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Representation of commutators on Schatten p-classes

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Abstract. Let $\mathcal{C}_p$ be the Schatten p-class of ℓ_2 for $1 < p < \infty$, and let $\mathcal{T}_p$ be the closed subspace of $\mathcal{C}_p$ consisting of all lower triangular matrices. In this paper, we show that a bounded linear operator T on $\mathcal{C}_p$ is a commutator if and only if T is not of the form $\lambda I + K$, where $0 \neq \lambda \in \mathbb{C}$ and K is a $\mathcal{C}_p$-strictly singular operator. It is done by showing that a bounded linear operator T on $\mathcal{T}_p$ is not a commutator if and only if T has the same form.

1. Introduction

When studying a general algebra, we are sure to meet a mathematical object — commutators, i.e., elements of the form $AB - BA$. Functional analysts are more willing to focus it on a Banach algebra. The following well-known obstruction was proven by Wintner [17] in 1947.

Theorem 1.1 (Wintner). *The identity in a unital Banach algebra is not a commutator.*

This immediately implies that no element of the form $\lambda I + K$ is a commutator in a Banach algebra $\mathcal{A}$, where K belongs to a norm closed (proper) ideal $\mathcal{I}$ of $\mathcal{A}$ and $\lambda \neq 0$ is a scalar. Usually, it is difficult to check whether an element of a general Banach algebra is a commutator.

If we consider the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on a Banach space X, and provided the space X has some "nice" property, then it allows one to tackle the problem successfully. Indeed, for example, if X is finite-dimensional, then the answer is classical and easily stated: a sufficient and necessary condition for $T \in \mathcal{L}(X)$ to be a commutator is that the trace of T is zero. The first profound result was given by Brown and Pearcy [8] in 1965 for $X = \ell_2$. They proved that an operator $T \in \mathcal{L}(\ell_2)$ is not a commutator if and only if T can be represented as $\lambda I + K$ for some compact operator K with $\lambda(\neq 0) \in \mathbb{C}$. In 1972–1973, Apostol [2,3] proved the same representation for $X = \ell_p$ $(1 < p < \infty)$ and c_0 and also presented some partial results for ℓ_1, ℓ_∞, and $C([0,1])$.

Thirty-five years later after Apostol's results, this topic was resuscitated by Dosev [12]. He developed Apostol's technique [2] and obtained complete classification of commutators on ℓ_1, which states that the representation of a commutator on ℓ_1 is the same as that

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of a commutator on ℓ_2. He applied the generalized technique to obtain partial results for commutators on spaces X of the form

$$X \approx \Big(\bigoplus_{i=1}^{\infty} X \Big)_p$$

if either $1 \le p < \infty$ or $p = 0$. In particular, he gave a characterization of commutators on $\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$. Dosev and Johnson [10] further showed that every noncommutator on $\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$ has the form $\lambda I + K$, where $\lambda \neq 0$ and K belongs to some proper ideal of $\mathcal{L}(\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n})$.

Wintner's theorem tells us that if one obtained complete representation of commutators in $\mathcal{L}(X)$ which is the same as mentioned above, that is, $T \in \mathcal{L}(X)$ is not a commutator if and only if $T = \lambda I + K$ with $\lambda \neq 0$, then there must be a largest nontrivial ideal of $\mathcal{L}(X)$ so that the operator K is in it. For example, the ideal $\mathcal{K}(X)$ of all compact operators on X is the largest nontrivial ideal in $\mathcal{L}(X)$ for $X = \ell_p$ $(1 \le p < \infty)$ or c_0. The situation for $X = \ell_\infty$ is different. The largest ideal in $\mathcal{L}(\ell_\infty)$ is the ideal of all strictly singular operators $\mathcal{S}(\ell_\infty)$ (incidentally, agrees with the ideal of all weakly compact operators), instead of $\mathcal{K}(\ell_\infty)$. Dosev and Johnson [10] proved that $T \in \mathcal{L}(\ell_\infty)$ is not commutator if and only if T has the form $\lambda I + S$ for some strictly singular operator S and $\lambda \neq 0$. It seemed that it is reasonable to consider a converse version of Wintner's theorem: for a Banach space X which satisfies that there is a largest nontrivial ideal $\mathcal{M}$ in $\mathcal{L}(X)$, does the following statement hold? $T \in \mathcal{L}(X)$ is not a commutator if and only if $T = \lambda I + K$ for some $K \in \mathcal{M}$ and $\lambda \neq 0$. Nevertheless, with the help of Tarbard's example [16], Dosev, Johnson, and Schechtman [11] found that the statement is not true. Thus, Dosev and Johnson [10] further proposed the following conjecture (which remains open now).

Conjecture (Dosev–Johnson). Let X be a Banach space such that $X \approx (\sum X)_p$, $1 \le p \le \infty$, or $p = 0$. Assume that $\mathcal{L}(X)$ has a largest nontrivial ideal $\mathcal{M}$. Then, $T \in \mathcal{L}(X)$ is not a commutator if and only if $T = \lambda I + K$ for some $K \in \mathcal{M}$ and $\lambda \neq 0$.

In order to obtain a complete classification of commutators in $\mathcal{L}(X)$, checking whether $\mathcal{L}(X)$ has a largest nontrivial ideal turns into an important step. Recall that an operator $S \in \mathcal{L}(X)$ is said to factor through $T \in \mathcal{L}(X)$ provided there are $A, B \in \mathcal{L}(X)$ such that

$$S = ATB.$$

The following useful set is defined by Dosev and Johnson [10] in a clever way. For a Banach space X, let

$$\mathcal{M}_X = \big\{ T \in \mathcal{L}(X) : I_X \text{ does not factor through } T \big\}. \tag{1.1}$$

Firstly, note that the set $\mathcal{M}_X$ is closed under left and right multiplication by operators in $\mathcal{L}(X)$. Therefore, the question whether $\mathcal{M}_X$ is an ideal is equivalent to that whether $\mathcal{M}_X$ is closed under addition. Secondly, note that if $\mathcal{M}_X$ is an ideal, then it is automatically the largest ideal in $\mathcal{L}(X)$. Finally, note that $T \notin \mathcal{M}_X$ if and only if there exists a subspace $Y \subset X$ so that the restriction $T|_Y$ of T is an isomorphism from Y onto its image TY such that TY is complemented in X and such that $Y \approx X$.

The next useful notion was used in [9, 11].

Definition 1.2. An infinite-dimensional Banach space X is said to be complementably homogeneous if every subspace of X isomorphic to X must contain a smaller subspace isomorphic to X and is complemented in X.

For example, $L_p([0,1])$ $(1 \leq p < \infty)$ (see [15]), $L_1([0,1])$ (see [13]) are complementably homogeneous spaces. Dosev, Johnson, and Schechtman [11] showed that $T \in \mathcal{L}(L_p[0,1])$ $(1 \leq p < \infty)$ is not commutator if and only if T has the form $\lambda I + K$, where K is $L_p[0,1]$-strictly singular and $\lambda \neq 0$. Chen, Johnson, and Zheng [9] proved that the same conclusion for

$$X = \Big(\sum \ell_q\Big)_{\ell_p}$$

with $1 \leq q < \infty$ and $1 < p < \infty$, and Zheng [18] further showed that it is again true for

$$X = \Big(\sum \ell_q\Big)_{\ell_1}$$

with $1 \leq q < \infty$.

The following property immediately follows from Definition 1.2.

Proposition 1.3. *Let X be a complementably homogeneous Banach space. Then, $\mathcal{M}_X$ is equal to the set of all X-strictly singular operators on X; i.e., those operators $T \in \mathcal{L}(X)$ satisfy that $T|_{X_0}$ is not an isomorphism for every subspace X_0 of X which is isomorphic to X.*

In this paper, we focus on the Schatten p-class of ℓ_2 and show the following theorem.

Theorem A. *$T \in \mathcal{L}(\mathcal{C}_p)$ $(1 < p < \infty)$ is a commutator if and only if T is not of the form $\lambda I + K$ with $\lambda \neq 0$, where K is $\mathcal{C}_p$-strictly singular.*

It is done by showing a more general result below.

Theorem B. *Let $\mathcal{T}_p$ be the closed subspace of $\mathcal{C}_p$ consisting of all lower triangle matrices; i.e.,*

$$\mathcal{T}_p = \big\{x \in \mathcal{C}_p : x(i,j) = 0 \text{ for } j > i\big\}.$$

Then, $T \in \mathcal{L}(\mathcal{T}_p)$ $(1 \leq p < \infty)$ is not a commutator if and only if T is of the form $\lambda I + K$ for some $\mathcal{T}_p$-strictly singular K and some scalar $\lambda \neq 0$.

Since $\mathcal{T}_p \approx \mathcal{C}_p$ for all $1 < p < \infty$, Theorem A is an immediate consequence of Theorem B.

This paper is organized as follows. In the second section, we present some notions, basic properties, and known results which will be used in the sequel. In the third section, we will give some properties of operators in $\mathcal{M}_{\mathcal{T}_p}$ for $1 \le p < 2$. In the fourth section, we will show that, for any fixed $1 \le p < 2$, every $\mathcal{T}_p$-strictly singular T is a commutator, which is equivalent to $T \in \mathcal{M}_{\mathcal{T}_p}$. We should mention that results in the third and fourth sections were motivated by Arazy [4, 5]. In the fifth section (the last section), motivated by Chen, Johnson, and Zheng [9], and applying the main results established in the third and the fourth sections, we will show the main theorem of this paper, i.e., Theorem B mentioned above.

2. Preliminaries

In this section, we recall some notions, basic properties, and known results which will be used in the sequel.

By a subspace, we will always mean a closed subspace. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is said to be semi-normalized if there exist positive numbers a and b such that

$$a \le \|x_n\| \le b, \quad n \in \mathbb{N}.$$

We denote by $[x_n]_{n=1}^{\infty}$ the closure of span$\{x_n\}_{n=1}^{\infty}$ in X. A basic sequence $\{x_n\}_{n=1}^{\infty}$ is said to be λ-equivalent to another basic sequence $\{y_n\}_{n=1}^{\infty}$ provided there exist $1 \le \lambda < \infty$ so that, for every scalar sequence $\{t_n\}_{n=1}^{\infty}$ with finitely many $t_n \ne 0$,

$$\lambda^{-1}\Big\|\sum_n t_n y_n\Big\| \le \Big\|\sum_n t_n x_n\Big\| \le \lambda\Big\|\sum_n t_n y_n\Big\|.$$

We say that a subspace Y of X is λ-complemented in X if there exists a projection P from X onto Y with $\|P\| \le \lambda$. For two Banach spaces X and Y, their Mazur's distance is defined by

$$d(X, Y) = \inf\big\{\|T\| \cdot \|T^{-1}\| : T \text{ is an isomorphism from } X \text{ onto } Y\big\}.$$

We use $X \approx Y$ to denote that X is linearly isomorphic to Y, and $X \simeq Y$ means that X is linearly isometric to Y.

Definition 2.1. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of nontrivial closed subspaces of a Banach space X, and let $K > 0$ be a constant. Then, $\{X_k\}_{k=1}^{\infty}$ is said to be a Schauder decomposition of the subspace

$$\Big[\bigcup_k X_k\Big]$$

with the decomposition constant K provided that

$$\Big\|\sum_{k=1}^{m} x_k\Big\| \le K\Big\|\sum_{k=1}^{n} x_k\Big\|$$

for every sequence $\{x_k\}_{k=1}^{\infty}$ with $x_k \in X_k$ and for all integers $m, n \in \mathbb{N}$ with $m \le n$.

The next theorem is classical. See, for instance, [1, Theorem 1.3.9].

