.*

Proof. First, we compute that for all $a \in A$ and all $b \in B$,

$P_{ab} =$  $= [\![x \in \mathcal{X} \mid F(x, `a) = `b_*]\!].$

Now we apply Lemma A.6.1 of Part I [17] twice to reason that

$$\begin{aligned}
\forall b \in B\; \bigvee_{a \in A} P_{ab} = \top_\mathcal{X} \quad &\Longleftrightarrow \quad \bigwedge_{b \in B} \bigvee_{a \in A} P_{ab} = \top_\mathcal{X} \\
&\Longleftrightarrow \quad \bigwedge_{b \in B} \bigvee_{a \in A} [\![x \in \mathcal{X} \mid F(x, `a) = `b_*]\!] = \top_\mathcal{X} \\
&\Longleftrightarrow \quad \bigwedge_{b \in B} [\![x \in \mathcal{X} \mid \exists a\; F(x, a) = `b_*]\!] = \top_\mathcal{X} \\
&\Longleftrightarrow \quad [\![x \in \mathcal{X} \mid \forall b_*\, \exists a\; F(x, a) = b_*]\!] = \top_\mathcal{X} \\
&\Longleftrightarrow \quad [\![\forall x\, \forall b_*\, \exists a\; F(x, a) = b_*]\!] = \top,
\end{aligned}$$

where the last equivalence follows from Lemma 3.2.1 of Part I [17]. ■

Proposition 7.4. *Let A and B be sets equipped with binary relations r and s, respectively, and let $R = `r$ and $S = `s$. Let $\mathcal{X}$ be a quantum set, and let $F : \mathcal{X} \times `A \to `B$ be a function. For each $a \in A$ and each $b \in B$, let P_{ab} be the relation of arity $(\mathcal{X})$ defined by*

$$P_{ab} = `b^\dagger \circ F \circ (I_\mathcal{X} \times `a).$$

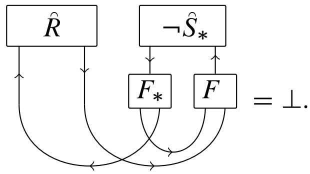
Then, $P_{a_1 b_1} \perp P_{a_2 b_2}$ for all $(a_1, a_2) \in r$ and $(b_1, b_2) \in \neg s$ if and only if

$$[\![\forall (x = x_*)\, \forall (a_1 = a_{1*})\, \forall (a_2 = a_{2*})\, (\hat{R}(a_1, a_{2*}) \to \hat{S}_*(F_*(x_*, a_{1*}), F(x, a_2)))]\!] = \top.$$

Proof. The equation in the statement of the proposition is equivalent to

$$[\![\exists (x = x_*)\, \exists (a_1 = a_{1*})\, \exists (a_2 = a_{2*})\, (\hat{R}(a_1, a_{2*}) \wedge \neg \hat{S}_*(F_*(x_*, a_{1*}), F(x, a_2)))]\!] = \bot,$$

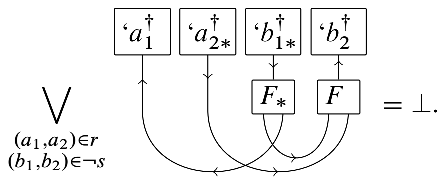
which may be rendered graphically as



Because A and B are ordinary sets, we have

$$\hat{R} = \sup\{`a_1^\dagger \times `a_{2*}^\dagger \mid (a_1, a_2) \in r\}$$

and $\neg\hat{S}_* = \sup\{`b_{1*}^\dagger \times `b_2 \mid (b_1, b_2) \in \neg s\}$. Thus, the equation in the statement of the proposition is equivalent to



Reasoning graphically, we conclude that this latter equation expresses the condition that $`b_1^\dagger \circ F \circ (I_{\mathcal{X}} \times `a_1)$ is orthogonal to $`b_2^\dagger \circ F \circ (I_{\mathcal{X}} \times `a_2)$ whenever $(a_1, a_2) \in r$ and $(b_1, b_2) \in \neg s$. ∎

Proposition 7.5. *Let A and B be ordinary sets. Let H be a nonzero finite-dimensional Hilbert space, and let $\mathcal{X}$ be the quantum set defined by*

$$\mathrm{At}(\mathcal{X}) = \{H\}.$$

Then, there is a one-to-one correspondence between families of projections $(p_{ab} \in L(H) \mid a \in A,\, b \in B)$ such that $\sum_{b\in B} p_{ab} = 1_H$ for all $a \in A$ and functions $F\colon \mathcal{X} \times `A \to `B$. This correspondence is obtained by combining the one-to-one correspondence of Lemma 7.2 with the canonical one-to-one correspondence between projection operators on H and relations of arity $(\mathcal{X})$. It is given by the equation

$$F(H \otimes \mathbb{C}_a, \mathbb{C}_b) = L(H \otimes \mathbb{C}_a, \mathbb{C}_b) \cdot (p_{ab} \otimes 1),$$

for $a \in A$ and $b \in B$.

The canonical one-to-one correspondence between projection operators p on H and relations P of arity $(\mathcal{X})$ is defined by

$$P(H, \mathbb{C}) = \{v \in L(H, \mathbb{C}) \mid vp = v\} = L(H, \mathbb{C}) \cdot p.$$

This is an isomorphism of ortholattices [16, App. B].

Proof of Proposition 7.5. For all $a \in A$ and $b \in B$, we calculate that

$$
\begin{aligned}
&F(H \otimes \mathbb{C}_a, \mathbb{C}_b) \\
&\quad = L(\mathbb{C}, \mathbb{C}_b) \cdot L(\mathbb{C}_b, \mathbb{C}) \cdot F(H \otimes \mathbb{C}_a, \mathbb{C}_b) \cdot (1_H \otimes L(\mathbb{C}, \mathbb{C}_a)) \cdot (1_H \otimes L(\mathbb{C}_a, \mathbb{C})) \\
&\quad = L(\mathbb{C}, \mathbb{C}_b) \cdot `b^\dagger(\mathbb{C}_b, \mathbb{C}) \cdot F(H \otimes \mathbb{C}_a, \mathbb{C}_b) \cdot (1_H \otimes `a(\mathbb{C}, \mathbb{C}_a)) \cdot (1_H \otimes L(\mathbb{C}_a, \mathbb{C})) \\
&\quad = L(\mathbb{C}, \mathbb{C}_b) \cdot (`b^\dagger \circ F \circ (I_{\mathcal{X}} \times `a))(H \otimes \mathbb{C}, \mathbb{C}) \cdot (1_H \otimes L(\mathbb{C}_a, \mathbb{C})) \\
&\quad = L(\mathbb{C}, \mathbb{C}_b) \cdot P_{ab}(H \otimes \mathbb{C}, \mathbb{C}) \cdot (1_H \otimes L(\mathbb{C}_a, \mathbb{C})) \\
&\quad = L(\mathbb{C}, \mathbb{C}_b) \cdot L(H \otimes \mathbb{C}, \mathbb{C}) \cdot (p_{ab} \otimes 1) \cdot (1_H \otimes L(\mathbb{C}_a, \mathbb{C})) \\
&\quad = L(H \otimes \mathbb{C}_a, \mathbb{C}_b) \cdot (p_{ab} \otimes 1).
\end{aligned}
$$

The unitor $H \otimes \mathbb{C} \to H$ has been suppressed. ∎

Theorem 7.6. *Let A and B be sets equipped with binary relations r and s, respectively, and let $R = `r$ and $S = `s$. Let H be a nonzero finite-dimensional Hilbert space, and let $\mathcal{X}$ be the quantum set defined by* $\mathrm{At}(\mathcal{X}) = \{H\}$*. Then, the one-to-one correspondence of Proposition* 7.5 *restricts to a one-to-one correspondence between families of projections $(p_{ab} \in L(H) \mid a \in A,\ b \in B)$ such that*

(1) $\sum_{b \in B} p_{ab} = 1_H$ *for all* $a \in A$;

(2) $p_{a_1 b_1} \perp p_{a_2 b_2}$ *for all* $(a_1, a_2) \in r$ *and* $(b_1, b_2) \in \neg s$,

and functions $F \colon \mathcal{X} \times `A \to `B$ such that

$$[\![\forall (x = x_*)\, \forall (a_1 = a_{1*})\, \forall (a_2 = a_{2*})\, (\hat{R}(a_1, a_{2*}) \to \hat{S}_*(F_*(x_*, a_{1*}), F(x, a_2)))]\!] = \top.$$

Proof. This follows immediately from Proposition 7.4 and Proposition 7.5. ∎

Definition 7.7. Let $(A, \sim_A)$ and $(B, \sim_B)$ be finite simple graphs, and let H be a nonzero finite-dimensional Hilbert space. We say that a family of projections $(p_{ab} \in L(H) \mid a \in A, b \in B)$ *witnesses* $A \xrightarrow{q} B$ if

(1) $\sum_{b \in B} p_{ab} = 1_H$ for all $a \in A$;

(2) $p_{a_1 b_1} \cdot p_{a_2 b_2} = 0$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ that satisfy either of the conditions $(a_1 =_A a_2) \wedge (b_1 \neq_B b_2)$ or $(a_1 \sim_A a_2) \wedge (b_1 \not\sim_B b_2)$.

