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Post-Hopf algebras, relative Rota–Baxter operators and solutions to the Yang–Baxter equation

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Abstract. In this paper, first, we introduce the notion of post-Hopf algebra, which gives rise to a post-Lie algebra on the space of primitive elements and the fact that there is naturally a post-Hopf algebra structure on the universal enveloping algebra of a post-Lie algebra. A novel property is that a cocommutative post-Hopf algebra gives rise to a generalized Grossman–Larson product, which leads to a subadjacent Hopf algebra and can be used to construct solutions to the Yang–Baxter equation. Then, we introduce the notion of relative Rota–Baxter operator on Hopf algebras. A cocommutative post-Hopf algebra gives rise to a relative Rota–Baxter operator on its subadjacent Hopf algebra. Conversely, a relative Rota–Baxter operator also induces a post-Hopf algebra. Finally, we show that relative Rota–Baxter operators give rise to matched pairs of Hopf algebras. Consequently, post-Hopf algebras and relative Rota–Baxter operators give solutions to the Yang–Baxter equation in certain cocommutative Hopf algebras.

1. Introduction

The notion of post-Lie algebra was introduced in [37]; it has important applications in geometric numerical integration [9, 32]. Recently, post-Lie algebras play an important role in regularity structures and planarly branched rough paths [5, 35]. If the Lie algebra in a post-Lie algebra is abelian, then we obtain a pre-Lie algebra [6]. Thus, a post-Lie algebra can be viewed as a nonabelian generalization of a pre-Lie algebra. People pay much attention to the studies of the universal enveloping algebras of a pre-Lie algebra as well as a post-Lie algebra. First, in [34], Oudom and Guin constructed an associative product on the symmetric module $\mathsf{S}(\mathfrak{h})$ of any pre-Lie algebra $\mathfrak{h}$. Then, this construction was generalized to the post-Lie case [12, 15, 17, 32, 33]. In particular, it was found that there is a new Hopf algebra structure on the Lie enveloping algebra of a post-Lie algebra, by which the Magnus expansions and Lie–Butcher series can be constructed. Moreover, Mencattini, Quesney, and Silva introduced the notion of D-bialgebra in [31] and showed that there is a functor from the category of post-Lie algebras to the category of D-bialgebras [31, Theorem 21]. This functor is full and faithful and provides an adjunction of categories whose adjoint is the primitive elements functor from the category of D-bialgebras to the category of post-Lie algebras.

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Motivated by the aforementioned studies on the universal enveloping algebras of a pre-Lie algebra as well as a post-Lie algebra, we introduce the notion of post-Hopf algebra. A post-Hopf algebra is a Hopf algebra H equipped with a coalgebra homomorphism from $H \otimes H$ to H, satisfying some compatibility conditions (see Definition 2.1). Magma algebras, in particular ordered rooted trees, provide a class of examples of post-Hopf algebras. A cocommutative post-Hopf algebra gives rise to a generalized Grossman–Larson product, which leads to a subadjacent Hopf algebra. Note that the classical Grossman–Larson product was defined in the context of polynomials of ordered rooted trees [33], and it has important applications in the studies of Magnus expansions [1, 7, 11, 13, 30, 31] and Lie–Butcher series [32, 33]. The terminology of post-Hopf algebras is justified by the fact that a post-Hopf algebra gives rise to a post-Lie algebra on the space of primitive elements.

Rota–Baxter operators on Lie algebras and associative algebras have important applications in various fields, such as Connes and Kreimer's algebraic approach to renormalization of quantum field theory [8, 26]. Rota–Baxter operators lead to the classical Yang–Baxter equation and integrable systems [27, 36], noncommutative symmetric functions and noncommutative Bohnenblust–Spitzer identities [14], splitting of operads [2], double Lie algebras [20], etc. See the book of Guo [23] for more details. Recently, the notion of Rota–Baxter operator on groups was introduced in [24] and further studied in [4]. One can obtain Rota–Baxter operators of weight 1 on Lie algebras from those on Lie groups by differentiation. Then, in the remarkable work of Goncharov [19], he succeeded in defining Rota–Baxter operators on cocommutative Hopf algebras such that many classical results still hold in the Hopf algebra level. In this paper, we introduce a more general notion of relative Rota–Baxter operator on Hopf algebras containing Goncharov's Rota–Baxter operators as special cases. A cocommutative post-Hopf algebra naturally gives rise to a relative Rota–Baxter operator on its subadjacent Hopf algebra, and conversely, a relative Rota–Baxter operator also induces a post-Hopf algebra.

The Yang–Baxter equation is an important subject in mathematical physics [25, 38]. Drinfeld highlighted the importance of the study of set-theoretical solutions to the Yang–Baxter equation in [10]. The pioneer works on set-theoretical solutions are those of Etingof–Schedler–Soloviev [16], Lu–Yan–Zhu [28], and Gateva-Ivanova–Van den Bergh [18]. In this paper, we provide another approach to understanding the structure of set-theoretical solutions to the Yang–Baxter equation in certain Hopf algebras. Note that a relative Rota–Baxter operator on a cocommutative Hopf algebra naturally gives rise to a matched pair of Hopf algebras. In particular, for a cocommutative post-Hopf algebra, the original Hopf algebra and the subadjacent Hopf algebra form a matched pair of Hopf algebras satisfying certain good properties. Based on this fact, we construct solutions to the Yang–Baxter equation in a Hopf algebra using post-Hopf algebras as well as relative Rota–Baxter operators and give explicit formulas of solutions for the post-Hopf algebras coming from ordered rooted trees.

The paper is organized as follows. In Section 2, first, we introduce the notion of post-Hopf algebra and show that a cocommutative post-Hopf algebra gives rise to a subadjacent Hopf algebra together with a module bialgebra structure on itself. In Section 3, we introduce the notion of relative Rota–Baxter operator and show that post-Hopf algebras are the underlying structures, which give rise to relative Rota–Baxter operators on the subadjacent Hopf algebras. In Section 4, we show that a relative Rota–Baxter operator gives rise to a matched pair of Hopf algebras. In particular, a cocommutative post-Hopf algebra gives rise to a matched pair of Hopf algebras. Consequently, one can construct solutions to the Yang–Baxter equation using post-Hopf algebras and relative Rota–Baxter operators.

Convention. In this paper, we fix an algebraically closed ground field $\mathbf{k}$ of characteristic 0. For any coalgebra (C, Δ, ε), we compress the Sweedler notation of the comultiplication Δ as

$$\Delta(x) = x_1 \otimes x_2$$

for simplicity. Furthermore, for $n \geq 1$, we write

$$\Delta^{(n)}(x) = (\Delta \otimes \mathrm{id}_C^{\otimes(n-1)}) \cdots (\Delta \otimes \mathrm{id}_C)\Delta(x) = x_1 \otimes \cdots \otimes x_{n+1}.$$

Let $(H, \cdot, 1, \Delta, \varepsilon, S)$ be a Hopf algebra. Denote by $G(H)$ the set of group-like elements in H, which is a group. Denote by $P_{g,h}(H)$ the subspace of (g, h)-primitive elements in H for $g, h \in G(H)$. Denote by $P(H)$ the subspace of primitive elements in H, which is a Lie algebra. For other basic notions in the theory of Hopf algebras, we follow the textbooks [29].

2. Post-Hopf algebras

In this section, first, we introduce the notion of a post-Hopf algebra and show that a cocommutative post-Hopf algebra gives rise to a subadjacent Hopf algebra together with a module bialgebra structure on itself. A post-Hopf algebra induces a post-Lie algebra structure on the space of primitive elements, and conversely, there is naturally a post-Hopf algebra structure on the universal enveloping algebra of a post-Lie algebra.

Recall from [32, 37] that a *post-Lie algebra* $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd)$ consists of a Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ and a binary product $\rhd : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ such that

$$x \rhd [y, z]_{\mathfrak{h}} = [x \rhd y, z]_{\mathfrak{h}} + [y, x \rhd z]_{\mathfrak{h}}, \tag{2.1}$$

$$([x, y]_{\mathfrak{h}} + x \rhd y - y \rhd x) \rhd z = x \rhd (y \rhd z) - y \rhd (x \rhd z). \tag{2.2}$$

Any post-Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd)$ has a *subadjacent Lie algebra*

$$\mathfrak{h}_{\rhd} := (\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}_{\rhd}})$$

defined by

$$[x, y]_{\mathfrak{h}_{\rhd}} := x \rhd y - y \rhd x + [x, y]_{\mathfrak{h}}, \quad \forall x, y \in \mathfrak{h},$$

and equations (2.1)-(2.2) equivalently mean that the linear map $L : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h})$ defined by $L_x y = x \rhd y$ is an action of the Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}_{\rhd}})$ on $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$.

A post-Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd)$ reduces to a pre-Lie algebra if the Lie bracket $[\cdot,\cdot]_{\mathfrak{h}}$ is abelian. More precisely, a *pre-Lie algebra* $(\mathfrak{h}, \rhd)$ is a vector space $\mathfrak{h}$ equipped with a binary product $\rhd\colon \mathfrak{h}\otimes\mathfrak{h}\to\mathfrak{h}$ such that

$$(x \rhd y - y \rhd x) \rhd z = x \rhd (y \rhd z) - y \rhd (x \rhd z), \quad \forall x, y, z \in \mathfrak{h}.$$

Recently, there are many fruitful studies on the universal enveloping algebra of a pre-Lie algebra and a post-Lie algebra due to various applications. First, Oudom and Guin constructed an associative product on the symmetric module $\mathsf{S}(\mathfrak{h})$ of any pre-Lie algebra $\mathfrak{h}$ in [34]. Then, this construction was generalized to the post-Lie case [12, 15, 17, 32, 33]. In particular, Mencattini, Quesney, and Silva introduced the notion of D-bialgebra and showed that there is a functor from the category of post-Lie algebras to the category of D-bialgebras [31, Theorem 21]. Recall from [31, Definition 19] that a D-bialgebra consists of a bialgebra $(D, \cdot, 1, \Delta, \varepsilon)$, a binary product $\rhd\colon D\otimes D\to D$, and an exhaustive, increasing filtration

$$\mathbf{k}1 = D^0 \subset D^1 \subset \cdots \subset D^n \subset \cdots$$

such that $D^i \cdot D^j \subset D^{i+j}$ and

(i) $1 \rhd X = X$ and $X \rhd 1 = \varepsilon(X)1$, for all $X \in D$,

(ii) $D_1 = \ker(\varepsilon) \cap D^1 = P(D)$ which generates $(D, \cdot, 1)$,

(iii) $\Delta(X \rhd Y) = (X_1 \rhd Y_1) \otimes (X_2 \rhd Y_2)$, for all $X, Y \in D$,

(iv) $X \rhd (Y \cdot Z) = (X_1 \rhd Y) \cdot (X_2 \rhd Z)$, for all $X, Y, Z \in D$,

(v) $(x \cdot X) \rhd y = x \rhd (X \rhd y) - (x \rhd X) \rhd y$, for all $x, y \in D_1$, $X \in D$,

(vi) D_1 is closed under the antisymmetrization of the associative product.

