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Monotonicity and phase transition for the VRJP and the ERRW

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Abstract. The vertex-reinforced jump process (VRJP), introduced by Davis and Volkov, is a continuous-time process that tends to come back to already visited vertices. It is closely linked to the edge-reinforced random walk (ERRW) introduced by Coppersmith and Diaconis in 1986 which is more likely to traverse edges it has already traversed. On $\mathbf{Z}^d$ for $d \geq 3$, both models were shown to be recurrent for small enough initial weights and transient for large enough initial weights. We show through a coupling of VRJPs for different weights that the VRJP (and the ERRW) exhibits some monotonicity. In particular, we show that increasing the initial weights of the VRJP and the ERRW makes them more transient, which means that the recurrence/transience phase transition is necessarily unique. Furthermore, by making the weights go to infinity, we show that the recurrence of the ERRW and the VRJP is implied by the recurrence of a random walk in a deterministic electrical network.

Keywords. Vertex reinforced jump process, edge reinforced random walk, monotonicity, coupling, phase transition

1. Introduction and results

1.1. Introduction

The edge-reinforced random walk (ERRW) was first introduced by Coppersmith and Diaconis in 1986 [2]. In this model, the more often the walk traverses an edge, the likelier it is to traverse it again in the future. This model was shown to be a random walk in random reversible environments [4, 10]. This representation led to several results on this model: first, recurrence and transience on trees depending on the reinforcement [12], then recurrence on the ladder [9] and $\mathbf{Z} \times G$ [14] for large enough reinforcement and on a modification of $\mathbf{Z}^2$ for large enough reinforcement [11]. It was then shown by two different techniques that the ERRW on $\mathbf{Z}^d$ is recurrent for large enough reinforcement (in [1] by Angel, Crawford and Kozma and in [16] by Sabot and Tarrès). The technique used in

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[16] was based on a link between the ERRW, the vertex-reinforced jump process (VRJP, introduced by Davis and Volkov [3]) and the super-symmetric hyperbolic sigma model (introduced in the context of Anderson localization in [7, 19] by Zirnbauer, Disertori and Spencer). This relation led to several other results for both the ERRW and the VRJP: the transience and a CLT in dimension 3 and higher for small enough reinforcements [5, 16, 18], a 0-1 law for recurrence on $\mathbf{Z}^d$ [18] and recurrence in dimension 2 [11, 15, 18]. This means that on the one hand, for $d \in \{1, 2\}$ the ERRW and the VRJP are recurrent for any reinforcement. On the other hand, for $d \geq 3$ both the ERRW and the VRJP are recurrent for large enough reinforcement and transient for small enough reinforcements. We know that in-between, the VRJP and the ERRW are recurrent or transient but it has not been known whether there is a unique phase transition.

In this paper we show that we can couple VRJPs for different weights (more precisely, we couple the β-fields associated to the VRJPs that were introduced in [17]). This coupling leads to a monotonicity for the VRJP similar to the Rayleigh monotonicity for electrical networks. This gives us the uniqueness of the recurrence/transience phase transition for the VRJP and the ERRW in dimension 3 and higher. This monotonicity can also be used to show that the VRJP and the ERRW with constant weights are recurrent on recurrent graphs by seeing random walks in electrical networks as VRJPs with infinite weights.

1.2. Statement of the results

Let $\mathcal{G} = (V, E)$ be a locally finite, non-directed graph. For simplicity, when $\mathcal{G}$ is finite with n vertices we will identify the set of vertices V with the set of integers from 1 to n that we will write as $[\![1, n]\!]$. To every edge $e \in E$ we associate a positive weight a_e. Let $x_0 \in V$ be a vertex of $\mathcal{G}$. The *edge-reinforced random walk* Y starting from x_0 is the random process with values in V defined by

$$Y_0 = x_0 \quad \text{a.s,}$$
$$\mathbb{P}(Y_{n+1} = y \mid Y_0, \dots, Y_n) = 1_{y \sim Y_n} \frac{a_{\{Y_n, y\}} + Z_n(\{Y_n, y\})}{\sum_{z \sim Y_n} [a_{\{Y_n, z\}} + Z_n(\{Y_n, z\})]},$$

where the random variables $(Z_n)_{n \in \mathbf{N}}$ are defined by

$$\forall e \in E, \quad Z_n(e) = \sum_{i=0}^{n-1} 1_{\{Y_i, Y_{i+1}\} = e}.$$

This means that at each step, the process chooses an edge e to traverse with a probability proportional to its initial weight a_e plus the number of times it has already traversed that undirected edge $Z_n(e)$. This means that the ERRW wants to go back through the edges it has visited often in the past. The smaller the initial weights, the less the process is likely to visit new edges and therefore the more it wants to go back through the already visited edges. Because of this, if the graph is $\mathbf{Z}^d$, this process can exhibit different behaviours depending on the initial weights. For small enough initial weights it is recurrent.

Theorem ([1, Theorem 1] and [16, Corollary 2]). *For any K there exists $a_0 > 0$ such that if $\mathcal{G}$ is a graph with all degrees bounded by K, then the edge-reinforced random walk on $\mathcal{G}$ with initial weights $a \in (0, a_0)$ is a mixture of positive recurrent Markov chains.*

For large enough initial weights, the process is transient.

Theorem ([5, Theorem 1]). *On $\mathbf{Z}^d$, $d \geq 3$, there exists $a_c(d) > 0$ such that, if $a_e > a_c(d)$ for all $e \in E$, then the ERRW with initial weights $(a_e)_{e\in E}$ is transient a.s.*

Note that the previous two theorems use results or ideas of [6,7]. The ERRW is linked to another random process, the *vertex-reinforced jump process* (VRJP). The VRJP on a locally finite graph $\mathcal{G} = (V, E)$ with positive weights $(W_e)_{e\in E}$ is the continuous-time process $(\widetilde{Y}_t)_{t\in\mathbf{R}^+}$ that starts at some vertex x_0 and that, conditionally on the past at time t, if $\widetilde{Y}_t = x$, jumps to a neighbour y of x at rate

$$W_{\{x,y\}}(1 + \ell_y(t)),$$

where $\ell_y(t)$ is the time already spent at y until time t:

$$\ell_y(t) := \int_0^t 1_{\widetilde{Y}_s = y}\, \mathrm{d}s.$$

Similarly to the ERRW, the VRJP wants to go back to places it has visited before. The VRJP is attracted to vertices it has already visited while the ERRW is attracted to edges it has already traversed but the behaviours are quite similar. The larger the initial weights, the less time it takes the process to jump and the less it feels the attraction to previously visited vertices. And just like for the ERRW, on $\mathbf{Z}^d$ the VRJP can exhibit different behaviours depending on the weights. The similarities in behaviour come from the following link between the two processes.

Theorem ([16, Theorem 1]). *The ERRW with weights $(a_e)_{e\in E}$ is equal in law to the discrete time process associated with a VRJP with random independent weights $W_e \sim$ Gamma$(a_e, 1)$.*

In this article we show, through a coupling, that the VRJP has a property similar to Rayleigh's monotonicity for electrical networks. This leads to several results for recurrence and transience. First, we show that the probability that the walk is recurrent is decreasing in the parameters of the VRJP. This is a corollary of our main theorem that will be stated at the end because it is technical and needs a few additional definitions.

Theorem 1. *Let $\mathcal{G} = (V, E)$ be an infinite, non-directed, connected graph without loops or multiple edges and $0 \in V$ a vertex in this graph. Let $(W_e^-)_{e\in E}$ and $(W_e^+)_{e\in E}$ be two families of positive weights such that for any $e \in E$, $0 < W_e^- \leq W_e^+$. The probability that the VRJP with weights W^- is recurrent is greater than or equal to the probability that the VRJP with weights W^+ is recurrent.*

It was already proved in [18] that the VRJP on $\mathbf{Z}^d$ with constant weights or weights invariant by translation is recurrent with probability 0 or 1. Together with our theorem this means that the VRJP and the ERRW are recurrent for small enough weights and then transient for larger weights. This means that the VRJP and the ERRW exhibit a phase transition for recurrence/transience on $\mathbf{Z}^d$ when all the edges have the same weight.

Theorem 2. *For $d \geq 3$ there exists $w_d \in (0,\infty)$ such that for any $w \in (0,\infty)$, the VRJP on $\mathbf{Z}^d$ with initial constant weight w ($W_e = w$ for all edges e) is recurrent if $w < w_d$ and transient if $w > w_d$.*

Theorem 3. *For $d \geq 3$ there exists $a_d \in (0,\infty)$ such that for any $a \in (0,\infty)$, the ERRW on $\mathbf{Z}^d$ with constant initial weight a ($a_e = a$ for all edges e) is recurrent if $a < a_d$ and transient if $a > a_d$.*

The link between the VRJP and electrical networks goes beyond this monotonicity property. The following theorem shows that recurrence of electrical networks, VRJP and ERRW are also closely linked.

Theorem 4. *Let $\mathcal{G} = (V,E)$ be an infinite, locally finite graph and $x_0 \in V$ a vertex. Let $(W_e)_{e\in E}$ be a family of positive weights. If the random walk on $\mathcal{G}$ starting at x_0 with deterministic conductances $(c_e)_{e\in E} = (W_e)_{e\in E}$ is recurrent then so are the ERRW and the VRJP starting at x_0 and with initial weights $(W_e)_{e\in E}$.*

To state our main technical theorem, we need some extra definitions and results related to the VRJP and the ERRW. First we need to recall the β-field (introduced in [17] by Tarrès, Sabot and Zeng), a random vector defined for weighted graphs. In the following, $M_{n,m}(\mathbf{R})$ will refer to real $n\times m$ matrices and $M_n(\mathbf{R}) := M_{n,n}(\mathbf{R})$.

Definition 1 (β-field). Let n be an integer, $(\eta_i)_{1\leq i\leq n}$ a family of non-negative parameters and $W \in M_n(\mathbf{R})$ a symmetric matrix with non-negative coefficients. Let $1_n \in \mathbf{R}^n$ be the vector $(1,\dots,1)$ and for any matrix X, let ${}^tX$ be its transpose. The measure $\nu_n^{W,\eta}$ on $(0,\infty)^n$ is defined by the following density:

$$\nu_n^{W,\eta}(\,\mathrm{d}\beta_1\dots\mathrm{d}\beta_n) := \left(\frac{2}{\pi}\right)^{n/2} e^{-\frac{1}{2}(1_n H_\beta {}^t1_n + \eta H_\beta^{-1}\,{}^t\eta - 2\sum_{1\leq i\leq n}\eta_i)} \frac{1}{\sqrt{\det(H_\beta)}} 1_{H_\beta>0}\,\mathrm{d}\beta_1\dots\mathrm{d}\beta_n,$$

where for all $i,j \in [\![1,n]\!]$,

$$H_\beta(i,i) = 2\beta_i - W(i,i), \quad H_\beta(i,j) = -W(i,j) \quad \text{if } i \neq j,$$

and $H_\beta > 0$ means that H_β is positive definite.

This family of measures is actually a family of probability measures, as was first proved in [17, Theorem 1] for $\eta = 0$ and then in [18, Lemma 4] for general η.

We write $\tilde{\nu}_n^{W,\eta}$ for the distribution of H_β when $(\beta_i)_{1\leq i\leq n}$ is distributed according to $\nu_n^{W,\eta}$.