Alice and Bob have a winning strategy for the (A, B)-homomorphism game, possibly using finite entangled quantum systems, if and only if there exists a family of projections on some nonzero finite-dimensional Hilbert space that witnesses $A \xrightarrow{q} B$ [26, Cor. 2.2].

Corollary 7.8. *Let $(A, \sim_A)$ and $(B, \sim_B)$ be finite simple graphs, and let $R = `(\sim_A)$ and $S = `(\sim_B)$. Let H be a nonzero finite-dimensional Hilbert space, and let $\mathcal{X}$ be the quantum set defined by* $\mathrm{At}(\mathcal{X}) = \{H\}$*. Then, the one-to-one correspondence of Proposition* 7.5 *restricts to a one-to-one correspondence between families of projections $(p_{ab} \in L(H) \mid a \in A,\ b \in B)$ witnessing $A \xrightarrow{q} B$ and functions $F \colon \mathcal{X} \times `A \to `B$ satisfying*

$$[\![\forall (x = x_*)\, \forall (a_1 = a_{1*})\, \forall (a_2 = a_{2*})\, (\hat{R}(a_1, a_{2*}) \to \hat{S}_*(F_*(x_*, a_{1*}), F(x, a_2)))]\!] = \top.$$

Proof. This corollary follows immediately from Theorem 7.6. ∎

Definition 7.9. Let A and B be finite sets, and let H be a nonzero finite-dimensional Hilbert space. We say that a family of projections $(p_{ab} \in L(H) \mid a \in A, b \in B)$ is a *magic unitary* if

(1) $\sum_{b \in B} p_{ab} = 1_H$ for all $a \in A$;

(2) $\sum_{a \in A} p_{ab} = 1_H$ for all $b \in B$.

A magic unitary in this sense is essentially a quantum family of bijections between A and B indexed by the quantum set whose only atom is H. In particular, if $A = B$, then it is a quantum family of bijections. Indeed, the universal C*-algebra generated by projections p_{ab}, for $a, b \in A$, satisfying conditions (1) and (2), is a well-established quantum generalization of the permutation group $\mathrm{Aut}(A)$, introduced in [44, Sec. 3].

Corollary 7.10. *Let A and B be sets. Let H be a nonzero finite-dimensional Hilbert space, and let $\mathcal{X}$ be the quantum set defined by $\mathrm{At}(\mathcal{X}) = \{H\}$. Then, the one-to-one correspondence of Proposition 7.5 restricts to a one-to-one correspondence between families of projections $(p_{ab} \in L(H) \mid a \in A,\ b \in B)$ that are magic unitaries and functions $F\colon \mathcal{X} \times `A \to `B$ such that*

(1) $[\![\forall x\, \forall b_*\, \exists a\, F(x,a) = b_*]\!] = \top$;

(2) $[\![\forall (x = x_*)\, \forall (a_1 = a_{1*})\, \forall (a_2 = a_{2*})\, (a_1 = a_{2*} \leftrightarrow F_*(x_*, a_{1*}) = F(x, a_2))]\!] = \top$.

Proof. Applying Theorem 7.6 with r equal to the identity binary relation on A and s equal to the identity binary relation on B, we find that

$$[\![\forall (x = x_*)\, \forall (a_1 = a_{1*})\, \forall (a_2 = a_{2*})\, (a_1 = a_{2*} \to F_*(x_*, a_{1*}) = F(x, a_2))]\!] = \top$$

for all functions $F\colon \mathcal{X} \times `A \to `B$. Applying Theorem 7.6 with r equal to the negation of the identity binary relation A and s equal to the negation of the identity binary relation on B, we find that the one-to-one correspondence of Proposition 7.5 restricts to a one-to-one correspondence between families of projections (p_{ab}) such that $\sum_{b \in B} p_{ab} = 1_H$ and $p_{a_1 b} \perp p_{a_2 b}$ for all distinct $a_1, a_2 \in A$ and all $b \in B$ and functions $F\colon \mathcal{X} \times `A \to `B$ such that

$$[\![\forall (x = x_*)\, \forall (a_1 = a_{1*})\, \forall (a_2 = a_{2*})\, (a_1 \neq a_{2*} \to F_*(x_*, a_{1*}) \neq F(x, a_2))]\!] = \top.$$

Applying Proposition 7.3, we also find that the one-to-one correspondence of Proposition 7.5 restricts to a one-to-one correspondence between families of projections (p_{ab}) such that $\sum_{b \in B} p_{ab} = 1_H$ for all $a \in A$ and $\bigvee_{a \in A} p_{ab} = 1_H$ for all $b \in B$ and functions $F\colon \mathcal{X} \times `A \to `B$ such that $[\![\forall x\, \forall b_*\, \exists a\, F(x,a) = b_*]\!] = \top$.

Combining these three observations and applying Proposition A.2, we conclude the statement of the corollary, because the atomic formulas $a_1 = a_{2*}$ and $F_*(x_*, a_{1*}) = F(x, a_2)$ have no variables in common and, thus,

$$[\![a_1 \neq a_{2*} \to F_*(x_*, a_{1*}) \neq F(x, a_2)]\!] = [\![F_*(x_*, a_{1*}) = F(x, a_2) \to a_1 = a_{2*}]\!].$$ ∎

Definition 7.11. Let $(A, \sim_A)$ and $(B, \sim_B)$ be finite simple graphs, and let H be a nonzero finite-dimensional Hilbert space. We say that a family of projections $(p_{ab} \in L(H) \mid a \in A, b \in B)$ *witnesses* $A \cong_q B$ if it witnesses both $A \xrightarrow{q} B$ and $B \xrightarrow{q} A$.

Alice and Bob have a winning strategy for the (A, B)-isomorphism game, possibly using finite entangled quantum systems, if and only if there exists a family of projections on some nonzero finite-dimensional Hilbert space that witnesses $A \cong_q B$ [2, Res. 2].

Corollary 7.12. *Let $(A, \sim_A)$ and $(B, \sim_B)$ be finite simple graphs, and let $R = `(\sim_A)$ and $S = `(\sim_B)$. Let H be a nonzero finite-dimensional Hilbert space, and let $\mathcal{X}$ be the quantum set defined by $\mathrm{At}(\mathcal{X}) = \{H\}$. Then, the one-to-one correspondence of Proposition 7.5 restricts to a one-to-one correspondence between families of projections $(p_{ab} \in L(H) \mid a \in A,\ b \in B)$ witnessing $A \cong_q B$ and functions $F \colon \mathcal{X} \times `A \to `B$ such that*

(1) $[\![\forall x\, \forall b_*\, \exists a\; F(x, a) = b_*]\!] = \top$;

(2) $[\![\forall (x = x_*)\, \forall (a_1 = a_{1*})\, \forall (a_2 = a_{2*})\, (a_1 = a_{2*} \leftrightarrow F_*(x_*, a_{1*}) = F(x, a_2))]\!] = \top$;

(3) $[\![\forall (x = x_*)\, \forall (a_1 = a_{1*})\, \forall (a_2 = a_{2*})\, (\hat{R}(a_1, a_{2*}) \leftrightarrow \hat{S}_*(F_*(x_*, a_{1*}), F(x, a_2)))]\!] = \top$.

Proof. We extend the proof of Corollary 7.10, by applying Theorem 7.6 first with $r = (\sim_A)$ and $s = (\sim_B)$ and then with $r = \neg(\sim_A)$ and $s = \neg(\sim_B)$, reasoning similarly. ∎

8. Quantum groups

In their essence, discrete quantum groups are the Pontryagin duals of compact quantum groups, and this is how they first arose; see [38, Sec. 3] and [52]. There are many equivalent definitions [8, 23, 43]. We could define a discrete quantum group structure on a quantum set $\mathcal{X}$ to be a suitable comultiplication on $c_0(\mathcal{X})$, as in [38], a suitable comultiplication on $c_c(\mathcal{X})$, as in [43], or a suitable comultiplication on $\ell^\infty(\mathcal{X})$, as implicitly in [23]. We work with Van Daele's definition [43, Def. 2.3], defining

$$c_c(\mathcal{X}) = \{a \in \ell^\infty(\mathcal{X}) \mid a(X) = 0 \text{ for cofinitely many } X \in \mathrm{At}(\mathcal{X})\}$$

for each quantum set $\mathcal{X}$.