Motivated by all the aforementioned studies on the universal enveloping algebra of a pre-Lie algebra and a post-Lie algebra, we propose the following definition of a post-Hopf algebra.

Definition 2.1. A *post-Hopf algebra* is a pair $(H, \rhd)$, where H is a Hopf algebra and $\rhd\colon H\otimes H\to H$ is a coalgebra homomorphism satisfying the following equalities:

$$x \rhd (y \cdot z) = (x_1 \rhd y) \cdot (x_2 \rhd z), \tag{2.3}$$
$$x \rhd (y \rhd z) = \big(x_1 \cdot (x_2 \rhd y)\big) \rhd z \tag{2.4}$$

for any $x, y, z \in H$, and the left multiplication $\alpha_{\rhd} : H \to \mathrm{End}(H)$ defined by

$$\alpha_{\rhd,x} y = x \rhd y, \quad \forall x, y \in H,$$

is convolution invertible in $\mathrm{Hom}(H, \mathrm{End}(H))$. Namely, there exists unique $\beta_{\rhd} : H \to \mathrm{End}(H)$ such that

$$\alpha_{\rhd,x_1}\beta_{\rhd,x_2} = \beta_{\rhd,x_1}\alpha_{\rhd,x_2} = \varepsilon(x)\,\mathrm{id}_H, \quad \forall x \in H. \tag{2.5}$$

A *homomorphism* from a post-Hopf algebra $(H, \rhd)$ to $(H', \rhd')$ is a Hopf algebra homomorphism $f : H \to H'$ satisfying

$$f(x \rhd y) = f(x) \rhd' f(y), \quad \forall x, y \in H.$$

It is obvious that post-Hopf algebras and homomorphisms between post-Hopf algebras form a category, which is denoted by PH. We denote by cocPH the subcategory of PH consisting of cocommutative post-Hopf algebras and homomorphisms between them.

Remark 2.2. By (ii) and (v) in the definition of a D-bialgebra, we deduce that

$$X \rhd (Y \rhd Z) = \big(X_1 \cdot (X_2 \rhd Y)\big) \rhd Z,$$

for all $X, Y, Z \in D$. Moreover, by the exhaustive and increasing filtration, we obtain that the left multiplication $\alpha_\rhd : D \to \mathrm{End}(D)$ defined by

$$\alpha_{\rhd,X} Y = X \rhd Y, \quad \forall X, Y \in D,$$

is convolution invertible in $\mathrm{Hom}(D, \mathrm{End}(D))$. Therefore, a D-bialgebra is a particular post-Hopf algebra.

Since a pre-Lie algebra can be viewed as a commutative post-Lie algebra, from this perspective, we introduce the notion of pre-Hopf algebra as a special post-Hopf algebra.

Definition 2.3. A post-Hopf algebra $(H, \rhd)$ is called a *pre-Hopf algebra* if H is a commutative Hopf algebra.

We have the following properties for post-Hopf algebras.

Lemma 2.4. *Let $(H, \rhd)$ be a post-Hopf algebra. Then, for all $x, y \in H$, we have*

$$x \rhd 1 = \varepsilon(x)1, \tag{2.6}$$

$$1 \rhd x = x, \tag{2.7}$$

$$S(x \rhd y) = x \rhd S(y). \tag{2.8}$$

Proof. Since $\rhd$ is a coalgebra homomorphism, we have

$$\begin{aligned} x \rhd 1 &= (x_1 \rhd 1)\varepsilon(x_2 \rhd 1) = (x_1 \rhd 1) \cdot (x_2 \rhd 1) \cdot S(x_3 \rhd 1) \\ &\overset{(2.3)}{=} (x_1 \rhd 1) \cdot S(x_2 \rhd 1) = \varepsilon(x \rhd 1)1 = \varepsilon(x)1. \end{aligned}$$

By equation (2.5), we have

$$\alpha_{\rhd,1}\beta_{\rhd,1} = \beta_{\rhd,1}\alpha_{\rhd,1} = \mathrm{id}_H,$$

which means that $\alpha_{\rhd,1}$ is a linear automorphism of H. On the other hand, we have

$$\alpha^2_{\rhd,1} x = 1 \rhd (1 \rhd x) \overset{(2.4)}{=} (1 \rhd 1) \rhd x \overset{(2.6)}{=} 1 \rhd x = \alpha_{\rhd,1} x.$$

Hence,

$$1 \rhd x = \alpha_{\rhd,1} x = x.$$

Finally, we have

$$\begin{aligned} S(x \rhd y) &= S(x_1 \rhd y_1)\varepsilon(x_2)\varepsilon(y_2) \overset{(2.6)}{=} S(x_1 \rhd y_1) \cdot (x_2 \rhd \varepsilon(y_2)1) \\ &= S(x_1 \rhd y_1) \cdot (x_2 \rhd (y_2 \cdot S(y_3))) \\ &\overset{(2.3)}{=} S(x_1 \rhd y_1) \cdot (x_2 \rhd y_2) \cdot (x_3 \rhd S(y_3)) \\ &= \varepsilon(x_1 \rhd y_1)(x_2 \rhd S(y_2)) = \varepsilon(x_1)\varepsilon(y_1)(x_2 \rhd S(y_2)) = x \rhd S(y). \end{aligned}$$ ■

Now, we give the main result in this section.

Theorem 2.5. *Let $(H, \rhd)$ be a cocommutative post-Hopf algebra. Then*

$$H_{\rhd} := (H, *_{\rhd}, 1, \Delta, \varepsilon, S_{\rhd})$$

is a Hopf algebra, which is called the subadjacent Hopf algebra, where for all $x, y \in H$,

$$x *_{\rhd} y := x_1 \cdot (x_2 \rhd y), \tag{2.9}$$

$$S_{\rhd}(x) := \beta_{\rhd,x_1}(S(x_2)). \tag{2.10}$$

Furthermore, $(H, \cdot, 1, \Delta, \varepsilon, S)$ is a left $H_{\rhd}$-module bialgebra via the action $\rhd$.

Proof. In order to show that $(H, *_{\rhd}, 1, \Delta, \varepsilon)$ is a cocommutative bialgebra, the simple computations analogous to those for the universal enveloping algebras of pre-Lie algebras and post-Lie algebras [12, 34] are enough. Here, we just emphasize on clarifying the antipode formula (2.10).

Since $\rhd$ is a coalgebra homomorphism and H is cocommutative, we know that

$$\Delta\beta_{\rhd,x} = (\beta_{\rhd,x_1} \otimes \beta_{\rhd,x_2})\Delta,$$

and $S_{\rhd}$ is a coalgebra homomorphism. Also, note that

$$\begin{aligned} x_1 *_{\rhd} S_{\rhd}(x_2) &\overset{(2.9)}{=} x_1 \cdot (x_2 \rhd S_{\rhd}(x_2)) \\ &\overset{(2.10)}{=} x_1 \cdot (\alpha_{\rhd,x_2}(\beta_{\rhd,x_3}(S(x_4)))) \\ &\overset{(2.5)}{=} x_1 \cdot (\varepsilon(x_2)S(x_3)) \\ &= \varepsilon(x)1. \end{aligned}$$

Using such an equality, we have

$$\alpha_{\rhd,x_1}\alpha_{\rhd,S_{\rhd}(x_2)} \overset{(2.4),(2.9)}{=} \alpha_{\rhd,x_1 *_{\rhd} S_{\rhd}(x_2)} = \alpha_{\rhd,\varepsilon(x)1} \overset{(2.7)}{=} \varepsilon(x)\mathrm{id}_H;$$

i.e.,

$$\beta_{\rhd,x} = \alpha_{\rhd,S_{\rhd}(x)},$$

as $\beta_{\triangleright}$ is also the convolution inverse of $\alpha_{\triangleright}$. So, equation (2.10) can be rewritten as

$$S_{\triangleright}(x) = S_{\triangleright}(x_1) \triangleright S(x_2), \quad \forall x \in H. \tag{2.11}$$

Again by the cocommutativity of H, we obtain that

$$\begin{aligned} S_{\triangleright}(x_1) *_{\triangleright} x_2 &\overset{(2.9)}{=} S_{\triangleright}(x_1) \cdot (S_{\triangleright}(x_2) \triangleright x_3) \\ &\overset{(2.11)}{=} (S_{\triangleright}(x_1) \triangleright S(x_2)) \cdot (S_{\triangleright}(x_3) \triangleright x_4) \\ &= (S_{\triangleright}(x_1) \triangleright S(x_3)) \cdot (S_{\triangleright}(x_2) \triangleright x_4) \\ &\overset{(2.3)}{=} S_{\triangleright}(x_1) \triangleright (S(x_2) \cdot x_3) \\ &\overset{(2.6)}{=} \varepsilon(S_{\triangleright}(x_1))\varepsilon(x_2)1 \\ &= \varepsilon(x)1. \end{aligned}$$

Therefore, $(H, *_{\triangleright}, 1, \Delta, \varepsilon, S_{\triangleright})$ is a cocommutative Hopf algebra.

