

# On the existence of global solutions for the 3D chemorepulsion system

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**Abstract.** In this paper, we give sufficient conditions for global-in-time existence of classical solutions for the fully parabolic chemorepulsion system posed on a convex, bounded three-dimensional domain. Our main result establishes global-in-time existence of regular nonnegative solutions provided that  $\nabla\sqrt{u} \in L^4(0, T; L^2(\Omega))$ . Our method is related to the Bakry–Émery calculation and appears to be new in this context.

## 1. Introduction

In this paper, we study the problem of global existence of solutions for the fully parabolic chemorepulsion system. The two-dimensional case was solved in [5]. Unlike in the more widely known chemoattraction case, 2D chemorepulsion leads to the global-in-time existence of classical solutions regardless of the size of the initial data. The question of global existence in three and higher dimensions remains open. In the present paper, we look into the 3D case and establish a conditional global regularity result for this model. First, we introduce the model.

Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded domain with a sufficiently smooth boundary. We consider the following fully parabolic chemorepulsion system

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u + u \nabla v) \\ \partial_t v = \Delta v - v + u \end{cases} \quad \text{in } (0, \infty) \times \Omega, \quad (1.1)$$

with homogeneous Neumann boundary conditions (no flux through the boundary)

$$\frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad \frac{\partial v}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad (1.2)$$

where  $\nu$  is the unit outward normal to the boundary, and with nonnegative initial conditions

$$u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0. \quad (1.3)$$

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The functions  $u$  and  $v$  describe densities of some living organisms and of a chemical substance which repels them, respectively. The “+” sign on the right-hand side of the first equation in (1.1) corresponds to the repulsion mechanism. The opposite phenomenon appears in the widely studied chemoattraction case described by the Keller–Segel system. For an overview of results for such systems, we refer to the surveys [2, 12]. The latter survey contains a chapter concerning the construction of solutions, including the irregular ones, to the fully parabolic chemorepulsion system.

Three- and higher-dimensional cases of the chemorepulsion system are still far from being understood. While we know that global weak solutions exist (see [5]), it is unclear whether regular bounded solutions exist for all  $t > 0$ . Some results concerning the perturbation of the parabolic-elliptic case are known, see [4]. Moreover, for the problem with nonlinear, sufficiently strong diffusion, see the global existence result in [7]. Similarly, it is known that nonlinear, sufficiently weak chemorepulsion leads to global-in-time solutions, see [17]. However, the main basic problem lacks a definitive answer. In the present note, we establish a conditional result.

We emphasise that our result is rather of methodological meaning. On the one hand, our method applied to the 3D chemorepulsion yields only a conditional result. Moreover, as communicated to us by M. Winkler and one of the referees, this result can be improved, see Appendix B. Indeed, the result in Appendix B covers our Theorem 1.1 and is applicable also in non-convex domains. Notice, however, that our method of estimating the Fisher information along the solution for a system of partial differential equations seems promising with other types of problems. The inequality from Appendix A turns out to be very helpful in such an approach. Indeed, in [3], Fisher’s information together with inequality (A.1) was successfully applied to obtain global-in-time unique regular solutions to the 1D thermoelasticity problem. Very recently, another application yielding progress in 1D combustion theory was performed in [13]. Last, but not least, notice that our approach gives also a qualitative conditional result, namely, it implies that concavity of  $v$  is sufficient to obtain global solution to (1.1), see Section 6.

The problem (1.1)–(1.3) captured attention of groups of researchers, in particular, due to its role in the attraction-repulsion competition, which appears to play a role in the modeling of, among other things, Alzheimer’s disease, see, for instance, [15]. The biological meaning of the attraction-repulsion competition was widely investigated, see, for instance, the contributions in [14, 16, 18].

Let us formulate our main result.

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^3$  be a convex, smooth bounded domain and let  $u_0, v_0 \in W^{1,p}(\Omega)$  for some  $p > 3$  with  $0 \not\equiv u_0 \geq 0$  and  $v_0 \geq 0$  in  $\Omega$ . Suppose that  $T_{\max} \in (0, \infty]$  is the maximal existence time of the classical solution to the system (1.1)–(1.3), constructed in [5, Theorem 2.1] (cf. Lemma 3.1 below). If  $\nabla \sqrt{u} \in L^4(0, T_{\max}; L^2(\Omega))$ , then  $T_{\max} = \infty$ .*

The paper is organised as follows: In Section 2, we introduce some technical tools such as the Winkler version of the Bernis-type inequality, Bochner’s formula, the behaviour of the normal derivative of the gradients of regular functions at the boundaries of convex

domains and further preparatory inequalities. In Section 3, we then recall well-known properties of solutions to (1.1)–(1.3).

Section 4 is devoted to our entropy estimate. It is a core and the main novelty of our approach. We estimate the time derivative of the entropy production term occurring in the Lyapunov functional. The latter entropy production term resembles the Fisher information along the heat flow and we utilise this similarity. Having this estimate established, in Section 5, we proceed with the proof of Theorem 1.1.

In Appendix A, we prove a new functional inequality, which we arrived at as a byproduct of our investigations. It seems interesting in its own right. Appendix B is devoted to the proof of an observation due to M. Winkler and one of the anonymous referees, which improves the conditional result.

## 2. Preliminaries

In this section, we collect some computations and known results which will be useful in the sequel.

We begin with the flat case of the well-known Bochner formula.

**Proposition 2.1.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \in \mathbb{N}$ , be a smooth domain and let  $u \in C^3(\bar{\Omega})$ . Then*

$$\frac{1}{2} \Delta |\nabla u|^2 = \nabla(\Delta u) \cdot \nabla u + |D^2 u|^2 \quad \text{in } \bar{\Omega}. \quad (2.1)$$

*Proof.* This can be checked by a direct calculation. ■

The following lemma informs us about the normal derivative of the square of the gradient of a function on the boundary of a convex set, provided the function's normal derivative vanishes.