Discrete quantum groups are undoubtedly the most compelling example illustrating the naturality of quantum logic to noncommutative mathematics. First, discrete quantum groups are a firmly established quantum generalization, as firmly established as any class of discrete quantum structures. They were first defined more than thirty years ago, and their place in the noncommutative dictionary appears to have never been in doubt. Second, the considerations that motivated their definition are far removed from the considerations that motivated the definition of the semantics considered here. Only the duality between operator algebras and quantum spaces is shared in common. Third, the example of discrete quantum groups carries empirical weight. While the semantics considered here can readily be motivated from first principles, the interpretation of equality that is given in this paper was in fact motivated by many of the examples that we have considered, so the incorporation of these examples is not so surprising. However, discrete quantum groups were not among the examples first considered by the author. The correct axiomatization of discrete quantum groups was naively hypothesized by the author and then established by Stefaan Vaes [42]. It is essentially his proof that is recorded here.

Let $\mathcal{X}$ be a quantum set. For each atom X of $\mathcal{X}$, we write J_X for the inclusion function of the atomic quantum set $\mathcal{Q}\{X\}$ into $\mathcal{X}$ [16, Def. 2.3], i.e., as an abbreviation for $J^{\mathcal{X}}_{\mathcal{Q}\{X\}}$ [16, Def. 8.2]. Similarly, we write $\top_X$ and $\bot_X$ as abbreviations for $\top_{\mathcal{Q}\{X\}}$ and $\bot_{\mathcal{Q}\{X\}}$, respectively. Let $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ be the category of representations of $\ell^\infty(\mathcal{X})$, implicitly of finite-dimensional nondegenerate normal $*$-representations. A morphism in $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ from a representation $\rho\colon \ell^\infty(\mathcal{X}) \to L(H)$ to a representation $\sigma\colon \ell^\infty(\mathcal{X}) \to L(K)$ is an intertwiner between the two representations, i.e., an operator $v \in L(H, K)$ such that $v \cdot \rho(a) = \sigma(a) \cdot v$ for all $a \in \ell^\infty(\mathcal{X})$. Up to isomorphism, the simple objects, i.e., the irreducible representations, are just the canonical projections $J_X^\star\colon \ell^\infty(\mathcal{X}) \to L(X)$ for $X \in \mathrm{At}(\mathcal{X})$ [16, Thm. 7.4], and every representation is a finite direct sum of these. The category $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ is a C*-category in the obvious way [31, Def. 2.1.1].

Let $\mathcal{X}$ be a quantum set, and let $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $C\colon \mathbf{1} \to \mathcal{X}$ be functions such that $F \circ (F \times I_{\mathcal{X}}) = F \circ (I_{\mathcal{X}} \times F)$ and $F \circ (C \times I_{\mathcal{X}}) = I_{\mathcal{X}} = F \circ (I_{\mathcal{X}} \times C)$. Then, $F^\star$ is a comultiplication on $\ell^\infty(\mathcal{X})$, and $C^\star$ is a counit for this comultiplication [16, Thm. 7.4]. In the usual way, we obtain a monoidal structure on the category $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$. For representations $\rho_1\colon \ell^\infty(\mathcal{X}) \to L(H_1)$ and $\rho_2\colon \ell^\infty(\mathcal{X}) \to L(H_2)$, we define $\rho_1 \boxtimes \rho_2\colon \ell^\infty(\mathcal{X}) \to L(H_1 \otimes H_2)$ by $\rho_1 \boxtimes \rho_2 = (\rho_1 \mathbin{\bar{\otimes}} \rho_2) \circ F^\star$. For intertwiners v_1, from $\rho_1\colon \ell^\infty(\mathcal{X}) \to L(H_1)$ to $\sigma_1\colon \ell^\infty(\mathcal{X}) \to L(K_1)$, and v_2, from $\rho_2\colon \ell^\infty(\mathcal{X}) \to L(H_2)$ to $\sigma_2\colon \ell^\infty(\mathcal{X}) \to L(K_2)$, we define $v_1 \boxtimes v_2 = v_1 \otimes v_2$. The standard computations show that $\mathrm{Rel}(\ell^\infty(\mathcal{X}))$ is a monoidal C*-category with product $\boxtimes$ and unit $C^\star\colon \ell^\infty(\mathcal{X}) \to L(\mathbb{C})$ [31, Def. 2.1.1]. We will show that it has conjugates [31, Def. 2.2.1].

We make a number of simplifying assumptions without loss of generality. First, we represent $\ell^\infty(\mathcal{X})$ in a small strict monoidal category of finite-dimensional Hilbert spaces [31, Sec. 2.1]. Second, we assume that each atom of $\mathcal{X}$ is a Hilbert space in this small category. Third, we assume that $C(\mathbb{C}, \mathbb{C}) \neq 0$. In other words, we assume that $\mathbb{C}$ is an atom of $\mathcal{X}$ and that furthermore it is the unique atom in the range of C [19, Def. 3.2]. Overall, we have that $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ is a small strict monoidal C*-category that contains $J_X^\star$ for all $X \in \mathrm{At}(\mathcal{X})$ and that has unit $C^\star = J_{\mathbb{C}}^\star$.

Proposition 8.1. *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be quantum sets, let P be a binary relation of arity $(\mathcal{X}, \mathcal{Y})$, and let Q be a binary relation of arity $(\mathcal{Y}, \mathcal{Z})$. Assume that*

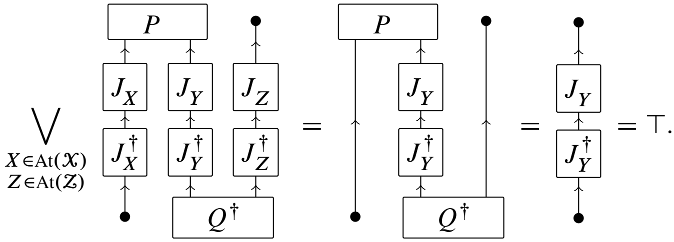
(1) $[\![\forall y\, \exists x\; P(x, y)]\!] = \top$;

(2) $[\![\forall y\, \exists z\; Q(y, z)]\!] = \top$.

Then, for all $Y \in \mathrm{At}(\mathcal{Y})$, there exist $X \in \mathrm{At}(\mathcal{X})$ and $Z \in \mathrm{At}(\mathcal{Z})$ such that $(P \times \top_{\mathcal{Z}}) \circ (J_X \times J_Y \times J_Z)$ is not orthogonal to $(\top_{\mathcal{X}} \times Q) \circ (J_X \times J_Y \times J_Z)$.

Proof. We are essentially given that

We now reason that



We have applied the equalities $\bigvee_{X\in\mathrm{At}(\mathcal{X})} J_X \circ J_X^\dagger = I_\mathcal{X}$ and $\bigvee_{Z\in\mathrm{At}(\mathcal{Z})} J_Z \circ J_Z^\dagger = I_\mathcal{Z}$ [19, Lem. A.4]. The last equality holds because $\top_\mathcal{Y} \circ J_Y = \top_Y \neq \bot_Y$ [16, Thm. B.8]. Hence, at least one term in the join on the left is equal to $\top$. We conclude that there exist $X \in \mathrm{At}(\mathcal{X})$ and $Z \in \mathrm{At}(\mathcal{Z})$ such that $((P \times \top_\mathcal{Z}) \circ (J_X \times J_Y \times J_Z)) \circ ((\top_\mathcal{X} \times Q) \circ (J_X \times J_Y \times J_Z))^\dagger = \top$, and therefore, the two relations are not orthogonal. ∎

For each atom X of $\mathcal{X}$, the inclusion function $J_X \colon \mathcal{Q}\{X\} \to \mathcal{X}$ is injective [16, Prop. 8.4], and therefore, the binary relation $J_X^\dagger \colon \mathcal{X} \to \mathcal{Q}\{X\}$ is a partial function. Because $J_X(X, X)$ is spanned by the identity operator on X, with all other components of J_X vanishing, it is easy to see that $(J_X^\dagger)^\star$ is equal to $[X]$, the minimal central projection in $\ell^\infty(\mathcal{X})$ corresponding to X. This is also the support projection of $J_X^\star$ [16, Lem. 8.3]. Thus, the support projection of $C^\star = J_\mathbb{C}^\star \colon \ell^\infty(\mathcal{X}) \to \mathbb{C}$ is $[\mathbb{C}] = (C^\dagger)^\star(1)$.

Lemma 8.2. *Let $\mathcal{X}$ be a quantum set, and let $F \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $C \colon \mathbf{1} \to \mathcal{X}$ be functions such that $F \circ (F \times I_\mathcal{X}) = F \circ (I_\mathcal{X} \times F)$ and $F \circ (C \times I_\mathcal{X}) = I_\mathcal{X} = F \circ (I_\mathcal{X} \times C)$. Assume that*

(1) $[\![\forall x_2\, \exists x_1\; F(x_1, x_2) = C_*]\!] = \top$;

(2) $[\![\forall x_2\, \exists x_3\; F(x_2, x_3) = C_*]\!] = \top$.