The link between the β-field and the VRJP is not obvious at first glance. It was shown in [17] (based on previous results in [16]) that the VRJP with weights W can be seen as a random walk in a random electrical network whose conductances are given by the weights W and the β-field. More precisely:

Theorem ([17, Theorem 3]). *Let $\mathcal{G} = (V, E)$ be a non-directed finite graph and $(W_e)_{e\in E}$ weights on the edges. We will identify V with the set of integers $[\![1, |V|]\!]$. Let H_β be distributed according to $\widetilde{\nu}_{|V|}^{W,0}$ and let G_β be the inverse of H_β. For any $x_0 \in V$ the discrete path of the VRJP (the sequence of vertices attained at successive jumps) on $\mathcal{G}$ with weights W, starting at x_0, is a random walk in a random electrical network where the conductances $(c_e)_{e\in E}$ are given by*

$$c_{\{x,y\}} = W_{\{x,y\}} G_\beta(x_0, x) G_\beta(x_0, y).$$

The reason we look at the β-field instead of the conductances is that the β-field has several interesting properties. First, the β-field does not depend on the starting point of the VRJP. Its Laplace transform has a simple expression and it is 1-dependent. But most importantly, the family of laws $\nu_n^{W,\eta}$ is stable by taking marginals or conditional distributions ([18, Lemma 5] and independently in [8]). More precisely:

Proposition 1.2.1. *Let n_1, n_2 be integers, and $n := n_1 + n_2$. Let $W \in M_n(\mathbf{R})$ be a symmetric matrix with non-negative coefficients and $(\eta_i)_{i\in[\![1,n_1+n_2]\!]}$ a family of non-negative coefficients. Let $(\beta_i)_{i\in[\![1,n_1+n_2]\!]}$ be random variables with $\nu_n^{W,\eta}$ distribution and $H_\beta \in M_n(\mathbf{R})$ the matrix as in Definition 1. We make the following bloc decompositions:*

$$W = \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix}, \quad H_\beta = \begin{pmatrix} H_\beta^{11} & H_\beta^{12} \\ H_\beta^{21} & H_\beta^{22} \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix},$$

where $W^{11}, H_\beta^{11} \in M_{n_1}(\mathbf{R})$, $W^{12}, H_\beta^{12} \in M_{n_1,n_2}(\mathbf{R})$, $W^{21}, H_\beta^{21} \in M_{n_2,n_1}(\mathbf{R})$, $W^{22}, H_\beta^{22} \in M_{n_2}(\mathbf{R})$, $\eta^1 \in \mathbf{R}^{n_1}$ and $\eta^2 \in \mathbf{R}^{n_2}$. Then the family $(\beta_i)_{1\le i\le n_1}$ is distributed according to $\nu_{n_1}^{W^{11},\hat{\eta}}$ where

$$\hat{\eta} \in \mathbf{R}^{n_1} \quad \textit{and} \quad \forall i \in [\![1, n_1]\!],\ \hat{\eta}_i := \eta_i + \sum_{k=1}^{n_2} W^{12}(i, k).$$

Conditionally on $(\beta_i)_{1\le i\le n_1}$, the family $(\beta_i)_{n_1+1\le i\le n_1+n_2}$ is distributed according to $\nu_{n_2}^{\check{W},\check{\eta}}$ where

$$\check{W} = W^{22} + W^{21}(H_\beta^{11})^{-1} W^{12},$$

and

$$\check{\eta} \in \mathbf{R}^{n_2} \quad \textit{and} \quad \check{\eta} = \eta^2 + W^{21}(H_\beta^{11})^{-1}\eta^1.$$

Definition 2. Let n be an integer and let $H \in M_n(\mathbf{R})$ be a symmetric matrix. We say that two integers $1 \le i, j \le n$ are *H-connected* if there exists a finite sequence $(k_1, \dots, k_m)$ such that $k_1 = i$, $k_m = j$ and for all $1 \le a \le m - 1$, $H(k_a, k_{a+1}) \neq 0$.

We can now state our (technical) main theorem which gives a coupling between VRJPs of different weights and a simpler corollary that is the equivalent of Rayleigh monotonicity for the VRJP.

Theorem 5. *Fix $n \in \mathbf{N}$. Let $W \in M_n(\mathbf{R})$ be a symmetric matrix with non-negative off-diagonal coefficients and null diagonal coefficients. Let $W^1, W^2 \in M_{n,1}(\mathbf{R})$ be matrices with non-negative coefficients and let $W^3 := W^1 + W^2$. Let $w^-, w^+ \in [0,\infty)$ with $w^- < w^+$. Define*

$$W^- := \begin{pmatrix} W & W^1 & W^2 \\ {}^tW^1 & 0 & w^- \\ {}^tW^2 & w^- & 0 \end{pmatrix}, \quad W^+ := \begin{pmatrix} W & W^1 & W^2 \\ {}^tW^1 & 0 & w^+ \\ {}^tW^2 & w^+ & 0 \end{pmatrix}, \quad W^\infty := \begin{pmatrix} W & W^3 \\ {}^tW^3 & 0 \end{pmatrix}.$$

If $n = 0$, we just have

$$W^- := \begin{pmatrix} 0 & w^- \\ w^- & 0 \end{pmatrix}, \quad W^+ := \begin{pmatrix} 0 & w^+ \\ w^+ & 0 \end{pmatrix}, \quad W^\infty := (0).$$

For any vector $X \in \mathbf{R}^{n+2}$ define $\overline{X} \in \mathbf{R}^{n+1}$ by

$$\forall i \in [\![1,n]\!],\ \overline{X}_i := X_i, \quad \overline{X}_{n+1} := X_{n+1} + X_{n+2}.$$

For any vector $X^1 \in [0,\infty)^{n+2}$, we introduce the following coupling between $\widetilde{\nu}_{n+2}^{W^-,0}$, $\widetilde{\nu}_{n+2}^{W^+,0}$ and $\widetilde{\nu}_{n+1}^{W^\infty,0}$. There exist random matrices H^-, H^+ and H^∞ (with inverses G^-, G^+ and G^∞ respectively) that are distributed according to $\widetilde{\nu}_{n+2}^{W^-,0}$, $\widetilde{\nu}_{n+2}^{W^+,0}$ and $\widetilde{\nu}_{n+1}^{W^\infty,0}$ respectively such that

$${}^tX^1 G^- X^1 = {}^tX^1 G^+ X^1 = {}^t\overline{X^1} G^\infty \overline{X^1} \quad \textit{almost surely,}$$

$H^-(i,i) = H^+(i,i) = H^\infty(i,i)$ for all $i \in [\![1,n]\!]$, and for any vector $X^2 \in [0,\infty)^{n+2}$ we have

$$\mathbb{E}({}^tX^1 G^+ X^2 \mid H^\infty) = {}^t\overline{X^1} G^\infty \overline{X^2},$$
$$\mathbb{E}({}^tX^1 G^- X^2 \mid H^+) = {}^tX^1 G^+ X^2 \quad \textit{if } n+1 \textit{ and } n+2 \textit{ are } H^-\textit{-connected.}$$

A special case of this theorem already appeared in [18], namely the case where $w^- = w^+$ and all the coefficients of X are 0 except for $X_{n+1} = 1$. This was used to define a non-negative martingale $(\psi_n)_{n\in\mathbf{N}}$ which was applied to give a characterization of recurrence and transience for the VRJP. More precisely, if the martingale converges to 0 then the process is recurrent, and if it converges to any other value, the process is transient. However, the case of $w^- \neq w^+$ and more general X was not settled. The more general result will allow us to compare the martingales $(\psi_n)_{n\in\mathbf{N}}$ for different choices of weights. Using this and the characterization of recurrence and transience for the VRJP with $(\psi_n)_{n\in\mathbf{N}}$ we will be able to show that the probability that the VRJP is transient is non-decreasing in the weights.

In practice, we will use the following slightly weaker version of this theorem to prove other results.

Theorem 6. *Let $n \geq 2$ be an integer, and let $W^-, W^+ \in M_n(\mathbf{R})$ be symmetric matrices with null diagonal coefficients and non-negative off-diagonal coefficients such that for any*

$i, j \in [\![1, n]\!]$, $W^-(i, j) \leq W^+(i, j)$ and i and j are W^--connected. Let H^- and H^+ be matrices distributed according to $\widetilde{\nu}_n^{W^-,0}$ and $\widetilde{\nu}_n^{W^+,0}$ respectively, and let their inverses be G^- and G^+ respectively. For any convex function f on $[0, \infty)$, any $i_0 \in [\![1, n]\!]$ and any deterministic vector $X \in [0, \infty)^n$,

$$\mathbb{E}\left(f\left(\frac{\sum_{j=1}^n X_j G^-(i_0, j)}{G^-(i_0, i_0)} \right) \right) \geq \mathbb{E}\left(f\left(\frac{\sum_{j=1}^n X_j G^+(i_0, j)}{G^+(i_0, i_0)} \right) \right).$$

1.3. Outline of the proof

There are two main steps of the proof. The first step (Section 2) is to show that if we prove the main theorem for a graph with two vertices then we can extend it to any finite graph. Fix a finite graph $G = (V, E)$ with positive weights $(W_e)_{e \in E}$ on the edges. The main idea is the following: for any edge $\{x, y\} \in E$, the law of the β-field on $V \backslash \{x, y\}$ does not depend on the weight $W_{\{x,y\}}$. This means that we can set the β-field on $V \backslash \{x, y\}$, and conditioned on this we can see how the modification of the weight $W_{\{x,y\}}$ impacts the β-field at the vertices x and y and in turn, how it affects the environment of the VRJP on the whole graph. Because of the nice restriction properties of the β-field, understanding this is exactly the same as understanding what happens when we modify the weight of the edge on a graph with two vertices.

The second step is to show that the main theorem is true for a graph with two vertices. This is the goal of Section 3, which is quite technical. We first make a change of variable to simplify the problem (in Subsection 3.1) and then we exhibit the coupling (in Subsection 3.2) used to prove our main theorem.

Finally, once we have proved our main theorem (in Section 4), we apply it to prove other results (in Section 5). Here the idea is to use the characterization of recurrence/transience in [18]. This characterization says that a process is recurrent if some non-negative martingale converges to 0. Our result tells us that if we decrease a weight, the probability that this martingale converges to 0 increases, which in turn means that the probability that the process is recurrent also increases. And therefore if we decrease weights, the probability that the process is recurrent increases. This is used jointly with a 0-1 law proved in [18] to conclude that both the VRJP and the ERRW are recurrent with probability 1 below some critical weight and transient with probability 1 above it.

2. A simplification

2.1. Schur's lemma

We will apply Schur's decomposition several times. It is useful because it behaves nicely under the marginal and conditional laws of the family of laws $\nu_n^{W,\eta}$.

Lemma 2.1.1 (Schur decomposition). *Let $n, m \in \mathbf{N}^*$. Let $H \in M_{n+m}(\mathbf{R})$ be a symmetric, positive definite matrix. Let $A \in M_n(\mathbf{R})$, $B \in M_{n,m}(\mathbf{R})$ and $C \in M_m(\mathbf{R})$ be such that*

H can be decomposed as

$$H = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}.$$

Its inverse is

$$H^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(C - {}^tBA^{-1}B)^{-1}\,{}^tBA^{-1} & -A^{-1}B(C - {}^tBA^{-1}B)^{-1} \\ -(C - {}^tBA^{-1}B)^{-1}\,{}^tBA^{-1} & (C - {}^tBA^{-1}B)^{-1} \end{pmatrix}.$$

2.2. *Reduction to two points*

The goal of this section is essentially to show that if we can prove our main theorem (Theorem 5) for a graph with two vertices then it is true for all finite graphs. First we need to prove a minor lemma.