Theorem 2.2 (Principle of small perturbations). *Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence with a basis constant K in a Banach space X. If $\{y_n\}_{n=1}^{\infty}$ is a sequence in X such that*

$$2K \sum_{n=1}^{\infty} \frac{\|x_n - y_n\|}{\|x_n\|} = \theta < 1,$$

then there exists $T \in \mathcal{L}(X)$ such that

$$T x_n = y_n, \quad n = 1, 2, \ldots,$$

and

$$\|T - \mathrm{id}_X\| \leq \theta.$$

Theorem 2.3. *Let X and Y be Banach spaces, and let $\{X_k\}_{k=1}^{\infty}$ be a Schauder decomposition of X with the decomposition constant K. Assume that $T \in \mathcal{L}(X, Y)$, $G_k \in \mathcal{L}(TX_k, Y)$, $k = 1, 2, \ldots$, such that*

$$2K \sum_{k=1}^{\infty} \|T|_{X_k}\| \cdot \|\mathrm{id}_{TX_k} - G_k\| \leq \varepsilon$$

for some $\varepsilon > 0$. Then, there exists $T_0 \in \mathcal{L}(X, Y)$ so that

$$T_0|_{X_k} = G_k T|_{X_k} \quad \text{and} \quad \|T_0 - T\| \leq \varepsilon.$$

Proof. For each $k \in \mathbb{N}$, let P_k be the natural projection from $X = \sum_{k=1}^{\infty} X_k$ onto X_k. Then

$$T_0 \equiv T + \sum_{k=1}^{\infty} (G_k - \mathrm{id}_{TX_k}) T P_k = \sum_{k=1}^{\infty} G_k T P_k$$

is strongly convergent. Clearly, T_0 has the property we desired. ■

We often apply the particular case of Theorem 2.3 that $X \subset Y$ and $T = \iota_{X \hookrightarrow Y}$ (the natural embedding).

Given $1 \leq p < \infty$, the Schatten p-class $\mathcal{C}_p$ of operators on the Hilbert space ℓ_2 is defined as follows.

Let $\mathcal{C}_p$ be the Banach space of all compact operators x on ℓ_2 so that

$$\|x\|_p = (\mathrm{trace}(x^* x)^{p/2})^{1/p} < \infty.$$

We use $\mathcal{C}_\infty$ to denote the Banach space of all compact operators on ℓ_2 with the usual operator-norm of the space $\mathcal{L}(\ell_2)$.

Now, we introduce some notations which will be used in the sequel. For two orthonormal bases $\{e_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$, we represent every $x \in \mathcal{L}(\ell_2)$ as a matrix:

$$x = (x(i, j))_{i,j=1}^{\infty}, \quad x(i, j) = (x f_j | e_i).$$

Let

$$e_{i,j} = (\cdot \,|\, f_j) e_i, \quad 1 \le i, j < \infty.$$

Note that

$$e_{i,j}(k,l) = \delta_{i,k} \cdot \delta_{j,l}.$$

Clearly, $\{e_{i,j}\}_{i,j=1}^{\infty}$ forms a Schauder basis of $\mathcal{C}_p$ for every such p, if we arrange it as follows:

$$\begin{aligned} &e_{1,1},\ e_{2.1},\ e_{2,2},\ e_{1,2},\ e_{3,1},\ e_{3,2},\ e_{3,3},\ e_{2,3},\ e_{1,3}, \dots, \\ &e_{n,1},\ e_{n,2}, \dots, e_{n,n},\ e_{n-1,n},\ e_{n-2,n}, \dots, e_{1,n}, \dots . \end{aligned}$$

For each $n \in \mathbb{N}$, we define two projections P_n and E_n on $\mathcal{C}_p$ as follows:

$$(P_n x)(i,j) = \begin{cases} x(i,j), & 1 \le i, j \le n, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

and

$$(E_n x)(i,j) = \begin{cases} x(i,j), & 1 \le \min\{i,j\} \le n, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

Let

$$P_{n,m} = P_m - P_n, \quad n < m \quad \text{and} \quad E^{(n)} = I - E_n, \quad n \in \mathbb{N}. \tag{2.3}$$

Clearly,

$$\|P_n\| = \|E^{(n)}\| = 1$$

for every n. We denote by $\mathcal{T}_p$ the subspace of $\mathcal{C}_p$ consisting of all lower triangular matrices:

$$\mathcal{T}_p = \big\{x \in \mathcal{C}_p : x(i,j) = 0, \infty > j > i \ge 1\big\}.$$

Note that the spaces $\mathcal{C}_p$ and $\mathcal{T}_p$ admit the following finite-dimensional Schauder decompositions:

$$\mathcal{C}_p = \sum_{n=1}^{\infty} (P_n - P_{n-1}) \mathcal{C}_p, \quad \mathcal{T}_p = \sum_{n=1}^{\infty} (P_n - P_{n-1}) \mathcal{T}_p, \quad \text{where } P_0 = 0.$$

The following property is classical. See, for instance, Gohberg and Krein's book [14, pp. 118–120].

Proposition 2.4. *For every $1 < p < \infty$, the triangular projection P_T defined by*

$$(P_T x)(i,j) = \begin{cases} x(i,j), & i \ge j, \\ 0, & j > i, \end{cases}$$

is a bounded projection from $\mathcal{C}_p$ onto $\mathcal{T}_p$.

The next property can be found in Arazy and Lindenstrauss's work [7, Proposition 1].

Proposition 2.5. *The space $\mathcal{C}_p$ is isomorphic to its subspace $\mathcal{T}_p$ if and only if $1 < p < \infty$.*

Another important projection, actually a whole class of projections, in $\mathcal{C}_p$ is defined as follows.

Definition 2.6. Let $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ be two sequences of mutually disjoint subsets of positive integers for $n \in \mathbb{N}$; i.e.,

$$A_j \cap A_k = \emptyset = B_j \cap B_k \quad \text{for all } 1 \le j \ne k \le n.$$

We define the projection $P_{(\{A_k\},\{B_k\})}$ by

$$(P_{(\{A_k\},\{B_k\})}x)(i,j) = \begin{cases} x(i,j), & \text{if } i \in A_k, j \in B_k, k = 1, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, for all $1 \le p \le \infty$ and every pair of such sequences $\{A_k\}$ and $\{B_k\}$, the projection $P_{(\{A_k\},\{B_k\})}$ is of norm one. It is easy to observe that

$$\|P_{(\{A_k\},\{B_k\})}x\|_p = \left(\sum_k \|x_k\|_p^p\right)^{1/p},$$

where

$$x_k(i,j) = x(i,j),$$

if $(i,j) \in A_k \times B_k$ and $= 0$, otherwise. If $p = \infty$, the sum in the right-hand side will be replaced by $\sup_k \|x_k\|_p$.

We will also apply the following theorems, which are due to Arazy [4] and Arazy and Lindenstrauss [7].

Theorem 2.7 ([4, Proposition 2.2]). *Assume that $X \subset \mathcal{C}_p$ $(1 \le p < 2)$ and that X is isomorphic to a Hilbert space. Then, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that*

$$\|E^{(n)}|_X\| \le \varepsilon.$$

Theorem 2.8 ([4, Lemma 2.4]). *Let $\{x_k\}_{k=1}^\infty$ be a normalized sequence in $\mathcal{C}_p$ $(1 \le p < 2)$ which is equivalent to the unit vector basis of ℓ_2. Then,*

(i) *for every $0 < \varepsilon < 1$, there exists a subsequence $\{x_{k_j}\}_{j=2}^\infty$ which is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_2 so that $[x_{k_j}]_{j=2}^\infty$ is $(1+\varepsilon)$-complemented in $\mathcal{C}_p$;*

(ii) *given any sequence $\{\alpha_i\}_{i=1}^\infty$ with $0 < \alpha_i < 1$, there exist a normalized sequence $\{u_i\}_{i=2}^\infty$ in $\mathcal{C}_p$ and three sequences $\{a_j\}_{j=1}^\infty$, $\{b_j\}_{j=1}^\infty$, and $\{c_i\}_{i=2}^\infty$ in $\mathcal{C}_p$ with*

$$u_i = \sum_{j=1}^{i-1}(e_{i,j} \otimes a_j + e_{j,i} \otimes b_j) + e_{i,i} \otimes c_i$$

so that

$$(\|a_j\|_p^p + \|b_j\|_p^p)^{1/p} \le \alpha_{j-1}, \quad \|c_j\|_p \le \alpha_{j-1} \quad \textit{for all } j \ge 2,$$
$$1 - \alpha_1 \le (\|a_1\|_p^p + \|b_1\|_p^p)^{1/p} \le 1,$$

and

$$\|u_i - x_{k_i}\|_p \le \alpha_i \quad \textit{for all } i \ge 2.$$

Theorem 2.9 ([7, Theorem 1]). *Let $1 \le p < \infty$, and let $\{x_n\}_{n=1}^{\infty}$ be a semi-normalized weakly null sequence in $\mathcal{C}_p$. Then, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which is equivalent to the unit vector basis of ℓ_2 or ℓ_p.*

Theorem 2.10 ([4, Theorem 3.2]). *For $1 \le p < \infty$, let X be a subspace of $\mathcal{T}_p$ isomorphic to $\mathcal{T}_p$, and let $0 < \theta < 1$. Then, there exists a subspace Y of X so that*

$$d(Y, \mathcal{T}_p) \le 1 + \theta$$

so that Y is $(2 + \theta)$-complemented in $\mathcal{T}_p$.

The following result follows from Proposition 2.5 and Theorem 2.10.

Corollary 2.11. *For every $1 < p < \infty$, $\mathcal{C}_p$ is complementably homogeneous.*

Now, we recall some known results related to tensor products of operators. Let H be a Hilbert space, let $\mathcal{C}_p(H)$ be the Schatten p-class of operators on H, and let $\ell_2 \otimes \ell_2$ be the tensor product of ℓ_2 by itself. If $x, y \in \mathcal{B}(\ell_2)$, then there exists a unique element $x \otimes y \in \mathcal{B}(\ell_2 \otimes \ell_2)$ satisfying

$$(x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta$$

for every pair $\xi, \ \eta \in \ell_2$. If $x, y \in \mathcal{C}_p(\ell_2)$ $(1 \le p \le \infty)$, then

$$x \otimes y \in \mathcal{C}_p(\ell_2 \otimes \ell_2)$$

and

$$\|x \otimes y\|_p = \|x\|_p \cdot \|y\|_p.$$

Moreover, $\mathcal{C}_p(\ell_2 \otimes \ell_2)$ is the closure of the linear hull of all elements $x \otimes y$ with $x, y \in \mathcal{C}_p(\ell_2)$. We therefore denote

$$\mathcal{C}_p(\ell_2 \otimes \ell_2) \ \text{by} \ \mathcal{C}_p(\ell_2) \otimes \mathcal{C}_p(\ell_2)$$

or simply by

$$\mathcal{C}_p(\ell_2 \otimes \ell_2) = \mathcal{C}_p \otimes \mathcal{C}_p.$$

Since we can identify $\mathcal{C}_p$ with $\mathcal{C}_p \otimes \mathcal{C}_p$, we call this identification a “tensor product representation” of $\mathcal{C}_p$ with $\mathcal{C}_p \otimes \mathcal{C}_p$. Obviously, we can identify $\mathcal{C}_p$ with $\mathcal{C}_p \otimes \mathcal{C}_p \otimes \mathcal{C}_p$ and with $\mathcal{C}_p \otimes \mathcal{C}_p \otimes \mathcal{C}_p \otimes \mathcal{C}_p, \ldots$, in an analogous way.

Let $p, q \in \mathcal{L}(\ell_2)$ be two projection operators. Then, $p \otimes q$ is again a projection operator on $\mathcal{L}(\ell_2)$, and

$$x \in \mathcal{C}_p \otimes \mathcal{C}_p \mapsto (p \otimes q)x \in \mathcal{C}_p \otimes \mathcal{C}_p,$$
$$x \in \mathcal{C}_p \otimes \mathcal{C}_p \mapsto x(p \otimes q) \in \mathcal{C}_p \otimes \mathcal{C}_p$$

induce two contractive projections on $\mathcal{C}_p \otimes \mathcal{C}_p$. Therefore, we have the following property.