**Lemma 2.2.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \in \mathbb{N}$ , be a convex bounded domain with smooth boundary. Suppose that a function  $u \in C^2(\bar{\Omega})$  satisfies  $\frac{\partial u}{\partial \nu} = 0$  on  $\partial\Omega$ . Then*

$$\frac{\partial |\nabla u|^2}{\partial \nu} \Big|_{\partial\Omega} \leq 0.$$

*Proof.* See [6, page 95]. ■

We will use a higher-dimensional version of the Bernis-type inequality given by Winkler [20, Lemma 3.3] (with  $h(\varphi) = \varphi$ ).

**Lemma 2.3.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \in \mathbb{N}$ , be a smooth, bounded domain. For all positive  $\varphi \in C^2(\bar{\Omega})$  with  $\frac{\partial \varphi}{\partial \nu} = 0$  on  $\partial\Omega$ , we have the following inequality:*

$$\int_{\Omega} \frac{|\nabla \varphi|^4}{\varphi^3} \leq (2 + \sqrt{n})^2 \int_{\Omega} \varphi |D^2 \log \varphi|^2. \quad (2.2)$$

Next, we prove two estimates holding in three-dimensional domains. The first one relates the Hessian of a function  $\varphi$  with  $\nabla \Delta \varphi$  (in contrast to the full third-order derivative).

**Lemma 2.4.** *Let  $\Omega \subset \mathbb{R}^3$  be a smooth, bounded domain. Then, there is a positive constant  $C$  such that for every  $\varphi \in C^3(\bar{\Omega})$  with  $\frac{\partial \varphi}{\partial \nu} = 0$  on  $\partial\Omega$  we have*

$$\|D^2\varphi\|_{L^6(\Omega)} \leq C \|\nabla \Delta \varphi\|_{L^2(\Omega)}.$$

*Proof.* According to [8, Theorem 19.1], there is  $c_1 > 0$  such that

$$\|D^2\varphi\|_{L^6} \leq c_1 \|\Delta \varphi\|_{L^6} + c_1 \|\varphi - \bar{\varphi}\|_{L^6}$$

for every  $\varphi \in C^2(\bar{\Omega})$  with  $\frac{\partial \varphi}{\partial \nu} = 0$  on  $\partial\Omega$ , where  $\bar{\varphi} = \frac{1}{|\Omega|} \int_{\Omega} \varphi$ . As  $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ , we can further estimate

$$\|D^2\varphi\|_{L^6} \leq c_2 (\|\nabla \Delta \varphi\|_{L^2}^2 + \|\Delta \varphi\|_{L^2}^2)^{1/2} + c_2 (\|\nabla \varphi\|_{L^2}^2 + \|\varphi - \bar{\varphi}\|_{L^2}^2)^{1/2}$$

for every  $\varphi \in C^3(\bar{\Omega})$  with  $\frac{\partial \varphi}{\partial \nu} = 0$  on  $\partial\Omega$  for some  $c_2 > 0$ . The statement then follows by the Poincaré inequality. See, for example, [9, Lemma A.1] which states that there exists a constant  $c > 0$  such that

$$\|\varphi - \bar{\varphi}\|_{L^2}^2 \leq c \|\nabla \varphi\|_{L^2}^2, \quad \|\nabla \varphi\|_{L^2}^2 \leq c \|\Delta \varphi\|_{L^2}^2, \quad \|\Delta \varphi\|_{L^2}^2 \leq c \|\nabla \Delta \varphi\|_{L^2}^2$$

for all  $\varphi \in C^3(\bar{\Omega})$ . ■

Finally, we combine several of the lemmata above to obtain an estimate required in the proof of our main result.

**Lemma 2.5.** *Let  $\Omega \subset \mathbb{R}^3$  be a convex, smooth bounded domain and let  $\varepsilon > 0$  and  $M > 0$ . Then, there exists  $C > 0$  such that for every  $0 < \varphi \in C^2(\bar{\Omega})$  with  $\int_{\Omega} \varphi = M > 0$  and  $\psi \in C^3(\bar{\Omega})$  that satisfy  $\partial_\nu \varphi = \partial_\nu \psi = 0$  on  $\partial\Omega$  we have*

$$\begin{aligned} & \int_{\Omega} |(\nabla \sqrt{\varphi})^T D^2 \psi (\nabla \sqrt{\varphi})| \\ & \leq C \left( \int_{\Omega} |\nabla \sqrt{\varphi}|^2 \right)^3 + C + \varepsilon \int_{\Omega} \varphi |D^2 \log \varphi|^2 + \varepsilon \int_{\Omega} |\nabla \Delta \psi|^2. \end{aligned} \quad (2.3)$$

*Proof.* By Hölder's inequality, we have

$$\begin{aligned} & \int_{\Omega} |(\nabla \sqrt{\varphi})^T D^2 \psi (\nabla \sqrt{\varphi})| \\ & \leq \frac{1}{2} \int_{\Omega} |\nabla \sqrt{\varphi}| |D^2 \psi| \frac{|\nabla \varphi|}{\varphi^{3/4}} \varphi^{1/4} \\ & \leq \frac{1}{2} \|\nabla \sqrt{\varphi}\|_{L^2(\Omega)} \|D^2 \psi\|_{L^6(\Omega)} \left\| \frac{\nabla \varphi}{\varphi^{3/4}} \right\|_{L^4(\Omega)} \|\varphi^{1/4}\|_{L^{12}(\Omega)} \end{aligned} \quad (2.4)$$

for all  $0 < \varphi \in C^2(\bar{\Omega})$  and  $\psi \in C^3(\bar{\Omega})$ .