Moreover, we have

$$(x *_{\triangleright} y) \triangleright z = (x_1 \cdot (x_2 \triangleright y)) \triangleright z = x \triangleright (y \triangleright z).$$

Then, by (2.3) and (2.6), $(H, \cdot, 1)$ is a left $H_{\triangleright}$-module algebra. Since $\triangleright$ is also a coalgebra homomorphism, $(H, \cdot, 1, \Delta, \varepsilon, S)$ is a left $H_{\triangleright}$-module bialgebra via the action $\triangleright$. ∎

Remark 2.6. The product (2.9) generalizes the Grossman–Larson product [21, 33, 34] defined in the context of (noncommutative) polynomials of (ordered) rooted trees. The Grossman–Larson product plays important roles in the theories of Magnus expansions [1, 7, 13, 31] and Lie–Butcher series [32, 33].

In the sequel, we study the relation between post-Hopf algebras and post-Lie algebras.

Theorem 2.7. *Let $(H, \triangleright)$ be a post-Hopf algebra. Then, its subspace $P(H)$ of primitive elements is a post-Lie algebra.*

Proof. Since $\triangleright$ is a coalgebra homomorphism, for all $x, y \in P(H)$, we have

$$\begin{aligned} \Delta(x \triangleright y) &= (x_1 \triangleright y_1) \otimes (x_2 \triangleright y_2) \\ &= (1 \triangleright 1) \otimes (x \triangleright y) + (1 \triangleright y) \otimes (x \triangleright 1) \\ &\quad + (x \triangleright 1) \otimes (1 \triangleright y) + (x \triangleright y) \otimes (1 \triangleright 1) \\ &\overset{(2.6),(2.7)}{=} 1 \otimes (x \triangleright y) + (x \triangleright y) \otimes 1. \end{aligned}$$

Thus, we obtain a linear map $\triangleright : P(H) \otimes P(H) \to P(H)$. By (2.3), for all $x, y \in P(H)$, we have

$$\begin{aligned} x \triangleright (y \cdot z) &= (1 \triangleright y) \cdot (x \triangleright z) + (x \triangleright y) \cdot (1 \triangleright z) \\ &\overset{(2.7)}{=} y \cdot (x \triangleright z) + (x \triangleright y) \cdot z. \end{aligned}$$

Thus, we have

$$\begin{aligned} x \rhd [y,z] &= x \rhd (y\cdot z) - x \rhd (z \cdot y) \\ &= y\cdot(x \rhd z) + (x \rhd y)\cdot z - z\cdot(x \rhd y) - (x \rhd z)\cdot y \\ &= [x \rhd y, z] + [y, x \rhd z]. \end{aligned}$$

By (2.4), we have

$$x \rhd (y \rhd z) = (1\cdot(x \rhd y)) \rhd z + (x\cdot(1 \rhd y)) \rhd z = (x \rhd y) \rhd z + (x\cdot y) \rhd z.$$

Thus, we have

$$\begin{aligned} [x,y] \rhd z &= (x\cdot y) \rhd z - (y \cdot x) \rhd z \\ &= x \rhd (y \rhd z) - (x \rhd y) \rhd z - y \rhd (x \rhd z) + (y \rhd x) \rhd z. \end{aligned}$$

Therefore, $(P(H), [\cdot,\cdot], \rhd)$ is a post-Lie algebra. ■

In [12, 34], the authors studied the universal enveloping algebra of a pre-Lie algebra and also of a post-Lie algebra. By [12, Proposition 3.1 and Theorem 3.4], the binary product $\rhd$ in a post-Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd)$ can be extended to its universal enveloping algebra and induces a subadjacent Hopf algebra structure isomorphic to the universal enveloping algebra $\mathrm{U}(\mathfrak{h}_{\rhd})$ of the subadjacent Lie algebra $\mathfrak{h}_{\rhd}$.

We summarize their result in the setting of post-Hopf algebras as follows. We do not claim any originality (see [12, 34] for details).

Theorem 2.8. *Let $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd)$ be a post-Lie algebra with its subadjacent Lie algebra $\mathfrak{h}_{\rhd}$. Then, $(\mathrm{U}(\mathfrak{h}), \bar{\rhd})$ is a post-Hopf algebra, where $\bar{\rhd}$ is the extension of $\rhd$ determined by*

$$1 \bar{\rhd} u = u, \quad x_1\cdots x_r \bar{\rhd} u = x_1 \bar{\rhd} (x_2\cdots x_r \bar{\rhd} u) - (x_1 \bar{\rhd} x_2 \cdots x_r) \bar{\rhd} u$$

for all $x_1, \dots, x_r \in \mathfrak{h}$ and $u \in \mathrm{U}(\mathfrak{h})$ with $r \geq 1$.

Moreover, the subadjacent Hopf algebra $\mathrm{U}(\mathfrak{h})_{\bar{\rhd}}$ is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathfrak{h}_{\rhd})$ of the subadjacent Lie algebra $\mathfrak{h}_{\rhd}$.

Theorem 2.9. *Let $(V, \rhd)$ be a magma algebra. Extend the magma operation $\rhd : V \otimes V \to V$ on the vector space V to the coshuffle Hopf algebra $(\mathrm{T}V, \cdot, \Delta^{\mathrm{cosh}})$ (using the same notation $\rhd$) as follows:*

$$\begin{aligned} 1 \rhd a &= a, \\ x \rhd a &= x \rhd a, \\ (x\otimes x_1) \rhd a &= x \rhd (x_1 \rhd a) - (x \rhd x_1) \rhd a, \\ &\ \ \vdots \\ (x\otimes x_1 \otimes\cdots\otimes x_n) \rhd a &= x \rhd ((x_1\otimes\cdots\otimes x_n) \rhd a) \\ &\quad - \sum_{i=1}^{n} (x_1\otimes\cdots\otimes x_{i-1}\otimes(x \rhd x_i)\otimes x_{i+1}\otimes\cdots\otimes x_n) \rhd a, \end{aligned}$$

and

$$
\begin{aligned}
1 \rhd 1 &= 1, \\
x \rhd 1 &= 0, \\
X \rhd (a_1 \otimes \cdots \otimes a_m) &= (X_1 \rhd a_1) \otimes \cdots \otimes (X_m \rhd a_m),
\end{aligned}
$$

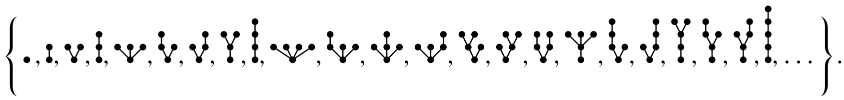
where $x, x_1, \ldots, x_n, a, a_1, \ldots, a_m \in V$ *and* $X \in \mathsf{T}V$,

$$\Delta^{\mathrm{cosh}(m-1)} X = X_1 \otimes \cdots \otimes X_m.$$

Then, $(\mathsf{T}V, \cdot, \Delta^{\mathrm{cosh}}, \rhd)$ *is a post-Hopf algebra.*

Proof. According to the discussion in [17, Proposition 1 and Lemma 1], it is straightforward to obtain that $(\mathsf{T}V, \cdot, \Delta^{\mathrm{cosh}}, \rhd)$ is a post-Hopf algebra. ■

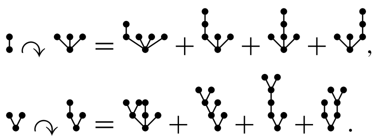
Example 2.10. Let $\mathcal{OT}$ be the set of isomorphism classes of ordered rooted trees, which is denoted by

$\mathcal{OT} =$ 

Let $\mathbf{k}\{\mathcal{OT}\}$ be the free $\mathbf{k}$-vector space generated by $\mathcal{OT}$. The *left grafting operator* $\curvearrowright$: $\mathbf{k}\{\mathcal{OT}\} \otimes \mathbf{k}\{\mathcal{OT}\} \to \mathbf{k}\{\mathcal{OT}\}$ is defined by

$$\tau \curvearrowright \omega = \sum_{s \in \mathrm{Nodes}(\omega)} \tau \circ_s \omega, \quad \forall \tau, \omega \in \mathcal{OT},$$

where $\tau \circ_s \omega$ is the ordered rooted tree resulting from attaching the root of τ to the node s of the tree ω from the left. For example, we have



It is obvious that $(\mathbf{k}\{\mathcal{OT}\}, \curvearrowright)$ is a magma algebra. By Theorem 2.9,

$$(\mathsf{T}\mathbf{k}\{\mathcal{OT}\}, \cdot, \Delta^{\mathrm{cosh}}, \rhd)$$

is a post-Hopf algebra, where the underlying coshuffle Hopf algebra

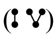
$$(\mathsf{T}\mathbf{k}\{\mathcal{OT}\}, \cdot, \Delta^{\mathrm{cosh}})$$

has the linear basis consisting of all ordered rooted forests, and its antipode S is given by

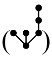
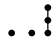
$$S(\tau_1 \tau_2 \cdots \tau_m) = (-1)^m \tau_m \tau_{m-1} \cdots \tau_1, \quad \forall \tau_1, \tau_2, \ldots, \tau_m \in \mathcal{OT}.$$

Moreover, it is the universal enveloping algebra of the free post-Lie algebra on one generator $\{\bullet\}$. See [17, 32, 37] for more details about free post-Lie algebras and their universal enveloping algebras.

Let $B^+ : \mathsf{T}\mathbf{k}\{\mathcal{OT}\} \to \mathbf{k}\{\mathcal{OT}\}$ be the linear map producing an ordered tree τ from any ordered rooted forest $\tau_1 \cdots \tau_m$ by grafting the m trees $\tau_1, \ldots, \tau_m$ on a new root $\bullet$ in order. For example, we have

$B^+($  $) =$ .

Let $B^- : \mathbf{k}\{\mathcal{OT}\} \to \mathsf{T}\mathbf{k}\{\mathcal{OT}\}$ be the linear map producing an ordered forest from any ordered rooted tree τ by removing its root. For example, we have

$B^-($  $) =$ .

