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Bases of tensor products and geometric Satake correspondence

To the memory of C. S. Seshadri

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Abstract. The geometric Satake correspondence can be regarded as a geometric construction of rational representations of a complex connected reductive group G. In their study of this correspondence, Mirković and Vilonen introduced algebraic cycles that provide a linear basis in each irreducible representation. Generalizing this construction, Goncharov and Shen define a linear basis in each tensor product of irreducible representations. We investigate these bases and show that they share many properties with the dual canonical bases of Lusztig.

Keywords. Affine Grassmannian, Mirković–Vilonen cycles, tensor product, fusion product, canonical bases

1. Introduction

Let G be a connected reductive group over the field of complex numbers, endowed with a Borel subgroup and a maximal torus. Let Λ^+ be the set of dominant weights relative to these data. Given $\lambda \in \Lambda^+$, denote by $V(\lambda)$ the irreducible rational representation of G with highest weight λ. Given a finite sequence $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ in Λ^+, define

$$V(\boldsymbol{\lambda}) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n).$$

A construction due to Mirković and Vilonen [39] in the context of the geometric Satake correspondence endows the spaces $V(\lambda)$ with linear bases. Specifically, $V(\lambda)$ is identified with the intersection homology of a parabolic affine Schubert variety $\overline{\mathrm{Gr}^\lambda}$, while (after an ingenious use of hyperbolic localization) the fundamental classes of so-called "Mirković–

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Vilonen cycles" contained in $\overline{\mathrm{Gr}^{\boldsymbol{\lambda}}}$ form a basis of this intersection homology. In [22], Goncharov and Shen extend this construction to the tensor products $V(\boldsymbol{\lambda})$. In the present paper, we investigate the properties of these bases, which we call MV bases.

We show that the MV basis of a representation $V(\boldsymbol{\lambda})$ is compatible with the isotypic filtration of this representation. This basis is also compatible with the action of the Chevalley generators of the Lie algebra $\mathfrak{g}$ of G, in the sense that the leading terms of the action define on the basis the structure of a crystal in the sense of Kashiwara. (For $n = 1$, this fact is due to Braverman and Gaitsgory [9].) Let us transport this crystal structure onto the set $\mathscr{Z}(\boldsymbol{\lambda})$ of MV cycles, which naturally indexes the MV basis of $V(\boldsymbol{\lambda})$. Then, with the help of the path model [33], we prove that there is a natural crystal isomorphism

$$\mathscr{Z}(\boldsymbol{\lambda}) \cong \mathscr{Z}(\lambda_1) \otimes \cdots \otimes \mathscr{Z}(\lambda_n).$$

(For $n = 2$, this isomorphism is again due to Braverman and Gaitsgory.)

We study the transition matrix between the MV basis of a tensor product $V(\boldsymbol{\lambda})$ and the tensor product of the MV bases of the different factors $V(\lambda_1), \ldots, V(\lambda_n)$. Using the fusion product in the sense of Beilinson and Drinfeld [6], we explain that the entries of this transition matrix can be computed as intersection multiplicities. As a consequence, the transition matrix is unitriangular with nonnegative integral entries.

The properties stated above are analogues of results obtained by Lusztig about the dual canonical bases. To be specific, Lusztig [37] defines a notion of based module (module endowed with a basis enjoying certain specific properties) over the quantized enveloping algebra $U_v(\mathfrak{g})$ and shows the following facts:

- A simple module over $U_v(\mathfrak{g})$, endowed with its canonical basis, is a based module.
- The tensor product of finitely many based modules can be endowed with a basis that makes it a based module. This basis is constructed from the tensor product of the bases of the factors by adding corrective terms in a unitriangular fashion.
- The basis of a based module is compatible with the decreasing isotypic filtration of the module. Each subquotient in this filtration, endowed with the induced basis, is isomorphic as a based module to the direct sum of copies of a simple module endowed with its canonical basis.

The dual canonical bases for the representations $V(\boldsymbol{\lambda})$ (see [14]) can then be defined by dualizing Lusztig's construction and specializing the quantum parameter at $v = 1$.

Because of its compatibility with the isotypic filtration, the dual canonical basis of a tensor product $V(\boldsymbol{\lambda})$ yields a linear basis of the invariant subspace $V(\boldsymbol{\lambda})^G$. Just as well, the MV basis provides a linear basis (sometimes called the Satake basis) of $V(\boldsymbol{\lambda})^G$. The Satake basis and the dual canonical basis of $V(\boldsymbol{\lambda})^G$ generally differ (the paper [13] provides a counterexample); nonetheless we show that the Satake basis enjoys the first two items in Khovanov and Kuperberg's list of properties for the dual canonical basis [30]. In particular, after restriction to the fixed-point subspaces, the signed permutation

$$V(\lambda_1) \otimes V(\lambda_2) \otimes \cdots \otimes V(\lambda_n) \xrightarrow{\simeq} V(\lambda_2) \otimes \cdots \otimes V(\lambda_n) \otimes V(\lambda_1)$$

maps the Satake basis of the domain to that of the codomain.

As explained in [3], the MV bases of the irreducible representations $V(\lambda)$ can be glued together to produce a basis of the algebra $\mathbb{C}[N]$ of regular functions on the unipotent radical N of B. Of particular interest would be any relation to the cluster algebra structure on $\mathbb{C}[N]$ [7, 21]. The methods developed in the present paper allow for explicit computations. For instance, we show that the cluster monomials attached to certain seeds belong to the MV basis. However, $\mathbb{C}[N]$ is not of finite cluster type in general, which means that the cluster monomials do not span the whole space. We compute in type D_4 the MV basis element at a specific position not covered by cluster monomials; at this spot, the MV basis, the dual canonical basis and the dual semicanonical basis pairwise differ.

2. Mirković–Vilonen cycles and bases

In the whole paper G is a connected reductive group over $\mathbb{C}$, endowed with a Borel subgroup B and a maximal torus $T \subseteq B$. We denote by Λ the character lattice of T, by $\Phi \subseteq \Lambda$ the root system of (G, T), by $\Phi^\vee$ the coroot system, and by W the Weyl group. The datum of B determines a set of positive roots in Φ. We denote the dominance order on Λ by $\leq$ and the cone of dominant weights by Λ^+. We denote the half-sum of the positive coroots by ρ and regard it as a linear form $\rho : \Lambda \to \mathbb{Q}$; then $\rho(\alpha) = 1$ for each simple root α. The Langlands dual of G is the connected reductive group $G^\vee$ over $\mathbb{C}$ built from the dual torus $T^\vee = \Lambda \otimes_\mathbb{Z} \mathbb{G}_\mathrm{m}$ and the root system $\Phi^\vee$. The positive coroots define a Borel subgroup $B^\vee \subseteq G^\vee$.

2.1. Recollection on the geometric Satake correspondence

The geometric Satake correspondence was devised by Lusztig [36] and given its definitive shape by Beilinson and Drinfeld [6] and Mirković and Vilonen [39]. Additional references for the material presented in this section are [46] and [4].

As recalled in the introduction of [41], loop groups appear under several guises across mathematics: there is the differential-geometric variant, the algebraic-geometric one, etc. We adopt the framework of Lie theory and Kac–Moody groups [31]. For instance, though the affine Grassmannian is a (generally nonreduced) ind-scheme, we will only look at its complex ind-variety structure.

Let $\mathcal{O} = \mathbb{C}[\![z]\!]$ be the ring of formal power series in z with complex coefficients and let $\mathcal{K} = \mathbb{C}(\!(z)\!)$ be the fraction field of $\mathcal{O}$. The affine Grassmannian of the Langlands dual $G^\vee$ is the homogeneous space $\mathrm{Gr} = G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$. This space, a partial flag variety for an affine Kac–Moody group, is endowed with the structure of an ind-variety.

Each weight $\lambda \in \Lambda$ gives a point z^λ in $T^\vee(\mathcal{K})$, whose image in Gr is denoted by L_λ. The $G^\vee(\mathcal{O})$-orbit through L_λ in Gr, denoted by Gr^λ, is a smooth connected simply-connected variety of dimension $2\rho(\lambda)$. The Cartan decomposition implies that

$$\mathrm{Gr} = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}^\lambda; \quad \text{moreover} \quad \overline{\mathrm{Gr}^\lambda} = \bigsqcup_{\substack{\mu \in \Lambda^+ \\ \mu \leq \lambda}} \mathrm{Gr}^\mu.$$

Let Perv(Gr) be the category of $G^\vee(\mathcal{O})$-equivariant perverse sheaves on Gr (for the middle perversity) supported on finitely many $G^\vee(\mathcal{O})$-orbits, with coefficients in $\mathbb{C}$. This is an abelian semisimple category. The simple objects in Perv(Gr) are the intersection cohomology sheaves

$$\mathscr{I}_\lambda = \mathrm{IC}\big(\overline{\mathrm{Gr}^\lambda}, \underline{\mathbb{C}}\big).$$

(By convention, IC sheaves are shifted so as to be perverse.)

Let $\theta \in \Lambda$ be a dominant and regular weight. The embedding

$$\mathbb{C}^\times \xrightarrow{\theta} T^\vee(\mathbb{C}) \to G^\vee(\mathcal{K})$$

gives rise to an action of $\mathbb{C}^\times$ on Gr. For each $\mu \in \Lambda$, the point L_μ is fixed by this action; we denote its stable and unstable sets by

$$S_\mu = \Big\{x \in \mathrm{Gr} \;\Big|\; \lim_{c \to 0} \theta(c) \cdot x = L_\mu\Big\} \quad \text{and} \quad T_\mu = \Big\{x \in \mathrm{Gr} \;\Big|\; \lim_{c \to \infty} \theta(c) \cdot x = L_\mu\Big\}$$

and denote the inclusion maps by $s_\mu : S_\mu \to \mathrm{Gr}$ and $t_\mu : T_\mu \to \mathrm{Gr}$.

Given $\mu \in \Lambda$ and $\mathscr{A} \in \mathrm{Perv(Gr)}$, Mirković and Vilonen [39, Theorem 3.5] identify the homology groups

$$H_c\big(S_\mu, (s_\mu)^* \mathscr{A}\big) \quad \text{and} \quad H\big(T_\mu, (t_\mu)^! \mathscr{A}\big)$$

via Braden's hyperbolic localization, show that these groups are concentrated in degree $2\rho(\mu)$, and define

$$F_\mu(\mathscr{A}) = H^{2\rho(\mu)}\big(T_\mu, (t_\mu)^! \mathscr{A}\big) \quad \text{and} \quad F(\mathscr{A}) = \bigoplus_{\mu \in \Lambda} F_\mu(\mathscr{A}).$$

Then F is an exact and faithful functor from Perv(Gr) to the category of finite-dimensional Λ-graded $\mathbb{C}$-vector spaces. Mirković and Vilonen prove that F induces an equivalence $\overline{F}$ from Perv(Gr) to the category $\mathrm{Rep}(G)$ of finite-dimensional rational representations of G, the Λ-grading on $F(\mathscr{A})$ giving rise to the decomposition of $\overline{F}(\mathscr{A})$ into weight subspaces. In the course of the proof, it is shown that $\overline{F}$ maps $\mathscr{I}_\lambda$ to the irreducible highest weight representation $V(\lambda)$.

The map $G^\vee(\mathcal{K}) \to \mathrm{Gr}$ is a principal $G^\vee(\mathcal{O})$-bundle. From the $G^\vee(\mathcal{O})$-space Gr, we form the associated bundle

$$\mathrm{Gr}_2 = G^\vee(\mathcal{K}) \times^{G^\vee(\mathcal{O})} \mathrm{Gr}.$$

This space is called the 2-*fold convolution variety* and has the structure of an ind-variety. The action of $G^\vee(\mathcal{K})$ on Gr defines a map $m : \mathrm{Gr}_2 \to \mathrm{Gr}$ of ind-varieties. Let $p : G^\vee(\mathcal{K}) \to \mathrm{Gr}$ and $q : G^\vee(\mathcal{K}) \times \mathrm{Gr} \to \mathrm{Gr}_2$ be the quotient maps. Given two equivariant perverse sheaves $\mathscr{A}_1$ and $\mathscr{A}_2$ on Gr, there is a unique equivariant perverse sheaf $\mathscr{A}_1 \tilde{\boxtimes} \mathscr{A}_2$ on Gr_2 such that

$$p^* \mathscr{A}_1 \boxtimes \mathscr{A}_2 = q^*(\mathscr{A}_1 \tilde{\boxtimes} \mathscr{A}_2)$$

in the equivariant derived category of constructible sheaves on $G^\vee(\mathcal{K}) \times \mathrm{Gr}$. We then define the convolution product of $\mathscr{A}_1$ and $\mathscr{A}_2$ to be

$$\mathscr{A}_1 * \mathscr{A}_2 = m_*(\mathscr{A}_1 \tilde{\boxtimes} \mathscr{A}_2).$$

Using Beilinson and Drinfeld's fusion product, one defines associativity and commutativity constraints and obtains a monoidal structure on $\mathrm{Perv}(\mathrm{Gr})$. Then F is a tensor functor; in particular, the fusion product imparts an explicit identification of Λ-graded vector spaces

$$F(\mathscr{A}_1 * \mathscr{A}_2) \cong F(\mathscr{A}_1) \otimes F(\mathscr{A}_2)$$

for any $(\mathscr{A}_1, \mathscr{A}_2) \in \mathrm{Perv}(\mathrm{Gr})^2$.

2.2. *Mirković–Vilonen cycles*

In this paper we study tensor products $V(\boldsymbol{\lambda}) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is a sequence of dominant weights. Accordingly, we want to consider convolution products

$$\mathscr{I}_{\boldsymbol{\lambda}} = \mathscr{I}_{\lambda_1} * \cdots * \mathscr{I}_{\lambda_n}$$

and therefore need the n-fold convolution variety

$$\mathrm{Gr}_n = \underbrace{G^\vee(\mathcal{K}) \times^{G^\vee(\mathcal{O})} \cdots \times^{G^\vee(\mathcal{O})} G^\vee(\mathcal{K})}_{n \text{ factors } G^\vee(\mathcal{K})} / G^\vee(\mathcal{O}).$$

As is customary, we denote elements in Gr_n as classes $[g_1, \ldots, g_n]$ of tuples of elements in $G^\vee(\mathcal{K})$. Plainly $\mathrm{Gr}_1 = \mathrm{Gr}$; hence, we write the quotient map $G^\vee(\mathcal{K}) \to \mathrm{Gr}$ as $g \mapsto [g]$. We define a map $m_n : \mathrm{Gr}_n \to \mathrm{Gr}$ by setting $m_n([g_1, \ldots, g_n]) = [g_1 \ldots g_n]$.

Given $G^\vee(\mathcal{O})$-stable subsets $K_1, \ldots, K_n$ of Gr, we define

$$K_1 \tilde{\times} \cdots \tilde{\times} K_n = \{[g_1, \ldots, g_n] \in \mathrm{Gr}_n \mid [g_1] \in K_1, \ldots, [g_n] \in K_n\}.$$

Alternatively, $K_1 \tilde{\times} \cdots \tilde{\times} K_n$ can be defined as

$$\widehat{K}_1 \times^{G^\vee(\mathcal{O})} \cdots \times^{G^\vee(\mathcal{O})} \widehat{K}_n / G^\vee(\mathcal{O})$$

where each $\widehat{K}_j$ is the preimage of K_j under the quotient map $G^\vee(\mathcal{K}) \to \mathrm{Gr}$.

For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$, we set

$$\mathrm{Gr}_n^{\boldsymbol{\lambda}} = \mathrm{Gr}^{\lambda_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}^{\lambda_n}.$$

Viewing $\mathrm{Gr}_n^{\boldsymbol{\lambda}}$ as an iterated fibration with base Gr^{λ_1} and successive fibers $\mathrm{Gr}^{\lambda_2}, \ldots, \mathrm{Gr}^{\lambda_n}$, we infer that it is a smooth connected simply-connected variety of dimension $2\rho(|\boldsymbol{\lambda}|)$, where

$$|\boldsymbol{\lambda}| = \lambda_1 + \cdots + \lambda_n.$$

Also

$$\mathrm{Gr}_n = \bigsqcup_{\boldsymbol{\lambda} \in (\Lambda^+)^n} \mathrm{Gr}_n^{\boldsymbol{\lambda}} \quad \text{and} \quad \overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} = \overline{\mathrm{Gr}^{\lambda_1}} \tilde{\times} \cdots \tilde{\times} \overline{\mathrm{Gr}^{\lambda_n}} = \bigsqcup_{\substack{\boldsymbol{\mu} \in (\Lambda^+)^n \\ \mu_1 \leq \lambda_1, \ldots, \mu_n \leq \lambda_n}} \mathrm{Gr}_n^{\boldsymbol{\mu}}.$$

Proposition 2.1. *Let $\boldsymbol{\lambda} \in (\Lambda^+)^n$. Then $\mathscr{I}_{\boldsymbol{\lambda}}$ is the direct image by m_n of the intersection cohomology sheaf of $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}$ with trivial local system, to wit*

$$\mathscr{I}_{\boldsymbol{\lambda}} = (m_n)_* \, \mathrm{IC}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big),$$

and the cohomology sheaves $\mathscr{H}^k \, \mathrm{IC}(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}})$ vanish unless k and $2\rho(|\boldsymbol{\lambda}|)$ have the same parity.

Proof. We content ourselves with the case $n = 2$; the proof is the same in the general case but requires more notation. Working out the technicalities explained in [4, §1.16.4], we get

$$\mathrm{IC}\big(\overline{\mathrm{Gr}_2^{(\lambda_1,\lambda_2)}}, \underline{\mathbb{C}}\big) = \mathrm{IC}\big(\overline{\mathrm{Gr}^{\lambda_1}}, \underline{\mathbb{C}}\big) \,\tilde{\boxtimes}\, \mathrm{IC}\big(\overline{\mathrm{Gr}^{\lambda_2}}, \underline{\mathbb{C}}\big).$$

Applying $(m_2)_*$ then gives the desired equality, while the parity property follows from [36, Sect. 11]. ∎

A key argument in Mirković and Vilonen's proof of the geometric Satake correspondence is the fact that for any $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, all the irreducible components of $\overline{\mathrm{Gr}^{\lambda}} \cap S_\mu$ (respectively, $\overline{\mathrm{Gr}^{\lambda}} \cap T_\mu$) have dimension $\rho(\lambda + \mu)$ (respectively, $\rho(\lambda - \mu)$) (see [39, Theorem 3.2]). We need a similar result for the intersections $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(S_\mu)$ and $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu)$ inside the n-fold convolution variety.

Let $N^\vee$ be the unipotent radical of $B^\vee$. Then S_μ is the $N^\vee(\mathcal{K})$-orbit through L_μ; this well-known fact follows from the easily proved inclusion $N^\vee(\mathcal{K})L_\mu \subseteq S_\mu$ and the Iwasawa decomposition

$$G^\vee(\mathcal{K}) = \bigsqcup_{\mu \in \Lambda} N^\vee(\mathcal{K}) z^\mu G^\vee(\mathcal{O}). \tag{1}$$

We record that

$$S_\mu = N^\vee(\mathcal{K})L_\mu = (N^\vee(\mathcal{K}) z^\mu N^\vee(\mathcal{O}))/N^\vee(\mathcal{O}) = (N^\vee(\mathcal{K}) z^\mu)/N^\vee(\mathcal{O})$$

and that for each $\lambda \in \Lambda^+$, the intersection $\overline{\mathrm{Gr}^{\lambda}} \cap S_\mu$ is stable under the action of the connected subgroup $N^\vee(\mathcal{O})$, hence so is each irreducible component of this intersection.

The construction of the n-fold convolution variety is functorial in the group $G^\vee$. Applied to the inclusion $B^\vee \to G^\vee$, this remark provides a natural map

$$\Psi : \bigsqcup_{(\mu_1,\ldots,\mu_n) \in \Lambda^n} (N^\vee(\mathcal{K}) z^{\mu_1}) \times^{N^\vee(\mathcal{O})} \cdots \times^{N^\vee(\mathcal{O})} (N^\vee(\mathcal{K}) z^{\mu_n})/N^\vee(\mathcal{O}) \to \mathrm{Gr}_n.$$

Using (1), we easily see that Ψ is bijective.

Given weights $\mu_1, \ldots, \mu_n$ and $N^\vee(\mathcal{O})$-stable subsets $Z_1 \subseteq S_{\mu_1}, \ldots, Z_n \subseteq S_{\mu_n}$, we define

$$Z_1 \ltimes \cdots \ltimes Z_n = \widetilde{Z}_1 \times^{N^\vee(\mathcal{O})} \cdots \times^{N^\vee(\mathcal{O})} \widetilde{Z}_n / N^\vee(\mathcal{O})$$

where each $\widetilde{Z}_j$ is the preimage of Z_j under the quotient map $N^\vee(\mathcal{K}) z^{\mu_j} \to S_{\mu_j}$. If $Z_1, \ldots, Z_n$ are varieties, then $Z_1 \ltimes \cdots \ltimes Z_n$ is an iterated fibration with base Z_1 and successive fibers $Z_2, \ldots, Z_n$ and Ψ induces a homeomorphism from $Z_1 \ltimes \cdots \ltimes Z_n$ onto its image. The bijectivity of Ψ implies that

$$\mathrm{Gr}_n = \bigsqcup_{(\mu_1,\ldots,\mu_n)\in\Lambda^n} \Psi(S_{\mu_1} \ltimes \cdots \ltimes S_{\mu_n}).$$

We use Irr(−) to designate the set of irreducible components of its argument. For $\lambda \in \Lambda^+$, $\boldsymbol{\lambda} \in (\Lambda^+)^n$, and $\mu \in \Lambda$, we define

$$_*\mathscr{Z}(\lambda)_\mu = \mathrm{Irr}\big(\overline{\mathrm{Gr}^\lambda} \cap S_\mu\big) \quad \text{and} \quad _*\mathscr{Z}(\boldsymbol{\lambda})_\mu = \mathrm{Irr}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(S_\mu)\big).$$

Proposition 2.2. *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$ and let $\mu \in \Lambda$.*

(i) *All the irreducible components of $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(S_\mu)$ have dimension $\rho(|\boldsymbol{\lambda}| + \mu)$.*

(ii) *The map $(Z_1, \ldots, Z_n) \mapsto \overline{\Psi(Z_1 \ltimes \cdots \ltimes Z_n)}$ induces a bijection*

$$\bigsqcup_{\substack{(\mu_1,\ldots,\mu_n)\in\Lambda^n \\ \mu_1+\cdots+\mu_n=\mu}} {}_*\mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times {}_*\mathscr{Z}(\lambda_n)_{\mu_n} \xrightarrow{\simeq} {}_*\mathscr{Z}(\boldsymbol{\lambda})_\mu.$$

(*The bar above $\Psi(Z_1 \ltimes \cdots \ltimes Z_n)$ means closure in $(m_n)^{-1}(S_\mu)$.*)

Proof. One easily checks that $(m_n \circ \Psi)(S_{\mu_1} \ltimes \cdots \ltimes S_{\mu_n}) \subseteq S_{\mu_1+\cdots+\mu_n}$, whence

$$(m_n)^{-1}(S_\mu) = \bigsqcup_{\substack{(\mu_1,\ldots,\mu_n)\in\Lambda^n \\ \mu_1+\cdots+\mu_n=\mu}} \Psi(S_{\mu_1} \ltimes \cdots \ltimes S_{\mu_n})$$

for any $\mu \in \Lambda$. Adding $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ to the mix, we see that

$$\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(S_\mu) = \bigsqcup_{\substack{(\mu_1,\ldots,\mu_n)\in\Lambda^n \\ \mu_1+\cdots+\mu_n=\mu}} \Psi\Big(\big(\overline{\mathrm{Gr}^{\lambda_1}} \cap S_{\mu_1}\big) \ltimes \cdots \ltimes \big(\overline{\mathrm{Gr}^{\lambda_n}} \cap S_{\mu_n}\big)\Big)$$

is the disjoint union over $(\mu_1, \ldots, \mu_n)$ of an iterated fibration with base $\overline{\mathrm{Gr}^{\lambda_1}} \cap S_{\mu_1}$ and successive fibers $\overline{\mathrm{Gr}^{\lambda_2}} \cap S_{\mu_2}$, …, $\overline{\mathrm{Gr}^{\lambda_n}} \cap S_{\mu_n}$. The proposition then follows from Mirković and Vilonen's dimension estimates. ■

For $\lambda \in \Lambda^+$, $\boldsymbol{\lambda} \in (\Lambda^+)^n$, and $\mu \in \Lambda$, we similarly define

$$\mathscr{Z}(\lambda)_\mu = \mathrm{Irr}\big(\overline{\mathrm{Gr}^\lambda} \cap T_\mu\big) \quad \text{and} \quad \mathscr{Z}(\boldsymbol{\lambda})_\mu = \mathrm{Irr}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu)\big).$$

Then all cycles in $\mathscr{Z}(\boldsymbol{\lambda})_\mu$ have dimension $\rho(|\boldsymbol{\lambda}| - \mu)$ and there is a natural bijection

$$\bigsqcup_{\substack{(\mu_1,\ldots,\mu_n)\in\Lambda^n \\ \mu_1+\cdots+\mu_n=\mu}} \mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathscr{Z}(\lambda_n)_{\mu_n} \xrightarrow{\simeq} \mathscr{Z}(\boldsymbol{\lambda})_\mu. \tag{2}$$

Elements in $_*\mathscr{Z}(\lambda)_\mu$, $\mathscr{Z}(\lambda)_\mu$, $_*\mathscr{Z}(\boldsymbol{\lambda})_\mu$ or $\mathscr{Z}(\boldsymbol{\lambda})_\mu$ are called *Mirković–Vilonen* (MV) *cycles*. For future use, we note that the map $Z \mapsto Z \cap \mathrm{Gr}_n^{\boldsymbol{\lambda}}$ provides a bijection from $\mathscr{Z}(\boldsymbol{\lambda})_\mu$ onto the set of irreducible components of $\mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\mu)$. (Each $Z \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$ meets the open subset $\mathrm{Gr}_n^{\boldsymbol{\lambda}}$ of $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}$, because the dimension of $(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \setminus \mathrm{Gr}_n^{\boldsymbol{\lambda}}) \cap (m_n)^{-1}(T_\mu)$ is smaller than the dimension of Z.)

2.3. Mirković–Vilonen bases

Following Goncharov and Shen [22, Sect. 2.4], we now define the MV basis of the tensor product representation

$$V(\boldsymbol{\lambda}) = F(\mathscr{I}_{\boldsymbol{\lambda}}) = \bigoplus_{\mu \in \Lambda} F_\mu(\mathscr{I}_{\boldsymbol{\lambda}}).$$

Let $\boldsymbol{\lambda} \in (\Lambda^+)^n$ and let $\mu \in \Lambda$. By base change in the Cartesian square

$$\begin{array}{ccc} \overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu) & \xrightarrow{\ f\ } & \overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \\ {\scriptstyle m_n}\big\downarrow & & \big\downarrow{\scriptstyle m_n} \\ T_\mu & \xrightarrow{\ t_\mu\ } & \mathrm{Gr} \end{array}$$

we compute

$$\begin{aligned} F_\mu(\mathscr{I}_{\boldsymbol{\lambda}}) &= H^{2\rho(\mu)}\big(T_\mu, (t_\mu)^!(m_n)_* \operatorname{IC}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big)\big) \\ &= H^{2\rho(\mu)}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu), f^! \operatorname{IC}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big)\big). \end{aligned}$$

Let $j : \mathrm{Gr}_n^{\boldsymbol{\lambda}} \to \overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}$ and $g : \mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\mu) \to \mathrm{Gr}_n^{\boldsymbol{\lambda}}$ be the inclusion maps. We can then look at the sequence of maps

$$\begin{aligned} F_\mu(\mathscr{I}_{\boldsymbol{\lambda}}) &\to H^{2\rho(\mu)}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu), f^! j_* j^* \operatorname{IC}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big)\big) \\ &= H^{2\rho(\mu)}\big(\mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\mu), g^! \underline{\mathbb{C}}_{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}[\dim \mathrm{Gr}_n^{\boldsymbol{\lambda}}]\big) \\ &\xrightarrow{\cap[\mathrm{Gr}_n^{\boldsymbol{\lambda}}]} H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\mu)}\big(\mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\mu)\big). \end{aligned}$$

Here the first two maps carry out the restriction to $\mathrm{Gr}_n^{\boldsymbol{\lambda}}$ (technically, an adjunction followed by a base change) and the last map is the Alexander duality.[1]

We claim that these maps are isomorphisms. For the Alexander duality, this comes from the smoothness of $\mathrm{Gr}_n^{\boldsymbol{\lambda}}$. For the restriction, consider a stratum $\mathrm{Gr}_n^{\boldsymbol{\eta}} \subseteq \overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}$ with $\boldsymbol{\eta} \neq \boldsymbol{\lambda}$; denoting the inclusion map by i and using the perversity condition, the parity property in Proposition 2.1, and the dimension estimate for $\overline{\mathrm{Gr}_n^{\boldsymbol{\eta}}} \cap (m_n)^{-1}(T_\mu)$, one checks that

$$H^k\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu), f^! i_* i^! \operatorname{IC}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big)\big)$$

vanishes if $k < 2\rho(\mu) + 2$; therefore the stratum $\mathrm{Gr}_n^{\boldsymbol{\eta}}$ does not contribute to $F_\mu(\mathscr{I}_{\boldsymbol{\lambda}})$.

To sum up there is a natural isomorphism

$$F_\mu(\mathscr{I}_{\boldsymbol{\lambda}}) \xrightarrow{\simeq} H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\mu)}\big(\mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\mu)\big). \tag{3}$$

The irreducible components $\mathrm{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\mu)$ all have dimension $\rho(|\boldsymbol{\lambda}| - \mu)$ and their fundamental classes provide a basis of the Borel–Moore homology group above. Gathering these bases for all weights $\mu \in \Lambda$ produces what we call the *MV basis* of $V(\boldsymbol{\lambda})$.

[1] Specifically, the generalization presented in [15, Sect. 19.1, equation (3)] or in [24, Theorem IX.4.7].

3. L-perfect bases

In this section we consider a general setup, which captures properties shared by both the MV bases and the dual canonical bases. As before, G is a connected reductive group over $\mathbb{C}$ endowed with a Borel subgroup B and a maximal torus $T \subseteq B$, Λ is the character lattice of T, and Φ and $\Phi^\vee$ are the root and coroot systems of (G, T). We denote by $\{\alpha_i \mid i \in I\}$ the set of simple roots defined by B and by $\{\alpha_i^\vee \mid i \in I\}$ the set of simple coroots. Again, $\leq$ is the dominance order on Λ and Λ^+ is the cone of dominant weights. We regard the Weyl group W as a subgroup of $\mathrm{Aut}(\Lambda)$; for $i \in I$, we denote by s_i the simple reflection along the root α_i. When needed, we choose simple root vectors e_i and f_i of weights $\pm\alpha_i$ in the Lie algebra of G in such a way that $[e_i, f_i] = -\alpha_i^\vee$.

3.1. Seminormal crystals

We start by recalling the following definitions due to Kashiwara [28]. A *seminormal crystal* is a set B endowed with a map $\mathrm{wt} : B \to \Lambda$ and, for each "color" $i \in I$, a partition into a collection of finite oriented strings. This latter structure is recorded by the datum of operators

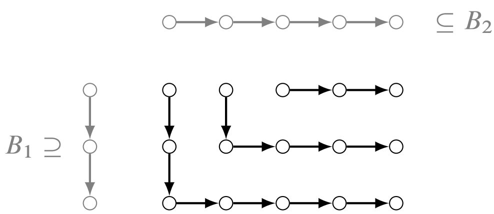
$$\tilde{e}_i : B \to B \sqcup \{0\} \quad \text{and} \quad \tilde{f}_i : B \to B \sqcup \{0\}$$

which move an element of $b \in B$ upwards and downwards, respectively, along the string of color i to which b belongs. The special value 0 is assigned to $\tilde{e}_i(b)$ or $\tilde{f}_i(b)$ when b is at the upper or lower end of a string of color i. For convenience one usually further sets $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$. The position of b in its string of color i is recorded by functions ε_i and φ_i which are defined as follows:

$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i(b) \neq 0\}, \quad \varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i(b) \neq 0\}.$$

Two compatibility conditions between the weight map wt and the datum of the partitions into oriented strings are required: first, $\mathrm{wt}(b)$ increases by α_i when b moves upwards the string of color i to which it belongs; second, $\varphi_i(b) - \varepsilon_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle$ for any $b \in B$ and any $i \in I$. These conditions imply that the image of a string of color i by the map wt is stable under the action of the simple reflection s_i. As a consequence, the set $\{\mathrm{wt}(b) \mid b \in B\}$ is stable under the action of the Weyl group W.

The direct sum of two seminormal crystals B_1 and B_2 is defined to be just the disjoint union of the underlying sets. The tensor product $B_1 \otimes B_2$ is defined to be the Cartesian product of the sets endowed with the maps given in [28, §7.3]. Notably, for each color, the strings in $B_1 \otimes B_2$ are created from the strings contained in B_1 and in B_2 by the process illustrated by the picture below.



A morphism $f : B \to C$ between two seminormal crystals is a map that preserves the weight and that commutes with all the operators $\tilde{e}_i$ and $\tilde{f}_i$. (In contrast with the more general definition given in [26], morphisms between seminormal crystals are necessarily strict.)

3.2. *L-perfect bases*

To a subset $J \subseteq I$ we attach the standard Levi subgroup M_J of G, the cone

$$\Lambda_J^+ = \{\lambda \in \Lambda \mid \forall j \in J,\ \langle \alpha_j^\vee, \lambda \rangle \geq 0\}$$

of *dominant weights* for M_J, and the J*-dominance order* $\leq_J$ on Λ defined by

$$\mu \leq_J \lambda \iff \lambda - \mu \in \operatorname{span}_{\mathbb{N}} \{\alpha_j \mid j \in J\}.$$

Given $\lambda \in \Lambda_J^+$ we denote by $V_J(\lambda)$ the irreducible rational representation of M_J with highest weight λ. Given a finite sequence $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in Λ_J^+ we define

$$V_J(\boldsymbol{\lambda}) = V_J(\lambda_1) \otimes \cdots \otimes V_J(\lambda_n).$$

For $J = I$ we recover the conventions previously used by dropping the decoration J in the notation Λ_J^+, $\leq_J$ or $V_J(\boldsymbol{\lambda})$.