Since  $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$  and  $\int_{\Omega} \varphi = M$ , there is  $c_1 > 0$  such that

$$\begin{aligned} \|\varphi^{1/4}\|_{L^{12}(\Omega)} &= \|\varphi^{1/2}\|_{L^6(\Omega)}^{1/2} \leq c_1 \|\varphi^{1/2}\|_{W^{1,2}(\Omega)}^{1/2} \\ &\leq c_1 \|\nabla \varphi^{1/2}\|_{L^2(\Omega)}^{1/2} + c_1 \|\varphi^{1/2}\|_{L^2(\Omega)}^{1/2} = c_1 (\|\nabla \sqrt{\varphi}\|_{L^2(\Omega)}^{1/2} + M^{1/4}) \end{aligned}$$

for all  $0 < \varphi \in C^2(\bar{\Omega})$  with  $\int_{\Omega} \varphi = M$ . In combination with (2.4), Lemma 2.4, Winkler's inequality (2.2), the elementary estimate

$$a(\sqrt{a} + \sqrt{b}) \leq 2(a + b)\sqrt{a + b} \quad \text{for } a, b \geq 0$$

and Young's inequality, we see that by taking  $a = \|\nabla \sqrt{\varphi}\|_{L^2(\Omega)}$ ,  $b = M^{1/2}$  and with some  $c_2 > 0$  and  $C > 0$ , we have

$$\begin{aligned} &\int_{\Omega} |(\nabla \sqrt{\varphi})^T D^2 \psi \nabla(\sqrt{\varphi})| \\ &\leq c_2 \|\nabla \sqrt{\varphi}\|_{L^2(\Omega)} \|\nabla \Delta \psi\|_{L^2(\Omega)} \left( \int_{\Omega} \varphi |D^2 \log \varphi|^2 \right)^{1/4} (\|\nabla \sqrt{\varphi}\|_{L^2(\Omega)}^{1/2} + M^{1/4}) \\ &\leq 2c_2 (\|\nabla \sqrt{\varphi}\|_{L^2} + M^{1/2})^{\frac{3}{2}} \left( \int_{\Omega} \varphi |D^2 \log \varphi|^2 \right)^{1/4} \|\nabla \Delta \psi\|_{L^2} \\ &\leq C \|\nabla \sqrt{\varphi}\|_{L^2}^6 + C + \varepsilon \int_{\Omega} \varphi |D^2 \log \varphi|^2 + \varepsilon \|\nabla \Delta \psi\|_{L^2}^2 \end{aligned}$$

for all  $0 < \varphi \in C^2(\bar{\Omega})$  and  $\psi \in C^3(\bar{\Omega})$  with  $\int_{\Omega} \varphi = M$  and  $\partial_\nu \varphi = \partial_\nu \psi = 0$  on  $\partial\Omega$ . ■

### 3. Known properties of the solutions

Next, we list some known properties of the solutions to (1.1)–(1.3) constructed in [5].

**Lemma 3.1.** *Let*

$$\Omega \subset \mathbb{R}^n, n \in \mathbb{N}, \text{ be a smooth, bounded domain,} \quad (3.1)$$

and let

$$u_0, v_0 \in W^{1,p}(\Omega) \text{ for some } p > n \quad \text{with } 0 \not\equiv u_0 \geq 0 \text{ and } v_0 \geq 0 \text{ in } \Omega. \quad (3.2)$$

Then, the system (1.1)–(1.3) has a maximal unique classical solution

$$(u, v) \in C^0([0, T_{\max}); W^{1,p}(\Omega)) \cap C^\infty(\bar{\Omega} \times (0, T_{\max})), \quad (3.3)$$

and if  $T_{\max} < \infty$ , then

$$\limsup_{t \nearrow T_{\max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.$$

Moreover,  $u$  and  $v$  are positive in  $\bar{\Omega} \times (0, T_{\max})$ .

*Proof.* The existence of a  $C^\infty(\bar{\Omega} \times (0, T_{\max}))$  solution that is nonnegative has been proved in [5, Theorem 2.1]. Applying the Hopf lemma to each of the equations in (1.1) separately (in view of the regularity (3.3),  $\Delta v$  is bounded on  $\bar{\Omega} \times [\tau, T]$  for every  $0 < \tau < T < T_{\max}$ ), due to zero Neumann boundary data (1.2), we arrive at the positivity of  $u$  and  $v$  for each  $t \in (0, T_{\max})$ . ■

As noted in [5, equation (3)], integrating both equations in (1.1) immediately ensures that both solution components are uniformly in time bounded in  $L^1(\Omega)$ .

**Lemma 3.2.** *Suppose that the assumptions of Lemma 3.1 hold. Then, the solution  $(u, v)$  of (1.1)–(1.3) given by Lemma 3.1 fulfills*

$$\begin{aligned} \|u(\cdot, t)\|_{L^1(\Omega)} &= \|u_0\|_{L^1(\Omega)}, \\ \|v(\cdot, t)\|_{L^1(\Omega)} &= e^{-t}(\|v_0\|_{L^1(\Omega)} - \|u_0\|_{L^1(\Omega)}) + \|u_0\|_{L^1(\Omega)} \end{aligned}$$

for all  $t \in (0, T_{\max})$ .

Moreover, [5] has identified a Lyapunov functional, which served as the main ingredient for solving the question of global existence the two-dimensional case.

**Lemma 3.3.** *Under the assumptions of Lemma 3.1 the solution  $(u, v)$  satisfies*

$$\frac{d}{dt} \left( \int_{\Omega} u \log u + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \right) = - \left( \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2 + \int_{\Omega} \frac{|\nabla u|^2}{u} \right) \quad (3.4)$$

for all  $t \in (0, T_{\max})$ . In particular,

$$\int_0^{T_{\max}} \left( \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2 + \int_{\Omega} \frac{|\nabla u|^2}{u} \right) < \infty. \quad (3.5)$$

*Proof.* The differential inequality (3.4) is entailed in [5, Lemma 2.2], upon which (3.5) results by an integration in time as the Lyapunov functional is bounded from below. ■

## 4. The main estimate

This section contains our main contribution, a calculation of the evolution of the Fisher information along the trajectories of (1.1)–(1.3). It is related to the Bakry–Émery calculation, see [1], applied however to a system of equations.