Moreover, the operation B^- extends to $\mathsf{T}\mathbf{k}\{\mathcal{OT}\}$ by

$$B^-(\tau_1 \cdots \tau_m) = B^-(\tau_1) \cdots B^-(\tau_m), \quad \forall \tau_1, \ldots, \tau_m \in \mathcal{OT}.$$

Note that the subadjacent Hopf algebra $(\mathsf{T}\mathbf{k}\{\mathcal{OT}\}, *_\rhd, \Delta^{\mathrm{cosh}}, S_\rhd)$ is isomorphic to the Grossman-Larson Hopf algebra of ordered rooted trees defined in [21]. Using the left grafting operation, the multiplication $*_\rhd$ is given by

$$\mathcal{X} *_\rhd \mathcal{Y} = B^-(\mathcal{X} \rhd B^+(\mathcal{Y}))$$

for all ordered rooted forests $\mathcal{X}, \mathcal{Y}$, and the antipode $S_\rhd$ can be recursively defined by

$$S_\rhd(1) = 1, \quad S_\rhd(\mathcal{X}) \overset{(2.7),(2.11)}{=} S(\mathcal{X}) + (\mathrm{id}_{\mathsf{T}\mathbf{k}\{\mathcal{OT}\}} - \mu\varepsilon)(S_\rhd(\mathcal{X}_1)) \rhd S(\mathcal{X}_2),$$

where μ is the unit map and ε is the counit map.

Next, we provide a class of examples of cocommutative post-Hopf algebras not coming from post-Lie algebras.

Example 2.11. In [24], the authors introduced the notion of a Rota–Baxter operator on a group. Namely, given any group G, a map $B : G \to G$ is a Rota–Baxter operator on G if

$$B(g)B(h) = B(gB(g)hB(g)^{-1}), \quad \forall g, h \in G.$$

A group endowed with a Rota–Baxter operator is called a Rota–Baxter group. Also, Rota–Baxter groups have been studied by Bardakov and Gubarev recently relating to skew left braces and the Yang–Baxter equation [4].

Given any Rota–Baxter group (G, B), define the binary operation $\rhd : G \times G \to G$ associated to B by

$$g \rhd h = B(g)hB(g)^{-1}, \quad \forall g, h \in G.$$

Now, consider the group algebra $\mathbf{k}[G]$ of a Rota–Baxter group G with the multiplication $\rhd$ linearly extending that on G. It is straightforward to check that $(\mathbf{k}[G], \rhd)$ is a cocommutative post-Hopf algebra. Especially when G is abelian, $(\mathbf{k}[G], \rhd)$ is a cocommutative pre-Hopf algebra.

Note that the post-Hopf algebra structure can be equipped on non-cocommutative Hopf algebras. The following example may give an illustration.

Example 2.12. We classify all post-Hopf algebra structures on the smallest noncommutative and non-cocommutative Hopf algebra, namely, Sweedler's 4-dimensional Hopf algebra:

$$H_4 = \mathbf{k}\langle 1, g, x, gx \mid g^2 = 1, x^2 = 0, gx = -xg\rangle,$$

with its coalgebra structure and its antipode given by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(x) &= x \otimes 1 + g \otimes x, \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0, \\ S(g) &= g, & S(x) &= -gx. \end{aligned}$$

Further,

$$G(H_4) = \{1, g\}, \quad P_{1,g}(H_4) = \mathbf{k}x, \quad P_{g,1}(H_4) = \mathbf{k}gx.$$

Let $(H_4, \triangleright)$ be a post-Hopf algebra structure on H_4. Then

$$\begin{aligned} \Delta(g \triangleright g) &= (g \triangleright g) \otimes (g \triangleright g), \\ \Delta(g \triangleright x) &= (g \triangleright x) \otimes (g \triangleright 1) + (g \triangleright g) \otimes (g \triangleright x) \\ &\overset{(2.6)}{=} (g \triangleright x) \otimes 1 + (g \triangleright g) \otimes (g \triangleright x). \end{aligned}$$

Namely,

$$g \triangleright g \in G(H_4) \quad \text{and} \quad g \triangleright x \in P_{1,g\triangleright g}(H_4).$$

Since $g \in G(H_4)$ implies that $\alpha_{\triangleright,g}$ is invertible by equation (2.5), we know that

$$g \triangleright g = g \quad \text{and} \quad g \triangleright x \in P_{1,g}(H_4) \setminus \{0\}.$$

Also,

$$g \triangleright (g \triangleright x) \overset{(2.4)}{=} (g(g \triangleright g)) \triangleright x = g^2 \triangleright x = 1 \triangleright x \overset{(2.7)}{=} x.$$

Therefore, $g \triangleright x = x$ or $-x$. On the other hand,

$$\Delta(x \triangleright g) = (x \triangleright g) \otimes (1 \triangleright g) + (g \triangleright g) \otimes (x \triangleright g) \overset{(2.7)}{=} (x \triangleright g) \otimes g + g \otimes (x \triangleright g).$$

Then, $x \triangleright g \in P_{g,g}(H_4)$, and thus,

$$x \triangleright g = 0.$$

So,

$$\begin{aligned} \Delta(x \triangleright x) &= (x \triangleright x) \otimes (1 \triangleright 1) + (g \triangleright x) \otimes (x \triangleright 1) \\ &\quad + (x \triangleright g) \otimes (1 \triangleright x) + (g \triangleright g) \otimes (x \triangleright x) \\ &\overset{(2.6),(2.7)}{=} (x \triangleright x) \otimes 1 + g \otimes (x \triangleright x). \end{aligned}$$

That is, $x \rhd x \in P_{1,g}(H_4)$, and we can set

$$x \rhd x = ax$$

for some $a \in \mathbf{k}$. Then

$$\begin{aligned} a(g \rhd x) = x \rhd (g \rhd x) &\overset{(2.4)}{=} (x(1 \rhd g) + g(x \rhd g)) \rhd x = xg \rhd x = -gx \rhd x \\ = g \rhd (x \rhd x) &\overset{(2.4)}{=} (g(g \rhd x)) \rhd x. \end{aligned}$$

It implies that

$$g \rhd x = -x$$

unless $a = 0$.

In summary, one can easily check that there is a post-Hopf algebra structure $(H_4, \rhd_a)$ for any $a \in \mathbf{k}$ defined as below such that $\alpha_{\rhd_a}$ is the convolution inverse of itself.

$\rhd_a$	1	g	x	gx
1	1	g	x	gx
g	1	g	$-x$	$-gx$
x	0	0	ax	agx
gx	0	0	ax	agx

Moreover, if $a \neq 0$, there is a post-Hopf algebra isomorphism from $(H_4, \rhd_a)$ to $(H_4, \rhd_1)$ mapping g to g and x to ax. Hence, the Sweedler 4-dimensional Hopf algebra has three non-isomorphic post-Hopf algebra structures:

$$(H_4, \varepsilon \otimes \mathrm{id}_{H_4}), \quad (H_4, \rhd_0), \quad (H_4, \rhd_1).$$

3. Relative Rota–Baxter operators on Hopf algebras

In this section, first, we recall relative Rota–Baxter operators on Lie algebras and groups and Rota–Baxter operators on cocommutative Hopf algebras. Then, we introduce a more general notion of relative Rota–Baxter operator of weight 1 on cocommutative Hopf algebras with respect to module bialgebras. We establish the relation between the category of relative Rota–Baxter operators of weight 1 on cocommutative Hopf algebras and the category of cocommutative post-Hopf algebras.

Let $\phi : \mathfrak{h} \to \mathrm{Der}(\mathfrak{k})$ be an action of a Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ on a Lie algebra $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}})$. A linear map $T : \mathfrak{k} \to \mathfrak{h}$ is called a *relative Rota–Baxter operator of weight* 1 on $\mathfrak{h}$ with respect to $(\mathfrak{k}; \phi)$ if

$$[T(u), T(v)]_{\mathfrak{h}} = T(\phi(T(u))v - \phi(T(v))u + [u, v]_{\mathfrak{k}}), \quad \forall u, v \in \mathfrak{k}.$$

Let $\Phi : \mathcal{H} \to \mathrm{Aut}(\mathcal{K})$ be an action of a group $\mathcal{H}$ on a group $\mathcal{K}$. A map $T : \mathcal{K} \to \mathcal{H}$ is called a *relative Rota–Baxter operator of weight* 1 if

$$T(h) \cdot_{\mathcal{H}} T(k) = T(h \cdot_{\mathcal{K}} \Phi(T(h))k), \quad \forall h, k \in \mathcal{K}.$$

Given any Hopf algebra $(H, \Delta, \varepsilon, S)$, define the adjoint action of H on itself by

$$\mathrm{ad}_x\, y = x_1 y S(x_2).$$

A *Rota–Baxter operator of weight* 1 on a cocommutative Hopf algebra H was defined by Goncharov in [19], which is a coalgebra homomorphism $B : H \to H$ satisfying

$$B(x)B(y) = B(x_1\, \mathrm{ad}_{B(x_2)}\, y) = B(x_1 B(x_2) y S(B(x_3))), \quad \forall\, x, y \in H.$$

In the sequel, all the relative Rota–Baxter operators under consideration are of weight 1, so we will not emphasize it anymore.

Now, we generalize the above adjoint action to arbitrary actions and introduce the notion of relative Rota–Baxter operator on cocommutative Hopf algebras.

Definition 3.1. Let H and K be two cocommutative Hopf algebras such that K is a (left) H-module bialgebra via an action $\rightharpoonup$. A coalgebra homomorphism $T : K \to H$ is called a *relative Rota–Baxter operator* (on the cocommutative Hopf algebra H) with respect to the cocommutative H-module bialgebra $(K, \rightharpoonup)$ if the following equality holds:

$$T(a)T(b) = T(a_1(T(a_2) \rightharpoonup b)), \quad \forall a, b \in K. \tag{3.1}$$

A *homomorphism* between two relative Rota–Baxter operators $T : K \to H$ and $T' : K' \to H'$ is a pair of Hopf algebra homomorphisms $f : H \to H'$ and $g : K \to K'$ such that

$$fT = T'g, \quad g(x \rightharpoonup a) = f(x) \rightharpoonup g(a), \quad \forall\, x \in H, a \in K. \tag{3.2}$$

It is obvious that relative Rota–Baxter operators on cocommutative Hopf algebras and homomorphisms between them form a category, which is denoted by rRB.

Let K be a cocommutative H-module bialgebra via an action $\rightharpoonup$. It is obvious that via the restrictions of the action $\rightharpoonup$, we obtain actions of $G(H)$ on $G(K)$ and of $P(H)$ on $P(K)$, for which we use the same notations. As expected, a relative Rota–Baxter operator with respect to a cocommutative H-module bialgebra $(K, \rightharpoonup)$ will naturally induce a relative Rota–Baxter operator on the group $G(H)$ and on the Lie algebra $P(H)$, respectively.