Let V be a rational representation of G. With respect to the action of M_J the space V can be uniquely written as a direct sum of isotypic components

$$V = \bigoplus_{\mu \in \Lambda_J^+} V_{J,\mu}$$

where $V_{J,\mu}$ is the sum of all subrepresentations of $\operatorname{res}_{M_J}^G(V)$ isomorphic to $V_J(\mu)$. We define

$$V_{J,\leq_J \mu} = \bigoplus_{\substack{\nu \in \Lambda_J^+ \\ \nu \leq_J \mu}} V_{J,\nu}.$$

We say that a linear basis B of V is *L-perfect*[2] if for each $J \subseteq I$ and each $\mu \in \Lambda_J^+$,

(P1) the subspace $V_{J,\leq_J \mu}$ is spanned by a subset of B;

(P2) the induced basis on the quotient $V_{J,\leq_J \mu}/V_{J,<_J \mu} \cong V_{J,\mu}$ is compatible with a decomposition of the isotypic component as a direct sum of irreducible representations.

Taking $J = \emptyset$, we see that an L-perfect basis of V consists of weight vectors (note that $\leq_\emptyset$ is the trivial order on Λ). Now let $i \in I$, and for each nonnegative integer ℓ, define

$$V_{\{i\},\leq \ell} = \bigoplus_{\substack{\mu \in \Lambda \\ 0 \leq \langle \alpha_i^\vee, \mu \rangle \leq \ell}} V_{\{i\},\mu},$$

[2] L stands for Levi. This notion of L-perfect basis appears unnamed (and in a dual form) in Braverman and Gaitsgory's paper [9, Sect. 4.3].

the sum of all irreducible subrepresentations of $\operatorname{res}^G_{M_{\{i\}}}(V)$ of dimension at most $\ell+1$. If B satisfies conditions (P1) and (P2) for $J=\{i\}$, then $V_{\{i\},\leq\ell}$ is spanned by $B\cap V_{\{i\},\leq\ell}$ and the induced basis on $V_{\{i\},\leq\ell}/V_{\{i\},\leq\ell-1}$ is compatible with a decomposition as a direct sum of irreducible representations. Therefore $(B\cap V_{\{i\},\leq\ell})\setminus(B\cap V_{\{i\},\leq\ell-1})$ decomposes as the disjoint union of oriented strings of length ℓ, in such a way that the simple root vector e_i or f_i acts on a basis vector of $V_{\{i\},\leq\ell}/V_{\{i\},\leq\ell-1}$ by moving it upwards or downwards along the string that contains it, up to a scalar.

We can sum up the discussion in the previous paragraph as follows: if B satisfies (P1) and (P2) for all J of cardinality ≤ 1, then B is endowed with the structure of crystal and is perfect in the sense of Berenstein and Kazhdan [8, Definition 5.30].

Lemma 3.1. *Let B be an L-perfect basis of a rational representation V of G and let $B'\subseteq B$. Assume that the linear space V' spanned by B' is a subrepresentation of V. Then B' is an L-perfect basis of V' and (the image of) $B\setminus B'$ is an L-perfect basis of the quotient V/V'.*

Proof. Let $J\subseteq I$ and $\mu\in\Lambda^+_J$. Then

$$V'_{J,\mu}=V'\cap V_{J,\mu}\quad\text{and}\quad V'_{J,\leq_J\mu}=V'\cap V_{J,\leq_J\mu}.$$

Now both spaces V' and $V_{J,\leq_J\mu}$ are spanned by a subset of the basis B, so their intersection $V'_{J,\leq_J\mu}$ is spanned by a subset of B, namely $B'\cap V_{J,\leq_J\mu}$.

Let C be the image of $(B\cap V_{J,\leq_J\mu})\setminus(B\cap V_{J,<_J\mu})$ in the quotient $V_{J,\leq_J\mu}/V_{J,<_J\mu}\cong V_{J,\mu}$. Then C can be viewed as a basis of $V_{J,\mu}$ and it can be partitioned into disjoint subsets $C_1,\dots,C_n$ so that each C_k spans an irreducible subrepresentation. By construction, the subspace $V'_{J,\mu}$ is spanned by a subset $C'\subseteq C$. Each subset C_k can be either contained in C' or disjoint from C', depending on whether the subrepresentation that it spans is contained in $V'_{J,\mu}$ or meets $V'_{J,\mu}$ trivially. Therefore C' is the disjoint union of some C_k, which means that C' is compatible with a decomposition of $V'_{J,\mu}$ as a direct sum of irreducible subrepresentations.

Thus, B' satisfies both conditions (P1) and (P2), and is therefore an L-perfect basis of V'. The proof that $B\setminus B'$ yields an L-perfect basis of the quotient V/V' rests on similar arguments and is left to the reader. ■

Under the assumptions of Lemma 3.1, the subset B' is a subcrystal of B. In other words, the crystal structure on B is the direct sum of the crystal structures on B' and $B\setminus B'$.

Proposition 3.2. *Let V be a rational representation of G. Up to isomorphism, the crystal of an L-perfect basis of V depends only on V, and not on the basis.*

Proof. Let B be an L-perfect basis of V. The conditions imposed on B with the choice $J=I$ imply the existence in V of a composition series compatible with B. By Lemma 3.1, the crystal B is the direct sum of the crystals of the L-perfect bases induced by B on the subquotients. It thus suffices to prove the desired uniqueness property in the particular case where V is an irreducible representation, which in fact is just Theorem 5.37 in [8]. ■

In particular, the crystal of an L-perfect basis of an irreducible representation $V(\lambda)$ is unique. We use henceforth the notation $B(\lambda)$ for the associated crystal.

Remark 3.3. The crystal $B(\lambda)$ of an irreducible representation $V(\lambda)$ was introduced by Kashiwara in the context of representations of quantum groups. The definition via crystallization at $q = 0$ and the definition via the combinatorics of L-perfect bases yield the same crystal; this follows from [27, Sect. 5].

Fortunately, this nice little theory is not empty. As mentioned in the introduction, any tensor product of irreducible representations has an L-perfect basis, namely its dual canonical basis. Another example of an L-perfect basis: it can be shown, in the case where G is simply laced, that the dual semicanonical basis of an irreducible representation is L-perfect.

Theorem 3.4. *The MV basis of a tensor product of irreducible representations is L-perfect.*

The end of Sect. 3 is devoted to the proof of this result. The case of an irreducible representation is handled in [9, Proposition 4.1]. The proof for an arbitrary tensor product follows the same lines. It is only sketched in *loc. cit.*, and we add quite a few details to Braverman and Gaitsgory's exposition.

3.3. Geometric Satake and restriction to a standard Levi subgroup

Consider a subset $J \subseteq I$. In this section we recall Beilinson and Drinfeld's geometric construction of the restriction functor $\operatorname{res}^G_{M_J}$ (see [6, Sect. 5.3]). Additional details can be found in [38, Sect. 8.6] and [4, Sect. 1.15].

Define the root and coroot systems

$$\Phi_J = \Phi \cap \operatorname{span}_{\mathbb{Z}} \{\alpha_j \mid j \in J\} \quad \text{and} \quad \Phi_J^\vee = \Phi^\vee \cap \operatorname{span}_{\mathbb{Z}} \{\alpha_j^\vee \mid j \in J\}$$

and denote by $\rho_J : \Lambda \to \mathbb{Q}$ the half-sum of the positive coroots in $\Phi_J^\vee$. Then $\rho - \rho_J$ vanishes on $\mathbb{Z}\Phi_J$, so induces a linear form $\rho_{I,J} : \Lambda/\mathbb{Z}\Phi_J \to \mathbb{Q}$.

To J we also attach the standard Levi subgroup $M_J^\vee$ of $G^\vee$. Choose a dominant $\theta_J \in \Lambda$ such that $\langle \alpha_j^\vee, \theta_J \rangle = 0$ for each $j \in J$ and $\langle \alpha_i^\vee, \theta_J \rangle > 0$ for each $i \in I \setminus J$. The embedding

$$\mathbb{C}^\times \xrightarrow{\theta_J} T^\vee(\mathbb{C}) \to G^\vee(\mathcal{K})$$

gives rise to an action of $\mathbb{C}^\times$ on Gr. Then the set Gr_J of fixed points under this action can be identified with the affine Grassmannian for $M_J^\vee$. We denote by $\operatorname{Perv}(\operatorname{Gr}_J)$ the category of $M_J^\vee(\mathcal{O})$-equivariant sheaves on Gr_J supported on finitely many $M_J^\vee(\mathcal{O})$-orbits.

Let $\zeta \in \Lambda/\mathbb{Z}\Phi_J$ be a coset. All the points L_μ for $\mu \in \zeta$ belong to the same connected component of Gr_J, which we denote by $\operatorname{Gr}_{J,\zeta}$. The map $\zeta \mapsto \operatorname{Gr}_{J,\zeta}$ is a bijection from $\Lambda/\mathbb{Z}\Phi_J$ onto $\pi_0(\operatorname{Gr}_J)$. We denote the stable and unstable sets of $\operatorname{Gr}_{J,\zeta}$ with respect to the $\mathbb{C}^\times$-action by $\operatorname{Gr}^+_{J,\zeta}$ and $\operatorname{Gr}^-_{J,\zeta}$ and form the diagram

$$\begin{array}{ccc} & \mathrm{Gr} & \\ {\scriptstyle s_\zeta^J}\nearrow & & \nwarrow{\scriptstyle t_\zeta^J} \\ \mathrm{Gr}^+_{J,\zeta} & & \mathrm{Gr}^-_{J,\zeta} \\ {\scriptstyle p_{J,\zeta}}\searrow & & \swarrow{\scriptstyle q_{J,\zeta}} \\ & \mathrm{Gr}_{J,\zeta} & \end{array} \tag{4}$$

where s_ζ^J and t_ζ^J are the inclusion maps and the maps $p_{J,\zeta}$ and $q_{J,\zeta}$ are defined by

$$p_{J,\zeta}(x) = \lim_{c\to 0} \theta_J(c)\cdot x \quad \text{and} \quad q_{J,\zeta}(x) = \lim_{c\to\infty} \theta_J(c)\cdot x.$$

Given $\zeta \in \Lambda/\mathbb{Z}\Phi_J$ and $\mathscr{A} \in \mathrm{Perv}(\mathrm{Gr})$, Beilinson and Drinfeld identify the two sheaves

$$(q_{J,\zeta})_*(t_\zeta^J)^!\mathscr{A} \quad \text{and} \quad (p_{J,\zeta})_!(s_\zeta^J)^*\mathscr{A}$$

on $\mathrm{Gr}_{J,\zeta}$ via Braden's hyperbolic localization and show that they live in perverse degree $2\rho_{I,J}(\zeta)$. Then they define a functor $r_J^I : \mathrm{Perv}(\mathrm{Gr}) \to \mathrm{Perv}(\mathrm{Gr}_J)$ by

$$r_J^I(\mathscr{A}) = \bigoplus_{\zeta\in\Lambda/\mathbb{Z}\Phi_J} (q_{J,\zeta})_*(t_\zeta^J)^!\mathscr{A}[2\rho_{I,J}(\zeta)].$$

For $\mu \in \Lambda$, let $T_{J,\mu}$ be the analog of the unstable subset T_μ for the affine Grassmannian Gr_J. Let ζ be the coset of μ modulo $\mathbb{Z}\Phi_J$ and let $t_{J,\mu} : T_{J,\mu} \to \mathrm{Gr}_{J,\zeta}$ be the inclusion map. Using the Iwasawa decomposition, one checks that $T_\mu \subseteq \mathrm{Gr}^-_{J,\zeta}$ and

$$T_\mu = (q_{J,\zeta})^{-1}(T_{J,\mu}). \tag{5}$$

Performing base change as indicated in the diagram

$$\begin{array}{ccccc} & & \overset{t_\mu}{\frown\!\!\frown\!\!\frown} & & \\ T_\mu & \longrightarrow & \mathrm{Gr}^-_{J,\zeta} & \xrightarrow[t_\zeta^J]{} & \mathrm{Gr} \\ \downarrow & & \downarrow{\scriptstyle q_{J,\zeta}} & & \\ T_{J,\mu} & \xrightarrow[t_{J,\mu}]{} & \mathrm{Gr}_{J,\zeta} & & \end{array}$$

we obtain, for any sheaf $\mathscr{A} \in \mathrm{Perv}(\mathrm{Gr})$, a canonical isomorphism

$$H^{2\rho(\mu)}(T_\mu, (t_\mu)^!\mathscr{A}) \cong H^{2\rho_J(\mu)}(T_{J,\mu}, (t_{J,\mu})^! r_J^I(\mathscr{A})). \tag{6}$$

For $\mathscr{B} \in \mathrm{Perv}(\mathrm{Gr}_J)$, define

$$F_{J,\mu}(\mathscr{B}) = H^{2\rho_J(\mu)}(T_{J,\mu}, (t_{J,\mu})^!\mathscr{B}) \quad \text{and} \quad F_J(\mathscr{B}) = \bigoplus_{\mu\in\Lambda} F_{J,\mu}(\mathscr{B}).$$

Then (6) can be rewritten as $F_\mu = F_{J,\mu} \circ r_J^I$. This equality can be refined in the following statement: the functor F_J induces an equivalence $\overline{F_J}$ from $\mathrm{Perv}(\mathrm{Gr}_J)$ to the category

$\operatorname{Rep}(M_J)$ of finite-dimensional rational representations of M_J and the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Perv}(\mathrm{Gr}) & \xrightarrow{\overline{F}} & \operatorname{Rep}(G) \\ \downarrow{\scriptstyle r_J^I} & & \downarrow{\scriptstyle \operatorname{res}^G_{M_J}} \\ \operatorname{Perv}(\mathrm{Gr}_J) & \xrightarrow{\overline{F_J}} & \operatorname{Rep}(M_J) \end{array}$$

3.4. The J-decomposition of an MV cycle

We fix a subset $J \subseteq I$. We denote by $P_J^{-,\vee}$ the parabolic subgroup of $G^\vee$ containing $M_J^\vee$ and the negative root subgroups.

The group $P_J^{-,\vee}(\mathcal{K})$ certainly acts on Gr; it also acts on Gr_J via the quotient morphism $P_J^{-,\vee}(\mathcal{K}) \to M_J^\vee(\mathcal{K})$. Given $\mu \in \Lambda_J^+$, we denote by Gr_J^μ the orbit of L_μ under the action of $M_J^\vee(\mathcal{O})$ (or $P_J^{-,\vee}(\mathcal{O})$) on Gr_J. Noting that

$$\lim_{a\to\infty} \theta_J(a) g \theta_J(a)^{-1} = 1$$

for all g in the unipotent radical of $P_J^{-,\vee}$, we see that for any $\zeta \in \Lambda/\mathbb{Z}\Phi_J$, the connected component $\mathrm{Gr}_{J,\zeta}$ of Gr_J and the unstable subset $\mathrm{Gr}^-_{J,\zeta}$ in Gr are stable under the action of $P_J^{-,\vee}(\mathcal{O})$ and that the map $q_{J,\zeta}$ is equivariant.

Let $\boldsymbol{\lambda} \in (\Lambda^+)^n$, let $\mu \in \Lambda_J^+$ and let ζ be the coset of μ modulo $\mathbb{Z}\Phi_J$. We consider the diagram

$$\begin{array}{ccccc} (m_n)^{-1}(\mathrm{Gr}^-_{J,\zeta}) & \xrightarrow{m_n} & \mathrm{Gr}^-_{J,\zeta} & \xrightarrow{q_{J,\zeta}} & \mathrm{Gr}_{J,\zeta} \\ \downarrow & & \downarrow{\scriptstyle t^J_\zeta} & & \\ \mathrm{Gr}_n & \xrightarrow{m_n} & \mathrm{Gr} & & \end{array}$$

The group $G^\vee(\mathcal{K})$ acts on Gr_n by left multiplication on the first factor and the action of the subgroup $G^\vee(\mathcal{O})$ leaves $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}$ stable. Let H be the stabilizer of L_μ with respect to the action of $P_J^{-,\vee}(\mathcal{O})$ on Gr_J. It acts on $E = \overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta} \circ m_n)^{-1}(L_\mu)$. Since $q_{J,\zeta} \circ m_n$ is equivariant under the action of $P_J^{-,\vee}(\mathcal{O})$, we can make the identification

$$\begin{array}{ccc} P_J^{-,\vee}(\mathcal{O}) \times^H E & \xrightarrow{\cong} & \overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta} \circ m_n)^{-1}(\mathrm{Gr}_J^\mu) \\ \downarrow & & \downarrow{\scriptstyle q_{J,\zeta}\circ m_n} \\ P_J^{-,\vee}(\mathcal{O})/H & \xrightarrow{\cong} & \mathrm{Gr}_J^\mu \end{array}$$

where the left vertical arrow is the projection along the first factor. We thereby see that the right vertical arrow is a locally trivial fibration.

In particular, all the fibers $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta} \circ m_n)^{-1}(x)$ with $x \in \mathrm{Gr}_J^\mu$ are isomorphic varieties. Recalling that $(q_{J,\zeta})^{-1}(L_\mu) \subseteq T_\mu$, we find the following bound for their dimension:

$$\dim\bigl(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta} \circ m_n)^{-1}(x)\bigr) = \dim E \le \dim\bigl(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\mu)\bigr) = \rho(|\boldsymbol{\lambda}| - \mu).$$

Therefore

$$\dim\bigl(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}\cap(q_{J,\zeta}\circ m_n)^{-1}(\mathrm{Gr}_J^{\mu})\bigr) \le \dim \mathrm{Gr}_J^{\mu}+\rho(|\boldsymbol{\lambda}|-\mu) = 2\rho_J(\mu)+\rho(|\boldsymbol{\lambda}|-\mu). \tag{7}$$

Since Gr_J^{μ} is connected and simply-connected, the fibration induces a bijection between the set of irreducible components of $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta}\circ m_n)^{-1}(\mathrm{Gr}_J^{\mu})$ and the set of irreducible components of any fiber $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta}\circ m_n)^{-1}(x)$.

We define

$$\mathscr{Z}^J(\boldsymbol{\lambda})_\mu = \bigl\{Z \in \mathrm{Irr}\bigl(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta}\circ m_n)^{-1}(\mathrm{Gr}_J^{\mu})\bigr) \bigm| \dim Z = 2\rho_J(\mu) + \rho(|\boldsymbol{\lambda}| - \mu)\bigr\}.$$

For $\nu \in \Lambda$, we define

$$\mathscr{Z}_J(\mu)_\nu = \mathrm{Irr}\bigl(\overline{\mathrm{Gr}_J^{\mu}} \cap T_{J,\nu}\bigr).$$

As we saw in Sect. 2.2, the map $Z \mapsto Z \cap \mathrm{Gr}_J^{\mu}$ is a bijection from $\mathscr{Z}_J(\mu)_\nu$ onto the set of irreducible components of $\mathrm{Gr}_J^{\mu} \cap T_{J,\nu}$.

Fix now $\boldsymbol{\lambda} \in (\Lambda^+)^n$ and $\nu \in \Lambda$. Following Braverman and Gaitsgory, we define a bijection

$$\mathscr{Z}(\boldsymbol{\lambda})_\nu \cong \bigsqcup_{\mu\in\Lambda_J^+} \mathscr{Z}^J(\boldsymbol{\lambda})_\mu \times \mathscr{Z}_J(\mu)_\nu.$$

The union above can be restricted to those weights μ such that $\mu - \nu \in \mathbb{Z}\Phi_J$, for otherwise $\mathscr{Z}_J(\mu)_\nu$ is empty. Let ζ denote the coset of ν modulo $\mathbb{Z}\Phi_J$.

First choose $\mu \in \Lambda_J^+ \cap \zeta$ and a pair $(Z^J, Z_J) \in \mathscr{Z}^J(\boldsymbol{\lambda})_\mu \times \mathscr{Z}_J(\mu)_\nu$. Using (5) and the fibration above, we see that $Z^J \cap (q_{J,\zeta}\circ m_n)^{-1}(Z_J \cap \mathrm{Gr}_J^{\mu})$ is an irreducible subset of

$$\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta}\circ m_n)^{-1}(T_{J,\nu}) = \overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (m_n)^{-1}(T_\nu)$$

of dimension

$$\dim Z^J - \dim \mathrm{Gr}_J^{\mu} + \dim Z_J = \rho(|\boldsymbol{\lambda}| - \mu) + \rho_J(\mu - \nu) = \rho(|\boldsymbol{\lambda}| - \nu).$$

Therefore there is a unique $Z \in \mathscr{Z}(\boldsymbol{\lambda})_\nu$ that contains $Z^J \cap (q_{J,\zeta}\circ m_n)^{-1}(Z_J \cap \mathrm{Gr}_J^{\mu})$ as a dense subset.

Conversely, start from $Z \in \mathscr{Z}(\boldsymbol{\lambda})_\nu$. Then $Z \subseteq T_\nu \subseteq \mathrm{Gr}_{J,\zeta}^-$. We can thus partition Z into locally closed subsets as follows:

$$Z = \bigsqcup_{\mu\in\Lambda_J^+\cap\zeta} \bigl(Z \cap (q_{J,\zeta}\circ m_n)^{-1}(\mathrm{Gr}_J^{\mu})\bigr).$$

Since Z is irreducible, there is a unique $\mu \in \Lambda_J^+ \cap \zeta$ such that $Z \cap (q_{J,\zeta}\circ m_n)^{-1}(\mathrm{Gr}_J^{\mu})$ is open dense in Z. That subset is certainly irreducible, hence contained in an irreducible component Z^J of $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}} \cap (q_{J,\zeta}\circ m_n)^{-1}(\mathrm{Gr}_J^{\mu})$. Also, the composition $q_{J,\zeta}\circ m_n$ maps $Z \cap (q_{J,\zeta}\circ m_n)^{-1}(\mathrm{Gr}_J^{\mu})$ to an irreducible subset of $\mathrm{Gr}_J^{\mu} \cap T_{J,\nu}$, which in turn is contained in an irreducible component $Z_J \in \mathscr{Z}_J(\mu)_\nu$. Then

$$Z \cap (q_{J,\zeta}\circ m_n)^{-1}(\mathrm{Gr}_J^{\mu}) \subseteq Z^J \cap (q_{J,\zeta}\circ m_n)^{-1}(Z_J).$$

The left-hand side has dimension $\rho(|\boldsymbol{\lambda}| - \nu)$ and the right-hand side has dimension

$$\dim Z^J - 2\rho_J(\mu) + \dim Z_J = \dim Z^J - 2\rho_J(\mu) + \rho(\mu - \nu).$$

Combining this with the bound (7) we get

$$\dim Z^J = \rho(|\boldsymbol{\lambda}| - \mu) + 2\rho_J(\mu)$$

and therefore $Z^J \in \mathscr{Z}^J(\boldsymbol{\lambda})_\mu$.

These two constructions define mutually inverse bijections; in particular,

$$Z^J \cap (q_{J,\zeta} \circ m_n)^{-1}(Z_J \cap \mathrm{Gr}_J^\mu) = Z \cap (q_{J,\zeta} \circ m_n)^{-1}(\mathrm{Gr}_J^\mu).$$

We record that to each MV cycle $Z \in \mathscr{Z}(\boldsymbol{\lambda})_\nu$ is assigned a weight $\mu \in \Lambda_J^+$ characterized by the conditions

$$(q_{J,\zeta} \circ m_n)(Z) \subseteq \overline{\mathrm{Gr}_J^\mu} \quad \text{and} \quad (q_{J,\zeta} \circ m_n)(Z) \cap \mathrm{Gr}_J^\mu \neq \emptyset.$$

This weight will be denoted by $\mu_J(Z)$.

3.5. *MV bases are L-perfect*

We now give the proof of Theorem 3.4, properly speaking. We fix a positive integer n and a tuple $\boldsymbol{\lambda} \in (\Lambda^+)^n$. We need two ingredients besides the constructions explained in Sects. 2.3 and 3.3.

(A) Take $\mathscr{A} \in \mathrm{Perv}(\mathrm{Gr})$ and write the sheaf

$$\mathscr{B} = r_J^I(\mathscr{A}) = \bigoplus_{\zeta \in \Lambda/\mathbb{Z}\Phi_J} (q_{J,\zeta})_* (t_\zeta^J)^! \mathscr{A}[2\rho_{I,J}(\zeta)]$$

in $\mathrm{Perv}(\mathrm{Gr}_J)$ as a direct sum of isotypic components,

$$\mathscr{B} = \bigoplus_{\mu \in \Lambda_J^+} \mathrm{IC}\big(\overline{\mathrm{Gr}_J^\mu}, \mathscr{L}_\mu\big). \tag{8}$$

The local systems $\mathscr{L}_\mu$ on Gr_J^μ that appear in (8) can be expressed as $\mathscr{L}_\mu = \mathscr{H}^k h^! \mathscr{B}$ where $h : \mathrm{Gr}_J^\mu \to \mathrm{Gr}_J$ is the inclusion map and $k = -\dim \mathrm{Gr}_J^\mu = -2\rho_J(\mu)$. With $e : \{x\} \to \mathrm{Gr}_J^\mu$ the inclusion of a point and ζ the coset of μ modulo $\mathbb{Z}\Phi_J$, the fiber of $\mathscr{L}_\mu$ is

$$(\mathscr{L}_\mu)_x \cong e^! \mathscr{L}_\mu[2 \dim \mathrm{Gr}_J^\mu] \cong H^{2\rho_J(\mu)}(\{x\}, e^! h^! \mathscr{B}).$$

For the specific case

$$\mathscr{A} = \mathscr{I}_{\boldsymbol{\lambda}} = (m_n)_* \mathrm{IC}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big),$$

noting the equality $\rho_J(\mu) + \rho_{I,J}(\zeta) = \rho(\mu)$, we get

$$(\mathscr{L}_\mu)_x \cong H^{2\rho(\mu)}\big((q_{J,\zeta} \circ m_n)^{-1}(x), f^! \mathrm{IC}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big)\big)$$

where f is the injection map depicted in the Cartesian diagram

$$\begin{array}{ccccc}
(q_{J,\zeta}\circ m_n)^{-1}(x) & & \xrightarrow{\quad\quad f\quad\quad} & & \mathrm{Gr}_n \\
\downarrow & & & & \downarrow{\scriptstyle m_n} \\
(q_{J,\zeta})^{-1}(x) & \longrightarrow & \mathrm{Gr}^-_{J,\zeta} & \xrightarrow{t^J_\zeta} & \mathrm{Gr} \\
\downarrow & & \downarrow{\scriptstyle q_{J,\zeta}} & & \\
\{x\} \xrightarrow{e} \mathrm{Gr}^\mu_J & \xrightarrow{h} & \mathrm{Gr}_{J,\zeta} & &
\end{array}$$

The same reasoning as in Sect. 2.3 proves that only the stratum $\mathrm{Gr}^{\boldsymbol{\lambda}}_n$ contributes to this cohomology group. Denoting by $g:\mathrm{Gr}^{\boldsymbol{\lambda}}_n\cap(q_{J,\zeta}\circ m_n)^{-1}(x)\to\mathrm{Gr}^{\boldsymbol{\lambda}}_n$ the inclusion map, this observation leads to an isomorphism

$$\begin{aligned}(\mathscr{L}_\mu)_x &= H^{2\rho_J(\mu)+2\rho_{I,J}(\zeta)}\big(\mathrm{Gr}^{\boldsymbol{\lambda}}_n\cap(q_{J,\zeta}\circ m_n)^{-1}(x),g^!\underline{\mathbb{C}}_{\mathrm{Gr}^{\boldsymbol{\lambda}}_n}[\dim\mathrm{Gr}^{\boldsymbol{\lambda}}_n]\big)\\ &\xrightarrow{\cap[\mathrm{Gr}^{\boldsymbol{\lambda}}_n]} H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\mu)}\big(\mathrm{Gr}^{\boldsymbol{\lambda}}_n\cap(q_{J,\zeta}\circ m_n)^{-1}(x)\big).\end{aligned}$$

Thus the local systems $\mathscr{L}_\mu$ appearing in (8) have a natural basis, namely the set $\mathscr{Z}^J(\boldsymbol{\lambda})_\mu$.

We record the following consequence of this discussion: given $(\boldsymbol{\lambda},\mu,\nu)\in(\Lambda^+)^n\times\Lambda^+_J\times\Lambda$ such that $\mu-\nu\in\mathbb{Z}\Phi_J$, we have

$$\begin{aligned}\dim H^{2\rho_J(\nu)}&\big(T_{J,\nu},(t_{J,\nu})^!\,\mathrm{IC}\big(\overline{\mathrm{Gr}^\mu_J},\mathscr{L}_\mu\big)\big)\\ &= \operatorname{rank}\mathscr{L}_\mu\times\dim H^{2\rho_J(\nu)}\big(T_{J,\nu},(t_{J,\nu})^!\,\mathrm{IC}\big(\overline{\mathrm{Gr}^\mu_J},\underline{\mathbb{C}}\big)\big)\\ &= \operatorname{Card}\mathscr{Z}^J(\boldsymbol{\lambda})_\mu\times\operatorname{Card}\mathscr{Z}_J(\mu)_\nu\\ &= \operatorname{Card}\{Z\in\mathscr{Z}(\boldsymbol{\lambda})_\nu\mid\mu_J(Z)=\mu\}.\end{aligned}\tag{9}$$

(B) Now let us start with a sheaf $\mathscr{B}$ in $\mathrm{Perv}(\mathrm{Gr}_J)$ and a weight $\mu\in\Lambda^+_J$. Let us denote by $i:\overline{\mathrm{Gr}^\mu_J}\to\mathrm{Gr}_J$ the inclusion map. By [5, Amplification 1.4.17.1], the largest subobject of $\mathscr{B}$ in $\mathrm{Perv}(\mathrm{Gr}_J)$ supported on $\overline{\mathrm{Gr}^\mu_J}$ is $\mathscr{B}_{\leq_J\mu}={}^p\tau_{\leq 0}i_*i^!\mathscr{B}$, where ${}^p\tau_{\leq 0}$ is the truncation functor for the perverse t-structure. From the distinguished triangle

$${}^p\tau_{\leq 0}i_*i^!\mathscr{B}\to i_*i^!\mathscr{B}\to{}^p\tau_{>0}i_*i^!\mathscr{B}\xrightarrow{+}$$

in the bounded derived category of constructible sheaves on Gr_J, we deduce the long exact sequence

$$\begin{aligned}H^{2\rho_J(\nu)-1}&(T_{J,\nu},(t_{J,\nu})^{!\,p}\tau_{>0}i_*i^!\mathscr{B})\to H^{2\rho_J(\nu)}(T_{J,\nu},(t_{J,\nu})^{!\,p}\tau_{\leq 0}i_*i^!\mathscr{B})\\ &\to H^{2\rho_J(\nu)}(T_{J,\nu},(t_{J,\nu})^!i_*i^!\mathscr{B})\to H^{2\rho_J(\nu)}(T_{J,\nu},(t_{J,\nu})^{!\,p}\tau_{>0}i_*i^!\mathscr{B}).\end{aligned}$$

Theorem 3.5 in [39] implies that the two extreme terms vanish, and therefore

$$F_{J,\nu}(\mathscr{B}_{\leq_J\mu})=H^{2\rho_J(\nu)}(T_{J,\nu},(t_{J,\nu})^!i_*i^!\mathscr{B}).$$

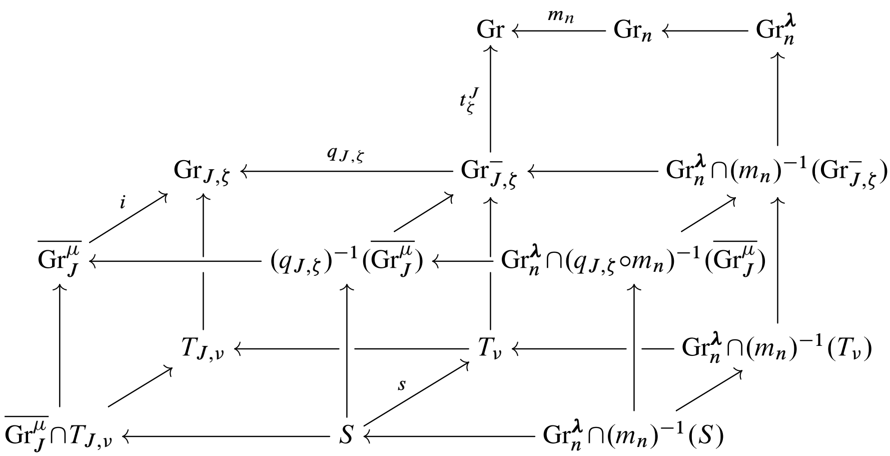
Let us patch all these pieces together. We take $(\boldsymbol{\lambda}, \mu, \nu) \in (\Lambda^+)^n \times \Lambda_J^+ \times \Lambda$ such that μ and ν belong to the same coset $\zeta \in Z/\mathbb{Z}\Phi_J$ and we consider

$$\mathscr{I}_{\boldsymbol{\lambda}} = (m_n)_* \operatorname{IC}\big(\overline{\operatorname{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big) \quad \text{and} \quad \mathscr{B} = r_J^I(\mathscr{I}_{\boldsymbol{\lambda}}).$$

Composing the isomorphisms given in (6) and (3), we get

$$\begin{aligned} H^{2\rho_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! \mathscr{B}) &\cong H^{2\rho(\nu)}(T_\nu, (t_\nu)^! \mathscr{I}_{\boldsymbol{\lambda}}) \\ &\cong H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\nu)}(\operatorname{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\nu)). \end{aligned} \tag{10}$$

To save space we set $S = (q_{J,\zeta})^{-1}(\overline{\operatorname{Gr}_J^\mu}) \cap T_\nu$ and denote by $s : S \to T_\nu$ the inclusion map. Chasing in the three-dimensional figure



(11)

we complete (10) to the following commutative diagram:

$$\begin{array}{ccccc} H^{2\rho_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! i_* i^! \mathscr{B}) & \xrightarrow{\simeq} & H^{2\rho(\nu)}(S, (t_\nu s)^! \mathscr{I}_{\boldsymbol{\lambda}}) & \xrightarrow{\simeq} & H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\nu)}(\operatorname{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(S)) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2\rho_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^! \mathscr{B}) & \xrightarrow{\simeq} & H^{2\rho(\nu)}(T_\nu, (t_\nu)^! \mathscr{I}_{\boldsymbol{\lambda}}) & \xrightarrow{\simeq} & H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\nu)}(\operatorname{Gr}_n^{\boldsymbol{\lambda}} \cap (m_n)^{-1}(T_\nu)) \end{array}$$

As explained in (B), the left vertical arrow of this diagram is the inclusion map

$$F_{J,\nu}(\mathscr{B}_{\leq_J \mu}) \to F_{J,\nu}(\mathscr{B}).$$

If an MV cycle $Z \in \mathscr{Z}(\boldsymbol{\lambda})_\nu$ satisfies

$$\mu_J(Z) \leq_J \mu,$$

then it is contained in $(m_n)^{-1}(S)$, so the fundamental class of $Z \cap \operatorname{Gr}_n^{\boldsymbol{\lambda}}$ belongs to $F_{J,\nu}(\mathscr{B}_{\leq_J \mu})$. Looking at equation (9), we see that there are just enough such MV cycles to span this subspace. Going through the geometric Satake correspondence, we conclude that the MV basis of $V(\boldsymbol{\lambda})$ satisfies condition (P1) for being L-perfect.