Throughout this section, we fix a domain and initial data fulfilling (3.1) and (3.2) as well as the solution  $(u, v)$  of (1.1)–(1.3), with maximal existence time  $T_{\max}$  given by Lemma 3.1. Moreover, we denote

$$\frac{d}{dt} \left( \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2 + \int_{\Omega} \frac{|\nabla u|^2}{u} \right) =: \frac{d}{dt} I(t). \quad (4.1)$$

Our aim is to obtain an estimate of  $I$ . Notice that  $I$  is an extended version of the Fisher information. Indeed, in the case of a single heat equation, the quantity  $\int_{\Omega} \frac{|\nabla u|^2}{u}$  is called Fisher's information. The following remark explains our strategy.

**Remark 4.1.** We note that the inequality  $\dot{I} \leq cI^2$  would imply boundedness of  $I$  due to (3.5). Indeed, using Ladyzhenskaya's trick (see [11]), we would have

$$\frac{d}{dt}(I(t)e^{-\int_0^t cI(s)ds}) \leq 0.$$

Below, we formulate and prove our main contribution. It extends the calculation controlling the evolution of the Fisher information to the case of a system of equations.

**Lemma 4.2.** *For all  $t \in (0, T_{\max})$ , the estimate*

$$\begin{aligned} \dot{I}(t) &\leq -2 \int_{\Omega} u |D^2 \log u|^2 + 8 \int_{\Omega} (\nabla \sqrt{u})^T D^2 v (\nabla \sqrt{u}) \\ &\quad - 2 \int_{\Omega} |\nabla \Delta v|^2 - 4 \int_{\Omega} |\Delta v|^2 - 2 \int_{\Omega} |\nabla v|^2 + 2 \int_{\Omega} \nabla u \cdot \nabla v \end{aligned}$$

holds.

*Proof.* We notice that

$$\frac{d}{dt} \int_{\Omega} |\Delta v|^2 = 2 \int_{\Omega} \Delta v_t \Delta v \quad \text{in } (0, T_{\max}).$$

From the second equation in (1.1), we can substitute  $\Delta v = v_t + v - u$  (equivalently, we can take the inner product of the second equation in (1.1) with  $\Delta v_t$ ) to get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\Delta v|^2 &= 2 \int_{\Omega} \Delta v_t (v_t + v - u) = -2 \int_{\Omega} \nabla v_t \cdot (\nabla v_t + \nabla v - \nabla u) \\ &= -2 \int_{\Omega} |\nabla v_t|^2 - \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + 2 \int_{\Omega} \nabla v_t \cdot \nabla u \quad \text{in } (0, T_{\max}). \end{aligned} \quad (4.2)$$

From (4.2), we get

$$\frac{d}{dt} \left( \int_{\Omega} |\Delta v|^2 + |\nabla v|^2 \right) = -2 \int_{\Omega} |\nabla v_t|^2 + 2 \int_{\Omega} \nabla v_t \cdot \nabla u \quad \text{in } (0, T_{\max}), \quad (4.3)$$

and from (4.1) and (4.3), we obtain

$$\dot{I}(t) = -2 \int_{\Omega} |\nabla v_t|^2 + 2 \int_{\Omega} \nabla v_t \cdot \nabla u + \frac{d}{dt} \int_{\Omega} \frac{|\nabla u|^2}{u} \quad \text{for all } t \in (0, T_{\max}). \quad (4.4)$$

Next, we compute the last term on the right-hand side,

$$\frac{d}{dt} \int_{\Omega} \frac{|\nabla u|^2}{u} = 4 \frac{d}{dt} \int_{\Omega} |\nabla \sqrt{u}|^2 = 8 \int_{\Omega} \nabla \varrho_t \cdot \nabla \varrho \quad \text{in } (0, T_{\max}), \quad (4.5)$$

where we applied the substitution  $\varrho := \sqrt{u}$ . From the first equation in (1.1) we have

$$\varrho_t = \frac{u_t}{2\sqrt{u}} = \frac{\Delta u + \nabla u \cdot \nabla v + u \Delta v}{2\sqrt{u}} = \frac{\Delta u}{2\sqrt{u}} + \nabla \varrho \cdot \nabla v + \frac{1}{2} \varrho \Delta v, \quad (4.6)$$

where

$$\Delta \varrho = \operatorname{div}(\nabla \varrho) = \operatorname{div}\left(\frac{\nabla u}{2\sqrt{u}}\right) = \frac{\Delta u}{2\sqrt{u}} - \frac{|\nabla u|^2}{4u^{3/2}} = \frac{\Delta u}{2\sqrt{u}} - \frac{|\nabla \varrho|^2}{\varrho} \quad (4.7)$$

in  $\Omega \times (0, T_{\max})$ . So, by plugging (4.7) into (4.6), we find that

$$\varrho_t = \Delta \varrho + \frac{|\nabla \varrho|^2}{\varrho} + \nabla \varrho \cdot \nabla v + \frac{1}{2} \varrho \Delta v \quad \text{in } \Omega \times (0, T_{\max}).$$

Hence, (4.5) becomes

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{|\nabla u|^2}{u} &= 8 \int_{\Omega} \nabla \varrho \cdot \nabla \left( \Delta \varrho + \frac{|\nabla \varrho|^2}{\varrho} + \nabla \varrho \cdot \nabla v + \frac{1}{2} \varrho \Delta v \right) \\ &= 8 \left[ \int_{\Omega} \nabla \varrho \cdot \nabla (\Delta \varrho) + \int_{\Omega} 2 \frac{(\nabla \varrho)^T D^2 \varrho (\nabla \varrho)}{\varrho} - \int_{\Omega} \frac{|\nabla \varrho|^4}{\varrho^2} \right] \\ &\quad + 8 \left[ \int_{\Omega} (\nabla \varrho)^T D^2 \varrho (\nabla v) + \int_{\Omega} (\nabla \varrho)^T D^2 v (\nabla \varrho) \right] \\ &\quad + 4 \int_{\Omega} \nabla (\varrho \Delta v) \cdot \nabla \varrho \quad \text{in } (0, T_{\max}). \end{aligned} \quad (4.8)$$