Then, every simple object of the strict monoidal C-category* $\mathrm{Rep}(\ell^\infty(X))$ *has a conjugate. In other words, for every atom $X \in \mathrm{At}(\mathcal{X})$, there exist an atom $\overline{X} \in \mathrm{At}(X)$ and intertwiners $v_X \colon C^\star \to J_X^\star \boxtimes J_{\overline{X}}^\star$ and $w_X \colon C^\star \to J_{\overline{X}}^\star \boxtimes J_X^\star$ such that $(v_X^\dagger \boxtimes 1_X)(1_X \boxtimes w_X) = 1_X$ and $(w_X^\dagger \boxtimes 1_{\overline{X}})(1_{\overline{X}} \boxtimes v_X) = 1_{\overline{X}}$. Therefore,* $\mathrm{Rep}(\ell^\infty(X))$ *is rigid.*

Proof. Let $X_2 \in \mathrm{At}(\mathcal{X})$. Let $P = [\![(x_1, x_2) \in \mathcal{X} \times \mathcal{X} \mid F(x_1, x_2) = C_*]\!] = C^\dagger \circ F$, and similarly, let $Q = [\![(x_2, x_3) \in \mathcal{X} \times \mathcal{X} \mid F(x_2, x_3) = C_*]\!] = C^\dagger \circ F$. We apply Proposition 8.1 to obtain atoms X_1 and X_3 such that

$$((C^\dagger \circ F) \times \top_\mathcal{X}) \circ (J_{X_1} \times J_{X_2} \times J_{X_3}) \not\perp (\top_\mathcal{X} \times (C^\dagger \circ F)) \circ (J_{X_1} \times J_{X_2} \times J_{X_3})$$

as binary relations from $\mathcal{Q}\{X_1 \otimes X_2 \otimes X_3\}$ to $\mathbf{1}$. Writing $G_{12} = C^\dagger \circ F \circ (J_{X_1} \times J_{X_2})$ and $G_{23} = C^\dagger \circ F \circ (J_{X_2} \times J_{X_3})$, we have that $G_{12} \times \top_{X_3} \not\perp \top_{X_1} \times G_{23}$. In other words,

$$G_{12}(X_1 \otimes X_2, \mathbb{C}) \otimes L(X_3, \mathbb{C}) \not\perp L(X_1, \mathbb{C}) \otimes G_{23}(X_2 \otimes X_3, \mathbb{C})$$

as subspaces of $L(X_1 \otimes X_2 \otimes X_3, \mathbb{C})$. Hence, let $v_{12} \in G_{12}(X_1 \otimes X_2, \mathbb{C})$, $\xi_3 \in L(X_3, \mathbb{C})$, $\xi_1 \in L(X_1, \mathbb{C})$, and $v_{23} \in G_{23}(X_2 \otimes X_3, \mathbb{C})$ be such that $v_{12} \otimes \xi_3$ is not orthogonal to $\xi_1 \otimes v_{23}$ as elements of $L(X_1 \otimes X_2 \otimes X_3, \mathbb{C})$. It certainly follows that

$$(v_{12} \otimes 1) \cdot (1 \otimes v_{23}^\dagger) \neq 0.$$

We now show that v_{12} is an intertwiner. To do so, we recall our previous observation that $C^\dagger\colon \mathcal{X} \to \mathbf{1}$ is a partial function and that $[\mathbb{C}] = (C^\dagger)^\star(1)$ is the support projection of $C^\star$. Hence, $G_{12}\colon \mathcal{Q}\{X_1 \otimes X_2\} \to \mathbf{1}$ is a partial function, and v_{12} satisfies $v_{12} = 1 \cdot v_{12} = v_{12} \cdot G_{12}^\star(1) = v_{12} \cdot (J_{X_1}^\star \,\bar{\otimes}\, J_{X_2}^\star)(F^\star([\mathbb{C}]))$ [16, Thm. 6.3]. We now calculate that for all $a \in \ell^\infty(\mathcal{X})$,

$$\begin{aligned} v_{12} \cdot (J_{X_1}^\star \boxtimes J_{X_2}^\star)(a) &= v_{12} \cdot (J_{X_1}^\star \,\bar{\otimes}\, J_{X_2}^\star)(F^\star(a)) \\ &= v_{12} \cdot (J_{X_1}^\star \,\bar{\otimes}\, J_{X_2}^\star)(F^\star([\mathbb{C}])) \cdot (J_{X_1}^\star \,\bar{\otimes}\, J_{X_2}^\star)(F^\star(a)) \\ &= v_{12} \cdot (J_{X_1}^\star \,\bar{\otimes}\, J_{X_2}^\star)(F^\star([\mathbb{C}] \cdot a)) = v_{12} \cdot (J_{X_1}^\star \,\bar{\otimes}\, J_{X_2}^\star)(F^\star([\mathbb{C}] \cdot C^\star(a))) \\ &= v_{12} \cdot (J_{X_1}^\star \,\bar{\otimes}\, J_{X_2}^\star)(F^\star([\mathbb{C}])) \cdot C^\star(a) = v_{12} \cdot C^\star(a) = C^\star(a) \cdot v_{12}. \end{aligned}$$

Therefore, v_{12} is an intertwiner from $J_{X_1}^\star \boxtimes J_{X_2}^\star$ to $C^\star$. Similarly, v_{23} is an intertwiner from $J_{X_2}^\star \boxtimes J_{X_3}^\star$ to $C^\star$. Furthermore, by our choice of v_{12} and v_{23}, we have that

$$(v_{12} \otimes 1)(1 \otimes v_{23}^\dagger) \neq 0.$$

Altogether, we find that $s := (v_{12} \otimes 1) \cdot (1 \otimes v_{23}^\dagger)$ is a nonzero intertwiner from $J_{X_1}^\star$ to $J_{X_3}^\star$. These are irreducible representations of $\ell^\infty(\mathcal{X})$, and thus, by Schur's Lemma, s is an isomorphism in $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$. Defining $v := (1 \otimes s^{-1})v_{23}^\dagger$ and $w := v_{12}^\dagger$, we obtain intertwiners $v\colon C^\star \to J_{X_2}^\star \boxtimes J_{X_1}^\star$ and $w\colon C^\star \to J_{X_1}^\star \boxtimes J_{X_2}^\star$ such that $(w^\dagger \otimes 1) \cdot (1 \otimes v) = 1$ and therefore also $(1 \otimes v^\dagger) \cdot (w \otimes 1) = 1$. We now calculate that

$$\begin{aligned} &(1 \otimes ((v^\dagger \otimes 1) \cdot (1 \otimes w))) \cdot w \\ &\quad = (1 \otimes v^\dagger \otimes 1) \cdot (1 \otimes 1 \otimes w) \cdot w = (1 \otimes v^\dagger \otimes 1) \cdot (w \otimes w) \\ &\quad = (1 \otimes v^\dagger \otimes 1) \cdot (w \otimes 1 \otimes 1) \cdot w = (((1 \otimes v^\dagger) \cdot (w \otimes 1)) \otimes 1) \cdot w \\ &\quad = (1 \otimes 1) \cdot w = w. \end{aligned}$$

Since w is nonzero, we find that $(v^\dagger \otimes 1) \cdot (1 \otimes w)$ is a nonzero intertwiner on $J_{X_2}^\star$. It is therefore a scalar, and it can be no scalar other than 1. We conclude that $J_{X_1}^\star$ and $J_{X_2}^\star$ are conjugate objects in $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$, and more generally, that every simple object in $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ has a conjugate. Because every object in $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ is a direct sum of simple objects, it follows that every object has a conjugate. In other words, $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ is rigid. ∎

Lemma 8.3. *Let $\mathcal{X}$ be a quantum set, and let $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $C\colon \mathbf{1} \to \mathcal{X}$ be functions such that $F \circ (F \times I_{\mathcal{X}}) = F \circ (I_{\mathcal{X}} \times F)$ and $F \circ (C \times I_{\mathcal{X}}) = I_{\mathcal{X}} = F \circ (I_{\mathcal{X}} \times C)$. Assume that* $\mathrm{Rep}(\ell^\infty(X))$ *is rigid. Then, $F^\star(a) \cdot (1 \otimes b)$ and $(a \otimes 1) \cdot F^\star(b)$ are both in the algebraic tensor product $c_c(\mathcal{X}) \odot c_c(\mathcal{X})$ for all $a, b \in c_c(\mathcal{X})$.*