Lemma 2.2.1. *Let n be an integer, and let $H \in M_n(\mathbf{R})$ be a symmetric, positive definite matrix with non-positive off-diagonal coefficients. For any integers $1 \le i, j \le n$, $H^{-1}(i, j) > 0$ iff i and j are H-connected. In particular, for any integer $1 \le i \le n$, $H^{-1}(i, i) > 0$.*

Proof. Since H is a symmetric, positive definite matrix, all the eigenvalues are positive reals. Let λ^- be the smallest eigenvalue of H and λ^+ the largest one. Since H is symmetric, all its diagonal coefficients $H(i, i)$ satisfy $\lambda^- \le H(i, i) \le \lambda^+$. This means that all the coefficients of $I_n - \frac{1}{\lambda^+}H$ are non-negative and its eigenvalues are between 0 and $1 - \lambda^-/\lambda^+ < 1$. Consequently,

$$H^{-1} = \frac{1}{\lambda^+}\left(I_n - \left(I_n - \frac{1}{\lambda^+}H\right)\right)^{-1} = \frac{1}{\lambda^+}\sum_{k\ge 0}\left(I_n - \frac{1}{\lambda^+}H\right)^k.$$

Integers i and j are H-connected iff there exists $m \ge 0$ such that $(I_n - \frac{1}{\lambda^+}H)^m > 0$ (since all the coefficients of $I_n - \frac{1}{\lambda^+}H$ are non-negative). Hence $H^{-1}(i, j) > 0$ iff i and j are H-connected. ■

We will use the following lemma to reduce our problem to the study of $\nu_1^{W,\eta}$ and $\nu_2^{W,0}$.

Lemma 2.2.2. *Let $n \in \mathbf{N}^*$. Let $H^{11} \in M_n(\mathbf{R})$ be a symmetric, positive definite matrix with non-positive off-diagonal coefficients. Let $H^{12} \in M_{n,2}(\mathbf{R})$ be a matrix with non-positive coefficients. Define $\bar{H}^{12} \in M_{n,1}(\mathbf{R})$ by*

$$\bar{H}^{12} = H^{12}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $H \in M_{n+2}(\mathbf{R})$ and $\bar{H} \in M_{n+1}(\mathbf{R})$ be symmetric, positive definite matrices with non-positive off-diagonal coefficients and with the following bloc decompositions:

$$H = \begin{pmatrix} H^{11} & H^{12} \\ {}^tH^{12} & H^{22} \end{pmatrix}, \quad \bar{H} = \begin{pmatrix} H^{11} & \bar{H}^{12} \\ {}^t\bar{H}^{12} & \bar{H}^{22} \end{pmatrix}.$$

Let G and $\overline{G}$ be the inverses of H and $\overline{H}$ respectively, with bloc decompositions

$$G = \begin{pmatrix} G^{11} & G^{12} \\ {}^tG^{12} & G^{22} \end{pmatrix}, \quad \overline{G} = \begin{pmatrix} G^{11} & \overline{G}^{12} \\ {}^t\overline{G}^{12} & \overline{G}^{22} \end{pmatrix}.$$

For any vector $X \in \mathbf{R}^{n+2}$ we define the vector $\overline{X} \in \mathbf{R}^{n+1}$ by

$$\forall i \in [\![1,n]\!],\ \overline{X}_i := X_i, \quad \overline{X}_{n+1} := X_{n+1} + X_{n+2}.$$

For any vectors $X^1, X^2 \in [0,\infty)^{n+2}$ there exist

- $\alpha_1(X^1) \geq 0$ *and* $\alpha_2(X^1) \geq 0$ *that only depend on* X^1, H^{11} *and* H^{12}*,*
- $\alpha_1(X^2) \geq 0$ *and* $\alpha_2(X^2) \geq 0$ *that only depend on* X^2, H^{11} *and* H^{12}*,*
- $C(X^1, X^2) \geq 0$ *that only depends on* X^1, X^2 *and* H^{11} (*but not* H^{12} *or* H^{22})*,*

such that

$$\begin{aligned} {}^tX^1GX^2 &= C(X^1,X^2) + \begin{pmatrix}\alpha_1(X^1) & \alpha_2(X^1)\end{pmatrix} G^{22} \begin{pmatrix}\alpha_1(X^2) \\ \alpha_2(X^2)\end{pmatrix}, \\ {}^t\overline{X}^1\overline{G}\,\overline{X}^2 &= C(X^1,X^2) + (\alpha_1(X^1)+\alpha_2(X^1))\overline{G}^{22}(\alpha_1(X^2)+\alpha_2(X^2)). \end{aligned}$$

The previous lemma allows us to transform the expression ${}^tX^1GX^2$ into the form $A + {}^tY^1G^{22}Y^2$. The properties of the family of laws $\nu_n^{W,\eta}$ (defined in Proposition 1.2.1) tell us that the study of G^{22} knowing A, Y^1 and Y^2 is the same as the study of $\nu_2^{W,0}$ for some parameter W. This means that if we get some monotonicity for the family of laws $\nu_2^{W,0}$ we should be able to get it for $\nu_n^{W,0}$ for any n.

Proof of Lemma 2.2.2. First we look at H. Let G be the inverse of H. We use the same bloc decomposition as for H:

$$G = \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix},$$

where $G^{11} \in M_n(\mathbf{R})$, $G^{12} \in M_{n,2}(\mathbf{R})$, $G^{21} \in M_{2,n}(\mathbf{R})$ and $G^{22} \in M_2(\mathbf{R})$. By Schur decomposition (Lemma 2.1.1) we have

$$\begin{aligned} G &= \begin{pmatrix} (H^{11})^{-1} + (H^{11})^{-1}H^{12}G^{22}\,{}^tH^{12}(H^{11})^{-1} & -(H^{11})^{-1}H^{12}G^{22} \\ -G^{22}\,{}^tH^{12}(H^{11})^{-1} & G^{22} \end{pmatrix} \\ &= \begin{pmatrix} I_n & -(H^{11})^{-1}H^{12} \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} (H^{11})^{-1} & 0 \\ 0 & G^{22} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -H^{21}(H^{11})^{-1} & I_2 \end{pmatrix}. \end{aligned}$$

Now we need to introduce the notion of M-matrices. There are multiple equivalent definitions (see [13]). According to one, an M-*matrix* is a matrix with non-negative off-diagonal coefficients and such that either all its principal minors are positive (which is the case of H^{11}) or its inverse only has non-negative coefficients. This means that $(H^{11})^{-1}$ is an M-matrix and therefore all its coefficients are non-negative. Since all the coefficients of $-H^{12}$ are also non-negative by definition, this means that all the coefficients of

$-(H^{11})^{-1}H^{12}$ are non-negative. Let $X^1, X^2 \in \mathbf{R}^{n+2}$ be two vectors with bloc decompositions

$$X^1 := \begin{pmatrix} X^{11} \\ X^{12} \end{pmatrix}, \quad X^2 := \begin{pmatrix} X^{21} \\ X^{22} \end{pmatrix},$$

where $X^{11}, X^{21} \in \mathbf{R}^n$ and $X^{12}, X^{22} \in \mathbf{R}^2$. Let $M := -H^{21}(H^{11})^{-1}$. We have

$$\begin{aligned}
&{}^tX^1GX^2 \\
&= \begin{pmatrix} {}^tX^{11} & {}^tX^{12} \end{pmatrix} \begin{pmatrix} I_n & -(H^{11})^{-1}H^{12} \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} (H^{11})^{-1} & 0 \\ 0 & G^{22} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -H^{21}(H^{11})^{-1} & I_2 \end{pmatrix} \begin{pmatrix} X^{21} \\ X^{22} \end{pmatrix} \\
&= \begin{pmatrix} {}^tX^{11} & {}^tX^{12} \end{pmatrix} \begin{pmatrix} I_n & {}^tM \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} (H^{11})^{-1} & 0 \\ 0 & G^{22} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ M & I_2 \end{pmatrix} \begin{pmatrix} X^{21} \\ X^{22} \end{pmatrix} \\
&= \begin{pmatrix} {}^tX^{11} & {}^tX^{11}\,{}^tM + {}^tX^{12} \end{pmatrix} \begin{pmatrix} (H^{11})^{-1} & 0 \\ 0 & G^{22} \end{pmatrix} \begin{pmatrix} X^{21} \\ MX^{21} + X^{22} \end{pmatrix} \\
&= {}^tX^{11}(H^{11})^{-1}X^{21} + ({}^tX^{11}\,{}^tM + {}^tX^{12})G^{22}(MX^{21} + X^{22}) \\
&= {}^tX^{11}(H^{11})^{-1}X^{21} + {}^t(MX^{11} + X^{12})G^{22}(MX^{21} + X^{22}).
\end{aligned}$$

Now we can define $\alpha_1(X^1), \alpha_2(X^1), \alpha_1(X^2)$ and $\alpha_2(X^2)$ by

$$\begin{pmatrix} \alpha_1(X^1) \\ \alpha_2(X^1) \end{pmatrix} := MX^{11} + X^{12}, \quad \begin{pmatrix} \alpha_1(X^2) \\ \alpha_2(X^2) \end{pmatrix} := MX^{21} + X^{22}.$$

We also define $C(X^1, X^2) := {}^tX^{11}(H^{11})^{-1}X^{21}$. We get

$${}^tX^1GX^2 = C(X^1, X^2) + \begin{pmatrix} \alpha_1(X^1) & \alpha_2(X^1) \end{pmatrix} G^{22} \begin{pmatrix} \alpha_1(X^2) \\ \alpha_2(X^2) \end{pmatrix}.$$

Similarly,

$$\begin{aligned}
&{}^t\bar{X}^1\bar{G}\bar{X}^2 \\
&= {}^tX^{11}(H^{11})^{-1}X^{21} + {}^t\big(-\bar{H}^{21}(H^{11})^{-1}X^{11} + \bar{X}^{12}\big)\bar{G}^{22}\big(-\bar{H}^{21}(H^{11})^{-1}X^{21} + \bar{X}^{22}\big) \\
&= C(X^1, X^2) + (\alpha_1(X^1) + \alpha_2(X^1))\bar{G}^{22}\big(\alpha_1(X^2) + \alpha_2(X^2)\big). \qquad \blacksquare
\end{aligned}$$

3. The coupling

3.1. A change of variables

When we look at $\nu_2^{W,0}$, instead of looking at the β-field (β_1, β_2) we will look at two other variables that will make our coupling and various calculations more explicit. In the following lemma we state this change of variables and some relevant properties of the new variables.

Lemma 3.1.1. *Let $\lambda \in [0,1]$ and $w \geq 0$ be such that if $w = 0$ then $\lambda \notin \{0,1\}$. Let $W := \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$. Let (β_1, β_2) be distributed according to $\nu_2^{W,0}$. Define*

$$\gamma := \frac{1}{\begin{pmatrix}\lambda & 1-\lambda\end{pmatrix}\left(\begin{smallmatrix} 2\beta_1 & -w \\ -w & 2\beta_2 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} \lambda \\ 1-\lambda \end{smallmatrix}\right)} = \frac{4\beta_1\beta_2 - w^2}{2w\lambda(1-\lambda) + 2\beta_2\lambda^2 + 2\beta_1(1-\lambda)^2},$$

$$Z := \frac{2\beta_1 - \lambda^2\gamma}{w + \lambda(1-\lambda)\gamma}.$$

Set $w_\gamma := w + \lambda(1-\lambda)\gamma$. Then both Z and γ are positive and

$$2\beta_1 = \lambda^2\gamma + w_\gamma Z, \quad 2\beta_2 = (1-\lambda)^2\gamma + w_\gamma \frac{1}{Z}.$$

The random variable γ is the only random variable such that

$$\begin{pmatrix} 2\beta_1 & -w \\ -w & 2\beta_2 \end{pmatrix} - \gamma \begin{pmatrix} \lambda^2 & \lambda(1-\lambda) \\ \lambda(1-\lambda) & (1-\lambda)^2 \end{pmatrix}$$

is of rank 1. *The law of γ is* Gamma$(1/2, 1/2)$. *The law of Z, knowing γ, is*

$$\frac{\sqrt{w_\gamma}}{\sqrt{2\pi}} \exp\left(-w_\gamma \frac{(z-1)^2}{2z}\right) \frac{1}{z}\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right) 1_{z>0}\, dz.$$

This law is a mixture of an inverse Gaussian law and its inverse.