Corollary 2.12. *Let the projections P_n and $E^{(n)}$ be defined on $\mathcal{C}_p$ by (2.1) and (2.3). Then, $I \otimes P_n$ and $I \otimes E^{(n)}$ induce two contractive projections on $\mathcal{C}_p \otimes \mathcal{C}_p$.*

The following property is due to Arazy and Friedman [6].

Proposition 2.13 ([6, Theorem 2.2]). *Let $x \in \mathcal{C}_p$ ($1 \le p \le \infty$), $\|x\|_p = 1$, and let $x_{i,j} = e_{i,j} \otimes x$ for $1 \le i, j < \infty$. Then, $\{x_{i,j}\}_{i,j=1}^{\infty}$ is isometrically equivalent to the standard unit matrices $\{e_{i,j}\}_{i,j=1}^{\infty}$ in $\mathcal{C}_p$, and there is a contractive projection from $\mathcal{C}_p$ onto $[x_{i,j}]_{i,j=1}^{\infty}$.*

A triangular sequence is a double sequence of the form $\{x_{i,j}\}_{1\le j\le i<\infty}$. In short, we denote it also by $\{x_{i,j}\}_{j\le i}$ and call it simply a *triangle*. A *subtriangle* of $\{x_{i,j}\}_{j\le i}$ is a triangle of the form $\{x_{i_k,j_l}\}_{l\le k}$, where $\{i_k\}_{k=1}^{\infty}$ and $\{j_l\}_{l=1}^{\infty}$ are increasing sequences of positive integers with $i_k \ge j_k$ for every k.

In what follows, we will use the phrase "by passing a subtriangle" $\{x_{i_k,j_l}\}_{l\le k}$ (which has some nice properties) starting with a triangle $\{x_{i,j}\}_{j\le i}$. The general scheme of a such procedure is the following. Let $j_1 \in \mathbb{N}$, and let $\{x_{i_k^{(1)},j_1}\}_{k=1}^{\infty}$ be a be a "nice" subsequence of $\{x_{i,j_1}\}_{i=j_1}^{\infty}$. Assume that $j_1 < j_2 < \cdots < j_m$ have been chosen and that we have already defined increasing sequences

$$\{i_k^{(l)}\}_{k=l}^{\infty}, \quad 1 \le l \le m,$$

so that

$$\{x_{i_k^{(l)},j_l}\}_{k=l}^{\infty}$$

is a "nice" subsequence of

$$\{x_{i_k^{(l-1)},j_l}\}_{j_l\le i_k^{(l-1)}}.$$

Choose $j_{m+1} \in \mathbb{N}$, which is greater than j_m, such that there is a "nice" subsequence

$$\{x_{i_k^{(m+1)},j_{m+1}}\}_{k=m+1}^{\infty} \text{ of } \{x_{i_k^{(m)},j_m}\}_{j_m\le i_k^{m}}.$$

If we write $i_k = i_k^{(k)}$, then $\{x_{i_k,j_l}\}_{l\le k}$ is clearly a subtriangle of $\{x_{i,j}\}_{j\le i}$, and each column $\{x_{i_k,j_l}\}_{k=l}^{\infty}$ is "nice".

3. Properties of operators in $\mathcal{M}_{\mathcal{T}_p}$

In this section, we will show that $\mathcal{M}_{\mathcal{T}_p}$ is the largest nontrivial ideal in $\mathcal{L}(\mathcal{T}_p)$ for $1 \le p < \infty$ and present a "local" property of operators in $\mathcal{M}_{\mathcal{T}_p}$ (Theorem 3.6).

The following property is included in [4, p. 300, lines 19–20].

Theorem 3.1. *For any fixed* $1 \leq p \leq \infty$*, we have*

$$\mathcal{C}_p \approx \Big(\sum \mathcal{C}_p\Big)_p \quad \textit{and} \quad \mathcal{T}_p \approx \Big(\sum \mathcal{T}_p\Big)_p .$$

Recall (1.1) that, for a Banach space X,

$$\mathcal{M}_X = \big\{T \in \mathcal{L}(X) : I_X \text{ does not factor through } T\big\}.$$

The following property follows immediately from Proposition 1.3 and Theorem 2.10.

Proposition 3.2. *For every* $1 \leq p < \infty$*,* $\mathcal{M}_{\mathcal{T}_p}$ *is the subspace of all* $\mathcal{T}_p$*-strictly singular operators on* $\mathcal{T}_p$*.*

Theorem 3.3. *For* $1 \leq p < \infty$*,* $\mathcal{M}_{\mathcal{T}_p}$ *is the largest ideal of* $\mathcal{L}(\mathcal{T}_p)$*.*

Proof. Let $T \in \mathcal{L}(\mathcal{T}_p)$. If there is a subtriangle $\{e_{i_k,j_l}\}_{l \leq k}$ of $\{e_{i,j}\}_{j \leq i}$ such that

$$\big|(Te_{i_k,j_l})(i_k, j_l)\big| \geq 1/2$$

for all $l \leq k$, then, by [5, Corollary 2.2], we obtain $T \notin \mathcal{M}_{T_p}$. Otherwise, there exist a positive integer j_0 and an increasing sequence of positive integers $\{i_k\}_{k=1}^{\infty}$ with $i_1 \geq j_0$ such that, for all $j > j_0$, there are only finitely many

$$|(Te_{i_k,j})(i_k, j)| \ \text{ of } \ \big\{|(Te_{i_k,j})(i_k, j)|\big\}_{k=1}^{\infty}$$

which are greater than or equal to $1/2$. Thus, there is a subtriangle $\{e_{i_{k_v},j_\mu}\}_{\mu \leq v}$ of $\{e_{i,j}\}_{j \leq i}$ with $j_1 > j_0$ such that

$$\big|(I - Te_{i_{k_v},j_\mu})(i_{k_v}, j_\mu)\big| \geq 1/2.$$

It follows again from [5, Corollary 2.2] that

$$I - T \notin \mathcal{M}_{\mathcal{T}_p}.$$

By [10, Proposition 5.1], $\mathcal{M}_{\mathcal{T}_p}$ is the largest ideal of $\mathcal{L}(\mathcal{T}_p)$. ■

Lemma 3.4. *Let* $T \in \mathcal{L}(\mathcal{T}_p)$ $(1 \leq p < 2)$*. Then, one of the following conditions is satisfied:*

(a) *For any* $\varepsilon > 0$*, there is a subtriangle* $\{e_{i_k,j_l}\}_{l \leq k}$ *of* $\{e_{i,j}\}_{j \leq i}$ *such that* $T|_{[e_{i_k,j_l}]_{l \leq k}}$ *is compact and*

$$\|T|_{[e_{i_k,j_l}]_{l \leq k}}\| \leq \varepsilon.$$

(b) *For any sequence* $\{\varepsilon_l\}_{l=1}^{\infty}$ *of positive numbers, there is a subtriangle* $\{e_{i_k,j_l}\}_{l \leq k}$ *of* $\{e_{i,j}\}_{j \leq i}$ *so that, for any fixed* l*,* $\{Te_{i_k,j_l}\}_{k=l}^{\infty}$ *is equivalent to the unit vector basis of* ℓ_2 *and*

$$\|T|_{[e_{i_k,j_l}]_{k=l}^{\infty}}\| \leq \varepsilon_l.$$

(c) *There exist* $K > 0$ *and a subtriangle* $\{e_{i_k,j_l}\}_{l \leq k}$ *of* $\{e_{i,j}\}_{j \leq i}$ *such that, for any fixed* l*,* $\{Te_{i_k,j_l}\}_{k=l}^{\infty}$ *is* K*-equivalent to the unit vector basis of* ℓ_2*.*

Proof. (a) We assume that A is an infinite set of natural numbers satisfying that for the first number j_1 of A, there is a subsequence $\{e_{i_k^{(1)},j_1}\}_{k=1}^{\infty}$ of $\{e_{i,j_1}\}_{i=j_1}^{\infty}$ such that

$$\lim_{k\to\infty} \|Te_{i_k^{(1)},j_1}\|_p = 0.$$

Assume that $j_1 < j_2 < \cdots < j_m$ have been chosen from A and that we have already defined m increasing sequences $\{i_k^{(l)}\}_{k=l}^{\infty}$, $1 \le l \le m$ satisfying that for each such l, $\{i_k^{(l)}\}_{k=l}^{\infty}$ is a subsequence of $\{i_k^{(l-1)}\}_{k=l-1}^{\infty}$ with $i_l^{(l)} \ge j_l$ and that

$$\lim_{k\to\infty} \|Te_{i_k^{(l)},j_l}\|_p = 0.$$

Then, let j_{m+1} be the first number of $A \setminus \{j_1, \ldots, j_m\}$ which is greater than j_m, and let $\{i_k^{(m+1)}\}_{k=m+1}^{\infty}$ be a subsequence of $\{i_k^{(m)}\}_{k=m}^{\infty}$ with $i_{m+1}^{(m+1)} \ge j_{m+1}$ and with

$$\lim_{k\to\infty} \|Te_{i_k^{(m+1)},j_{m+1}}\|_p = 0.$$

We write $i_k = i_k^{(k)}$. Then, $\{e_{i_k,j_l}\}_{l\le k}$ is a subtriangle of $\{e_{i,j}\}_{j\le i}$. For any fixed sequence $\{c_{k,l}\}_{l\le k}$ of positive numbers, we can assume that

$$\|Te_{i_k,j_l}\|_p \le c_{k,l}.$$

For any fixed $\varepsilon > 0$, since $\{c_{k,l}\}_{l\le k}$ is arbitrary, we can claim that $T|_{[e_{i_k,j_l}]_{l\le k}}$ is compact and that $\|T|_{[e_{i_k,j_l}]_{l\le k}}\| \le \varepsilon$.

(b) If the process in (a) cannot be continued, then there exist a positive integer j_0 and an increasing sequence $\{i_s\}_{s=1}^{\infty}$ of positive integers so that, for every $j \ge j_0$, $\{Te_{i_s,j}\}_{s=1}^{\infty}$ does not admit a null subsequence. Therefore, there is a subtriangle $\{e_{i_k,j_l}\}_{l\le k}$ of $\{e_{i,j}\}_{j\le i}$ such that, for each fixed l,

$$\liminf_{k\to\infty} \|Te_{i_k,j_l}\|_p > 0.$$

By passing a subtriangle of $\{e_{i_k,j_l}\}_{l\le k}$, without loss of generality, we can assume that $\delta_l \le \|Te_{i_k,j_l}\|_p \le 2\delta_l$, where $\{\delta_l\}_{l=1}^{\infty}$ is a sequence of positive integers. Since $\{e_{i_k,j_l}\}_{k=l}^{\infty}$ is weakly null, by Theorem 2.9 and by passing a subtriangle of $\{e_{i_k,j_l}\}_{l\le k}$ (if necessary), we can further claim that $\{Te_{i_k,j_l}\}_{k=l}^{\infty}$ is equivalent to the unit vector basis of ℓ_2 or ℓ_p. If it is M_l-equivalent to the unit vector basis of ℓ_p, then for every sequence $\{t_k\}_{k=l}^{\infty}$ of scalars with finitely many $t_k \ne 0$,

$$\begin{aligned}
\delta_l {M_l}^{-1}\left(\sum_{k=l}^{\infty} |t_k|^p\right)^{1/p} &\le \left\| \sum_{k=l}^{\infty} t_k Te_{i_k,j_l} \right\|_p \\
&\le \|T\| \left\| \sum_{k=l}^{\infty} t_k e_{i_k,j_l} \right\|_p \\
&\le \|T\| \left(\sum_{k=l}^{\infty} |t_k|^2 \right)^{1/2}.
\end{aligned}$$

This is a contradiction to the fact that $1 \le p < 2$ and that $\{t_k\}_{k=l}^{\infty}$ is arbitrary. Hence, $\{Te_{i_k,j_l}\}_{k=l}^{\infty}$ is equivalent to the unit vector basis of ℓ_2. By Theorem 2.8, we can claim that

$$\{Te_{i_k,j_l}/\|Te_{i_k,j_l}\|_p\}_{k=l}^{\infty}$$

is 2-equivalent to the unit vector basis of ℓ_2.