Theorem 3.2. *Let $T : K \to H$ be a relative Rota–Baxter operator with respect to a cocommutative H-module bialgebra $(K, \rightharpoonup)$.*

(i) *$T|_{G(K)}$ is a relative Rota–Baxter operator on the group $G(H)$ with respect to the action $(G(K), \rightharpoonup)$;*

(ii) *$T|_{P(K)}$ is a relative Rota–Baxter operator on the Lie algebra $P(H)$ with respect to the action $(P(K), \rightharpoonup)$.*

Proof. Since T is a coalgebra homomorphism, it follows that $T|_{G(K)}$ is a map from $G(K)$ to $G(H)$, and $T|_{P(K)}$ is a map from $P(K)$ to $P(H)$.

For any $a, b \in G(K)$, we have

$$T(a)T(b) = T(a(T(a) \rightharpoonup b)),$$

which implies that $T|_{G(K)}$ is a relative Rota–Baxter operator on the group $G(H)$ with respect to the action $(G(K), \rightharpoonup)$.

For any $a, b \in P(K)$, we have

$$T(a)T(b) = T(ab) + T(T(a) \rightharpoonup b),$$

and thus,

$$[T(a), T(b)] = T\big(T(a) \rightharpoonup b\big) - T\big(T(b) \rightharpoonup a\big) + T([a,b]).$$

Hence, $T|_{P(K)}$ is a relative Rota–Baxter operator on the Lie algebra $P(H)$ with respect to the action $(P(K), \rightharpoonup)$. ∎

It was proved in [3] that a relative Rota–Baxter operator $T : \mathfrak{k} \to \mathfrak{h}$ on a Lie algebra $\mathfrak{h}$ with respect to an action $(\mathfrak{k}; \phi)$ endows $\mathfrak{k}$ with the following post-Lie algebra structure $\rhd$:

$$u \rhd v = \phi(T(u))v, \quad \forall u, v \in \mathfrak{k}. \tag{3.3}$$

Moreover, associated to a post-Lie algebra, the identity map is naturally a relative Rota–Baxter operator on the subadjacent Lie algebra. In the sequel, we generalized this important relationship to the context of Hopf algebras. First, we show that a cocommutative post-Hopf algebra naturally gives rise to a relative Rota–Baxter operator.

Proposition 3.3. *Let $(H, \rhd)$ be a cocommutative post-Hopf algebra and $H_\rhd$ the subadjacent Hopf algebra. Then, the identity map $\mathrm{id}_H : H \to H_\rhd$ is a relative Rota–Baxter operator on $H_\rhd$ with respect to the $H_\rhd$-module bialgebra $(H, \rhd)$.*

Moreover, if $g : H \to H'$ is a cocommutative post-Hopf algebra homomorphism from $(H, \rhd)$ to $(H', \rhd')$, then (g, g) is a homomorphism from the relative Rota–Baxter operator $\mathrm{id}_H : H \to H_\rhd$ to $\mathrm{id}_{H'} : H' \to H'_{\rhd'}$. Consequently, we obtain a functor Υ : $\mathsf{cocPH} \to \mathsf{rRB}$ from the category of cocommutative post-Hopf algebras to the category of relative Rota–Baxter operators with respect to cocommutative module bialgebras.

Proof. For any $x, y, z \in H$, we have

$$\mathrm{id}_H(x) *_\rhd \mathrm{id}_H(y) = x *_\rhd y = x_1 \cdot (x_2 \rhd y) = \mathrm{id}_H(x_1 \cdot (\mathrm{id}_H(x_2) \rhd y)),$$

so $\mathrm{id}_H : H \to H_\rhd$ is a relative Rota–Baxter operator with respect to the $H_\rhd$-module bialgebra $(H, \rhd)$.

Let $g : H \to H'$ be a cocommutative post-Hopf algebra homomorphism from $(H, \rhd)$ to $(H', \rhd')$. Then, (g, g) obviously satisfy equation (3.2). Since g is a coalgebra homomorphism and

$$g(x *_\rhd y) = g(x_1 \cdot (x_2 \rhd y)) = g(x_1) \cdot' (g(x_2) \rhd' g(y)) = g(x) *_{\rhd'} g(y),$$

we deduce that g is a homomorphism from the Hopf algebra $H_{\rhd}$ to $H'_{\rhd'}$. Therefore, (g, g) is a homomorphism from the relative Rota–Baxter operator $\mathrm{id}_H : H \to H_{\rhd}$ to $\mathrm{id}_{H'} : H' \to H'_{\rhd'}$. It is straightforward to check that this provides the stated functor Υ : cocPH $\to$ rRB. ∎

Theorem 3.4. *Let $T : K \to H$ be a relative Rota–Baxter operator on a cocommutative Hopf algebra H with respect to a cocommutative H-module bialgebra $(K, \rightharpoonup)$. Then, there exists a post-Hopf algebra structure $\rhd_T : K \otimes K \to K$ on K given by*

$$a \rhd_T b = T(a) \rightharpoonup b. \tag{3.4}$$

Let $T : K \to H$ and $T' : K' \to H'$ be two relative Rota–Baxter operators and (f, g) a homomorphism between them. Then, g is a homomorphism from the post-Hopf algebra $(K, \rhd_T)$ to $(K', \rhd_{T'})$. Consequently, we obtain a functor Ξ : rRB $\to$ cocPH from the category of relative Rota–Baxter operators on cocommutative Hopf algebras to the category of cocommutative post-Hopf algebras.

Moreover, the functor Ξ is right adjoint to the functor Υ given in Proposition 3.3.

Proof. Since T is a coalgebra homomorphism and $\rightharpoonup$ is the module bialgebra action compatible with each other as in (3.1), it is straightforward to check that $\rhd_T$ is a coalgebra homomorphism satisfying (2.3) and (2.4). Define linear map $S_T : K \to K$ by

$$S_T(a) = S_H(T(a_1)) \rightharpoonup S_K(a_2), \tag{3.5}$$

which actually satisfies

$$T(S_T(a)) = S_H(T(a)), \quad \forall a \in K. \tag{3.6}$$

Furthermore, define $\beta_{\rhd_T} \in \mathrm{Hom}(K, \mathrm{End}(K))$ by

$$\beta_{\rhd_T, a} := \alpha_{\rhd_T, S_T(a)} \quad \text{for } a \in K.$$

That is,

$$\beta_{\rhd_T, a} b = \alpha_{\rhd_T, S_T(a)} b = S_T(a) \rhd_T b.$$

Then, applying (3.1), (3.6), one can see that $\beta_{\rhd_T}$ is the convolution inverse of $\alpha_{\rhd_T}$. Hence, $(K, \rhd_T)$ is a post-Hopf algebra.

Let (f, g) be a homomorphism from the relative Rota–Baxter operator T to T'. Then, we have

$$g(a \rhd_T b) = g(T(a) \rightharpoonup b) = f(T(a)) \rightharpoonup g(b) = T'(g(a)) \rightharpoonup g(b) = g(a) \rhd_{T'} g(b),$$

which implies that g is a homomorphism from the post-Hopf algebra $(K, \rhd_T)$ to $(K', \rhd_{T'})$. So, it clearly provides the desired functor Ξ : rRB $\to$ cocPH.

Finally, we prove that Ξ : rRB $\to$ cocPH is right adjoint to Υ : cocPH $\to$ rRB. Namely,

$$\mathrm{Hom}_{\mathsf{rRB}}(\mathrm{id}_{H'} : H' \to H'_{\rhd'}, T : K \to H) \simeq \mathrm{Hom}_{\mathsf{cocPH}}((H', \rhd'), (K, \rhd_T)),$$

where $T:K\to H$ is a relative Rota–Baxter operator with respect to a cocommutative H-module bialgebra $(K,\rightharpoonup)$ and $(H',\rhd')$ is a cocommutative post-Hopf algebra.

Indeed, for any post-Hopf algebra homomorphism $g:(H',\rhd')\to(K,\rhd_T)$, let $f=Tg$, and then, (f,g) is a homomorphism between the relative Rota–Baxter operators $\mathrm{id}_{H'}: H'\to H'_{\rhd'}$ and $T:K\to H$. Conversely, if (f,g) is a homomorphism between the relative Rota–Baxter operators $\mathrm{id}_{H'}:H'\to H'_{\rhd'}$ and $T:K\to H$, we have $f=Tg$ and $g:(H',\rhd')\to(K,\rhd_T)$ is a post-Hopf algebra homomorphism. ■

By Theorem 3.4 and Theorem 2.5, we immediately get the following result.

Corollary 3.5. *Let $T:K\to H$ be a relative Rota–Baxter operator on a cocommutative Hopf algebra H with respect to a cocommutative H-module bialgebra $(K,\rightharpoonup)$. Then*

$$(K,*_T,1,\Delta,\varepsilon,S_T)$$

is a Hopf algebra, which is called the descendent Hopf algebra and denoted by K_T, where the antipode S_T is given by (3.5) and the multiplication $*_T$ is given by

$$a*_T b=a_1(T(a_2)\rightharpoonup b).$$

Moreover, $T:K_T\to H$ is a Hopf algebra homomorphism.

Let $\phi:\mathfrak{h}\to\mathrm{Der}(\mathfrak{k})$ be an action of a Lie algebra $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$ on $(\mathfrak{k},[\cdot,\cdot]_{\mathfrak{k}})$. Then, ϕ can be extended to a module bialgebra action $\bar{\phi}:\mathrm{U}(\mathfrak{h})\to\mathrm{End}(\mathsf{T}(\mathfrak{k}))$ by

$$\bar{\phi}(x)(1)=0,\ \bar{\phi}(x)(y_1\cdots y_r)=\sum_{i=1}^{r}y_1\cdots y_{i-1}\phi(x)(y_i)y_{i+1}\cdots y_r,$$

where $\mathsf{T}(\mathfrak{k})$ is the tensor $\mathbf{k}$-algebra of $\mathfrak{k}$, $x\in\mathfrak{h}$ and $y_1,\dots,y_r\in\mathfrak{k}$, $r\geq 1$. As $\mathfrak{h}$ acts on $\mathfrak{k}$ by derivations, it induces a module bialgebra action $\bar{\phi}$ of $\mathrm{U}(\mathfrak{h})$ on $\mathrm{U}(\mathfrak{k})$.