If $U := \sqrt{Z} - \frac{1}{\sqrt{Z}}$, then its density, knowing γ, is

$$\frac{\sqrt{w_\gamma}}{\sqrt{2\pi}} \exp\left(-w_\gamma \frac{u^2}{2}\right)\left(1 - (2\lambda - 1)\frac{u}{\sqrt{u^2+4}}\right) du.$$

This law is similar to a Gaussian, in particular the law of $|U|$ is that of the absolute value of a Gaussian.

We also have the following equality:

$$\det \begin{pmatrix} 2\beta_1 & -w \\ -w & 2\beta_2 \end{pmatrix} = 4\beta_1\beta_2 - w^2 = w_\gamma\gamma\left((1-\lambda)\sqrt{Z} + \frac{\lambda}{\sqrt{Z}}\right)^2.$$

The random variable γ is a generalization of the random variable γ defined in [17], in which it is only defined for $\lambda \in \{0,1\}$. It is used to make a link between the β-field and the VRJP starting at a specific point.

Proof of Lemma 3.1.1. In the proof, when talking about densities, c, z and b_i will correspond to the random variables γ, Z and β_i respectively. We also set, for any $c \in [0,\infty)$, $w_c := w + \lambda(1-\lambda)c$ (it represents w_γ when talking about densities).

Define

$$\mathcal{H} := \left\{(b_1, b_2) \in (0,\infty)^2 : \begin{pmatrix} 2b_1 & -w \\ -w & 2b_2 \end{pmatrix} > 0\right\}.$$

Let $f:(0,\infty)^2 \to \mathbf{R}^2$ be defined by

$$f(c,z) := \left(\frac{\lambda^2 c + w_c z}{2}, \frac{(1-\lambda)^2 c + w_c \frac{1}{z}}{2}\right).$$

The first step is to show that f is a bijection from $(0,\infty)^2$ to $\mathcal{H}$.

We start by checking that $f((0,\infty)^2) \subset \mathcal{H}$. First, $\frac{\lambda^2 c+w_c z}{2} > 0$ and $\frac{(1-\lambda)^2 c + w_c \frac{1}{z}}{2} > 0$. Then

$$4\frac{\lambda^2 c + w_c z}{2}\,\frac{(1-\lambda)^2 c + w_c \frac{1}{z}}{2} - w^2 > wzw\frac{1}{z} - w^2 = 0.$$

This means that $f((0,\infty)^2) \subset \mathcal{H}$.

Now we need a minor result on matrices that will make calculations with f simpler. Let $Y := \left(\begin{smallmatrix}\lambda\\ 1-\lambda\end{smallmatrix}\right)$. For any $(a_1,a_2) \in \mathcal{H}$ and $s \in \mathbf{R}$, we have

$$\det\left(\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix} - sY\,{}^tY\right) = \det\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix}\det\left(I_2 - s\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix}^{-1} Y\,{}^tY\right).$$

The matrix $s\left(\begin{smallmatrix}2a_1 & -w\\ -w & 2a_2\end{smallmatrix}\right)^{-1} Y\,{}^tY$ is of rank 1 and its only non-zero eigenvalue is therefore equal to its trace, which is $s\,{}^tY\left(\begin{smallmatrix}2a_1 & -w\\ -w & 2a_2\end{smallmatrix}\right)^{-1} Y$. Therefore

$$\det\left(I_2 - s\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix}^{-1} Y\,{}^tY\right) = 1 - s\,{}^tY\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix}^{-1} Y,$$

and thus

$$\det\left(\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix} - sY\,{}^tY\right) = \det\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix}\left(1 - s\,{}^tY\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix}^{-1} Y\right).$$

This means that

$$\det\left(\begin{pmatrix}2a_1 & -w\\ -w & 2a_2\end{pmatrix} - sY\,{}^tY\right) = 0 \iff s = \frac{1}{{}^tY\left(\begin{smallmatrix}2a_1 & -w\\ -w & 2a_2\end{smallmatrix}\right)^{-1} Y}.$$

Now we notice that if $(b_1,b_2) := f(c,z)$ then

$$\begin{pmatrix}2b_1 & -w\\ -w & 2b_2\end{pmatrix} - cY\,{}^tY = \begin{pmatrix}w_c z & -w_c\\ -w_c & w_c\frac{1}{z}\end{pmatrix},$$

which is of rank 1, and the eigenvector for the non-zero eigenvalue is $\left(\begin{smallmatrix}\sqrt{z}\\ -1/\sqrt{z}\end{smallmatrix}\right)$. Therefore if we know that $(b_1,b_2) = f(c,z)$ then

$$c = \frac{1}{{}^tY\left(\begin{smallmatrix}2b_1 & -w\\ -w & 2b_2\end{smallmatrix}\right)^{-1} Y} = \frac{4b_1b_2 - w^2}{2b_2\lambda^2 + 2b_1(1-\lambda)^2 + 2\lambda(1-\lambda)w},$$

$$z = \frac{2b_1 - \lambda^2 c}{w_c} = \frac{w_c}{2b_2 - (1-\lambda)^2 c}.$$

This means that as a function from $(0,\infty)^2$ to $\mathscr{H}$, f is injective and its inverse is the one we want. Conversely, f is surjective (as a function from $(0,\infty)^2$ to $\mathscr{H}$) by using the same formula. Hence f is indeed a bijection from $(0,\infty)^2$ to $\mathscr{H}$.

We also have

$$\begin{aligned}2\beta_1 &= \lambda^2\gamma + w_\gamma Z,\\ 2\beta_2 &= (1-\lambda)^2\gamma + w_\gamma \frac{1}{Z},\end{aligned}$$

and γ is the only random variable such that

$$\begin{pmatrix}2\beta_1 & -w\\ -w & 2\beta_2\end{pmatrix} - \gamma\begin{pmatrix}\lambda^2 & \lambda(1-\lambda)\\ \lambda(1-\lambda) & (1-\lambda)^2\end{pmatrix}$$

is of rank 1.

Now, for the second step, we can deduce the law of γ, Z from that of β_1,β_2 by a simple change of variable. The law of (β_1,β_2) is $\nu_2^{W,0}$ (see Definition 1), with density

$$\nu_2^{W,0}(\mathrm{d}\beta_1,\mathrm{d}\beta_2) := \frac{2}{\pi}e^{-\frac{1}{2}(2\beta_1+2\beta_2-2w)}\frac{1}{\sqrt{4\beta_1\beta_2-w^2}}1_{H_\beta>0}\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2,$$

where

$$H_\beta := \begin{pmatrix}2\beta_1 & -w\\ -w & 2\beta_2\end{pmatrix}.$$

The Jacobian J_f of the change of variables f is

$$J_f(c,z) = \begin{pmatrix}(\lambda^2+\lambda(1-\lambda)z)\frac{1}{2} & ((1-\lambda)^2+\lambda(1-\lambda)\frac{1}{z})\frac{1}{2}\\ (w+\lambda(1-\lambda)c)\frac{1}{2} & -(w+\lambda(1-\lambda)c)\frac{1}{2z^2}\end{pmatrix}$$

and its determinant is

$$\begin{aligned}D_f(c,z) &= -\frac{w_c}{4}\left((1-\lambda)^2+\lambda(1-\lambda)\frac{1}{z}+\lambda^2\frac{1}{z^2}+\lambda(1-\lambda)\frac{1}{z}\right)\\ &= -\frac{w_c}{4}\left(1-\lambda+\frac{\lambda}{z}\right)^2\\ &= -\frac{w_c}{4}\frac{1}{z}\left((1-\lambda)\sqrt{z}+\frac{\lambda}{\sqrt{z}}\right)^2.\end{aligned}$$

Now we can change variables (β_1,β_2) such that $H_\beta = \left(\begin{smallmatrix}2\beta_1 & -w\\ -w & 2\beta_2\end{smallmatrix}\right) > 0$ into variables (γ, z) defined by

$$\begin{aligned}\gamma &:= \frac{4\beta_1\beta_2-w^2}{2w\lambda(1-\lambda)+2\beta_2\lambda^2+2\beta_1(1-\lambda)^2},\\ z &:= \frac{2\beta_1-\lambda^2\gamma}{w+\lambda(1-\lambda)\gamma}.\end{aligned}$$

We need to make a few calculations before we can express the law of (γ, Z). First, for any $(c,z) \in (0,\infty)^2$, with $(b_1, b_2) := f(c,z)$,

$$\begin{aligned}
4b_1b_2 - w^2 &= (w_c z + \lambda^2 c)\left(w_c \frac{1}{z} + (1-\lambda)^2 c\right) - w^2 \\
&= w_c^2 + w_c\left(\lambda^2 c \frac{1}{z} + (1-\lambda)^2 cz\right) + \lambda^2(1-\lambda)^2c^2 - w^2 \\
&= (\lambda(1-\lambda)c)^2 + 2w\lambda(1-\lambda)c + w_c\left(\lambda^2 c\frac{1}{z} + (1-\lambda)^2 cz\right) + \lambda^2(1-\lambda)^2c^2 \\
&= 2c\lambda(1-\lambda)(w + \lambda(1-\lambda)c) + w_c c\left(\lambda^2 \frac{1}{z} + (1-\lambda)^2 z\right) \\
&= w_c c\left(\lambda^2\frac{1}{z} + (1-\lambda)^2 z + 2\lambda(1-\lambda)\right) = w_c c\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right)^2. \qquad (3.1)
\end{aligned}$$

Therefore

$$\frac{|D_f(c,z)|}{\sqrt{4b_1b_2 - w^2}} = \frac{\sqrt{w + \lambda(1-\lambda)c}}{4\sqrt{c}} \frac{1}{z}\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right).$$

We also have the following equality:

$$\begin{aligned}
b_1 + b_2 - w &= \lambda^2 \frac{c}{2} + w_c \frac{z}{2} + (1-\lambda)^2\frac{c}{2} + w_c\frac{1}{2z} - (w_c - \lambda(1-\lambda)c) \\
&= \left(\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\right)\frac{c}{2} + \frac{1}{2}w_c\left(z + \frac{1}{z} - 2\right) \\
&= \frac{c}{2} + \frac{1}{2}w_c\frac{1}{z}(z-1)^2.
\end{aligned}$$

Hence we get the following joint law for γ and Z (c represents γ and z represents Z):

$$\frac{2}{\pi}\frac{\sqrt{w_c}}{4\sqrt{c}}\frac{1}{z}\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right)\exp\left(-\frac{c}{2} - w_c\frac{(z-1)^2}{2z}\right)\mathrm{d}z\,\mathrm{d}c.$$

In particular, the law of Z, knowing γ, is

$$\frac{\sqrt{w_\gamma}}{\sqrt{2\pi}}\exp\left(-w_\gamma\frac{(z-1)^2}{2z}\right)\frac{1}{z}\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right)\mathrm{d}z.$$

It is indeed a density since it is a mixture of an inverse Gaussian and the inverse of an inverse Gaussian.

By (3.1) we also have

$$\det\begin{pmatrix} 2\beta_1 & -w \\ -w & 2\beta_2 \end{pmatrix} = 4\beta_1\beta_2 - w^2 = w_\gamma\gamma\left((1-\lambda)\sqrt{Z} + \frac{\lambda}{\sqrt{Z}}\right)^2.$$

Now, for the third and final step, we can look at the law of U, knowing γ. By definition, $U = \sqrt{Z} - \frac{1}{\sqrt{Z}}$. This means that $\sqrt{Z} = \frac{\sqrt{U^2+4}+U}{2}$ and $\frac{1}{\sqrt{Z}} = \frac{\sqrt{U^2+4}-U}{2}$.