Now, we assume that

$$\lim_{l\to\infty} \delta_l = 0.$$

For any sequence $\{\varepsilon_l\}_{l=1}^{\infty}$ of positive numbers, up to passing a subtriangle of $\{e_{i_k,j_l}\}_{l\le k}$, without loss of generality, we can assume that $\delta_l \le \varepsilon_l/4$. For every sequence $\{t_k\}_{k=l}^{\infty}$ of scalars with finitely many $t_k \neq 0$,

$$\begin{aligned}
\left\| T\left(\sum_{k=l}^{\infty} t_k e_{i_k,j_l}\right)\right\|_p &= \left\| \sum_{k=l}^{\infty} t_k \|Te_{i_k,j_l}\|_p \cdot Te_{i_k,j_l}/\|Te_{i_k,j_l}\|_p \right\|_p \\
&\le 2\left(\sum_{k=l}^{\infty} (|t_k\|Te_{i_k,j_l}\|_p)^2\right)^{1/2} \\
&\le 4\delta_l\left(\sum_{k=l}^{\infty} |t_k|^2\right)^{1/2} \\
&\le \varepsilon_l \left\| \sum_{k=l}^{\infty} t_k e_{i_k,j_l}\right\|_p .
\end{aligned}$$

Hence, $\|T|_{[e_{i_k,j_l}]_{k=l}^{\infty}}\| \le \varepsilon_l$.

(c) Suppose that $\limsup_{l\to\infty} \delta_l > 0$ in (b). Again by passing a subtriangle, we can assume that

$$M \le \|Te_{i_k,j_l}\|_p \le 2M \quad \text{for some } M > 0.$$

By an argument similar to that of (b) and passing a subtriangle of $\{e_{i_k,j_l}\}_{l\le k}$, we can assume that there is a positive number K such that, for each l, $\{Te_{i_k,j_l}\}_{k=l}^{\infty}$ is K-equivalent to the unit vector basis of ℓ_2. ∎

The following is the proof line of lemma; it was deeply motivated by the procedure of the proof of Arazy [4, Theorem 4.6]

Lemma 3.5. *For $1 \le p < 2$, let $\{x_{i,j}\}_{j\le i}$ be a triangle with entries in $\mathcal{T}_p$. Assume that*

(i) *$K, M > 0$ such that, for any fixed j, $\{x_{i,j}\}_{i=j}^{\infty}$ is K-equivalent to the unit vector basis of ℓ_2, and*

(ii) *for every finite sequence $\{t_j\}_{j=1}^{i}$ of scalars,*

$$\left\|\sum_{j=1}^{i} t_j x_{i,j}\right\|_p \le M\left(\sum_{j=1}^{i} |t_j|^2\right)^{1/2}.$$

Then, for all $0 < \theta < 1$, *there exist*

(a) *a tensor product representation of* $\mathcal{C}_p \otimes \mathcal{C}_p$ *as a subspace of* $\mathcal{C}_p$,

(b) *a subtriangle* $\{x_{i_k,j_l}\}_{l\le k}$ *of* $\{x_{i,j}\}_{j\le i}$,

(c) *an increasing sequence* $\{s_l\}_{l=1}^{\infty}$ *of positive integers,*

(d) *a sequence* $\{z_l\}_{l=1}^{\infty} \subset \mathcal{C}_p$ *of nonzero elements of the form* $z_l = P_{s_l,s_{l+1}} z_l$, *which is* $(K+1)$*-equivalent to the unit vector basis of* ℓ_2,

(e) *a linearly isomorphic embedding* V *from* $\overline{\operatorname{span}}\{x_{i_k,j_l}\}_{l\le k}$ *into* $\mathcal{C}_p$ *so that, for every* $x \in \overline{\operatorname{span}}\{x_{i_k,j_l}\}_{l\le k}$,

$$\|Vx - x\|_p \le \theta\|x\|_p \quad \text{and} \quad Vx_{i_k,j_l} = e_{k,1}\otimes z_l.$$

Proof. Since for each fixed j, $\{x_{i,j}\}_{i=j}^{\infty}$ is K-equivalent to the unit vector basis of ℓ_2, for any sequence of positive numbers $\{\varepsilon_j\}_{j=1}^{\infty}$ satisfying the fact that $\sum_{j=1}^{\infty}\varepsilon_j$ is small enough, applying Theorem 2.7, we can construct an increasing sequence of positive integers $\{n_j\}_{j=1}^{\infty}$ so that

$$\|E^{(n_j)}|_{[x_{i,j}]_{i=j}^{\infty}}\| < \varepsilon_j,$$

where $E^{(n)}$ is defined by (2.3). On the other hand, since $E_{n_j}x_{i,j} \xrightarrow{w} 0$ $(i\to\infty)$, for any sequence $\{c_{k,j}\}_{j\le k}$ of positive numbers so that $\sum_{j\le k} c_{k,j}$ is small enough, by a routine perturbation argument (say, by using Theorems 2.2 and 2.3), there are two increasing sequences $\{m_k\}_{k=1}^{\infty}$ and $\{i_k\}_{k=1}^{\infty}$ of positive integers with $m_k > n_k$ satisfying the fact that, for every $j \le k$,

$$\left\|P_{m_k,m_{k+1}}E_{n_j}x_{i_k,j} - E_{n_j}x_{i_k,j}\right\|_p < c_{k,j},$$

where E_n is defined by (2.2). For each $j\in\mathbb{N}$, we denote by

$$X_j = [x_{i_k,j}]_{k=j}^{\infty}.$$

Without loss of generality, we can assume that for every j there exists $T_j \in \mathcal{L}(X_j,\mathcal{T}_p)$ with

$$\|T_j - \mathrm{id}_{X_j}\| < \varepsilon_j \tag{3.1}$$

and with

$$T_j x_{i_k,j} = P_{m_k,m_{k+1}}E_{n_j}x_{i_k,j}.$$

Write

$$y_{k,j} = P_{m_k,m_{k+1}}E_{n_j}x_{i_k,j} \quad \text{and} \quad \mu_k = \sum_{j=1}^{k} n_j.$$

We can claim that

$$y_{k,j} = P_{m_k,m_k+\mu_j}y_{k,j}, \quad k,j\in\mathbb{N}. \tag{3.2}$$

Otherwise, we can substitute a new orthonormal basis for the original basis of the range space of the operators so that the representation above holds. Indeed, note that, for every

$j \leq k$, $E_{n_j} y_{k,j} = y_{k,j}$. Then, $\operatorname{rank}(y_{k,j}) \leq n_j$. For $x \in \mathcal{L}(\ell_2)$, we denote by $R(x)$ the range of x. Let

$$[f_i^{(k)}]_{i=1}^{\mu_1'}, \quad k = 1, 2, 3, \ldots, \; 1 \leq \mu_1' \leq \mu_1,$$

be orthonormal sequences so that

$$R(y_{k,1}) \subseteq [f_i^{(k)}]_{i=1}^{\mu_1'}$$

and so that

$$\bigcup_{k=1}^{\infty} \{f_i^{(k)}\}_{i=1}^{\mu_1'}$$

is again an orthonormal sequence. Since $\operatorname{rank}(y_{k,2}) \leq n_2$, $k = 2, 3, 4, \ldots$, there exist orthonormal sequences

$$\{f_i^{(k)}\}_{i=\mu_1'+1}^{\mu_2'}, \quad \mu_1' < \mu_2' \leq \mu_2,$$

so that

$$\{f_i^{(1)}\}_{i=1}^{\mu_1'} \bigcup \Big(\bigcup_{k=2}^{\infty} \{f_i^{(k)}\}_{i=1}^{\mu_2'} \Big)$$

is also an orthonormal sequence. We have finished the proof of the claim (3.2) by induction. Therefore,

$$y_{k,j} = P_{m_k, m_k+\mu_j} y_{k,j} = P_{m_k, m_{k+1}} E_{n_j} y_{i_k,j}, \quad k, j \in \mathbb{N}.$$

Consequently, for every $j \leq k$, we can choose an appropriate tensor product representation $\mathcal{C}_p$ as $\mathcal{C}_p^{(1)} \otimes \mathcal{C}_p^{(2)}$ so that

$$y_{k,j} = e_{k,1} \otimes a_{k,j}, \quad a_{k,j} \in \mathcal{C}_p^{\mu_j, n_j} \subseteq \mathcal{C}_p^{(2)},$$

where $\mathcal{C}_p^{(i)}$ are copies of $\mathcal{C}_p$ for $i = 1, 2$, and $\mathcal{C}_p^{m,n}$ denotes the space of all $m \times n$ scalar matrices with the norm induced from $\mathcal{C}_p$.

For a sequence $\{\alpha_{k,j}\}_{j \leq k}$ of positive numbers satisfying the fact that $\sum_{j \leq k} \alpha_{k,j}$ is small enough, by an argument of passing a subtriangle, we can assume that there exist $a_j \in \mathcal{C}_p^{(2)}$ for $1 \leq j \leq k$ such that

$$\|a_{k,j} - a_j\|_p = \|e_{k,1} \otimes a_{k,j} - e_{k,1} \otimes a_j\|_p < \alpha_{k,j}. \tag{3.3}$$

We can further assume that $\{a_j\}_{j=1}^{\infty}$ is semi-normalized, and for every finite sequence $\{t_k\}_{j=1}^{k}$ of scalars,

$$\Big\| \sum_{j=1}^{k} t_j a_j \Big\|_p = \Big\| \sum_{j=1}^{k} t_j e_{i_k,1} \otimes a_j \Big\|_p \leq (M+1) \Big(\sum_{j=1}^{i} |t_j|^2 \Big)^{1/2}.$$

By an argument similar to the proof of Lemma 3.4 (b), there is an increasing sequence of positive integers $\{j_l\}_{l=1}^{\infty}$ so that $\{a_{j_l}\}_{l=1}^{\infty}$ is $(K+1/2)$-equivalent to the unit vector basis of ℓ_2. We can also assume that $i_k \geq j_k$ for all $k \in \mathbb{N}$. For a sequence $\{\delta_l\}_{l=1}^{\infty}$ of positive numbers satisfying the fact that $\sum_{l=1}^{\infty} \delta_l$ is small enough, again by an argument of passing a subsequence, we may assume that there is an increasing sequence $\{s_l\}_{l=1}^{\infty}$ of positive integers such that

$$\|a_{j_l} - P_{s_l,s_{l+1}} a_{j_l}\|_p < \delta_l,$$

and $\{P_{s_l,s_{l+1}} a_{j_l}\}_{l=1}^{\infty}$ is $(K+1)$-equivalent to the unit vector basis of ℓ_2.