Due to the Bochner formula (2.1),

$$\nabla \varrho \cdot \nabla (\Delta \varrho) = -|D^2 \varrho|^2 + \frac{1}{2} \Delta (|\nabla \varrho|^2) \quad \text{in } \Omega \times (0, T_{\max}),$$

we get

$$\begin{aligned} \int_{\Omega} \nabla \varrho \cdot \nabla (\Delta \varrho) + \int_{\Omega} 2 \frac{(\nabla \varrho)^T D^2 \varrho (\nabla \varrho)}{\varrho} - \int_{\Omega} \frac{|\nabla \varrho|^4}{\varrho^2} \\ = - \int_{\Omega} \left| D^2 \varrho - \frac{\nabla \varrho \otimes \nabla \varrho}{\varrho} \right|^2 + \frac{1}{2} \int_{\Omega} \Delta (|\nabla \varrho|^2) \quad \text{in } (0, T_{\max}). \end{aligned}$$

We note that the boundary condition

$$\frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = 0$$

implies that

$$\frac{\partial \varrho}{\partial \nu} \Big|_{\partial \Omega} = \frac{\frac{\partial u}{\partial \nu}}{2\sqrt{u}} \Big|_{\partial \Omega} = 0$$

so that an integration by parts and an application of Lemma 2.2, which is possible thanks to the convexity of the domain  $\Omega$ , yield

$$\int_{\Omega} \Delta (|\nabla \varrho|^2) = \int_{\partial \Omega} \frac{\partial (|\nabla \varrho|^2)}{\partial \nu} \leq 0 \quad \text{in } (0, T_{\max}).$$



Plugging the above into (4.8), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{|\nabla u|^2}{u} &\leq -2 \int_{\Omega} u |D^2 \log u|^2 \\ &\quad + 8 \left[ \int_{\Omega} (\nabla \varrho)^T D^2 \varrho (\nabla v) + \int_{\Omega} (\nabla \varrho)^T D^2 v (\nabla \varrho) \right] \\ &\quad + 4 \int_{\Omega} \nabla(\varrho \Delta v) \cdot \nabla \varrho \quad \text{in } (0, T_{\max}), \end{aligned} \quad (4.9)$$

where we also used the relations

$$\int_{\Omega} \left| D^2 \varrho - \frac{\nabla \varrho \otimes \nabla \varrho}{\varrho} \right|^2 = \int_{\Omega} \varrho^2 |D^2 \log \varrho|^2 = \frac{1}{4} \int_{\Omega} u |D^2 \log u|^2$$

in the first term on the right-hand side.

We now focus on the last term in (4.9),

$$4 \int_{\Omega} \nabla(\varrho \Delta v) \cdot \nabla \varrho = 4 \int_{\Omega} |\nabla \varrho|^2 \Delta v + 4 \int_{\Omega} \varrho \nabla \varrho \cdot \nabla(\Delta v). \quad (4.10)$$

Integration by parts yields

$$4 \int_{\Omega} |\nabla \varrho|^2 \Delta v = -4 \int_{\Omega} \nabla(|\nabla \varrho|^2) \cdot \nabla v = -8 \int_{\Omega} (\nabla \varrho)^T D^2 \varrho (\nabla v) \quad (4.11)$$

in  $(0, T_{\max})$ . For the second term in (4.10) we substitute  $\Delta v = v_t + v - u$  from the second equation in (1.1) to obtain

$$\begin{aligned} 4 \int_{\Omega} \varrho \nabla \varrho \cdot \nabla(\Delta v) &= 2 \int_{\Omega} \nabla \varrho^2 \cdot \nabla(v_t + v - u) \\ &= 2 \int_{\Omega} \nabla u \cdot \nabla v_t + 2 \int_{\Omega} \nabla u \cdot \nabla v - 2 \int_{\Omega} |\nabla u|^2 \quad \text{in } (0, T_{\max}). \end{aligned} \quad (4.12)$$

Inserting (4.11) and (4.12) in (4.9) gives

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{|\nabla u|^2}{u} &\leq -2 \int_{\Omega} u |D^2 \log u|^2 + 8 \int_{\Omega} (\nabla \varrho)^T D^2 v (\nabla \varrho) \\ &\quad + 2 \int_{\Omega} \nabla u \cdot \nabla v_t + 2 \int_{\Omega} \nabla u \cdot \nabla v - 2 \int_{\Omega} |\nabla u|^2 \quad \text{in } (0, T_{\max}). \end{aligned}$$

Therefore, going back to (4.4), we have

$$\begin{aligned} \dot{I}(t) &\leq -2 \int_{\Omega} u |D^2 \log u|^2 + 8 \int_{\Omega} (\nabla \sqrt{u})^T D^2 v (\nabla \sqrt{u}) \\ &\quad - 2 \int_{\Omega} |\nabla v_t|^2 + 4 \int_{\Omega} \nabla v_t \cdot \nabla u + 2 \int_{\Omega} \nabla u \cdot \nabla v - 2 \int_{\Omega} |\nabla u|^2 \end{aligned}$$

for all  $t \in (0, T_{\max})$ . Since

$$\begin{aligned} & -2 \int_{\Omega} |\nabla v_t|^2 + 4 \int_{\Omega} \nabla v_t \cdot \nabla u - 2 \int_{\Omega} |\nabla u|^2 \\ & = -2 \int_{\Omega} |\nabla(v_t - u)|^2 = -2 \int_{\Omega} |\nabla(\Delta v - v)|^2 \\ & = -2 \int_{\Omega} |\nabla \Delta v|^2 - 4 \int_{\Omega} |\Delta v|^2 - 2 \int_{\Omega} |\nabla v|^2 \quad \text{in } (0, T_{\max}), \end{aligned}$$

we obtain the desired estimate.  $\blacksquare$

Next, we simplify the previous differential inequality, which will allow us to argue in a more straightforward manner in the sequel.