Therefore $Z = \frac{U^2+2+U\sqrt{U^2+4}}{2}$. The density of U, knowing γ, is thus

$$\begin{aligned}
&\frac{1}{2}\left(2u+\sqrt{u^2+4}+\frac{u^2}{\sqrt{u^2+4}}\right)\frac{\sqrt{w_\gamma}}{\sqrt{2\pi}}\exp\left(-w_\gamma\frac{u^2}{2}\right)\\
&\qquad\times\frac{2}{u^2+2+u\sqrt{u^2+4}}\left((1-\lambda)\frac{\sqrt{u^2+4}+u}{2}+\lambda\frac{\sqrt{u^2+4}-u}{2}\right)\mathrm{d}u\\
&\qquad=\frac{2u\sqrt{u^2+4}+2u^2+4}{2\sqrt{u^2+4}}\frac{\sqrt{w_\gamma}}{\sqrt{2\pi}}\exp\left(-w_\gamma\frac{u^2}{2}\right)\\
&\qquad\times\frac{2}{u^2+2+u\sqrt{u^2+4}}\left((1-\lambda)\frac{\sqrt{u^2+4}+u}{2}+\lambda\frac{\sqrt{u^2+4}-u}{2}\right)\mathrm{d}u\\
&\qquad=\frac{\sqrt{w_\gamma}}{\sqrt{2\pi}}\exp\left(-w_\gamma\frac{u^2}{2}\right)\left(1-(2\lambda-1)\frac{u}{\sqrt{u^2+4}}\right)\mathrm{d}u. \qquad\blacksquare
\end{aligned}$$

3.2. The tilted Gaussian law

Definition 3. For any $(K,\delta)\in(0,\infty)\times[-1,1]$ we define the tilted Gaussian law $\widetilde{\mathcal{N}}(K,\delta)$ by the following density:

$$\sqrt{\frac{K}{2\pi}}\exp\left(-\frac{Ku^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right)\mathrm{d}u.$$

It is indeed a density since it is the density of a Gaussian plus an antisymmetric term that is smaller than the Gaussian term.

Lemma 3.2.1. *Let $K>0$ and $\delta,\delta'\in[-1,1]$. Let U be a random variable distributed according to $\widetilde{\mathcal{N}}(K,\delta)$. Then*

$$\mathbb{E}\left(\frac{1+\delta'\frac{U}{\sqrt{U^2+4}}}{1+\delta\frac{U}{\sqrt{U^2+4}}}\right)=1.$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left(\frac{1+\delta'\frac{U}{\sqrt{U^2+4}}}{1+\delta\frac{U}{\sqrt{U^2+4}}}\right)&=\int_{\mathbf{R}}\sqrt{\frac{K}{2\pi}}\exp\left(-\frac{Ku^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right)\left(\frac{1+\delta'\frac{u}{\sqrt{u^2+4}}}{1+\delta\frac{u}{\sqrt{u^2+4}}}\right)\mathrm{d}u\\
&=\int_{\mathbf{R}}\sqrt{\frac{K}{2\pi}}\exp\left(-\frac{Ku^2}{2}\right)\left(1+\delta'\frac{u}{\sqrt{u^2+4}}\right)\mathrm{d}u=1. \qquad\blacksquare
\end{aligned}$$

Lemma 3.2.2. *Let $0<K^-\le K^+$ and $\delta\in[-1,1]$. There exist random variables U^- and U^+ distributed according to $\widetilde{\mathcal{N}}(K^-,\delta)$ and $\widetilde{\mathcal{N}}(K^+,\delta)$ respectively such that*

$$\forall\delta'\in[-1,1],\quad\mathbb{E}\left(\frac{1+\delta'\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}}\,\middle|\,U^+\right)=\frac{1+\delta'\frac{U^+}{\sqrt{(U^+)^2+4}}}{1+\delta\frac{U^+}{\sqrt{(U^+)^2+4}}},$$

and

$$K^-(U^-)^2 = K^+(U^+)^2 \quad a.s.$$

Proof. Let $K := \sqrt{\frac{K^+}{K^-}}$. Let U^+ be a random variable distributed according to $\widetilde{\mathcal{N}}(K^+,\delta)$. First we define

$$V^+ := \frac{U^+}{\sqrt{(U^+)^2+4}}, \quad V^- := \frac{KU^+}{\sqrt{K^2(U^+)^2+4}}.$$

We notice that $0 \le |V^+| \le |V^-| < 1$. Let $p_1, p_2 \in \mathbf{R}$ be defined by

$$p^+ := \frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+}, \quad p^- := \frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+}.$$

Both p^+ and p^- are non-negative. We also have

$$\begin{aligned} p^+ + p^- &= \frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+} + \frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+} \\ &= \frac{1+\delta V^- + 1 - \delta V^- + \frac{V^+}{V^-}(1+\delta V^- - 1 + \delta V^-)}{2(1+\delta V^+)} = \frac{2+\frac{V^+}{V^-}2\delta V^-}{2(1+\delta V^+)} = 1. \end{aligned}$$

Now, let U^- be the random variable such that, knowing U^+,

$$U^- := \begin{cases} KU^+ & \text{with probability } p^+, \\ -KU^+ & \text{with probability } p^-. \end{cases}$$

We want to show that U^- is distributed according to $\widetilde{\mathcal{N}}(K^-,\delta)$. For any test function f we have

$$\begin{aligned} \mathbb{E}(f(U^-)) &= \mathbb{E}(\mathbb{E}(f(U^-) \mid U^+)) \\ &= \mathbb{E}\left(\frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+}f(KU^+) + \frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+}f(-KU^+)\right). \end{aligned}$$

First, we get

$$\begin{aligned} &\mathbb{E}\left(\frac{1}{2}\left(1 \pm \frac{V^+}{V^-}\right)\frac{1 \pm \delta V^-}{1+\delta V^+} f(\pm KU^+)\right) \\ &= \int_{\mathbf{R}} \sqrt{\frac{K^+}{2\pi}} \exp\left(-\frac{K^+u^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right) \\ &\qquad \times \left(\frac{1}{2}\left(1\pm\frac{\sqrt{K^2u^2+4}}{K\sqrt{u^2+4}}\right)\frac{1\pm\delta K\frac{u}{\sqrt{K^2u^2+4}}}{1+\delta\frac{u}{\sqrt{u^2+4}}} f(\pm Ku)\right) \mathrm{d}u \\ &= \int_{\mathbf{R}} \sqrt{\frac{K^+}{2\pi}} \exp\left(-\frac{K^+u^2}{2}\right)\left(1\pm\delta K\frac{u}{\sqrt{K^2u^2+4}}\right)\left(\frac{1}{2}\left(1\pm\frac{\sqrt{K^2u^2+4}}{K\sqrt{u^2+4}}\right)f(\pm Ku)\right)\mathrm{d}u \\ &= \int_{\mathbf{R}} \sqrt{\frac{K^-}{2\pi}} \exp\left(-\frac{K^-v^2}{2}\right)\left(1+\delta\frac{v}{\sqrt{v^2+4}}\right)\left(\frac{1}{2}\left(1\pm\frac{\sqrt{v^2+4}}{\sqrt{v^2+4K}}\right)f(v)\right)\mathrm{d}v \end{aligned}$$

with $v = \pm Ku$.

If we put both equalities together, we get, for any test function f,

$$\mathbb{E}(f(U^-)) = \int_{\mathbf{R}} \sqrt{\frac{K^-}{2\pi}} \exp\Big(-\frac{K^- u^2}{2}\Big)\Big(1 + \delta \frac{u}{\sqrt{u^2+4}}\Big) f(u)\,\mathrm{d}u.$$

This means that U^- is indeed distributed according to $\widetilde{\mathcal{N}}(K^-, \delta)$.

Now we only need to show that U^+ and U^- satisfy the equality we want. First we notice that, for any $x \in (-1, 1)$,

$$\frac{1+\delta' x}{1+\delta x} = 1 + (\delta' - \delta)\frac{x}{1+\delta x}.$$

This means that we only need to show that

$$\mathbb{E}\left(\frac{\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}}\,\middle|\, U^+\right) = \frac{\frac{U^+}{\sqrt{(U^+)^2+4}}}{1+\delta\frac{U^+}{\sqrt{(U^+)^2+4}}},$$

which is the same as showing

$$\mathbb{E}\left(\frac{\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}}\,\middle|\, U^+\right) = \frac{V^+}{1+\delta V^+}.$$

By definition of U^-, V^- and V^+, we have

$$\begin{aligned}
&\mathbb{E}\left(\frac{\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}}\,\middle|\, U^+\right)\\
&= \frac{1}{2}\Big(1+\frac{V^+}{V^-}\Big)\frac{1+\delta V^-}{1+\delta V^+}\frac{\frac{KU^+}{\sqrt{(KU^+)^2+4}}}{1+\delta\frac{KU^+}{\sqrt{(KU^+)^2+4}}} + \frac{1}{2}\Big(1-\frac{V^+}{V^-}\Big)\frac{1-\delta V^-}{1+\delta V^+}\frac{\frac{-KU^+}{\sqrt{(-KU^+)^2+4}}}{1+\delta\frac{-KU^+}{\sqrt{(-KU^+)^2+4}}}\\
&= \frac{1}{2}\Big(1+\frac{V^+}{V^-}\Big)\frac{1+\delta V^-}{1+\delta V^+}\frac{V^-}{1+\delta V^-} + \frac{1}{2}\Big(1-\frac{V^+}{V^-}\Big)\frac{1-\delta V^-}{1+\delta V^+}\frac{-V^-}{1-\delta V^-}\\
&= \frac{1}{2}\Big(1+\frac{V^+}{V^-}\Big)\frac{V^-}{1+\delta V^+} + \frac{1}{2}\Big(1-\frac{V^+}{V^-}\Big)\frac{-V^-}{1+\delta V^+} = \frac{V^+}{1+\delta V^+}.
\end{aligned}$$

■

Lemma 3.2.3. *Let $w > 0$ and $W := \left(\begin{smallmatrix} 0 & w \\ w & 0\end{smallmatrix}\right)$. Fix $\lambda, \theta \in [0, 1]$. Let (β_1, β_2) be distributed according to $\nu_2^{W,0}$. Let H_β be the random matrix defined by*

$$H_\beta := \begin{pmatrix} 2\beta_1 & -w \\ -w & 2\beta_2 \end{pmatrix}.$$

Let G_β be the inverse of H_β. As in Lemma 3.1.1, set

$$\gamma := \frac{4\beta_1\beta_2 - w^2}{2w\lambda(1-\lambda) + 2\beta_2\lambda^2 + 2\beta_1(1-\lambda)^2}, \quad Z := \frac{2\beta_1 - \lambda^2\gamma}{w + \lambda(1-\lambda)\gamma}.$$

Then

$$\begin{pmatrix}\lambda & 1-\lambda\end{pmatrix} G_\beta \begin{pmatrix}\theta \\ 1-\theta\end{pmatrix} = \frac{\theta\frac{1}{\sqrt{Z}} + (1-\theta)\sqrt{Z}}{\gamma\big((1-\lambda)\sqrt{Z} + \lambda\frac{1}{\sqrt{Z}}\big)}.$$

Proof. First, by Lemma 3.1.1 we have

$$\begin{aligned} 2\beta_1 &= (w + \lambda(1-\lambda)\gamma)Z + \lambda^2\gamma, \\ 2\beta_2 &= (w + \lambda(1-\lambda)\gamma)\frac{1}{Z} + (1-\lambda)^2\gamma, \\ w &= (w + \lambda(1-\lambda)\gamma) - \lambda(1-\lambda)\gamma. \end{aligned}$$

To simplify notation, let $\widetilde{w} := w + \lambda(1-\lambda)\gamma$. A quantity that will be important in the following is the determinant of H_β, $4\beta_1\beta_2 - w^2$. By Lemma 3.1.1, we have

$$4\beta_1\beta_2 - w^2 = \widetilde{w}\gamma\left((1-\lambda)\sqrt{Z} + \lambda\frac{1}{\sqrt{Z}}\right)^2.$$