Turning now to condition (P2), we consider the diagram below, consisting of inclusion maps:

$$\begin{array}{ccccc} \overline{\mathrm{Gr}^\mu_J}\setminus \mathrm{Gr}^\mu_J & \xrightarrow{\;f\;} & \overline{\mathrm{Gr}^\mu_J} & \xleftarrow{\;g\;} & \mathrm{Gr}^\mu_J \\ & & {\scriptstyle i}\downarrow & \swarrow{\scriptstyle h} & \\ & & \mathrm{Gr}_{J,\zeta} & & \end{array}$$

For any sheaf $\mathscr{F} \in \mathrm{Perv}(\mathrm{Gr}_J)$ supported on $\overline{\mathrm{Gr}^\mu_J} \setminus \mathrm{Gr}^\mu_J$, we have

$$\mathrm{Hom}(\mathscr{F}, {}^p\tau_{\leq 0}(h_*h^!\mathscr{B})) = \mathrm{Hom}(\mathscr{F}, h_*h^!\mathscr{B}) = \mathrm{Hom}(h^*\mathscr{F}, h^!\mathscr{B}) = 0$$

in the bounded derived category of constructible sheaves over Gr_J (the first two equalities by adjunction, the last one because $h^*\mathscr{F} = 0$). Since $h_*h^!\mathscr{B}$ is concentrated in nonnegative perverse degrees (see [5, Proposition 1.4.16]), the sheaf ${}^p\tau_{\leq 0}(h_*h^!\mathscr{B})$ is perverse, and from the semisimplicity of $\mathrm{Perv}(\mathrm{Gr}_J)$ we conclude that

$$\mathrm{Hom}({}^p\tau_{\leq 0}(h_*h^!\mathscr{B}), \mathscr{F}) = 0.$$

Again, in the distinguished triangle

$$i_*f_*f^!i^!\mathscr{B} \to i_*i^!\mathscr{B} \to i_*g_*g^!i^!\mathscr{B} \xrightarrow{+}$$

all sheaves are concentrated in nonnegative perverse degrees. Denoting by ${}^pH^1$ the first homology group for the perverse t-structure, we obtain the exact sequence

$$0 \to {}^p\tau_{\leq 0}(i_*f_*f^!i^!\mathscr{B}) \to {}^p\tau_{\leq 0}(i_*i^!\mathscr{B}) \to {}^p\tau_{\leq 0}(h_*h^!\mathscr{B}) \to {}^pH^1(i_*f_*f^!i^!\mathscr{B}).$$

The perverse sheaf on the right is supported on $\overline{\mathrm{Gr}^\mu_J} \setminus \mathrm{Gr}^\mu_J$, so the right arrow is zero by the previous step. The resulting short exact sequence can be identified with

$$0 \to \mathscr{B}_{<_J\mu} \to \mathscr{B}_{\leq_J\mu} \to \mathscr{B}_{\leq_J\mu}/\mathscr{B}_{<_J\mu} \to 0.$$

With the same arguments as in point (B) above, we deduce that

$$F_{J,\nu}(\mathscr{B}_{\leq_J\mu}/\mathscr{B}_{<_J\mu}) = F_{J,\nu}({}^p\tau_{\leq 0}h_*h^!\mathscr{B}) = H^{2\rho_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^!h_*h^!\mathscr{B}).$$

In (11), we replace $\overline{\mathrm{Gr}^\mu_J}$ by Gr^μ_J; the same chasing as before now leads to the isomorphism

$$H^{2\rho_J(\nu)}(T_{J,\nu}, (t_{J,\nu})^!h_*h^!\mathscr{B}) \xrightarrow{\simeq} H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\nu)}\big(\mathrm{Gr}^{\boldsymbol{\lambda}}_n \cap (q_{J,\zeta} \circ m_n)^{-1}(\mathrm{Gr}^\mu_J) \cap (m_n)^{-1}(T_\nu)\big). \tag{12}$$

Now point (A) at the beginning of this section explains that $\mathscr{H}^k h^!\mathscr{B}$, where $k = -2\rho_J(\mu)$, is the local system $\mathscr{L}_\mu$ and it comes with a natural basis, namely $\mathscr{Z}^J(\boldsymbol{\lambda})_\mu$. This basis induces a decomposition of $\mathscr{B}_{\leq_J\mu}/\mathscr{B}_{<_J\mu}$ into a sum of simple objects in $\mathrm{Perv}(\mathrm{Gr}_J)$. On the one hand, this decomposition can be followed through the geometric Satake correspondence, where it gives a decomposition of the subquotient of the isotypic filtration of $\operatorname{res}^G_{M_J} V(\boldsymbol{\lambda})$ into a direct sum of irreducible representations. On the other hand, it can also be tracked through the isomorphism (12):

$$F_{J,\nu}(\mathscr{B}_{\leq_J \mu}/\mathscr{B}_{<_J \mu}) \cong \bigoplus_{Y \in \mathscr{Z}^J(\boldsymbol{\lambda})_\mu} H^{\mathrm{BM}}_{2\rho(|\boldsymbol{\lambda}|-\nu)}(\mathrm{Gr}^{\boldsymbol{\lambda}}_n \cap Y \cap (m_n)^{-1}(T_\nu)). \qquad (13)$$

From Sect. 3.4, we see that the irreducible components of

$$\overline{\mathrm{Gr}^{\boldsymbol{\lambda}}_n} \cap (q_{J,\xi} \circ m_n)^{-1}(\mathrm{Gr}^\mu_J) \cap (m_n)^{-1}(T_\nu)$$

of dimension $\rho(|\boldsymbol{\lambda}| - \nu)$ are the cycles $Z^J \cap (q_{J,\xi} \circ m_n)^{-1}(Z_J \cap \mathrm{Gr}^\mu_J)$ with $(Z^J, Z_J) \in \mathscr{Z}^J(\boldsymbol{\lambda})_\mu \times \mathscr{Z}_J(\mu)_\nu$. The basis of the right-hand side of (12) afforded by the fundamental classes of these irreducible components is thus compatible with the decomposition (13). Therefore, the MV basis of $V(\boldsymbol{\lambda})$ satisfies condition (P2) for being L-perfect.

The proof of Theorem 3.4 is now complete.

Remark 3.5. The proof establishes that the MV basis of $V(\boldsymbol{\lambda})$ satisfies a stronger property than (P2): there exists an isomorphism of the isotypic component $V(\boldsymbol{\lambda})_{J,\mu}$ with a direct sum of copies of the irreducible representation $V_J(\mu)$ such that the induced basis on $V(\boldsymbol{\lambda})_{J,\mu}$ matches the direct sum of the MV bases of the summands.

3.6. *Crystal structure on MV cycles*

Let $\boldsymbol{\lambda} \in (\Lambda^+)^n$. The MV basis of $V(\boldsymbol{\lambda})$ defined in Sect. 2.3 is indexed by

$$\mathscr{Z}(\boldsymbol{\lambda}) = \bigsqcup_{\nu \in \Lambda} \mathscr{Z}(\boldsymbol{\lambda})_\nu$$

and is L-perfect. Thus, the set $\mathscr{Z}(\boldsymbol{\lambda})$ is endowed with the structure of a crystal, as explained in Sect. 3.2. Obviously the weight of an MV cycle $Z \in \mathscr{Z}(\boldsymbol{\lambda})_\nu$ is simply $\mathrm{wt}(Z) = \nu$. The aim of this section is to characterize the action on $\mathscr{Z}(\boldsymbol{\lambda})$ of the operators $\tilde{e}_i$ and $\tilde{f}_i$.

In semisimple rank 1, one can give an explicit analytical description of the MV cycles, as follows.

Proposition 3.6. *Assume that G has semisimple rank 1 and denote by α and $\alpha^\vee$ the positive root and coroot. Let $y : \mathbb{G}_\mathrm{a} \to G^\vee$ be the additive one-parameter subgroup for the root $-\alpha^\vee$. Let $(\mu, \nu) \in \Lambda^+ \times \Lambda$ and set $r = \langle \alpha^\vee, \mu \rangle$. Then $\overline{\mathrm{Gr}^\mu} \cap T_\nu$ is nonempty if and only if there exists $p \in \{0, 1, \ldots, r\}$ such that $\nu = \mu - p\alpha$; in this case, the map $a \mapsto y(az^{p-r})L_\nu$ induces an isomorphism of algebraic varieties*

$$\mathcal{O}/z^p\mathcal{O} \xrightarrow{\simeq} \overline{\mathrm{Gr}^\mu} \cap T_\nu,$$

so $\overline{\mathrm{Gr}^\mu} \cap T_\nu$ is an affine space of dimension p and $\mathscr{Z}(\mu)_\nu$ is a singleton.

We skip the proof since this proposition is well-known; compare for instance with [2, Proposition 3.10]. We can now describe the crystal structure on $\mathscr{Z}(\boldsymbol{\lambda})$, which extends [2, Proposition 4.2].

Proposition 3.7. *Let* $(\boldsymbol{\lambda}, \nu) \in (\Lambda^+)^n \times \Lambda$, *let* $i \in I$ *and let* $Z \in \mathscr{Z}(\boldsymbol{\lambda})_\nu$.

(i) *We have* $\mathrm{wt}(Z) = \nu$, $\varepsilon_i(Z) = \frac{1}{2}\langle \alpha_i^\vee, \mu_{\{i\}}(Z) - \nu\rangle$ *and* $\varphi_i(Z) = \frac{1}{2}\langle \alpha_i^\vee, \mu_{\{i\}}(Z) + \nu\rangle$.

(ii) *Let* $Y \in \mathscr{Z}(\boldsymbol{\lambda})_{\nu+\alpha_i}$. *Then* $Y = \tilde{e}_i Z$ *if and only if* $Y \subseteq \overline{Z}$ *and* $\mu_{\{i\}}(Y) = \mu_{\{i\}}(Z)$.

Proof. Let $\boldsymbol{\lambda}$, ν, i, Z as in the statement and set $\mu = \mu_{\{i\}}(Z)$. By definition, the MV cycles $\tilde{e}_i Z$ and $\tilde{f}_i Z$ (if nonzero) are obtained by letting the Chevalley generators e_i and f_i act on (the basis element indexed by) Z in the appropriate subquotient of the isotypic filtration of $\mathrm{res}^G_{M_{\{i\}}} V(\boldsymbol{\lambda})$. According to (13), this entails that

$$\mu_{\{i\}}(Z) = \mu_{\{i\}}(\tilde{e}_i Z) = \mu_{\{i\}}(\tilde{f}_i Z) \quad \text{and} \quad Z^{\{i\}} = (\tilde{e}_i Z)^{\{i\}} = (\tilde{f}_i Z)^{\{i\}}.$$

In addition, $\mathscr{Z}_{\{i\}}(\mu)_{\nu+\alpha_i}$ and $\mathscr{Z}_{\{i\}}(\mu)_{\nu-\alpha_i}$ are empty or singletons, and the MV cycles $(\tilde{e}_i Z)_{\{i\}}$ and $(\tilde{f}_i Z)_{\{i\}}$ in the affine Grassmannian $\mathrm{Gr}_{\{i\}}$ are uniquely determined by weight considerations.

The statements can be deduced from these facts by using the explicit description provided by Proposition 3.6 and the construction of the map $(Z^J, Z_J) \mapsto Z$ in Sect. 3.4. ∎

4. The path model and MV cycles

In the previous section, we defined the structure of a crystal on the set $\mathscr{Z}(\boldsymbol{\lambda})$. In this section, we turn to Littelmann's path model [33] to study this structure. This combinatorial device can be used to effectively assemble MV cycles. Our construction is inspired by the results presented in [17] but is more flexible, for it relaxes the restriction to minimal galleries.

In this paper, piecewise linear means continuous piecewise linear. We keep the notation set up at the opening of Sect. 3.

4.1. Recollections on the path model

Let $\Lambda_\mathbb{R} = \Lambda \otimes_\mathbb{Z} \mathbb{R}$ be the real vector space spanned by the weight lattice and let $\Lambda^+_\mathbb{R}$ be the dominant cone inside $\Lambda_\mathbb{R}$.

A *path* is a piecewise linear map $\pi : [0, 1] \to \Lambda_\mathbb{R}$ such that $\pi(0) = 0$ and $\pi(1) \in \Lambda$. We denote by $\widetilde{\Pi}$ the set of all paths. The concatenation $\pi * \eta$ of two paths is defined in the usual way: $\pi * \eta(t) = \pi(2t)$ for $0 \le t \le 1/2$, and $\pi * \eta(t) = \eta(2t - 1) + \pi(1)$ for $1/2 \le t \le 1$.

In [33], the third author associates to each simple root α of Φ a pair (e_α, f_α) of "root operators" from $\widetilde{\Pi}$ to $\widetilde{\Pi} \sqcup \{0\}$ and shows that the construction yields a seminormal crystal structure on $\widetilde{\Pi}$. Here the weight map is given by $\mathrm{wt}(\pi) = \pi(1)$. To agree with the notation in Sect. 3.1, we will write $\tilde{e}_i$ and $\tilde{f}_i$ instead of e_{α_i} and f_{α_i} for each $i \in I$.

Let $\ell : [0,1] \to \mathbb{R}$ be a piecewise linear function. We say that $p \in \mathbb{R}$ is a *local absolute minimum* of ℓ if there exists a compact interval $[a,b] \subseteq [0,1]$ over which ℓ takes the value p, and there exists an $\epsilon > 0$ such that $\ell(x) > p$ for all $x \in (a-\epsilon, a) \cap [0,1]$ and all $x \in (b, b+\epsilon) \cap [0,1]$.

Given $\pi \in \widetilde{\Pi}$, we denote by $\mathcal{A}\pi$ the set of all paths $\eta \in \widetilde{\Pi}$ that can be obtained from π by applying a finite sequence of root operators $\tilde{e}_i$ or $\tilde{f}_i$. We say that a path $\pi \in \widetilde{\Pi}$ is *integral* if for each $\eta \in \mathcal{A}\pi$ and each $i \in I$, all local absolute minima of the function $t \mapsto \langle \alpha_i^\vee, \eta(t) \rangle$ are integers.

We denote the set of all integral paths by Π. Obviously, Π is a subcrystal of $\widetilde{\Pi}$. Moreover, the general definition of the root operators in [33, Sect. 1] simplifies in the case of integral paths, which only need to be cut into three parts: the initial part is left invariant, the second part is reflected, and the third part is translated. Specifically, given $(\pi, \eta) \in \Pi^2$ and $i \in I$, we have $\eta = \tilde{e}_i \pi$ if and only if there exist a negative integer $p \in \mathbb{Z}$ and two reals a and b with $0 \le a < b \le 1$ such that the function $t \mapsto \langle \alpha_i^\vee, \pi(t) \rangle$ is weakly decreasing on $[a,b]$, and for each $t \in [0,1]$,

- if $t \le a$, then $\langle \alpha_i^\vee, \pi(t) \rangle \ge p+1$ and $\eta(t) = \pi(t)$;
- if $t = a$, then $\langle \alpha_i^\vee, \pi(t) \rangle = p+1$;
- if $a < t < b$, then $p \le \langle \alpha_i^\vee, \pi(t) \rangle < p+1$ and $\eta(t) = \pi(t) - (\langle \alpha_i^\vee, \pi(t) \rangle - p - 1)\alpha_i$;
- if $t = b$, then $\langle \alpha_i^\vee, \pi(t) \rangle = p$;
- if $t \ge b$, then $\langle \alpha_i^\vee, \pi(t) \rangle \ge p$ and $\eta(t) = \pi(t) + \alpha_i$.

We say that an integral path $\pi \in \Pi$ is *dominant* if its image is contained in $\Lambda_{\mathbb{R}}^+$.

Remark 4.1. Let Γ be the group of all strictly increasing piecewise linear maps from $[0,1]$ onto itself, the product being the composition of functions. This group acts on the set of all paths by right composition: $\pi \mapsto \pi \circ \gamma$ for a path π and $\gamma \in \Gamma$. We say that $\pi \circ \gamma$ is obtained from π by a *piecewise linear reparameterization*. Visibly, the set Π of integral paths is stable under this action, the weight map wt is invariant, and the root operators are equivariant. We can thus safely consider all our previous constructions modulo this action. We will sometimes implicitly assume that this quotient has been taken, i.e. among the possible parameterizations we choose one which is appropriate for the application in view.

The first two items in the following proposition ensure that there is an abundance of integral paths.

Proposition 4.2. (i) *A dominant path π is integral whenever for each $i \in I$, the function $t \mapsto \langle \alpha_i^\vee, \pi(t) \rangle$ is weakly increasing.*

(ii) *The set Π is stable under concatenation of paths and the map $\pi \otimes \eta \mapsto \pi * \eta$ is a morphism of crystals from $\Pi \otimes \Pi$ to Π.*

(iii) *Let $\pi \in \Pi$. Then $\mathcal{A}\pi$ contains a unique dominant path η and is isomorphic as a crystal to $B(\mathrm{wt}(\eta))$.*

Proposition 4.2 follows from the results in [35]. Two lemmas will help us bridge the gap.

We fix a scalar product $(\cdot,\cdot)$ on $\Lambda_{\mathbb{R}}$ and let $d(\cdot,\cdot)$ be the corresponding distance function. We fix a basis $\mathbb{B}$ of Λ and let $\mathbb{L}_1 \subset \Lambda_{\mathbb{R}}$ be the associated unit cube, i.e. the set of points in $\Lambda_{\mathbb{R}}$ which can be written as a linear combination of $\mathbb{B}$ with coefficients in the interval $[0,1]$. Let M be the maximal distance between two points in $\mathbb{L}_1$. Let $P \in \Lambda_{\mathbb{Q}}^+$ be a dominant rational point and let $S(P,1)$ be the sphere with center P and radius 1. Let g be a ray starting at P and let g_1 be the intersection point of this ray with the sphere $S(P,1)$.

Lemma 4.3. *For any $\epsilon > 0$ there is a ray f starting at P such that $(f \setminus \{P\}) \cap \Lambda \neq \emptyset$, and for $\{f_1\} = f \cap S(P,1)$ we have $d(g_1, f_1) < \epsilon$.*

Proof. Parameterize g by $g(t) = P + t(g_1 - P)$ for $t \geq 0$. Choose $t_2 \gg 0$ and pick $\lambda \in \Lambda$ such that $g(t_2) \in \lambda + \mathbb{L}_1$. Let f be the ray starting at P passing through λ. Let f_1 be the intersection point of this ray with $S(P,1)$; then $f(t) = P + t(f_1 - P)$ for $t \geq 0$ is a parameterization of f. Set $g_2 = g(t_2)$ and $f_2 = f(t_2)$. Noting that $d(P, f_2) = t_2 = d(P, g_2)$ and using the triangular inequality, we get

$$d(f_2, \lambda) = |d(P, f_2) - d(P, \lambda)| = |d(P, g_2) - d(P, \lambda)| \leq d(g_2, \lambda) \leq M,$$

whence $d(g_2, f_2) \leq 2M$. By the intercept theorem $d(f_1, g_1)/d(g_2, f_2) = 1/t_2$, and hence $d(f_1, g_1) \leq 2M/t_2$. For t_2 large enough we obtain $d(g_1, f_1) < \epsilon$. ∎

In [35], a seemingly complicated definition of *locally integral concatenation* is introduced; it is a generalization of the concept of LS-paths [33]. This notion provides a sufficient condition for a path to be integral. Let us review it in the case of a rather special class of paths for which the property of being a locally integral concatenation reduces to condition $(*)$ below.

We extend the concatenation operation $*$ to paths that do not necessarily end at an integral weight. For $\mu \in \Lambda_{\mathbb{R}}$, let π_μ be the map $[0,1] \to \Lambda_{\mathbb{R}}$, $t \mapsto t\mu$. A path $\pi \in \widetilde{\Pi}$ is said to be *dominant rational* if it is of the form $\pi = \pi_{\mu_1} * \cdots * \pi_{\mu_s}$, where $(\mu_1, \ldots, \mu_s) \in (\Lambda_{\mathbb{Q}}^+)^s$ and $\mu_1 + \cdots + \mu_s \in \Lambda^+$. For such a path, being a locally integral concatenation (see [35, Definition 5.3]) means:

$(*)$ For each $j \in \{1, \ldots, s\}$ such that $\mu_j \neq 0$, the affine line passing through $\mu_1 + \cdots + \mu_{j-1}$ and $\mu_1 + \cdots + \mu_j$ meets at least two lattice points.

An equivalent formulation: the affine line meets at least one rational point and one lattice point.

Lemma 4.4. *A dominant rational path can be approximated by a locally integral concatenation.*

Proof. Let $\pi = \pi_{\mu_1} * \cdots * \pi_{\mu_s}$ be a dominant rational path ending in $\mu \in \Lambda^+$. We define the support of an element $\mu \in \Lambda_{\mathbb{R}}$ as the set of indices $i \in I$ such that $\langle \alpha_i^\vee, \mu \rangle \neq 0$. We can assume that the support of each μ_j is the same as the support of μ; otherwise we approximate π by a path we get by slightly perturbing $\mu_1, \dots, \mu_s$, for instance by replacing each μ_j by $\mu_j + \epsilon(\mu/s - \mu_j)$ for some rational $0 < \epsilon \ll 1$. We can also assume the support of μ is I, otherwise we work within the subspace $\bigcap_{i \in I \setminus \mathrm{supp}(\mu)} \ker \alpha_i^\vee$.

Under these assumptions, each weight μ_j is regular dominant. Then small perturbations $\mu_1', \dots, \mu_{s-1}'$ of the directions $\mu_1, \dots, \mu_{s-1}$ remain dominant, and so does $\mu_s' = \mu - (\mu_1' + \cdots + \mu_{s-1}')$. By Lemma 4.3, one can perturb in such a way that the new path $\eta := \pi_{\mu_1'} * \cdots * \pi_{\mu_s'}$ is a dominant rational path and the first $s-1$ line segments of η satisfy the affine line condition $(*)$. The last line segment of η meets the lattice point μ and a rational point, and thus satisfies $(*)$ too. Hence η is a locally integral concatenation, approximating the dominant rational path π. ■

Proof of Proposition 4.2. A path in statement (i) is of the form $\pi = \pi_{\mu_1} * \cdots * \pi_{\mu_s}$, where $(\mu_1, \dots, \mu_s) \in (\Lambda_{\mathbb{R}}^+)^s$ are dominant. Such a path can be approximated by a dominant rational path without altering $\mu_1 + \cdots + \mu_s$. In turn, a rational dominant path can be approximated by a locally integral concatenation by Lemma 4.4. Lastly, a locally integral concatenation is an integral path by [35, Lemma 5.6] and Proposition 5.9. The integrality property in (i) follows now by the continuity of the root operators [35, property (v) CONTINUITY].

It remains to prove the other two statements. The endpoint of a path is by definition an element of the lattice, so the concatenation of integral paths is an integral path. Moreover, by [35, Lemma 6.12], concatenation defines a morphism of crystals $\Pi \otimes \Pi \to \Pi$. This shows (ii). Statement (iii) follows by [35, Lemma 6.11]. ■

4.2. *From paths to MV cycles*

We need additional terminology before we proceed to the main construction of this section.

To each coroot $\alpha^\vee \in \Phi^\vee$ corresponds an additive one-parameter subgroup $x_{\alpha^\vee} : \mathbb{G}_a \to G^\vee$. Given additionally an integer $p \in \mathbb{Z}$, we define a map

$$x_{(\alpha^\vee, p)} : \mathbb{C} \to G^\vee(\mathcal{K}), \quad a \mapsto x_{\alpha^\vee}(az^p).$$

An *affine coroot* is a pair $(\alpha^\vee, p)$ consisting of a coroot $\alpha^\vee \in \Phi^\vee$ and an integer p. The *direction* of an affine coroot $(\alpha^\vee, p)$ is $\alpha^\vee$. An affine coroot is said to be *positive* if its direction is so. We denote the set of affine coroots by $\Phi_a^\vee$ and the set of positive affine coroots by $\Phi_a^{\vee,+}$.

To an affine coroot β, besides the map x_β defined above, we attach a hyperplane H_β and a negative closed half-space H_β^- in $\Lambda_{\mathbb{R}}$ as follows:

$$H_{(\alpha^\vee,p)} = \{x \in \Lambda_{\mathbb{R}} \mid \langle \alpha^\vee, x\rangle = p\}, \quad H_{(\alpha^\vee,p)}^- = \{x \in \Lambda_{\mathbb{R}} \mid \langle \alpha^\vee, x\rangle \le p\}.$$

Let s_β be the reflection across the hyperplane H_β; concretely,

$$s_{(\alpha^\vee,p)}(x) = x - (\langle \alpha^\vee, x\rangle - p)\alpha$$

for any $x \in \Lambda_{\mathbb{R}}$. In addition, we denote by τ_λ the translation $x \mapsto x + \lambda$ by the element $\lambda \in \Lambda$. The subgroup of $\mathrm{Aut}(\Lambda_{\mathbb{R}})$ generated by all the reflections s_β is the affine Weyl group W_a. When we add the translations τ_λ, we obtain the extended affine Weyl group $\widetilde{W}_a$. Then $\tau_\lambda \in W_a$ if and only if $\lambda \in \mathbb{Z}\Phi$.

The group $\widetilde{W}_a$ acts on the set $\Phi_a^\vee$ of affine roots: one demands that $w(H_\beta^-) = H_{w\beta}^-$ for each element $w \in \widetilde{W}_a$ and each affine coroot $\beta \in \Phi_a^\vee$. Then for each $\beta \in \Phi_a^\vee$ and each $\lambda \in \Lambda$, we have $x_{\tau_\lambda \beta}(a) = z^\lambda x_\beta(a) z^{-\lambda}$ for all $a \in \mathbb{C}$.

We denote by $\mathfrak{H}$ the arrangement formed by the hyperplanes H_β, where $\beta \in \Phi_a^\vee$. It divides the vector space $\Lambda_\mathbb{R}$ into faces. The closure of a face is the disjoint union of faces of smaller dimension. Endowed with the set of all faces, $\Lambda_\mathbb{R}$ becomes a polysimplicial complex, called the *affine Coxeter complex*.

For each face $\mathfrak{f}$ of the affine Coxeter complex, we denote by $N^\vee(\mathfrak{f})$ the subgroup of $N^\vee(\mathcal{K})$ generated by the elements $x_{\alpha^\vee}(az^p)$, where $a \in \mathcal{O}$ and $(\alpha^\vee, p)$ is a positive affine coroot such that $\mathfrak{f} \subseteq H^-_{(\alpha^\vee, p)}$. We note that $N^\vee(\tau_\lambda \mathfrak{f}) = z^\lambda N^\vee(\mathfrak{f}) z^{-\lambda}$ for each face $\mathfrak{f}$ and each $\lambda \in \Lambda$ and that $N^\vee(\mathfrak{f}) \subseteq N^\vee(\mathcal{O})$ if $\mathfrak{f} \subseteq \Lambda_\mathbb{R}^+$.

For $x \in \Lambda_\mathbb{R}$, we denote by $\mathfrak{f}_x$ the face in the affine Coxeter complex that contains the point x. We use the symbol $\prod'$ to denote the restricted product of groups, consisting of families involving only finitely many nontrivial terms.

Using these conventions and the notation introduced in Sect. 2.2, given $(\pi_1, \dots, \pi_n) \in \Pi^n$, we define $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$ as the subset of Gr_n of all elements

$$\left[\Big(\prod_{t_1 \in [0,1]} v_{1,t_1}\Big) z^{\mathrm{wt}(\pi_1)}, \dots, \Big(\prod_{t_n \in [0,1]} v_{n,t_n}\Big) z^{\mathrm{wt}(\pi_n)}\right]$$

with

$$((v_{1,t_1}), \dots, (v_{n,t_n})) \in \prod_{t_1 \in [0,1]}{}' N^\vee(\mathfrak{f}_{\pi_1(t_1)}) \times \cdots \times \prod_{t_n \in [0,1]}{}' N^\vee(\mathfrak{f}_{\pi_n(t_n)}).$$

Proposition 4.5. *Let* $(\pi_1, \dots, \pi_n) \in \Pi^n$.

(i) *The set* $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$ *is stable under left multiplication by* $N^\vee(\mathcal{O})$.

(ii) *Let* $\mu = \mathrm{wt}(\pi_1) + \cdots + \mathrm{wt}(\pi_n)$. *Then the set* $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$ *is an irreducible constructible subset of* $(m_n)^{-1}(S_\mu)$.

(iii) *We have*

$$\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n) = \Psi(\mathring{\mathbf{Z}}(\pi_1) \ltimes \cdots \ltimes \mathring{\mathbf{Z}}(\pi_n)),$$
$$\mathring{\mathbf{Z}}(\pi_1 * \cdots * \pi_n) = m_n(\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)).$$

(iv) *Let* $i \in I$ *and compute* $\eta_1 \otimes \cdots \otimes \eta_n = \tilde{e}_i(\pi_1 \otimes \cdots \otimes \pi_n)$ *in the crystal* $\Pi^{\otimes n}$, *provided that this operation is defined. Then* $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$ *is contained in the closure of* $\mathring{\mathbf{Z}}(\eta_1 \otimes \cdots \otimes \eta_n)$ *in* Gr_n.

Proof. Assertion (i) is a direct consequence of the equality $N^\vee(\mathfrak{f}_{\pi_1(0)}) = N^\vee(\mathfrak{f}_0) = N^\vee(\mathcal{O})$.

Assertion (ii) comes from general principles once we have replaced the restricted infinite product by a finite one.

The first equation in (iii) is tautological. In the second one, we view the concatenation $\pi = \pi_1 * \cdots * \pi_n$ as a map from $[0, n]$ to $\Lambda_{\mathbb{R}}$, each path $\pi_1, \ldots, \pi_n$ being travelled at nominal speed. We set $\nu_0 = 0$, and for $j \in \{2, \ldots, n\}$ we set $\nu_{j-1} = \mathrm{wt}(\pi_1) + \cdots + \mathrm{wt}(\pi_{j-1})$. Then for $j \in \{1, \ldots, n\}$ and $t \in [0, 1]$, we have $\pi(t + (j-1)) = \nu_{j-1} + \pi_j(t)$, and accordingly $N^\vee(\mathfrak{f}_{\pi(t+(j-1))}) = z^{\nu_{j-1}} N^\vee(\mathfrak{f}_{\pi_j(t)}) z^{-\nu_{j-1}}$. A banal calculation then yields the desired result.

The proof of (iv) is much more involved. We defer its presentation to Sect. 4.5. ■

For $(\pi_1, \ldots, \pi_n) \in \Pi^n$, we denote by $\mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n)$ the closure of $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$ in $(m_n)^{-1}(S_\mu)$, where $\mu = \mathrm{wt}(\pi_1) + \cdots + \mathrm{wt}(\pi_n)$.

Theorem 4.6. *Let $(\pi_1, \ldots, \pi_n) \in \Pi^n$, set $\mu = \mathrm{wt}(\pi_1) + \cdots + \mathrm{wt}(\pi_n)$ and for $j \in \{1, \ldots, n\}$, let λ_j be the weight of the unique dominant path in $\mathcal{A}\pi_j$. Then $\mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n)$ is an MV cycle; specifically, $\mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n) \in {}_*\mathscr{Z}(\lambda_1, \ldots, \lambda_n)_\mu$.*

Proof. We start with the particular case $n = 1$. Let $\pi \in \Pi$, let η be the unique dominant path in $\mathcal{A}\pi$, set $\lambda = \mathrm{wt}(\eta)$ and $\mu = \mathrm{wt}(\pi)$, and set $p = 2\rho(\lambda)$ and $k = \rho(\lambda + \mu)$. By Proposition 4.2 (iii), the crystal $\mathcal{A}\pi$ is isomorphic to $B(\lambda)$, so it contains a unique lowest weight element ξ. Then π can be reached by applying a sequence of root operators $\tilde{f}_i$ to η or by applying a sequence of root operators $\tilde{e}_i$ to ξ. Thus, there exists a finite sequence $(\pi_0, \ldots, \pi_p)$ of elements in $\mathcal{A}\pi$ such that $\pi_0 = \xi$, $\pi_k = \pi$, $\pi_p = \eta$ and each π_{j+1} is obtained from π_j by applying a root operator $\tilde{e}_i$.

Since η is dominant, each face $\mathfrak{f}_{\eta(t)}$ is contained in $\Lambda^+_{\mathbb{R}}$, so each group $N^\vee(\mathfrak{f}_{\eta(t)})$ is contained in $N^\vee(\mathcal{O})$. Then by construction $\mathring{\mathbf{Z}}(\eta) \subseteq N^\vee(\mathcal{O})L_\lambda$, and therefore $\overline{\mathbf{Z}(\eta)}$ (the closure of $\mathbf{Z}(\eta)$ in Gr) is contained in $\overline{\mathrm{Gr}^\lambda}$. Also, Proposition 4.5 (iv) implies that

$$\overline{\mathbf{Z}(\pi_0)} \subseteq \overline{\mathbf{Z}(\pi_1)} \subseteq \cdots \subseteq \overline{\mathbf{Z}(\pi_p)}. \tag{14}$$

These inclusions are strict because $\overline{\mathbf{Z}(\pi_j)}$ is contained in the closure of $\overline{\mathrm{Gr}^\lambda} \cap S_{\mathrm{wt}(\pi_j)}$, which is disjoint from $S_{\mathrm{wt}(\pi_{j+1})}$ by [39, Proposition 3.1 (a)], while $\mathbf{Z}(\pi_{j+1})$ is contained in $S_{\mathrm{wt}(\pi_{j+1})}$. Thus (14) is a strictly increasing chain of closed irreducible subsets of $\overline{\mathrm{Gr}^\lambda}$. As $\overline{\mathrm{Gr}^\lambda}$ has dimension p, we see that each $\overline{\mathbf{Z}(\pi_j)}$ has dimension j.

In particular, $\overline{\mathbf{Z}(\pi)}$ has dimension k. But $\mathbf{Z}(\pi)$ is locally closed, because it is a closed subset of S_μ which is locally closed. So $\mathbf{Z}(\pi)$ has dimension k. At this point, we know that $\mathbf{Z}(\pi)$ is a closed irreducible subset of $\overline{\mathrm{Gr}^\lambda} \cap S_\mu$ of dimension $k = \rho(\lambda + \mu)$. Therefore $\mathbf{Z}(\pi)$ belongs to $\mathscr{Z}(\lambda)_\mu$.