**Lemma 4.3.** *Throughout  $(0, T_{\max})$ , it holds that*

$$\begin{aligned} & \frac{d}{dt} \left( 4 \int_{\Omega} |\nabla \sqrt{u}|^2 + \int_{\Omega} |\Delta v|^2 \right) \\ & \leq -2 \int_{\Omega} u |D^2 \log u|^2 - 2 \int_{\Omega} |\nabla \Delta v|^2 - 2 \int_{\Omega} |\Delta v|^2 \\ & \quad + 8 \int_{\Omega} (\nabla \sqrt{u})^T D^2 v \nabla \sqrt{u}. \end{aligned}$$

*Proof.* This follows immediately from Lemma 4.2 and the fact that

$$\frac{d}{dt} \int_{\Omega} |\nabla v|^2 = -2 \int_{\Omega} \Delta v v_t = -2 \int_{\Omega} |\Delta v|^2 - 2 \int_{\Omega} |\nabla v|^2 + 2 \int_{\Omega} \nabla v \cdot \nabla v$$

in  $(0, T_{\max})$ .  $\blacksquare$

## 5. Proof of the main theorem

We are now in a position to utilise our calculation from Lemma 4.2 and complete the proof of the announced result. As in the previous section, we fix a domain  $\Omega$  and initial data  $u_0, v_0$  satisfying (3.1) and (3.2) as well as the solution  $(u, v)$  of (1.1)–(1.3) given by Lemma 3.1. Moreover, as the solution is unique by Lemma 3.1,  $T_{\max}$  is infinite if and only if the solution with initial data  $(u(\cdot, t_0), v(\cdot, t_0))$  for some  $t_0 \in (0, T_{\max})$  exists globally. Thus, by switching to the solution with these initial data and recalling (3.3), we may assume  $u, v \in C^\infty(\bar{\Omega} \times [0, T_{\max}))$ .

The following lemma is the first step in a bootstrapping procedure yielding the required regularity of the solution.

**Lemma 5.1.** *Suppose that  $n = 3$  and that  $\Omega$  is convex. Let  $T \in (0, T_{\max}] \cap (0, \infty)$  and suppose that  $\nabla \sqrt{u} \in L^4(0, T; L^2(\Omega))$ . Then, there is  $C > 0$  such that*

$$\int_{\Omega} u^3(\cdot, t) \leq C \quad \text{for all } t \in (0, T).$$

*Proof.* We define

$$J(t) := 4 \int_{\Omega} |\nabla \sqrt{u}|^2 + \int_{\Omega} |\Delta v|^2.$$

Taking  $\varphi = u$ ,  $\psi = v$  and  $\varepsilon = \frac{1}{4}$  in Lemma 2.5 and making use of Lemma 4.3, we arrive at

$$\dot{J}(t) \leq c_1 \left( \int_{\Omega} |\nabla \sqrt{u}|^2 \right)^3 + c_1 \leq c_1 \left( \int_{\Omega} |\nabla \sqrt{u}|^2 \right)^2 J(t) + c_1$$

in  $(0, T)$  for some  $c_1 > 0$ . Thus, with

$$K(t) := c_1 \int_0^t \left( \int_{\Omega} |\nabla \sqrt{u}|^2 \right)^2, \quad t \in (0, T),$$

we have

$$J(t) \leq e^{K(t)} J(0) + c_1 \int_0^t e^{K(t-s)} ds \leq e^{K(T)} J(0) + c_1 T e^{K(T)}$$

for all  $t \in (0, T)$ . Since  $K(T) < \infty$  by assumption, we obtain boundedness of the quantity  $\sup_{t \in (0, T)} \|\nabla \sqrt{u(\cdot, t)}\|_{L^2(\Omega)}$ , which, in conjunction with Lemma 3.2, implies the desired estimate as  $W^{1,2}(\Omega)$  embeds continuously into  $L^6(\Omega)$ . ■

Next, we show the higher regularity of the obtained solution.

**Lemma 5.2.** *Under the assumptions of Lemma 5.1 there is  $C > 0$  such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T).$$

*Proof.* We fix  $3 < r < q < \infty$ . Making use of well-known semigroup estimates (cf. [19, Lemma 1.3 (ii) and (iii)]), we obtain

$$\begin{aligned} & \|\nabla v(\cdot, t)\|_{L^q(\Omega)} \\ & \leq \|\nabla e^{t(\Delta-1)} v_0\|_{L^q(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)} u(\cdot, s)\|_{L^q(\Omega)} ds \\ & \leq c_1 e^{-t} \|\nabla v_0\|_{L^q(\Omega)} + c_2 \int_0^t \left( 1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{3} - \frac{1}{q})} \right) e^{-(t-s)} \|u(\cdot, s)\|_{L^3(\Omega)} ds \\ & \leq c_1 \|\nabla v_0\|_{L^q(\Omega)} + c_2 \sup_{s \in (0, T)} \|u(\cdot, s)\|_{L^3(\Omega)} \int_0^T \left( 1 + s^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{3} - \frac{1}{q})} \right) ds \end{aligned}$$

for all  $t \in (0, T)$  and some  $c_1, c_2 > 0$ . Since  $-\frac{1}{2} - \frac{3}{2}(\frac{1}{3} - \frac{1}{q}) > -1$  and recalling Lemma 5.1, we conclude that there is  $c_3 > 0$  such that

$$\|\nabla v(\cdot, t)\|_{L^q(\Omega)} \leq c_3 \quad \text{for all } t \in (0, T).$$

Since  $q > 3$ ,  $W^{1,q}(\Omega)$  embeds continuously into  $L^\infty(\Omega)$ , and so, the above estimate in conjunction with Lemma 3.2 implies that  $\sup_{t \in (0, T)} \|v(\cdot, t)\|_{L^\infty(\Omega)}$  is finite as well.