The following extension theorem of relative Rota–Baxter operators from Lie algebras to their universal enveloping algebras generalizes the case of Rota–Baxter operators given in [19, Theorem 2]. See also [12,15] for more details about the Hopf algebra of a post-Lie algebra.

Theorem 3.6. *Any relative Rota–Baxter operator $T:\mathfrak{k}\to\mathfrak{h}$ on a Lie algebra $\mathfrak{h}$ with respect to an action $(\mathfrak{k};\phi)$ can be extended to a unique relative Rota–Baxter operator $\bar{T}:\mathrm{U}(\mathfrak{k})\to\mathrm{U}(\mathfrak{h})$ with respect to the extended $\mathrm{U}(\mathfrak{h})$-module bialgebra $(\mathrm{U}(\mathfrak{k}),\bar{\phi})$ by*

$$\begin{aligned}&\bar{T}(y_1\cdots y_n)\\&\quad=\big(T(y_1)\bar{T}-\bar{T}\bar{\phi}(T(y_1))\big)\cdots\big(T(y_n)\bar{T}-\bar{T}\bar{\phi}(T(y_n))\big)(1),\quad\forall y_1,\dots,y_n\in\mathfrak{k},\ n\geq 1,\end{aligned}$$

where those $T(y_k)$'s left to $\bar{T}$ are interpreted as the left multiplication by them.

Furthermore, the post-Hopf algebra $(\mathrm{U}(\mathfrak{k}),\rhd_{\bar{T}})$ induced by the relative Rota–Baxter operator $\bar{T}:\mathrm{U}(\mathfrak{k})\to\mathrm{U}(\mathfrak{h})$ as in Theorem 3.4 coincides with the extended post-Hopf *algebra* $(\mathrm{U}(\mathfrak{k}), \bar{\triangleright}_T)$ *from* $(\mathfrak{k}, \triangleright_T)$ *given in Theorem* 2.8. *Namely, we have the following diagram:*

$$\begin{array}{ccc} (\mathfrak{k}, \triangleright_T) & \xrightarrow{\textit{extension}} & (\mathrm{U}(\mathfrak{k}), \triangleright_{\bar{T}}) \\ \textit{Rota–Baxter action}\ \Big\uparrow & & \Big\uparrow\ \textit{Rota–Baxter action} \\ \mathfrak{k} \xrightarrow{T} \mathfrak{h} & \xrightarrow{\textit{extension}} & \mathrm{U}(\mathfrak{k}) \xrightarrow{\bar{T}} \mathrm{U}(\mathfrak{h}). \end{array}$$

Proof. Let

$$J_{\mathfrak{k}} = (yz - zy - [y, z]_{\mathfrak{k}} \mid y, z \in \mathfrak{k})$$

be the ideal of $\mathsf{T}(\mathfrak{k})$ such that

$$\mathrm{U}(\mathfrak{k}) \simeq \mathsf{T}(\mathfrak{k})/J_{\mathfrak{k}}.$$

We recursively define a linear map $\bar{T} : \mathsf{T}(\mathfrak{k}) \to \mathrm{U}(\mathfrak{h})$ by

$$\bar{T}(1) = 1, \quad \bar{T}(yu) = T(y)\bar{T}(u) - \bar{T}\big(\bar{\phi}(T(y))u\big), \quad \forall y \in \mathfrak{k},\ u \in \mathfrak{k}^{\otimes n},\ n \geq 0. \tag{3.7}$$

Then, it is straightforward to deduce that $\bar{T}(J_{\mathfrak{k}}) = 0$, and we have the induced linear map $\bar{T} : \mathrm{U}(\mathfrak{k}) \to \mathrm{U}(\mathfrak{h})$.

Next, we prove that $\bar{T} : \mathrm{U}(\mathfrak{k}) \to \mathrm{U}(\mathfrak{h})$ is a relative Rota–Baxter operator. Namely,

$$\bar{T}(u)\bar{T}(v) = \bar{T}\big(u_1\bar{\phi}(\bar{T}(u_2))v\big)$$

for any $u \in \mathrm{U}(\mathfrak{k})_m, v \in \mathrm{U}(\mathfrak{k})_n$. It can be done by induction on m. The case when $m = 1$ is due to the recursive definition (3.7) of $\bar{T}$. For $yu \in \mathrm{U}(\mathfrak{k})_{m+1}$, since $\bar{\phi}$ is a module bialgebra action, we have

$$\begin{aligned} \bar{T}(yu)\bar{T}(v) &= T(y)\bar{T}(u)\bar{T}(v) - \bar{T}\big(\bar{\phi}(T(y))u\big)\bar{T}(v) \\ &= T(y)\bar{T}\big(u_1\bar{\phi}(\bar{T}(u_2))v\big) - \bar{T}\Big(\big(\bar{\phi}(T(y))u_1\big)\big(\bar{\phi}(\bar{T}(u_2))v\big)\Big) \\ &\quad - \bar{T}\Big(u_1\Big(\bar{\phi}\Big(\bar{T}\big(\bar{\phi}(T(y))u_2\big)\Big)v\Big)\Big) \\ &= \bar{T}\big(yu_1\bar{\phi}(\bar{T}(u_2))v\big) + \bar{T}\Big(\bar{\phi}(T(y))\big(u_1\bar{\phi}(\bar{T}(u_2))v\big)\Big) \\ &\quad - \bar{T}\Big(\big(\bar{\phi}(T(y))u_1\big)\big(\bar{\phi}(\bar{T}(u_2))v\big)\Big) - \bar{T}\Big(u_1\Big(\bar{\phi}\Big(\bar{T}\big(\bar{\phi}(T(y))u_2\big)\Big)v\Big)\Big) \\ &= \bar{T}\big(yu_1\bar{\phi}(\bar{T}(u_2))v\big) + \bar{T}\Big(u_1\bar{\phi}\big(T(y)\bar{T}(u_2)\big)v\Big) \\ &\quad - \bar{T}\Big(u_1\Big(\bar{\phi}\Big(\bar{T}\big(\bar{\phi}(T(y))u_2\big)\Big)v\Big)\Big) \\ &= \bar{T}\big(yu_1\bar{\phi}(\bar{T}(u_2))v\big) + \bar{T}\Big(u_1\bar{\phi}\big(\bar{T}(yu_2)\big)v\Big) \\ &= \bar{T}\big((yu)_1\bar{\phi}(\bar{T}((yu)_2))v\big), \end{aligned}$$

which implies that $\bar{T} : \mathrm{U}(\mathfrak{k}) \to \mathrm{U}(\mathfrak{h})$ is a relative Rota–Baxter operator. The above procedure also implies that the extension from $T : \mathfrak{k} \to \mathfrak{h}$ to $\bar{T} : \mathrm{U}(\mathfrak{k}) \to \mathrm{U}(\mathfrak{h})$ is unique.

By (3.3), the induced post-Lie product $\triangleright_T$ on $\mathfrak{k}$ is given by

$$y \triangleright_T z = \phi(T(y))z, \quad \forall y, z \in \mathfrak{k}.$$

Then, by Theorem 2.8, the extended post-Hopf product $\bar{\rhd}_T$ on $\mathrm{U}(\mathfrak{k})$ is recursively defined by

$$y\bar{\rhd}_T 1 = 0, \quad y\bar{\rhd}_T zv = (y \rhd_T z)v + z(y\bar{\rhd}_T v),$$
$$1\bar{\rhd}_T v = v, \quad yu\bar{\rhd}_T v = y\bar{\rhd}_T(u\bar{\rhd}_T v) - (y\bar{\rhd}_T u)\bar{\rhd}_T v,$$

for any $y, z \in \mathfrak{k}$, $u, v \in \mathrm{U}(\mathfrak{k})$. On the other hand, by (3.4), we know that

$$u \rhd_{\bar{T}} v = \bar{\phi}(\bar{T}(u))v, \quad \forall u, v \in \mathrm{U}(\mathfrak{k}).$$

In particular, $y \rhd_{\bar{T}} 1 = 0$, $1 \rhd_{\bar{T}} v = v$ and

$$\begin{aligned}
y \rhd_{\bar{T}} zv &= \bar{\phi}(T(y))(zv) = (\phi(T(y))z)v + z\bar{\phi}(T(y))v \\
&= (y \rhd_T z)v + z(y \rhd_{\bar{T}} v), \\
yu \rhd_{\bar{T}} v &= \bar{\phi}(\bar{T}(yu))v = \bar{\phi}\big(T(y)\bar{T}(u) - \bar{T}\big(\bar{\phi}(T(y))u\big)\big)v \\
&= \bar{\phi}(T(y))\big(\bar{\phi}(\bar{T}(u))v\big) - \bar{\phi}\big(\bar{T}\big(\bar{\phi}(T(y))u\big)\big)v \\
&= y \rhd_{\bar{T}} (u \rhd_{\bar{T}} v) - (y \rhd_{\bar{T}} u) \rhd_{\bar{T}} v.
\end{aligned}$$

Therefore, the two post-Hopf products on $\mathrm{U}(\mathfrak{k})$ coincide, and we get the desired diagram. ∎

4. Matched pairs of Hopf algebras and solutions to the Yang–Baxter equation

In this section, we show that a relative Rota–Baxter operator on cocommutative Hopf algebras naturally gives rise to a matched pair of Hopf algebras. As applications, we construct solutions to the Yang–Baxter equation using post-Hopf algebras and relative Rota–Baxter operators on cocommutative Hopf algebras.

