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Subconvexity for L-functions on GL_3 over number fields

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Abstract. In this paper, over an arbitrary number field, we prove subconvexity bounds for self-dual GL_3 L-functions in the t-aspect and for self-dual $\mathrm{GL}_3 \times \mathrm{GL}_2$ L-functions in the GL_2 Archimedean aspect.

Keywords. L-functions, subconvexity

Contents

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1. Introduction

There is a great interest in upper bounds for the central values of L-functions. The subconvexity problem is concerned with improving over their convexity bound resulting from the Phragmén–Lindelöf convexity principle.

The subconvexity problem for GL_1 and GL_2 over arbitrary number fields was completely solved in the seminal work of Michel and Venkatesh [39]. More recent work on the subconvexity for GL_2 over number fields may be found in [7, 35, 36, 46, 66, 67].

Xiaoqing Li [30] made the first progress on the subconvexity problem for GL_3 in the t-aspect and $\mathrm{GL}_3 \times \mathrm{GL}_2$ in the GL_2 spectral aspect. For a *self-dual* Hecke–Maass form π for $\mathrm{SL}_3(\mathbb{Z})$ and the family $\mathscr{B}$ given by an orthonormal basis of even Hecke–Maass cusp forms for $\mathrm{SL}_2(\mathbb{Z})$, she established the averaged Lindelöf hypothesis for the first moment:

$$\sum_{f \in \mathscr{B}} e^{-(t_f - T)^2/M^2} L\big(\tfrac12, \pi \otimes f\big) + \frac{1}{4\pi} \int_{\mathbb{R}} e^{-(t-T)^2/M^2} \big|L\big(\tfrac12 + it, \pi\big)\big|^2 \mathrm{d}t \ll_{\pi,\varepsilon} M T^{1+\varepsilon} \tag{1.1}$$

for $T^{3/8+\varepsilon} \leqslant M \leqslant T^{1/2}$, where $\frac14 + t_f^2$ is the Laplace eigenvalue for f. As a consequence of (1.1) and the non-negativity of $L\big(\frac12, \pi \otimes f\big)$ due to the self-dual assumption, she obtained the subconvexity bounds

$$L\big(\tfrac12 + it, \pi\big) \ll_{\pi,\varepsilon} (1 + |t|)^{11/16+\varepsilon}, \quad L\big(\tfrac12, \pi \otimes f\big) \ll_{\pi,\varepsilon} (1 + |t_f|)^{11/8+\varepsilon}. \tag{1.2}$$

In a similar framework, Blomer [5] proved the subconvexity for twisted GL_3 and $\mathrm{GL}_3 \times \mathrm{GL}_2$ L-functions in the q-aspect:

$$L\big(\tfrac12 + it, \pi \otimes \chi\big) \ll_{t,\pi,\varepsilon} q^{5/8+\varepsilon}, \quad L\big(\tfrac12, \pi \otimes f \otimes \chi\big) \ll_{f,\pi,\varepsilon} q^{5/4+\varepsilon}, \tag{1.3}$$

where χ is a quadratic Dirichlet character of prime modulus q. The work of Blomer clearly stems from the remarkable paper of Conrey and Iwaniec [12] on the cubic moment of twisted GL_2 L-functions. A recent advance on the path of Conrey and Iwaniec is the work of Young [68], in which he introduced new analytic techniques, which are quite different from those of Xiaoqing Li, to prove the hybrid Weyl-type subconvexity bound

$$L\big(\tfrac12 + it, \chi\big) \ll_{\varepsilon} (q(1 + |t|))^{1/6+\varepsilon}, \quad L\big(\tfrac12, f \otimes \chi\big) \ll_{\varepsilon} (q(1 + |t_f|))^{1/3+\varepsilon}. \tag{1.4}$$

Later, in the spirit of Blomer and Young, Nunes [47] improved Xiaoqing Li's exponent $\frac{11}{16}$ in (1.2) to Blomer's $\frac58$ in (1.3).

In this paper, we shall prove subconvexity results in the t-aspect for GL_3 and the GL_2 Archimedean aspect for $\mathrm{GL}_3 \times \mathrm{GL}_2$ over arbitrary number fields. The Voronoï summation formula for GL_3 of Ichino and Templier [20] is used in its full generality, with the aid of the asymptotic formulae of Bessel functions for GL_3 in [57].

As explained in §9, there are technical issues to generalize the analysis of Xiaoqing Li to number fields other than $\mathbb{Q}$, so, instead, our approach is inspired by the works of Blomer, Young, and Nunes. As for the strategy, briefly speaking, Xiaoqing Li uses the Voronoï summation twice, while we use the Voronoï summation once, followed by the large sieve.

For other related works, see for example [18, 21, 38, 51–54, 56].

For subconvexity results for GL_3 over $\mathbb{Q}$ without the self-dual assumption, we refer the reader to the papers of Munshi, Holowinsky, Nelson, Yongxiao Lin et al. [2, 16, 31, 32, 42–45, 58, 59, 63].

1.1. Statement of results

Let F be a fixed number field of degree N. Let S_∞ denote the set of Archimedean places of F. As usual, write $v \mid \infty$ in place of $v \in S_\infty$. For $v \mid \infty$, let N_v be the degree of $F_v/\mathbb{R}$.

Let π be a fixed *self-dual* spherical automorphic cuspidal representation of PGL_3 over F. Let $\mathcal{B}$ be an orthonormal basis consisting of Hecke–Maass cusp forms for the spherical cuspidal spectrum for PGL_2 over F. For $f \in \mathcal{B}$, let $\nu_f \in \mathbb{C}^{|S_\infty|}$ be its Archimedean parameter such that either $\nu_{f,v}$ is real or $i\nu_{f,v} \in (-\frac{1}{2}, \frac{1}{2})$ for every $v \mid \infty$; we may readily assume that $\nu_{f,v} \in [0, \infty) \cup i[0, \frac{1}{2})$.

We are concerned with the $\mathrm{GL}_3 \times \mathrm{GL}_2$ Rankin–Selberg L-functions $L(s, \pi \otimes f)$ for f in the family $\mathcal{B}$ and the GL_3 L-function $L(s, \pi)$. The following is our main theorem.

Theorem 1.1. *Let notation be as above. Let $\varepsilon > 0$. Let $T, M \in \mathbb{R}_+^{|S_\infty|}$ be such that $1 \leqslant T_v^\varepsilon \leqslant M_v \leqslant T_v^{1-\varepsilon}$ for every $v \mid \infty$. Set $\mathrm{N}(T) = \prod_{v|\infty} T_v^{N_v}$ and $\mathrm{N}^\natural(M) = \prod_{v|\infty} M_v$. Assume that $T_v \geqslant \mathrm{N}(T)^\varepsilon$ for every $v \mid \infty$. Define $\mathcal{B}_{T,M}$ to be the collection of $f \in \mathcal{B}$ satisfying $|\nu_{f,v} - T_v| \leqslant M_v$ for all $v \mid \infty$, and define $\mathbb{R}_{T,M}$ to be the intersection of the intervals $[T_v - M_v, T_v + M_v]$ for all $v \mid \infty$. Then*

$$\sum_{f \in \mathcal{B}_{T,M}} L\big(\tfrac{1}{2}, \pi \otimes f\big) + \int_{\mathbb{R}_{T,M}} \big|L\big(\tfrac{1}{2} + it, \pi\big)\big|^2 \,\mathrm{d}t \ll_{\varepsilon,\pi,F} \mathrm{N}^\natural(M)\mathrm{N}(T)^{5/4+\varepsilon}, \tag{1.5}$$

with the implied constant depending only on ε, π, and F.

Remark 1.2. The assumption $T_v \geqslant \mathrm{N}(T)^\varepsilon$ $(v \mid \infty)$ is used to make the contribution from exceptional forms negligible and to address some issues with the infinitude of units.

For $f \in \mathcal{B}$, define its Archimedean conductor $\mathrm{C}_\infty(f) = \mathrm{C}(\nu_f)^2$ by

$$\mathrm{C}_\infty(f) = \mathrm{N}(1 + |\nu_f|)^2 = \prod_{v|\infty} (1 + |\nu_{f,v}|)^{2N_v}.$$

Since π is self-dual, by the non-negativity theorem of Lapid [29], we have

$$L\big(\tfrac{1}{2}, \pi \otimes f\big) \geqslant 0. \tag{1.6}$$

As a consequence of (1.6), we derive from (1.5) the following subconvexity bounds by taking $M = T^\varepsilon$.

Corollary 1.3. *Let notation be as above. We have*

$$L\big(\tfrac{1}{2}+it,\pi\big) \ll_{\varepsilon,\pi,F} (1+|t|^N)^{5/8+\varepsilon}, \tag{1.7}$$

and

$$L\big(\tfrac{1}{2},\pi\otimes f\big) \ll_{\varepsilon,\pi,F} \mathrm{C}_\infty(f)^{5/8+\varepsilon} \tag{1.8}$$

if $|\nu_{f,v}| \geqslant \mathrm{C}_\infty(f)^\varepsilon$ *for all* $v\,|\,\infty$.

1.2. Subconvexity for GL_2

With the Archimedean analysis of this paper, following Young, one should be able to establish the Weyl-type bound (or even the hybrid Weyl-type bound)

$$L\big(\tfrac{1}{2},f\big) \ll_{\varepsilon,F} \mathrm{C}_\infty(f)^{1/6+\varepsilon}. \tag{1.9}$$

For $F=\mathbb{Q}$, this is a result of Ivić [21]. For arbitrary F, Han Wu [67] has a uniform subconvexity bound with a weaker exponent.

Let ϕ be a fixed spherical Hecke–Maass cusp form for PGL_2 over F. By combining (1.8) and (1.9), with $\pi=\mathrm{Sym}^2\phi$, we obtain

$$L\big(\tfrac{1}{2},\phi\otimes\phi\otimes f\big) \ll_{\varepsilon,\phi,F} \mathrm{C}_\infty(f)^{19/24+\varepsilon}. \tag{1.10}$$

For comparison, when $F=\mathbb{Q}$, Bernstein and Reznikov [4] proved

$$L\big(\tfrac{1}{2},\phi\otimes\phi'\otimes f\big) \ll_{\varepsilon,\phi,\phi'} |t_f|^{5/3+\varepsilon}. \tag{1.11}$$

1.3. Remarks on hybrid subconvexity

Let χ be a quadratic Hecke character for F of prime conductor $\mathfrak{q}$. In light of the works of Young [68] and Blomer [5] in the case $F=\mathbb{Q}$, the following hybrid subconvexity bounds should hold:

$$\begin{aligned} L\big(\tfrac{1}{2}+it,\pi\otimes\chi\big) &\ll_{\varepsilon,\pi,F} (\mathrm{N}(\mathfrak{q})(1+|t|^N))^{5/8+\varepsilon}, \\ L\big(\tfrac{1}{2},\pi\otimes f\otimes\chi\big) &\ll_{\varepsilon,\pi,F} \big(\mathrm{N}(\mathfrak{q})^2(1+C_\infty(f))\big)^{5/8+\varepsilon}, \end{aligned} \tag{1.12}$$

at least when $|t|, C_\infty(f) \geqslant \mathrm{N}(\mathfrak{q})^\varepsilon$. This is believed by Nunes [47] for $F=\mathbb{Q}$, and has been confirmed by the author for $F=\mathbb{Q}$ or $\mathbb{Q}(i)$ (unpublished). It seems that (1.12) can be verified whenever the class number h_F is 1, though the group of units might cause some trouble. In general, one has to compute certain non-Archimedean local integrals which should be transformed eventually into the character sums studied by Conrey and Iwaniec. The transformation could be quite intricate in view of Blomer's computations in [5, §6]. Recently, Nelson [46] generalized Conrey–Iwaniec to general number fields, and his work might provide some hint for this problem.