The reasoning above establishes the case $n = 1$ of the theorem. The general case then follows from Propositions 2.2 (ii) and 4.5 (iii). ■

4.3. *A more economical definition*

For the proofs in the following sections, it will be convenient to have a more economical presentation of the sets $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$. We need a few additional pieces of notation.

When $\mathfrak{f}$ and $\mathfrak{f}'$ are two faces of the affine Coxeter complex such that $\mathfrak{f}$ is contained in the closure $\overline{\mathfrak{f}'}$ of $\mathfrak{f}'$, we denote by $\Phi_a^{\vee,+}(\mathfrak{f},\mathfrak{f}')$ the set of all positive affine coroots β such that $\mathfrak{f} \subseteq H_\beta$ and $\mathfrak{f}' \not\subseteq H_\beta^-$. We denote by $\mathscr{N}^\vee(\mathfrak{f},\mathfrak{f}')$ the subgroup of $N^\vee(\mathcal{K})$ generated by the elements $x_\beta(a)$ with $\beta \in \Phi_a^{\vee,+}(\mathfrak{f},\mathfrak{f}')$ and $a \in \mathbb{C}$.

We recall that a group Γ is the (*internal*) *Zappa–Szép product* of two subgroups Γ' and Γ'' if the product in Γ induces a bijection $\Gamma' \times \Gamma'' \to \Gamma$. We indicate this situation with the notation $\Gamma = \Gamma' \bowtie \Gamma''$.

The following result is Proposition 19 (ii) in [2].

Lemma 4.7. *Let $\mathfrak{f}$ and $\mathfrak{f}'$ be two faces of the affine Coxeter complex such that $\mathfrak{f} \subseteq \overline{\mathfrak{f}'}$. Then $N^\vee(\mathfrak{f}) = \mathscr{N}^\vee(\mathfrak{f},\mathfrak{f}') \bowtie N^\vee(\mathfrak{f}')$, and the map*

$$\mathbb{C}^{\Phi_a^{\vee,+}(\mathfrak{f},\mathfrak{f}')} \to \mathscr{N}^\vee(\mathfrak{f},\mathfrak{f}'), \quad (a_\beta) \mapsto \prod_{\beta \in \Phi_a^{\vee,+}(\mathfrak{f},\mathfrak{f}')} x_\beta(a_\beta),$$

is bijective, whichever order on $\Phi_a^{\vee,+}(\mathfrak{f},\mathfrak{f}')$ is used to compute the product.

Given $\pi \in \Pi$ and $t \in [0,1[$, we denote by $\mathfrak{f}_{\pi(t+0)}$ the face in the affine Coxeter complex that contains the points $\pi(t+h)$ for all small enough $h > 0$. Obviously, its closure meets, hence contains, the face $\mathfrak{f}_{\pi(t)}$. We set $\Phi_a^{\vee,+}(\pi,t) = \Phi_a^{\vee,+}(\mathfrak{f}_{\pi(t)},\mathfrak{f}_{\pi(t+0)})$ and $\mathscr{N}^\vee(\pi,t) = \mathscr{N}^\vee(\mathfrak{f}_{\pi(t)},\mathfrak{f}_{\pi(t+0)})$.

Concretely, $\Phi_a^{\vee,+}(\pi,t)$ is the set of all $\beta \in \Phi_a^{\vee,+}$ such that π quits the half-space H_β^- at time t and $\mathscr{N}^\vee(\pi,t)$ is the subgroup of $N^\vee(\mathcal{K})$ generated by the elements $x_\beta(a)$ with $\beta \in \Phi_a^{\vee,+}(\pi,t)$ and $a \in \mathbb{C}$. Note that $\Phi_a^{\vee,+}(\pi,t)$ is empty save for finitely many t.

Proposition 4.8. *Let $(\pi_1,\dots,\pi_n) \in \Pi^n$. Then $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$ is the set of all elements*

$$\Bigl[\Bigl(\prod_{t_1 \in [0,1[} v_{1,t_1}\Bigr) z^{\mathrm{wt}(\pi_1)}, \dots, \Bigl(\prod_{t_n \in [0,1[} v_{n,t_n}\Bigr) z^{\mathrm{wt}(\pi_n)}\Bigr]$$

with

$$((v_{1,t_1}),\dots,(v_{n,t_n})) \in \prod_{t_1 \in [0,1[} \mathscr{N}^\vee(\pi_1,t_1) \times \cdots \times \prod_{t_n \in [0,1[} \mathscr{N}^\vee(\pi_n,t_n).$$

Proof. Let $\pi \in \Pi$. Let $(t_1,\dots,t_m)$ be the ordered list of all elements $t \in [0,1[$ such that $\Phi_a^{\vee,+}(\pi,t) \neq \emptyset$ and set $t_{m+1} = 1$.

Pick $\ell \in \{1,\dots,m\}$; between the times t_ℓ and $t_{\ell+1}$, the path π never quits the half-space H_β^- of a positive affine coroot β; as a consequence, the map $t \mapsto N^\vee(\mathfrak{f}_{\pi(t)})$ is nondecreasing on $]t_\ell, t_{\ell+1}]$. It is also nondecreasing on $[0,t_1]$ if $t_1 > 0$. However, when t goes past the point t_ℓ, the group $N^\vee(\mathfrak{f}_{\pi(t)})$ loses the first factor of the Zappa–Szép product $N^\vee(\mathfrak{f}_{\pi(t_\ell)}) = \mathscr{N}^\vee(\pi,t_\ell) \bowtie N^\vee(\mathfrak{f}_{\pi(t_\ell+0)})$.

It follows that for any family (u_t) in $\prod'_{t \in [t_\ell,t_{\ell+1}]} N^\vee(\mathfrak{f}_{\pi(t)})$, there exists $(v,u) \in \mathscr{N}^\vee(\pi,t_\ell) \times N^\vee(\mathfrak{f}_{\pi(t_{\ell+1})})$ such that $\prod_{t \in [t_\ell,t_{\ell+1}]} u_t = vu$. To see this, one decomposes u_{t_ℓ} as vu' according to the Zappa–Szép product and one defines u as the product of u' and of the u_t for $t \in]t_\ell, t_{\ell+1}]$. Assembling these pieces (and an analogous statement over the interval $[0, t_1]$ if $t_1 > 0$) from left to right, and noting that $N^\vee(\mathfrak{f}_{\pi(1)})$ stabilizes $L_{\mathrm{wt}(\pi)}$, we deduce that $\mathring{\mathbf{Z}}(\pi)$ is the image of the map

$$\prod_{t\in[0,1[} \mathscr{N}^\vee(\pi, t) \to \mathrm{Gr}, \quad (v_t) \mapsto \Big(\prod_{t\in[0,1[} v_t\Big) L_{\mathrm{wt}(\pi)}.$$

This proves our statement in the case of just one path. The general case then follows from Proposition 4.5 (iii). ∎

4.4. *Isomorphisms of crystals*

In the previous section we explained how to build elements in ${}_*\mathscr{Z}(\boldsymbol{\lambda})_\mu$, while in Sect. 3 we were dealing with MV cycles in $\mathscr{Z}(\boldsymbol{\lambda})$. This clumsiness is due to a mismatch between the definition of the path model and the conventions in [3, 39]. To mitigate the disagreement, we define a crystal structure on

$${}_*\mathscr{Z}(\boldsymbol{\lambda}) = \bigsqcup_{\mu\in\Lambda} {}_*\mathscr{Z}(\boldsymbol{\lambda})_\mu.$$

Recall the setup of Sect. 3.3: we consider a subset $J \subseteq I$, define an action of $\mathbb{C}^\times$ on Gr given by a special dominant weight θ_J, and get the diagram (4). Now let $(\boldsymbol{\lambda}, \nu) \in (\Lambda^+)^n \times \Lambda$, let ζ be the coset of ν modulo $\mathbb{Z}\Phi_J$, and let $Z \in {}_*\mathscr{Z}(\boldsymbol{\lambda})_\nu$. Then $m_n(Z) \subseteq S_\nu \subseteq \mathrm{Gr}^+_{J,\zeta}$ and there is a unique weight $\mu \in \Lambda^+_J$ characterized by the conditions

$$(p_{J,\zeta} \circ m_n)(Z) \subseteq \overline{\mathrm{Gr}^\mu_J} \quad \text{and} \quad (p_{J,\zeta} \circ m_n)(Z) \cap \mathrm{Gr}^\mu_J \neq \emptyset.$$

We denote this weight by $\mu_J(Z)$.

By analogy with Proposition 3.7, we can then claim the existence of a crystal structure on ${}_*\mathscr{Z}(\boldsymbol{\lambda})$ such that for all $\nu \in \Lambda$, $i \in I$ and $Z \in \mathscr{Z}(\boldsymbol{\lambda})_\nu$,

- we have $\mathrm{wt}(Z) = \nu$, $\varepsilon_i(Z) = \frac{1}{2}\langle\alpha_i^\vee, \mu_{\{i\}}(Z) - \nu\rangle$ and $\varphi_i(Z) = \frac{1}{2}\langle\alpha_i^\vee, \mu_{\{i\}}(Z) + \nu\rangle$;
- if $Y \in {}_*\mathscr{Z}(\boldsymbol{\lambda})_{\nu+\alpha_i}$, then $Y = \tilde{e}_i Z$ if and only if $\overline{Y} \supseteq Z$ and $\mu_{\{i\}}(Y) = \mu_{\{i\}}(Z)$.

Theorem 4.9. *Let $(\lambda_1, \ldots, \lambda_n) \in (\Lambda^+)^n$, and for each $j \in \{1, \ldots, n\}$ choose a subcrystal Π_j of Π isomorphic to $B(\lambda_j)$. Then the map $(\pi_1, \ldots, \pi_n) \mapsto \mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n)$ is an isomorphism of crystals*

$$\Pi_1 \otimes \cdots \otimes \Pi_n \xrightarrow{\simeq} {}_*\mathscr{Z}(\lambda_1, \ldots, \lambda_n).$$

Proof. Let $i \in I$ and let $(\pi_1, \ldots, \pi_n) \in \Pi_1 \times \cdots \times \Pi_n$. Set $\nu = \mathrm{wt}(\pi_1) + \cdots + \mathrm{wt}(\pi_n)$, let ζ be the coset of ν modulo $\mathbb{Z}\alpha_i$, and set $\pi = \pi_1 * \cdots * \pi_n$,

$$p = \min\{\langle\alpha_i^\vee, \pi(t)\rangle \mid t \in [0, 1]\} \quad \text{and} \quad q = \langle\alpha_i^\vee, \nu\rangle = \langle\alpha_i^\vee, \pi(1)\rangle.$$

For any $a \in \mathbb{C}[z, z^{-1}]$ and any positive coroot $\alpha^\vee$, we have

$$\lim_{c\to 0} \theta_{\{i\}}(c) x_{\alpha^\vee}(a) \theta_{\{i\}}(c)^{-1} = \begin{cases} x_{\alpha^\vee}(a) & \text{if } \alpha^\vee = \alpha_i^\vee, \\ 1 & \text{otherwise.} \end{cases}$$

Using Proposition 4.8, we see that $p_{\{i\},\zeta}(\mathring{\mathbf{Z}}(\pi))$ is the set of all elements of the form

$$\lim_{c\to 0} \prod_{t\in[0,1[} \Big(\prod_{\beta\in\Phi_a^{\vee,+}(\pi,t)} \theta_{\{i\}}(c) x_\beta(a_{t,\beta}) \theta_{\{i\}}(c)^{-1} \Big) L_\nu$$

where $a_{t,\beta}$ are complex numbers. All factors in the product disappear in the limit $c \to 0$, except those for the affine roots β of direction $\alpha_i^\vee$. Let $(\alpha_i^\vee, p_1)$, ..., $(\alpha_i^\vee, p_s)$ be these affine roots. Since the function $t \mapsto \langle \alpha_i^\vee, \pi(t)\rangle$ assumes the value p and thereafter reaches the value q, the path π must, at some point, quit each half-space $H^-_{(\alpha_i^\vee,p)}$, $H^-_{(\alpha_i^\vee,p+1)}, \dots, H^-_{(\alpha_i^\vee,q-1)}$, so

$$\{p, p+1, \dots, q-1\} \subseteq \{p_1, \dots, p_s\} \subseteq \{p, p+1, \dots\}.$$

We conclude that

$$p_{\{i\},\zeta}(\mathring{\mathbf{Z}}(\pi)) = \{x_{\alpha_i^\vee}(az^p)L_\nu \mid a \in \mathcal{O}/z^{q-p}\mathcal{O}\}.$$

Proposition 4.5 (iii) and a variant of Proposition 3.6 then jointly imply

$$(p_{\{i\},\zeta} \circ m_n)(\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)) = p_{\{i\},\zeta}(\mathring{\mathbf{Z}}(\pi)) = \overline{\mathrm{Gr}^\mu_{\{i\}}} \cap S_{\{i\},\nu}$$

where $\mu = \nu - p\alpha_i$. Thus,

$$\mu_{\{i\}}(\mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n)) = \nu - p\alpha_i \tag{15}$$

and

$$\varepsilon_i(\mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n)) = -p \quad \text{and} \quad \varphi_i(\mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n)) = q - p.$$

These equations show that the map $(\pi_1, \dots, \pi_n) \mapsto \mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n)$ is compatible with the functions ε_i and φ_i.

Now compute

$$\eta_1 \otimes \cdots \otimes \eta_n = \tilde{e}_i(\pi_1 \otimes \cdots \otimes \pi_n)$$

in the crystal $\Pi_1 \otimes \cdots \otimes \Pi_n$, assuming this is doable. By Proposition 4.5 (iv),

$$\overline{\mathbf{Z}(\eta_1 \otimes \cdots \otimes \eta_n)} \supseteq \mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n). \tag{16}$$

Let $\eta = \eta_1 * \cdots * \eta_n$. Then $\eta = \tilde{e}_i\pi$ by Proposition 4.2 (ii), and therefore

$$\mathrm{wt}(\eta) = \nu + \alpha_i \quad \text{and} \quad \min\{\langle\alpha_i^\vee, \eta(t)\rangle \mid t \in [0,1]\} = p + 1.$$

Repeating the arguments above, we get

$$\mu_{\{i\}}(\mathbf{Z}(\eta_1 \otimes \cdots \otimes \eta_n)) = (\nu + \alpha_i) - (p+1)\alpha_i = \nu - p\alpha_i.$$

Together with (15) and (16), this gives

$$\mathbf{Z}(\eta_1 \otimes \cdots \otimes \eta_n) = \tilde{e}_i \mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n).$$

We conclude that the map $(\pi_1, \dots, \pi_n) \mapsto \mathbf{Z}(\pi_1 \otimes \cdots \otimes \pi_n)$ has the required compatibility with the operations $\tilde{e}_i$. ∎

Corollary 4.10. *Let* $(\lambda_1, \ldots, \lambda_n) \in (\Lambda^+)^n$. *Then the map* $(Z_1, \ldots, Z_n) \mapsto \Psi(Z_1 \ltimes \cdots \ltimes Z_n)$ *from Proposition* 2.2 (ii) *is an isomorphism of crystals*

$$_*\mathscr{Z}(\lambda_1) \otimes \cdots \otimes {}_*\mathscr{Z}(\lambda_n) \xrightarrow{\simeq} {}_*\mathscr{Z}(\lambda_1, \ldots, \lambda_n).$$

The crystals $\mathscr{Z}(\boldsymbol{\lambda})$ enjoy a factorization property analogous to Corollary 4.10; **one must however use the opposite tensor product on crystals.**

Remark 4.11. The plactic algebra [32] is an algebraic combinatorial tool invented by Lascoux and Schützenberger long before the notion of a crystal basis of a representation was introduced. Loosely speaking, for a complex reductive algebraic group, the plactic algebra is the algebra having as basis the union $\bigcup_{\lambda \in \Lambda^+} B(\lambda)$ of the crystal bases $B(\lambda)$ for all irreducible representations, the product being given by the tensor product of crystals. For $G = \mathrm{SL}_n(\mathbb{C})$, Lascoux and Schützenberger give a description of such an algebra in terms of the word algebra modulo the Knuth relations, and it was shown later that this algebra is isomorphic to the one given by the crystal basis. A combinatorial Lascoux–Schützenberger type description for the other types was given in [34]; this description uses the path model.

It is natural to ask whether it is possible to do the same with MV cycles: endow the set of all MV cycles for all dominant weights $\lambda \in \Lambda^+$ with the structure of a crystal and define (in a geometric way) a multiplication on the cycles which mimics the plactic algebra. For $G = \mathrm{SL}_n(\mathbb{C})$, a positive answer was given in [18]. This approach was adapted to the symplectic case in [44].

The results in this section can be naturally viewed as a generalization of [18] to arbitrary connected reductive groups. Using [34] and Proposition 4.2, one can use the set Π to construct the plactic algebra so that it has as basis equivalence classes (generalized Knuth relations) of elements in Π. The sets $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$ (Proposition 4.5) replace in the general setting the Białynicki-Birula cells in [18]. By combining Proposition 4.5 (iii) and Theorem 4.9, we see that the closure of $m_n(\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n))$ is an MV cycle which depends only on the class of the path $\pi_1 * \cdots * \pi_n$ modulo the generalized Knuth relations. In particular, the main result of [18] follows as a special case.

A different approach to this problem was taken by Xiao and Zhu [45]. They define a set of 'elementary Littelmann paths', modeled over minuscule or quasi-minuscule representations, use the methods from [40] to assign an MV cycle to each concatenation of elementary Littelmann paths, and show that the resulting map factorizes through the generalized Knuth relations.

4.5. Proof of Proposition 4.5 (iv)

This section can be skipped without substantial loss of appreciation of our main storyline. We follow the same method as in [2, proof of Proposition 5.11].

The group $G^\vee(\mathbb{C})$ is generated by elements $x_{\alpha^\vee}(a)$ and c^λ, where $(a, \alpha^\vee) \in \mathbb{C} \times \Phi^\vee$ and $(c, \lambda) \in \mathbb{C}^\times \times \Lambda$, which obey the following relations:

- For any $(a, \alpha^\vee) \in \mathbb{C} \times \Phi^\vee$ and any $(c, \lambda) \in \mathbb{C}^\times \times \Lambda$,

$$c^\lambda x_{\alpha^\vee}(a) c^{-\lambda} = x_{\alpha^\vee}(c^{\langle \alpha^\vee, \lambda \rangle} a).$$

- Given two linearly independent elements $\alpha^\vee$ and $\beta^\vee$ in $\Phi^\vee$, there exist constants $C_{i,j}$ such that

$$x_{\alpha^\vee}(a) x_{\beta^\vee}(b) x_{\alpha^\vee}(a)^{-1} x_{\beta^\vee}(b)^{-1} = \prod_{(i,j)} x_{i\alpha^\vee + j\beta^\vee}(C_{i,j} a^i b^j) \tag{17}$$

for any $(a, b) \in \mathbb{C}^2$. The product on the right-hand side is taken over all pairs of positive integers (i, j) for which $i\alpha^\vee + j\beta^\vee \in \Phi^\vee$, in the order of increasing $i + j$.

Further, the one-parameter subgroups $x_{\alpha^\vee}$ can be normalized so that for any root $\alpha \in \Phi$,

- for any $(a, b) \in \mathbb{C}^2$ such that $1 - ab \neq 0$,

$$x_{\alpha^\vee}(a) x_{-\alpha^\vee}(b) = x_{-\alpha^\vee}(b/(1 - ab))(1 - ab)^\alpha x_{\alpha^\vee}(a/(1 - ab)); \tag{18}$$

- there exists an element $\overline{s_\alpha} \in G^\vee(\mathbb{C})$ such that for any $a \in \mathbb{C}^\times$,

$$x_{\alpha^\vee}(a) x_{-\alpha^\vee}(a^{-1}) x_{\alpha^\vee}(a) = x_{-\alpha^\vee}(a^{-1}) x_{\alpha^\vee}(a) x_{-\alpha^\vee}(a^{-1}) = a^\alpha\, \overline{s_\alpha} = \overline{s_\alpha}\, a^{-\alpha}. \tag{19}$$

This element $\overline{s_\alpha}$ lifts to the normalizer of $T^\vee(\mathbb{C})$ the reflection $s_\alpha \in W$ along the root α. All the above relations also hold for scalars b, c in $\mathcal{K}$, provided of course that we regard them in $G^\vee(\mathcal{K})$.

The Chevalley commutation relation (17) implies the following easy lemma.

Lemma 4.12. *Let $\mathfrak{f}$ be a face of the affine Coxeter complex, let $(\alpha^\vee, p)$ and $(\beta^\vee, q)$ be two positive affine coroots, and let $(a, b) \in \mathcal{O}^2$. Assume that $\alpha^\vee$ is simple, $\alpha^\vee \neq \beta^\vee$, and*

$$\mathfrak{f} \subseteq H^-_{(-\alpha^\vee, -p)} \cap H^-_{(\beta^\vee, q)}.$$

Then

$$x_{-\alpha^\vee}(az^{-p}) x_{\beta^\vee}(bz^q) x_{-\alpha^\vee}(-az^{-p}) \in N^\vee(\mathfrak{f}).$$

Proof. We consider the situation set forth in the statement of the lemma. Using (17), we write

$$x_{-\alpha^\vee}(az^{-p}) x_{\beta^\vee}(bz^q) x_{-\alpha^\vee}(-az^{-p}) x_{\beta^\vee}(-bz^q) = \prod_{(i,j)} x_{-i\alpha^\vee + j\beta^\vee}(C_{i,j} a^i b^j z^{-ip+jq}) \tag{20}$$

where the product on the right-hand side is taken over all pairs of positive integers (i, j) for which $-i\alpha^\vee + j\beta^\vee$ is a coroot.

Consider such a pair (i, j). In view of our assumptions, the coroot $-i\alpha^\vee + j\beta^\vee$ is necessarily positive. Moreover, for any $x \in \mathfrak{f}$ we have

$$\langle -i\alpha^\vee + j\beta^\vee, x \rangle = i \langle -\alpha^\vee, x \rangle + j \langle \beta^\vee, x \rangle \leq i(-p) + jq,$$

so $\mathfrak{f} \subseteq H^-_{(-i\alpha^\vee + j\beta^\vee, -ip+jq)}$. It follows that the right-hand side of (20) lies in $\mathcal{N}^\vee(\mathfrak{f})$, which readily implies the statement. ∎

Given $g \in N^\vee(\mathcal{K})$, there is a unique tuple $(a_i) \in \mathcal{K}^I$ such that

$$g \equiv \prod_{i \in I} x_{\alpha_i^\vee}(a_i) \bmod (N^\vee(\mathcal{K}), N^\vee(\mathcal{K}));$$

looking at a specific $i \in I$, we denote by $\mathbf{a}_{i,p}(g)$ the coefficient of z^p in the Laurent series a_i. This procedure defines a morphism of groups $\mathbf{a}_{i,p} : N^\vee(\mathcal{K}) \to \mathbb{C}$ for each pair $(i,p) \in I \times \mathbb{Z}$.

Lemma 4.13. *Let $\pi \in \Pi$ and let $(t_1, \ldots, t_m)$ be the ordered list of all elements $t \in [0,1[$ such that $\Phi_a^{\vee,+}(\pi,t) \neq \emptyset$. Set $t_{m+1} = 1$. Let $i \in I$ and set*

$$p = \min\{\langle \alpha_i^\vee, \pi(t)\rangle \mid t \in [0,1]\}.$$

Let $r \in \{1, \ldots, m+1\}$ and let $(v_\ell) \in \prod_{\ell=r}^m \mathcal{N}^\vee(\pi, t_\ell)$.

(i) *Let r^+ be the smallest element in*

$$\bigl\{\ell \in \{r, \ldots, m\} \bigm| (\alpha_i^\vee, p) \in \Phi_a^{\vee,+}(\pi, t_\ell)\bigr\},$$

assuming that this set is nonempty. Then for any $u \in N^\vee(\mathfrak{f}_{\pi(t_r)})$ there exists $(v'_\ell) \in \prod_{\ell=r}^m \mathcal{N}^\vee(\pi, t_\ell)$ such that

$$v'_r \cdots v'_m L_{\mathrm{wt}(\pi)} = u v_r \cdots v_m L_{\mathrm{wt}(\pi)}$$

and

$$\mathbf{a}_{i,p}(v'_\ell) = \begin{cases} \mathbf{a}_{i,p}(u) + \mathbf{a}_{i,p}(v_\ell) & \text{if } \ell = r^+, \\ \mathbf{a}_{i,p}(v_\ell) & \text{for all other } \ell \in \{r, \ldots, m\}. \end{cases}$$

(ii) *For any $c \in 1 + z\mathcal{O}$ and any $\lambda \in \Lambda$, there exists $(v'_\ell) \in \prod_{\ell=r}^m \mathcal{N}^\vee(\pi, t_\ell)$ such that*

$$v'_r \cdots v'_m L_{\mathrm{wt}(\pi)} = c^\lambda v_r \cdots v_m L_{\mathrm{wt}(\pi)}$$

and $\mathbf{a}_{i,p}(v'_\ell) = \mathbf{a}_{i,p}(v_\ell)$ for all $\ell \in \{r, \ldots, m\}$.

(iii) *For any $b \in \mathbb{C}$ not in*

$$\{0\} \cup \bigl\{\mathbf{a}_{i,p}(v_r) + \cdots + \mathbf{a}_{i,p}(v_\ell) \mid \ell \in \{r, \ldots, m\}\bigr\},$$

there exists $(v'_\ell) \in \prod_{\ell=r}^m \mathcal{N}^\vee(\pi, t_\ell)$ such that

$$v'_r \cdots v'_m L_{\mathrm{wt}(\pi)} = x_{(-\alpha_i^\vee, -p)}(1/b) v_r \cdots v_m L_{\mathrm{wt}(\pi)}.$$

Proof. The lemma is trivial for $r = m+1$. Proceeding by decreasing induction, we choose $r \in \{1, \ldots, m\}$, assume that statements (i)–(iii) hold for $r+1$, and show that they also hold for r. We recall (see the proof of Proposition 4.8) that

$$N^\vee(\mathfrak{f}_{\pi(t_r)}) = \mathcal{N}^\vee(\pi, t_r) \ltimes N^\vee(\mathfrak{f}_{\pi(t_r+0)}) \quad \text{and} \quad N^\vee(\mathfrak{f}_{\pi(t_r+0)}) \subseteq N^\vee(\mathfrak{f}_{\pi(t_{r+1})}).$$

Let $(v_\ell) \in \prod_{\ell=r}^m \mathcal{N}^\vee(\pi, t_\ell)$.

We start with (i). Let $u \in N^\vee(\mathfrak{f}_{\pi(t_r)})$. We can write $uv_r \in N^\vee(\mathfrak{f}_{\pi(t_r)})$ as a product $v_r'u'$ with $(v_r', u') \in \mathscr{N}^\vee(\pi, t_r) \times N^\vee(\mathfrak{f}_{\pi(t_r+0)})$. Then

$$\mathbf{a}_{i,p}(u) + \mathbf{a}_{i,p}(v_r) = \mathbf{a}_{i,p}(v_r') + \mathbf{a}_{i,p}(u').$$

Noting that $u' \in N^\vee(\mathfrak{f}_{\pi(t_{r+1})})$, we make use of the inductive assumption: there exists $(v_\ell') \in \prod_{\ell=r+1}^m \mathscr{N}^\vee(\pi, t_\ell)$ such that

$$v_{r+1}' \cdots v_m' L_{\mathrm{wt}(\pi)} = u' v_{r+1} \cdots v_m L_{\mathrm{wt}(\pi)}$$

and

$$\mathbf{a}_{i,p}(v_\ell') = \begin{cases} \mathbf{a}_{i,p}(u') + \mathbf{a}_{i,p}(v_\ell) & \text{if } \ell = (r+1)^+, \\ \mathbf{a}_{i,p}(v_\ell) & \text{for all other } \ell \in \{r+1, \dots, m\}. \end{cases}$$

We distinguish two cases. If $(\alpha_i^\vee, p) \in \Phi_a^{\vee,+}(\pi, t_r)$, then $\mathfrak{f}_{\pi(t_r+0)} \not\subseteq H^-_{(\alpha_i^\vee, p)}$, whence $\mathbf{a}_{i,p}(u') = 0$; also $r^+ = r$ in this case. If $(\alpha_i^\vee, p) \notin \Phi_a^{\vee,+}(\pi, t_r)$, then $\mathbf{a}_{i,p}(v_r) = \mathbf{a}_{i,p}(v_r') = 0$; here $r^+ = (r+1)^+$. In both cases, routine checks conclude the proof of (i).

We now turn to statement (ii). Let $c \in 1 + z\mathcal{O}$ and let $\lambda \in \Lambda$. One easily checks that any subgroup of the form $N^\vee(\mathfrak{f})$, in particular $N^\vee(\mathfrak{f}_{\pi(t_r)})$, is stable under conjugation by c^λ. Additionally, for any $v \in N^\vee(\mathfrak{f}_{\pi(t_r)})$, when we write

$$v \equiv \prod_{i \in I} x_{\alpha_i^\vee}(a_i) \bmod (N^\vee(\mathcal{K}), N^\vee(\mathcal{K})),$$

the Laurent series a_i has valuation at least p. This series is multiplied by $c^{\langle \alpha_i^\vee, \lambda \rangle}$ when one conjugates v by c^λ. Looking at the coefficient of z^p then gives $\mathbf{a}_{i,p}(v) = \mathbf{a}_{i,p}(c^\lambda v c^{-\lambda})$.

Write $c^\lambda v_r c^{-\lambda} \in N^\vee(\mathfrak{f}_{\pi(t_r)})$ as a product $v_r' u$ with $(v_r', u) \in \mathscr{N}^\vee(\pi, t_r) \times N^\vee(\mathfrak{f}_{\pi(t_r+0)})$. Then

$$\mathbf{a}_{i,p}(v_r) = \mathbf{a}_{i,p}(c^\lambda v_r c^{-\lambda}) = \mathbf{a}_{i,p}(v_r') + \mathbf{a}_{i,p}(u).$$

By induction, there exists $(v_\ell') \in \prod_{\ell=r+1}^m \mathscr{N}^\vee(\pi, t_\ell)$ such that

$$v_{r+1}' \cdots v_m' L_{\mathrm{wt}(\pi)} = u c^\lambda v_{r+1} \cdots v_m L_{\mathrm{wt}(\pi)}$$

and

$$\mathbf{a}_{i,p}(v_\ell') = \begin{cases} \mathbf{a}_{i,p}(u) + \mathbf{a}_{i,p}(v_\ell) & \text{if } \ell = (r+1)^+, \\ \mathbf{a}_{i,p}(v_\ell) & \text{for all other } \ell \in \{r+1, \dots, m\}. \end{cases}$$

Again we distinguish two cases. If $(\alpha_i^\vee, p) \in \Phi_a^{\vee,+}(\pi, t_r)$, then $\mathfrak{f}_{\pi(t_r+0)} \not\subseteq H^-_{(\alpha_i^\vee, p)}$ and therefore $\mathbf{a}_{i,p}(u) = 0$. If $(\alpha_i^\vee, p) \notin \Phi_a^{\vee,+}(\pi, t_r)$, then $\mathbf{a}_{i,p}(v_r) = \mathbf{a}_{i,p}(v_r') = 0$ and anew $\mathbf{a}_{i,p}(u) = 0$. Thus, $\mathbf{a}_{i,p}(u) = 0$ holds unconditionally, which concludes the proof of (ii).

Lastly, let us deal with statement (iii). We distinguish three cases.

Suppose first that $(\alpha_i^\vee, p) \in \Phi_a^{\vee,+}(\pi, t_r)$. We write $v_r = x_{(\alpha_i^\vee, p)}(a)\widetilde{v}_r$ where $a = \mathbf{a}_{i,p}(v_r)$ and $\widetilde{v}_r$ is a product of elements $x_\beta(a_\beta)$ with $\beta \in \Phi_a^{\vee,+}(\pi, t_r) \setminus \{(\alpha_i^\vee, p)\}$ and

$a_\beta \in \mathbb{C}$. From (18) we get

$$x_{(-\alpha_i^\vee,-p)}(1/b)x_{(\alpha_i^\vee,p)}(a) = (1-a/b)^{-\alpha_i}x_{(\alpha_i^\vee,p)}(a(1-a/b))x_{(-\alpha_i^\vee,-p)}(1/(b-a)).$$

By Lemma 4.12,

$$x_{(-\alpha_i^\vee,-p)}(1/(b-a))\widetilde{v}_r x_{(-\alpha_i^\vee,-p)}(-1/(b-a))$$

belongs to $N^\vee(\mathfrak{f}_{\pi(t_r)})$. We write it as a product $\widetilde{v}'_r u$ with $(\widetilde{v}'_r, u) \in \mathscr{N}^\vee(\pi, t_r) \times N^\vee(\mathfrak{f}_{\pi(t_r+0)})$. By induction, there exists $(v'_\ell) \in \prod_{\ell=r+1}^m \mathscr{N}^\vee(\pi, t_\ell)$ such that

$$v'_{r+1}\cdots v'_m L_{\mathrm{wt}(\pi)} = ux_{(-\alpha_i^\vee,-p)}(1/(b-a))v_{r+1}\cdots v_m L_{\mathrm{wt}(\pi)}.$$

Then

$$x_{(-\alpha_i^\vee,-p)}(1/b)v_r\cdots v_m L_{\mathrm{wt}(\pi)} = (1-a/b)^{-\alpha_i}[x_{(\alpha_i^\vee,p)}(a(1-a/b))\widetilde{v}'_r]v'_{r+1}\cdots v'_m L_{\mathrm{wt}(\pi)}.$$

Denoting the element in square brackets by v'_r, we get the desired expression, up to the inconsequential left multiplication by $(1-a/b)^{-\alpha_i}$.