For each fixed l and for every sequence $\{t_k\}_{k=1}^{\infty}$ of scalars with finitely many $t_k \neq 0$, we have

$$\left\| \sum_{k=l}^{\infty} t_k e_{k,1} \otimes a_{j_l} - \sum_{k=l}^{\infty} t_k e_{k,1} \otimes P_{s_l,s_{l+1}} a_{j_l} \right\|_p < \delta_l \left(\sum_{l=1}^{\infty} |t_j|^2 \right)^{1/2}. \tag{3.4}$$

Denote by

$$Y_l = [e_{k,1} \otimes P_{s_l,s_{l+1}} a_{j_l}]_{k=l}^{\infty}.$$

Then, due to Definition 2.1 and Corollary 2.12, $\{Y_l\}_{l=1}^{\infty}$ is a 1-Schauder decomposition of

$$\overline{\operatorname{span}}\left\{ \bigcup_l Y_l \right\}.$$

By (3.1), (3.3), and (3.4), we may assume that, for every l, there exists $S_l \in \mathcal{L}(X_{j_l}, \mathcal{C}_p)$ satisfying

$$\|S_l - \mathrm{id}_{X_{j_l}}\| < \varepsilon_{j_l} \quad \text{and} \quad S_l x_{i_k,j_l} = e_{k,1} \otimes P_{s_l,s_{l+1}} a_{j_l}.$$

Put $z_l = P_{s_l,s_{l+1}} a_{j_l}$. Then, by Theorem 2.3, for every $0 < \theta < 1$, there exists a linear isomorphic embedding V from $\overline{\operatorname{span}}\{x_{i_k,j_l}\}_{l \leq k}$ into $\mathcal{C}_p$ so that, for every $x \in \overline{\operatorname{span}}\{x_{i_k,j_l}\}_{l \leq k}$,

$$\|Vx - x\|_p \leq \theta \|x\|_p \quad \text{and} \quad V x_{i_k,j_l} = e_{k,1} \otimes z_l. \qquad \blacksquare$$

Now, we are ready to state and prove the main theorem of this section as follows.

Theorem 3.6. *Let $1 \leq p < 2$. Then, $T \in \mathcal{L}(\mathcal{T}_p)$ is $\mathcal{T}_p$-strictly singular, or equivalently, $T \in \mathcal{M}_{\mathcal{T}_p}$, if and only if, for every $X \subset \mathcal{T}_p$ with $X \approx \mathcal{T}_p$ and for all $\varepsilon > 0$, there exists a subspace $Y \subseteq X$ such that $Y \approx \mathcal{T}_p$ and $\|T|_Y\| < \varepsilon$.*

Proof. Sufficiency. It follows from the definition of $\mathcal{T}_p$-strictly singular operators.

Necessity. Suppose that $T \in \mathcal{M}_{\mathcal{T}_p}$ and that $X \subset \mathcal{T}_p$ with $X \approx \mathcal{T}_p$. Let $S : \mathcal{T}_p \to X$ be a linear isomorphism, and $\varepsilon > 0$. We will apply Lemmas 3.4 and 3.5 to the proof.

Case 1. If TS satisfies (a) of Lemma 3.4, then there is a subtriangle

$$\{e_{i_k,j_l}\}_{l \leq k} \text{ of } \{e_{i,j}\}_{j \leq i}$$

so that

$$\|TS|_{[e_{i_k,j_l}]_{l \leq k}}\| \leq \varepsilon / \|S^{-1}\|.$$

Let

$$Y = S([e_{i_k,j_l}]_{l\leq k}) \subseteq X.$$

Then

$$\|T|_Y\| = \|TS|_{[e_{i_k,j_l}]_{l\leq k}} S^{-1}|_Y\| \leq \frac{\varepsilon}{\|S^{-1}\|} \cdot \|S^{-1}\| = \varepsilon.$$

Case 2. If TS satisfies (b) of Lemma 3.4, then for every sequence of positive numbers $\{\varepsilon_l\}_{l=1}^{\infty}$ so that $\sum_{l=1}^{\infty} \varepsilon_l < \infty$, there is a subtriangle $\{e_{i_k,j_l}\}_{l\leq k}$ of $\{e_{i,j}\}_{j\leq i}$, satisfying, for each fixed l, $\|TS|_{[e_{i_k,j_l}]_{k=l}^{\infty}}\| \leq \varepsilon_l$. Choose an integer number l_0 so that

$$\sum_{l=l_0}^{\infty} \varepsilon_l < \varepsilon/\|S^{-1}\|.$$

Since $\sum_{l=1}^{\infty}[e_{i_k,j_l}]_{k=l}^{\infty}$ is a Schauder decomposition of itself,

$$\|TS|_{[e_{i_k,j_l}]_{l_0\leq l\leq k}}\| \leq \varepsilon/\|S^{-1}\|.$$

Put $Y = S([e_{i_k,j_l}]_{l_0\leq l\leq k}) \subseteq X$. Then, $\|T|_Y\| < \varepsilon$.

Case 3. If TS satisfies (c) of Lemma 3.4, then there exist a positive constant K and a subtriangle $\{e_{i_k,j_l}\}_{l\leq k}$ of $\{e_{i,j}\}_{j\leq i}$ such that, for each fixed l, $\{TSe_{i_k,j_l}\}_{k=l}^{\infty}$ is K-equivalent to the unit vector basis of ℓ_2. Meanwhile, for every finite sequence$\{t_l\}_{l=1}^{k}$ of scalars,

$$\Big\|\sum_{l=1}^{k} t_l TSe_{i_k,j_l}\Big\|_p \leq \|TS\| \Big\|\sum_{l=1}^{k} t_l e_{i_k,i_l}\Big\|_p = \|TS\|\Big(\sum_{l=1}^{k}|t_l|^2\Big)^{1/2}.$$

Applying Lemma 3.5 and by an argument of passing a subtriangle of $\{TSe_{i_k,j_l}\}_{l\leq k}$, we can assume that there exist a tensor product representation of $\mathcal{C}_p \otimes \mathcal{C}_p$ as $\mathcal{C}_p$, an increasing sequence $\{s_l\}_{l=1}^{\infty}$ of positive integers, a sequence $\{z_l\}_{l=1}^{\infty}$ of nonzero elements of $\mathcal{C}_p$ satisfying

$$z_l = P_{s_l,s_{l+1}} z_l,$$

which is $(K+1)$-equivalent to the unit vector basis of ℓ_2, and a linear isomorphic embedding V from $\overline{\text{span}}\{TSe_{i_k,j_l}\}_{l\leq k}$ into $\mathcal{C}_p$ so that

$$\|V\|,\ \|V^{-1}\| < 2 \quad \text{and} \quad VTSe_{i_k,j_l} = e_{k,1} \otimes z_l.$$

Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of positive numbers with $0 < \alpha_i < 1$ so that $\sum_{i=1}^{\infty} \alpha_i$ is small enough. By Theorem 2.8, there exist a sequence $\{u_l\}_{l=2}^{\infty} \subset \mathcal{C}_p$ with

$$\|u_l\|_p \equiv M \in [(K+1)^{-1}, K+1]$$

and three sequences $\{a_i\}_{i=1}^{\infty}$, $\{b_j\}_{i=1}^{\infty}$, and $\{c_k\}_{l=1}^{\infty}$ in $\mathcal{C}_p$ satisfying

$$u_k = \sum_{i=1}^{k-1}(e_{k,i} \otimes a_i + e_{i,k} \otimes b_i) + e_{k,k} \otimes c_k$$

so that for $i \geq 2$

$$(\|a_i\|_p^p + \|b_i\|_p^p)^{1/p} \leq \alpha_{i-1}, \quad \|c_i\|_p \leq \alpha_{i-1},$$
$$M - \alpha_1 \leq (\|a_1\|_p^p + \|b_1\|_p^p)^{1/p} \leq M,$$

and there is a subsequence of $\{z_l\}_{l=1}^{\infty}$ (again denoted by $\{z_l\}_{l=2}^{\infty}$) satisfying

$$\|z_l - u_l\|_p \leq \alpha_l, \quad l = 2, 3, \ldots .$$

For each fixed l, and for each sequence $\{t_k\}_{k=1}^{\infty}$ of scalars with finitely many $t_k \neq 0$,

$$\left\| \sum_{k=l}^{\infty} t_k e_{k,1} \otimes z_l - \sum_{k=l}^{\infty} t_k e_{k,1} \otimes u_l \right\|_p < \alpha_l \left(\sum_{l=1}^{\infty} |t_j|^2 \right)^{1/2}.$$

Denote by

$$Z_l = [e_{k,1} \otimes z_l]_{k=l}^{\infty}.$$

Then, $\{Z_l\}_{l=1}^{\infty}$ is a 1-Schauder decomposition of $\overline{\operatorname{span}}\{\bigcup_l Z_l\}$. By Theorem 2.3, there exists a linear isomorphic embedding U from $\overline{\operatorname{span}}\{\bigcup_l Z_l\}$ into $\mathcal{C}_p$ so that

$$\|U\|, \ \|U^{-1}\| < 2 \quad \text{and} \quad U e_{k,1} \otimes z_l = e_{k,1} \otimes u_l.$$

If there is an integer i' such that $b_{i'} \neq 0$, then we can choose a projection operator $q \in \mathcal{L}(\ell_2)$ so that $q u_l = e_{i',l} \otimes b_{i'}$ for $l \geq i' + 1$. By Proposition 2.13, for every sequence $\{t_{k,l}\}_{i'+1 \leq l \leq k}$ of scalars with finitely many $t_{k,l} \neq 0$,

$$\begin{aligned}
\|b_{i'}\|_p \left\| \sum_{i'+1 \leq l \leq k} t_{k,l} e_{i_k, j_l} \right\|_p &= \left\| \sum_{i'+1 \leq l \leq k} t_{k,l} e_{k,1} \otimes e_{i',l} \otimes b_{i'} \right\|_p \\
&= \left\| (I \otimes q) \cdot UVTS \left(\sum_{i'+1 \leq l \leq k} t_{k,l} e_{i_k, j_l} \right) \right\|_p \\
&\leq \left\| UVTS \left(\sum_{i'+1 \leq l \leq k} t_{k,l} e_{i_k, j_l} \right) \right\|_p \\
&\leq \|UVTS\| \left\| \sum_{i'+1 \leq l \leq k} t_{k,l} e_{i_k, j_l} \right\|_p .
\end{aligned}$$

This says that T is not $\mathcal{T}_p$-strictly singular. Thus, $b_i = 0$ for all i.

Next, we will use average skill to "kill this theorem". (Such an idea can be seen in [4].) Given $m \in \mathbb{N}$ and for $1 \leq \mu \leq v < \infty$, we write

$$h_{v,\mu} = \sum_{j=1}^{m} e_{vm+j,1} \otimes u_{\mu m + j} / m^{1/p}.$$

For every sequence $\{t_{v,\mu}\}_{\mu\le v}$ of scalars with finitely many $t_{v,\mu}\neq 0$, we obtain

$$\begin{aligned}
\Big\|\sum_{\mu\le v} t_{v,\mu}h_{v,\mu}\Big\|_p &\le m^{-1/p}\bigg\{\Big\|\sum_{\mu\le v} t_{v,\mu}\sum_{j=1}^{m}\sum_{i=1}^{\mu m+j-1} e_{vm+j,1}\otimes e_{\mu m+j,i}\otimes a_i\Big\|_p \\
&\quad + \Big\|\sum_{\mu\le v} t_{v,\mu}\sum_{j=1}^{m} e_{vm+j,1}\otimes e_{\mu m+j,\mu m+j}\otimes c_{\mu m+j}\Big\|_p\bigg\} \\
&= m^{-1/p}\bigg\{\Big\|\sum_{i=1}^{\infty}\sum_{\substack{\mu\le v\\ i-\mu m+1\le j\le m}} t_{v,\mu}e_{vm+j,1}\otimes e_{\mu m+j,i}\otimes a_i\Big\|_p \\
&\quad + \Big\|\sum_{\mu\le v}\sum_{j=1}^{m} t_{v,\mu}e_{vm+j,1}\otimes e_{\mu m+j,\mu m+j}\otimes c_{\mu m+j}\Big\|_p\bigg\} \\
&\le m^{-1/p}\bigg\{\sum_{i=1}^{\infty}\|a_i\|_p\Big\|\sum_{\mu\le v}\sum_{j=1}^{m} t_{v,\mu}e_{vm+j,1}\otimes e_{\mu m+j,i}\Big\|_p \\
&\quad + \Big\|\sum_{j=1}^{m}\sum_{\mu=1}^{\infty}\sum_{v=\mu}^{\infty} t_{v,\mu}e_{vm+j,1}\otimes e_{\mu m+j,\mu m+j}\otimes c_{\mu m+j}\Big\|_p\bigg\} \\
&\le m^{-1/p}\bigg\{\sum_{i=1}^{\infty}\|a_i\|_p m^{1/2}\Big(\sum_{\mu\le v}|t_{v,\mu}|^2\Big)^{1/2} \\
&\quad + \sum_{j=1}^{m}\sum_{\mu=1}^{\infty}\|c_{\mu m+j}\|_p\Big(\sum_{v=\mu}^{\infty}|t_{v,\mu}|^2\Big)^{1/2}\bigg\} \\
&\le m^{-1/p}\bigg\{\sum_{i=1}^{\infty}\|a_i\|_p m^{1/2} + \sum_{j=m+1}^{\infty}\|c_j\|_p\bigg\}\Big(\sum_{\mu\le v}|t_{v,\mu}|^2\Big)^{1/2} \\
&\le m^{1/2-1/p}\bigg\{M + 2\sum_{i=1}^{\infty}\alpha_i\bigg\}\Big(\sum_{\mu\le v}|t_{v,\mu}|^2\Big)^{1/2}. \qquad (3.5)
\end{aligned}$$