First, we recall the smash product and matched pairs of Hopf algebras. Let H and K be two Hopf algebras such that K is a left H-module algebra via an action $\rightharpoonup$. There is the following *smash product* on $K \otimes H$:

$$(a\#x)(a'\#x') = a(x_1 \rightharpoonup a')\#x_2x'$$

for any $x, x' \in H$, $a, a' \in K$, where $a \otimes x \in K \otimes H$ is rewritten as $a\#x$ to emphasize this smash product. We denote such a smash product algebra by $K \rtimes H$. Especially, if H is cocommutative and K is a left H-module bialgebra via $\rightharpoonup$, then $K \rtimes H$ becomes a Hopf algebra with the usual tensor product comultiplication and the antipode defined by

$$S(a\#x) = (S_H(x_1) \rightharpoonup S_K(a))\#S_H(x_2).$$

Definition 4.1. A *matched pair of Hopf algebras* is a 4-tuple $(H, K, \rightharpoonup, \leftharpoonup)$, where H and K are Hopf algebras, $\rightharpoonup: H \otimes K \to K$ and $\leftharpoonup: H \otimes K \to H$ are linear maps such that

K is a left H-module coalgebra, and H is a right K-module coalgebra and the following compatibility conditions hold:

$$\begin{aligned} x \rightharpoonup (ab) &= (x_1 \rightharpoonup a_1)((x_2 \leftharpoonup a_2) \rightharpoonup b) \\ x \rightharpoonup 1_K &= \varepsilon_H(x)1_K \\ (xy) \leftharpoonup a &= (x \leftharpoonup (y_1 \rightharpoonup a_1))(y_2 \leftharpoonup a_2) \\ 1_H \leftharpoonup a &= \varepsilon_K(a)1_H \\ (x_1 \leftharpoonup a_1) \otimes (x_2 \rightharpoonup a_2) &= (x_2 \leftharpoonup a_2) \otimes (x_1 \rightharpoonup a_1) \end{aligned}$$

for all $x, y \in H$ and $a, b \in K$.

Let $(H, K, \rightharpoonup, \leftharpoonup)$ be a matched pair of Hopf algebras. The *double crossproduct* $K \bowtie H$ of K and H is the **k**-vector space $K \otimes H$ with the unit $1_K \otimes 1_H$ such that its product, coproduct, counit, and antipode are given by

$$\begin{aligned} (a \otimes x)(b \otimes y) &= a(x_1 \rightharpoonup b_1) \otimes (x_2 \leftharpoonup b_2)y, \\ \Delta(a \otimes x) &= a_1 \otimes x_1 \otimes a_2 \otimes x_2, \\ \varepsilon(a \otimes x) &= \varepsilon_K(a)\varepsilon_H(x), \\ S(a \otimes x) &= (S_H(x_2) \rightharpoonup S_K(a_2)) \otimes (S_H(x_1) \leftharpoonup S_K(a_1)), \end{aligned}$$

for all $a, b \in K$ and $x, y \in H$. See [29] for further details of the double crossproducts.

By [29, Proposition 21.6], we have the following proposition.

Proposition 4.2. *With the above notations,* $(H, K, \rightharpoonup, \leftharpoonup)$ *is a matched pair of Hopf algebras if and only if there exist a Hopf algebra* A *and injective Hopf algebra homomorphisms* $i_K : K \to A$, $i_H : H \to A$ *such that the map*

$$\xi : K \otimes H \to A,\ a \otimes x \mapsto i_K(a)i_H(x)$$

is a linear isomorphism.

Let $T : K \to H$ be a relative Rota–Baxter operator on a cocommutative Hopf algebra H with respect to a cocommutative H-module bialgebra $(K, \rightharpoonup)$. Define a linear map $\leftharpoonup: H \otimes K \to H$ by

$$x \leftharpoonup a = S_H\big(T(x_1 \rightharpoonup a_1)\big)x_2T(a_2). \tag{4.1}$$

Theorem 4.3. *With the above notations,* H *is a right* K_T*-module coalgebra via the action* $\leftharpoonup$ *given in equation (4.1). Moreover, the* 4*-tuple* $(H, K_T, \rightharpoonup, \leftharpoonup)$ *is a matched pair of cocommutative Hopf algebras.*

Proof. We define a linear map $\Phi_T : K \otimes H \to K \otimes H$ as follows:

$$\Phi_T(a \otimes x) = a_1 \otimes T(a_2)x,\ \forall x \in H,\ a \in K. \tag{4.2}$$

Since T is a coalgebra homomorphism, the linear map Φ_T is invertible. Moreover, we have

$$\Phi_T^{-1}(a \otimes x) = a_1 \otimes S_H(T(a_2))x, \ \forall x \in H, \ a \in K.$$

Transfer the smash product Hopf algebra structure $K \rtimes H$ to $K \otimes H$ via the linear isomorphism $\Phi_T : K \otimes H \to K \rtimes H$, we obtain a Hopf algebra $(K \otimes H, \cdot_T, 1_T, \Delta_T, \varepsilon_T, \mathfrak{S}_T)$. Denote elements in $K \otimes H$ by $a \bowtie x$, $b \bowtie y$ for $x, y \in H$, $a, b \in K$; by the cocommutativity of K, we have

$$\begin{aligned}
&(a \bowtie x) \cdot_T (b \bowtie y)\\
&\quad = \Phi_T^{-1}\big(\Phi_T(a \bowtie x)\Phi_T(b \bowtie y)\big)\\
&\quad = \Phi_T^{-1}\big(a_1(T(a_2)x_1 \rightharpoonup b_1)\#T(a_3)x_2T(b_2)y\big)\\
&\quad = a_1(T(a_2)x_1 \rightharpoonup b_1) \bowtie S_H\big(T(a_3(T(a_4)x_2 \rightharpoonup b_2))\big)T(a_5)x_3T(b_3)y\\
&\quad = a_1(T(a_2) \rightharpoonup (x_1 \rightharpoonup b_1)) \bowtie S_H\big(T(a_3(T(a_4) \rightharpoonup (x_2 \rightharpoonup b_2)))\big)T(a_5)x_3T(b_3)y\\
&\quad \overset{(3.1)}{=} a_1 *_T (x_1 \rightharpoonup b_1) \bowtie S_H\big(T(a_2)T(x_2 \rightharpoonup b_2)\big)T(a_3)x_3T(b_3)y\\
&\quad = a_1 *_T (x_1 \rightharpoonup b_1) \bowtie S_H\big(T(x_2 \rightharpoonup b_2)\big)S_H(T(a_2))T(a_3)x_3T(b_3)y\\
&\quad = a *_T (x_1 \rightharpoonup b_1) \bowtie S_H\big(T(x_2 \rightharpoonup b_2)\big)x_3T(b_3)y\\
&\quad = a *_T (x_1 \rightharpoonup b_1) \bowtie (x_2 \leftharpoonup b_2)y,
\end{aligned}$$

and

$$\begin{aligned}
\Delta_T(a \bowtie x) &= (\Phi_T^{-1} \otimes \Phi_T^{-1})(\Delta\Phi_T(a \bowtie x))\\
&= (\Phi_T^{-1} \otimes \Phi_T^{-1})\Delta(a_1\#T(a_2)x)\\
&= \Phi_T^{-1}(a_1\#T(a_3)x_1) \otimes \Phi_T^{-1}(a_2\#T(a_4)x_2)\\
&= (a_1 \bowtie S_H(T(a_2))T(a_5)x_1) \otimes (a_3 \bowtie S_H(T(a_4))T(a_6)x_2)\\
&= (a_1 \bowtie S_H(T(a_2))T(a_3)x_1) \otimes (a_4 \bowtie S_H(T(a_5))T(a_6)x_2)\\
&= (a_1 \bowtie x_1) \otimes (a_2 \bowtie x_2),\\
\mathfrak{S}_T(a \bowtie x) &= \Phi_T^{-1}(S_{\rtimes}\Phi_T(a \bowtie x))\\
&= \Phi_T^{-1}(S_{\rtimes}(a_1\#T(a_2)x))\\
&= \Phi_T^{-1}((S_H(T(a_1)x_1) \rightharpoonup S_K(a_2))\#S_H(T(a_3)x_2))\\
&= \Phi_T^{-1}((S_H(x_1) \rightharpoonup (S_H(T(a_1)) \rightharpoonup S_K(a_2)))\#S_H(T(a_3)x_2))\\
&\overset{(3.5)}{=} \Phi_T^{-1}((S_H(x_1) \rightharpoonup S_T(a_1))\#S_H(x_2))S_H(T(a_2))\\
&= (S_H(x_1) \rightharpoonup S_T(a_1)) \bowtie S_H(T(S_H(x_2) \rightharpoonup S_T(a_2)))S_H(x_3)S_H(T(a_3))\\
&\overset{(3.6)}{=} (S_H(x_1) \rightharpoonup S_T(a_1)) \bowtie S_H(T(S_H(x_2) \rightharpoonup S_T(a_2)))S_H(x_3)T(S_T(a_3))\\
&\overset{(4.1)}{=} (S_H(x_1) \rightharpoonup S_T(a_1)) \bowtie (S_H(x_2) \leftharpoonup S_T(a_2)).
\end{aligned}$$

Moreover, it is obvious that $1_T = 1 \bowtie 1$ and

$$\varepsilon_T(a \bowtie x) = \varepsilon_K(a_1)\varepsilon_H(T(a_2)x) = \varepsilon_K(a)\varepsilon_H(x).$$

Define linear maps $\mathrm{i}_{K_T} : K_T \to K \otimes H$ and $\mathrm{i}_H : H \to K \otimes H$ by

$$\mathrm{i}_{K_T}(a) = a \bowtie 1, \quad \mathrm{i}_H(x) = 1 \bowtie x.$$

Then, it is obvious that i_{K_T} and i_H are injective Hopf algebra homomorphisms, and

$$\begin{aligned}(a \bowtie 1) \cdot_T (1 \bowtie x) &= a *_T (1 \rightharpoonup 1) \bowtie (1 \leftharpoonup 1)x = a_1(T(a_2) \rightharpoonup 1) \bowtie x \\ &= a_1 \varepsilon_H(T(a_2)) \bowtie x = a \bowtie x.\end{aligned}$$

Therefore, we obtain that $(K \otimes H, \cdot_T, 1_T, \Delta_T, \varepsilon_T, \mathfrak{S}_T)$ is a Hopf algebra that can be factorized into Hopf algebras K_T and H. Thus, we deduce that H is a right K_T-module coalgebra via the action $\leftharpoonup$ and K_T is a left H-module coalgebra via the action $\rightharpoonup$, and the 4-tuple $(H, K_T, \rightharpoonup, \leftharpoonup)$ is a matched pair of Hopf algebras by Proposition 4.2. Moreover, the Hopf algebra

$$(K \otimes H, \cdot_T, 1_T, \Delta_T, \varepsilon_T, \mathfrak{S}_T)$$

is exactly the double crossproduct $K_T \bowtie H$. ■

Conversely, let H and K be two cocommutative Hopf algebras such that K is an H-module bialgebra via an action $\rightharpoonup$. Let $T : K \to H$ be a coalgebra homomorphism, and $(K \otimes H, \cdot_T, 1_T, \Delta_T, \varepsilon_T, \mathfrak{S}_T)$ the Hopf algebra obtained from the smash product $K \rtimes H$ via the linear isomorphism Φ_T given in (4.2).