Proof. Let $a, b \in c_c(\mathcal{X})$. Let X_0, X_1, and X_2 be atoms of $\mathcal{X}$, and assume that $J^\star_{X_0}(b) \neq 0$ and that $\mathrm{Mor}(J^\star_{X_0}, J^\star_{X_1} \boxtimes J^\star_{X_2}) \neq 0$. Then, there exists an operator $v \in L(X_0, X_1 \otimes X_2)$ such that $(J^\star_{X_1} \boxtimes J^\star_{X_2})(b) \cdot v = v \cdot J^\star_{X_0}(b) \neq 0$, and hence, $(J^\star_{X_1} \boxtimes J^\star_{X_2})(b) \neq 0$. Therefore, for all atoms X_0, X_1, and X_2, the conditions $J^\star_{X_0}(b) \neq 0$ and $\mathrm{Mor}(J^\star_{X_0}, J^\star_{X_1} \boxtimes J^\star_{X_2}) \neq 0$ together imply that

$$(J^\star_{X_1} \boxtimes J^\star_{X_2})(b) \neq 0.$$

Let X_0 and X_1 be atoms of $\mathcal{X}$. For each atom $X \in \mathrm{At}(\mathcal{X})$, let $\bar{X} \in \mathrm{At}(\mathcal{X})$ be the unique atom such that the representations $J^\star_X$ and $J^\star_{\bar{X}}$ are conjugate. Applying Frobenius reciprocity [31, Thm. 2.2.6], we compute that for each atom X_2,

$$\mathrm{Mor}(J^\star_{X_0}, J^\star_{X_1} \boxtimes J^\star_{X_2}) = \mathrm{Mor}(J^\star_{X_1} \boxtimes J^\star_{X_2}, J^\star_{X_0})^\dagger = \mathrm{Mor}(J^\star_{X_2}, J^\star_{\bar{X}_1} \boxtimes J^\star_{X_0})^\dagger.$$

Because $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ is a rigid C*-category, there are only finitely many atoms X_2 such that $\mathrm{Mor}(J^\star_{X_2}, J^\star_{\bar{X}_1} \boxtimes J^\star_{X_0}) \neq 0$ [31, Cor. 2.2.9]. Therefore, for all atoms X_0 and X_1, there are only finitely many atoms X_2 such that $\mathrm{Mor}(J^\star_{X_0}, J^\star_{X_1} \boxtimes J^\star_{X_2}) \neq 0$.

We now compute that for all atoms X_1 and X_2,

$$\begin{aligned}(J^\star_{X_1} \mathbin{\bar{\otimes}} J^\star_{X_2})((a \otimes 1) \cdot F^\star(b)) &= (J^\star_{X_1} \mathbin{\bar{\otimes}} J^\star_{X_2})(a \otimes 1) \cdot (J^\star_{X_1} \mathbin{\bar{\otimes}} J^\star_{X_2})(F^\star(b)) \\ &= (J^\star_{X_1}(a) \otimes 1) \cdot (J^\star_{X_1} \boxtimes J^\star_{X_2})(b).\end{aligned}$$

Because $a, b \in c_c(\mathcal{X})$, we have that $J^\star_X(a) \neq 0$ and $J^\star_X(b) \neq 0$ for only finitely many atoms X. Thus, $J^\star_{X_1}(a) \otimes 1 \neq 0$ for only finitely many atoms X_1, and for each of those atoms, $(J^\star_{X_1} \boxtimes J^\star_{X_2})(b) \neq 0$ for only finitely many atoms X_2. We conclude that there are only finitely many pairs $(X_1, X_2) \in \mathrm{At}(\mathcal{X}) \times \mathrm{At}(\mathcal{X})$ such that $(J^\star_{X_1} \mathbin{\bar{\otimes}} J^\star_{X_2})((a \otimes 1) \cdot F^\star(b))$ is nonzero. Therefore, $(a \otimes 1) \cdot F^\star(b) \in c_c(\mathcal{X}) \odot c_c(\mathcal{X})$ as claimed. Similarly, $F^\star(a) \cdot (1 \otimes b) \in c_c(\mathcal{X}) \odot c_c(\mathcal{X})$. ∎

Lemma 8.4. *Let $\mathcal{X}$ be a quantum set, and let $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $C\colon \mathbf{1} \to \mathcal{X}$ be functions such that $F \circ (F \times I_{\mathcal{X}}) = F \circ (I_{\mathcal{X}} \times F)$ and $F \circ (C \times I_{\mathcal{X}}) = I_{\mathcal{X}} = F \circ (I_{\mathcal{X}} \times C)$. Assume that* $\mathrm{Rep}(\ell^\infty(\mathcal{X}))$ *is rigid. Let T_1 and T_2 be the functions on the algebraic tensor product $c_c(\mathcal{X}) \odot c_c(\mathcal{X})$ that are defined by the equations $T_1(a \otimes b) = F^\star(a) \cdot (1 \otimes b)$ and $T_2(a \otimes b) = (a \otimes 1) \cdot F^\star(b)$, respectively. Then, T_1 and T_2 are both surjective.*

Proof. We use the notation of Lemma 8.2. Let $c \in c_c(\mathcal{X})$, and let $X_0 \in \mathrm{At}(\mathcal{X})$. Then, $F^\star(c) \cdot (1 \otimes [\bar{X}_0])$ is in $c_c(\mathcal{X}) \odot c_c(\mathcal{X})$, so it can be written as

$$F^\star(c) \cdot (1 \otimes [\bar{X}_0]) = \sum_{i=1}^n a_i \otimes c_i$$

for some operators $a_1, \ldots, a_n$ and $c_1, \ldots, c_n$ in $c_c(\mathcal{X})$. For each index i, let

$$b_i = \iota^{X_0}((1 \otimes w^\dagger_{X_0}) \cdot (1 \otimes J^\star_{\bar{X}_0}(c_i) \otimes 1) \cdot (v_{X_0} \otimes 1)),$$

where ι^{X_0} is the canonical inclusion of $L(X_0)$ into $\ell^\infty(\mathcal{X})$. We now compute that for all atoms X_1 and X_2,

$$\begin{aligned}
&(J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2})\left(T_1\left(\sum_{i=1}^n a_i \otimes b_i\right)\right) \\
&= \sum_{i=1}^n (J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2})(F^\star(a_i) \cdot (1 \otimes b_i)) \\
&= \sum_{i=1}^n (J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2})(F^\star(a_i)) \cdot (1 \otimes J^\star_{X_2}(b_i)) \\
&= \delta_{X_0 X_2} \cdot \sum_{i=1}^n (J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2})(F^\star(a_i)) \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \\
&\quad \cdot (1 \otimes 1 \otimes J^\star_{\bar{X}_2}(c_i) \otimes 1) \cdot (1 \otimes v_{X_2} \otimes 1) \\
&= \delta_{X_0 X_2} \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \cdot \left(\sum_{i=1}^n (J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2})(F^\star(a_i)) \otimes J^\star_{\bar{X}_2}(c_i) \otimes 1\right) \\
&\quad \cdot (1 \otimes v_{X_2} \otimes 1) \\
&= \delta_{X_0 X_2} \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \cdot \left(\sum_{i=1}^n (J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2} \mathbin{\bar\otimes} J^\star_{\bar{X}_2})((F^\star \mathbin{\bar\otimes} 1)(a_i \otimes c_i)) \otimes 1\right) \\
&\quad \cdot (1 \otimes v_{X_2} \otimes 1) \\
&= \delta_{X_0 X_2} \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \cdot \big((J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2} \mathbin{\bar\otimes} J^\star_{\bar{X}_2})((F^\star \mathbin{\bar\otimes} 1)(F^\star(c) \cdot (1 \otimes [\bar{X}_2]))) \otimes 1\big) \\
&\quad \cdot (1 \otimes v_{X_2} \otimes 1) \\
&= \delta_{X_0 X_2} \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \cdot \big((J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2} \mathbin{\bar\otimes} J^\star_{\bar{X}_2})((F^\star \mathbin{\bar\otimes} 1)(F^\star(c))) \otimes 1\big) \\
&\quad \cdot \big((J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2} \mathbin{\bar\otimes} J^\star_{\bar{X}_2})((F^\star \mathbin{\bar\otimes} 1)(1 \otimes [\bar{X}_2])) \otimes 1\big) \cdot (1 \otimes v_{X_2} \otimes 1) \\
&= \delta_{X_0 X_2} \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \cdot \big((J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2} \mathbin{\bar\otimes} J^\star_{\bar{X}_2})((F^\star \mathbin{\bar\otimes} 1)(F^\star(c))) \otimes 1\big) \cdot (1 \otimes v_{X_2} \otimes 1) \\
&= \delta_{X_0 X_2} \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \cdot \big((J^\star_{X_1} \boxtimes J^\star_{X_2} \boxtimes J^\star_{\bar{X}_2})(c) \otimes 1\big) \cdot ((1 \boxtimes v_{X_2}) \otimes 1) \\
&= \delta_{X_0 X_2} \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \cdot ((1 \boxtimes v_{X_2}) \otimes 1) \cdot \big((J^\star_{X_1} \boxtimes C^\star)(c) \otimes 1\big) \\
&= \delta_{X_0 X_2} \cdot (1 \otimes 1 \otimes w^\dagger_{X_2}) \cdot (1 \otimes v_{X_2} \otimes 1) \cdot \big(J^\star_{X_1}(c) \otimes 1\big) \\
&= \delta_{X_0 X_2} \cdot \big(J^\star_{X_1}(c) \otimes 1\big) = J^\star_{X_1}(c) \otimes J^\star_{X_2}([X_0]) = (J^\star_{X_1} \mathbin{\bar\otimes} J^\star_{X_2})(c \otimes [X_0]).
\end{aligned}$$