1.4. Features of analysis over complex numbers

The main difficulty in the analysis over $\mathbb{C}$ is the lack of suitable stationary phase results in two dimensions. Considerably more efforts are needed particularly for the Hankel transform and the Mellin transform. We would rather not discuss the technical details—it is more worthwhile and interesting to present here the features of certain trigonometric-hyperbolic functions arising in the phases.

To start with, the function $\mathrm{trh}(r,\omega)=\rho(r,\omega)e^{i\theta(r,\omega)}$ that occurs in the GL_2 Bessel integral on $\mathbb{C}$ is given by

$$\rho(r,\omega)=\sqrt{\frac{\cosh 2r+\cos 2\omega}{2}},\quad \tan\theta(r,\omega)=\tanh r\tan\omega.$$

Since $\mathrm{trh}(r,0)=\cosh r$ and $\mathrm{trh}(r,\pi/2)=i\sinh r$, one expects that the r-integral behaves like the Bessel integral on $\mathbb{R}_+$ or $\mathbb{R}_-$ when $\sin\omega$ or $\cos\omega$ is small, respectively.

After performing the GL_3 Hankel transform, we obtain a new function $\mathrm{trh}^\natural(r,\omega)=\rho^\natural(r,\omega)e^{i\theta^\natural(r,\omega)}$ defined by

$$\rho^\natural(r,\omega)=\frac{\cosh 2r-\cos 2\omega}{\cosh 2r+\cos 2\omega},\quad \tan(\theta^\natural(r,\omega)/2)=\frac{\sin 2\omega}{\sinh 2r}.$$

It is certainly a pleasure to see the square-root sign gone. More important is the symmetry in $\mathrm{trh}^\natural(r,\omega)$ reflected by the identities

$$\frac{\partial^2\log\rho^\natural}{\partial r^2}=-\frac{\partial^2\log\rho^\natural}{\partial\omega^2}=\frac{\partial^2\theta^\natural}{\partial r\partial\omega},\quad \frac{\partial^2\theta^\natural}{\partial r^2}=-\frac{\partial^2\theta^\natural}{\partial\omega^2}=-\frac{\partial^2\log\rho^\natural}{\partial r\partial\omega},$$

which play a critical role in our analysis after applying the Mellin technique.

Finally, we remark that, at the stage of the Mellin technique, new phenomena emerge in our analysis for $\cos 2\omega$ in the vicinity of 0 (for $|\sin\omega|$ and $|\cos\omega|$ nearly equal).

Notation

By writing $X\ll Y$ or $X=O(Y)$ we mean that $|X|\leqslant cY$ for some constant $c>0$, and by $X\asymp Y$ we mean that $X\ll Y$ and $Y\ll X$. We write $X\ll_{P,Q,\dots}Y$ or $X=O_{P,Q,\dots}(Y)$ if the implied constant c depends on $P,Q,\dots$. Throughout this article $\mathrm{N}(T)\gg 1$ and each $T_v\geqslant\mathrm{N}(T)^\varepsilon\gg 1$ will be large, and we say that X is *negligibly small* if $X=O_A(\mathrm{N}(T)^{-A})$ (or $O_A(T_v^{-A})$) for arbitrarily large but fixed $A>0$.

We adopt the usual ε-convention of analytic number theory: the value of ε may differ from one occurrence to another.

Part I. Number-theoretic preliminaries

2. Notation over number fields

Basic notions

Let F be a number field of degree N. Let $\mathcal{O}$ be its ring of integers and $\mathcal{O}^\times$ be the group of units. Let $\mathfrak{D}$ be the different ideal of F. Let N and Tr denote the norm and the trace for F, respectively. Let d_F be the discriminant of F. Denote by $\mathbb{A}$ the adele ring of F and by $\mathbb{A}^\times$ the idele group of F.

For any place v of F, we denote by F_v the corresponding local field. When v is non-Archimedean, let $\mathfrak{p}_v$ be the corresponding prime ideal of $\mathfrak{O}$ and let ord_v or v itself denote the additive valuation; occasionally, $\mathfrak{p}_v$ also stands for the prime ideal in $\mathfrak{O}_v$. Denote by $\mathfrak{D}_v$ the local different ideal. Let N_v be the local degree of F_v; in particular, $N_v = 1$ if $F_v = \mathbb{R}$ and $N_v = 2$ if $F_v = \mathbb{C}$. Let $|\ |_v$ denote the normalized absolute value on F_v. We have $|\ |_v = |\ |$ if $F_v = \mathbb{R}$ and $|\ |_v = |\ |^2$ if $F_v = \mathbb{C}$, where $|\ |$ is the usual absolute value. Let r_1 and r_2 be the number of real and complex places of F, respectively.

Let S_∞ and S_f denote the sets of Archimedean and non-Archimedean places of F, respectively. Write $v\,|\,\infty$ and $v \nmid \infty$ as an abbreviation for $v \in S_\infty$ and $v \in S_f$, respectively. For a finite set S of places, denote by $\mathbb{A}^S$, respectively F_S, the subring of adeles with trivial component above S, respectively above the complement of S. The absolute values on $\mathbb{A}^{\times S}$ and F_S will be denoted by $|\ |^S$ and $|\ |_S$ respectively. For brevity, write $\mathbb{A}_f = \mathbb{A}^{S_\infty}$ and $F_\infty = F_{S_\infty}$.

Additive characters and Haar measures

Fix the (non-trivial) standard additive character $\psi = \otimes_v \psi_v$ on $\mathbb{A}/F$ as in [28, §XIV.1] such that $\psi_v(x) = e(-x)$ if $F_v = \mathbb{R}$, $\psi_v(z) = e(-(z+\overline{z}))$ if $F_v = \mathbb{C}$, and ψ_v has conductor $\mathfrak{D}_v^{-1}$ for any non-Archimedean F_v. For a finite set S of places, denote $\psi^S = \prod_{v\notin S} \psi_v$ and $\psi_S = \prod_{v\in S} \psi_v$. We split $\psi = \psi_\infty \psi_f$ so that $\psi_\infty(x) = e(-\mathrm{Tr}_{F_\infty}(x))$ $(x \in F_\infty)$.

We choose the Haar measure $\mathrm{d}x$ of F_v self-dual with respect to ψ_v as in [28, §XIV.1]; the Haar measure is the ordinary Lebesgue measure on the real line if $F_v = \mathbb{R}$, twice the ordinary Lebesgue measure on the complex plane if $F_v = \mathbb{C}$, and the measure for which $\mathfrak{O}_v$ has measure $\mathrm{N}(\mathfrak{D}_v)^{-1/2}$ if F_v is non-Archimedean. We slightly modify the Haar measure $\mathrm{d}^\times x$ of $F_v^\times$ defined in [28, §XIV.2]: $\mathrm{d}^\times x = \mathrm{d}x/|x|_v$ if $v\,|\,\infty$, and $\mathrm{d}^\times x = \mathrm{N}(\mathfrak{D}_v)^{1/2}\mathrm{N}(\mathfrak{p}_v)/(\mathrm{N}(\mathfrak{p}_v)-1)\cdot \mathrm{d}x/|x|_v$ if $v \nmid \infty$, so that $\mathfrak{O}_v^\times$ has mass 1 (it is $\mathrm{N}(\mathfrak{D}_v)^{-1/2}$ in [28]).

We remark that the absolute value, measure and additive character on $F_\infty = \prod_{v|\infty} F_v$ are chosen differently in [9]. For example, they use $|\ |$ for both real and complex v, their additive measure differs from ours by a factor of $1/\pi$ or $1/2\pi$ if v is real or complex, respectively, and their additive character is $e(\mathrm{Tr}_{F_\infty}(x))$ instead of $e(-\mathrm{Tr}_{F_\infty}(x))$.

Ideals

In general, we use Gothic letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ to denote *non-zero* fractional ideals of F, while we reserve $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{d}$, $\mathfrak{f}$, $\mathfrak{q}$, and $\mathfrak{r}$ for non-zero integral ideals of F. Let $\mathfrak{p}$ always stand for a prime ideal. Let $\mathrm{N}(\mathfrak{a})$ denote the norm of $\mathfrak{a}$. If $\alpha \in F^\times$, we denote by (α) the corresponding principal ideal. If $\mathfrak{a}$ is a fractional ideal, we shall often write just $\alpha\mathfrak{a}$ for the product $(\alpha)\mathfrak{a}$.

Given $\theta \in \mathbb{R}_+^{|S_\infty|}$ with $\sum_{v|\infty} N_v\theta_v = 1$, for each principal fractional ideal $\mathfrak{a}$ we choose once and for all a generator, which we denote $[\mathfrak{a}]$, so that

$$|[\mathfrak{a}]|_v \asymp \mathrm{N}(\mathfrak{a})^{N_v\theta_v}, \quad v\,|\,\infty; \tag{2.1}$$

such a choice is guaranteed by Dirichlet's units theorem (see [28, §V.1]). Later, we shall set $\theta_v = \log T_v / \log \mathrm{N}(T)$.

For each non-zero fractional ideal $\mathfrak{a}$, we fix a corresponding $\pi_{\mathfrak{a}} \in \mathbb{A}_f^\times$ so that $\mathrm{ord}_v(\mathfrak{a}) = \mathrm{ord}_v(\pi_{\mathfrak{a},v})$ for all $v \nmid \infty$. Set $\delta = \pi_{\mathfrak{D}}$. For brevity, write $\mathfrak{a}_v = \mathfrak{a}\mathcal{O}_v$.

Let C_F be the class group and h_F be the class number of F. We shall write $\mathfrak{a} \sim \mathfrak{b}$ when $\mathfrak{a}$ and $\mathfrak{b}$ are in the same ideal class. We choose a set $\widetilde{C}_F$ of integral ideals that represent the class group.

Characters and Mellin transforms

For v Archimedean, define the (unitary) character $\chi_{i\nu,m}(x) = |x|_v^{i\nu}(x/|x|)^m$ $(x \in F_v^\times)$ for $\nu \in \mathbb{R}$ and $m \in \{0,1\}$ $(= \mathbb{Z}/2\mathbb{Z})$ if F is real, and $m \in \mathbb{Z}$ if F is complex. Let $\widehat{\mathfrak{a}}_v$ denote the unitary dual of $F_v^\times$. We shall identify $\widehat{\mathfrak{a}}_v$ with $\mathbb{R} \times \{0,1\}$ or $\mathbb{R} \times \mathbb{Z}$ according as v is real or complex. Let $\mathrm{d}\mu(\nu,m)$ denote the usual Lebesgue measure on $\mathbb{R} \times \{0,1\}$ or $\mathbb{R} \times \mathbb{Z}$. For notational simplicity, we shall write summation on $\{0,1\}$ or $\mathbb{Z}$ as integration.

For $f \in L^1(F_v^\times) \cap L^2(F_v^\times)$, define its (local) Mellin transform $\breve{f}(\nu,m)$ by

$$\breve{f}(\nu,m) = \int_{F_v^\times} f(x)\chi_{i\nu,m}(x)\,\mathrm{d}^\times x. \tag{2.2}$$

The Mellin inversion formula reads

$$f(x) = \frac{1}{2\pi c_v}\iint_{\widehat{\mathfrak{a}}_v} \breve{f}(\nu,m)\overline{\chi_{i\nu,m}(x)}\,\mathrm{d}\mu(\nu,m), \tag{2.3}$$

where $c_v = 2$ if v is real and $c_v = 2\pi$ if v is complex. Moreover, by Plancherel's theorem,

$$\int_{F_v^\times} |f(x)|^2\,\mathrm{d}^\times x = \frac{1}{2\pi c_v}\iint_{\widehat{\mathfrak{a}}_v} |\breve{f}(\nu,m)|^2\,\mathrm{d}\mu(\nu,m). \tag{2.4}$$

Let $\widehat{\mathfrak{a}} = \prod_{v|\infty} \widehat{\mathfrak{a}}_v$ be the unitary dual of $F_\infty^\times$. For $(\nu,m) \in \widehat{\mathfrak{a}}$, define $\chi_{i\nu,m}$ to be the product of $\chi_{i\nu_v,m_v}$. Let $\mathrm{d}\mu(\nu,m)$ be the Lebesgue measure on $\widehat{\mathfrak{a}}$. In an obvious way, formulae (2.2)–(2.4) extend to $F_\infty^\times$ and $\widehat{\mathfrak{a}}$.