We know that

$$\begin{aligned} G_\beta(1,1) &= \frac{2\beta_2}{4\beta_1\beta_2 - w^2}, \quad G_\beta(2,2) = \frac{2\beta_1}{4\beta_1\beta_2 - w^2}, \\ G_\beta(1,2) &= G_\beta(2,1) = \frac{w}{4\beta_1\beta_2 - w^2}. \end{aligned}$$

Therefore

$$\begin{pmatrix}\lambda & 1-\lambda\end{pmatrix} G_\beta \begin{pmatrix}\theta \\ 1-\theta\end{pmatrix} = \frac{\lambda\theta 2\beta_2 + (\lambda(1-\theta) + (1-\lambda)\theta)w + (1-\lambda)(1-\theta)2\beta_1}{4\beta_1\beta_2 - w^2}.$$

Now we also have

$$\begin{aligned} &\lambda\theta 2\beta_2 + \big(\lambda(1-\theta) + (1-\lambda)\theta\big)w + (1-\lambda)(1-\theta)2\beta_1 \\ &\quad = \lambda\theta\left(\widetilde{w}\frac{1}{Z} + (1-\lambda)^2\gamma\right) + \big(\lambda(1-\theta) + (1-\lambda)\theta\big)(\widetilde{w} - \lambda(1-\lambda)\gamma) \\ &\qquad + (1-\lambda)(1-\theta)(\widetilde{w}Z + \lambda^2\gamma) \\ &\quad = \lambda\theta\widetilde{w}\frac{1}{Z} + \big(\lambda(1-\theta) + (1-\lambda)\theta\big)\widetilde{w} + (1-\lambda)(1-\theta)\widetilde{w}Z \\ &\quad = \widetilde{w}\left(\lambda\frac{1}{\sqrt{Z}} + (1-\lambda)\sqrt{Z}\right)\left(\theta\frac{1}{\sqrt{Z}} + (1-\theta)\sqrt{Z}\right). \end{aligned}$$

We therefore get

$$\begin{aligned} \begin{pmatrix}\lambda & 1-\lambda\end{pmatrix} G_\beta \begin{pmatrix}\theta \\ 1-\theta\end{pmatrix} &= \frac{\widetilde{w}\big(\lambda\frac{1}{\sqrt{Z}} + (1-\lambda)\sqrt{Z}\big)\big(\theta\frac{1}{\sqrt{Z}} + (1-\theta)\sqrt{Z}\big)}{\widetilde{w}\gamma\big((1-\lambda)\sqrt{Z} + \lambda\frac{1}{\sqrt{Z}}\big)^2} \\ &= \frac{\theta\frac{1}{\sqrt{Z}} + (1-\theta)\sqrt{Z}}{\gamma\big((1-\lambda)\sqrt{Z} + \lambda\frac{1}{\sqrt{Z}}\big)}. \end{aligned}$$

■

4. Main theorem

Some of the results are based on manipulations on graphs; mostly we will use restriction of graphs with wired boundary condition.

Definition 4. Let $\mathcal{G} = (V, E)$ be a locally finite, non-directed graph. Let $(W_e)_{e\in E}$ be a family of weights on the edges of $\mathcal{G} = (V, E)$. Let A be a finite subset of V. The *restriction with wired boundary condition* $(\widetilde{V}^A, \widetilde{E}^A)$, $\widetilde{W}^A$ of the weighted graph $\mathcal{G}, W$ to the subset A of vertices is defined by

$$\widetilde{V}^A := A \cup \{x_A\},$$
$$\widetilde{E}^A := \big\{\{x, y\} \in E : x, y \in A\big\} \cup \big\{\{x_A, y\} \subset \widetilde{V}^A : \exists x \in V\backslash A, \{x, y\} \in E\big\},$$
$$\forall \{x, y\} \in \widetilde{E}^A, x, y \in A, W^A_{\{x,y\}} := W_{\{x,y\}},$$
$$\forall x \in \widetilde{V}^A\backslash\{x_A\} \text{ such that } \{x_A, x\} \in \widetilde{E}^A, W^A_{\{x_A,a\}} := \sum_{y\in v\backslash A} 1_{\{x,y\}\in E} W_{\{x,y\}}.$$

We can now prove our main theorem.

Proof of Theorem 5. According to Proposition 1.2.1, the marginal law of $(\beta_i)_{1\le i\le n}$ is the same under $\nu^{W^-,0}_{n+2}$, $\nu^{W^+,0}_{n+2}$ and $\nu^{W^\infty,0}_{n+1}$ and is equal to $\nu^{W,\eta}_n$ for some $\eta \in \mathbf{R}^n$. Let H be distributed according to $\widetilde{\nu}^{W,\eta}_n$. Let $K \in [0, \infty)$ be the random variable defined by

$$K := {}^tW^2 H^{-1} W^1,$$

and $\widetilde{K}$ the random matrix defined by

$$\widetilde{K} = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}.$$

Let $X^1 \in [0, \infty)^{n+2}$. Let $\alpha_1(X^1)$ and $\alpha_2(X^1)$ be the numbers defined in Lemma 2.2.2 and $\alpha(X^1) := \alpha_1(X^1) + \alpha_2(X^1)$. Let $\lambda \in [0, 1]$ be the random variable defined by

$$\lambda := \begin{cases} \frac{\alpha_1(X^1)}{\alpha_1(X^1)+\alpha_2(X^1)} & \text{if } \alpha_1(X^1) + \alpha_2(X^1) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\delta \in [-1, 1]$ the random variable defined by $\delta := 1 - 2\lambda$.

If $n + 1$ and $n + 2$ are H^--connected then $K + w^- > 0$. Let γ be a random variable distributed according to a Gamma(1/2, 1/2) distribution and independent of H. Now, knowing H, let U^- and U^+ be random variables distributed according to $\widetilde{\mathcal{N}}(K + w^-, \delta)$ and $\widetilde{\mathcal{N}}(K + w^+, \delta)$ respectively and such that

$$\forall \delta' \in [-1, 1], \quad \mathbb{E}\left(\frac{1 + \delta' \frac{U^-}{\sqrt{(U^-)^2+4}}}{1 + \delta \frac{U^-}{\sqrt{(U^-)^2+4}}} \,\middle|\, U^+\right) = \frac{1 + \delta' \frac{U^+}{\sqrt{(U^+)^2+4}}}{1 + \delta \frac{U^+}{\sqrt{(U^+)^2+4}}}.$$

Such random variables exist by Lemma 3.2.2. We define the positive random variables Z^- and Z^+ by

$$U^- = \sqrt{Z^-} - \frac{1}{\sqrt{Z^-}}, \quad U^+ = \sqrt{Z^+} - \frac{1}{\sqrt{Z^+}}.$$

Now, we define the random variables $\widetilde{\beta}^-_{n+1}, \widetilde{\beta}^-_{n+2}, \widetilde{\beta}^+_{n+1}$ and $\widetilde{\beta}^+_{n+2}$ by

$$\begin{aligned}
2\widetilde{\beta}^-_{n+1} &= \big(K + w^- + \lambda(1-\lambda)\gamma\big)Z^- + \lambda^2\gamma,\\
2\widetilde{\beta}^-_{n+2} &= \big(K + w^- + \lambda(1-\lambda)\gamma\big)/Z^- + (1-\lambda)^2\gamma,\\
2\widetilde{\beta}^+_{n+1} &= \big(K + w^+ + \lambda(1-\lambda)\gamma\big)Z^+ + \lambda^2\gamma,\\
2\widetilde{\beta}^+_{n+2} &= \big(K + w^+ + \lambda(1-\lambda)\gamma\big)/Z^+ + (1-\lambda)^2\gamma.
\end{aligned}$$

Let

$$\widetilde{K}^- := \begin{pmatrix} 0 & w^- + K \\ w^- + K & 0 \end{pmatrix}, \quad \widetilde{K}^+ := \begin{pmatrix} 0 & w^+ + K \\ w^+ + K & 0 \end{pmatrix}.$$

By Lemma 3.1.1, knowing K and δ, $(\widetilde{\beta}^-_{n+1}, \widetilde{\beta}^-_{n+2})$ and $(\widetilde{\beta}^-_{n+1}, \widetilde{\beta}^-_{n+2})$ are distributed according to $\nu_2^{\widetilde{K}^-,0}$ and $\nu_2^{\widetilde{K}^+,0}$ respectively. Now we can define

$$H^\pm = \begin{pmatrix} H & -W^1 & -W^2 \\ -{}^tW^1 & 2\widetilde{\beta}^\pm_{n+1} + {}^tW^1H^{-1}W^1 & -w^\pm \\ -{}^tW^2 & -w^\pm & 2\widetilde{\beta}^\pm_{n+2} + {}^tW^2H^{-1}W^2 \end{pmatrix},$$

$$H^\infty = \begin{pmatrix} H & -W^1 - W^2 \\ -{}^tW^1 - {}^tW^2 & \gamma + ({}^tW^1 + {}^tW^2)H^{-1}(W^1 + W^2) \end{pmatrix}.$$

By Proposition 1.2.1 and Lemma 2.1.1, H^-, H^+ and H^∞ are distributed according to $\widetilde{\nu}^{W^-,0}_{n+2}, \widetilde{\nu}^{W^+,0}_{n+2}$ and $\widetilde{\nu}^{W^\infty,0}_{n+1}$ respectively. Let G^-, G^+ and G^∞ be the inverses of H^-, H^+ and H^∞ respectively. Let

$$G^{22,\pm} := \begin{pmatrix} G^\pm(n+1,n+1) & G^\pm(n+1,n+2) \\ G^\pm(n+2,n+1) & G^\pm(n+2,n+2) \end{pmatrix},$$

$$G^{22,\infty} := \big(G^\infty(n+1,n+1)\big).$$

Notice that by Schur's lemma (Lemma 2.1.1), $G^\infty(n+1,n+1) = 1/\gamma$.

For any $X^2 \in [0,\infty)^{n+2}$, by Lemma 2.2.2 there exist non-negative random variables $C(X^1,X^2), \alpha_1(X^2)$ and $\alpha_2(X^2)$ that only depend on H, W^1 and W^2 such that

$${}^tX^1G^-X^2 = C(X^1,X^2) + \big(\alpha_1(X^1) \quad \alpha_2(X^1)\big)\, G^{22,-} \begin{pmatrix} \alpha_1(X^2) \\ \alpha_2(X^2) \end{pmatrix},$$

$${}^tX^1G^+X^2 = C(X^1,X^2) + \big(\alpha_1(X^1) \quad \alpha_2(X^1)\big)\, G^{22,+} \begin{pmatrix} \alpha_1(X^2) \\ \alpha_2(X^2) \end{pmatrix},$$

$${}^t\overline{X}^1G^\infty\overline{X}^2 = C(X^1,X^2) + (\alpha_1(X^1) + \alpha_2(X^1))G^{22,\infty}(\alpha_1(X^2) + \alpha_2(X^2)).$$

Let $\alpha(X^2) := \alpha_1(X^2) + \alpha_2(X^2)$ and let $\theta \in [0, 1]$ be the random variable defined by

$$\theta := \begin{cases} \frac{\alpha_1(X^2)}{\alpha_1(X^2)+\alpha_2(X^2)} & \text{if } \alpha_1(X^2) + \alpha_2(X^2) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} {}^tX^1G^-X^2 &= C(X^1, X^2) + \alpha(X^1)\alpha(X^2)\big(\lambda \quad 1-\lambda\big)G^{22,-}\begin{pmatrix}\theta \\ 1-\theta\end{pmatrix}, \\ {}^tX^1G^+X^2 &= C(X^1, X^2) + \alpha(X^1)\alpha(X^2)\big(\lambda \quad 1-\lambda\big)G^{22,+}\begin{pmatrix}\theta \\ 1-\theta\end{pmatrix}, \\ {}^t\overline{X}^1G^\infty\overline{X}^2 &= C(X^1, X^2) + \alpha(X^1)\alpha(X^2)G^{22,\infty}. \end{aligned}$$