Proposition 4.4. *If $K \otimes 1$ is a subalgebra of the Hopf algebra*

$$(K \otimes H, \cdot_T, 1_T, \Delta_T, \varepsilon_T, \mathfrak{S}_T),$$

then T is a relative Rota–Baxter operator with respect to the H-module bialgebra $(K, \rightharpoonup)$.

Proof. Since $K \otimes 1$ is a subalgebra of $(K \otimes H, \cdot_T, 1_T, \Delta_T, \varepsilon_T, \mathfrak{S}_T)$, for any $a, b \in K$, we have

$$\begin{aligned}&(a \bowtie 1) \cdot_T (b \bowtie 1) \\ &\quad = \Phi_T^{-1}\big(\Phi_T(a \bowtie 1)\Phi_T(b \bowtie 1)\big) \\ &\quad = \Phi_T^{-1}\big(a_1(T(a_2) \rightharpoonup b_1)\# T(a_3)T(b_2)\big) \\ &\quad = a_1(T(a_2) \rightharpoonup b_1) \bowtie S_H\big(T(a_3(T(a_4) \rightharpoonup b_2))\big)T(a_5)T(b_3) \in K \otimes 1.\end{aligned}$$

Applying $m_H(T \otimes \mathrm{id}_H)$ and $T \otimes \varepsilon_H$ to it, respectively, we obtain that

$$\begin{aligned}T(a)T(b) &= T(a_1(T(a_2) \rightharpoonup b_1))S_H\big(T(a_3(T(a_4) \rightharpoonup b_2))\big)T(a_5)T(b_3) \\ &= T(a_1(T(a_2) \rightharpoonup b_1))\varepsilon_H(S_H\big(T(a_3(T(a_4) \rightharpoonup b_2))\big)T(a_5)T(b_3)) \\ &= T(a_1(T(a_2) \rightharpoonup b_1)).\end{aligned}$$

Namely, (3.1) holds, and T is a relative Rota–Baxter operator. ■

Let $(H, \triangleright)$ be a cocommutative post-Hopf algebra and

$$H_{\triangleright} := (H, *_{\triangleright}, 1, \Delta, \varepsilon, S_{\triangleright})$$

the subadjacent Hopf algebra given in Theorem 2.5. By Proposition 3.3, the identity map $\mathrm{id}_H : H \to H_{\triangleright}$ is a relative Rota–Baxter operator. By Theorem 4.3, we have the following corollary.

Corollary 4.5. *Let $(H, \triangleright)$ be a cocommutative post-Hopf algebra. Then, the 4-tuple $(H_{\triangleright}, H_{\triangleright}, \triangleright, \triangleleft)$ is a matched pair of cocommutative Hopf algebras, where $\triangleleft$ is given by*

$$a \triangleleft b = S_{\triangleright}\big(a_1 \triangleright b_1\big) *_{\triangleright} a_2 *_{\triangleright} b_2.$$

Moreover, we have the compatibility condition

$$a *_{\triangleright} b = (a_1 \triangleright b_1) *_{\triangleright} (a_2 \triangleleft b_2). \tag{4.3}$$

Proof. We only need to check the stated compatibility condition, which follows from

$$\begin{aligned}(a_1 \triangleright b_1) *_{\triangleright} (a_2 \triangleleft b_2) &= (a_1 \triangleright b_1) *_{\triangleright} \big(S_{\triangleright}\big(a_2 \triangleright b_2\big) *_{\triangleright} a_3 *_{\triangleright} b_3\big)\\ &= \big((a_1 \triangleright b_1) *_{\triangleright} S_{\triangleright}\big(a_2 \triangleright b_2\big)\big) *_{\triangleright} a_3 *_{\triangleright} b_3\\ &= \varepsilon(a_1 \triangleright b_1) a_2 *_{\triangleright} b_2\\ &= \varepsilon(a_1)\varepsilon(b_1) a_2 *_{\triangleright} b_2\\ &= a *_{\triangleright} b.\end{aligned}$$

■

At the end of this section, we show that post-Hopf algebras and relative Rota–Baxter operators on cocommutative Hopf algebras give rise to solutions to the Yang–Baxter equation.

Definition 4.6. A solution of the Yang–Baxter equation on a vector space V is an invertible linear endomorphism $R : V \otimes V \to V \otimes V$ such that

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$

Theorem 4.7 ([22, Theorem 5.11]). *Let H be a cocommutative Hopf algebra and $(H, H, \rightharpoonup, \leftharpoonup)$ a matched pair of cocommutative Hopf algebras such that*

$$xy = (x_1 \rightharpoonup y_1)(x_2 \leftharpoonup y_2), \quad \forall x, y \in H.$$

Then

$$R(x \otimes y) = (x_1 \rightharpoonup y_1) \otimes (x_2 \leftharpoonup y_2)$$

is a solution to the Yang–Baxter equation on the vector space H.

Theorem 4.8. *Let $(H, \triangleright)$ be a cocommutative post-Hopf algebra. Then, $R : H \otimes H \to H \otimes H$ defined by*

$$R(a \otimes b) = (a_1 \triangleright b_1) \otimes \big(S_{\triangleright}(a_2 \triangleright b_2) *_{\triangleright} a_3 *_{\triangleright} b_3\big)$$

is a solution to the Yang–Baxter equation on the vector space H.

Proof. By Corollary 4.5, we deduce that $(H_{\rhd}, H_{\rhd}, \rhd, \lhd)$ is a matched pair of cocommutative Hopf algebras and satisfies the condition (4.3). Moreover, by Theorem 4.7, we obtain that R is a solution of the Yang–Baxter equation on the vector space H. ■

Example 4.9. Consider the post-Hopf algebra $(\mathrm{T}\mathbf{k}\{\mathcal{OT}\}, \Delta^{\cosh}, \rhd)$ given in Example 2.10. Then

$$R : \mathrm{T}\mathbf{k}\{\mathcal{OT}\} \otimes \mathrm{T}\mathbf{k}\{\mathcal{OT}\} \to \mathrm{T}\mathbf{k}\{\mathcal{OT}\} \otimes \mathrm{T}\mathbf{k}\{\mathcal{OT}\}$$

defined by

$$R(\mathcal{X} \otimes \mathcal{Y}) = (\mathcal{X}_1 \rhd \mathcal{Y}_1) \otimes (\mathcal{X}_2 \lhd \mathcal{Y}_2), \quad \mathcal{X}, \mathcal{Y} \in \mathrm{T}\mathbf{k}\{\mathcal{OT}\}$$

is a solution to the Yang–Baxter equation on the vector space $\mathrm{T}\mathbf{k}\{\mathcal{OT}\}$. More precisely, we have

$$R(\mathcal{X} \otimes \mathcal{Y}) = (\mathcal{X}_1 \rhd \mathcal{Y}_1) \otimes B^{-}\big(S_{\rhd}(\mathcal{X}_2 \rhd \mathcal{Y}_2) \rhd (\mathcal{X}_3 \rhd B^{+}(\mathcal{Y}_3))\big),$$

where

$$\Delta^{\cosh^{(2)}}\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{X}_3 \quad \text{and} \quad \Delta^{\cosh^{(2)}}\mathcal{Y} = \mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3.$$

Let $T : K \to H$ be a relative Rota–Baxter operator on H with respect to a cocommutative H-module bialgebra $(K, \rightharpoonup)$. By Theorem 3.4, $(K, \rhd_T)$ is a cocommutative post-Hopf algebra. By Corollary 3.5, there is a descendent Hopf algebra

$$K_T = (K, *_T, \Delta, \varepsilon, S_T)$$

such that K is a K_T-module bialgebra via the action $\rhd_T$ defined in (3.4). By Corollary 4.5, we have the following corollary.

Corollary 4.10. *The* 4*-tuple* $(K_T, K_T, \rhd_T, \lhd_T)$ *is a matched pair of cocommutative Hopf algebras; here,* $\rhd_T$ *is given by* (3.4)*, and* $\lhd_T$ *is given by*

$$a \lhd_T b = S_T\big(a_1 \rhd_T b_1\big) *_T a_2 *_T b_2. \tag{4.4}$$

Moreover, we have the compatibility condition

$$a *_T b = (a_1 \rhd_T b_1) *_T (a_2 \lhd_T b_2).$$

By Theorem 4.8, we have the following corollary.

Corollary 4.11. *Let* $T : K \to H$ *be a relative Rota–Baxter operator with respect to a cocommutative* H*-module bialgebra* $(K, \rightharpoonup)$*. Then,* $R : K \otimes K \to K \otimes K$ *defined by*

$$R(a \otimes b) = (a_1 \rhd_T b_1) \otimes (a_2 \lhd_T b_2)$$

is a solution to the Yang–Baxter equation on the vector space K*, where* $\rhd_T$ *and* $\lhd_T$ *are defined by* (3.4) *and* (4.4)*, respectively.*

In this paper, we introduce the notions of post-Hopf algebra and relative Rota–Baxter operator on cocommutative Hopf algebras. A cocommutative post-Hopf algebra gives rise to a generalized Grossman–Larson product, which leads to a subadjacent Hopf algebra and can be used to construct solutions to the Yang–Baxter equation. Moreover, a relative Rota–Baxter operator on cocommutative Hopf algebras naturally induces a cocommutative post-Hopf algebra, and conversely, the identity map is a relative Rota–Baxter operator on the subadjacent Hopf algebra of a cocommutative post-Hopf algebra. Note that a relative Rota–Baxter operator is defined on a cocommutative Hopf algebra; how to define it on a general Hopf algebra is still an interesting question. On the other hand, since the universal enveloping algebra of a post-Lie algebra is a post-Hopf algebra, it is natural to expect further applications of post-Hopf algebras in Magnus expansions and related areas.

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