3. Automorphic forms on GL_2

In this section, we briefly recollect some notation and preliminaries, mostly for the statement of the Kuznetsov formula of Bruggeman and Miatello for PGL_2. For simplicity, only spherical automorphic forms on $\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})$ are considered. The reader is referred to [64] for further discussions.

Define

$$N = \left\{\begin{pmatrix} 1 & r \\ & 1\end{pmatrix}\right\}, \quad A = \left\{\begin{pmatrix} x & \\ & 1\end{pmatrix}\right\}, \quad P = \left\{\begin{pmatrix} x & r \\ & y\end{pmatrix}\right\}.$$

We denote by $K = K_f K_\infty$ the standard maximal compact subgroup of $\mathrm{PGL}_2(\mathbb{A})$. For each non-Archimedean v, let $K_v = \mathrm{PGL}_2(\mathcal{O}_v)$. Note that $\mathrm{PGL}_2(F) \cap K_f = \mathrm{PGL}_2(\mathcal{O})$ (the intersection is taken in $\mathrm{GL}_2(\mathbb{A}_f)$). Let $K_v = \mathrm{O}_2(\mathbb{R})/\{\pm 1_2\}$ if v is real, and $K_v = \mathrm{U}_2(\mathbb{C})/\{\pm 1_2\}$ if v is complex.

We identify $N(F_\infty)$ with F_∞ and $A(F_\infty)$ with $F_\infty^\times$, and define their measures accordingly. For $v \mid \infty$, we normalize the Haar measure on K_v so that $K_v/A(F_v) \cap K_v$ has measure 1. Thus the measure of K_v is 2 or 2π according as v is real or complex. The Haar measure on $\mathrm{PGL}_2(F_\infty)$ is defined via the Iwasawa decomposition $\mathrm{PGL}_2(F_\infty) = N(F_\infty)A(F_\infty)K_\infty$. Again, our measure on the hyperbolic space $\mathrm{PGL}_2(F_v)/K_v$ is different from that in [9] or [64].

3.1. Archimedean representations

In this paper, we shall be concerned only with spherical representations of $\mathrm{PGL}_2(F_\infty)$.

Let $\mathfrak{a}$ be the vector space $\mathbb{R}^{r_1+r_2}$. We usually identify $\mathbb{R}$ with its image under the diagonal embedding $\mathbb{R} \hookrightarrow \mathfrak{a}$. Let $\mathfrak{a}_\mathbb{C}$ be the complexification of $\mathfrak{a}$. Let $Y \subset \mathfrak{a}_\mathbb{C}$ be the set of $\nu = (\nu_v)_{v|\infty}$ such that $\nu_v \in \mathbb{R}$ or $i\nu_v \in (-\frac{1}{2}, \frac{1}{2})$. We associate to ν in Y a unique spherical unitary irreducible representation $\pi(i\nu)$ of $\mathrm{PGL}_2(F_\infty)$. Namely, ν determines a character of the diagonal torus $A(F_\infty)$ via

$$\begin{pmatrix} x & \\ & 1 \end{pmatrix} \mapsto \prod_{v|\infty} |x|_v^{i\nu_v}, \quad x \in F_\infty^\times,$$

and we let $\pi(i\nu)$ be the irreducible spherical constituent of the representation unitarily induced from this character. The spherical $\pi(i\nu)$ is tempered if and only if $\nu \in \mathfrak{a}$. Let $\mathrm{d}\nu$ be the usual Lebesgue measure on $\mathfrak{a}$. We equip $\mathfrak{a}$ with the Plancherel measure $\mathrm{d}\mu(\nu)$ defined as the product of

$$\mathrm{d}\mu(\nu_v) = \begin{cases} \nu_v \tanh(\pi\nu_v)\, \mathrm{d}\nu_v & \text{if } v \text{ is real}, \\ \nu_v^2 \mathrm{d}\nu_v & \text{if } v \text{ is complex}. \end{cases} \tag{3.1}$$

Moreover, we define the function $\mathrm{Pl}(\nu)$ to be the product of

$$\mathrm{Pl}_v(\nu_v) = \begin{cases} \cosh(\pi\nu_v) & \text{if } v \text{ is real}, \\ \sinh(2\pi\nu_v)/\nu_v & \text{if } v \text{ is complex}. \end{cases} \tag{3.2}$$

Note that $2^{r_2-r_1} \cdot \mathrm{Pl}(\nu)\, \mathrm{d}\mu(\nu)$ is the measure used in [9, 64].

We must fix, once and for all, a spherical Whittaker vector corresponding to each $\pi(i\nu)$, with respect to the character ψ_∞ on $N(F_\infty)$. We choose $W_{i\nu}$ to be the product of $W_{i\nu_v}$ with

$$W_{i\nu_v}\begin{pmatrix} x_v & \\ & 1 \end{pmatrix} = \begin{cases} |x_v|_v^{1/2} K_{i\nu_v}(2\pi|x_v|) & \text{if } v \text{ is real}, \\ |x_v|_v^{1/2} K_{2i\nu_v}(4\pi|x_v|) & \text{if } v \text{ is complex}. \end{cases} \tag{3.3}$$

Finally, for $V \in \mathfrak{a}_+ = \mathbb{R}_+^{r_1+r_2}$, we define

$$\mathrm{N}(V) = \prod_{v|\infty} V_v^{N_v}, \quad \mathrm{N}^\natural(V) = \prod_{v|\infty} V_v. \tag{3.4}$$

We define the conductor of $\nu \in Y$ to be $\mathrm{C}(\nu) = \mathrm{N}(1 + |\nu|)$, that is,

$$\mathrm{C}(\nu) = \prod_{v|\infty} (1 + |\nu_v|)^{N_v}. \tag{3.5}$$

3.2. Automorphic forms

We shall be interested in the space of spherical automorphic forms, that is, functions in $L^2(\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A}))$ that are (right) invariant under K.

We fix an orthonormal basis $\mathscr{B}$ for the cuspidal subspace that consists of eigenforms for the Hecke algebra as well as the Laplacian operators (Hecke–Maass cusp forms). Each $f \in \mathscr{B}$ transforms under a certain representation $\pi(i\nu_f)$ of $\mathrm{PGL}_2(F_\infty)$, for some $\nu_f \in Y$. We have the Kim–Sarnak bound [6] over the field F:

$$|\mathrm{Im}(\nu_{f,v})| \leqslant \tfrac{7}{64}, \quad v\,|\,\infty. \tag{3.6}$$

The Fourier coefficients $a_f(\mathfrak{a}, \alpha)$ are indexed by pairs consisting of an ideal class and an element of F. To be precise, $a_f(\mathfrak{a}, \alpha)$ are defined so that

$$f\left(\begin{pmatrix}\pi_{\mathfrak{a}} & \\ & 1\end{pmatrix} g_\infty\right) = \sum_{\alpha\in F^\times} \frac{a_f(\mathfrak{a},\alpha)}{\sqrt{\mathrm{N}(\alpha\mathfrak{a}\mathfrak{D})}} W_{i\nu_f}\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} g_\infty\right), \quad g_\infty \in \mathrm{GL}_2(F_\infty), \tag{3.7}$$

where $\pi_{\mathfrak{a}} \in \mathbb{A}_f^\times$ is a representative of $\mathfrak{a}$. It should be kept in mind that $a_f(\mathfrak{a}, \alpha)$ vanishes unless $\alpha \in \mathfrak{a}^{-1}\mathfrak{D}^{-1}$ and only depends on the ideal $\alpha\mathfrak{a}$. We therefore set $a_f(\mathfrak{m}) = a_f(\mathfrak{a}, \alpha)$ if $\mathfrak{m} = \alpha\mathfrak{a}\mathfrak{D}$. These $a_f(\mathfrak{m})$ may be interpreted in terms of the non-Archimedean spherical Whittaker function with respect to the additive character ψ_f on $N(\mathbb{A}_f)$. We denote by $\lambda_f(\mathfrak{m})$ the $\mathfrak{m}$-th Hecke eigenvalue of f. We normalize in such a way that the Ramanujan conjecture corresponds to $|\lambda_f(\mathfrak{p})| \leqslant 2$. As usual, there is a constant C_f such that

$$a_f(\mathfrak{m}) = C_f \lambda_f(\mathfrak{m}) \tag{3.8}$$

for any non-zero integral ideal $\mathfrak{m}$. See [64, §2.5] for more details.

For $s \in \mathbb{C}$, define via the Iwasawa decomposition the function

$$f_s\left(\begin{pmatrix}x & r \\ & y\end{pmatrix} k\right) = |x/y|^{s+1/2}, \quad x, y \in \mathbb{A}^\times,\ r \in \mathbb{A},\ k \in K,$$

and define the Eisenstein series $E(g; s)$ by

$$E(g;s) = \sum_{\gamma\in P(F)\backslash\mathrm{GL}_2(F)} f_s(\gamma g), \quad g \in \mathrm{GL}_2(\mathbb{A}),$$

for $\mathrm{Re}(s) > \frac{1}{2}$ and by a process of meromorphic continuation in general. For our purpose, we only need the knowledge of its Fourier coefficients $a_{E(s)}(\mathfrak{m}) = a_{E(s)}(\mathfrak{a}, \alpha)$ ($\mathfrak{m} = \alpha\mathfrak{a}\mathfrak{D}$) for $\mathfrak{m}$ non-zero. More precisely, we have

$$a_{E(s)}(\mathfrak{m}) = C_{E(s)} \tau_s(\mathfrak{m}), \tag{3.9}$$

where

$$\tau_s(\mathfrak{m}) = \sum_{\mathfrak{b}|\mathfrak{m}} \mathrm{N}(\mathfrak{m}\mathfrak{b}^{-2})^s, \quad C_{E(s)} = \frac{P(s)}{\zeta_F(2s+1)\mathrm{N}(\mathfrak{D})^s}, \tag{3.10}$$

$\zeta_F(s)$ is the Dedekind ζ function for F, and $P(s)$ is the product of

$$P_v(s) = \begin{cases} 2\pi^{s+1/2}/\Gamma\big(s+\frac{1}{2}\big) & \text{if } v \text{ is real,} \\ 2(2\pi)^{2s+1}/\Gamma(2s+1) & \text{if } v \text{ is complex.} \end{cases} \tag{3.11}$$

See for example [11, §§3.7, 4.6]. A subtle issue is that the results in [11, §4.6] are proven for ψ_v of conductor $\mathfrak{O}_v$, but this may be easily addressed by re-scaling the character $\psi_v(x_v)$ and the Haar measure $\mathrm{d}x_v$ (say, $\psi_v(x_v) = \psi_v^\natural(\delta_v x_v)$ and $\mathrm{d}x_v = \sqrt{|\delta_v|_v}\,\mathrm{d}^\natural x_v$). Moreover, the $P_v(s)$ in (3.11) arises from a computation of Jacquet's integral. Note that the definition of P may be extended from $\mathbb{C}$ to $\mathfrak{a}_\mathbb{C} = \mathbb{C}^{r_1+r_2}$ and that

$$|P(i\nu)|^2 = 2^{2r_1+3r_2}\pi^{r_2}\cdot \mathrm{Pl}(\nu), \quad \nu \in \mathfrak{a}. \tag{3.12}$$

3.3. *Kuznetsov–Bruggeman–Miatello formula*

We first make a preliminary definition.