Relying on the maximum principle and again on well-known semigroup estimates (cf. [19, Lemma 1.3 (iv)]), we further estimate

$$\begin{aligned}
& \|u(\cdot, t)\|_{L^\infty(\Omega)} \\
& \leq \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}\nabla \cdot (u\nabla v)(\cdot, s) \, ds\|_{L^\infty(\Omega)} \\
& \leq \|u_0\|_{L^\infty(\Omega)} + c_4 \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{r}-\frac{1}{\infty})}) \|(u\nabla v)(\cdot, s)\|_{L^r(\Omega)} \, ds \\
& \leq \|u_0\|_{L^\infty(\Omega)} + c_4 \sup_{s \in (0, t)} \|(u\nabla v)(\cdot, s)\|_{L^r(\Omega)} \int_0^t (1 + s^{-\frac{1}{2}-\frac{3}{2r}}) \, ds
\end{aligned}$$

for all  $t \in (0, T)$  and some  $c_4 > 0$ . Since with  $\lambda := \frac{rq}{q-r}$  and  $\theta := \frac{\lambda-1}{\lambda} \in (0, 1)$ , we have

$$\begin{aligned}
\|(u\nabla v)(\cdot, s)\|_{L^r(\Omega)} & \leq \|u(\cdot, s)\|_{L^\lambda(\Omega)} \|\nabla v(\cdot, s)\|_{L^q(\Omega)} \\
& \leq \|u(\cdot, s)\|_{L^\infty(\Omega)}^\theta \|u(\cdot, s)\|_{L^1(\Omega)}^{1-\theta} \|\nabla v(\cdot, s)\|_{L^q(\Omega)}
\end{aligned}$$

for all  $s \in (0, T)$ , we see that there is  $c_5 > 0$  such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_5 + c_5 \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^\infty(\Omega)}^\theta$$

for all  $t \in (0, T)$ . For  $t \in (0, T)$ , we set  $A(t) := \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^\infty(\Omega)}$ . Since  $\theta \in (0, 1)$ , by means of Young's inequality we obtain  $A(t) \leq c_5 + c_5 A^\theta(t) \leq c_6 + \frac{1}{2}A(t)$  for some  $c_6 > 0$  (not depending on  $t$ ) and hence also  $A(t) \leq 2c_6$  for all  $t \in (0, T)$ . Taking  $t \nearrow T$  shows that also  $u$  remains bounded in  $\Omega \times (0, T)$ . ■

We are now in position to prove Theorem 1.1.

*Proof of Theorem 1.1.* Suppose that  $T_{\max} < \infty$ , then Lemma 5.2 asserts boundedness of  $u$  and  $v$  in  $\Omega \times (0, T_{\max})$ . However, this contradicts the extensibility criterion in Lemma 3.1. ■

## 6. Conclusion

On the one hand, we obtained a condition which guarantees global existence of solutions for the chemorepulsion system in three-dimensional space. This result can be improved, as suggested to us by M. Winkler and an anonymous referee, see Appendix B. On the other hand, we notice from our computations that the concavity of the function  $v$  would greatly simplify our argument. It would lead to boundedness of the function  $I(t)$ , and hence to the global existence of solutions, as shown in this paper. Indeed, we see from (2.3) that if the function  $v$  is concave, i.e., its Hessian is negative-semidefinite,

$$x^T D^2 v x \leq 0 \quad \text{for every } x \in \mathbb{R}^n,$$

we have the following differential inequality for  $I(t)$ :

$$\dot{I}(t) \leq 0.$$

From equation (1.1), we see that  $v$  is not far from being concave. Taking a simplified version of the equation for  $v$  [assuming  $v_t = 0$  and neglecting  $v$  on the right-hand side of the second equation in (1.1)], we have

$$\Delta v = -u \leq 0,$$

which would also hold if the Hessian of  $v$  was negative-semidefinite.

Verifying the concavity of a solution of a parabolic boundary value problem posed on a convex domain has been studied before. In the context of one parabolic equation of certain type, some positive results can be found in [10], for example. However, we are not aware of any result in this direction for systems of equations.

## A. A new inequality

As a byproduct of our arguments, we discovered a differential inequality relating the second norm of the Hessian of the square-root of a positive function with the dissipation of the Fisher information along the heat flow. Due to the fact that both of these quantities appear in the calculation of the evolution of the Fisher information along the heat flow, the following inequality is interesting in its own right and may have further applications. In particular, it was used by the first author in [3], where global-in-time regular unique solution to a 1D thermoelasticity system is obtained. Next, a very recent application of the inequality (A.1) led to the interesting result in the theory of 1D combustion, see [13].

**Lemma A.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a smooth bounded domain. For every positive function  $u \in C^2(\bar{\Omega})$  with the boundary condition  $\frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = 0$ , we have*

$$\int_{\Omega} |D^2 \sqrt{u}|^2 \leq C \int_{\Omega} u |D^2 \log u|^2, \quad (\text{A.1})$$

where  $C = 1 + \frac{\sqrt{n}}{2} + \frac{n}{8}$ .