Suppose now that there exists $q > p$ such that $(\alpha_i^\vee, q) \in \Phi_a^{\vee,+}(\pi, t_r)$; then $\mathbf{a}_{i,p}(v_r) = 0$. We write $v_r = x_{(\alpha_i^\vee,q)}(a)\widetilde{v}_r$ where $a \in \mathbb{C}$ and $\widetilde{v}_r$ is a product of elements $x_\beta(a_\beta)$ with $\beta \in \Phi_a^{\vee,+}(\pi, t_r) \setminus \{(\alpha_i^\vee, q)\}$ and $a_\beta \in \mathbb{C}$. Let c be a square root in $1 + t\mathscr{O}$ of $1-(a/b)t^{q-p}$. From (18) we get

$$x_{(-\alpha_i^\vee,-p)}(1/b)x_{(\alpha_i^\vee,q)}(a) = c^{-\alpha_i}x_{(\alpha_i^\vee,q)}(a)x_{(-\alpha_i^\vee,-p)}(1/b)c^{-\alpha_i}.$$

By Lemma 4.12,

$$x_{(-\alpha_i^\vee,-p)}(1/b)(c^{-\alpha_i}\widetilde{v}_r c^{\alpha_i})x_{(-\alpha_i^\vee,-p)}(-1/b)$$

belongs to $N^\vee(\mathfrak{f}_{\pi(t_r)})$; we write it as a product $\widetilde{v}'_r u$ with $(\widetilde{v}'_r, u) \in \mathscr{N}^\vee(\pi, t_r) \times N^\vee(\mathfrak{f}_{\pi(t_r+0)})$. By induction, there exists $(v'_\ell) \in \prod_{\ell=r+1}^m \mathscr{N}^\vee(\pi, t_\ell)$ such that

$$v'_{r+1}\cdots v'_m L_{\mathrm{wt}(\pi)} = ux_{(-\alpha_i^\vee,-p)}(1/b)c^{-\alpha_i}v_{r+1}\cdots v_m L_{\mathrm{wt}(\pi)}.$$

Then

$$x_{(-\alpha_i^\vee,-p)}(1/b)v_r\cdots v_m L_{\mathrm{wt}(\pi)} = c^{-\alpha_i}[x_{(\alpha_i^\vee,q)}(a)\widetilde{v}'_r]v'_{r+1}\cdots v'_m L_{\mathrm{wt}(\pi)}.$$

Denoting the element in square brackets by v'_r, we get the desired expression, up to the inopportune left multiplication by $c^{-\alpha_i}$. The latter can however be wiped off by a further use of the inductive assumption.

Last, suppose that no affine coroot of direction $\alpha_i^\vee$ occurs in $\Phi_a^{\vee,+}(\pi, t_r)$; then $\mathbf{a}_{i,p}(v_r) = 0$. By Lemma 4.12,

$$x_{(-\alpha_i^\vee,-p)}(1/b)v_r x_{(-\alpha_i^\vee,-p)}(-1/b)$$

belongs to $N^\vee(\mathfrak{f}_{\pi(t_r)})$. We write it as a product $v'_r u$ with $(v'_r, u) \in \mathscr{N}^\vee(\pi, t_r) \times N^\vee(\mathfrak{f}_{\pi(t_r+0)})$. By induction, there exists $(v'_\ell) \in \prod_{\ell=r+1}^m \mathscr{N}^\vee(\pi, t_\ell)$ such that

$$v'_{r+1}\cdots v'_m L_{\mathrm{wt}(\pi)} = ux_{(-\alpha_i^\vee,-p)}(1/b)v_{r+1}\cdots v_m L_{\mathrm{wt}(\pi)}.$$

Then

$$x_{(-\alpha_i^\vee,-p)}(1/b)v_r\cdots v_m L_{\mathrm{wt}(\pi)} = v'_r v'_{r+1}\cdots v'_m L_{\mathrm{wt}(\pi)},$$

as desired, which concludes the proof of (iii). ■

Let us now consider $i \in I$ and two integral paths π and η related by the equation $\eta = \tilde{e}_i\pi$. We denote by p the minimum of the function $t \mapsto \langle\alpha_i^\vee, \pi(t)\rangle$ over the interval $[0, 1]$ and by a and b the two points in time where π is bent to produce η. Noting that the conditions spelled out in Sect. 4.1 do not uniquely determine b, we choose it to be the largest possible: either $b = 1$ or $\langle\alpha_i^\vee, \pi(b+h)\rangle > p$ for all small enough $h > 0$.

Let $(t_1, \dots, t_m)$ be the ordered list of all elements in $[0, 1[$ such that $\Phi_a^{\vee,+}(\pi, t) \neq \emptyset$. We set $t_{m+1} = 1$. The set $\Phi_a^{\vee,+}(\pi, a)$ may be empty; if this happens, we insert a in the list $(t_1, \dots, t_m)$, for it will simplify the notation hereafter. On the contrary, the above condition imposed on b ensures that either $b = 1$ or $(\alpha_i^\vee, p) \in \Phi_a^{\vee,+}(\pi, b)$, so b automatically appears in the list $(t_1, \dots, t_{m+1})$. We denote by r and s the indices in $\{1, \dots, m+1\}$ such that $a = t_r$ and $b = t_s$. By design $t_r = a < t_{r+1} \leq t_s = b$.

Lemma 4.14. *Adopt the setting described in the preceding two paragraphs. Choose $(v_\ell) \in \prod_{\ell=1}^m \mathscr{N}^\vee(\pi, t_\ell)$ such that $\mathbf{a}_{i,p}(v_s) + \cdots + \mathbf{a}_{i,p}(v_\ell) \neq 0$ for each $\ell \in \{s, \dots, m\}$. Then for any $h \in \mathbb{C}^\times$, there exists $(w_\ell) \in \prod_{\ell=1}^m \mathscr{N}^\vee(\eta, t_\ell)$ such that*

$$v_1\cdots v_{r-1}x_{(-\alpha_i^\vee,-p-1)}(h)v_r\cdots v_m L_{\mathrm{wt}(\pi)} = w_1\cdots w_m L_{\mathrm{wt}(\eta)}.$$

Proof. Let (v_ℓ) be as in the statement and let $h \in \mathbb{C}^\times$. We set

$$A = v_1\cdots v_{r-1} \quad \text{and} \quad B = v_r\cdots v_m.$$

We note that $\mathfrak{f}_{\pi(t_r)} \subseteq H_{(\alpha_i^\vee,p+1)}$, so $x_{(\alpha_i^\vee,p+1)}(-1/h) \in N^\vee(\mathfrak{f}_{\pi(t_r)})$.

Using Lemma 4.13 (i), we find $(v'_{r+1}, \dots, v'_m) \in \prod_{\ell=r}^m \mathscr{N}^\vee(\pi, t_\ell)$ such that

$$x_{(\alpha_i^\vee,p+1)}(-1/h)BL_{\mathrm{wt}(\pi)} = v'_r\cdots v'_m L_{\mathrm{wt}(\pi)}$$

and $\mathbf{a}_{i,p}(v'_\ell) = \mathbf{a}_{i,p}(v_\ell)$ for all $\ell \in \{r, \dots, m\}$. We set $c = \mathbf{a}_{i,p}(v_s)$ and write $v'_s = x_{(\alpha_i^\vee,p)}(c)\tilde{v}'_s$. Then $\tilde{v}'_s \in \mathscr{N}^\vee(\pi, t_s)$ and $\mathbf{a}_{i,p}(\tilde{v}'_s) = 0$. We also set

$$C = v'_r\cdots v'_{s-1} \quad \text{and} \quad D = \tilde{v}'_s v'_{s+1}\cdots v'_m.$$

Using Lemma 4.13 (iii), we find $(\tilde{v}''_s, v''_{s+1}, \dots, v''_m) \in \prod_{\ell=s}^m \mathscr{N}^\vee(\pi, t_\ell)$ such that

$$x_{(-\alpha_i^\vee,-p)}(-1/c)DL_{\mathrm{wt}(\pi)} = \tilde{v}''_s v''_{s+1}\cdots v''_m L_{\mathrm{wt}(\pi)}.$$

Last, we set

$$E = x_{(\alpha_i^\vee,p)}(c)x_{(-\alpha_i^\vee,-p)}(1/c)x_{(\alpha_i^\vee,p)}(c),$$
$$F = x_{(\alpha_i^\vee,p)}(-c)\tilde{v}''_s v''_{s+1}\cdots v''_m,$$
$$K = x_{(-\alpha_i^\vee,-p-1)}(h)x_{(\alpha_i^\vee,p+1)}(1/h).$$

Then

$$Ax_{(-\alpha_i^\vee,-p-1)}(h)BL_{\mathrm{wt}(\pi)} = AKCEFL_{\mathrm{wt}(\pi)}. \tag{21}$$

Observing that

$$\Phi_a^{\vee,+}(\eta,t_\ell) = \begin{cases} \Phi_a^{\vee,+}(\pi,t_\ell) & \text{if } 1\le \ell < r, \\ \{(\alpha_i^\vee, p+1)\} \sqcup s_{(\alpha_i^\vee,p+1)}(\Phi_a^{\vee,+}(\pi,t_r)) & \text{if } \ell = r, \\ s_{(\alpha_i^\vee,p+1)}(\Phi_a^{\vee,+}(\pi,t_\ell)) & \text{if } r<\ell<s, \\ \tau_{\alpha_i}(\Phi_a^{\vee,+}(\pi,t_\ell)) & \text{if } s\le \ell\le m, \end{cases}$$

we check that the sequence

$$\begin{aligned}\big(v_1,\dots,v_{r-1},x_{(\alpha_i^\vee,p+1)}(-h)(z^{(p+1)\alpha_i}\overline{s_i})v_r'(z^{(p+1)\alpha_i}\overline{s_i})^{-1},\\ (z^{(p+1)\alpha_i}\overline{s_i})v_{r+1}'(z^{(p+1)\alpha_i}\overline{s_i})^{-1},\dots,(z^{(p+1)\alpha_i}\overline{s_i})v_{s-1}'(z^{(p+1)\alpha_i}\overline{s_i})^{-1},\\ z^{\alpha_i}x_{(\alpha_i^\vee,p)}(-c)\widetilde{v}_s''z^{-\alpha_i},z^{\alpha_i}v_{s+1}''z^{-\alpha_i},\dots,z^{\alpha_i}v_m''z^{-\alpha_i}\big)\end{aligned} \tag{22}$$

belongs to $\prod_{\ell=1}^m \mathcal{N}^\vee(\eta,t_\ell)$. In addition, the product of the elements in this sequence is

$$Ax_{(\alpha_i^\vee,p+1)}(-h)(z^{(p+1)\alpha_i}\overline{s_i})C(z^{(p+1)\alpha_i}\overline{s_i})^{-1}z^{\alpha_i}Fz^{-\alpha_i}.$$

We now apply two transformations to the sequence (22): we conjugate the last $m-s+1$ terms by $(-c)^{-\alpha_i}$, and we conjugate the last $m-r+1$ by $h^{-\alpha_i}$. The resulting sequence, denoted by (w_ℓ), still belongs to $\prod_{\ell=1}^m \mathcal{N}^\vee(\eta,t_\ell)$, because all our constructions are $T^\vee(\mathbb{C})$-equivariant.

Observing that

$$K = h^{-\alpha_i}x_{(\alpha_i^\vee,p+1)}(-h)(z^{(p+1)\alpha_i}\overline{s_i}) \quad \text{and} \quad E = (z^{(p+1)\alpha_i}\overline{s_i})^{-1}(-c)^{-\alpha_i}z^{\alpha_i}$$

(see (19)), we obtain

$$w_1\cdots w_m = AKCEFz^{-\alpha_i}(-ch)^{\alpha_i},$$

and a comparison with (21) yields

$$Ax_{(-\alpha_i^\vee,-p-1)}(h)BL_{\mathrm{wt}(\pi)} = AKCEFz^{-\alpha_i}L_{\mathrm{wt}(\eta)} = w_1\cdots w_m L_{\mathrm{wt}(\eta)},$$

as desired. ∎

We can now prove Proposition 4.5 (iv). We consider the situation

$$\eta_1\otimes\cdots\otimes\eta_n = \tilde{e}_i(\pi_1\otimes\cdots\otimes\pi_n)$$

in the crystal $\Pi^{\otimes n}$, and our aim is to show that $\mathring{\mathbf{Z}}(\pi_1\otimes\cdots\otimes\pi_n)$ is contained in the closure of $\mathring{\mathbf{Z}}(\eta_1\otimes\cdots\otimes\eta_n)$ in Gr_n.

As in the proof of Proposition 4.5 (iii), we regard the concatenation $\pi = \pi_1 * \cdots * \pi_n$ as a map from $[0,n]$ to $\Lambda_{\mathbb{R}}$, each path $\pi_1,\dots,\pi_n$ being travelled at nominal speed, and the same for $\eta = \eta_1 * \cdots * \eta_n$. Thus, for each $j \in \{1, \ldots, n\}$ the restriction of π to the interval $[j-1, j]$ is π_j, up to the obvious shifts in time and space.

By Proposition 4.2 (ii), we have $\eta = \tilde{e}_i \pi$. We denote by a and b the two points in time where π is bent to produce η. Let $(t_1, \ldots, t_m)$ be the ordered list of all elements in $[0, n[$ such that $\Phi_a^{\vee,+}(\pi, t) \neq \emptyset$. We insert a in this list if it does not already appear there. We set $t_0 = 0$ and $t_{m+1} = n$. We denote by r and s the indices in $\{1, \ldots, m+1\}$ such that $a = t_r$ and $b = t_s$.

There is a unique integer $k \in \{1, \ldots, n\}$ such that a and b both belong to $[k-1, k]$. Plainly, $\eta_k = \tilde{e}_i \pi_k$ and $\eta_j = \pi_j$ for all $j \in \{1, \ldots, n\} \setminus \{k\}$. We record that $\eta_1 * \cdots * \eta_j = \tilde{e}_i(\pi_1 * \cdots * \pi_j)$ if $j \in \{k, \ldots, n\}$.

For $j \in \{1, \ldots, n\}$, we set $\nu_j = \mathrm{wt}(\pi_1) + \cdots + \mathrm{wt}(\pi_j)$ and denote by m_j the largest $\ell \in \{0, \ldots, m\}$ such that $t_\ell \in [0, j[$. Then $\mathring{\mathbf{Z}}(\pi_1 \otimes \cdots \otimes \pi_n)$ is the set of all elements

$$\left[\left(\prod_{\ell=1}^{m_1} v_\ell\right) z^{\nu_1}, z^{-\nu_1}\left(\prod_{\ell=m_1+1}^{m_2} v_\ell\right) z^{\nu_2}, \ldots, z^{-\nu_{n-1}}\left(\prod_{\ell=m_{n-1}+1}^{m_n} v_\ell\right) z^{\nu_n}\right] \tag{23}$$

with $(v_\ell) \in \prod_{\ell=1}^{m} \mathscr{N}^\vee(\pi, t_\ell)$.

Now assume that (v_ℓ) is chosen so that $\mathbf{a}_{i,p}(v_s) + \cdots + \mathbf{a}_{i,p}(v_\ell) \neq 0$ for each $\ell \in \{s, \ldots, m\}$ and pick $h \in \mathbb{C}^\times$. Lemma 4.14 provides a sequence $(w_\ell) \in \prod_{\ell=1}^{m} \mathscr{N}^\vee(\eta, t_\ell)$ such that

$$v_1 \cdots v_{r-1} x_{(-\alpha_i^\vee, -p-1)}(h) v_r \cdots v_m L_{\mathrm{wt}(\pi)} = w_1 \cdots w_m L_{\mathrm{wt}(\eta)}.$$

However, (w_ℓ) satisfies more equations: for $j \in \{1, \ldots, n\}$, we have

$$\begin{cases} v_1 \cdots v_{m_j} L_{\nu_j} = w_1 \cdots w_{m_j} L_{\nu_j} & \text{if } j < k, \\ v_1 \cdots v_{r-1} x_{(-\alpha_i^\vee, -p-1)}(h) v_r \cdots v_{m_j} L_{\nu_j} = w_1 \cdots w_{m_j} L_{\nu_j + \alpha_i} & \text{if } j \geq k, \end{cases} \tag{24}$$

in the first case because $w_\ell = v_\ell$ for all $\ell \in \{1, \ldots, m_{k-1}\}$, in the second case because Lemma 4.14 would have returned the subsequence $(w_\ell)_{1 \leq \ell \leq m_j}$ if we had fed it with the paths $\pi_1 * \cdots * \pi_j$ and $\eta_1 * \cdots * \eta_j$ and the datum $(v_\ell)_{1 \leq \ell \leq m_j}$ and h.

The system (24) translates to a single equation in Gr_n, which demonstrates that the element obtained by inserting $x_{(-\alpha_i^\vee, -p-1)}(h)$ just before v_r in (23) belongs to the set $\mathring{\mathbf{Z}}(\eta_1 \otimes \cdots \otimes \eta_n)$. Letting h tend to 0, we conclude that (23) lies in the closure of this set. To be sure, this conclusion has been reached under the assumption that $\mathbf{a}_{i,p}(v_s) + \cdots + \mathbf{a}_{i,p}(v_\ell) \neq 0$ for each $\ell \in \{s, \ldots, m\}$, but this restriction can be removed by a small perturbation of $\mathbf{a}_{i,p}(v_s)$.

Thus, Proposition 4.5 (iv) is, at last, fully proven.

5. Comparison with the tensor product basis

We keep the notation from Sect. 2. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$. The tensor product $V(\boldsymbol{\lambda})$ can be endowed on the one hand with its MV basis (Sect. 2.3), on the other hand with the tensor product of the MV bases of the factors $V(\lambda_1)$, …, $V(\lambda_n)$. In this section, we compare these two bases through the explicit identification

$$F(\mathscr{I}_{\lambda_1} * \cdots * \mathscr{I}_{\lambda_n}) \cong F(\mathscr{I}_{\lambda_1}) \otimes \cdots \otimes F(\mathscr{I}_{\lambda_n})$$

afforded by Beilinson and Drinfeld's fusion product. We show that the transition matrix is upper unitriangular and that its entries are intersection multiplicities. The order relation needed to convey the triangularity involves the inclusion of cycles.

5.1. Deformations

The Beilinson–Drinfeld Grassmannian $\mathscr{G}r^{\mathrm{BD}}$ is a relative version of the affine Grassmannian where the base is the space of effective divisors on a smooth curve. The choice of the affine line amply satisfies our needs and offers three advantages: there is a natural global coordinate on $\mathbb{A}^1$, every G-torsor on $\mathbb{A}^1$ is trivializable, and the monodromy of any local system is trivial. Rather than looking for more generality, we will pragmatically stick with this choice. Consistent with Sect. 2, the coordinate on $\mathbb{A}^1$ is denoted by z.

Formally, the Beilinson–Drinfeld Grassmannian $\mathscr{G}r_n^{\mathrm{BD}}$ is defined as the functor on the category of commutative $\mathbb{C}$-algebras that assigns to an algebra R the set of isomorphism classes of triples $(x_1, \ldots, x_n; \mathscr{F}, \beta)$, where $(x_1, \ldots, x_n) \in \mathbb{A}^n(R)$, $\mathscr{F}$ is a $G^\vee$-torsor over $\mathbb{A}^1_R$ and β is a trivialization of $\mathscr{F}$ away from the points $x_1, \ldots, x_n$ ([6, Sect. 5.3.10]; [43, Definition 3.3]; [46, Definition 3.1.1]). We denote by $\pi : \mathscr{G}r_n^{\mathrm{BD}} \to \mathbb{A}^n$ the morphism to the base which forgets $\mathscr{F}$ and β. It is known that $\mathscr{G}r_n^{\mathrm{BD}}$ is representable by an ind-scheme and π is ind-proper.

We are only interested in the set of $\mathbb{C}$-points, endowed with its ind-variety structure. Using a trivialization of $\mathscr{F}$, we can thus adopt the following simplified definition: $\mathscr{G}r_n^{\mathrm{BD}}$ is the set of pairs $(x_1, \ldots, x_n; [\beta])$, where $(x_1, \ldots, x_n) \in \mathbb{C}^n$ and $[\beta]$ belongs to the homogeneous space

$$G^\vee(\mathbb{C}[z, (z-x_1)^{-1}, \ldots, (z-x_n)^{-1}])/G^\vee(\mathbb{C}[z]).$$

This set is endowed with the structure of an ind-variety.

Example 5.1 ([6, Remark in Sect. 5.3.10]). We consider the case $G^\vee = \mathrm{GL}_N$. Here the datum of $[\beta]$ is equivalent to the datum of the $\mathbb{C}[z]$-lattice $\beta(L_0)$ in $\mathbb{C}(z)^N$, where $L_0 = \mathbb{C}[z]^N$ is the standard lattice. Let us write $\mathbf{x}$ for the point $(x_1, \ldots, x_n)$ and set $f_{\mathbf{x}} = (z-x_1)\cdots(z-x_n)$. Then a lattice L is of this form $\beta(L_0)$ if and only if there exists a positive integer k such that $f_{\mathbf{x}}^k L_0 \subseteq L \subseteq f_{\mathbf{x}}^{-k} L_0$. For each positive integer k, define $(\mathscr{G}r_n^{\mathrm{BD}})_k$ to be the subset of $\mathscr{G}r_n^{\mathrm{BD}}$ consisting of all pairs $(\mathbf{x}; L)$ with $f_{\mathbf{x}}^k L_0 \subseteq L \subseteq f_{\mathbf{x}}^{-k} L_0$. We identify $\mathbb{C}[z]/(f_{\mathbf{x}}^{2k})$ with the vector space V of polynomials of degree strictly less than $2kn$, and subsequently identify $L_0/f_{\mathbf{x}}^{2k} L_0$ with V^N. The space $(\mathscr{G}r_n^{\mathrm{BD}})_k$ can then be realized as a Zariski-closed subset of

$$\mathbb{C}^n \times \bigcup_{d=0}^{2knN} \mathbb{G}_d(V^N)$$

where $\mathbb{G}_d(V^N)$ denotes the Grassmannian of d-planes in V^N. In this way, $\mathcal{G}r_n^{\mathrm{BD}}$ is the inductive limit of a system of algebraic varieties and closed embeddings, in other words, an ind-variety.

We also want to deform the n-fold convolution variety Gr_n. Accordingly, we define $\mathcal{G}r_n$ as the set of pairs $(x_1,\dots,x_n;[\beta_1,\dots,\beta_n])$, where $(x_1,\dots,x_n)\in\mathbb{C}^n$ and $[\beta_1,\dots,\beta_n]$ belongs to

$$G^\vee(\mathbb{C}[z,(z-x_1)^{-1}])\times^{G^\vee(\mathbb{C}[z])}\cdots\times^{G^\vee(\mathbb{C}[z])}G^\vee(\mathbb{C}[z,(z-x_n)^{-1}])/G^\vee(\mathbb{C}[z])$$

(see [43, Definition 3.8] or [46, (3.1.21)]). The set $\mathcal{G}r_n$ is endowed with the structure of an ind-variety; it comes with a map $m_n:\mathcal{G}r_n\to\mathcal{G}r_n^{\mathrm{BD}}$ defined by

$$m_n(x_1,\dots,x_n;[\beta_1,\dots,\beta_n])=(x_1,\dots,x_n;[\beta_1\cdots\beta_n]).$$

Example 5.2. We again consider the case $G^\vee=\mathrm{GL}_N$. Then an element in $\mathcal{G}r_n$ is the datum of a point $(x_1,\dots,x_n)\in\mathbb{C}^n$ and a sequence $(L_1,\dots,L_n)$ of $\mathbb{C}[z]$-lattices in $\mathbb{C}(z)^N$ for which there exists a positive integer k such that

$$(z-x_j)^kL_{j-1}\subseteq L_j\subseteq(z-x_j)^{-k}L_{j-1}$$

for all $j\in\{1,\dots,n\}$; here again $L_0=\mathbb{C}[z]^N$ is the standard lattice and $L_j=(\beta_1\cdots\beta_j)(L_0)$.

In the above example, we can partition $\mathcal{G}r_n$ into cells by specifying the relative positions of the pairs of lattices (L_{j-1},L_j) in terms of invariant factors. This construction can be generalized to an arbitrary group G as follows: given $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_n)$ in $(\Lambda^+)^n$, we define $\mathcal{G}r_n^{\boldsymbol{\lambda}}$ as the subset of $\mathcal{G}r_n$ consisting of all pairs $(x_1,\dots,x_n;[\beta_1,\dots,\beta_n])$ with

$$\beta_j\in G^\vee(\mathbb{C}[z])(z-x_j)^{\lambda_j}G^\vee(\mathbb{C}[z])$$

for $j\in\{1,\dots,n\}$. The Cartan decomposition

$$G^\vee(\mathbb{C}[z,(z-x_j)^{-1}])=\bigsqcup_{\lambda_j\in\Lambda^+}G^\vee(\mathbb{C}[z])(z-x_j)^{\lambda_j}G^\vee(\mathbb{C}[z])$$

yields

$$\mathcal{G}r_n=\bigsqcup_{\boldsymbol{\lambda}\in(\Lambda^+)^n}\mathcal{G}r_n^{\boldsymbol{\lambda}}$$

and it can be checked that

$$\overline{\mathcal{G}r_n^{\boldsymbol{\lambda}}}=\bigsqcup_{\substack{\boldsymbol{\mu}\in(\Lambda^+)^n\\ \mu_1\leq\lambda_1,\dots,\mu_n\leq\lambda_n}}\mathcal{G}r_n^{\boldsymbol{\mu}}. \tag{25}$$

In addition, the maps $(x_1,\dots,x_j;[\beta_1,\dots,\beta_j])\mapsto(x_1,\dots,x_{j-1};[\beta_1,\dots,\beta_{j-1}])$ exhibit $\mathcal{G}r_n^{\boldsymbol{\lambda}}$ as the total space of an iterated fibration with base $\mathcal{G}r_1^{\lambda_1}$ and successive fibers $\mathcal{G}r_1^{\lambda_2}, \dots, \mathcal{G}r_1^{\lambda_n}$. It follows that $\mathcal{G}r_n^{\boldsymbol{\lambda}}$ is a smooth connected variety of dimension $2\rho(|\boldsymbol{\lambda}|) + n$.

Let us now investigate the fibers of the map $\pi \circ m_n : \mathcal{G}r_n \to \mathbb{C}^n$. Given $x \in \mathbb{C}$, we set $\mathcal{O}_x = \mathbb{C}[\![z - x]\!]$ and $\mathcal{K}_x = \mathbb{C}(\!(z - x)\!)$. We identify $\mathcal{O}$ and $\mathcal{K}$ with $\mathcal{O}_x$ and $\mathcal{K}_x$ by means of the map $z \mapsto z - x$.

We fix $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{C}^n$. Let $\operatorname{supp}(\mathbf{x})$ be the set of values $y \in \mathbb{C}$ that appear in the tuple $\mathbf{x}$. For $y \in \operatorname{supp}(\mathbf{x})$, denote by m_y the number of indices $j \in \{1, \dots, n\}$ such that $x_j = y$ and choose an increasing sequence $(p_0 = 0, p_1, p_2, \dots, p_{m_y} = n)$ in such a way that each interval $[p_{k-1} + 1, p_k]$ contains exactly one index j such that $x_j = y$. For $\boldsymbol{\beta} = [\beta_1, \dots, \beta_n]$ in the fiber

$$(\mathcal{G}r_n)_{\mathbf{x}} = G^\vee(\mathbb{C}[z, (z-x_1)^{-1}]) \times^{G^\vee(\mathbb{C}[z])} \cdots \times^{G^\vee(\mathbb{C}[z])} G^\vee(\mathbb{C}[z, (z-x_n)^{-1}]) / G^\vee(\mathbb{C}[z]),$$

we define $\Theta(\boldsymbol{\beta})_y$ as the point $[(\beta_1 \cdots \beta_{p_1}), (\beta_{p_1+1} \cdots \beta_{p_2}), \dots, (\beta_{p_{m_y-1}+1} \cdots \beta_n)]$ in

$$\underbrace{G^\vee(\mathcal{K}_y) \times^{G^\vee(\mathcal{O}_y)} \cdots \times^{G^\vee(\mathcal{O}_y)} G^\vee(\mathcal{K}_y)}_{m_y \text{ factors } G^\vee(\mathcal{K}_y)} / G^\vee(\mathcal{O}_y) \cong \mathrm{Gr}_{m_y}$$

(note that $\Theta(\boldsymbol{\beta})_y$ does not depend on this choice, because $\beta_j \in G^\vee(\mathcal{O}_y)$ if $x_j \neq y$).

Proposition 5.3. *The map $\boldsymbol{\beta} \mapsto (\Theta(\boldsymbol{\beta})_y)$ is a bijection*

$$(\mathcal{G}r_n)_{\mathbf{x}} \xrightarrow{\simeq} \prod_{y \in \operatorname{supp}(\mathbf{x})} \mathrm{Gr}_{m_y}.$$

Proof. Combining the Iwasawa decomposition (1) with the easily proven equality

$$N^\vee(\mathcal{K}_x) = N^\vee(\mathbb{C}[z, (z-x)^{-1}]) N^\vee(\mathcal{O}_x), \tag{26}$$

we obtain the well-known equality

$$G^\vee(\mathcal{K}_x) = G^\vee(\mathbb{C}[z, (z-x)^{-1}]) G^\vee(\mathcal{O}_x),$$

for each $x \in \mathbb{C}$.

The case $n = 1$ of the proposition is banal. Assume that $n \geq 2$, and for $y \in \operatorname{supp}(\mathbf{x})$, pick $\boldsymbol{\gamma}_y \in \mathrm{Gr}_{m_y}$. Set $\mathbf{x}' = (x_1, \dots, x_{n-1})$ and $m = m_{x_n}$, and write $\boldsymbol{\gamma}_{x_n} = [\gamma_1, \dots, \gamma_m]$. Reasoning by induction on n, we know that there is a unique $\boldsymbol{\beta}' = [\beta_1, \dots, \beta_{n-1}]$ in $(\mathcal{G}r_{n-1})_{\mathbf{x}'}$ such that

$$\Theta(\boldsymbol{\beta}')_y = \begin{cases} \boldsymbol{\gamma}_y & \text{if } x_n \neq y, \\ [\gamma_1, \dots, \gamma_{m-1}] & \text{if } x_n = y. \end{cases}$$

The elements $\gamma_1, \dots, \gamma_m$ belong to $G^\vee(\mathcal{K})$, which we identify with $G^\vee(\mathcal{K}_{x_n})$. We choose $\beta_n \in G^\vee(\mathbb{C}[z, (z - x_n)^{-1}])$ such that

$$(\beta_1 \dots \beta_{n-1})^{-1}(\gamma_1 \dots \gamma_m) \in \beta_n G^\vee(\mathcal{O}_{x_n}).$$

Then $[\beta_1, \dots, \beta_{n-1}, \beta_n]$ is the unique element $\boldsymbol{\beta}$ in $(\mathcal{G}r_n)_{\mathbf{x}}$ such that $\Theta(\boldsymbol{\beta})_y = \boldsymbol{\gamma}_y$ for all y. ■

Keep the notation above for $\mathbf{x}$ and the integers m_y and let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in $(\Lambda^+)^n$. For each $y \in \operatorname{supp}(\mathbf{x})$, define $\boldsymbol{\lambda}_y \in (\Lambda^+)^{m_y}$ as the ordered tuple formed by the weights λ_j for $j \in \{1, \ldots, n\}$ such that $x_j = y$. Then, under the bijection given in Proposition 5.3, the fiber $(\mathcal{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}$ identifies with

$$\prod_{y \in \operatorname{supp}(\mathbf{x})} \operatorname{Gr}_{m_y}^{\boldsymbol{\lambda}_y}.$$

5.2. *Global cycles*

Recall our notation $N^\vee$ for the unipotent radical of $B^\vee$. For $\mu \in \Lambda$ and $x \in \mathbb{C}$, we define

$$\widetilde{S}_{\mu|x} = (z-x)^\mu N^\vee(\mathbb{C}[z, (z-x)^{-1}]) = N^\vee(\mathbb{C}[z, (z-x)^{-1}])(z-x)^\mu.$$

Equation (26) expresses that the natural map

$$N^\vee(\mathbb{C}[z, (z-x)^{-1}])/N^\vee(\mathbb{C}[z]) \to N^\vee(\mathcal{K}_x)/N^\vee(\mathcal{O}_x)$$

is bijective; composing with the natural map $N^\vee(\mathcal{K})/N^\vee(\mathcal{O}) \to \operatorname{Gr}$, we obtain, after left multiplication by $(z-x)^\mu$, a bijection

$$\widetilde{S}_{\mu|x}/N^\vee(\mathbb{C}[z]) \xrightarrow{\simeq} S_\mu.$$

For $(\mu_1, \ldots, \mu_n) \in \Lambda^n$, let $S_{\mu_1} \propto \cdots \propto S_{\mu_n}$ be the set of all pairs $(x_1, \ldots, x_n; [\beta_1, \ldots, \beta_n])$ with $(x_1, \ldots, x_n)$ in $\mathbb{C}^n$ and $[\beta_1, \ldots, \beta_n]$ in

$$\widetilde{S}_{\mu_1|x_1} \times^{N^\vee(\mathbb{C}[z])} \cdots \times^{N^\vee(\mathbb{C}[z])} \widetilde{S}_{\mu_n|x_n} / N^\vee(\mathbb{C}[z]).$$

Rewriting the Iwasawa decomposition as

$$G^\vee(\mathbb{C}[z, (z-x)^{-1}]) = \bigsqcup_{\mu \in \Lambda} N^\vee(\mathbb{C}[z, (z-x)^{-1}])(z-x)^\mu G^\vee(\mathbb{C}[z]),$$

we then see that the natural map

$$\Psi : \bigsqcup_{(\mu_1, \ldots, \mu_n) \in \Lambda^n} S_{\mu_1} \propto \cdots \propto S_{\mu_n} \to \mathcal{G}r_n$$

is bijective. Here Ψ is regarded as the calligraphic variant of the letter Ψ used in Sect. 2.2; these two glyphs may be hard to distinguish, but hopefully this choice will not lead to any confusion.

More generally, given $(\mu_1, \ldots, \mu_n) \in \Lambda^n$ and $N^\vee(\mathcal{O})$-stable subsets $Z_1 \subseteq S_{\mu_1}, \ldots, Z_n \subseteq S_{\mu_n}$, we define $Z_1 \propto \cdots \propto Z_n$ to be the subset of all pairs $(x_1, \ldots, x_n; [\beta_1, \ldots, \beta_n])$ with $(x_1, \ldots, x_n) \in \mathbb{C}^n$ and

$$[\beta_1, \ldots, \beta_n] \in \widetilde{Z}_{1|x_1} \times^{N^\vee(\mathbb{C}[z])} \cdots \times^{N^\vee(\mathbb{C}[z])} \widetilde{Z}_{n|x_n} / N^\vee(\mathbb{C}[z])$$

where each $\widetilde{Z}_{j|x_j}$ is the preimage of Z_j under the map $\widetilde{S}_{\mu_j|x_j} \to S_{\mu_j}$.

For $\mu \in \Lambda$, we define

$$\dot{S}_\mu = \bigcup_{\substack{(\mu_1, \ldots, \mu_n) \in \Lambda^n \\ \mu_1 + \cdots + \mu_n = \mu}} \Psi(S_{\mu_1} \propto \cdots \propto S_{\mu_n}).$$

Proposition 5.4. *Let* $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ *in* $(\Lambda^+)^n$ *and let* $\mu \in \Lambda$.