Definition 3.1. Let $\mathfrak{b}$, $\mathfrak{q}$ be fractional ideals with $\mathfrak{b} \mid \mathfrak{q}$. We set $(\mathfrak{b}/\mathfrak{q})^\times$ to be those elements $x \in \mathfrak{b}/\mathfrak{q}$ which generate $\mathfrak{b}/\mathfrak{q}$ as an $\mathfrak{O}$-module. For $x \in (\mathfrak{b}/\mathfrak{q})^\times$ define x^{-1} to be the unique class $y \in (\mathfrak{b}^{-1}/\mathfrak{q}\mathfrak{b}^{-2})^\times$ such that $xy \in 1 + \mathfrak{q}\mathfrak{b}^{-1}$.

We now define the Kloosterman sum. Here, we must include ideal classes as parameters.

Definition 3.2 (Kloosterman sum). Let $\mathfrak{a}_1, \mathfrak{a}_2$ be non-zero fractional ideals of F, and $\mathfrak{c}$ be any ideal such that $\mathfrak{c}^2 \sim \mathfrak{a}_1\mathfrak{a}_2$. Let $c \in \mathfrak{c}^{-1}$, $\alpha_1 \in \mathfrak{a}_1^{-1}\mathfrak{D}^{-1}$, and $\alpha_2 \in \mathfrak{a}_1\mathfrak{c}^{-2}\mathfrak{D}^{-1}$. We define the *Kloosterman sum*

$$\mathrm{KS}(\alpha_1, \mathfrak{a}_1; \alpha_2, \mathfrak{a}_2; c, \mathfrak{c}) = \sum_{x \in (\mathfrak{a}_1\mathfrak{c}^{-1}/\mathfrak{a}_1(c))^\times} \psi_\infty\left(\frac{\alpha_1 x + \alpha_2 x^{-1}}{c}\right), \tag{3.13}$$

where $(\mathfrak{a}_1\mathfrak{c}^{-1}/\mathfrak{a}_1(c))^\times$ and $x^{-1} \in (\mathfrak{a}_1^{-1}\mathfrak{c}/\mathfrak{a}_1^{-1}(c)\mathfrak{c}^2)^\times$ are defined as in Definition 3.1.

We should view the ideals $\alpha_1\mathfrak{a}_1$ and $\alpha_2\mathfrak{a}_2$ as the parameters of this Kloosterman sum, and the ideal $c\mathfrak{c}$ as the modulus. However, KS does depend on the choice of generator; it is not invariant under the substitution $\alpha \to \epsilon\alpha$ if $\epsilon \in \mathfrak{O}^\times$ is a unit. To relate the definition to the usual Kloosterman sum, we note if $\mathfrak{a}_1 = \mathfrak{a}_2 = \mathfrak{c} = \mathfrak{O}$, then for $\alpha_1, \alpha_2 \in \mathfrak{D}^{-1}$ and $c \in \mathfrak{O}$ we have

$$\mathrm{KS}(\alpha_1, \mathfrak{a}_1; \alpha_2, \mathfrak{a}_2; c, \mathfrak{c}) = \sum_{x \in (\mathfrak{O}/c)^\times} \psi_\infty\left(\frac{\alpha_1 x + \alpha_2 x^{-1}}{c}\right).$$

We have the Weil bound for Kloosterman sums:

$$\mathrm{KS}(\alpha_1, \mathfrak{a}_1; \alpha_2, \mathfrak{a}_2; c, \mathfrak{c}) \ll \mathrm{N}((\alpha_1\mathfrak{a}_1\mathfrak{D}, \alpha_2\mathfrak{a}_1^{-1}\mathfrak{c}\mathfrak{D}, c\mathfrak{c}))^{1/2}\mathrm{N}(c\mathfrak{c})^{1/2+\varepsilon}, \tag{3.14}$$

where $(\cdot,\cdot,\cdot)$ denotes greatest common divisor (of ideals).

Definition 3.3 (Space of test functions). Let $S > \frac{1}{2}$.[1] We set $\mathscr{H}(S)$ to be the space of functions $h : \mathfrak{a} \to \mathbb{C}$ of the form $h(\nu) = \prod_{v|\infty} h_v(\nu_v)$, where each $h_v : \mathbb{R} \to \mathbb{C}$ extends to an even holomorphic function on the strip $\{s : |\mathrm{Im}(s)| \leqslant S\}$ such that, on the horizontal line $\mathrm{Im}(s) = \sigma$ ($|\sigma| \leqslant S$), we have uniformly

$$h_v(t + i\sigma) \ll e^{-\pi|t|}(|t| + 1)^{-N} \quad \text{for some } N > 6.$$

Next, we define the Bessel kernel.

Definition 3.4 (Bessel kernel). Let $\nu \in \mathfrak{a}_{\mathbb{C}}$.

(1) When $F_v = \mathbb{R}$, for $x \in \mathbb{R}_+$ we define

$$\begin{aligned} B_{\nu_v}(x) &= \frac{\pi}{\sin(\pi \nu_v)}\big(J_{-2\nu_v}(4\pi\sqrt{x}) - J_{2\nu_v}(4\pi\sqrt{x})\big), \\ B_{\nu_v}(-x) &= \frac{\pi}{\sin(\pi \nu_v)}\big(I_{-2\nu_v}(4\pi\sqrt{x}) - I_{2\nu_v}(4\pi\sqrt{x})\big) \\ &= 4\cos(\pi\nu_v)K_{2\nu_v}(4\pi\sqrt{x}). \end{aligned}$$

(2) When $F_v = \mathbb{C}$, for $z \in \mathbb{C}^\times$ we define

$$B_{\nu_v}(z) = \frac{2\pi^2}{\sin(2\pi\nu_v)}\big(J_{-2\nu_v}(4\pi\sqrt{z})J_{-2\nu_v}(4\pi\sqrt{\bar{z}}) - J_{2\nu_v}(4\pi\sqrt{z})J_{2\nu_v}(4\pi\sqrt{\bar{z}})\big).$$

For $x \in F_\infty^\times$, we define

$$\mathcal{B}_\nu(x) = \prod_{v|\infty} B_{\nu_v}(x_v).$$

Note that the I-Bessel functions in [9] or [64] should be changed to J-Bessel functions in the complex case. Moreover, according to [57], we have normalized the Bessel kernel by a factor of π or $2\pi^2$ for real or complex v, respectively.

Proposition 3.5 (Kuznetsov formula). *Let h be a test function on $\mathfrak{a}_{\mathbb{C}}$ belonging to $\mathscr{H}(S)$ (see Definition 3.3) and define*

$$\mathcal{H} = \int_{\mathfrak{a}} h(\nu)\,\mathrm{d}\mu(\nu), \quad \mathcal{H}(x) = \int_{\mathfrak{a}} h(\nu)\mathcal{B}_{i\nu}(x)\,\mathrm{d}\mu(\nu), \quad x \in F_\infty^\times, \tag{3.15}$$

where $\mathcal{B}_\nu(x)$ is the Bessel kernel defined in Definition 3.4 and $\mathrm{d}\mu(\nu)$ is the Plancherel measure in (3.1). Let $\mathcal{B}$ be an orthonormal basis of the spherical cuspidal spectrum on $\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})/K$, so that each $f \in \mathcal{B}$ has Archimedean parameter $\nu_f \in Y$ (f transforms under the $\mathrm{GL}_2(F_\infty)$-representation $\pi(i\nu_f)$) and Hecke eigenvalues $\lambda_f(\mathfrak{m})$. Let $\mathfrak{a}_1, \mathfrak{a}_2$ be fractional ideals. Let $\alpha_1 \in \mathfrak{a}_1^{-1}\mathfrak{D}^{-1}$, $\alpha_2 \in \mathfrak{a}_2^{-1}\mathfrak{D}^{-1}$. Set $\mathfrak{m}_1 = \alpha_1\mathfrak{a}_1\mathfrak{D}$, $\mathfrak{m}_2 = \alpha_2\mathfrak{a}_2\mathfrak{D}$, and

$$\beta = \beta_{\mathfrak{c},\mathfrak{a}_1\mathfrak{a}_2} = [\mathfrak{c}^2(\mathfrak{a}_1\mathfrak{a}_2)^{-1}] \tag{3.16}$$

[1] The condition $S > \frac{1}{2}$ is borrowed from [9], while [64] requires $S > 2$ for some convergence issues. Note that the space of test functions in [10, 27, 34] is much larger.

for every $\mathfrak{c} \in \widetilde{C}_F$ *with* $\mathfrak{c}^2 \sim \mathfrak{a}_1\mathfrak{a}_2$ *(here* $[\mathfrak{c}^2(\mathfrak{a}_1\mathfrak{a}_2)^{-1}]$ *is a chosen generator for this principal ideal). We have*

$$\sum_{f \in \mathcal{B}} \omega_f h(\nu_f)\lambda_f(\mathfrak{m}_1)\lambda_f(\mathfrak{m}_2) + \frac{1}{4\pi} c_0 \int_{-\infty}^{\infty} \omega(t) h(t) \tau_{it}(\mathfrak{m}_1)\tau_{it}(\mathfrak{m}_2) dt$$
$$= c_1 \delta_{\mathfrak{m}_1,\mathfrak{m}_2} \mathcal{H} + c_2 \sum_{\substack{\mathfrak{c} \in \widetilde{C}_F \\ \mathfrak{c}^2 \sim \mathfrak{a}_1\mathfrak{a}_2}} \sum_{\epsilon \in \mathcal{O}^\times/\mathcal{O}^{\times 2}} \sum_{c \in \mathfrak{c}^{-1}} \frac{\mathrm{KS}(\alpha_1, \mathfrak{a}_1; \epsilon\alpha_2/\beta, \mathfrak{a}_2; c, \mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} \mathcal{H}\left(\frac{\epsilon\alpha_1\alpha_2}{\beta c^2}\right), \tag{3.17}$$

where

$$\omega_f = \frac{|C_f|^2}{\mathrm{Pl}(\nu_f)}, \quad \omega(t) = \frac{2^{2r_1+3r_2}\pi^{r_2}}{|\zeta_F(1+2it)|^2} \tag{3.18}$$

(see (3.2) *and* (3.8) *for the definitions of* $\mathrm{Pl}(\nu_f)$ *and* C_f*),* $\tau_s(\mathfrak{m})$ *is defined in* (3.10)*,* $\delta_{\mathfrak{m}_1,\mathfrak{m}_2}$ *is the Kronecker* δ *that detects* $\mathfrak{m}_1 = \mathfrak{m}_2$*,* KS *is the Kloosterman sum as in Definition* 3.2*, and the constants* c_0*,* c_1 *and* c_2 *are given by*

$$c_0 = \frac{2^{r_1}(2\pi)^{r_2} R_F}{w_F\sqrt{|d_F|}}, \quad c_1 = \frac{2^{r_2}\sqrt{|d_F|}}{2\pi^{2r_1+2r_2}h_F}, \quad c_2 = \frac{2^{r_2}}{4\pi^{2r_1+2r_2}h_F}, \tag{3.19}$$

in which w_F*,* d_F*,* R_F *and* h_F *are the number of roots of unity in* F*, the discriminant, the regulator and the class number of* F*, respectively.*

Formula (3.17) is just a rewriting of (15) in [64] in the fashion of [12, (3.17)]. Some remarks are in order. The test function in [64] has been modified here by $\mathrm{Pl}(\nu)$ (see (3.2) and (3.12)). The $1/4\pi$ in the first line of (3.17) is adopted from [23, (7.15)], and in the adelic setting it also arises from applying [15, (5.16)] to [15, (4.21), (4.25)] . The constant c_0 accounts for the translation of the spectral decomposition from the adelic to the classical setting. The constants c_1 and c_2 are adapted from [64, (16)], with extra factors due to our normalization of measures, absolute values and Bessel kernels.