By Lemma 3.2.3, we have

$$\big(\lambda \quad 1-\lambda\big)G^{22,\pm}\begin{pmatrix}\theta \\ 1-\theta\end{pmatrix} = \frac{\theta\frac{1}{\sqrt{Z^\pm}} + (1-\theta)\sqrt{Z^\pm}}{\gamma\big(\lambda\frac{1}{\sqrt{Z^\pm}} + (1-\lambda)\sqrt{Z^\pm}\big)}.$$

Define $\delta' := 1 - 2\theta$. By definition of Z^- and Z^+, we also have

$$\big(\lambda \quad 1-\lambda\big)G^{22,\pm}\begin{pmatrix}\theta \\ 1-\theta\end{pmatrix} = \frac{1 + \delta'\frac{U^\pm}{\sqrt{(U^\pm)^2+4}}}{\gamma\big(1 + \delta\frac{U^\pm}{\sqrt{(U^\pm)^2+4}}\big)}.$$

Finally, by definition of U^- and U^+, we have

$$\mathbb{E}\left(\big(\lambda \quad 1-\lambda\big)G^{22,-}\begin{pmatrix}\theta \\ 1-\theta\end{pmatrix} \,\middle|\, H^+\right) = \big(\lambda \quad 1-\lambda\big)G^{22,+}\begin{pmatrix}\theta \\ 1-\theta\end{pmatrix}$$

and therefore

$$\mathbb{E}({}^tX^1G^-X^2 \mid H^+) = {}^tX^1G^+X^2.$$

Similarly, by Lemma 3.2.1,

$$\mathbb{E}({}^tX^1G^+X^2 \mid H^\infty) = {}^t\overline{X^1}G^\infty\overline{X^2}.$$

We also have

$$\begin{aligned} {}^tX^1G^-X^1 &= C(X^1, X^1) + (\alpha_1(X^1) + \alpha_2(X^1))^2\frac{1}{\gamma} \\ &= {}^tX^1G^+X^1 = {}^t\overline{X^1}G^\infty\overline{X^1}. \end{aligned}$$ ■

Proof of Theorem 6. Let $i \in [\![1, n]\!]$. We will only show this result when W^- and W^+ differ by only two symmetric coefficients (i.e. one edge): (k, l) and (l, k). We can assume that $W^-(n-1, n) < W^+(n-1, n)$ because of the symmetries of the family of laws $\widetilde{\nu}_n^{W,0}$.

For any $j_1, j_2 \in [\![1,n]\!]$, j_1 and j_2 are W^--connected. This means that by the main theorem (where X^1 is defined by $X^1_i = 1$ and $X^1_j = 0$ for $j \neq i$), there exist matrices H^- and H^+ distributed according to $\widetilde{\nu}_n^{W^-,0}$ and $\widetilde{\nu}_n^{W^+,0}$ respectively, with inverses G^- and G^+ respectively, and such that

- $G^-(i,i) = G^+(i,i)$ almost surely,
- $\forall X \in [0,\infty)^n$, $\mathbb{E}(\sum_{j=1}^n X_j G^-(i,j) \mid H^+) = \sum_{j=1}^n X_j G^+(i,j)$.

This means that for any convex function f and any vector $X \in [0,\infty)^n$,

$$\mathbb{E}\left(f\left(\frac{\sum_{j=1}^n X_j G^-(i,j)}{G^-(i,i)}\right)\right) \geq \mathbb{E}\left(f\left(\frac{\sum_{j=1}^n X_j G^+(i,j)}{G^+(i,i)}\right)\right). \qquad \blacksquare$$

5. Proofs of Theorems 1–3

5.1. *Proof of Theorem 1*

Let $d_{\mathcal{G}}(\cdot,\cdot)$ be the graph distance on $\mathcal{G}$. Let $\mathcal{G}_n$ be the restriction with wired boundary condition of the graph $\mathcal{G}$ to the vertices at distance less than n from the origin (see Definition 4) . This means that $\mathcal{G}_n = (V_n, E_n)$ with

$$\begin{aligned} V_n &= \{x \in V : d_{\mathcal{G}}(0,x) < n\} \cup \{\delta_n\}, \\ E_n &= \{\{x,y\} \in E : (x,y) \in V_n^2\} \\ &\quad \cup \{\{x,\delta_n\} : d_{\mathcal{G}}(0,x) = n-1, \exists y \in V \setminus V_n, d_{\mathcal{G}}(x,y) = 1\}. \end{aligned}$$

Let $|V_n|$ be the number of vertices in V_n. Let $W_n^- \in M_{|V_n|}(\mathbf{R})$ and $W_n^+ \in M_{|V_n|}(\mathbf{R})$ be the symmetric matrices defined by:

- for any $x, y \in V_n$ such that $\{x,y\} \notin E_n$, $W_n^-(x,y) = W_n^+(x,y) := 0$,
- for any $x, y \in V_n \setminus \{\delta_n\}$, $W_n^-(x,y) := W^-_{\{x,y\}}$ and $W_n^+(x,y) := W^+_{\{x,y\}}$,
- for any $x \in V_n \setminus \{\delta_n\}$, $W_n^-(x,\delta_n) = W_n^-(\delta_n,x) := \sum_{y \in V, \{x,y\} \in E} W^-_{\{x,y\}} 1_{y \notin V_n}$,
- for any $x \in V_n \setminus \{\delta_n\}$, $W_n^+(x,\delta_n) = W_n^+(\delta_n,x) := \sum_{y \in V, \{x,y\} \in E} W^+_{\{x,y\}} 1_{y \notin V_n}$.

This means that for any $x, y \in V_n$, $W_n^-(x,y) \leq W_n^+(x,y)$. Let H_n^- and H_n^+ be random matrices distributed according to $\widetilde{\nu}_{|V_n|}^{W_n^-,0}$ and $\widetilde{\nu}_{|V_n|}^{W_n^+,0}$ respectively. Let G_n^- and G_n^+ be the inverses of H_n^- and H_n^+ respectively. By [18, Theorem 1], there exist non-negative random variables $\psi^-(0)$ and $\psi^+(0)$ such that

$$\frac{G_n^-(0,\delta_n)}{G_n^-(\delta_n,\delta_n)} \xrightarrow[n\to\infty]{} \psi^-(0) \text{ in law}, \quad \frac{G_n^+(0,\delta_n)}{G_n^+(\delta_n,\delta_n)} \xrightarrow[n\to\infty]{} \psi^+(0) \text{ in law}.$$

Furthermore, still by [18, Theorem 1], we have

$$\begin{aligned} \mathbb{P}(\text{VRJP with weights } W^- \text{ is recurrent}) &= \mathbb{P}(\psi^-(0) = 0), \\ \mathbb{P}(\text{VRJP with weights } W^+ \text{ is recurrent}) &= \mathbb{P}(\psi^+(0) = 0). \end{aligned}$$

Let $f : [0,\infty) \to \mathbf{R}$ be a continuous, bounded, convex function. By Theorem 6, for any $n \geq 1$,

$$\mathbb{E}\left(f\left(\frac{G_n^-(0,\delta_n)}{G_n^-(\delta_n,\delta_n)}\right)\right) \geq \mathbb{E}\left(f\left(\frac{G_n^+(0,\delta_n)}{G_n^+(\delta_n,\delta_n)}\right)\right).$$

This means that $\mathbb{E}(f(\psi^-(0))) \geq \mathbb{E}(f(\psi^+(0)))$. For any $n \geq 1$, let $f_n : [0,\infty) \to \mathbf{R}$ be defined by

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leq x \leq 1/n, \\ 0 & \text{if } x > 1/n. \end{cases}$$

For any $n \geq 1$, the function f_n is continuous, bounded and convex, so $\mathbb{E}(f_n(\psi^-(0))) \geq \mathbb{E}(f_n(\psi^+(0)))$. We notice that

$$\mathbb{E}(f_n(\psi^-(0))) \xrightarrow[n\to\infty]{} \mathbb{P}(\psi^-(0) = 0), \quad \mathbb{E}(f_n(\psi^+(0))) \xrightarrow[n\to\infty]{} \mathbb{P}(\psi^+(0) = 0).$$

This means that $\mathbb{P}(\psi^-(0) = 0) \geq \mathbb{P}(\psi^+(0) = 0)$ and therefore the probability that the VRJP with weights w^- is recurrent is greater than the probability that the VRJP with weights w^+ is recurrent. ■

5.2. *Proof of Theorem 2*

Fix $d \geq 3$. By [18, Proposition 3], for any $w \in (0,\infty)$, the VRJP on $\mathbf{Z}^d$ is either almost surely recurrent or almost surely transient. Furthermore, by Theorem 1, the probability that the VRJP is recurrent is non-increasing in the initial weight. Therefore, there exists $w_d \in [0,\infty]$ such that the VRJP on $\mathbf{Z}^d$ with initial weight $w \in (0,\infty)$ is recurrent if $w < w_d$ and transient if $w > w_d$. Since the VRJP is recurrent in dimension 3 for small enough weights [16, Corollary 3], we have $w_d \neq 0$, and since it is transient for large enough weights [18, Lemma 9], $w_d \neq \infty$. ■

5.3. *Proof of Theorem 3*

Fix $d \geq 3$. Let E^d be the set of vertices in $\mathbf{Z}^d$. Let $0 < a^- < a^+$. Let $(W_e^-)_{e\in E}$ be iid random Gamma variables with parameter a^- and let $(W_e')_{e\in E}$ be iid random Gamma variables with parameter $a^+ - a^-$, independent of $(W_e^-)_{e\in E}$. By [16, Theorem 1], the ERRW on $\mathbf{Z}^d$ with initial weight $a^- \in (0,\infty)$ is a mixture of VRJPs on $\mathbf{Z}^d$ where the weights are $(W_e^-)_{e\in E}$, and the ERRW on $\mathbf{Z}^d$ with initial weight $a^+ \in (0,\infty)$ is a mixture of VRJPs on $\mathbf{Z}^d$ where the weights are $(W_e^- + W_e')_{e\in E}$. Now, by Theorem 6, the VRJP with weights $(W_e^-)_{e\in E}$ has a higher probability of being recurrent than the VRJP with weights $(W_e^- + W_e')_{e\in E}$. Therefore the probability that the ERRW with constant weight equal to a is recurrent is non-increasing in a. By [18, Proposition 5], the ERRW with initial weight a is either almost surely transient or almost surely recurrent. Therefore, there exists $a_d \in [0,\infty]$ such that the ERRW on $\mathbf{Z}^d$ with initial weight $a \in (0,\infty)$ is recurrent if $a < a_d$ and transient if $a > a_d$. Since the ERRW is recurrent in dimension 3 for small enough weights, we have $a_d \neq 0$, and since it is transient for large enough weights, $a_d \neq \infty$. ■

6. Proof of Theorem 4

6.1. Preliminaries

Definition 5. Let $\mathcal{G} = (V, E)$ be a finite graph and $(W_e)_{e\in E}$ be positive weights. Let H_β be the random matrix distributed according to $\widetilde{\nu}_n^{W,0}$ and G_β its inverse. Let $x, y \in V$ be distinct vertices of $\mathcal{G}$. The *effective weight* between x and y, $w^{\text{eff}}_{x,y}$, is the random variable defined by

$$w^{\text{eff}}_{x,y} := \frac{G_\beta(x,y)}{G_\beta(x,x)G_\beta(y,y) - G_\beta(x,y)^2}.$$

Remark 1. Let $\mathcal{G} = (V, E)$ be a finite graph and $(W_e)_{e\in E}$ be positive weights. Let $(\beta_i)_{i\in V}$ be random variables distributed according to $\nu_n^{W,0}$, H_β the corresponding matrix (distributed according to $\widetilde{\nu}_n^{W,0}$) and G_β its inverse. Let $x, y \in V$ be distinct vertices of $\mathcal{G}$ and w^{eff} the effective weight between x and y. Let $V_1 := V\backslash\{x, y\}$ and $V_2 := \{x, y\}$ be two subsets of V. The corresponding decomposition of H_β is

$$H_\beta = \begin{pmatrix} H_\beta^{V_1} & -W^{V_1,V_2} \\ -{}^tW^{V_1,V_2} & H_\beta^{V_2} \end{pmatrix}.$$

By Lemma 2.1.1,

$$w^{\text{eff}} = W_{x,y} + \left({}^tW^{V_1,V_2}(H_\beta^{V_1})^{-1}W^{V_1,V_2}\right)(x,y). \tag{6.1}$$

Furthermore, by Lemmas 1.2.1 and 2.1.1, the law of $\frac{G_\beta(x,y)}{G_\beta(y,y)}$ knowing the β-field on V_1 is the same as the law of $\frac{G_\beta(z_1,z_2)}{G_\beta(z_2,z_2)}$ on a two-vertex graph $\{z_1, z_2\}$ where $W_{z_1,z_2} = w^{\text{eff}}$.