Let

$$w_{v,\mu} = \sum_{j'=1}^{m} e_{i_{vm+j'},j_{\mu m+j'}}/m^{1/p}.$$

Then, again by Proposition 2.13, for all $m\in\mathbb{N}$ satisfying

$$m^{1/2-1/p}\bigg\{M + 2\sum_{i=1}^{\infty}\alpha_i\bigg\} < \varepsilon/(4\|S^{-1}\|), \qquad (3.6)$$

we obtain

$$\Big\|\sum_{\mu\le v} t_{v,\mu}w_{v,\mu}\Big\|_p = \Big\|\sum_{\mu\le v} t_{v,\mu}e_{v,\mu}\Big\|_p \ge \Big\|\sum_{\mu\le v} t_{v,\mu}e_{v,\mu}\Big\|_2 = \Big(\sum_{\mu\le v}|t_{v,\mu}|^2\Big)^{1/2}. \qquad (3.7)$$

Since

$$h_{v,\mu} = UVTSw_{v,\mu},$$

by (3.5), (3.6), and (3.7),

$$\|UVTS|_{[w_{v,\mu}]_{\mu\le v}}\| < \varepsilon/(4\|S^{-1}\|).$$

Let

$$Y = S([w_{v,\mu}]_{\mu\le v})(\subseteq X).$$

Then

$$\|T|_Y\| \le \|V^{-1}\|\,\|U^{-1}\|\,\|UVTS|_{[w_{v,\mu}]_{\mu\le v}}\|\,\|S^{-1}\| < \varepsilon. \qquad \blacksquare$$

Remark 3.7. In Theorem 3.6, we cannot claim that the restriction $T|_Y$ of T is compact. For example, let T satisfy $Te_{i,j} = e_{2^i(2j+1),1}$. Then, by [4, Proposition 2.2] and

$$\|TE_n x\| \le \left(\|TE_n x\|^2 + \|TE^{(n)}x\|^2\right)^{\frac{1}{2}} = \|Tx\|,$$

we get that $T|_Y$ is not compact for every subspace $Y \subset \mathcal{T}_p$ isomorphic to $\mathcal{T}_p$.

4. Every $T \in \mathcal{M}_{\mathcal{T}_p}$ is a commutator

In this section, we will show that every $T \in \mathcal{M}_{\mathcal{T}_p}$ is a commutator for all $1 \le p < 2$.

Definition 4.1. A sequence $\{X_i\}_{i=0}^{\infty}$ of closed subspaces of a Banach space X is said to be an ℓ_p-decomposition of X for $1 \le p < \infty$ or $p = 0$, provided the following three conditions are satisfied:

(1) $\{X_i\}_{i=0}^{\infty}$ is a Schauder decomposition of X.

(2) X_i $(i = 0, 1, 2, \ldots)$ are uniformly linear isomorphic to X.

(3) There is a positive constant K, such that for every convergent series $\sum_{i=0}^{\infty} x_i \in X$ with $x_i \in X_i$,

$$\frac{1}{K}\left(\sum_{i=0}^{\infty}\|x_i\|^p\right)^{1/p} \le \left\|\sum_{i=0}^{\infty} x_i\right\| \le K\left(\sum_{i=0}^{\infty}\|x_i\|^p\right)^{1/p}.$$

The next property follows immediately.

Proposition 4.2. *Let $\{X_i\}_{i=0}^{\infty}$ be an ℓ_p-decomposition of a Banach space X. Then, for every strictly increasing sequence $\{m_j\}_{j=0}^{\infty}$ of positive integers, $\{\widetilde{X}_j\}_{j=0}^{\infty}$ is again an ℓ_p-decomposition of X, where*

$$\widetilde{X}_0 = \sum_{i=0}^{m_0} X_i \quad \textit{and} \quad \widetilde{X}_j = \sum_{i=m_{j-1}+1}^{m_j} X_i \quad \textit{for } j > 0.$$

Assume that $\{X_i\}_{i=0}^{\infty}$ is an ℓ_p-decomposition of X. For each $j \in \{0, 1, 2, \ldots\}$, we denote by $P_{\mathfrak{D},j}$ the natural projection from $X = \sum_{i=0}^{\infty} X_i$ onto X_j. Obviously, $X \approx (\sum X)_p$. Let $\{\psi_i\}_{i=0}^{\infty}$ be a sequence of uniform isomorphisms $\psi_i : X_i \to X$; i.e., both $\{\psi_i\}_{i=0}^{\infty}$ and $\{\psi_i^{-1}\}_{i=0}^{\infty}$ are uniformly bounded. Next, let

$$U : \sum_{i=0}^{\infty} x_i \in X \mapsto (\psi_0(x_0), \psi_1(x_1), \ldots), \quad x_i \in X_i.$$

Then, U is an isomorphism from X onto $(\sum X)_p$. Let $\mathcal{U}$ be the set of all such isomorphisms U. We denote by L (resp., R) the left (resp., right) shift operator, i.e., for $y = (y_i)_{i=0}^{\infty} \in (\sum X)_p$,

$$L(y) = (y_1, y_2, \ldots) \quad \text{(resp., } R(y) = (0, y_0, y_1, \ldots)\text{)}.$$

Next, let

$$L_{\mathfrak{D}} = U^{-1}LU, \quad R_{\mathfrak{D}} = U^{-1}RU.$$

Finally, let

$$\mathcal{L}_{\mathfrak{D}} = \{L_{\mathfrak{D}} = U^{-1}RU : U \in \mathcal{U}\}, \quad \mathcal{R}_{\mathfrak{D}} = \{R_{\mathfrak{D}} = U^{-1}RU : U \in \mathcal{U}\}.$$

We denote by D_S the inner derivation determined by S in $\mathcal{L}(X)$, i.e.,

$$D_S T = ST - TS.$$

Keep these notations just mentioned above in mind. Then, we have the following property.

Proposition 4.3. *An operator $T \in \mathcal{L}(X)$ is a commutator if and only if there exists $S \in \mathcal{L}(X)$ such that $T \in D_S\mathcal{L}(X)$.*

The following theorem is due to Dosev [12, Corollary 7].

Theorem 4.4 (Dosev). *Let $\mathfrak{D} = \{X_i\}$ be a ℓ_p-decomposition of a Banach space X. Then, for all $T \in \mathcal{L}(X)$, $R \in \mathcal{R}_{\mathfrak{D}}$, and for $L \in \mathcal{L}_{\mathfrak{D}}$, we have*

$$TP_{\mathfrak{D},0} \in \operatorname{Im}(D_R) \quad \textit{and} \quad P_{\mathfrak{D},0}T \in \operatorname{Im}(D_L),$$

where $P_{\mathfrak{D},0}$ is the natural projection from $X = \sum_{i=0}^{\infty} X_i$ onto X_0, and $\operatorname{Im}(D_R)$ (resp., $\operatorname{Im}(D_L)$) denotes the image of D_R (resp., D_L).

The theorem below follows from a quick observation of the proof of Dosev [12, Theorem 8].

Theorem 4.5. *Let $\mathfrak{D} = \{X_i\}$ be an ℓ_p-decomposition of a Banach space X and $\widetilde{P}_n = \sum_{i=0}^{n} P_{\mathfrak{D},i}$. Assume that $T \in \mathcal{L}(X)$ such that*

$$\sum_{n=0}^{\infty} \|(I - \widetilde{P}_n)T\| + \sum_{n=0}^{\infty} \|T(I - \widetilde{P}_n)\| + \sum_{m,n=0}^{\infty} \|(I - \widetilde{P}_m)T(I - \widetilde{P}_n)\| < \infty.$$

Then, for all $R \in \mathcal{R}_{\mathfrak{D}}$ and $L \in \mathcal{L}_{\mathfrak{D}}$, we have

$$T \in \operatorname{Im}(D_R) \cap \operatorname{Im}(D_L).$$

The next result is also due to Dosev [12, Lemma 5].

Lemma 4.6 (Dosev). *Let $\mathfrak{D} = \{X_i\}$ be an ℓ_p-decomposition of a Banach space X and $\widetilde{P}_n = \sum_{i=0}^n P_{\mathfrak{D},i}$, where $P_{\mathfrak{D},i}$ is the natural projection from $X = \sum_{i=0}^\infty X_i$ onto X_i. Suppose that $T \in \mathcal{L}(X)$ satisfies*

$$\lim_{n\to\infty} \|(I - \widetilde{P}_n)T\| = \lim_{n\to\infty} \|T(I - \widetilde{P}_n)\| = 0.$$

Then, there exists an increasing sequence $\{m_j\}_{j=0}^\infty$ of positive integers such that

$$\sum_{j=0}^\infty \|(I - \widetilde{P}_{m_j})T\| + \sum_{j=0}^\infty \|T(I - \widetilde{P}_{m_j})\| + \sum_{i,j=0}^\infty \|(I - \widetilde{P}_{m_i})T(I - \widetilde{P}_{m_j})\| < \infty.$$

We can further show the following result.

Lemma 4.7. *Let $\mathfrak{D} = \{X_i\}$ be an ℓ_p-decomposition of a Banach space X and $\widetilde{P}_n = \sum_{i=0}^n P_{\mathfrak{D},i}$. Suppose that $T \in \mathcal{L}(X)$ satisfies*

$$\lim_{n\to\infty} \|(I - \widetilde{P}_n)T(I - P_{\mathfrak{D},0})\| = \lim_{n\to\infty} \|T(I - \widetilde{P}_n)\| = 0.$$

Then, T is a commutator.