Lemma 3.6. *Let* f *be a Hecke–Maass cusp form whose* L^2 *norm is* 1*. Let* ω_f *and* $\omega(t)$ *be defined as in* (3.18)*. Then*

$$\omega_f \gg \mathrm{C}(\nu_f)^{-\varepsilon}, \quad \omega(t) \gg (1+|t|)^{-\varepsilon}, \tag{3.20}$$

where $\mathrm{C}(\nu_f)$ *is the conductor of* ν_f *as defined in* (3.5)*, and the implied constants depend only on* ε *and* F*.*

By the Rankin–Selberg method, ω_f is a multiple of $1/L(1, \mathrm{Sym}^2 f)$:

$$\omega_f = \frac{2^{2r_1+3r_2}\pi^{r_2}}{2L(1, \mathrm{Sym}^2 f)}. \tag{3.21}$$

Then (3.20) follows from [41, Theorem 1]. For $F = \mathbb{Q}$, the lower bound for ω_f was first proven by Iwaniec [22].

4. Voronoï summation for GL_3

The purpose of this section is to derive a Voronoï summation formula for GL_3 in the classical terms from the formula of Ichino and Templier [20] in the adelic setting (see Appendix A).

Notation in the adelic setting

We first recollect some notation from [20]. Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A})$. Let $\widetilde{\pi} = \otimes_v \widetilde{\pi}_v$ be the contragradient representation of π. Let ω denote the central character of π.

Let S be a finite set of places of F including the ramified places of π and all the Archimedean places. Denote by $W_{\mathrm{o}}^S = \prod_{v \notin S} W_{\mathrm{o}v}$ the normalized unramified Whittaker function of $\pi^S = \otimes_{v\notin S} \pi_v$ above the complement of S. Let $\widetilde{W}_{\mathrm{o}}^S = \prod_{v\notin S} \widetilde{W}_{\mathrm{o}v}$ be the unramified (spherical) Whittaker function of $\widetilde{\pi}^S = \otimes_{v\notin S} \widetilde{\pi}_v$.

For any place v of F, to a smooth compactly supported function $w_v \in C_c^\infty(F_v^\times)$ is associated a dual function $\widetilde{w}_v$ of w_v such that

$$\int_{F_v^\times} \widetilde{w}_v(x)\chi(x)^{-1}|x|_v^{s-1}\,\mathrm{d}^\times x = \gamma(1-s, \pi_v \otimes \chi, \psi_v) \int_{F_v^\times} w_v(x)\chi(x)|x|_v^{-s}\,\mathrm{d}^\times x \tag{4.1}$$

for all s of real part sufficiently large and all unitary multiplicative characters χ of $F_v^\times$. The equality (4.1) is independent of the chosen Haar measure $\mathrm{d}^\times x$ on $F_v^\times$ and defines $\widetilde{w}_v$ uniquely in terms of π_v, ψ_v and w_v. For S as above, we put $w_S = \prod_{v\in S} w_v$, $\widetilde{w}_S = \prod_{v\in S} \widetilde{w}_v$.

Let v be an unramified place of π. It should be kept in mind that, since the additive character ψ_v has conductor $\mathfrak{D}_v^{-1}$, $W_{\mathrm{o}v}(a(x_1, x_2))$ vanishes unless both x_1, x_2 are in $\mathfrak{D}_v^{-1}$, where

$$a(x_1, x_2) = \begin{pmatrix} x_1x_2 & & \\ & x_1 & \\ & & 1 \end{pmatrix}.$$

Definition 4.1 (Kloosterman sum on a local field). Let v be non-Archimedean. For $\alpha, 1/\nu \in \mathcal{O}_v$ and $\beta \in \nu^2\mathfrak{D}_v^{-2}$, define the *local Kloosterman sum*

$$\mathrm{Kl}_v(\alpha, \beta; \nu) = \sum_{\delta_v x \in \nu\mathcal{O}_v^\times/\mathcal{O}_v} \psi_v(\alpha x + \beta x^{-1}). \tag{4.2}$$

In the quotient $\nu\mathcal{O}_v^\times/\mathcal{O}_v$ above, the group $\mathcal{O}_v$ acts additively on $\nu\mathcal{O}_v^\times$ if $|\nu|_v > 1$, and $\nu\mathcal{O}_v^\times/\mathcal{O}_v = \{1\}$ if $|\nu|_v = 1$ so that the Kloosterman sum is equal to 1.

Let R be a finite set of places where π is unramified. We define $\widetilde{W}_{\mathrm{o}R} = \prod_{v\in R} \widetilde{W}_{\mathrm{o}v}$ and $\mathrm{Kl}_R(\alpha, \beta; \nu) = \prod_{v\in R} \mathrm{Kl}_v(\alpha_v, \beta_v; \nu_v)$ for $\alpha, \beta, \nu \in F_R$ satisfying $|\alpha|_v \leqslant 1 \leqslant |\nu|_v$ and $|\beta|_v \leqslant |\nu/\delta|_v^2$ for all $v \in R$.

4.1. *Adelic Voronoï summation for* GL_3

Note that it is required in [20] that S contains the ramified places of ψ. In order to make their Voronoï summation useful when the class number h_F is not 1 (or when $\mathfrak{D}$ is not principal), one has to relax this condition. For this, we shall outline a proof of the following Voronoï summation for GL_3 in Appendix A.

Proposition 4.2. *Let notation be as above. Let $\zeta \in \mathbb{A}^S$ and $\alpha \in \mathbb{A}^{\times S}$. Let R be the set of places v such that $|\zeta/\alpha|_v > 1$. Let $w_v \in C_c^\infty(F_v^\times)$ for all $v \in S$. Then we have the identity*

$$\sum_{\gamma \in F^\times} \psi^S(\gamma\zeta) W_{\mathrm{o}}^S(a(1/\delta, \alpha\gamma)) w_S(\gamma) = \frac{\overline{\omega}_R(\zeta/\alpha)|\zeta|_R|\alpha|^{S\cup R}}{\omega^S(\delta)\sqrt{|\delta|^S}} \sum_{\gamma\in F^\times} K_R(\gamma,\zeta,\widetilde{W}_{\mathrm{o}R})\widetilde{W}_{\mathrm{o}}^{S\cup R}(a(1/\delta,\gamma\delta/\alpha))\widetilde{w}_S(\gamma), \tag{4.3}$$

where $K_R(\gamma,\zeta,\widetilde{W}_{\mathrm{o}R})$ is defined to be the sum

$$\sum_{\nu\in F_R^\times/\mathcal{O}_R^\times} \widetilde{W}_{\mathrm{o}R}\big(a(\nu\alpha/\zeta\delta, \gamma\delta/\nu^2\zeta)\big)\,\mathrm{Kl}_R(1,-\gamma/\zeta;\nu), \tag{4.4}$$

with ν subject to

$$1 \leqslant |\nu|_v \leqslant |\zeta/\alpha|_v, \quad |\gamma/\zeta|_v \leqslant |\nu/\delta|_v^2 \quad \text{for all } v \in R. \tag{4.5}$$

Notation in the classical setting

Henceforth, we shall assume that π is *unramified* (*spherical*) at every non-Archimedean place and that its central character ω is *trivial*. We may thus choose $S = S_\infty$.

For non-zero *integral* ideals $\mathfrak{n}_1, \mathfrak{n}_2$, we define the Fourier coefficient

$$A(\mathfrak{n}_1,\mathfrak{n}_2) = \mathrm{N}(\mathfrak{n}_1\mathfrak{n}_2\mathfrak{D}^{-2}) W_{\mathrm{o}}^{S_\infty}\begin{pmatrix}\mathfrak{n}_1\mathfrak{n}_2\mathfrak{D}^{-2} & & \\ & \mathfrak{n}_1\mathfrak{D}^{-1} & \\ & & 1\end{pmatrix}. \tag{4.6}$$

Normalize the Fourier coefficients so that $A(1,1) = 1$. We have the multiplicative relation

$$A(\mathfrak{n}_1\mathfrak{m}_1,\mathfrak{n}_2\mathfrak{m}_2) = A(\mathfrak{n}_1,\mathfrak{n}_2)A(\mathfrak{m}_1,\mathfrak{m}_2), \quad (\mathfrak{n}_1\mathfrak{n}_2,\mathfrak{m}_1\mathfrak{m}_2) = (1), \tag{4.7}$$

and the Hecke relation

$$A(\mathfrak{n}_1,\mathfrak{n}_2) = \sum_{\mathfrak{d}|\mathfrak{n}_1,\mathfrak{d}|\mathfrak{n}_2} \mu(\mathfrak{d})A(\mathfrak{n}_1\mathfrak{d}^{-1},1)A(1,\mathfrak{n}_2\mathfrak{d}^{-1}), \tag{4.8}$$

where μ is the Möbius function for F. Moreover, it is known that $\widetilde{A}(\mathfrak{n}_1,\mathfrak{n}_2) = A(\mathfrak{n}_2,\mathfrak{n}_1)$ if $\widetilde{A}(\mathfrak{n}_1,\mathfrak{n}_2)$ are the Fourier coefficients for $\widetilde{\pi}$.

It is known from [57, §17] that for each $v\,|\,\infty$, $\widetilde{f}_v(y) = |y|_v^{-1}\widetilde{w}_v(-y)$ is the Hankel integral transform of $f_v(x) = |x|_v^{-1}w_v(x)$ integrated against the Bessel kernel $J_{\pi_v}(xy)$ attached to π_v (see [57, (17.20)]):

$$\widetilde{f}_v(y) = \int_{F_v^\times} f_v(x)J_{\pi_v}(xy)\,\mathrm{d}x. \tag{4.9}$$

The asymptotic expansion for $J_{\pi_v}(x)$ will be given in §4.4.

Remark 4.3. When v is real, $\widetilde{w}_v(y) = |y|_v \widetilde{f}_v(-y)$ is equal to the $F(y)$ in [40, Theorem 1.18] (if $f_v(x)$ is their $f(x)$). The reason for our normalization of Hankel transforms is to get the Fourier transform and the classical Hankel transform in the GL_1 and GL_2 settings, respectively.

Definition 4.4 (Bessel kernel and Hankel transform). For $x \in F_\infty$, we define the *Bessel kernel*

$$\mathcal{J}(x) = \prod_{v|\infty} J_{\pi_v}(x_v).$$

Let $\mathscr{C}_c^\infty(F_\infty^\times)$ denote the space of compactly supported smooth functions $f : F_\infty^\times \to \mathbb{C}$ that are of the product form $f(x) = \prod_{v|\infty} f_v(x_v)$. For $f \in \mathscr{C}_c^\infty(F_\infty^\times)$, we define its *Hankel transform* $\widetilde{f}$ by

$$\widetilde{f}(y) = \int_{F_\infty^\times} f(x)\mathcal{J}(xy)\,\mathrm{d}x, \quad y \in F_\infty^\times. \tag{4.10}$$

Definition 4.5. For $\mathfrak{b} \subset \mathcal{O}$, define the ring

$$F_{\mathfrak{b}} = \{\alpha \in F : \alpha \in \mathcal{O}_v \text{ for all } \mathfrak{p}_v \mid \mathfrak{b}\}.$$

Define $\psi_{\mathfrak{b}} = \prod_{\mathfrak{p}_v|\mathfrak{b}} \psi_v$ and $\mathfrak{a}_{\mathfrak{b}} = (\mathfrak{a}, \mathfrak{b}^\infty) = \prod_{\mathfrak{p}_v|\mathfrak{b}} \mathfrak{p}_v^{\mathrm{ord}_v(\mathfrak{a})}$.