Lemma 6.1.1. *Let $\mathcal{G} = (V, E)$ be a finite graph and $x_0, \delta \in V$ two distinct vertices. Let $(c_e)_{e\in E}$ be a family of random (not necessarily independent) positive conductances. Let c^{eff} be the (random) effective conductance between x_0 and δ for the electrical network with initial conductances $(c_e)_{e\in E}$. Let $\overline{c}^{\text{eff}}$ be the equivalent conductance between x_0 and δ if we set conductances $(\overline{c}_e)_{e\in E}$ defined by $\overline{c}_e := \mathbb{E}(c_e)$ on $\mathcal{G}$. Then*

$$\mathbb{E}(c^{\text{eff}}) \le \overline{c}^{\text{eff}}.$$

Proof. Let $(V_x)_{x\in V}$ be the (random) potential with $V_{x_0} = 1$ and $V_\delta = 0$ that minimizes the energy

$$\mathcal{E} := \frac{1}{2}\sum_{\{x,y\}\in E} c_e(V_x - V_y)^2.$$

This potential is harmonic on $V\backslash\{x_0, \delta\}$ by the Dirichlet principle and therefore $(V_x - V_y)_{(x,y)\in E}$ is the flow that minimizes the energy and we get

$$\mathcal{E} := \frac{1}{2}c^{\text{eff}}.$$

Now let $(\overline{V}_x)_{x\in V}$ be the potential with $\overline{V}_{x_0} = 1$ and $\overline{V}_\delta = 0$ that minimizes the energy

$$\overline{\mathcal{E}} := \frac{1}{2} \sum_{\{x,y\}\in E} \overline{c}_e(\overline{V}_x - \overline{V}_y)^2.$$

We have

$$\overline{\mathcal{E}} := \frac{1}{2}\overline{c}^{\text{eff}}.$$

Now since V minimizes $\mathcal{E}$, we have

$$\mathcal{E} \le \frac{1}{2} \sum_{\{x,y\}\in E} c_e(\overline{V}_x - \overline{V}_y)^2.$$

By taking the expectation we get

$$\mathbb{E}(\mathcal{E}) \le \frac{1}{2} \sum_{\{x,y\}\in E} \mathbb{E}(c_e)(\overline{V}_x - \overline{V}_y)^2.$$

Therefore

$$\frac{1}{2}\mathbb{E}(c^{\text{eff}}) \le \frac{1}{2} \sum_{\{x,y\}\in E_{n+1}} \mathbb{E}(c_e)(\overline{V}_x - \overline{V}_y)^2.$$

Thus, $\frac{1}{2}\mathbb{E}(c^{\text{eff}}) \le \frac{1}{2}\overline{c}^{\text{eff}}$, so $\mathbb{E}(c^{\text{eff}}) \le \overline{c}^{\text{eff}}$. ■

Proposition 6.1.2. *Let $\mathcal{G} = (V, E)$ be a finite graph and $x_0, \delta \in V$ two distinct vertices. Let $(W_e)_e$ be a family of random (not necessarily independent) positive weights. Let w^{eff} be the (random) effective weight between x_0 and δ for the VRJP with weights $(W_e)_{e\in E}$. Let c^{eff} be the effective conductance between x_0 and δ if we set conductances $(c_e)_{e\in E}$ defined by $c_e := \mathbb{E}(W_e)$ on $\mathcal{G}$. We have the following inequality:*

$$\mathbb{E}(w^{\text{eff}}) \le c^{\text{eff}}.$$

Proof. We will show the result by induction on the number of vertices of the graph. If the graph has two vertices $\{x_0, \delta\}$ (and therefore only one edge) the result is obvious.

Now we assume that the result is true for all graphs with n vertices or fewer, and we will show it for any graph with $n + 1$ vertices.

Let $\mathcal{G}_{n+1} = (V_{n+1}, E_{n+1})$ be a finite graph with exactly $n + 1$ vertices, including x_0 and δ. Let $(W_e^{n+1})_{e\in E_{n+1}}$ be random weights on E_{n+1}. Let H_β be a random matrix distributed according to $\widetilde{\nu}_n^{W,0}$. Let w_{n+1}^{eff} be the (random) effective weight between x_0 and δ. Let $(c_e^{n+1})_{e\in E_{n+1}}$ be the deterministic conductances defined by $c_e^{n+1} = \mathbb{E}(W_e^{n+1})$. We define two effective conductances between x_0 and δ on $\mathcal{G}_{n+1}$: one for random conductances $W^{n+1}(\overline{c}_{n+1}^{\text{eff}})$ and the other for deterministic conductances $(c_e)_{e\in E}$ (c_{n+1}^{eff}). By Lemma 6.1.1,

$$\mathbb{E}(\overline{c}_{n+1}^{\text{eff}}) \le \mathbb{E}(\overline{c}_{n+1}^{\text{eff}}). \tag{6.2}$$

Now, let $y \in V_{n+1}$ be a vertex that is neither x_0 nor δ. Let $\mathcal{G}_n^y = (V_n^y, E_n^y)$ be the complete graph with n elements with $V_n^y = V_{n+1}\backslash\{y\}$. We can decompose V_{n+1} into V_n^y and $\{y\}$; the corresponding decomposition of H_β is given by

$$H_\beta := \begin{pmatrix} H_\beta^{V_n} & -W^{V_1,y} \\ -{}^tW^{V_1,y} & 2\beta_y \end{pmatrix}.$$

By Lemma 2.1.1, w_{n+1}^{eff} knowing H_β is equal to the effective weight w_n^{eff} on the graph $\mathcal{G}_n^y$ for weights and the β-field given by the matrix $H_\beta^{V_n^y} - \frac{1}{2\beta_y} W^{V_1^y,y}\, {}^tW^{V_1^y,y}$. This matrix knowing β_y and W^{n+1} is distributed according to $\nu_n^{W'}$ with $W'_{x_1,x_2} = W^{n+1}_{x_1,x_2} + \frac{W^{n+1}_{x_1,y}W^{n+1}_{y,x_2}}{2\beta_y}$. By Proposition 1.2.1, if $K_y := \sum_{x,\{x,y\}\in E_{n+1}} W^{n+1}_{y,x}$, the expectation of $\frac{1}{2\beta_y}$ knowing W^{n+1} is given by

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{2\beta_y}\right) &= \int_0^\infty \frac{1}{2b}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2b}}\exp\left(-\frac{1}{2}\left(2b + \frac{K_y^2}{2b} - 2K_y\right)\right)\mathrm{d}b \\
&= \frac{1}{K_y}\int_0^\infty \frac{1}{\sqrt{2s}}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2}\left(\frac{K_y^2}{2s} + 2s - 2K_y\right)\right)\mathrm{d}s \quad \left(\text{with } s = \frac{K_y^2}{4b}\right) \\
&= \frac{1}{K_y} \quad \text{by definition of } \nu_1^{0,K_y}.
\end{aligned}$$

Therefore for any $x_1, x_2 \in V_n^y$,

$$\mathbb{E}(W'_{x_1,x_2} \mid W^{n+1}) = W^{n+1}_{x_1,x_2} + \frac{W^{n+1}_{x_1,y}W^{n+1}_{y,x_2}}{\sum_x W^{n+1}_{y,x}}.$$

Similarly the effective conductance $\overline{c}_{n+1}^{\text{eff}}$ between x_0 and δ on $\mathcal{G}_{n+1}$ with conductances W^{n+1} is equal to the effective conductance $\overline{c}_n^{\text{eff}}$ between x_0 and δ on $\mathcal{G}_n^y$ with conductances $\overline{c}'_{x_1,x_2} := W^{n+1}_{x_1,x_2} + \frac{W^{n+1}_{x_1,y}W^{n+1}_{y,x_2}}{\sum_x W_{y,x}}$. This means that, for any $e \in E_n^y$,

$$\mathbb{E}(W'_e \mid W^{n+1}) = \overline{c}'_e,$$

so by the inductive hypothesis

$$\mathbb{E}(w_n^{\text{eff}}) \leq \mathbb{E}(\overline{c}_n^{\text{eff}}),$$

which implies that

$$\mathbb{E}(w_{n+1}^{\text{eff}}) \leq \mathbb{E}(c_{n+1}^{\text{eff}}).$$

■

6.2. *Proof of Theorem 4*

Once we can compare the effective weight for the VRJP to effective conductance for an electrical network, the proof is quite straightforward. Let $\widetilde{W}_e$ be weights and let $c_e := \mathbb{E}(\widetilde{W}_e)$ be conductances. For any $n > 0$ we define $\overline{S}_n$ to be the vertices of V at distance n or more from x_0. Then $\mathcal{G}_n, \widetilde{W}^n$ (with $\mathcal{G}_n := (V_n, E_n)$) is the restriction with wired boundary condition of the weighted graph $\mathcal{G}, \widetilde{W}$ to $V\backslash\overline{S}_n$ (see Definition 4) and δ_n is the point obtained by fusing all points of $\overline{S}_n$ into one. For any n, let H_n be distributed according to $\widetilde{\nu}_{|V_n|}^{\widetilde{W}^n,\mathbf{0}}$ and let G_n be its inverse. By [18, Theorem 1], to show that the VRJP with weights $\widetilde{W}_e$ is recurrent, we only need to show that $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)} \to 0$ in probability. By Remark 1, the law of $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$ is entirely determined by the law of the effective weight. Since the effective conductive converges to 0, the effective weights converges to 0 in probability by Lemma 6.1.2. Then, by Remark 1, the law of $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$ knowing the effective weight is the same as if the graph had only two points x_0 and δ, with a weight equal to the effective weight between them. Now let (β_1, β_2) be distributed according to $\nu_2^{w^{\text{eff}},\mathbf{0}}$. The law of $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$ is the same as the law of

$$\frac{\frac{w^{\text{eff}}}{4\beta_1\beta_2-(w^{\text{eff}})^2}}{\frac{2\beta_1}{4\beta_1\beta_2-(w^{\text{eff}})^2}} = \frac{w^{\text{eff}}}{2\beta_1}.$$

By taking $\lambda = 1$ in Lemma 3.1.1, we get

$$\frac{w^{\text{eff}}}{2\beta_1} = \frac{w^{\text{eff}}}{W^{\text{eff}}\frac{1}{z}} = Z,$$

where the law of Z (knowing w^{eff}) is given by

$$\sqrt{\frac{w^{\text{eff}}}{2\pi}}\,\frac{1}{z\sqrt{z}}\exp\left(-\frac{w^{\text{eff}}}{2}\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)^2\right)1_{z>0}\,dz.$$

If w^{eff} goes to 0 then Z converges to 0 in probability and therefore $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$ converges to 0 in probability and we get the result we want. ∎

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