(i) *All the irreducible components of* $\overline{\mathcal{G}r_n^{\boldsymbol{\lambda}}} \cap \dot{S}_\mu$ *have dimension* $\rho(|\boldsymbol{\lambda}| + \mu) + n$.

(ii) *The map* $(Z_1, \dots, Z_n) \mapsto \overline{\Psi(Z_1 \propto \cdots \propto Z_n)}$ *induces a bijection*

$$\bigsqcup_{\substack{(\mu_1,\dots,\mu_n)\in\Lambda^n\\ \mu_1+\cdots+\mu_n=\mu}} {}_*\mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times {}_*\mathscr{Z}(\lambda_n)_{\mu_n} \xrightarrow{\simeq} \operatorname{Irr}\big(\overline{\mathcal{G}r_n^{\boldsymbol{\lambda}}} \cap \dot{S}_\mu\big).$$

(*The bar above* $\Psi(Z_1 \propto \cdots \propto Z_n)$ *means closure in* $\dot{S}_\mu$.)

Proof. Let $(\mu_1, \dots, \mu_n) \in \Lambda^n$ be such that $\mu_1 + \cdots + \mu_n = \mu$ and let $(Z_1, \dots, Z_n) \in {}_*\mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times {}_*\mathscr{Z}(\lambda_n)_{\mu_n}$. Then the set $\Psi(Z_1 \propto \cdots \propto Z_n)$ is irreducible. By Proposition 5.3 and its proof, the fiber of this set over a point $\mathbf{x} \in \mathbb{C}^n$ is isomorphic to the product, over all $y \in \operatorname{supp}(\mathbf{x})$, of cycles

$$\Psi(Z_{j_1} \ltimes \cdots \ltimes Z_{j_m}) \subseteq \mathrm{Gr}_m$$

where $j_1, \dots, j_m$ are the indices $j \in \{1, \dots, n\}$ such that $x_j = y$. We remark that if we set $\boldsymbol{\lambda}_y = (\lambda_{j_1}, \dots, \lambda_{j_m})$ and $\mu_y = \mu_{j_1} + \cdots + \mu_{j_m}$, then this cycle belongs to ${}_*\mathscr{Z}(\boldsymbol{\lambda}_y)_{\mu_y}$. By Proposition 2.2 (i), the dimension of the fiber of $\Psi(Z_1 \propto \cdots \propto Z_n)$ over $\mathbf{x}$ is therefore

$$\sum_{y\in\operatorname{supp}(\mathbf{x})} \rho(|\boldsymbol{\lambda}_y| + \mu_y) = \rho(|\boldsymbol{\lambda}| + \mu)$$

and we conclude that $\Psi(Z_1 \propto \cdots \propto Z_n)$ has dimension $\rho(|\boldsymbol{\lambda}| + \mu) + n$.

To finish the proof, we observe that the sets $\Psi(Z_1 \propto \cdots \propto Z_n)$ cover $\overline{\mathcal{G}r_n^{\boldsymbol{\lambda}}} \cap \dot{S}_\mu$ and are not redundant. ∎

Our MV bases are defined with the help of the unstable subsets T_μ instead of the stable subsets S_μ. We can easily adapt the constructions of this subsection to this case by replacing the Borel subgroup $B^\vee$ with its opposite with respect to $T^\vee$, and replacing similarly its unipotent radical $N^\vee$. We shall do this while keeping the notation $\propto$ and Ψ. Note that when we replace $\dot{S}_\mu$ by

$$\dot{T}_\mu = \bigcup_{\substack{(\mu_1,\dots,\mu_n)\in\Lambda^n\\ \mu_1+\cdots+\mu_n=\mu}} \Psi(T_{\mu_1} \propto \cdots \propto T_{\mu_n})$$

in Proposition 5.4, $\rho(|\boldsymbol{\lambda}| + \mu) + n$ must be replaced by $\rho(|\boldsymbol{\lambda}| - \mu) + n$ and the sets ${}_*\mathscr{Z}(\lambda_j)_{\mu_j}$ must be replaced by their unstarred counterparts.

5.3. *The fusion product*

For any $x \in \mathbb{C}$, the fibers of $\mathcal{G}r_n^{\mathrm{BD}}$ and $\mathcal{G}r_n$ over $(x, \dots, x)$ are isomorphic to Gr and Gr_n, respectively. Thus,

$$\mathcal{G}r_n^{\mathrm{BD}}|_\Delta \xrightarrow{\simeq} \Delta \times \mathrm{Gr} \quad \text{and} \quad \mathcal{G}r_n|_\Delta \xrightarrow{\simeq} \Delta \times \mathrm{Gr}_n,$$

where Δ is the small diagonal, defined as the image of the map $x \mapsto (x, \ldots, x)$ from $\mathbb{C}$ to $\mathbb{C}^n$. At the other extreme, the morphism $m_n : \mathcal{G}r_n \to \mathcal{G}r_n^{\mathrm{BD}}$ is an isomorphism after restriction to the open locus $U \subseteq \mathbb{C}^n$ of points with pairwise different coordinates (see [46, Lemma 3.1.23]), and by Proposition 5.3, $\mathcal{G}r_n|_U$ is isomorphic to $U \times (\mathrm{Gr})^n$. We define maps τ, i, j and ζ according to the diagram

$$
\begin{array}{ccccccccc}
\mathrm{Gr}_n & \xleftarrow{\ \tau\ } & \Delta \times \mathrm{Gr}_n & \xrightarrow{\ i\ } & \mathcal{G}r_n & \xleftarrow{\ j\ } & U \times (\mathrm{Gr})^n & \xrightarrow{\ \zeta\ } & (\mathrm{Gr})^n \\
 & & \downarrow & & \downarrow{\scriptstyle m_n} & & \downarrow{\scriptstyle \simeq} & & \\
 & & \Delta \times \mathrm{Gr} & & \mathcal{G}r_n^{\mathrm{BD}} & & \mathcal{G}r_n^{\mathrm{BD}}|_U & & \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
 & & \Delta & \longrightarrow & \mathbb{C}^n & \longleftarrow & U & &
\end{array}
$$

Let $\boldsymbol{\lambda} \in (\Lambda^+)^n$ and $\mu \in \Lambda$, set

$$\mathscr{B}(\boldsymbol{\lambda}) = \mathrm{IC}\big(\overline{\mathcal{G}r_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big), \quad d = \dim \mathcal{G}r_n^{\boldsymbol{\lambda}} = 2\rho(|\boldsymbol{\lambda}|) + n, \quad k = 2\rho(\mu) - n$$

and denote the inclusion $\dot{T}_\mu \to \mathcal{G}r_n$ by $\dot{i}_\mu$. The next statement is due to Mirković and Vilonen.

Proposition 5.5. (i) *There are natural isomorphisms*

$$i^! \mathscr{B}(\boldsymbol{\lambda})[n] \cong \tau^! \,\mathrm{IC}\big(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}}\big)$$

and

$$j^! \mathscr{B}(\boldsymbol{\lambda})[n] \cong \zeta^! \big(\mathrm{IC}\big(\overline{\mathrm{Gr}^{\lambda_1}}, \underline{\mathbb{C}}\big) \boxtimes \cdots \boxtimes \mathrm{IC}\big(\overline{\mathrm{Gr}^{\lambda_n}}, \underline{\mathbb{C}}\big)\big).$$

(ii) *Each cohomology sheaf of $(\pi \circ m_n)_* \mathscr{B}(\boldsymbol{\lambda})$ is a local system on $\mathbb{C}^n$.*

(iii) *The complex of sheaves $(\pi \circ m_n \circ \dot{i}_\mu)_* (\dot{i}_\mu)^! \mathscr{B}(\boldsymbol{\lambda})$ is concentrated in degree k and its k-th cohomology sheaf is a local system on $\mathbb{C}^n$.*

Proof. To prove statement (i), one follows the reasoning in [4, Sect. 1.7.5], noting that $\mathscr{B}(\boldsymbol{\lambda})$ and $\mathrm{IC}(\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}, \underline{\mathbb{C}})$ are the sheaves denoted by $(\tau^\circ \mathscr{I}_{\lambda_1}) \tilde{\boxtimes} \cdots \tilde{\boxtimes} (\tau^\circ \mathscr{I}_{\lambda_n})$ and $\mathscr{I}_{\lambda_1} \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathscr{I}_{\lambda_n}$ in *loc. cit.* Statement (ii) is [39, (6.4)]. Statement (iii) is contained in the proof of [39, Proposition 6.4], up to a base change in the Cartesian square

$$
\begin{array}{ccc}
\dot{T}_\mu & \xrightarrow{\ \dot{i}_\mu\ } & \mathcal{G}r_n \\
\downarrow & & \downarrow{\scriptstyle m_n} \\
T_\mu(\mathbb{A}^n) & \xrightarrow{\ k_\mu\ } & \mathcal{G}r_n^{\mathrm{BD}}
\end{array}
$$

∎

Combining Propositions 2.1 and 5.5 (i), we see that the total cohomology of the stalk of the complex $(\pi \circ m_n)_* \mathscr{B}(\boldsymbol{\lambda})$ identifies with $F(\mathscr{I}_{\boldsymbol{\lambda}})$ at any point in Δ, and with $F(\mathscr{I}_{\lambda_1}) \otimes \cdots \otimes F(\mathscr{I}_{\lambda_n})$ at any point in U. Statement (ii) in Proposition 5.5 thus provides the identification

$$F(\mathscr{I}_{\boldsymbol{\lambda}}) \cong F(\mathscr{I}_{\lambda_1}) \otimes \cdots \otimes F(\mathscr{I}_{\lambda_n})$$

required to compare the two bases of $V(\boldsymbol{\lambda})$. Statement (iii) further identifies the weight spaces

$$F_\mu(\mathscr{I}_{\boldsymbol{\lambda}}) \cong \bigoplus_{\substack{(\mu_1,\dots,\mu_n)\in\Lambda^n \\ \mu_1+\cdots+\mu_n=\mu}} F_{\mu_1}(\mathscr{I}_{\lambda_1}) \otimes \cdots \otimes F_{\mu_n}(\mathscr{I}_{\lambda_n}).$$

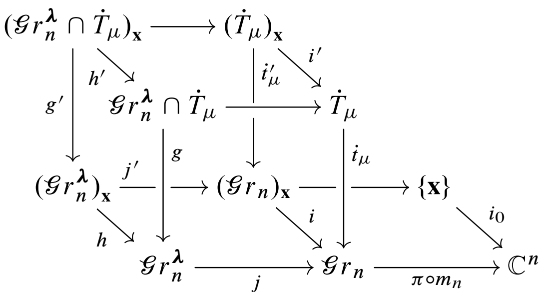
5.4. *Intersection multiplicities*

We keep the setup introduced in the previous section. In addition, we denote by

$$\mathscr{L}_\mu(\boldsymbol{\lambda}) = \mathscr{H}^k(\pi \circ m_n \circ \dot{\iota}_\mu)_*(\dot{\iota}_\mu)^!\mathscr{B}(\boldsymbol{\lambda})$$

the local system appearing in Proposition 5.5 (iii).

For each point $\mathbf{x} \in \mathbb{C}^n$, we define maps as indicated below:



where for instance $(\mathscr{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}$ is the fiber of $\mathscr{G}r_n^{\boldsymbol{\lambda}}$ over $\mathbf{x}$. (The notation i and j does not designate the same maps as in the previous subsection.) We then construct the following diagram, referred to as (♡) below:

$$\begin{array}{ccccc}
H^k(\dot{T}_\mu, (\dot{\iota}_\mu)^!\mathscr{B}(\boldsymbol{\lambda})) & \xrightarrow{\simeq} & H^{k+d}(\mathscr{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu, g^!\underline{\mathbb{C}}_{\mathscr{G}r_n^{\boldsymbol{\lambda}}}) & \xrightarrow[\simeq]{\cap[\mathscr{G}r_n^{\boldsymbol{\lambda}}]} & H^{\mathrm{BM}}_{d-k}(\mathscr{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu) \\
\simeq\downarrow & & \downarrow & & \downarrow (g^*u_{\mathbf{x}})\cap \\
H^k((\dot{T}_\mu)_{\mathbf{x}}, (\dot{\iota}'_\mu)^! i^*\mathscr{B}(\boldsymbol{\lambda})) & \xrightarrow{\simeq} & H^{k+d}((\mathscr{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}}, g'^!\underline{\mathbb{C}}_{(\mathscr{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}}) & \xrightarrow{\cap[(\mathscr{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}]} & H^{\mathrm{BM}}_{d-k-2n}((\mathscr{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}})
\end{array}$$

The left vertical arrow in (♡) is the restriction of the cohomology with support in $\dot{T}_\mu$ from $\mathscr{G}r_n$ to $(\mathscr{G}r_n)_{\mathbf{x}}$. In other words, it is the image by the functor $H^k(\dot{T}_\mu, (\dot{\iota}_\mu)^!-)$ of the adjunction morphism $\mathscr{B}(\boldsymbol{\lambda}) \to i_*i^*\mathscr{B}(\boldsymbol{\lambda})$. Lemma 5.7 below implies that it is an isomorphism. Likewise, the middle vertical arrow is the restriction from $\mathscr{G}r_n^{\boldsymbol{\lambda}}$ to $(\mathscr{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}$, afforded by the adjunction morphism $j^*\mathscr{B}(\boldsymbol{\lambda}) \to h_*h^*j^*\mathscr{B}(\boldsymbol{\lambda})$.

On the top line, the left arrow is the restriction from $\mathscr{G}r_n$ to $\mathscr{G}r_n^{\boldsymbol{\lambda}}$, fulfilled by the adjunction morphism $\mathscr{B}(\boldsymbol{\lambda}) \to j_*j^*\mathscr{B}(\boldsymbol{\lambda}) = j_*\underline{\mathbb{C}}_{\mathscr{G}r_n}[d]$. On the bottom line, it is the restriction from $(\mathscr{G}r_n)_{\mathbf{x}}$ to $(\mathscr{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}$, achieved by $i^*\mathscr{B}(\boldsymbol{\lambda}) \to (j')_*(j')^*i^*\mathscr{B}(\boldsymbol{\lambda})$. Mirković and Vilonen's argument (reproduced in Sect. 2.3) shows that these two arrows are isomorphisms.

The two paths around the left square in (♡) are two different expressions for the restriction from $\mathcal{G}r_n$ to $(\mathcal{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}$; therefore this square commutes.

In both lines of (♡) the right arrow is Alexander duality. We note that $H_{d-k}^{\mathrm{BM}}(\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)$ and $H_{d-k-2n}^{\mathrm{BM}}((\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}})$ are the top-dimensional Borel–Moore homology groups.

The map h is a regular embedding of codimension n. Its orientation class (generalized Thom class) is an element

$$u_{\mathbf{x}} \in H^{2n}((\mathcal{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}, h^! \underline{\mathbb{C}}_{\mathcal{G}r_n^{\boldsymbol{\lambda}}}).$$

The right vertical arrow in (♡) is the cap product with

$$g^* u_{\mathbf{x}} \in H^{2n}\big((\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}}, (h')^! \underline{\mathbb{C}}_{\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu}\big),$$

the restriction of $u_{\mathbf{x}}$ to $\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu$.

Lemma 5.6. *In the diagram (♡), the square on the right commutes.*

Proof. Applying formula IX.4.9 in [24], we get $u_{\mathbf{x}} \cap [\mathcal{G}r_n^{\boldsymbol{\lambda}}] = [(\mathcal{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}]$.

Formula (8) in [15, Sect. 19.1] (or formula IX.3.4 in [24]) asserts that given a topological manifold X and inclusions of closed subsets $a : A \to X$ and $b : B \to X$, for any

$$\alpha \in H^\bullet(A, a^! \underline{\mathbb{C}}_X), \quad \beta \in H^\bullet(B, b^! \underline{\mathbb{C}}_X), \quad C \in H_\bullet^{\mathrm{BM}}(X)$$

one has

$$(b^*\alpha) \cap (\beta \cap C) = (\alpha \cup \beta) \cap C. \tag{27}$$

Using the six operations formalism, one checks without much trouble that this result is also valid if A and B are only locally closed.

Now pick

$$\xi \in H^{k+d}(\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu, g^! \underline{\mathbb{C}}_{\mathcal{G}r_n^{\boldsymbol{\lambda}}}).$$

Applying (27) twice and using the fact that $u_{\mathbf{x}}$ has even degree, we compute

$$(h^*\xi) \cap (u_{\mathbf{x}} \cap [\mathcal{G}r_n^{\boldsymbol{\lambda}}]) = (\xi \cup u_{\mathbf{x}}) \cap [\mathcal{G}r_n^{\boldsymbol{\lambda}}] = (u_{\mathbf{x}} \cup \xi) \cap [\mathcal{G}r_n^{\boldsymbol{\lambda}}] = (g^* u_{\mathbf{x}}) \cap (\xi \cap [\mathcal{G}r_n^{\boldsymbol{\lambda}}]).$$

This equality means precisely that ξ has the same image under the two paths in (♡) that circumscribe the square on the right. ∎

Lemma 5.7. *There are natural isomorphisms*

$$H^k(\dot{T}_\mu, (\dot{\imath}_\mu)^! \mathscr{B}(\boldsymbol{\lambda})) \cong H^0(\mathbb{C}^n, \mathscr{L}_\mu(\boldsymbol{\lambda})) \quad \textit{and} \quad H^k((\dot{T}_\mu)_{\mathbf{x}}, (\dot{\imath}'_\mu)^! i^* \mathscr{B}(\boldsymbol{\lambda})) \cong (\mathscr{L}_\mu(\boldsymbol{\lambda}))_{\mathbf{x}}$$

and the left vertical arrow in (♡) is the stalk map $H^0(\mathbb{C}^n, \mathscr{L}_\mu(\boldsymbol{\lambda})) \to (\mathscr{L}_\mu(\boldsymbol{\lambda}))_{\mathbf{x}}$.

Proof. The first isomorphism is

$$H^0(\mathbb{C}^n, \mathscr{L}_\mu(\boldsymbol{\lambda})) = H^k\big(\mathbb{C}^n, (\pi \circ m_n)_* (\dot{\imath}_\mu)_* (\dot{\imath}_\mu)^! \mathscr{B}(\boldsymbol{\lambda})\big) = H^k(\dot{T}_\mu, (\dot{\imath}_\mu)^! \mathscr{B}(\boldsymbol{\lambda})).$$

The second one requires the notion of a universally locally acyclic complex (see [10, Sect. 5.1]). Specifically, $\mathscr{B}(\boldsymbol{\lambda})$ is $(\pi \circ m_n)$-ULA ([42, proof of Proposition IV.3.4] or [43, Lemma 3.20]), so there is an isomorphism

$$i^*\mathscr{B}(\boldsymbol{\lambda}) \to i^!\mathscr{B}(\boldsymbol{\lambda})[2n].$$

Then

$$\begin{aligned} H^k((\dot{T}_\mu)_{\mathbf{x}}, (\dot{i}'_\mu)^! i^* \mathscr{B}(\boldsymbol{\lambda})) &= H^k((\dot{T}_\mu)_{\mathbf{x}}, (\dot{i}'_\mu)^! i^! \mathscr{B}(\boldsymbol{\lambda})[2n]) \\ &= H^k\big(\{\mathbf{x}\}, (\pi \circ m_n)_*(\dot{i}'_\mu)_*(\dot{i}'_\mu)^! i^! \mathscr{B}(\boldsymbol{\lambda})[2n]\big) \\ &= H^k\big(\{\mathbf{x}\}, (\pi \circ m_n)_*(\dot{i}'_\mu)_* i'^! (\dot{i}_\mu)^! \mathscr{B}(\boldsymbol{\lambda})[2n]\big) \\ &= H^k\big(\{\mathbf{x}\}, (i_0)^!(\pi \circ m_n)_*(\dot{i}_\mu)_*(\dot{i}_\mu)^! \mathscr{B}(\boldsymbol{\lambda})[2n]\big), \end{aligned}$$

the last step being proper base change. Now $(\pi \circ m_n)_*(\dot{i}_\mu)_*(\dot{i}_\mu)^!\mathscr{B}(\boldsymbol{\lambda})$ is the local system $\mathscr{L}_\mu(\boldsymbol{\lambda})$ shifted by $-k$, and therefore

$$\begin{aligned} H^k((\dot{T}_\mu)_{\mathbf{x}}, (\dot{i}'_\mu)^! i^* \mathscr{B}(\boldsymbol{\lambda})) = H^0(\{\mathbf{x}\}, (i_0)^! \mathscr{L}_\mu(\boldsymbol{\lambda})[2n]) = H^0(\{\mathbf{x}\}, (i_0)^* \mathscr{L}_\mu(\boldsymbol{\lambda})) \\ = (\mathscr{L}_\mu(\boldsymbol{\lambda}))_{\mathbf{x}} \end{aligned}$$

as desired. ■

By Proposition 5.4, the irreducible components of $\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu$ are all top-dimensional and can be indexed by

$$\bigsqcup_{\substack{(\mu_1,\ldots,\mu_n)\in\Lambda^n \\ \mu_1+\cdots+\mu_n=\mu}} \mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathscr{Z}(\lambda_n)_{\mu_n}; \tag{28}$$

namely, to a tuple $\mathbf{Z} = (Z_1, \ldots, Z_n)$ is assigned the component

$$\mathcal{X}(\mathbf{Z}) = \overline{\Psi(Z_1 \propto \cdots \propto Z_n)} \cap \mathcal{G}r_n^{\boldsymbol{\lambda}},$$

the bar denoting closure in $\dot{T}_\mu$. From now on, to simplify the writing, we will substitute $\mathscr{Z}(\boldsymbol{\lambda})_\mu$ for the cumbersome compound (28), using implicitly the bijection (2).

The proof of Proposition 5.4 shows that for any $\mathbf{x} \in \mathbb{C}^n$, the irreducible components of the fiber $(\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}}$ all have the same dimension and can be indexed by $\mathscr{Z}(\boldsymbol{\lambda})_\mu$. Let us look more closely at two particular cases.

If $\mathbf{x} \in \mathbb{C}^n$ lies in the open locus U of points with pairwise different coordinates, then, under the bijection $(\mathcal{G}r_n)_{\mathbf{x}} \cong (\mathrm{Gr})^n$ from Proposition 5.3, the irreducible components of $(\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}}$ are identified with the sets

$$\mathcal{X}(\mathbf{Z})_{\mathbf{x}} \cong (Z_1 \cap \mathrm{Gr}^{\lambda_1}) \times \cdots \times (Z_n \cap \mathrm{Gr}^{\lambda_n}) \tag{29}$$

with $\mathbf{Z} = (Z_1, \ldots, Z_n)$ in $\mathscr{Z}(\boldsymbol{\lambda})_\mu$.

On the other hand, recalling that an element $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$ is a subset of $\overline{\mathrm{Gr}_n^{\boldsymbol{\lambda}}}$, we may consider the preimage $\mathcal{Y}(\mathbf{Z})$ of $\Delta \times (\mathbf{Z} \cap \mathrm{Gr}_n^{\boldsymbol{\lambda}})$ under the isomorphism $\mathcal{G}r_n|_\Delta \xrightarrow{\simeq} \Delta \times \mathrm{Gr}_n$.

Then for any $\mathbf{x} \in \Delta$, the irreducible components of the fiber $(\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}}$ are the sets $\mathcal{Y}(\mathbf{Z})_{\mathbf{x}}$ for $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$.

Let us introduce a last piece of notation before stating the next theorem. In Sect. 2.3, we explained the construction of the MV basis of the μ-weight space of $V(\boldsymbol{\lambda})$. This basis is in bijection with $\mathscr{Z}(\boldsymbol{\lambda})_\mu$ and we denote by $\langle \mathbf{Z} \rangle$ the element indexed by $\mathbf{Z}$. On the other hand, given $\mathbf{Z} = (Z_1, \dots, Z_n)$ in $\mathscr{Z}(\lambda_1) \times \cdots \times \mathscr{Z}(\lambda_n)$, we can look at $\langle\!\langle \mathbf{Z} \rangle\!\rangle = \langle Z_1 \rangle \otimes \cdots \otimes \langle Z_n \rangle$, another element in $V(\boldsymbol{\lambda})$.

Theorem 5.8. *Let $(\mathbf{Z}', \mathbf{Z}'') \in (\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$. The coefficient $a_{\mathbf{Z}',\mathbf{Z}''}$ in the expansion*

$$\langle\!\langle \mathbf{Z}'' \rangle\!\rangle = \sum_{\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu} a_{\mathbf{Z},\mathbf{Z}''} \langle \mathbf{Z} \rangle$$

is the multiplicity of $\mathcal{Y}(\mathbf{Z}')$ in the intersection product $\mathcal{X}(\mathbf{Z}'') \cdot (\mathcal{G}r_n^{\boldsymbol{\lambda}})|_\Delta$ computed in the ambient space $\mathcal{G}r_n^{\boldsymbol{\lambda}}$.

Proof. Taking into account Lemma 5.7, the diagram (♡) can be rewritten as follows:

$$\begin{array}{ccc} H^0(\mathbb{C}^n, \mathscr{L}_\mu(\boldsymbol{\lambda})) & \xrightarrow{\simeq} & H_{\mathrm{top}}^{\mathrm{BM}}(\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu) \\ \simeq \downarrow & & \downarrow (g^* u_{\mathbf{x}}) \cap \\ (\mathscr{L}_\mu(\boldsymbol{\lambda}))_{\mathbf{x}} & \xrightarrow{\simeq} & H_{\mathrm{top}}^{\mathrm{BM}}((\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}}) \end{array}$$

The fundamental classes of the irreducible components of $\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu$ and $(\mathcal{G}r_n^{\boldsymbol{\lambda}} \cap \dot{T}_\mu)_{\mathbf{x}}$ provide bases of the two Borel–Moore homology groups, both indexed by $\mathscr{Z}(\boldsymbol{\lambda})_\mu$. In these bases, the right vertical arrow can be regarded as a matrix, say $Q_{\mathbf{x}}$. This matrix can be computed by intersection theory: applying [15, Theorem 19.2], we see that if $\mathbf{x} \in U$ (respectively, $\mathbf{x} \in \Delta$), then the entry in $Q_{\mathbf{x}}$ at position $(\mathbf{Z}', \mathbf{Z}'')$ is the multiplicity of $\mathcal{X}(\mathbf{Z}')_{\mathbf{x}}$ (respectively, $\mathcal{Y}(\mathbf{Z}')_{\mathbf{x}}$) in the intersection product

$$\mathcal{X}(\mathbf{Z}'') \cdot (\mathcal{G}r_n^{\boldsymbol{\lambda}})_{\mathbf{x}}$$

computed in the ambient space $\mathcal{G}r_n^{\boldsymbol{\lambda}}$. Identifying $\mathcal{G}r_n|_U$ with $U \times (\mathrm{Gr})^n$ by virtue of Proposition 5.3 and using the description over U of $\mathcal{X}(\mathbf{Z}')$ and $\mathcal{X}(\mathbf{Z}'')$ afforded by (29), we see that $Q_{\mathbf{x}}$ is the identity matrix for each point $\mathbf{x} \in U$.

According to the discussion at the end of Sect. 5.3, the geometric Satake correspondence identifies $V(\boldsymbol{\lambda})_\mu$ with each fiber of the local system $\mathscr{L}_\mu(\boldsymbol{\lambda})$. The basis element $\langle \mathbf{Z} \rangle$ is the fundamental class of $\mathcal{X}(\mathbf{Z})_{\mathbf{x}}$ when $\mathbf{x} \in \Delta$, and the basis element $\langle\!\langle \mathbf{Z} \rangle\!\rangle$ is the fundamental class of $\mathcal{X}(\mathbf{Z})_{\mathbf{x}}$ when $\mathbf{x} \in U$. Therefore, the coefficient $a_{\mathbf{Z}',\mathbf{Z}''}$ in the statement of the theorem is the entry at position $(\mathbf{Z}', \mathbf{Z}'')$ in the product $Q_{\mathbf{x}_\Delta} \times (Q_{\mathbf{x}_U})^{-1}$, for any choice of $(\mathbf{x}_\Delta, \mathbf{x}_U) \in \Delta \times U$. ∎

In particular, the entries $a_{\mathbf{Z}',\mathbf{Z}''}$ of the transition matrix between our two bases are nonnegative integers.

Proposition 5.9. *In the setup of Theorem 5.8, the diagonal entry $a_{\mathbf{Z}'',\mathbf{Z}''}$ is equal to* 1.

Proof. Write $\mathbf{Z}'' = (Z_1, \dots, Z_n)$ in $\mathscr{Z}(\lambda_1) \times \cdots \times \mathscr{Z}(\lambda_n)$. By the slice theorem applied to the quotient map $G^\vee(\mathbb{C}[z, z^{-1}]) \to \mathrm{Gr}$ (or, in this concrete situation, using [17, Remark 15 and Corollary 5]), we can find, for each $j \in \{1, \dots, n\}$, an affine variety U_j and a map $\phi_j : U_j \to G^\vee(\mathbb{C}[z, z^{-1}])$ such that $u \mapsto [\phi_j(u)]$ sends U_j isomorphically to an open subset of Gr^{λ_j} which meets Z_j.

For $x \in \mathbb{C}$ and $u \in U_j$, let $\phi_j(u)_{|x}$ denote the result of substituting $z - x$ for z in $\phi_j(u)$. We can then define an open embedding Φ as in the diagram

$$\begin{array}{ccc} \mathbb{C}^n \times (U_1 \times \cdots \times U_n) & \xrightarrow{\Phi} & \mathcal{G}r_n^{\boldsymbol{\lambda}} \\ \downarrow & & \downarrow{\scriptstyle \pi \circ m_n} \\ \mathbb{C}^n & = & \mathbb{C}^n \end{array}$$

by setting

$$\Phi(x_1, \dots, x_n; u_1, \dots, u_n) = (x_1, \dots, x_n; [\phi_1(u_1)_{|x_1}, \dots, \phi_n(u_n)_{|x_n}]).$$

Since intersection multiplicities are of local nature, $a_{\mathbf{Z}'', \mathbf{Z}''}$ can be computed after restriction to the image of Φ, where the situation is that of a trivial bundle. ■

5.5. *An example*

It is possible to put coordinates on $\mathcal{G}r_n^{\boldsymbol{\lambda}}$ and to effectively compute the intersection multiplicities mentioned in Theorem 5.8. In this section, we look at the case of the group $G = \mathrm{SL}_3$. We adopt the usual description

$$\Lambda = (\mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2 \oplus \mathbb{Z}\varepsilon_3)/\mathbb{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

of the weight lattice, so that $V(\varepsilon_1)$ is the defining representation of G and $V(-\varepsilon_3)$ is its dual.

We consider the sequence of dominant weights $\boldsymbol{\lambda} = (\varepsilon_1, -\varepsilon_3)$. The basic MV cycles are $Z_i = \overline{\mathrm{Gr}^{\varepsilon_1}} \cap T_{\varepsilon_i}$ and $Z_{-i} = \overline{\mathrm{Gr}^{-\varepsilon_3}} \cap T_{-\varepsilon_i}$ for $i \in \{1, 2, 3\}$, and with this notation

$$\mathscr{Z}(\boldsymbol{\lambda}) = \big\{(Z_i, Z_{-j}) \bigm| (i, j) \in \{1, 2, 3\}^2\big\}.$$

To abbreviate, we set $\mathbf{Z}_{i,-j} = (Z_i, Z_{-j})$. For weight reasons, $\langle\!\langle \mathbf{Z}_{i,-j} \rangle\!\rangle = \langle \mathbf{Z}_{i,-j} \rangle$ if $i \neq j$. The rest of the transition matrix between the two bases is given as follows:

$$\begin{aligned} \langle\!\langle \mathbf{Z}_{1,-1} \rangle\!\rangle &= \langle \mathbf{Z}_{1,-1} \rangle, \\ \langle\!\langle \mathbf{Z}_{2,-2} \rangle\!\rangle &= \langle \mathbf{Z}_{2,-2} \rangle + \langle \mathbf{Z}_{1,-1} \rangle, \\ \langle\!\langle \mathbf{Z}_{3,-3} \rangle\!\rangle &= \langle \mathbf{Z}_{3,-3} \rangle + \langle \mathbf{Z}_{2,-2} \rangle. \end{aligned}$$

From these relations, we get $\langle \mathbf{Z}_{3,-3} \rangle = \langle\!\langle \mathbf{Z}_{3,-3} \rangle\!\rangle - \langle\!\langle \mathbf{Z}_{2,-2} \rangle\!\rangle + \langle\!\langle \mathbf{Z}_{1,-1} \rangle\!\rangle$. This allows one to check that $\langle \mathbf{Z}_{3,-3} \rangle$ is G-invariant, which in truth is a consequence of the compatibility of the MV basis of $V(\boldsymbol{\lambda})$ with the isotypic filtration (Theorem 3.4).

As an example, let us sketch out a computation which justifies that $\langle \mathbf{Z}_{1,-1} \rangle$ appears with coefficient 1 in $\langle\!\langle \mathbf{Z}_{2,-2} \rangle\!\rangle$. We consider two charts on $\mathcal{G}r_2^{\boldsymbol{\lambda}}$, both with $\mathbb{C}^6$ as domain:

$$\phi_1 : (x_1, x_2, a, b, c, d) \mapsto \left(x_1, x_2; \left[\left[\begin{pmatrix} z - x_1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ c & z - x_2 & 0 \\ d & 0 & z - x_2 \end{pmatrix} \right] \right] \right),$$

$$\phi_2 : (x_1, x_2, a', b', c', d') \mapsto \left(x_1, x_2; \left[\left[\begin{pmatrix} 1 & 0 & 0 \\ a' & z - x_1 & b' \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} z - x_2 & c' & 0 \\ 0 & 1 & 0 \\ 0 & d' & z - x_2 \end{pmatrix} \right] \right] \right).$$

(The matrices here belong to the group $\mathrm{PGL}_3(\mathbb{C}[z - x_1, z - x_2].)$ One easily computes the transition map between these two charts:

$$a' = 1/a, \quad b' = -b/a, \quad c' = -a(ac + bd + x_2 - x_1), \quad d' = -ad.$$

In the chart ϕ_1, the cycle $\mathcal{Y}(\mathbf{Z}_{1,-1})$ is defined by the equations $a = b = x_2 - x_1 = 0$. In the chart ϕ_2, the cycle $\mathcal{X}(\mathbf{Z}_{2,-2})$ is defined by the equations $b' = c' = 0$. Thus, the ideals in $R = \mathbb{C}[x_1, x_2, a, b, c, d]$ of the subvarieties

$$V = \phi_1^{-1}(\mathcal{Y}(\mathbf{Z}_{1,-1})) \quad \text{and} \quad X = \phi_1^{-1}(\mathcal{X}(\mathbf{Z}_{2,-2}))$$

are respectively

$$\mathfrak{p} = (a, b, x_2 - x_1) \quad \text{and} \quad \mathfrak{q} = (b, ac + x_2 - x_1).$$

Since $\mathfrak{q} \subseteq \mathfrak{p}$, we have $V \subseteq X$; in fact, V is a subvariety of X of codimension 1. The local ring $A = \mathcal{O}_{V,X}$ of X along V is the localization of $R/\mathfrak{q}$ at the ideal $\mathfrak{p}/\mathfrak{q}$. Observing that c is not in $\mathfrak{p}$, we see that its image in A is invertible, and then that $x_2 - x_1$ generates the maximal ideal of A. As a consequence, the order of vanishing of $x_2 - x_1$ along V (see [15, Sect. 1.2]) is 1. By definition, this is the multiplicity of $\mathcal{Y}(\mathbf{Z}_{1,-1})$ in the intersection product $\mathcal{X}(\mathbf{Z}_{2,-2}) \cdot \mathcal{G}r_2^{\boldsymbol{\lambda}}|_\Delta$.