Therefore, $T_1(\sum_{i=1}^n a_i \otimes b_i) = c \otimes [X_0]$. More generally, for all $c, d \in c_c(\mathcal{X})$ and all atoms $X_0 \in \mathrm{At}(\mathcal{X})$, we have that

$$T_1\left(\left(\sum_{i=1}^n a_i \otimes b_i\right) \cdot (1 \otimes d)\right) = (c \otimes [X_0]) \cdot (1 \otimes d) = c \otimes (d \cdot [X_0]),$$

so $c \otimes (d \cdot [X_0]) \in \mathrm{ran}(T_1)$. The expression $d \cdot [X_0]$ is nonzero for only finitely many atoms X_0, so we conclude that $c \otimes d \in \mathrm{ran}(T_1)$ for all $c, d \in c_c(\mathcal{X})$. Therefore, T_1 is surjective. Similarly, T_2 is surjective. ■

Theorem 8.5 (Vaes, cf. [42]). *Let $\mathcal{X}$ be a quantum set, and let $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $C: \mathbf{1} \to \mathcal{X}$ be functions such that $F \circ (F \times I_\mathcal{X}) = F \circ (I_\mathcal{X} \times F)$ and $F \circ (C \times I_\mathcal{X}) = I_\mathcal{X} = F \circ (I_\mathcal{X} \times C)$. Assume that*

(1) $[\![\forall x_1 \, \exists x_2 \; F(x_1, x_2) = C_*]\!] = \top$;

(2) $[\![\forall x_2 \, \exists x_1 \; F(x_1, x_2) = C_*]\!] = \top$.

Let $\Delta: c_c(\mathcal{X}) \to \mathrm{Mult}(c_c(\mathcal{X}) \odot c_c(\mathcal{X}))$ be defined by $\Delta = F^\star|_{c_c(\mathcal{X})}$. Then, $(c_c(\mathcal{X}), \Delta)$ is a discrete quantum group in the sense of [43, Def. 2.3].

Proof. Via the duality between quantum sets and hereditarily atomic von Neumann algebras [16, Thm. 7.4], we obtain unital normal $*$-homomorphisms $F^\star: \ell^\infty(\mathcal{X}) \mathbin{\bar\otimes} \ell^\infty(\mathcal{X}) \to \ell^\infty(\mathcal{X})$ and $C^\star: \ell^\infty(\mathcal{X}) \to \mathbb{C}$ that satisfy $(F^\star \mathbin{\bar\otimes} 1) \circ F^\star = (1 \mathbin{\bar\otimes} F^\star) \circ F^\star$, $(C^\star \mathbin{\bar\otimes} 1) \circ F^\star = 1$ and $(1 \mathbin{\bar\otimes} C^\star) \circ F^\star = 1$. The hereditarily atomic von Neumann algebra $\ell^\infty(\mathcal{X}) \mathbin{\bar\otimes} \ell^\infty(\mathcal{X})$ may be naturally regarded as an algebra of multipliers on $c_c(\mathcal{X}) \odot c_c(\mathcal{X})$, because $c_c(\mathcal{X}) \odot c_c(\mathcal{X})$ is a two-sided ideal in $\ell^\infty(\mathcal{X}) \mathbin{\bar\otimes} \ell^\infty(\mathcal{X})$. Thus, we may define unital $*$-homomorphisms $\Delta: c_c(\mathcal{X}) \to \mathrm{Mult}(c_c(\mathcal{X}) \odot c_c(\mathcal{X}))$ by $\Delta = F^\star|_{c_c(\mathcal{X})}$ and $\varepsilon: c_c(\mathcal{X}) \to \mathbb{C}$ by $\varepsilon = C^\star|_{c_c(\mathcal{X})}$.

We now observe that Δ is a comultiplication on $c_c(\mathcal{X})$. By Lemmas 8.2 and 8.3, $\Delta(a) \cdot (1 \otimes b)$ and $(a \otimes 1) \cdot \Delta(b)$ are both in $c_c(\mathcal{X}) \odot c_c(\mathcal{X})$ for all $a, b \in c_c(\mathcal{X})$. Furthermore, for all $a, b, c \in c_c(\mathcal{X})$,

$$\begin{aligned}
(a \otimes 1 \otimes 1) \cdot (\Delta \otimes 1)(\Delta(b) \cdot (1 \otimes c)) &= (a \otimes 1 \otimes 1) \cdot (F^\star \mathbin{\bar\otimes} 1)(F^\star(b) \cdot (1 \otimes c)) \\
&= (a \otimes 1 \otimes 1) \cdot (F^\star \mathbin{\bar\otimes} 1)(F^\star(b)) \cdot (F^\star \mathbin{\bar\otimes} 1)(1 \otimes c) \\
&= (1 \mathbin{\bar\otimes} F^\star)(a \otimes 1) \cdot (1 \mathbin{\bar\otimes} F^\star)(F^\star(b)) \cdot (1 \otimes 1 \otimes c) \\
&= (1 \mathbin{\bar\otimes} F^\star)((a \otimes 1) \cdot F^\star(b)) \cdot (1 \otimes 1 \otimes c) \\
&= (1 \otimes \Delta)((a \otimes 1) \cdot \Delta(b)) \cdot (1 \otimes 1 \otimes c).
\end{aligned}$$

Hence, Δ is indeed a comultiplication on $c_c(\mathcal{X})$.

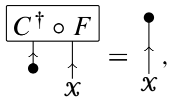
Let $a \in c_c(\mathcal{X})$, and assume that $\Delta([\mathbb{C}]) \cdot (1 \otimes a) = 0$. It follows that $\Delta([\mathbb{C}]) \cdot (1 \otimes aa^\dagger) = 0$ and thus that $\Delta([\mathbb{C}]) \cdot (1 \otimes [aa^\dagger]) = 0$, where $[aa^\dagger]$ is of course the support projection of the self-adjoint operator $aa^\dagger$. We now observe that the projection $\Delta([\mathbb{C}]) \in \ell^\infty(\mathcal{X}) \mathbin{\bar\otimes} \ell^\infty(\mathcal{X})$ corresponds to the relation

$$[\![F(x_1, x_2) = C_*]\!] = C^\dagger \circ F \in \mathrm{Rel}(\mathcal{X}, \mathcal{X})$$

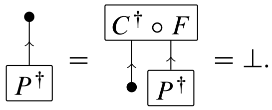
under the canonical correspondence [16, Thm. B.8]. Indeed,

$$(C^\dagger \circ F)^\star(1) = F^\star((C^\dagger)^\star(1)) = F^\star([\mathbb{C}]) = \Delta([\mathbb{C}]).$$

Similarly, the projection $[aa^\dagger]$ corresponds to some relation $P \in \mathrm{Rel}(\mathcal{X})$. The projections $\Delta([\mathbb{C}])$ and $(1 \otimes [aa^\dagger])$ are orthogonal, and thus, the relations $C^\dagger \circ F$ and $\top_\mathcal{X} \times P$ are orthogonal. Condition (2) may be rendered as



so we calculate that



We conclude that $P = \bot_\mathcal{X}$. Thus, $[aa^\dagger] = 0$, and therefore, $a = 0$. We have shown that for all $a \in c_c(\mathcal{X})$, the equation $\Delta([\mathbb{C}]) \cdot (1 \otimes a) = 0$ implies that $a = 0$.