Proof. Since

$$\lim_{n\to\infty} \|(I-\widetilde{P}_n)T(I-P_{\mathfrak{D},0})\| = \lim_{n\to\infty} \|T(I - \widetilde{P}_n)\| = \lim_{n\to\infty} \|T(I - P_{\mathfrak{D},0})(I - \widetilde{P}_n)\| = 0,$$

by Lemma 4.6, there exists an increasing sequence $\{m_j\}_{j=0}^\infty$ of positive integers such that

$$\sum_{j=0}^\infty \|(I - \widetilde{P}_{m_j})T(I - P_{\mathfrak{D},0})\| + \sum_{j=0}^\infty \|T(I - P_{\mathfrak{D},0})(I - \widetilde{P}_{m_j})\| + \sum_{i,j=0}^\infty \|(I - \widetilde{P}_{m_i})T(I - P_{\mathfrak{D},0})(I - \widetilde{P}_{m_j})\| < \infty.$$

Note that

$$\sum_{j=0}^\infty \|(I - \widetilde{P}_{m_j})T(I - \widetilde{P}_{m_0})\| \le \|I - \widetilde{P}_{m_0}\| \sum_{j=0}^\infty \|(I - \widetilde{P}_{m_j})T(I - P_{\mathfrak{D},0})\|.$$

Then, we have

$$\sum_{j=0}^\infty \|(I - \widetilde{P}_{m_j})T(I - \widetilde{P}_{m_0})\| + \sum_{j=0}^\infty \|T(I - \widetilde{P}_{m_0})(I - \widetilde{P}_{m_j})\| + \sum_{i,j=0}^\infty \|(I - \widetilde{P}_{m_i})T(I - \widetilde{P}_{m_0})(I - \widetilde{P}_{m_j})\| < \infty.$$

Let $\mathcal{D}' = \{\widetilde{X}_j\}_{j=0}^{\infty}$, where $\widetilde{X}_0 = \sum_{i=0}^{m_0} X_i$ and $\widetilde{X}_j = \sum_{i=m_{j-1}+1}^{m_j} X_i$ for $j > 0$. Then, by Proposition 4.2, $\mathcal{D}'$ is an ℓ_p-decomposition of X. Due to Theorems 4.4 and 4.5, we can choose $R \in \mathcal{R}_{\mathcal{D}'}$ such that

$$T\widetilde{P}_{m_0},\ T(I - \widetilde{P}_{m_0}) \in \operatorname{Im}(D_R).$$

Consequently,

$$T = T\widetilde{P}_{m_0} + T(I - \widetilde{P}_{m_0}) \in \operatorname{Im}(D_R).$$

Therefore, T is a commutator. ∎

The next result follows from [1, Theorem 2.2.3].

Lemma 4.8. *Given $1 \leq p < \infty$, let X be a Banach space satisfying $X \approx (\sum X)_p$. If Y and Z are closed subspaces of X with $Y \approx X$, and with $Y \oplus Z = X$, then for every complemented subspace W of Z, $Y \oplus W \approx X$.*

Lemma 4.9. *Given $1 \leq p \leq \infty$, let $\{e_{i_k,j_l}\}_{l \leq k}$ be a subtriangle of $\{e_{i,j}\}_{j \leq i}$. Then, the natural projection P from $\mathcal{T}_p$ onto $[e_{i_k,j_l}]_{l \leq k}$ satisfies $\|P\| \leq 2$.*

Proof. For each $k \in \mathbb{N}$, put

$$A = \{i_k\}_{k=1}^{\infty}, \quad B = \{j_l\}_{l=1}^{\infty},$$
$$B_k = \{j \in \mathbb{N} : j_k + 1 \leq j \leq j_{k+1}\}.$$

Then, $(P_{A,B} - P_{(\{i_k\},\{B_k\})})|_{\mathcal{T}_p}$ is just a reformulation of the natural projection P from $\mathcal{T}_p$ onto $[e_{i_k,j_l}]_{l \leq k}$. ∎

Note that a commutator is not invariant under small perturbations. For example, 0 is a commutator, but εI is not a commutator for all $\varepsilon > 0$. To show the main result of this section, we require the next lemma, which was motivated by the proof of [4, Theorem 4.6] and Arazy's another important result [5, Lemma 2.1]. It can also be regarded as a further representation of "small perturbations" of the target operator in Arazy's lemma.

Lemma 4.10. *Let $T \in \mathcal{M}_{\mathcal{T}_p}$ $(1 \leq p < 2)$. Then, there exist a subtriangle $\{e_{i_k,j_l}\}_{l \leq k}$ of $\{e_{i,j}\}_{j \leq i}$, a scalar α, and two operators $T_0 \in \mathcal{L}(\mathcal{T}_p)$ and $T_1 \in \mathcal{L}([e_{i_k,j_l}]_{l \leq k}, \mathcal{T}_p)$ such that*

$$T = T_0 + T_1 P, \tag{4.1}$$

$$(T_0 e_{i_k,j_l})(i_{k'}, j_{l'}) = \delta_{k,k'} \cdot \delta_{l,l'} \cdot \alpha, \tag{4.2}$$

and

$$\lim_{n\to\infty} \|E^{(n)} T_1\| = 0, \tag{4.3}$$

where P is the natural projection from $\mathcal{T}_p$ onto $[e_{i_k,j_l}]_{l \leq k}$ defined as in Lemma 4.9, and $E^{(n)}$ is defined by (2.3).

Proof. By Lemma 3.4, we can assume that there is a subtriangle $\{e_{i_k,j_l}\}_{l \leq k}$ of $\{e_{i,j}\}_{j \leq i}$ so that either $T|_{[e_{i_k,j_l}]_{l \leq k}}$ is compact, or for each fixed l, $\{Te_{i_k,j_l}\}_{k=l}^{\infty}$ is equivalent to the unit vector basis of ℓ_2.

Case 1. $T|_{[e_{i_k,j_l}]_{l\le k}}$ is compact. Let

$$T_0 = T(I-P), \quad T_1 = T|_{[e_{i_k,j_l}]_{l\le k}},$$

and $\alpha = 0$. Then, compactness of T_1 entails that

$$\lim_{n\to\infty} \|E^{(n)}T_1\| = 0.$$

Case 2. For each fixed l, $\{Te_{i_k,j_l}\}_{k=l}^{\infty}$ is equivalent to the unit vector basis of ℓ_2. By the same procedure in the proof of [4, Theorem 4.6], for every sequence $\{\varepsilon_l\}_{l=1}^{\infty}$ of positive numbers with $\sum_{l=1}^{\infty}\varepsilon_l < \infty$, we may assume that there are two increasing sequences $\{m_k\}_{k=1}^{\infty}$ and $\{n_l\}_{l=1}^{\infty}$ of positive integers with $m_k < i_k \le m_{k+1}$, $n_l < j_l \le n_{l+1}$, and three operators $S \in \mathcal{L}(\mathcal{T}_p)$, $S_1, S_2 \in \mathcal{L}([e_{i_k,j_l}]_{l\le k}, \mathcal{T}_p)$ so that S_1 is compact such that

$$T = S + S_1P + S_2P, \tag{4.4}$$

and such that the following formulas hold:

$$\begin{aligned} Se_{i_k,j_l} &= E_{n_{l+1}}P_{m_k,m_{k+1}}Se_{i_k,j_l}, \\ S_2|_{[e_{i_k,j_l}]_{k=l}^{\infty}} &= E^{(n_{l+1})}T|_{[e_{i_k,j_l}]_{k=l}^{\infty}}, \end{aligned}$$

and

$$\|E^{(n_{l+1})}|_{[Te_{i_k,j_l}]_{k=l}^{\infty}}\| \le \varepsilon_l.$$

By a routine diagonal process of passing subsequence, we can assume that the following limits exist for all $l' \le l$:

$$\lim_{k\to\infty}(Se_{i_k,j_l})(i_k, j_{l'}) = \alpha_{l,l'}. \tag{4.5}$$

Again by a diagonal process, we may assume that the following equations hold:

$$\alpha = \lim_{l\to\infty}\alpha_{l,l}, \tag{4.6}$$

$$\alpha_{l'} = \lim_{l\to\infty}\alpha_{l,l'}, \quad l' = 1, 2, \ldots. \tag{4.7}$$

For an arbitrary double sequence $\{\delta_{l,l'}\}_{l'\le l}$ of positive numbers so that $\sum_{l'\le l}\delta_{l,l'}$ is small enough, up to passing subsequence, we can assume that

$$\begin{aligned} |\alpha_{l,l'} - \alpha_{l'}| &< \delta_{l,l'}, \quad l' < l, \\ |\alpha_{l,l} - \alpha| &< \delta_{l,l}. \end{aligned}$$

Now, up to passing a subtriangle and by a perturbation argument, (4.5), (4.6), and (4.7), it follows from (4.4) that we can simply assume that

$$T = S_0 + S_1P + S_2P + S_3P,$$

where $S_0 = S - S_3 P$ satisfying

$$(S_0 e_{i_k, j_l})(i_k, j_{l'}) = \alpha_{l'}, \quad l' < l \le k,$$
$$(S_0 e_{i_k, j_l})(i_k, j_l) = \alpha, \quad l \le k,$$

and $S_3 \in \mathcal{L}([e_{i_k, j_l}]_{l \le k})$ satisfies

$$S_3 e_{i_k, j_l} = \sum_{l'=1}^{k} a_{l'}^{(k,l)} e_{i_k, j_{l'}} \tag{4.8}$$

and

$$|a_{l'}^{(k,l)}| < \delta_{l,l'}.$$

Compactness of S_1 entails that

$$\lim_{n\to\infty} \|E^{(n)} S_1\| = 0. \tag{4.9}$$

On the other hand, since for each positive integer l,

$$S_2|_{[e_{i_k, j_l}]_{k=l}^{\infty}} = E^{(n_{l+1})} T|_{[e_{i_k, j_l}]_{k=l}^{\infty}},$$

and since $[Te_{i_k, j_l}]_{k=l}^{\infty}$ is isomorphic to a Hilbert space, by Arazy's lemma (Theorem 2.8),

$$\lim_{n\to\infty} \sum_{l=1}^{m} \left\| E^{(n)} S_2|_{[e_{i_k, j_l}]_{k=l}^{\infty}} \right\| = 0, \quad m \in \mathbb{N},$$

and

$$\sum_{l=m+1}^{\infty} \left\| E^{(n)} S_2|_{[e_{i_k, j_l}]_{k=l}^{\infty}} \right\| \le \|T\| \sum_{l=m+1}^{\infty} \varepsilon_l, \quad m \in \mathbb{N}.$$

Therefore,

$$\lim_{n\to\infty} \|E^{(n)} S_2\| = 0. \tag{4.10}$$

By (4.8), for every sequence $\{t_k\}_{k=l}^{\infty}$ of scalars with finitely many $t_k \neq 0$,

$$\begin{aligned}
\left\| S_3\Big(\sum_{k=l}^{\infty} t_k e_{i_k, j_l}\Big) \right\|_p &= \left\| \sum_{k=l}^{\infty} t_k \sum_{l'=1}^{l} a_{l'}^{(k,l)} e_{i_k, j_l} \right\|_p \\
&\le \sum_{l'=1}^{l} \left\| \sum_{k=l}^{\infty} t_k a_{l'}^{(k,l)} e_{i_k, j_l} \right\|_p \\
&\le \sum_{l'=1}^{l} \delta_{l,l'} \Big(\sum_{k=l}^{\infty} |t_v|^2 \Big)^{1/2}.
\end{aligned}$$

Therefore,

$$\|S_3|_{[e_{i_k,j_l}]_{k=l}^{\infty}}\| \le \sum_{l'=1}^{l} \delta_{l,l'}.$$

For all positive numbers l_0 and $n > n_{l_0}$,

$$\|E^{(n)} S_3\| \le \sum_{l=l_0}^{\infty} \|S_3|_{[e_{i_k,j_l}]_{k=l}^{\infty}}\| \le \sum_{l=l_0}^{\infty} \sum_{l'=1}^{l} \delta_{l,l'}.$$

Therefore,

$$\lim_{n\to\infty} \|E^{(n)} S_3\| = 0. \tag{4.11}$$

Put $T_0 = S_0$ and $T_1 = S_1 + S_2 + S_3$. By (4.9), (4.10), and (4.11), we obtain

$$\lim_{n\to\infty} \|E^{(n)} T_1\| = 0.$$

Therefore, (4.1) and (4.3) hold.

In order to show (4.2), it suffices to prove that $\alpha_{l'} = 0$ for every l'.