Definition 4.6 (Kloosterman sum). Let $\mathfrak{b} \subset \mathfrak{q} \subset \mathcal{O}$. For $\alpha \in F_{\mathfrak{b}}$ and $\beta \in (\mathfrak{q}\mathfrak{D}_{\mathfrak{b}})^{-2}F_{\mathfrak{b}}$, define the *Kloosterman sum*

$$\mathrm{Kl}_{\mathfrak{b}}(\alpha, \beta; \mathfrak{q}) = \sum_{x \in ((\mathfrak{q}\mathfrak{D}_{\mathfrak{b}})^{-1}/\mathfrak{D}_{\mathfrak{b}}^{-1})^\times} \psi_{\mathfrak{b}}(\alpha x + \beta x^{-1}), \tag{4.11}$$

where $((\mathfrak{q}\mathfrak{D}_{\mathfrak{b}})^{-1}/\mathfrak{D}_{\mathfrak{b}}^{-1})^\times$ and $x^{-1} \in (\mathfrak{q}\mathfrak{D}_{\mathfrak{b}}/\mathfrak{q}^2\mathfrak{D}_{\mathfrak{b}})^\times$ are defined as in Definition 3.1.

We have $\psi_{\mathfrak{b}} = \psi_R$ and $\mathrm{Kl}_{\mathfrak{b}}(\alpha, \beta; \mathfrak{q}) = \mathrm{Kl}_R(\alpha, \beta; \nu)$ if R is the set of primes dividing $\mathfrak{b}$ and $\mathfrak{q}$ is the integral ideal generated by $1/\nu$. Moreover, it will be convenient to write $\mathrm{Kl}_{\mathfrak{b}}(\alpha, \beta; \mathfrak{q})$ as an integral on a certain compact homogeneous subspace of $F_R^\times$.

Lemma 4.7. *Let notation be as above. Define the Euler totient function $\varphi(\mathfrak{q})$ as usual by*

$$\varphi(\mathfrak{q}) = \mathrm{N}(\mathfrak{q})\prod_{\mathfrak{p}|\mathfrak{q}}(1 - \mathrm{N}(\mathfrak{p})^{-1}). \tag{4.12}$$

Define $\widehat{\mathcal{O}}_{\mathfrak{b}}^\times = \prod_{\mathfrak{p}_v|\mathfrak{b}} \mathcal{O}_v^\times$. Let $\pi_{(\mathfrak{q}\mathfrak{D})^{-1}} \in \mathbb{A}_f^\times$ be the chosen generator for $(\mathfrak{q}\mathfrak{D})^{-1}$. Then

$$\mathrm{Kl}_{\mathfrak{b}}(\alpha, \beta; \mathfrak{q}) = \varphi(\mathfrak{q}) \int_{\pi_{(\mathfrak{q}\mathfrak{D})^{-1}}\widehat{\mathcal{O}}_{\mathfrak{b}}^\times} \psi_{\mathfrak{b}}(\alpha x + \beta x^{-1})\,\mathrm{d}^\times x, \tag{4.13}$$

Proof. With our choice of Haar measure, the space $(\mathfrak{q}\mathfrak{D})^{-1}\widehat{\mathcal{O}}_{\mathfrak{b}}^\times$ $(= (\mathfrak{q}\mathfrak{D}_{\mathfrak{b}})^{-1}\widehat{\mathcal{O}}_{\mathfrak{b}}^\times)$ has total mass 1. On the other hand, the x-sum in (4.11) ranges over a set of size $\varphi(\mathfrak{q})$. Then (4.13) follows from the definitions. ■

4.2. *Classical Voronoï summation for* GL_3

In practice, we shall let $\zeta \in \mathbb{A}^{S_\infty} = \mathbb{A}_f$ be the diagonal embedding of a fraction $a/c \in F$, and it is preferable to have a classical formulation of the Voronoï summation in terms of Fourier coefficients, exponential factors, Kloosterman sums, and Hankel transforms.

Proposition 4.8. *Let notation be as above. Let* $a \in F$, $c \in F^\times$, *and* $\mathfrak{a} \subset \mathfrak{O}$. *Let* $R = \{v \nmid \infty : \mathrm{ord}_v(a/c) < \mathrm{ord}_v(\mathfrak{a}\mathfrak{D}^{-1})\}$, *and set* $\mathfrak{b} = \prod_{v\in R} \mathfrak{p}_v^{\mathrm{ord}_v((c/a)\mathfrak{a}\mathfrak{D}^{-1})}$ *(it is understood that* $\mathfrak{b} = \mathfrak{O}$ *if* $R = \emptyset$*). For* $f \in \mathscr{C}_c^\infty(F_\infty^\times)$ *let its Hankel transform* $\widetilde{f}$ *be given by* (4.10) *in Definition* 4.4. *Then*

$$\sum_{\gamma\in\mathfrak{a}^{-1}} \psi_\infty\left(-\frac{a\gamma}{c}\right) A(1, \mathfrak{a}\gamma) f(\gamma) = \frac{\mathrm{N}(\mathfrak{a})}{\mathrm{N}(\mathfrak{D})^{3/2}} \sum_{\mathfrak{b}\subset\mathfrak{q}\subset\mathfrak{O}} \frac{1}{\mathrm{N}(\mathfrak{b}\mathfrak{q})} \cdot \sum_{\gamma\in\mathfrak{a}(\mathfrak{b}\mathfrak{q}^2\mathfrak{D}^3)^{-1}} A(\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{q}^2\mathfrak{D}^3\gamma, \mathfrak{b}\mathfrak{q}^{-1})\, \mathrm{Kl}_{\mathfrak{b}}(1, \gamma c/a; \mathfrak{q})\, \widetilde{f}(\gamma). \tag{4.14}$$

Proof. Let $\zeta \in \mathbb{A}^{S_\infty} = \mathbb{A}_f$ be the diagonal embedding of $a/c \in F$. Let $\alpha \in \mathbb{A}_f^\times$ generate the ideal $\mathfrak{a}\mathfrak{D}^{-1}$. In the above settings, the left-hand side of (4.3) is translated into that of (4.14) up to the constant $\mathrm{N}(\mathfrak{a}^{-1}\mathfrak{D}^2)$. Note that $\psi_f(a\gamma/c) = \psi_\infty(-a\gamma/c)$ as ψ is trivial on F. For the right-hand side of (4.3), we set $\mathfrak{q}$ to be the ideal generated by $1/\nu$, then the conditions in (4.5) amount to $\mathfrak{b} \subset \mathfrak{q} \subset \mathfrak{O}$ and $\gamma\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{D} \subset \mathfrak{q}^{-2}\mathfrak{D}^{-2}$. The Kloosterman sum is $\mathrm{Kl}_{\mathfrak{b}}(1, \gamma c/a; \mathfrak{q})$ as defined in Definition 4.6. After changing γ to $-\gamma$, we arrive at the right-hand side of (4.14). ■

Remark 4.9. When $h_F = 1$, we may choose $a \in \mathfrak{O}$ and $c \in \mathfrak{O} \smallsetminus \{0\}$ such that $(a, c) = (1)$ and let $\mathfrak{a} = \mathfrak{D}$, $\mathfrak{b} = (c)$, and $\mathfrak{q} = (c/d)$ with $d \mid c$. In this way, upon changing γ to n or d^2n/c^3 on the left or right, respectively, we obtain the classical Voronoï summation formula as in [40].

4.3. *Averages of Fourier coefficients*

We recollect here some results from [55, §4]. First, as a consequence of the Rankin–Selberg theory [25], it is well-known that for $X \geqslant 1$,

$$\sum\sum_{\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2)\leqslant X} |A(\mathfrak{n}_1, \mathfrak{n}_2)|^2 = O_\pi(X). \tag{4.15}$$

As a consequence (see [55, (4.4)]), for $0 \leqslant c < 1$ we have

$$\sum_{\mathrm{N}(\mathfrak{n}_2)\leqslant X} \frac{|A(\mathfrak{n}_1, \mathfrak{n}_2)|}{\mathrm{N}(\mathfrak{n}_2)^c} = O_{c,\pi}(\mathrm{N}(\mathfrak{n}_1)X^{1-c}). \tag{4.16}$$

Moreover, we have the following lemma for the average over $\gamma \in \mathfrak{a}^{-1} \smallsetminus \{0\}$.

Lemma 4.10. *For $V \in \mathfrak{a}_+$ and $S \subset S_\infty$, define $|V|_S = \prod_{v \in S} V_v^{N_v}$ and*

$$F_\infty^S(V) = \{x \in F_\infty : |x|_v > V_v^{N_v} \text{ for all } v \in S, |x|_v \leqslant V_v^{N_v} \text{ for all } v \in S_\infty \smallsetminus S\}. \tag{4.17}$$

Let $0 \leqslant c < 1 < d$. Then for any $0 < \varepsilon < d - 1$ we have

$$\sum_{\substack{\gamma \in F^\times \cap F_\infty^S(V) \\ \gamma\mathfrak{a} \subset \mathcal{O}}} \frac{|A(\mathfrak{n}, \gamma\mathfrak{a})|}{|\mathrm{N}\gamma|^c |\gamma|_S^{d-c}} = O_{\varepsilon,c,d,\pi}\left(\frac{\mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{a})^{1+\varepsilon}\mathrm{N}(V)^{1-c+\varepsilon}}{|V|_S^{d-c}}\right). \tag{4.18}$$

Proof. When $\mathrm{N}(\mathfrak{a})\mathrm{N}(V) \geqslant 1$ is assumed, (4.18) follows from Lemma 4.2 or 4.3 in [55] in the case $S = \emptyset$ or $S \neq \emptyset$, respectively. However, this assumption may be safely and conveniently removed. For example, the sum in (4.18) has no terms if $S = \emptyset$ and $\mathrm{N}(\mathfrak{a})\mathrm{N}(V) < 1$. ∎

The next lemma is a generalization of [5, (10)].

Lemma 4.11. *Let $\theta \leqslant \frac{1}{2}$ be an exponent such that*

$$|A(\mathfrak{n}_1, \mathfrak{n}_2)| \leqslant \mathrm{N}(\mathfrak{n}_1\mathfrak{n}_2)^{\theta+\varepsilon}. \tag{4.19}$$

Define $F_\infty^{\emptyset}(V)$ as in (4.17). For $\mathfrak{f} \subset \mathcal{O}$, we have

$$\sum_{\substack{\gamma \in F^\times \cap F_\infty^{\emptyset}(V) \\ \gamma\mathfrak{a} \subset \mathfrak{f}}} |A(\mathfrak{n}, \gamma\mathfrak{a})|^2 \ll_{\varepsilon,\pi} \mathrm{N}(\mathfrak{f}\mathfrak{n})^{\theta+\varepsilon}\mathrm{N}(\mathfrak{a}\mathfrak{f}^{-1})^{1+\varepsilon}\mathrm{N}(V). \tag{4.20}$$

Proof. First of all, one can apply [55, Lemma 4.1] with (4.15) to prove

$$\sum_{\substack{\gamma \in F^\times \cap F_\infty^{\emptyset}(V) \\ \gamma\mathfrak{a} \subset \mathcal{O}}} |A(1, \gamma\mathfrak{a})|^2 \ll_{\varepsilon,\pi} \mathrm{N}(\mathfrak{a})^{1+\varepsilon}\mathrm{N}(V); \tag{4.21}$$

the proof is similar to that of [55, Lemma 4.2]. The left-hand side of (4.20) is bounded by

$$\begin{aligned} &\sum_{\mathfrak{m} | (\mathfrak{f}\mathfrak{n})^\infty} \sum_{\substack{\gamma \in F^\times \cap F_\infty^{\emptyset}(V) \\ \gamma\mathfrak{a} \subset \mathfrak{f}\mathfrak{m} \\ (\gamma\mathfrak{a}(\mathfrak{f}\mathfrak{m})^{-1}, \mathfrak{f}\mathfrak{m}\mathfrak{n}) = (1)}} |A(\mathfrak{n}, \gamma\mathfrak{a})|^2 \\ &\qquad \leqslant \sum_{\mathfrak{m} | (\mathfrak{f}\mathfrak{n})^\infty} |A(\mathfrak{n}, \mathfrak{f}\mathfrak{m})|^2 \sum_{\substack{\gamma \in F^\times \cap F_\infty^{\emptyset}(V) \\ \gamma\mathfrak{a}(\mathfrak{f}\mathfrak{m})^{-1} \subset \mathcal{O}}} |A(1, \gamma\mathfrak{a}(\mathfrak{f}\mathfrak{m})^{-1})|^2, \end{aligned}$$

and (4.20) then follows from (4.19) and (4.21). ∎

4.4. *Asymptotics for* GL_3*-Bessel kernels*

Assume further that π_∞ is a *spherical* representation of $\mathrm{PGL}_3(F_\infty)$ so that its Archimedean Langlands parameter is given by a triple $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ in $\mathfrak{a}_{\mathbb{C}}^3$, with $\mu_1 + \mu_2 + \mu_3 = 0$. For each $v \,|\, \infty$, the Bessel kernel $J_{\pi_v}(x) = J_{\boldsymbol{\mu}_v}(x)$ depends only on $\boldsymbol{\mu}_v$. A main result of [57] is the following asymptotic expansions for $J_{\boldsymbol{\mu}_v}(x)$ when x is large (see [57, Theorems 14.1, 16.6]).