5.6. *Factorizations*

A nice feature of the Beilinson–Drinfeld Grassmannian is its factorizable structure (see for instance [42, Proposition II.1.13]). On the other side of the geometric Satake equivalence, this corresponds to associativity properties of partial tensor products.

Let $\mathbf{n} = (n_1, \ldots, n_r)$ be a composition of n with r parts. We define the partial diagonal

$$\Delta_{\mathbf{n}} = \{(\underbrace{x_1, \ldots, x_1}_{n_1 \text{ times}}, \ldots, \underbrace{x_r, \ldots, x_r}_{n_r \text{ times}}) \mid (x_1, \ldots, x_r) \in \mathbb{C}^r\}.$$

We write $\boldsymbol{\lambda}$ as a concatenation $(\boldsymbol{\lambda}_{(1)}, \ldots, \boldsymbol{\lambda}_{(r)})$, where each $\boldsymbol{\lambda}_{(j)}$ belongs to $(\Lambda^+)^{n_j}$, and similarly we write each $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$ as $(\mathbf{Z}_{(1)}, \ldots, \mathbf{Z}_{(r)})$ with $\mathbf{Z}_{(j)} \in \mathscr{Z}(\boldsymbol{\lambda}_{(j)})$. Then

$$V(\boldsymbol{\lambda}) = V(\boldsymbol{\lambda}_{(1)}) \otimes \cdots \otimes V(\boldsymbol{\lambda}_{(r)}) \quad \text{and} \quad \langle \mathbf{Z}_{(j)} \rangle \in V(\boldsymbol{\lambda}_{(j)}).$$

Further, define

$$\mathcal{X}(\mathbf{Z}, \mathbf{n}) = \overline{\Psi(Z_1 \propto \cdots \propto Z_n)|_{\Delta_{\mathbf{n}}}} \cap \mathcal{G}r_n^{\boldsymbol{\lambda}}$$

where the bar means closure in $(\dot{T}_\mu)|_{\Delta_\mathbf{n}}$. The $\mathcal{X}(\mathbf{Z}, \mathbf{n})$ generalize the set $\mathcal{X}(\mathbf{Z})$ defined in Sect. 5.4, as the latter corresponds to the composition $(1, \dots, 1)$.

Theorem 5.8 can then be extended to this context in a straightforward fashion, as demonstrated by the following statement.

Proposition 5.10. *Let $(\mathbf{Z}', \mathbf{Z}'') \in (\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$. The coefficient $b_{\mathbf{Z}',\mathbf{Z}''}$ in the expansion*

$$\langle \mathbf{Z}''_{(1)} \rangle \otimes \cdots \otimes \langle \mathbf{Z}''_{(r)} \rangle = \sum_{\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu} b_{\mathbf{Z},\mathbf{Z}''} \langle \mathbf{Z} \rangle$$

is the multiplicity of $\mathcal{Y}(\mathbf{Z}')$ in the intersection product $\mathcal{X}(\mathbf{Z}'', \mathbf{n}) \cdot (\mathcal{G}r_n^{\boldsymbol{\lambda}})|_\Delta$ computed in the ambient space $\mathcal{G}r_n^{\boldsymbol{\lambda}}|_{\Delta_\mathbf{n}}$.

The proof does not require any new ingredient and is left to the reader.

5.7. Triangularity

In this section, we show that the transition matrix described in Theorem 5.8 is unitriangular with respect to a suitable order on $\mathscr{Z}(\boldsymbol{\lambda})_\mu$.

Proposition 5.11. *Let $(\mu_1, \dots, \mu_n)$ and $(\nu_1, \dots, \nu_n)$ in Λ^n and let $\mathcal{S}$ be a stratum for the ind-structure of $\mathcal{G}r_n$. If $\Psi(T_{\nu_1} \propto \cdots \propto T_{\nu_n})$ meets the closure of $\mathcal{S} \cap \Psi(T_{\mu_1} \propto \cdots \propto T_{\mu_n})$, then*

$$\nu_1 \geq \mu_1, \quad \nu_1 + \nu_2 \geq \mu_1 + \mu_2, \quad \dots, \quad \nu_1 + \cdots + \nu_n \geq \mu_1 + \cdots + \mu_n.$$

Proof. Given a tuple $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ in $(\Lambda/\mathbb{Z}\Phi)^n$, we set

$$\mathcal{G}r_{n,\boldsymbol{\zeta}} = \bigsqcup_{\substack{\boldsymbol{\lambda} \in (\Lambda^+)^n \\ \lambda_1 \in \zeta_1, \dots, \lambda_n \in \zeta_n}} \mathcal{G}r_n^{\boldsymbol{\lambda}}.$$

From (25), we deduce that each $\mathcal{G}r_{n,\boldsymbol{\zeta}}$ is closed and connected in the ind-topology. As these subsets form a finite partition of the space $\mathcal{G}r_n$, they are its connected components. We easily verify that a subset of the form $\Psi(T_{\mu_1} \propto \cdots \propto T_{\mu_n})$ is contained in $\mathcal{G}r_{n,\boldsymbol{\zeta}}$ if each ζ_j is the coset of μ_j modulo $\mathbb{Z}\Phi$. Therefore, a necessary condition for the set $\Psi(T_{\nu_1} \propto \cdots \propto T_{\nu_n})$ to meet the closure of $\mathcal{S} \cap \Psi(T_{\mu_1} \propto \cdots \propto T_{\mu_n})$ is that $\mu_j - \nu_j \in \mathbb{Z}\Phi$ for each $j \in \{1, \dots, n\}$.

Let $\lambda^\vee \in \operatorname{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$ be a dominant integral weight for the group $G^\vee$ and let V be the finite-dimensional irreducible representation of $G^\vee$ of highest weight $\lambda^\vee$. Then $G^\vee(\mathbb{C}(z))$ acts on $V \otimes \mathbb{C}(z)$. The standard lattice $L_0 = V \otimes \mathbb{C}[z]$ is left stable by $G^\vee(\mathbb{C}[z])$.

We choose a nonzero linear form $p : V \to \mathbb{C}$ that vanishes on all weight subspaces of V but the highest weight subspace. Extending the scalars, we regard p as a linear form $V \otimes \mathbb{C}(z) \to \mathbb{C}(z)$.

For $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{C}^n$, we set $f_\mathbf{x} = (z - x_1) \cdots (z - x_n)$. Let $\mathcal{S}$ be a stratum for the ind-structure of $\mathcal{G}r_n$. There exists a positive integer k such that $f_\mathbf{x}^k L_0 \subseteq \beta_1 \cdots \beta_n(L_0) \subseteq f_\mathbf{x}^{-k} L_0$ for each $(x_1, \dots, x_n; [\beta_1, \dots, \beta_n]) \in \mathcal{S}$.

Now we take $(\mu_1, \dots, \mu_n) \in \Lambda^n$ and $(x_1, \dots, x_n; [\beta_1, \dots, \beta_n])$ in the intersection $\mathcal{S} \cap \Psi(T_{\mu_1} \propto \cdots \propto T_{\mu_n})$. Then $p(\beta_1 \cdots \beta_n(L_0))$ is the fractional ideal

$$(z - x_1)^{\langle \lambda^\vee, \mu_1 \rangle} \cdots (z - x_n)^{\langle \lambda^\vee, \mu_n \rangle} \mathbb{C}[z],$$

and therefore

$$\dim(p(\beta_1 \cdots \beta_n(L_0))/f_{\mathbf{x}}^k \mathbb{C}[z]) = kn - \langle \lambda^\vee, \mu_1 + \cdots + \mu_n \rangle.$$

If the point $(x_1, \dots, x_n; [\beta_1, \dots, \beta_n])$ degenerates to

$$(y_1, \dots, y_n; [\gamma_1, \dots, \gamma_n]) \in \Psi(T_{\nu_1} \propto \cdots \propto T_{\nu_n}),$$

then

$$\dim(p(\gamma_1 \cdots \gamma_n(L_0))/f_{\mathbf{y}}^k \mathbb{C}[z]) \leq \dim(p(\beta_1 \cdots \beta_n(L_0))/f_{\mathbf{x}}^k \mathbb{C}[z])$$

which translates to

$$\langle \lambda^\vee, \nu_1 + \cdots + \nu_n \rangle \geq \langle \lambda^\vee, \mu_1 + \cdots + \mu_n \rangle.$$

This inequality holds for any dominant coweight $\lambda^\vee$, hence $\nu_1 + \cdots + \nu_n \geq \mu_1 + \cdots + \mu_n$.

This proves the last of the stated inequalities. The other ones can be obtained in a similar way, by taking the image under the obvious truncation map $\mathcal{G}r_n \to \mathcal{G}r_j$ for each $j \in \{1, \dots, n\}$. ∎

Corollary 5.12. *In the setup of Theorem 5.8, let $(\mu_1, \dots, \mu_n)$ and $(\nu_1, \dots, \nu_n)$ in Λ^n be such that $\mathbf{Z}' \in \mathscr{Z}(\lambda_1)_{\nu_1} \times \cdots \times \mathscr{Z}(\lambda_n)_{\nu_n}$ and $\mathbf{Z}'' \in \mathscr{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathscr{Z}(\lambda_n)_{\mu_n}$. A necessary condition for $a_{\mathbf{Z}',\mathbf{Z}''} \neq 0$ is that*

$$\nu_1 \geq \mu_1, \quad \nu_1 + \nu_2 \geq \mu_1 + \mu_2, \quad \dots, \quad \nu_1 + \cdots + \nu_{n-1} \geq \mu_1 + \cdots + \mu_{n-1}.$$

We can obtain more stringent conditions regarding the transition matrix by looking at the associativity properties from Sect. 5.6. The sharpest result is obtained with a composition (n_1, n_2) of n with two parts. Accordingly, we write $\boldsymbol{\lambda}$ as a concatenation $(\boldsymbol{\lambda}_{(1)}, \boldsymbol{\lambda}_{(2)})$ and similarly write each $\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})$ as $(\mathbf{Z}_{(1)}, \mathbf{Z}_{(2)})$. Here $\mathbf{Z}_{(1)}$ is an element in $\mathscr{Z}(\lambda_1) \times \cdots \times \mathscr{Z}(\lambda_{n_1})$, but owing to the bijection (2) it can also be regarded as a cycle in $\overline{\mathrm{Gr}_{n_1}^{\boldsymbol{\lambda}_{(1)}}}$.

Theorem 5.13. *Let $(\mathbf{Z}', \mathbf{Z}'') \in (\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$. Consider the expansion*

$$\langle \mathbf{Z}''_{(1)} \rangle \otimes \langle \mathbf{Z}''_{(2)} \rangle = \sum_{\mathbf{Z} \in \mathscr{Z}(\boldsymbol{\lambda})_\mu} b_{\mathbf{Z},\mathbf{Z}''} \langle \mathbf{Z} \rangle.$$

If $b_{\mathbf{Z}',\mathbf{Z}''} \neq 0$, then either $\mathbf{Z}' = \mathbf{Z}''$ or $\mathbf{Z}'_{(1)} \subsetneq \overline{\mathbf{Z}''_{(1)}}$ as cycles in $\overline{\mathrm{Gr}_{n_1}^{\boldsymbol{\lambda}_{(1)}}}$. In addition, $b_{\mathbf{Z}'',\mathbf{Z}''} = 1$.

Proof. Let $\mathbf{Z}' = (Z'_1, \dots, Z'_n)$ and $\mathbf{Z}'' = (Z''_1, \dots, Z''_n)$ in $\mathscr{Z}(\boldsymbol{\lambda})_\mu$.

For $j \in \{1, \dots, n\}$, let μ_j be the weight such that $Z''_j \in \mathscr{Z}(\lambda_j)_{\mu_j}$. Using the gallery models from [17] (or Theorem 4.6 and Proposition 4.8), we find a nonnegative integer d_j and construct a map $\phi_j : \mathbb{C}^{d_j} \to N^{-,\vee}(\mathbb{C}[z, z^{-1}])$ such that $\{[\phi_j(\mathbf{a})z^{\mu_j}] \mid \mathbf{a} \in \mathbb{C}^{d_j}\}$ is a dense subset of Z''_j. Then

$$\Phi : (\mathbf{x}; \mathbf{a}_1, \dots, \mathbf{a}_n) \mapsto \big(\mathbf{x}; [\phi_1(\mathbf{a}_1)_{|x_1}(z - x_1)^{\mu_1}, \dots, \phi_n(\mathbf{a}_n)_{|x_n}(z - x_n)^{\mu_n}]\big)$$

maps $\mathbb{C}^n \times \mathbb{C}^{d_1} \times \dots \times \mathbb{C}^{d_n}$ onto a dense subset of $\Psi(Z''_1 \propto \dots \propto Z''_n)$, where the notation $(\dots)_{|x}$ means the result of substituting $z - x$ for z in $(\dots)$.

Assume that $b_{\mathbf{Z}',\mathbf{Z}''} \neq 0$. By Proposition 5.10, $\mathcal{Y}(\mathbf{Z}')$ is contained in $\mathcal{X}(\mathbf{Z}'', (n_1, n_2))$, hence in the closure of $\Psi(Z''_1 \propto \dots \propto Z''_n)|_{\Delta_{(n_1,n_2)}}$.

Take a point in $\mathbf{Z}'_{(1)} \cap \mathrm{Gr}_{n_1}^{\boldsymbol{\lambda}(1)}$, written as $[g_1, \dots, g_{n_1}]$ where each g_j is in $G^\vee(\mathbb{C}[z, z^{-1}])$. We can complete this datum to get an element

$$\Gamma = (0, \dots, 0; [g_1, \dots, g_n])$$

of $\mathcal{Y}(\mathbf{Z}')$. Working in the analytic topology for expositional simplicity, we see that Γ is the limit of a sequence $(\Phi(\mathbf{x}_p; \mathbf{a}_{1,p}, \dots, \mathbf{a}_{n,p}))_{p\in\mathbb{N}}$ with $\mathbf{x}_p \in \Delta_{(n_1,n_2)}$ and $(\mathbf{a}_{1,p}, \dots, \mathbf{a}_{n,p}) \in \mathbb{C}^{d_1} \times \dots \times \mathbb{C}^{d_n}$. We write

$$\mathbf{x}_p = (\underbrace{x_{1,p}, \dots, x_{1,p}}_{n_1 \text{ times}}, \underbrace{x_{2,p}, \dots, x_{2,p}}_{n_2 \text{ times}}) \quad \text{with of course} \quad \lim_{p\to\infty} x_{1,p} = \lim_{p\to\infty} x_{2,p} = 0. \tag{30}$$

Then

$$\begin{aligned}[g_1, \dots, g_{n_1}] &= \lim_{p\to\infty} [\phi_1(\mathbf{a}_{1,p})z^{\mu_1}, \dots, \phi_{n_1}(\mathbf{a}_{1,n_1})z^{\mu_{n_1}}]_{|x_{1,p}} \\ &= \lim_{p\to\infty} [\phi_1(\mathbf{a}_{1,p})z^{\mu_1}, \dots, \phi_{n_1}(\mathbf{a}_{1,n_1})z^{\mu_{n_1}}]\end{aligned}$$

is the limit of a sequence of points in $\mathbf{Z}''_{(1)}$. Therefore $\mathbf{Z}'_{(1)} \cap \mathrm{Gr}_{n_1}^{\boldsymbol{\lambda}(1)} \subseteq \overline{\mathbf{Z}''_{(1)}}$, whence the inclusion $\mathbf{Z}'_{(1)} \subseteq \overline{\mathbf{Z}''_{(1)}}$.

In addition to $b_{\mathbf{Z}',\mathbf{Z}''} \neq 0$, assume that the last inclusion is an equality. Then we have $\mathbf{Z}'_{(1)} = \mathbf{Z}''_{(1)}$ because these two MV cycles are irreducible components of the same $\overline{\mathrm{Gr}_{n_1}^{\boldsymbol{\lambda}(1)}} \cap (m_{n_1})^{-1}(T_{\mu(1)})$, with indeed $\mu(1) = \mu_1 + \dots + \mu_{n_1}$. We regard $\mathbf{Z}'_{(2)}$ and $\mathbf{Z}''_{(2)}$ as cycles in $\overline{\mathrm{Gr}_{n_2}^{\boldsymbol{\lambda}(2)}}$. Take a point in $\mathbf{Z}'_{(2)} \cap \mathrm{Gr}_{n_2}^{\boldsymbol{\lambda}(2)}$, written as $[g_{n_1+1}, \dots, g_n]$ where each g_j is in $G^\vee(\mathbb{C}[z, z^{-1}])$. We can then look at the element

$$\Gamma = (0, \dots, 0; [z^{\mu_1}, \dots, z^{\mu_{n_1}}, g_{n_1+1}, \dots, g_n])$$

of $\mathcal{Y}(\mathbf{Z}')$. Again Γ is the limit of a sequence $(\Phi(\mathbf{x}_p; \mathbf{a}_{1,p}, \dots, \mathbf{a}_{n,p}))_{p\in\mathbb{N}}$ with $\mathbf{x}_p \in \Delta_{(n_1,n_2)}$ and $(\mathbf{a}_{1,p}, \dots, \mathbf{a}_{n,p}) \in \mathbb{C}^{d_1} \times \dots \times \mathbb{C}^{d_n}$. We set

$$B_p = z^{-\mu(1)}\phi_1(\mathbf{a}_{1,p})z^{\mu_1} \cdots \phi_{n_1}(\mathbf{a}_{n_1,p})z^{\mu_{n_1}}.$$

Writing again (30), we have

$$L_{\mu_{(1)}} = \lim_{p\to\infty} [z^{\mu_{(1)}} B_p]_{|x_{1,p}} = \lim_{p\to\infty} [z^{\mu_{(1)}} B_p] \tag{31}$$

and

$$\begin{aligned} z^{\mu_{(1)}}&[g_{n_1+1},\dots,g_n] \\ &= \lim_{p\to\infty} (z^{\mu_{(1)}} B_p)_{|x_{1,p}} [\phi_{n_1+1}(\mathbf{a}_{n_1+1,p}) z^{\mu_{n_1+1}},\dots,\phi_n(\mathbf{a}_{n,p}) z^{\mu_n}]_{|x_{2,p}}. \end{aligned} \tag{32}$$

Let K be the kernel of the evaluation map $N^{-,\vee}(\mathbb{C}[z^{-1}]) \to N^{-,\vee}(\mathbb{C})$ at $z=\infty$. The multiplication induces a bijection

$$K \times N^{-,\vee}(\mathbb{C}[z]) \xrightarrow{\simeq} N^{-,\vee}(\mathbb{C}[z,z^{-1}]).$$

We decompose B_p as a product $B_{-,p}B_{+,p}$ according to this bijection. Using (31) and identifying the ind-variety T_0 with K, we find that $B_{-,p}\to 1$ as $p\to\infty$. Inserting this information in (32), we obtain

$$[g_{n_1+1},\dots,g_n] = \lim_{p\to\infty} B_{+,p}[\phi_{n_1+1}(\mathbf{a}_{n_1+1,p}) z^{\mu_{n_1+1}},\dots,\phi_n(\mathbf{a}_{n,p}) z^{\mu_n}],$$

so $[g_{n_1+1},\dots,g_n]$ is the limit of a sequence of points in $\mathbf{Z}''_{(2)}$. We conclude that $\mathbf{Z}'_{(2)} \subseteq \overline{\mathbf{Z}''_{(2)}}$, and since these two cycles have the same dimension, actually $\mathbf{Z}'_{(2)} = \mathbf{Z}''_{(2)}$.

To sum up: if $b_{\mathbf{Z}',\mathbf{Z}''} \neq 0$, then $\mathbf{Z}'_{(1)} \subseteq \overline{\mathbf{Z}''_{(1)}}$, and in case $\mathbf{Z}'_{(1)} = \mathbf{Z}''_{(1)}$, we additionally have $\mathbf{Z}'_{(2)} = \mathbf{Z}''_{(2)}$. This proves the first statement in the theorem. The second one is proved in the same manner as Proposition 5.9. ∎

Remark 5.14. Using Theorem 5.13, one easily sharpens Corollary 5.12: with the notation of the latter, if $a_{\mathbf{Z}',\mathbf{Z}''} \neq 0$, then either $\mathbf{Z}' = \mathbf{Z}''$ or one of the displayed inequalities is strict. The proof is left to the reader.

Application to standard monomial theory. Let $\lambda \in \Lambda^+$ and let $\ell \subseteq V(\lambda)^*$ be the line spanned by a highest weight vector. The group G acts on the projective space $\mathbb{P}(V(\lambda)^*)$; let Q be the stabilizer of ℓ, a parabolic subgroup of G. The map $g \mapsto g\ell$ induces an embedding of the partial flag variety $X = G/Q$ in $\mathbb{P}(V(\lambda)^*)$. We denote by $\mathscr{L}$ the pull-back of the line bundle $\mathscr{O}(1)$ by this embedding. Then the homogeneous coordinate ring of X is

$$R_\lambda = \bigoplus_{m\geq 0} H^0(X, \mathscr{L}^{\otimes m});$$

here $H^0(X,\mathscr{L}^{\otimes m})$ is isomorphic to $V(m\lambda)$ and the multiplication in R_λ is given by the projection onto the Cartan component

$$V(m\lambda)\otimes V(n\lambda) \to V((m+n)\lambda).$$

The algebra R_λ is endowed with an MV basis, obtained by gathering the MV bases of the summands $V(m\lambda)$.

Each MV cycle $Z \in \mathscr{Z}(\lambda)$ defines a basis element $\langle Z\rangle \in V(\lambda)$. Given an m-tuple $\mathbf{Z} = (Z_1, \dots, Z_m)$ of elements of $\mathscr{Z}(\lambda)$, the product $\langle Z_1\rangle \cdots \langle Z_m\rangle$ in the algebra R_λ is the image of $\langle\langle \mathbf{Z}\rangle\rangle = \langle Z_1\rangle \otimes \cdots \otimes \langle Z_m\rangle$ under the projection $V(\lambda)^{\otimes m} \to V(m\lambda)$. This product is called *standard* if $\mathbf{Z}$ lies in the Cartan component of the crystal $\mathscr{Z}(\lambda)^{\otimes m}$.

Remark 3.5 implies that the MV basis element $\langle \mathbf{Z}\rangle \in V(\lambda)^{\otimes m}$ goes, under the projection $V(\lambda)^{\otimes m} \to V(m\lambda)$, either to an element in the MV basis of $V(m\lambda)$ or to 0, depending on whether $\mathbf{Z}$ lies or not in the Cartan component of $\mathscr{Z}(\lambda)^{\otimes m}$.

Using Corollary 5.12 and Remark 5.14, we can then endow, for each degree m, the Cartan component of $\mathscr{Z}(\lambda)^{\otimes m}$ with an order, so that the transition matrix expressing the standard monomials in the MV basis of R_λ is unitriangular. In particular, the standard monomials form a basis for the algebra R_λ too, and straightening laws can be obtained from Theorem 5.8.

The dual of the MV basis is compatible with the Demazure modules contained in $V(m\lambda)^*$; this property is recorded in [3, Remark 2.6 (ii)] but the crux of the argument is due to Kashiwara [26]. This implies that for any Schubert variety $Y \subseteq X$, the kernel of the restriction map

$$\bigoplus_{m\geq 0} H^0(X, \mathscr{L}^{\otimes m}) \to \bigoplus_{m\geq 0} H^0(Y, \mathscr{L}^{\otimes m})$$

is spanned by a subset of the MV basis of R_λ. Therefore the homogeneous coordinate ring of Y is also endowed with an MV basis.

These observations suggest that the MV basis could be a relevant tool for the study of the standard monomial theory.

5.8. A conjectural symmetry

Recall the notation set up in Sects. 3.1–3.2. Given $\lambda \in \Lambda^+$, we set $\lambda^* = -w_0\lambda$, where as usual w_0 denotes the longest element in the Weyl group W. As is well-known, there exists a unique bijection $\sigma : B(\lambda) \to B(\lambda^*)$ which for each $i \in I$ exchanges the actions of $\tilde{e}_i$ and $\tilde{f}_i$. In our context, we will regard σ as a bijection $\mathscr{Z}(\lambda) \to \mathscr{Z}(\lambda^*)$ and may define it by means of [33, Lemma 2.1 (e)] and Theorem 4.9.

Now let $n \geq 1$ and let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ in $(\Lambda^+)^n$. We set $\boldsymbol{\lambda}^* = (\lambda_n^*, \dots, \lambda_1^*)$ and define a bijection

$$\sigma : \mathscr{Z}(\lambda_1) \times \cdots \times \mathscr{Z}(\lambda_n) \to \mathscr{Z}(\lambda_n^*) \times \cdots \times \mathscr{Z}(\lambda_1^*)$$

by $\sigma(Z_1, \dots, Z_n) = (\sigma(Z_n), \dots, \sigma(Z_1))$. (Using the same symbol σ to denote different bijections is certainly abusive, but adding extra indices to disambiguate would overload the notation without clear benefit.) The Cartesian products above are in fact tensor product of crystals, and here again σ exchanges the actions of $\tilde{e}_i$ and $\tilde{f}_i$ for each $i \in I$ [23, Theorem 2].

Let $\mu \in \Lambda$ and choose $(\mathbf{Z}', \mathbf{Z}'') \in (\mathscr{Z}(\boldsymbol{\lambda})_\mu)^2$; we then obtain $\sigma(\mathbf{Z}')$ and $\sigma(\mathbf{Z}'')$ in $\mathscr{Z}(\boldsymbol{\lambda}^*)_{-\mu}$. Recall the notation introduced in Theorem 5.8 to denote the entries of the transition matrix between the two bases of $V(\boldsymbol{\lambda})$ and adopt a similar notation for $V(\boldsymbol{\lambda}^*)$.

Conjecture 5.15. The equality $a_{\mathbf{Z}',\mathbf{Z}''} = a_{\sigma(\mathbf{Z}'),\sigma(\mathbf{Z}'')}$ holds.

In type A_1, this conjecture follows from the results in [1]. Its general validity would have two interesting consequences.

Firstly, one could then strengthen Theorem 5.13. Indeed, $b_{\mathbf{Z}',\mathbf{Z}''} \neq 0$ would imply not only $\mathbf{Z}'_{(1)} \subseteq \overline{\mathbf{Z}''_{(1)}}$, but also $\sigma(\mathbf{Z}'_{(2)}) \subseteq \overline{\sigma(\mathbf{Z}''_{(2)})}$, restoring the symmetry between the two tensor factors.

Secondly, the MV basis of an irreducible representation $V(\lambda)$ would then satisfy the analogue of [37, Proposition 21.1.2]. In fact, one easily verifies that the MV basis enjoys this property if λ is minuscule or quasi-minuscule. Our conjecture would allow one to deduce the general case by taking suitable tensor products, mimicking the strategy of proof from [40].

6. The basis of the invariant subspace

Let $n \geq 1$ and let $\boldsymbol{\lambda} \in (\Lambda^+)^n$. The MV basis of $V(\boldsymbol{\lambda})$ is compatible with the isotypic filtration, hence provides a basis of the fixed-point subspace $V(\boldsymbol{\lambda})^G$, called the *Satake basis* in [13]. In this section we study two properties of this basis.

6.1. Cyclic permutations

Let us write $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and consider the rotated sequence $\boldsymbol{\lambda}^{[1]} = (\lambda_2, \dots, \lambda_n, \lambda_1)$. Thus,

$$V(\boldsymbol{\lambda}) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n) \quad \text{and} \quad V(\boldsymbol{\lambda}^{[1]}) = V(\lambda_2) \otimes \cdots \otimes V(\lambda_n) \otimes V(\lambda_1).$$

The signed cyclic permutation

$$x_1 \otimes \cdots \otimes x_n \mapsto (-1)^{2\rho(\lambda_1)} x_2 \otimes \cdots \otimes x_n \otimes x_1,$$

defines an isomorphism of G-modules $R : V(\boldsymbol{\lambda}) \to V(\boldsymbol{\lambda}^{[1]})$. In particular, R induces a linear bijection between the invariant subspaces.

Theorem 6.1. *The signed cyclic permutation R maps the Satake basis of $V(\boldsymbol{\lambda})^G$ to the Satake basis of $V(\boldsymbol{\lambda}^{[1]})^G$.*

Theorem 6.1 replicates a similar result for the dual canonical basis due to Lusztig [37, 28.2.9], and our proof below mirrors Lusztig's argument. In the case where all the weights λ_j are minuscule, Theorem 6.1 has been proved by Fontaine, Kamnitzer and Kuperberg [13, Theorem 4.5]. The bijection induced by R between the two Satake bases has a nice interpretation, both in terms of crystals (see [12]) and in terms of cluster combinatorics (see [22, Sect. 2.1.6]).

The rest of this section is devoted to the proof of Theorem 6.1.

As in Sect. 3, we denote by $\{\alpha_i \mid i \in I\}$ the set of simple roots and choose simple root vectors e_i and f_i in the Lie algebra of G of weights $\pm\alpha_i$ such that $[e_i, f_i] = -\alpha_i^\vee$. The Weyl group W is generated by the simple reflections s_i and contains a longest element w_0.

Given $\lambda \in \Lambda^+$ and $w \in W$, we can pick a reduced word $(i_1, \dots, i_\ell)$ of w and form the product of divided powers

$$\theta(w, \lambda) = f_{i_1}^{(n_1)} \cdots f_{i_\ell}^{(n_\ell)}, \quad \text{where} \quad n_j = \langle \alpha_{i_j}^\vee, s_{i_{j+1}} \cdots s_{i_\ell} \lambda \rangle.$$

This element does not depend on the choice of $(i_1, \dots, i_\ell)$ (see [37, Proposition 28.1.2]), which legitimizes the notation. We note that $\theta(w_0, \lambda)$ acts on $V(\lambda)$ by mapping highest weight vectors to lowest weight vectors.

We set $\lambda = \lambda_1$, the first element in the sequence $\boldsymbol{\lambda}$. With the notation of Sect. 5.4, the highest and lowest weight elements in the MV basis of $V(\lambda)$ are

$$v_\lambda = \langle \{L_\lambda\} \rangle \quad \text{and} \quad v_{w_0\lambda} = \langle \overline{\mathrm{Gr}^\lambda} \rangle.$$

Under suitable normalizations in the geometric Satake equivalence, these two elements are related by $v_{w_0\lambda} = \theta(w_0, \lambda) \cdot v_\lambda$ (see [3, Theorem 5.2 and Remark 2.10]). We define v_λ^* to be the linear form on $V(\lambda)$ such that $\langle v_\lambda^*, v_\lambda \rangle = 1$ and v_λ^* vanishes on all weight subspaces of weight different from λ. Similarly, we define $v_{w_0\lambda}^*$ to be the linear form on $V(\lambda)$ such that $\langle v_{w_0\lambda}^*, v_{w_0\lambda} \rangle = 1$ and $v_{w_0\lambda}^*$ vanishes on all weight subspaces of weight different from $w_0\lambda$.

Let M be a representation of G. The assignment $v \otimes m \mapsto (-1)^{2\rho(\lambda)} m \otimes v$ defines an isomorphism $P : V(\lambda) \otimes M \to M \otimes V(\lambda)$.

We set $\lambda^* = -w_0\lambda$. Let M° be the isotypic component of M corresponding to the highest weight λ^*, namely, the sum of all subrepresentations isomorphic to $V(\lambda^*)$. Given a weight $\mu \in \Lambda$, we denote by M_μ the corresponding weight subspace of M and set $M_\mu^\circ = M^\circ \cap M_\mu$. Then $M_{\lambda^*}^\circ$ is the set of all vectors in M_{λ^*} that are annihilated by all the root vectors e_i, and $M_{w_0\lambda^*}^\circ$ is the set of all vectors in $M_{w_0\lambda^*}$ that are annihilated by all the root vectors f_i.

Lemma 6.2. *The following diagram commutes and consists of isomorphisms of vector spaces:*

$$\begin{array}{ccc} (V(\lambda) \otimes M)^G & \xrightarrow{P} & (M \otimes V(\lambda))^G \\ \Big\downarrow {\scriptstyle v_{w_0\lambda}^* \otimes \mathrm{id}_M} & & \Big\downarrow {\scriptstyle \mathrm{id}_M \otimes v_\lambda^*} \\ M_{\lambda^*}^\circ & \xrightarrow{\theta(w_0, \lambda^*)} & M_{w_0\lambda^*}^\circ \end{array} \tag{33}$$

Proof. By additivity, we can reduce to the case where M is a simple representation. If M is not isomorphic to the dual of $V(\lambda)$, then all four spaces are zero and the statement is banal. We therefore assume that $M \cong V(\lambda^*)$; in this case, all four spaces are one-dimensional.

Let m_{λ^*} be a highest weight vector in M and set $m_{w_0\lambda^*} = \theta(w_0, \lambda^*) \cdot m_{\lambda^*}$. There exists a unique G-invariant bilinear form $\Phi : V(\lambda) \times M \to \mathbb{C}$ such that $\Phi(v_{w_0\lambda}, m_{\lambda^*}) = 1$. This form Φ is nondegenerate and a standard computation gives $\Phi(v_\lambda, m_{w_0\lambda^*}) = (-1)^{2\rho(\lambda)}$.

The assignment $v \otimes m \mapsto \Phi(v, ?)m$ defines a G-equivariant isomorphism $V(\lambda) \otimes M \to \mathrm{End}(M)$. The preimage x of id_M by this bijection spans the vector space $(V(\lambda) \otimes M)^G$. By construction, $(v^*_{w_0\lambda} \otimes \mathrm{id}_M)(x) = m_{\lambda^*}$ and $(v^*_\lambda \otimes \mathrm{id}_M)(x) = (-1)^{2\rho(\lambda)} m_{w_0\lambda^*}$. Thus, both paths around the diagram map x to $m_{w_0\lambda^*}$. ■

We take $M = V(\lambda_2) \otimes \cdots \otimes V(\lambda_n)$. We define $M^\bullet$ to be the step in the isotypic filtration of M where the component M° is appended to smaller ones. There is a natural quotient map $p : M^\bullet \to M^\circ$.