Fix $l', N \in \mathbb{N}$ and $k \ge N + l'$. Then

$$\|S_0\| N^{1/2} \ge \left\| \sum_{l=l'+1}^{l'+N} S_0 e_{i_k,j_l} \right\|_p \ge \left| \sum_{l=l'+1}^{l'+N} (S_0 e_{i_k,j_l})(i_k, j_{l'}) \right| = |\alpha_{l'}| N.$$

Since N is arbitrary, $\alpha_{l'} = 0$. ∎

Theorem 4.11. *Every $T \in \mathcal{M}_{\mathcal{T}_p}$ $(1 \le p < 2)$ is a commutator.*

Proof. By Lemma 4.10, there exists a subtriangle $\{e_{i_k,j_l}\}_{l\le k}$ of $\{e_{i,j}\}_{j\le i}$, a scalar α, and two operators $T_0 \in \mathcal{L}(\mathcal{T}_p)$ and $T_1 \in \mathcal{L}\big([e_{i_k,j_l}]_{l\le k}, \mathcal{T}_p\big)$ such that

$$T = T_0 + T_1 P,$$
$$(T_0 e_{i_k,j_l})(i_{k'}, j_{l'}) = \delta_{k,k'} \cdot \delta_{l,l'} \cdot \alpha,$$
$$\lim_{n\to\infty} \|E^{(n)} T_1\| = 0,$$

where P is the natural projection P from $\mathcal{T}_p$ onto $[e_{i_k,j_l}]_{l\le k}$ defined as in Lemma 4.9.

Note that

$$\|T_1 P - E_n T_1 P\| = \|E^{(n)} T_1 P\| \to 0, \quad \text{as } v \to \infty,$$

and

$$E_n T P \in \mathcal{M}_{\mathcal{T}_p}.$$

Since $\mathcal{M}_{\mathcal{T}_p}$ is the largest closed ideal of $\mathcal{L}(\mathcal{T}_p)$ (Theorem 3.3), $T_1 P \in \mathcal{M}_{T_p}$. Consequently, $T_0 = T - T_1 P \in \mathcal{M}_{T_p}$. It follows from [5, Corollary 2.2] that $\alpha = 0$.

By Lemma 4.8, we can assume that $(I - P)\mathcal{T}_p \approx \mathcal{T}_p$. For each $v \in \mathbb{N}$, put

$$A_v = \{i_{2^v(2\mu+1)}\}_{\mu=1}^{\infty} \quad \text{and} \quad B_v = \{j_{2^v(2\mu+1)}\}_{\mu=1}^{\infty}.$$

Then, $P_{(\{A_v\},\{B_v\})}P$ is again a projection satisfying for all $v \in \mathbb{N}$,

$$P_{(\{A_v\},\{B_v\})}P\mathcal{T}_p = \Big(\sum_{v=1}^{\infty}[e_{i_{2^v(2s+1)},j_{2^v(2t+1)}}]_{t\le s}\Big)_p,$$

and

$$P_{A_v,B_v}P\mathcal{T}_p = [e_{i_{2^v(2s+1)},j_{2^v(2t+1)}}]_{t\le s}.$$

For every sequence $\{\delta_v\}_{v=1}^{\infty}$ of positive numbers with $\sum_{v=1}^{\infty}\delta_v < \infty$ and for every $v \in \mathbb{N}$, by Theorems 2.10 and 3.6, we can choose a closed subspace X_v of $[e_{i_{2^v(2s+1)},j_{2^v(2t+1)}}]_{t\le s}$ such that X_v is 2-isomorphic to $\mathcal{T}_p$ with $\|T|_{X_v}\| \le \delta_v$ and such that X_v is 3-complemented in $[e_{i_{2^v(2s+1)},\, j_{2^v(2t+1)}}]_{t\le s}$. Let $Q_v : [e_{i_{2^v(2s+1)},\, j_{2^v(2t+1)}}]_{t\le s} \to X_v$ be a projection with $\|Q_v\| \le 3$. Then

$$\sum_{v=1}^{\infty}X_v \simeq \Big(\sum_{v=1}^{\infty}\bigoplus X_v\Big)_p \approx \Big(\sum_{v=1}^{\infty}\bigoplus \mathcal{T}_p\Big)_p,$$

the series $\sum_{v=1}^{\infty}Q_vP_{A_v,B_v}P$ is strongly convergent and induces a projection from $\mathcal{T}_p$ onto $\sum_{v=1}^{\infty}X_v$. Let

$$X_0 = \ker\Big(\sum_{v=1}^{\infty}Q_vP_{A_v,B_v}P\Big).$$

By Lemma 4.8, X_0 is isomorphic to $\mathcal{T}_p$. Thus,

$$\mathcal{D} = \{X_v\}_{v=0}^{\infty}$$

is a ℓ_p-decomposition of $\mathcal{T}_p$ and satisfies the fact that, for all $v \in \mathbb{N}$,

$$P_{\mathcal{D},0} = I - \sum_{v=1}^{\infty}Q_vP_{A_v,B_v}P \quad \text{and} \quad P_{\mathcal{D},v} = Q_vP_{A_v,B_v}P.$$

Denote by

$$\widetilde{P}_w = \sum_{v=0}^{w}P_{\mathcal{D},v}.$$

Then

$$\|T(I-\widetilde{P}_w)\| \le \sum_{v=w+1}^{\infty}\|TP_{\mathcal{D},v}\| \le 6\sum_{v=w+1}^{\infty}\delta_v.$$

Therefore,

$$\lim_{w\to\infty}\|T(I-\widetilde{P}_w)\| = 0. \tag{4.12}$$

On the other hand, it follows from

$$\begin{aligned}(I-\widetilde{P}_w)T(I-P_{\mathcal{D},0}) &= (I-\widetilde{P}_w)T_1P(I-P_{\mathcal{D},0})\\ &= (I-\widetilde{P}_w)T_1(I-P_{\mathcal{D},0})\\ &= \Big(\sum_{v=w+1}^{\infty}Q_vP_{A_v,B_v}P\Big)T_1\Big(\sum_{v=1}^{\infty}Q_vP_{A_v,B_v}P\Big)\end{aligned}$$

that

$$\|(I - \widetilde{P}_w)T_1(I - P_{\mathfrak{D},0})\| \le 36\|E^{(j_{2w+1})}T_1\|.$$

Therefore,

$$\lim_{w\to\infty} \|(I - \widetilde{P}_w)T(I - P_{\mathfrak{D},0})\| = 0. \tag{4.13}$$

By (4.12), (4.13), and Lemma 4.7, T is a commutator. ∎

5. A characterization of commutators on $\mathcal{L}(\mathcal{T}_p)$ $(1 \le p < \infty)$

In this section, we will show Theorem B mentioned in the end of the first section; that is, let $1 \le p < \infty$, $T \in \mathcal{L}(\mathcal{T}_p)$ is a commutator if and only if $T - \lambda I$ is not $\mathcal{T}_p$-strictly singular for each $\lambda \neq 0 \in \mathbb{C}$.

For X and Y be two subspaces of a Banach space Z, let

$$d(S_X, Y) = \inf\{\|x - y\| : x \in S_X, y \in Y\},$$

where S_X is the unit sphere of X.

Note that if both X and Y are closed with $X \cap Y = 0$, then $X + Y$ is a closed subspace of Z if and only if $d(S_X, Y) > 0$. Note also that

$$1/2d(S_X, Y) \le d(S_Y, X) \le 2d(S_X, Y).$$

The following two theorems are due to Chen, Johnson, and Zheng [9, Lemmas 2.13, 2.14, and Theorem 2.15].

Theorem 5.1 (Chen–Johnson–Zheng). *Let $p \in [1, \infty] \cup \{0\}$, and let X be a complementably homogeneous Banach space isomorphic to $(\sum X)_p$ and T a bounded linear operator on X for which there is a subspace Y of X isomorphic to X such that $T|_Y$ is an isomorphism and $d(S_Y, TY) > 0$. Then, T is a commutator.*

Theorem 5.2 (Chen–Johnson–Zheng). *Let $p \in [1, \infty] \cup \{0\}$, and let X be a complementably homogeneous Banach space isomorphic to $(\sum X)_p$. Suppose that the set of all X-strictly singular operators on X form an ideal in $\mathcal{L}(X)$. Let $T : X \to X$ be a bounded linear operator such that, for every $\lambda' \in \mathbb{C}$, $T - \lambda' I$ is not X-strictly singular. If there is a $\lambda \in \mathbb{C}$ and a subspace Y of X isomorphic to X and such that $(T - \lambda I)|_Y$ is X-strictly singular, then T is a commutator.*

Theorem 5.3. *Given $1 \le p < 2$, let $T \in \mathcal{L}(\mathcal{T}_p)$ such that $T - \lambda I$ is not $\mathcal{T}_p$-strictly singular for all $\lambda \in \mathbb{C}$. Then, T is a commutator.*

Proof. By Lemma 4.10, we can assume that there exist $T_0 \in \mathcal{L}(\mathcal{T}_p)$, $T_1 \in \mathcal{M}_{\mathcal{T}_p}$, and a scalar α such that

$$T = T_0 + T_1,$$

and

$$(T_0 e_{i_k,j_l})(i_{k'}, j_{l'}) = \delta_{k,k'} \cdot \delta_{l,l'} \cdot \alpha.$$

Let P be the natural projection from $\mathcal{T}_p$ onto $[e_{i_k,j_l}]_{l \le k}$ defined as in Lemma 4.9. Then

$$PT_0P = \alpha P.$$

Therefore,

$$(I - P)TP = TP - P(T_0 + T_1)P = (T - \alpha I)P - PT_1P.$$

If $(I - P)TP$ is not $\mathcal{T}_p$-strictly singular, then there is a subspace X of $\mathcal{T}_p$ isomorphic to $\mathcal{T}_p$ so that $(I - P)TP$ is an isomorphism on X. Denote by $Y = PX$. Then, $Y \approx \mathcal{T}_p$. Since $(I - P)T$ is an isomorphism on Y, there is a positive number c so that

$$\|(I - P)T(y)\| \ge c\|y\|, \quad y \in Y.$$

For all $Ty \in S_{TY}$ and $y' \in Y$,

$$\|Ty - y'\| \ge \frac{\|(I - P)Ty\|}{\|I - P\|} \ge \frac{c\|y\|}{\|I - P\|} \ge \frac{c}{\|T\|\|I - P\|}.$$

Therefore, $d(S_{TY}, Y) > 0$. By Theorem 5.1, T is a commutator.

In the case that $(I - P)TP$ is $\mathcal{T}_p$-strictly singular, since

$$(T - \alpha I)P = (I - P)TP + PT_1P,$$

$(T - \alpha I)|_{[e_{i_k,j_l}]_{l \le k}}$ is $\mathcal{T}_p$-strictly singular. It follows from Theorem 5.2 that T is a commutator. ∎

Theorem 5.4. *Let $1 \le p < \infty$. Then, an operator $T \in \mathcal{L}(\mathcal{T}_p)$ is a commutator if and only if $T - \lambda I \notin \mathcal{M}_{\mathcal{T}_p}$ for all $\lambda \ne 0$.*

Proof. According to Brown and Pearcy [8], the conclusion is true for $p = 2$. By a duality argument, it suffices to show that it is true for $1 \le p < 2$.

Necessity. It follows immediately from Wintner's theorem (i.e., Theorem 1.1).

Sufficiency. Suppose that $T - \lambda I \notin \mathcal{M}_{\mathcal{T}_p}$ for all $\lambda \ne 0$. If $T \in \mathcal{M}_{\mathcal{T}_p}$, then, by Theorem 4.11, T is a commutator. If $T \notin \mathcal{M}_{\mathcal{T}_p}$, then, by Theorem 5.3, $T - \lambda I \notin \mathcal{M}_{\mathcal{T}_p}$ for all $\lambda \in \mathbb{C}$. Consequently, T is a commutator. ∎

Corollary 5.5. *Let $1 < p < \infty$. Then, an operator $T \in \mathcal{L}(\mathcal{C}_p)$ is a commutator if and only if $T - \lambda I \notin \mathcal{M}_{\mathcal{C}_p}$ for any $\lambda \ne 0$.*

Proof. By Theorem 5.4, if suffices to note that $\mathcal{T}_p \approx \mathcal{C}_p$ for all $1 < p < \infty$ (Proposition 2.5). ∎

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