Lemma 4.12. *Let K be a non-negative integer.*

(1) *When v is real, for $x \gg_{K,\boldsymbol{\mu}_v} 1$ we have the asymptotic expansion*

$$J_{\boldsymbol{\mu}_v}(\pm x) = \frac{e(\pm 3x^{1/3})}{x^{1/3}} \sum_{k=0}^{K-1} \frac{B_k^{\pm}}{x^{k/3}} + O_{K,\boldsymbol{\mu}_v}\left(\frac{1}{x^{(K+1)/3}}\right), \tag{4.22}$$

with the coefficients $B_k^{\pm}$ depending only on $\boldsymbol{\mu}_v$.

(2) *When v is complex, for $|z| \gg_{K,\boldsymbol{\mu}_v} 1$ we have the asymptotic expansion*

$$J_{\boldsymbol{\mu}_v}(z) = \sum_{\xi^3=1} \frac{e(3(\xi z^{1/3} + \overline{\xi}\overline{z}^{1/3}))}{|z|^{2/3}} \sum\sum_{k+l \leqslant K-1} \frac{B_k B_l}{\xi^{k-l} z^{k/3} \overline{z}^{l/3}} + O_{K,\boldsymbol{\mu}_v}\left(\frac{1}{|z|^{(K+2)/3}}\right), \tag{4.23}$$

with the coefficients B_k depending only on $\boldsymbol{\mu}_v$.

The self-dual assumption

Subsequently, we shall assume that π is a *self-dual* spherical automorphic cuspidal representation of $\mathrm{PGL}_3(\mathbb{A})$. In particular, we have $A(\mathfrak{n}_1, \mathfrak{n}_2) = A(\mathfrak{n}_2, \mathfrak{n}_1)$ $(= \overline{A(\mathfrak{n}_1, \mathfrak{n}_2)})$ and $\boldsymbol{\mu} = (\mu, 0, -\mu)$ with $\mu \in iY \subset \mathfrak{a}_{\mathbb{C}}$ ($\mu_v \in i\mathbb{R}$ or $\mu_v \in (-\frac{1}{2}, \frac{1}{2})$ for every $v \,|\, \infty$; see §3.1 for the definitions).

It is known by [14] that π comes from the symmetric square lift of a Hecke–Maass form for GL_2. Thus the Kim–Sarnak bound [6] for GL_2 implies that

$$|A(\mathfrak{n}_1, \mathfrak{n}_2)| \leqslant \mathrm{N}(\mathfrak{n}_1\mathfrak{n}_2)^{7/32+\varepsilon}, \tag{4.24}$$

and

$$|\mathrm{Re}(\mu_v)| \leqslant \tfrac{7}{32}, \quad v \,|\, \infty. \tag{4.25}$$

5. Preliminaries on L-functions

Let f be a spherical Hecke–Maass cusp form on $\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})$ with Hecke eigenvalues $\lambda_f(\mathfrak{n})$ and Archimedean parameter $\nu_f \in Y$. Let $E(s)$ be the spherical Eisenstein series on $\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})$. Let π be a fixed self-dual spherical automorphic cuspidal representation of $\mathrm{PGL}_3(\mathbb{A})$ with Fourier coefficients $A(\mathfrak{n}_1, \mathfrak{n}_2)$ and Archimedean parameter $(\mu, 0, -\mu)$ $(\mu \in iY)$.

5.1. L-functions $L(s, \pi)$, $L(s, \pi \otimes f)$, and $L(s, \pi \otimes E(it))$

The L-function attached to π is defined by

$$L(s, \pi) = \sum_{\mathfrak{n} \subset \mathcal{O}} \frac{A(1, \mathfrak{n})}{\mathrm{N}(\mathfrak{n})^s}. \tag{5.1}$$

The Rankin–Selberg L-function $L(s, \pi \otimes f)$ is defined by

$$L(s, \pi \otimes f) = \sum_{\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathcal{O}} \sum \frac{A(\mathfrak{n}_1, \mathfrak{n}_2)\lambda_f(\mathfrak{n}_2)}{\mathrm{N}(\mathfrak{n}_1^2 \mathfrak{n}_2)^s}. \tag{5.2}$$

The completed L-function for π is $\Lambda(s, \pi) = \mathrm{N}(\mathfrak{D})^{3s/2}\gamma(s, \pi)L(s, \pi)$, where $\gamma(s, \pi) = \gamma(s)$ is the product of

$$\gamma_v(s) = (N_v \pi)^{-3N_v s/2}\Gamma\left(\frac{N_v(s - \mu_v)}{2}\right)\Gamma\left(\frac{N_v s}{2}\right)\Gamma\left(\frac{N_v(s + \mu_v)}{2}\right). \tag{5.3}$$

Recall that $N_v = 1$ if v is real and $N_v = 2$ if v is complex. It is known that $\Lambda(s, \pi)$ is entire and has the functional equation

$$\Lambda(s, \pi) = \Lambda(1 - s, \pi).$$

We define $\gamma(s, \nu)$ to be the product of

$$\gamma_v(s, \nu_v) = \gamma_v(s - i\nu_v)\gamma_v(s + i\nu_v). \tag{5.4}$$

Let $\gamma(s, \pi \otimes f) = \gamma(s, \nu_f)$. Then $\Lambda(s, \pi \otimes f) = \mathrm{N}(\mathfrak{D})^{3s}\gamma(s, \pi \otimes f)L(s, \pi \otimes f)$ is also entire and satisfies the functional equation

$$\Lambda(s, \pi \otimes f) = \Lambda(1 - s, \pi \otimes f).$$

Similar to (5.2), we define

$$L(s, \pi \otimes E(it)) = \sum_{\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathcal{O}} \sum \frac{A(\mathfrak{n}_1, \mathfrak{n}_2)\tau_{it}(\mathfrak{n}_2)}{\mathrm{N}(\mathfrak{n}_1^2 \mathfrak{n}_2)^s}, \tag{5.5}$$

where $\tau_s(\mathfrak{n})$ is defined as in (3.10). We have

$$L(s, \pi \otimes E(it)) = L(s + it, \pi)L(s - it, \pi), \tag{5.6}$$

and hence

$$L\left(\tfrac{1}{2}, \pi \otimes E(it)\right) = \left|L\left(\tfrac{1}{2} + it, \pi\right)\right|^2. \tag{5.7}$$

5.2. *Approximate functional equations for $L(s, \pi \otimes f)$ and $L(s, \pi)$*

Following Blomer [5], for a large even integer $A' > 0$, we introduce the polynomial $p(s, \nu)$ as the product of $p_v(s, \nu_v)$ $(v \mid \infty)$ defined by

$$\prod_{k=0}^{N_v A'/2-1} \left((s + 2k/N_v - \mu_v)^2 + \nu_v^2\right)\left((s + 2k/N_v)^2 + \nu_v^2\right)\left((s + 2k/N_v + \mu_v)^2 + \nu_v^2\right) \tag{5.8}$$

so that $p_v(s, \nu_v)$ annihilates the rightmost $N_v A'/2$ poles of each of the gamma factors in $\gamma_v(s, \nu_v)$ defined by (5.3)–(5.4). This polynomial will eventually be used to overcome the obstacle posed by the presence of an infinite group of units.

We have the following approximate functional equation for $L(s, \pi \otimes f)$ (see [24, Theorem 5.3]):

$$L\big(\tfrac{1}{2}, \pi \otimes f\big) = 2 \sum\sum_{\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathfrak{O}} \frac{A(\mathfrak{n}_1, \mathfrak{n}_2)\lambda_f(\mathfrak{n}_2)}{\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2)^{1/2}} V(\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2\mathfrak{D}^{-3}); \nu_f) \tag{5.9}$$

with

$$V(y;\nu) = \frac{1}{2\pi i}\int_{(3)} G(u,\nu) y^{-u} \frac{\mathrm{d}u}{u}, \quad y > 0, \tag{5.10}$$

in which $G(u, \nu)$ is the product of

$$G_v(u,\nu_v) = \frac{\gamma_v\big(\tfrac{1}{2}+u,\nu_v\big)}{\gamma_v\big(\tfrac{1}{2},\nu_v\big)} \cdot \frac{p_v\big(\tfrac{1}{2}+u,\nu_v\big)p_v\big(\tfrac{1}{2}-u,\nu_v\big)e^{N_v u^2/N}}{p_v\big(\tfrac{1}{2},\nu_v\big)^2}. \tag{5.11}$$

Note that the second quotient in (5.11) is even in u and is equal to 1 when $u = 0$. Similarly, the approximate functional equation for $L(s, \pi \otimes E(it))$, along with (5.7), yields

$$\big|L\big(\tfrac{1}{2}+it,\pi\big)\big|^2 = 2 \sum\sum_{\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathfrak{O}} \frac{A(\mathfrak{n}_1, \mathfrak{n}_2)\tau_{it}(\mathfrak{n}_2)}{\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2)^{1/2}} V(\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2\mathfrak{D}^{-3}); t). \tag{5.12}$$

Properties of $V(y;\nu)$ and $G(u,\nu)$ are given in the following lemma (see [5, Lemma 1] and [56, Lemma 3.7]).