Let T_1 and T_2 be the linear maps on $c_c(\mathcal{X}) \odot c_c(\mathcal{X})$ defined by $T_1(a \otimes b) = \Delta(a) \cdot (1 \otimes b)$ and $T_2(a \otimes b) = (a \otimes 1) \cdot \Delta(b)$, respectively. By Lemmas 8.2 and 8.4, both T_1 and T_2 are surjective. For all $a \in c_c(\mathcal{X})$, we compute that

$$\Delta(a) \cdot (1 \otimes [\mathbb{C}]) = F^\star(a) \cdot (1 \otimes [\mathbb{C}]) = (1 \mathbin{\bar{\otimes}} C^\star)(F^\star(a)) \otimes [\mathbb{C}] = a \otimes [\mathbb{C}],$$

because $[\mathbb{C}]$ is the support projection of the unital normal $*$-homomorphism

$$C^\star : c_c(\mathcal{X}) \to \mathbb{C}.$$

Furthermore, we have already shown that for all $a \in c_c(\mathcal{X})$, we have that

$$\Delta([\mathbb{C}]) \cdot (1 \otimes a) = 0$$

only if $a = 0$. We conclude by [43, Thm. 3.4] that $(c_c(\mathcal{X}), \Delta)$ is a discrete quantum group. ■

Corollary 8.6 (Vaes). *Let $\mathcal{X}$ be a quantum set. Then, there is a one-to-one correspondence between $*$-homomorphisms*

$$\Delta : c_c(\mathcal{X}) \to \mathrm{Mult}(c_c(\mathcal{X}) \odot c_c(\mathcal{X}))$$

such that $(c_c(\mathcal{X}), \Delta)$ is a discrete quantum group in the sense of [43, Def. 2.3] *and pairs of functions, $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $C : \mathbf{1} \to \mathcal{X}$, such that*

(1) $[\![\forall(x_1 = x_{1*})\, \forall(x_2 = x_{2*})\, \forall(x_3 = x_{3*})\, F(F(x_1, x_2), x_3) = F_*(x_{1*}, F_*(x_{2*}, x_{3*}))]\!] = \top$*;*

(2) $[\![\forall(x = x_*)\, F(x, C) = x_*]\!] = \top$*;*

(3) $[\![\forall (x = x_*)\, F(C, x) = x_*]\!] = \top$;

(4) $[\![\forall x_1\, \exists x_2\, F(x_1, x_2) = C_*]\!] = \top$;

(5) $[\![\forall x_2\, \exists x_1\, F(x_1, x_2) = C_*]\!] = \top$.

This correspondence is given by $\Delta = F^\star|_{c_c(\mathcal{X})}$ *and* $\varepsilon = C^\star|_{c_c(\mathcal{X})}$, *where* $\varepsilon\colon c_c(\mathcal{X}) \to \mathbb{C}$ *is the counit of* $(c_c(\mathcal{X}), \Delta)$ [43, Sec. 3].

Proof (cf. [6]*).* Let F be a function $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$, and let C be a function $\mathbf{1} \to \mathcal{X}$. By Lemma 3.5.3 and Proposition 3.5.4 of Part I [17], conditions (1), (2), and (3) are equivalent to $F \circ (F \times I_\mathcal{X}) = F \circ (I_\mathcal{X} \times F)$, $F \circ (I_\mathcal{X} \times C) = I_\mathcal{X}$ and $F \circ (C \times I_\mathcal{X}) = I_\mathcal{X}$, respectively. Therefore, by Theorem 8.5, conditions (1)–(5) imply that $(c_c(\mathcal{X}), F^\star|_{c_c(\mathcal{X})})$ is a discrete quantum group.

Conversely, let $\Delta\colon c_c(\mathcal{X}) \to \mathrm{Mult}(c_c(\mathcal{X}) \odot c_c(\mathcal{X}))$ be a $*$-homomorphism, and assume that $(c_c(\mathcal{X}), \Delta)$ is a discrete quantum group. Let $a_0 \in c_c(\mathcal{X})$. Let $A \subseteq c_c(\mathcal{X})$ be the $*$-algebra generated by a_0. Because $a_0(X) = 0$ for all but finitely many atoms X, we know that A is a finite-dimensional C*-algebra. As observed in [43, Sec. 2], the multiplier algebra $\mathrm{Mult}(c_c(\mathcal{X}) \odot c_c(\mathcal{X}))$ is canonically isomorphic to $\ell(\mathcal{X} \times \mathcal{X})$ [16, Def. 5.1]. For all atoms $X_1, X_2 \in \mathrm{At}(\mathcal{X})$, the function $A \to L(X_1 \otimes X_2)$ that is defined by

$$a \mapsto \Delta(a)(X_1 \otimes X_2)$$

is a $*$-homomorphism between finite-dimensional C*-algebras, and therefore,

$$\|\Delta(a_0)(X_1 \otimes X_2)\| \leq \|a_0\|.$$

We conclude that $\|\Delta(a_0)\| \leq \|a_0\|$. Therefore, $\Delta(c_c(\mathcal{X})) \subseteq \ell^\infty(\mathcal{X} \times \mathcal{X})$.

The $*$-homomorphism $\Delta\colon c_c(\mathcal{X}) \to \ell^\infty(\mathcal{X} \times \mathcal{X})$ is bounded, and thus, it extends uniquely to a $*$-homomorphism $\Delta_0\colon c_0(\mathcal{X}) \to \ell^\infty(\mathcal{X} \times \mathcal{X})$. This is a nondegenerate representation of the C*-algebra $c_0(\mathcal{X})$ because $(c_c(\mathcal{X}) \odot 1) \cdot \Delta(c_c(\mathcal{X})) = c_c(\mathcal{X}) \odot c_c(\mathcal{X})$ by the definition of a discrete quantum group. The enveloping von Neumann algebra of $c_0(\mathcal{X})$ is of course $\ell^\infty(\mathcal{X})$, and hence, Δ_0 extends to a unital normal $*$-homomorphism $\Delta_1\colon \ell^\infty(\mathcal{X}) \to \ell^\infty(\mathcal{X} \times \mathcal{X})$ [36, Thm. 3.7.7]. Thus, we obtain a function $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ such that $F^\star$ extends Δ.

Let $b \in c_c(\mathcal{X})$. Appealing to the definition of a comultiplication [43, Def. 2.1], we calculate that for all atoms X_1 and X_2,

$$\begin{aligned}
&([X_1] \otimes 1 \otimes 1) \cdot (F^\star \mathbin{\bar{\otimes}} 1)(F^\star(b)) \cdot (1 \otimes 1 \otimes [X_2]) \\
&\quad = ([X_1] \otimes 1 \otimes 1) \cdot (F^\star \mathbin{\bar{\otimes}} 1)(F^\star(b) \cdot (1 \otimes [X_2])) \\
&\quad = (1 \mathbin{\bar{\otimes}} F^\star)(([X_1] \otimes 1) \cdot F^\star(b)) \cdot (1 \otimes 1 \otimes [X_2]) \\
&\quad = ([X_1] \otimes 1 \otimes 1) \cdot (1 \mathbin{\bar{\otimes}} F^\star)(F^\star(b)) \cdot (1 \otimes 1 \otimes [X_2]).
\end{aligned}$$

Therefore, for all $b \in c_c(\mathcal{X})$, $(F^\star \mathbin{\bar{\otimes}} 1)(F^\star(b)) = (1 \mathbin{\bar{\otimes}} F^\star)(F^\star(b))$. Because $c_c(\mathcal{X})$ is ultraweakly dense in $\ell^\infty(\mathcal{X})$, we conclude that $(F \times I_\mathcal{X}) \circ F = (I_\mathcal{X} \times F) \circ F$, establishing condition (1).

The discrete quantum group has a counit $\varepsilon\colon c_c(\mathcal{X}) \to \mathbb{C}$ [43, Sec. 3]. By elementary algebra, $\varepsilon = J_X^\star|_{c_c(\mathcal{X})}$ for some one-dimensional atom X, and without loss of generality, we may assume that $\varepsilon = J_{\mathbb{C}}^\star|_{c_c(\mathcal{X})}$. Let $C = J_{\mathbb{C}}$ so that $\varepsilon = C^\star|_{c_c(\mathcal{X})}$. Appealing to the definition of a counit, we calculate that for all $a \in c_c(\mathcal{X})$ and all $X \in \mathrm{At}(\mathcal{X})$,

$$(C^\star \mathbin{\overline{\otimes}} 1)(F^\star(a)) \cdot [X] = (C^\star \mathbin{\overline{\otimes}} 1)(F^\star(a) \cdot (1 \otimes [X])) = a \cdot [X].$$

Therefore, for all $a \in c_c(\mathcal{X})$, $(C^\star \mathbin{\overline{\otimes}} 1)(F^\star(a)) = a$. Because $c_c(\mathcal{X})$ is ultraweakly dense in $\ell^\infty(\mathcal{X})$, we conclude that $F \circ (C \times I_{\mathcal{X}}) = I_{\mathcal{X}}$. Similarly, $F \circ (I_{\mathcal{X}} \times C) = I_{\mathcal{X}}$. We have established conditions (2) and (3).

Let $p \in \ell^\infty(\mathcal{X})$ be a nonzero projection. Hence, there exists an atom X such that $p \cdot [X] \neq 0$. It certainly follows that $[\mathbb{C}] \otimes (p \cdot [X]) \neq 0$. Appealing directly to the definition of a discrete quantum group [43, Def. 2.3], we infer that $\Delta([\mathbb{C}])(1 \otimes (p \cdot [X])) \neq 0$ and thus that $\Delta([\mathbb{C}]) \cdot (1 \otimes p) \neq 0$. Therefore, $\Delta([\mathbb{C}])$ is not orthogonal to $1 \otimes p$ for any projection $p \neq 0$. Equivalently, $\Delta([\mathbb{C}])$ is not below $1 \otimes r$ for any projection $r \neq 1$.