**Remark A.2.** We note that the inequality (A.1) does not hold pointwise, i.e., there is no constant  $C > 0$  such that, for every positive  $u \in C^2(\bar{\Omega})$ ,

$$|D^2 \sqrt{u}|^2 \leq C u |D^2 \log u|^2 \quad \text{in } \bar{\Omega}.$$

*Proof of Lemma A.1.* We first note that

$$[D^2 \sqrt{u}]_{ij}^2 = \left( \frac{\partial_{x_i x_j} u}{2u^{1/2}} - \frac{\partial_{x_i} u \partial_{x_j} u}{4u^{3/2}} \right)^2 = \frac{1}{4} \left( \frac{\partial_{x_i x_j} u}{u^{1/2}} - \frac{1}{2} \frac{\partial_{x_i} u \partial_{x_j} u}{u^{3/2}} \right)^2$$

and

$$u[D^2 \log u]_{ij}^2 = u \left( \frac{\partial_{x_i x_j} u}{u} - \frac{\partial_{x_i} u \partial_{x_j} u}{u^2} \right)^2 = \left( \frac{\partial_{x_i x_j} u}{u^{1/2}} - \frac{\partial_{x_i} u \partial_{x_j} u}{u^{3/2}} \right)^2$$

in  $\bar{\Omega}$ . Using the simple fact that  $(a + b)^2 \leq 2(a^2 + b^2)$ , we get

$$\begin{aligned} [D^2 \sqrt{u}]_{ij}^2 &= \frac{1}{4} \left( \frac{\partial_{x_i x_j} u}{u^{1/2}} - \frac{\partial_{x_i} u \partial_{x_j} u}{u^{3/2}} + \frac{1}{2} \frac{\partial_{x_i} u \partial_{x_j} u}{u^{3/2}} \right)^2 \\ &\leq \frac{1}{4} \left[ 2 \left( \frac{\partial_{x_i x_j} u}{u^{1/2}} - \frac{\partial_{x_i} u \partial_{x_j} u}{u^{3/2}} \right)^2 + \frac{1}{2} \left( \frac{\partial_{x_i} u \partial_{x_j} u}{u^{3/2}} \right)^2 \right] \\ &= \frac{1}{2} u [D^2 \log u]_{ij}^2 + \frac{1}{8} \left( \frac{\partial_{x_i} u \partial_{x_j} u}{u^{3/2}} \right)^2. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} |D^2 \sqrt{u}|^2 &= \sum_{i,j=1}^n [D^2 \sqrt{u}]_{ij}^2 \leq \sum_{i,j=1}^n \left[ \frac{1}{2} u [D^2 \log u]_{ij}^2 + \frac{1}{8} \left( \frac{\partial_{x_i} u \partial_{x_j} u}{u^{3/2}} \right)^2 \right] \\ &= \frac{1}{2} u |D^2 \log u|^2 + \frac{1}{8} \frac{|\nabla u|^4}{u^3} \quad \text{in } \bar{\Omega}. \end{aligned} \tag{A.2}$$

Applying Lemma 2.3 to (A.2), we get

$$\int_{\Omega} |D^2 \sqrt{u}|^2 \leq \left( \frac{1}{2} + \frac{1}{8} (2 + \sqrt{n})^2 \right) \int_{\Omega} u |D^2 \log u|^2,$$

as required. ■

## B. Alternative proof

This section is devoted to the presentation of the result communicated to us by M. Winkler as well as one of the anonymous referees. It gives an alternative conditional result, extending our Theorem 1.1. On the one hand, no convexity of the domain is required, on the other hand, only a zero-order estimate of  $u$  is required.

The proof is based on the well-known fact that in dimension 3, bounding the  $L^\infty((0, T_{\max}); L^p(\Omega))$  norm of  $u$  for any  $p > \frac{3}{2}$  allows prolongation of a solution to (1.1), see, for instance, [2].

**Lemma B.1.** *Let  $\Omega \subset \mathbb{R}^3$  be a smooth, bounded domain and let  $u_0, v_0$  be as in (3.2). Suppose that the maximal existence time  $T_{\max}$  of the solution  $(u, v)$  to (1.1)–(1.3) given by Lemma 3.1 is finite. Then*

$$\int_0^{T_{\max}} \|u(t)\|_{L^3(\Omega)}^2 dt = \infty. \tag{B.1}$$

In particular,

$$\int_0^{T_{\max}} \|\nabla \sqrt{u(t)}\|_{L^2(\Omega)}^4 dt = \infty.$$

*Proof.* In view of the Gagliardo–Nirenberg inequality and conservation of mass, we only need to verify (B.1). To this end, we suppose that (B.1) does not hold, which allows us to first apply maximal Sobolev regularity theory to the second equation in (1.1) to see that

$$\int_{\frac{T_{\max}}{2}}^{T_{\max}} \|\Delta v(t)\|_{L^3(\Omega)}^2 dt < \infty. \quad (\text{B.2})$$

Then, we use the first equation in (1.1) along with an integration by parts to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 + \int_{\Omega} u^2 &\leq \frac{1}{2} \int_{\Omega} u^2 \Delta v + \int_{\Omega} u^2 \\ &\leq \frac{1}{2} \|u\|_{L^3(\Omega)}^2 \|\Delta v\|_{L^3(\Omega)} + \int_{\Omega} u^2 \end{aligned}$$

in  $(0, \infty)$ . Next, applications of the Gagliardo–Nirenberg interpolation and Young’s inequality give for all  $t \in (0, T_{\max})$ ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 + \int_{\Omega} u^2 \\ \leq \int_{\Omega} (|\nabla u|^2 + u^2) + c_1 (\|\Delta v\|_{L^3(\Omega)}^2 + 1) \|u\|_{L^2(\Omega)}^2 \end{aligned}$$

for some constant  $c_1 > 0$ .

Writing

$$y(t) := \int_{\Omega} u^2(\cdot, t) \quad \text{and} \quad h(t) := 2c_1 (\|\Delta v(\cdot, t)\|_{L^3(\Omega)}^2 + 1) \quad \text{for } t \in (0, T_{\max}),$$

we arrive at a differential inequality

$$\dot{y}(t) \leq h(t)y(t) \quad \text{for all } t \in (0, T_{\max}). \quad (\text{B.3})$$

Since  $h \in L^1(\frac{T_{\max}}{2}, T_{\max})$  by (B.2), integration in time of (B.3) shows boundedness of  $u$  in  $L^\infty((0, T_{\max}); L^2(\Omega))$ , contradicting the finiteness of  $T_{\max}$ . ■

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