We set $\mathscr{M} = \mathscr{Z}(\lambda_2) \times \cdots \times \mathscr{Z}(\lambda_n)$. Using the notation introduced in Sect. 5.4, the MV basis of M consists of elements $\langle \mathbf{Z} \rangle$ for $\mathbf{Z} \in \mathscr{M}$. Let $\mathscr{M}^\bullet$ be the set of all $Z \in \mathscr{M}$ such that $\langle \mathbf{Z} \rangle \in M^\bullet$; since MV bases are L-perfect, $\{\langle \mathbf{Z} \rangle \mid \mathbf{Z} \in \mathscr{M}^\bullet\}$ is a basis of $M^\bullet$. Let $\mathscr{M}^\circ$ be the set of all $Z \in \mathscr{M}^\bullet$ such that $\langle \mathbf{Z} \rangle \notin \ker p$; then $\{p(\langle \mathbf{Z} \rangle) \mid \mathbf{Z} \in \mathscr{M}^\circ\}$ is a basis of M°. In consequence, each weight subspace of M° is endowed with a basis.

As a crystal, $\mathscr{M}$ decomposes as the disjoint union (direct sum) of its connected components, and $\mathscr{M}^\circ$ is the union of the connected components of $\mathscr{M}$ that are isomorphic to $\mathscr{Z}(\lambda^*)$. For each connected component $\mathscr{C} \subseteq \mathscr{M}^\circ$, the subspace of M° spanned by $B_{\mathscr{C}} = \{p(\langle \mathbf{Z} \rangle) \mid \mathbf{Z} \in \mathscr{C}\}$ is a subrepresentation isomorphic to $V(\lambda^*)$, and by Remark 3.5, $B_{\mathscr{C}}$ identifies with the MV basis of $V(\lambda^*)$. The action of $\theta(w_0, \lambda^*)$ therefore maps the highest weight element in $B_{\mathscr{C}}$ to the lowest element in $B_{\mathscr{C}}$. We conclude that the bottom horizontal arrow in (33) maps the basis of $M^\circ_{\lambda^*}$ to the basis of $M^\circ_{w_0\lambda^*}$.

Each element in the MV basis of $V(\boldsymbol{\lambda}^{[1]}) = M \otimes V(\lambda)$ is of the form $\langle \mathbf{Z} \rangle$ with $\mathbf{Z}$ in $\mathscr{Z}(\boldsymbol{\lambda}^{[1]}) = \mathscr{M} \times \mathscr{Z}(\lambda)$. Let $V(\lambda)_{\neq\lambda}$ be the sum of all the weight subspaces of $V(\lambda)$ other than the higher weight subspace. Theorem 5.13 implies that for each $\mathbf{Z}_{(1)} \in \mathscr{M}$,

$$\langle \mathbf{Z}_{(1)} \rangle \otimes \langle \{L_\lambda\} \rangle \equiv \langle (\mathbf{Z}_{(1)}, \{L_\lambda\}) \rangle \; (\mathrm{mod}\, M \otimes V(\lambda)_{\neq\lambda}).$$

Thus, for $\mathbf{Z}_{(1)} \in \mathscr{M}$ and $\mathbf{Z} = (\mathbf{Z}_{(1)}, \{L_\lambda\})$, we have $(\mathrm{id}_M \otimes v^*_\lambda)(\langle \mathbf{Z} \rangle) = \langle \mathbf{Z}_{(1)} \rangle$.

As evidenced by the crystal structure on $\mathscr{M} \otimes \mathscr{Z}(\lambda)$, the Satake basis of the space $(M \otimes V(\lambda))^G$ consists of the vectors $\langle \mathbf{Z} \rangle$ for the pairs $\mathbf{Z} = (\mathbf{Z}_{(1)}, \{L_\lambda\})$ such that $\mathbf{Z}_{(1)} \in \mathscr{M}^\circ_{w_0\lambda^*}$. Noting that $\langle \mathbf{Z}_{(1)} \rangle \in M^\circ_{w_0\lambda^*}$ for those $\mathbf{Z}_{(1)}$, we conclude that the right vertical arrow in (33) maps basis elements to basis elements.

Similarly, the left vertical arrow in (33) maps the Satake basis of $(V(\lambda) \otimes M)^G$ to the basis of $M^\circ_{\lambda^*}$. Lemma 6.2 then concludes the proof of Theorem 6.1.

6.2. *Tensor product with an invariant element*

Let (n', n'') be a composition of n with two parts. Correspondingly, we write $\boldsymbol{\lambda} \in (\Lambda^+)^n$ as a concatenation $(\boldsymbol{\lambda}', \boldsymbol{\lambda}'')$ and view each element in $\mathscr{Z}(\boldsymbol{\lambda})$ as a pair $(\mathbf{Z}', \mathbf{Z}'') \in \mathscr{Z}(\boldsymbol{\lambda}') \times \mathscr{Z}(\boldsymbol{\lambda}'')$.

The following proposition implies that the Satake basis of the invariant subspace of $V(\boldsymbol{\lambda})$ satisfies the second item in Khovanov and Kuperberg's list of properties for the dual canonical basis (see the introduction of [30]).

Proposition 6.3. *Let* $(\mathbf{Z}', \mathbf{Z}'') \in \mathscr{Z}(\boldsymbol{\lambda})$. *If* $\langle \mathbf{Z}' \rangle \in V(\boldsymbol{\lambda}')^G$, *then* $\langle \mathbf{Z}' \rangle \otimes \langle \mathbf{Z}'' \rangle = \langle (\mathbf{Z}', \mathbf{Z}'') \rangle$.

Proof. Let $\mathbf{Z}' \in \mathscr{Z}(\boldsymbol{\lambda}')$. Recall the map $m_{n'} : \mathrm{Gr}_{n'} \to \mathrm{Gr}$ defined in Sect. 2.2 and the notation μ_I from Sect. 3.4 and set $\mu = \mu_I(\mathbf{Z}')$. Then $m_{n'}(\mathbf{Z}') \subseteq \overline{\mathrm{Gr}^\mu}$ and $\langle \mathbf{Z}' \rangle$ appear in the isotypic filtration of $V(\boldsymbol{\lambda}')$ at the step where the component of type $V(\mu)$ is appended.

If $\langle \mathbf{Z}' \rangle \in V(\boldsymbol{\lambda}')^G$, then $\mu = 0$, accordingly $\overline{\mathrm{Gr}^\mu} = \{L_0\}$, and as a result

$$\overline{\mathbf{Z}'} \subseteq (m_{n'})^{-1}(\{L_0\}) \subseteq (m_{n'})^{-1}(T_0).$$

This implies that no MV cycle in $\mathscr{Z}(\boldsymbol{\lambda}')$ can be strictly contained in $\overline{\mathbf{Z}'}$. (Such a cycle would be contained in $(m_{n'})^{-1}(T_0)$, so would be an irreducible component of $\overline{\mathrm{Gr}_{n'}^{\boldsymbol{\lambda}'}} \cap (m_{n'})^{-1}(T_0)$, and would end up having dimension $\rho(|\boldsymbol{\lambda}'|)$, the same as $\mathbf{Z}'$.) The desired result now directly follows from Theorem 5.13. ■

7. Applications to the MV basis of $\mathbb{C}[N]$

We adopt the notation set up in the preamble of Sect. 3. Let N be the unipotent radical of the Borel subgroup B and let $\mathbb{C}[N]$ be the algebra of regular functions on N. At the expense of an isogeny, which does not alter N, we can assume that G is simply-connected.

For each dominant weight $\lambda \in \Lambda^+$, we can choose a highest weight vector v_λ in the representation $V(\lambda)$ and define the linear form $v_\lambda^* : V(\lambda) \to \mathbb{C}$ such that $\langle v_\lambda^*, v_\lambda \rangle = 1$ and $\langle v_\lambda^*, v \rangle = 0$ for all weight vectors v of weight other than λ. This yields an embedding $\Psi_\lambda : v \mapsto \langle v_\lambda^*, ?v \rangle$ of $V(\lambda)$ into $\mathbb{C}[N]$, where $\langle v_\lambda^*, ?v \rangle$ stands for the function $n \mapsto \langle v_\lambda^*, nv \rangle$. The MV bases of the representations $V(\lambda)$ can be transported to $\mathbb{C}[N]$ through these maps Ψ_λ, and they glue together to form a basis of $\mathbb{C}[N]$, which we call the MV basis of $\mathbb{C}[N]$ (see [3]).

The algebra $\mathbb{C}[N]$ comes with several remarkable bases: the MV basis, subject of our current investigation, but also the dual canonical basis of Lusztig also konwn as the upper global basis of Kashiwara, and (in simply laced type) the dual semicanonical basis. The theory of cluster algebras was developed in order to compute effectively these bases (or at least, the dual canonical basis). Concretely, the cluster structure of $\mathbb{C}[N]$ allows one to define specific elements, called cluster monomials, which are linearly independent and easily amenable to calculations. It is known that both the dual canonical and the dual semicanonical bases contain all the cluster monomials [20, 25], but also that these bases differ (except when cluster monomials span $\mathbb{C}[N]$).

The methods developed in Sect. 5 allow us to effectively compute products of elements of the MV basis of $\mathbb{C}[N]$. This allows us to prove that this basis contains quite a few cluster monomials (Proposition 7.2) and that it generally differs from both the dual canonical and the dual semicanonical bases (Proposition 7.3).

7.1. *Cluster monomials*

As explained in [21, Sect. 4.3], each reduced word $(i_1, \ldots, i_\ell)$ of the longest element w_0 in the Weyl group W yields a seed of the cluster structure of $\mathbb{C}[N]$. The main result of this section, Proposition 7.2, presents a sufficient condition for the cluster monomials built from one of these seeds to belong to the MV basis of $\mathbb{C}[N]$.

Set $\mathfrak{t}_{\mathbb{R}}^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R})$ and let $C = \{x \in \mathfrak{t}_{\mathbb{R}}^{\vee} \mid \forall i \in I,\ \langle x, \alpha_i \rangle > 0\}$ be the Weyl chamber in $\mathfrak{t}_{\mathbb{R}}^{\vee}$. We consider the following condition on a reduced word $(i_1, \ldots, i_\ell)$:

(A) There exist $x_1 \in s_{i_1}(C)$, $x_2 \in (s_{i_1} s_{i_2})(C)$, …, $x_\ell \in (s_{i_1} \cdots s_{i_\ell})(C)$ such that $x_k - x_{k+1} \in C$ for each $k \in \{1, \ldots, \ell - 1\}$.

For instance, choose $(x, y) \in C^2$ in such a way that the straight line joining x to $-y$ avoids all the two-codimensional faces of the Weyl fan in $\mathfrak{t}_{\mathbb{R}}^{\vee}$. List in order the chambers successively crossed by this line: $C, s_{i_1}C, (s_{i_1}s_{i_2})(C), \ldots$. The word $(i_1, i_2, \ldots)$ produced in this manner is then reduced and obviously satisfies condition (A).

Let $Q \subseteq \Lambda$ be the root lattice. We denote by Q_+ the positive cone in Q with respect to the dominance order, that is, the set of all linear combinations of the simple roots α_i with nonnegative integral coefficients. We set $Q_- = -Q_+$.

Lemma 7.1. *Let $(i_1, \ldots, i_\ell)$ be a reduced word, set $w_k = s_{i_1} \cdots s_{i_k}$ for $k \in \{1, \ldots, \ell\}$, and let $(\nu_1, \ldots, \nu_\ell) \in w_1(Q_-) \times \cdots \times w_\ell(Q_-)$. Assume that $\nu_1 + \cdots + \nu_k \in Q_+$ for all $k \in \{1, \ldots, \ell - 1\}$, and $\nu_1 + \cdots + \nu_\ell = 0$, and $(i_1, \ldots, i_\ell)$ satisfies condition* (A)*. Then $\nu_1 = \cdots = \nu_\ell = 0$.*

Proof. We set $\mu_0 = 0$ and $\mu_k = \nu_1 + \cdots + \nu_k$ for $k \in \{1, \ldots, \ell\}$. We pick $x_1, \ldots, x_\ell$ as stated in condition (A). Then

$$\sum_{k=1}^{\ell} \langle x_k, \nu_k \rangle = \sum_{k=1}^{\ell} \langle x_k, \mu_k - \mu_{k-1} \rangle = \sum_{k=1}^{\ell-1} \langle x_k - x_{k+1}, \mu_k \rangle.$$

From $x_k \in w_k(C)$ and $\nu_k \in w_k(Q_-)$, we deduce that $\langle x_k, \nu_k \rangle \leq 0$ for each $k \in \{1, \ldots, \ell\}$. On the other hand, from $x_k - x_{k+1} \in C$ and $\mu_k \in Q_+$, we deduce that $\langle x_k - x_{k+1}, \mu_k \rangle \geq 0$ for each $k \in \{1, \ldots, \ell - 1\}$. We conclude that each $\langle x_k, \nu_k \rangle$ is indeed zero, which implies $\nu_k = 0$. ∎

In Sect. 6.1, we defined, for each $(\lambda, w) \in \Lambda^+ \times W$, a product $\theta(\lambda, w)$ of divided powers of the root vectors f_i. We can then set $v_{w\lambda} = \theta(\lambda, w) \cdot v_\lambda$; this is a vector of weight $w\lambda$ in $V(\lambda)$. We define $\Delta_{\lambda, w\lambda} = \Psi_\lambda(v_{w\lambda})$, usually called a *flag minor* if λ is minuscule. We denote by $\{\varpi_i \mid i \in I\}$ the set of fundamental weights.

Proposition 7.2. *Let $(i_1, \ldots, i_\ell)$ be a reduced word and define $x_k = \Delta_{\varpi_{i_k}, s_{i_1} \cdots s_{i_k} \varpi_{i_k}}$ for each $k \in \{1, \ldots, \ell\}$. If $(i_1, \ldots, i_\ell)$ satisfies condition* (A)*, then any monomial in $x_1, \ldots, x_\ell$ belongs to the MV basis of $\mathbb{C}[N]$.*

Proof. We choose $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ in $(\Lambda^+)^\ell$. For $k \in \{1, \ldots, \ell\}$, we set $w_k = s_{i_1} \cdots s_{i_k}$. The extremal weight vector $v_{w_k \lambda_k} \in V(\lambda_k)$ belongs to the MV basis [3, Remark 2.10 and Theorem 5.2], so $v_{w_k \lambda_k} = \langle Z_k \rangle$ where Z_k is the cycle $\overline{\mathrm{Gr}^{\lambda_k}} \cap T_{w_k \lambda_k}$. We set $\mu = w_1 \lambda_1 + \cdots + w_\ell \lambda_\ell$ and $\mathbf{Z} = (Z_1, \ldots, Z_\ell)$. We adopt the convention of Sect. 5.4 and regard $\mathbf{Z}$ as an element of $\mathscr{Z}(\boldsymbol{\lambda})_\mu$; then $\langle\!\langle \mathbf{Z} \rangle\!\rangle = v_{w_1 \lambda_1} \otimes \cdots \otimes v_{w_\ell \lambda_\ell}$.

Let us expand this element relative to the MV basis of $V(\boldsymbol{\lambda})$. As in Theorem 5.8, we write

$$\langle\!\langle \mathbf{Z} \rangle\!\rangle = \sum_{\mathbf{Z}' \in \mathscr{Z}(\boldsymbol{\lambda})_\mu} a_{\mathbf{Z}',\mathbf{Z}} \langle \mathbf{Z}' \rangle. \tag{34}$$

Suppose $\mathbf{Z}' \in \mathscr{Z}(\boldsymbol{\lambda})_\mu$ satisfies $a_{\mathbf{Z}',\mathbf{Z}} \neq 0$. Let $(\nu_1, \ldots, \nu_\ell) \in \Lambda^\ell$ be the tuple of weights such that $\mathbf{Z}' \in \mathscr{Z}(\lambda_1)_{\nu_1} \times \cdots \times \mathscr{Z}(\lambda_\ell)_{\nu_\ell}$. For each $k \in \{1, \ldots, \ell\}$, we have $\mathscr{Z}(\lambda_k)_{\nu_k} \neq \emptyset$, so $w_k^{-1}\nu_k$ is a weight of $V(\lambda_k)$, whence $(\nu_k - w_k\lambda_k) \in w_k(Q_-)$. From $\nu_1 + \cdots + \nu_\ell = \mu$, we deduce that $(\nu_1 - w_1\lambda_1) + \cdots + (\nu_\ell - w_\ell\lambda_\ell) = 0$. And by Corollary 5.12, we get

$$(\nu_1 - w_1\lambda_1) + \cdots + (\nu_k - w_k\lambda_k) \in Q_+$$

for each $k \in \{1, \ldots, \ell - 1\}$. Then, assuming that $(i_1, \ldots, i_\ell)$ satisfies condition (A) and applying Lemma 7.1, we find $\nu_k = w_k\lambda_k$ for each $k \in \{1, \ldots, \ell - 1\}$. In other words, none of the inequalities given in Corollary 5.12 is strict. By Remark 5.14, this forces $\mathbf{Z}' = \mathbf{Z}$. Thus, the expansion (34) contains a single term, namely $\langle \mathbf{Z} \rangle$.

Set $\lambda = \lambda_1 + \cdots + \lambda_\ell$ and let $p : V(\boldsymbol{\lambda}) \to V(\lambda)$ be the unique morphism that maps $v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$ to v_λ. Noting that p is the quotient map to the top factor in the isotypic filtration of $V(\boldsymbol{\lambda})$ and applying Remark 3.5, we find that $p(\langle \mathbf{Z} \rangle)$ belongs to the MV basis of $V(\lambda)$. From the equality $v_{w_1\lambda_1} \otimes \cdots \otimes v_{w_\ell\lambda_\ell} = \langle \mathbf{Z} \rangle$, we deduce that

$$\Delta_{\lambda_1, w_1\lambda_1} \cdots \Delta_{\lambda_\ell, w_\ell\lambda_\ell} = \langle v_\lambda^*, p(\text{?}\langle \mathbf{Z} \rangle) \rangle = \Psi_\lambda(p(\langle \mathbf{Z} \rangle))$$

belongs to the MV basis of $\mathbb{C}[N]$. The claim in the proposition is the particular case where each λ_k is a multiple of ϖ_{i_k}. ■

7.2. *A computation in type D_4*

In [29], Kashiwara and Saito found an example in type A_5 where the singular support of a simple perverse sheaf related to the canonical basis is not irreducible. Looking again at this situation, Geiß, Leclerc and Schröer [19] computed the dual canonical and dual semicanonical elements and found that they were different. They also observed that a similar phenomenon occurs in type D_4. In [3, Sect. 2.7], this setting in type D_4 was examined anew: the MV basis is a third basis, different from the other ones. In an appendix to [3], Dranowski, Kamnitzer and Morton-Ferguson extended this observation to the spot in type A_5 not covered by Kashiwara and Saito.

Let us have a closer look at the D_4 case. As usual, we label the vertices of the Dynkin diagram from 1 to 4, with 2 for the central node. Our three bases are indexed by the crystal $B(\infty)$: given $b \in B(\infty)$, we denote the corresponding dual semicanonical basis element by $C(b)$, the dual canonical basis element by $C'(b)$, and the MV basis element by $C''(b)$. Letting b_0 be the highest weight element in $B(\infty)$, we set

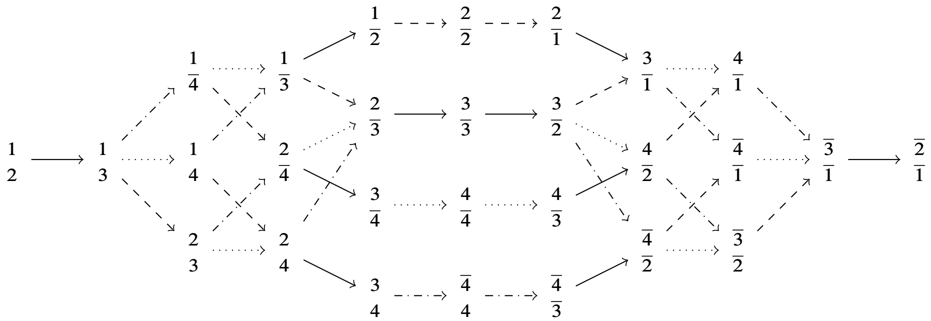
$$b_1 = (\tilde{f}_2(\tilde{f}_1\tilde{f}_3\tilde{f}_4)\tilde{f}_2)^2 b_0 \quad \text{and} \quad b_{12} = (\tilde{f}_2)^2(\tilde{f}_1\tilde{f}_3\tilde{f}_4)^2(\tilde{f}_2)^2 b_0.$$

Proposition 7.3. *The basis elements are related by the equations*

$$C(b_{12}) = C''(b_{12}) + 2C(b_1) \quad \textit{and} \quad C'(b_{12}) = C''(b_{12}) + C(b_1).$$

The proof is given in [3], except for one justification left to the present paper. We here fill the gap.

The fundamental weight ϖ_2 is the highest root $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. The crystal of the representation $V(\varpi_2)$ (the adjoint representation) is pictured below. Highest weights are towards the left, vertices are represented as keys $\tfrac{p}{q}$ with p, q in $\{1, 2, 3, 4, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$, and operators $\tilde{f_1}$, $\tilde{f_2}$, $\tilde{f_3}$ and $\tilde{f_4}$ are indicated by dashed, solid, dotted and dash-dotted arrows, respectively.



If we endow the weight lattice Λ with its usual basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, then the weight of the element $\tfrac{p}{q}$ is simply $\varepsilon_p + \varepsilon_q$, with the convention that $\varepsilon_{\overline{i}} = -\varepsilon_i$ for $i \in \{1, 2, 3, 4\}$. The crystal contains four elements of weight zero, namely $\tfrac{2}{2}$, $\tfrac{3}{3}$, $\tfrac{4}{4}$ and $\tfrac{\overline{4}}{4}$.

We set $\boldsymbol{\lambda} = (\varpi_2, \varpi_2)$ and look at the tensor square $V(\boldsymbol{\lambda}) = V(\varpi_2)^{\otimes 2}$. As in Sect. 5.4, its MV basis consists of symbols $\langle \mathbf{Z} \rangle$, where $\mathbf{Z} = (Z_1, Z_2)$ is a pair in $\mathscr{Z}(\varpi_2) \times \mathscr{Z}(\varpi_2)$. In addition, $V(\boldsymbol{\lambda})$ is endowed with the tensor product basis. To keep the notation straightforward, we indicate MV cycles by the keys $\tfrac{p}{q}$, making use of the isomorphism between $\mathscr{Z}(\varpi_2)$ and the crystal pictured above.

We claim that

$$\left\langle \tfrac{2}{3} \right\rangle \otimes \left\langle \tfrac{3}{2} \right\rangle = 2\left\langle \left(\tfrac{1}{2}, \tfrac{\overline{2}}{1} \right) \right\rangle + \left\langle \left(\tfrac{1}{3}, \tfrac{\overline{3}}{1} \right) \right\rangle + \left\langle \left(\tfrac{2}{3}, \tfrac{\overline{3}}{2} \right) \right\rangle + \left\langle \left(\tfrac{1}{4}, \tfrac{\overline{4}}{1} \right) \right\rangle$$
$$+ \left\langle \left(\tfrac{1}{4}, \tfrac{4}{1} \right) \right\rangle + \left\langle \left(\tfrac{2}{4}, \tfrac{\overline{4}}{2} \right) \right\rangle + \left\langle \left(\tfrac{2}{4}, \tfrac{4}{2} \right) \right\rangle + \left\langle \left(\tfrac{1}{3}, \tfrac{3}{1} \right) \right\rangle + \left\langle \left(\tfrac{2}{3}, \tfrac{3}{2} \right) \right\rangle. \qquad (35)$$

Let $p : V(\varpi_2)^{\otimes 2} \to V(2\varpi_2)$ be the unique morphism that maps $v_{\varpi_2} \otimes v_{\varpi_2}$ to $v_{2\varpi_2}$. Applying $\Psi_{2\varpi_2} \circ p$ to (35), we obtain the equality

$$C''(b_{13})C''(b_{14}) = 2C''(b_1) + \sum_{i=2}^{8} C''(b_i) + C''(b_{12})$$

asserted without proof in [3]. Establishing (35) will therefore complete the proof of Proposition 7.3. Actually, an inspection of the proof in *loc. cit.* reveals that it is enough to justify that the coefficient of $\left\langle \left(\tfrac{1}{2}, \tfrac{\overline{2}}{1} \right) \right\rangle$ is strictly larger than 1.

We will use Theorem 5.8 to prove this fact. Here the group $G^\vee$ is SO_8. For $(i, j) \in \{1, \ldots, 8\}$, we denote by $E_{i,j}$ the 8×8 matrix with zeros everywhere except for a 1 at position (i, j). For each coroot $\alpha^\vee \in \Phi^\vee$, we define a subgroup $x_{\alpha^\vee} : \mathbb{C} \to G^\vee$ by the following formulas, where I is the identity matrix, $a \in \mathbb{C}$, and i, j are elements in $\{1,2,3,4\}$ such that $i < j$:

$$x_{(\varepsilon_i-\varepsilon_j)^\vee}(a) = I + a(E_{i,j} - E_{9-j,9-i}), \qquad x_{(\varepsilon_i+\varepsilon_j)^\vee}(a) = I + a(E_{i,9-j} - E_{j,9-i}),$$
$$x_{(\varepsilon_j-\varepsilon_i)^\vee}(a) = I + a(E_{9-i,9-j} - E_{j,i}), \quad x_{(-\varepsilon_i-\varepsilon_j)^\vee}(a) = I + a(E_{9-i,j} - E_{9-j,i}).$$

For each root α, we define a map $\chi_\alpha : \mathbb{C}^{10} \to G^\vee(\mathbb{C}[z,z^{-1}])$ by the formula

$$\chi_\alpha(\mathbf{a})(z) = \Big(\prod_{k=1}^{8} x_{\beta_k^\vee}(a_k)\Big)\, x_{\alpha^\vee}(a_9 + za_{10}) z^\alpha$$

where $\mathbf{a}$ stands for the tuple $(a_1,\ldots,a_{10}) \in \mathbb{C}^{10}$ and where $\beta_1^\vee,\ldots,\beta_8^\vee$ are the coroots $\beta^\vee$ such that $\langle\beta^\vee,\alpha\rangle = 1$. We specify the enumeration in our cases of interest as follows.

α	$\beta_1^\vee$	$\beta_2^\vee$	$\beta_3^\vee$	$\beta_4^\vee$	$\beta_5^\vee$	$\beta_6^\vee$	$\beta_7^\vee$	$\beta_8^\vee$
$\varepsilon_1+\varepsilon_2$	$(\varepsilon_1-\varepsilon_3)^\vee$	$(\varepsilon_1-\varepsilon_4)^\vee$	$(\varepsilon_1+\varepsilon_4)^\vee$	$(\varepsilon_1+\varepsilon_3)^\vee$	$(\varepsilon_2-\varepsilon_3)^\vee$	$(\varepsilon_2-\varepsilon_4)^\vee$	$(\varepsilon_2+\varepsilon_4)^\vee$	$(\varepsilon_2+\varepsilon_3)^\vee$
$-\varepsilon_1-\varepsilon_2$	$(-\varepsilon_2-\varepsilon_3)^\vee$	$(-\varepsilon_2-\varepsilon_4)^\vee$	$(\varepsilon_4-\varepsilon_2)^\vee$	$(\varepsilon_3-\varepsilon_2)^\vee$	$(-\varepsilon_1-\varepsilon_3)^\vee$	$(-\varepsilon_1-\varepsilon_4)^\vee$	$(\varepsilon_4-\varepsilon_1)^\vee$	$(\varepsilon_3-\varepsilon_1)^\vee$
$\varepsilon_2-\varepsilon_3$	$(\varepsilon_2-\varepsilon_1)^\vee$	$(\varepsilon_2-\varepsilon_4)^\vee$	$(\varepsilon_2+\varepsilon_4)^\vee$	$(\varepsilon_1+\varepsilon_2)^\vee$	$(-\varepsilon_1-\varepsilon_3)^\vee$	$(-\varepsilon_3-\varepsilon_4)^\vee$	$(\varepsilon_4-\varepsilon_3)^\vee$	$(\varepsilon_1-\varepsilon_3)^\vee$
$\varepsilon_3-\varepsilon_2$	$(\varepsilon_3-\varepsilon_1)^\vee$	$(\varepsilon_3-\varepsilon_4)^\vee$	$(\varepsilon_3+\varepsilon_4)^\vee$	$(\varepsilon_1+\varepsilon_3)^\vee$	$(-\varepsilon_1-\varepsilon_2)^\vee$	$(-\varepsilon_2-\varepsilon_4)^\vee$	$(\varepsilon_4-\varepsilon_2)^\vee$	$(\varepsilon_1-\varepsilon_2)^\vee$

We now define two charts on $\mathcal{G}r_2^{\boldsymbol{\lambda}}$, both with $\mathbb{C}^{22}$ as domain:

$$\phi_1 : (x_1,x_2,\mathbf{a},\mathbf{b}) \mapsto \big(x_1,x_2;[\chi_{\varepsilon_1+\varepsilon_2}(\mathbf{a})(z-x_1),\chi_{-\varepsilon_1-\varepsilon_2}(\mathbf{b})(z-x_2)]\big),$$
$$\phi_2 : (x_1,x_2,\mathbf{a}',\mathbf{b}') \mapsto \big(x_1,x_2;[\chi_{\varepsilon_2-\varepsilon_2}(\mathbf{a}')(z-x_1),\chi_{\varepsilon_3-\varepsilon_2}(\mathbf{b}')(z-x_2)]\big).$$

One can then compute the transition map between these two charts. (The calculations were actually carried out with SINGULAR [11].) One finds the variables a_1', …, b_{10}' as rational functions in $x_2-x_1,a_1,\ldots,b_{10}$. We denote by f the l.c.m. of the denominators.

Recall the notation used in Sect. 5.4. In the chart ϕ_1, the cycle $\mathcal{Y}\big(\begin{smallmatrix}1\\2\end{smallmatrix},\begin{smallmatrix}\bar{2}\\1\end{smallmatrix}\big)$ is defined by the equations $a_1=\cdots=a_{10}=x_2-x_1=0$, so the ideal in $R=\mathbb{C}[x_1,x_2,a_1,\ldots,a_{10},b_1,\ldots,b_{10}]$ of

$$V = \phi_1^{-1}\big(\mathcal{Y}\big(\begin{smallmatrix}1\\2\end{smallmatrix},\begin{smallmatrix}\bar{2}\\1\end{smallmatrix}\big)\big)$$

is

$$\mathfrak{p} = (a_1,\ldots,a_{10},x_2-x_1).$$

In the chart ϕ_2, the cycle $\mathcal{X}\big(\begin{smallmatrix}2\\3\end{smallmatrix},\begin{smallmatrix}3\\2\end{smallmatrix}\big)$ is defined by the equations $a_2'=a_3'=a_4'=a_8'=a_9'=a_{10}'=b_2'=b_3'=b_4'=b_8'=0$. Since the zero locus of f contains the locus where the transition map between the charts is not defined, the ideal $\mathfrak{q}$ of the subvariety

$$X = \phi_1^{-1}\big(\mathcal{X}\big(\begin{smallmatrix}2\\3\end{smallmatrix},\begin{smallmatrix}3\\2\end{smallmatrix}\big)\big)$$

is the preimage in R of the ideal $\mathfrak{q}_f=(a_2',a_3',a_4',a_8',a_9',a_{10}',b_2',b_3',b_4',b_8')$ of the localized ring R_f. SINGULAR gives the following expression:

$$\mathfrak{q} = (a_1a_4 + a_2a_3, a_1a_6 - a_2a_5, a_3a_6 + a_4a_5, a_1a_7 - a_3a_5, a_2a_7 + a_4a_5, a_1a_8 - a_4a_5,$$
$$a_2a_8 - a_4a_6, a_3a_8 - a_4a_7, a_5a_8 + a_6a_7, a_9, a_{10}, a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1,$$
$$a_5b_4+a_6b_3+a_7b_2+a_8b_1, a_3b_6+a_4b_5+a_7b_2+a_8b_1-(x_2-x_1), a_2b_7-a_3b_6+a_6b_3-a_7b_2,$$
$$a_1b_8 + a_3b_6 + a_5b_4 + a_7b_2, a_2b_8 - a_4b_6 + a_6b_4 - a_8b_2, a_3b_8 - a_4b_7 + a_7b_4 - a_8b_3).$$

We observe that $\mathfrak{q} \subseteq \mathfrak{p}$, hence $V \subseteq X$.

Let a and x be two indeterminates. Let B be the field $\mathbb{C}(x, b_1, \dots, b_{10})$. Extract the last seven equations from $\mathfrak{q}$ and remove the term $x_2 - x_1$ in the third one; we then deal with seven linear equations with coefficients in B in the eight variables $a_1, \dots, a_8$. This system has a nonzero solution $(c_1, \dots, c_8) \in B^8$. We can then define an algebra morphism $u : R/\mathfrak{q} \to B[a]/(a^2)$ by

$$u(x_1) = u(x_2) = x, \quad u(b_i) = b_i \quad \text{for } i \in \{1, \dots, 10\},$$
$$u(a_9) = u(a_{10}) = 0, \quad u(a_i) = c_i a \quad \text{for } i \in \{1, \dots, 8\}.$$

The ring $B[a]/(a^2)$ is local with maximal ideal (a) and the preimage of this ideal by u is the ideal $\mathfrak{p}/\mathfrak{q}$ of $R/\mathfrak{q}$.

Let A be the localization of $R/\mathfrak{q}$ at $\mathfrak{p}/\mathfrak{q}$. Then u extends to an algebra morphism $\overline{u} : A \to B[a]/(a^2)$. By construction, the kernel of $\overline{u}$ contains $x_2 - x_1$ but not all of $a_1, \dots, a_8$. Therefore $x_2 - x_1$ does not generate the maximal ideal of A. Since A is the local ring $\mathscr{O}_{V,X}$ of X along V, this means that the order of vanishing of $x_2 - x_1$ along V is larger than 1. In other words, the multiplicity of $\mathcal{Y}\left(\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}\overline{2}\\1\end{smallmatrix}\right)$ in the intersection product $\mathcal{X}\left(\begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}\right) \cdot \mathcal{G}r_2^{\boldsymbol{\lambda}}|_\Delta$ is larger than 1. Applying Theorem 5.8, we conclude that in (35) the coefficient of $\left\langle\left(\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}\overline{2}\\1\end{smallmatrix}\right)\right\rangle$ is strictly larger than 1.

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