We have already observed in the proof of Theorem 8.5 that the projection $\Delta([\mathbb{C}])$ corresponds to the relation $[\![F(x_1, x_2) = C_*]\!]$ in the sense of [16, Thm. B.8], so $[\![F(x_1, x_2) = C_*]\!]$ is not below $\top_{\mathcal{X}} \times R$ for any $R \neq \top_{\mathcal{X}}$. Therefore, by Proposition 2.4.2 (2) of Part I [17],

$$\begin{aligned}&[\![x_2 \in \mathcal{X} \mid \exists x_1\, F(x_1, x_2) = C_*]\!]\\ &\quad = \inf\{R \in \mathrm{Rel}(\mathcal{X}) \mid \top_{\mathcal{X}} \times R \geq [\![F(x_1, x_2) = C_*]\!]\} = \top_{\mathcal{X}}.\end{aligned}$$

We conclude by Lemma 3.2.1 of Part I [17] that $[\![\forall x_2\, \exists x_1\, F(x_1, x_2) = C_*]\!] = \top$. Similarly, $[\![\forall x_1\, \exists x_2\, F(x_1, x_2) = C_*]\!] = \top$. We have established conditions (4) and (5).

The $*$-algebra $c_c(\mathcal{X})$ is ultraweakly dense in $\ell^\infty(\mathcal{X})$, and thus, the equation $\Delta = F^\star|_{c_c(\mathcal{X})}$ defines a bijection between

- $*$-homomorphisms $\Delta\colon c_c(\mathcal{X}) \to \mathrm{Mult}(c_c(\mathcal{X}) \odot c_c(\mathcal{X}))$ such that $(c_c(\mathcal{X}), \Delta)$ is a discrete quantum group and
- functions $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ for which there exists a function $C\colon \mathbf{1} \to \mathcal{X}$ such that F and C together satisfy conditions (1)–(5).

The function $C\colon \mathbf{1} \to \mathcal{X}$ is easily seen to be unique because conditions (2) and (3) imply that $F \circ (I_{\mathcal{X}} \times C) = I_{\mathcal{X}}$ and $F \circ (C \times I_{\mathcal{X}}) = I_{\mathcal{X}}$. We constructed C to satisfy $\varepsilon = C^\star|_{c_c(\mathcal{X})}$, where ε is the counit of the discrete quantum group $(c_c(\mathcal{X}), \Delta)$. Hence, the theorem is proved. ∎

A. Appendix. Quantifier laws

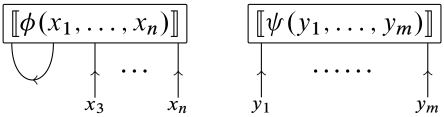
We show that the existential quantifier distributes over disjunction, just as it does classically, and consequently, the universal quantifier distributes over conjunction, just as it does classically. We also show that the existential quantifier commutes with conjunction by a second formula, subject to the restriction that the two formulas have no free variables in common at all, and consequently, the universal quantifier commutes with disjunction by a second formula, subject to the same restriction.

Proposition A.1. *Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ and $\mathcal{Y}_1, \dots, \mathcal{Y}_m$ be quantum sets. Let $x_1, \dots, x_n$ and $y_1, \dots, y_m$ be distinct variables of sorts $\mathcal{X}_1, \dots, \mathcal{X}_n$ and $\mathcal{Y}_1, \dots, \mathcal{Y}_m$, respectively. Let $\phi(x_1, \dots, x_n)$ and $\psi(y_1, \dots, y_m)$ be nonduplicating formulas. If $\mathcal{X}_2 = \mathcal{X}_1^*$, then*

$$\begin{aligned}
&[\![(x_3, \dots, x_n, y_1, \dots, y_m) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m \\
&\qquad \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, (\phi(x_1, \dots, x_n) \wedge \psi(y_1, \dots, y_m))]\!] \\
&= [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \phi(x_1, \dots, x_n)]\!] \\
&\qquad \wedge [\![(y_1, \dots, y_m) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m \mid \psi(y_1, \dots, y_m)]\!].
\end{aligned}$$

Proof. We reason graphically, applying Theorem 3.3.2 of Part I [17]:

$$[\![(\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, (\phi(x_1, \dots, x_n) \wedge \psi(y_1, \dots, y_m))]\!]$$

$=$ 

$$= [\![(\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \phi(x_1, \dots, x_n)]\!] \wedge [\![\psi(y_1, \dots, y_m)]\!]. \qquad \blacksquare$$

Proposition A.2. *Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be quantum sets, and let $x_1, \dots, x_n$ be distinct variables of sorts $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. Let $\phi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ be nonduplicating formulas. If $\mathcal{X}_2 = \mathcal{X}_1^*$, then*

$$\begin{aligned}
&[\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*) \\
&\qquad\qquad (\phi(x_1, \dots, x_n) \vee \psi(x_1, \dots, x_n))]\!] \\
&= [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \phi(x_1, \dots, x_n)]\!] \\
&\quad \vee [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \psi(x_1, \dots, x_n)]\!].
\end{aligned}$$

Proof. We appeal to Theorem 3.3.2 of Part I [17] and to the fact that composition distributes over the join of binary relations between quantum sets:

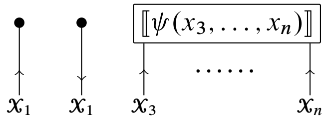
$$\begin{aligned}
&[\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*) \\
&\qquad\qquad (\phi(x_1, \dots, x_n) \vee \psi(x_1, \dots, x_n))]\!] \\
&= ([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!] \\
&\quad \vee [\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \psi(x_1, \dots, x_n)]\!]) \circ (E_{\mathcal{X}_1}^\dagger \times I_{\mathcal{X}_3} \times \cdots \times I_{\mathcal{X}_n}) \\
&= ([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!] \circ (E_{\mathcal{X}_1}^\dagger \times I_{\mathcal{X}_3} \times \cdots \times I_{\mathcal{X}_n})) \\
&\quad \vee ([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \psi(x_1, \dots, x_n)]\!] \circ (E_{\mathcal{X}_1}^\dagger \times I_{\mathcal{X}_3} \times \cdots \times I_{\mathcal{X}_n})) \\
&= [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \phi(x_1, \dots, x_n)]\!] \\
&\quad \vee [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \psi(x_1, \dots, x_n)]\!]. \qquad \blacksquare
\end{aligned}$$

Lemma A.3. *Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be quantum sets, and let $x_1, \dots, x_n$ be distinct variables of sorts $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. Let $\psi(x_3, \dots, x_n)$ be a nonduplicating formula. If $\mathcal{X}_2 = \mathcal{X}_1^*$ and $\mathcal{X}_1 \neq `\emptyset$, then*

$$\begin{aligned}&[\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \psi(x_3, \dots, x_n)]\!] \\ &\quad = [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid \psi(x_3, \dots, x_n)]\!].\end{aligned}$$

Proof. We reason graphically:

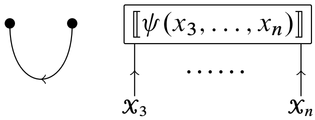
$$[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \psi(x_3, \dots, x_n)]\!] =$$



,

$$\begin{aligned}&[\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \psi(x_3, \dots, x_n)]\!] \\ &\quad = [\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \psi(x_3, \dots, x_n)]\!] \circ (E_{\mathcal{X}_1}^\dagger \times I_{\mathcal{X}_3} \times \cdots \times I_{\mathcal{X}_n})\end{aligned}$$

=



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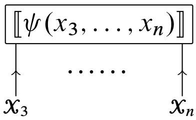

$$= [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid \psi(x_3, \dots, x_n)]\!].$$

The assumption that $\mathcal{X}_1 \neq `\emptyset$ is used in the second-to-last equality. ∎

Proposition A.4. *Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be quantum sets, and let $x_1, \dots, x_n$ be distinct variables of sorts $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. Let $\phi(x_1, \dots, x_n)$ and $\psi(x_3, \dots, x_n)$ be nonduplicating formulas. If $\mathcal{X}_2 = \mathcal{X}_1^*$ and $\mathcal{X}_1 \neq `\emptyset$, then*

$$\begin{aligned}&[\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*) \\ &\qquad\qquad (\phi(x_1, \dots, x_n) \vee \psi(x_3, \dots, x_n))]\!] \\ &\quad = [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid (\exists (x_1 = x_2) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, \phi(x_1, \dots, x_n)]\!] \\ &\qquad \vee [\![(x_3, \dots, x_n) \in \mathcal{X}_3 \times \cdots \times \mathcal{X}_n \mid \psi(x_3, \dots, x_n)]\!].\end{aligned}$$

Proof. We combine Proposition A.2 with Lemma A.3. ∎

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