Lemma 5.1. *Let $U > 1$, $A > 0$, and $\varepsilon > 0$. Suppose that $\nu \in Y$ satisfies the Kim–Sarnak bounds (3.6). Let $\mathrm{C}(\nu)$ be defined as in (3.5).*

(1) *We have*

$$V(y;\nu) \ll_{A,A'} \left(1 + \frac{y}{\mathrm{C}(\nu)^3}\right)^{-A} \tag{5.13}$$

and

$$V(y;\nu) = \frac{1}{2\pi i}\int_{\varepsilon - iU}^{\varepsilon + iU} G(u,\nu) y^{-u} \frac{\mathrm{d}u}{u} + O_{\varepsilon,A'}\left(\frac{\mathrm{C}(\nu)^{3\varepsilon}}{y^\varepsilon e^{U^2/2}}\right). \tag{5.14}$$

(2) *When $\mathrm{Re}(u) > 0$, the function $H_v(u,\nu_v) = G_v(u,\nu_v)p_v(\tfrac{1}{2},\nu_v)^2$ is even and holomorphic on the region $|\mathrm{Im}(\nu_v)| \leqslant A' + \frac{9}{32} = A' + \frac{1}{2} - \frac{7}{32}$ (see (4.25)), and it satisfies in this region the uniform bound*

$$H_v(u,\nu_v) \ll_{A',\mathrm{Re}(u)} (1+|\nu_v|)^{3N_v(\mathrm{Re}(u)+2A')}, \tag{5.15}$$

and, more generally,

$$\frac{\partial^j}{\partial \nu_v^j} H_v(u,\nu_v) \ll_{j,A',\mathrm{Re}(u)} (1+|\nu_v|)^{3N_v(\mathrm{Re}(u)+2A')-j}(1+\mathrm{Im}(u))^j. \tag{5.16}$$

Proof. The estimate in (5.13) may be found in [24, Proposition 5.4]. The expression of $V(y;\nu)$ in (5.14) is essentially due to Blomer [5, Lemma 1]. The bounds in (5.15) and (5.16) follow readily from Stirling's formulae for $\log\Gamma$ and its derivatives (see for example [37, §§1.1, 1.2]). ∎

6. Choice of the test function

Definition 6.1. Let $T, M \in \mathfrak{a}_+$ be such that $1 \ll T_v^{\varepsilon} \leqslant M_v \leqslant T_v^{1-\varepsilon}$ for each $v \mid \infty$. Define the function $k(\nu) = k_{T,M}(\nu)$ to be the product of

$$k_v(\nu_v) = e^{-(\nu_v - T_v)^2/M_v^2} + e^{-(\nu_v + T_v)^2/M_v^2}. \tag{6.1}$$

We modify $k(\nu)$ slightly, and let $k^{\natural}(\nu) = k^{\natural}_{T,M}(\nu)$ be the product of

$$k_v^{\natural}(\nu_v) = k_v(\nu_v) p_v\big(\tfrac{1}{2}, \nu_v\big)^2 / T_v^{6N_v A'}, \tag{6.2}$$

with $p_v\big(\frac{1}{2}, \nu_v\big)$ defined as in (5.8). For $\nu \in \mathfrak{a}$, we have $k^{\natural}(\nu) > 0$, and $k^{\natural}(\nu) \gg 1$ if $|\nu_v - T_v| \leqslant M_v$ for all $v \mid \infty$. Note also that $k_v^{\natural}(\nu_v) = o(e^{-T_v^2/M_v^2})$ if $i\nu_v \in (-\frac{1}{2}, \frac{1}{2})$.

For $\mathrm{Re}(u) = \varepsilon$, the function

$$h(\nu) = h_{T,M}(\nu) = G(u, \nu) k^{\natural}_{T,M}(\nu) \tag{6.3}$$

lies in the space $\mathscr{H}(S)$ of Definition 3.3 with $S = A' + \frac{9}{32}$. It is clear that $h(\nu)$ is the product of

$$h_v(\nu_v) = k_v(\nu_v) H_v(u, \nu_v) / T_v^{6N_v A'}, \tag{6.4}$$

and it follows from Lemma 5.1 (2) that

$$h_v(\nu_v) \ll_{A',\varepsilon} (1 + |\nu_v|)^{\varepsilon} k_v(\nu_v), \quad \nu_v \in \mathbb{R}. \tag{6.5}$$

Henceforth, in view of (5.14), we shall assume that $\mathrm{Re}(u) = \varepsilon$ and $|\mathrm{Im}(u)| \leqslant \log \mathrm{N}(T)$.

Appendix A. Proof of the adelic Voronoï summation for GL_3

The purpose of this appendix is to prove the adelic Voronoï summation for GL_3 in Proposition 4.2 without the condition in [20] that S contains the ramified places of ψ. By directly modifying [20, Theorem 1], such a Voronoï summation for GL_2 and GL_3 was formulated in [55, §3.1], but it is not entirely correct in the GL_3 case. A particular issue with the direct modification is that the identity (2.4) in their proof,

$$\int_{F_v^{n-2}} \widetilde{W}_{\mathrm{o}v}\left(\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix}\right) \mathrm{d}x = \widetilde{W}_{\mathrm{o}v}\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix}, \tag{A.1}$$

is no longer valid if $n = 3$ and $\mathfrak{D}_v \neq \mathfrak{O}_v$ (though this is always true if $n = 2$ as the integration would disappear). To rectify this, one must replace the

$$W_{\mathrm{o}}^S \begin{pmatrix} \gamma & \\ & 1_2 \end{pmatrix} = W_{\mathrm{o}}^S(a(1, \gamma))$$

in the left-hand side of [20, (1.2)] (for $n = 3$) by

$$W_{\mathrm{o}}^S \begin{pmatrix} \gamma/\delta & & \\ & 1/\delta & \\ & & 1 \end{pmatrix} = W_{\mathrm{o}}^S(a(1/\delta, \gamma))$$

and thus resort to [20, Theorem 3] rather than [20, Theorem 1]. Nevertheless, the main results in [55] are not invalidated, for only several changes involving $\mathfrak{D}$ or δ are needed.

Proof of Proposition 4.2

We retain the adelic notation of §4. Let

$$w' = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad w_2 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix},$$

$$n(x) = \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad n^-(x) = \begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix}.$$

By abuse of notation, we shall denote by γ, ζ, δ, α their local components γ_v, ζ_v, δ_v, α_v respectively. According to the proof of [20, Theorem 3], the local integral at a place $v \notin S$ that we need to consider is

$$I_v^\sharp(\gamma) = \int_{F_v} \widetilde{W}_{ov}(a(1,\gamma)n^-(x)w'n^-(-\zeta)a(1/\delta,\alpha)^{-1})\,\mathrm{d}x,$$

while at the places in S we have the transform defined by (4.1),

$$I_v^\sharp(\gamma) = \widetilde{w}_v(\gamma); \tag{A.2}$$

see [20, §§2.7, 5.3]. Our goal is to prove that the sum over $\gamma \in F^\times$ of the product of $I_v^\sharp(\gamma)$ is equal to the right-hand side of (4.3).

For $v \notin S \cup R$, we have $|\zeta/\alpha|_v \leqslant 1$. It follows that

$$a(1/\delta,\alpha)w' \cdot w'n^-(-\zeta)a(1/\delta,\alpha)^{-1} = n^-(-\zeta/\alpha) \in \mathrm{GL}_3(\mathcal{O}_v).$$

Thus

$$\begin{aligned} I_v^\sharp(\gamma) &= \int_{F_v} \widetilde{W}_{ov}\left(\begin{pmatrix} \gamma & \\ & 1_2 \end{pmatrix}\begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \delta/\alpha & & \\ & 1 & \\ & & \delta \end{pmatrix}\right)\mathrm{d}x \\ &= \overline{\omega}_v(\delta)|\alpha/\delta|_v \int_{F_v} \widetilde{W}_{ov}\left(\begin{pmatrix} \gamma/\alpha & & \\ & 1/\delta & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix}\right)\mathrm{d}x. \end{aligned}$$

For $|x|_v > 1$, we have the Iwasawa decomposition

$$\begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/x & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1/x & & \\ & x & \\ & & 1 \end{pmatrix}\begin{pmatrix} & -1 & \\ 1 & 1/x & \\ & & 1 \end{pmatrix}.$$

Therefore the integrand above is equal to

$$\overline{\psi}(\gamma\delta/\alpha x)\widetilde{W}_{ov}\begin{pmatrix} \gamma/\alpha x & & \\ & x/\delta & \\ & & 1 \end{pmatrix},$$

and it vanishes as $|x/\delta|_v > 1/|\delta|_v$. Then we infer that the integrand is non-zero only if $x \in \mathcal{O}_v$. Consequently,

$$I_v^\sharp(\gamma) = \frac{\overline{\omega}_v(\delta)|\alpha|_v}{\sqrt{|\delta|_v}} \cdot \widetilde{W}_{\mathrm{o}v} \begin{pmatrix} \gamma/\alpha & & \\ & 1/\delta & \\ & & 1 \end{pmatrix}. \tag{A.3}$$

Note that (A.3) is reduced to (A.1) if $\delta = \alpha = 1$.

Now let $v \in R$ with $|\zeta/\alpha|_v > 1$. It is the second case in the proof of [20, Theorem 3]. We adapt their computations for the GL_3 case as follows.

We start by rewriting

$$I_v^\sharp(\gamma) = |\gamma|_v \int_{F_v} \widetilde{W}_{\mathrm{o}v}\big(n^-(x)w'a(1,\gamma)n^-(-\zeta)a(1/\delta,\alpha)^{-1}\big)\,\mathrm{d}x.$$

The Iwasawa decomposition of $n^-(-\zeta/\alpha)$ yields

$$\begin{aligned} I_v^\sharp(\gamma) &= |\gamma|_v \int_{F_v} \widetilde{W}_{\mathrm{o}v}\big(n^-(x)w'a(\delta,\gamma/\alpha)n(-\alpha/\zeta)a(\zeta/\alpha,\alpha^2/\zeta^2)\big)\,\mathrm{d}x \\ &= |\gamma|_v \int_{F_v} \widetilde{W}_{\mathrm{o}v}\big(n^-(x)w'n(-\gamma/\zeta)a(\zeta\delta/\alpha,\gamma\alpha/\zeta^2)\big)\,\mathrm{d}x. \end{aligned}$$

From

$$n^-(x)w'n(-\gamma/\zeta)w' = \begin{pmatrix} 1 & & -\gamma/\zeta \\ & 1 & -x\gamma/\zeta \\ & & 1 \end{pmatrix} n^-(x),$$

we have

$$I_v^\sharp(\gamma) = |\gamma|_v \int_{F_v} \overline{\psi}_v(-x\gamma/\zeta)\widetilde{W}_{\mathrm{o}v}\big(n^-(x)w'a(\zeta\delta/\alpha,\gamma\alpha/\zeta^2)\big)\,\mathrm{d}x.$$

Since

$$\begin{aligned} n^-(x)w'a(\zeta\delta/\alpha,\gamma\alpha/\zeta^2) &= w_2 n(x)w_2 w'a(\zeta\delta/\alpha,\gamma\alpha/\zeta^2) \\ &= w_2 \begin{pmatrix} 1 & & \\ & \gamma\delta/\zeta & \\ & & \zeta\delta/\alpha \end{pmatrix} n(x\gamma\delta/\zeta)w_2 w' \end{aligned}$$

and $w_2 w' \in \mathrm{GL}_3(\mathcal{O})$, we have

$$I_v^\sharp(\gamma) = |\zeta/\delta|_v \int_{F_v} \psi_v(x/\delta)\widetilde{W}_{\mathrm{o}v}\left(w_2 \begin{pmatrix} 1 & & \\ & \gamma\delta/\zeta & \\ & & \zeta\delta/\alpha \end{pmatrix} n(x)\right)\,\mathrm{d}x. \tag{A.4}$$

To compute the Kloosterman integral in (A.4), we invoke the following result adopted from [20, §6] in the GL_3 setting (see in particular (6.3) and Corollary 6.7 there).

Lemma A.1. *Let ψ_v be unramified and ψ'_v be trivial on $\mathcal{O}_v$. Let $\widetilde{W}_v$ be a $\overline{\psi}_v$-Whittaker function invariant under* $\mathrm{GL}_3(\mathcal{O}_v)$. *Let* $\mathrm{d}x$ *be the Haar measure self-dual with respect to ψ_v (the measure of $\mathcal{O}_v$ is* 1*). Let $\beta, \zeta \in F_v^\times$. Then*

$$\int_{F_v} \psi'_v(x)\widetilde{W}_v\left(w_2\begin{pmatrix}1 & & \\ & \beta & \\ & & \zeta\end{pmatrix} n(x)\right)\mathrm{d}x = \sum_{\substack{\nu\in F_v^\times/\mathcal{O}_v^\times \\ 1\leqslant|\nu|_v\leqslant|\zeta|_v \\ |\beta|_v\leqslant|\nu|_v^2}} \widetilde{W}_v\begin{pmatrix}\beta/\nu & & \\ & \nu & \\ & & \zeta\end{pmatrix}\mathrm{Kl}(\beta;\nu;\psi_v,\psi'_v), \tag{A.5}$$

with

$$\mathrm{Kl}(\beta;\nu;\psi_v,\psi'_v) = \sum_{x\in\nu\mathcal{O}_v^\times/\mathcal{O}_v} \psi'_v(x)\psi_v(-\beta/x).$$

It is required that ψ_v is *unramified* in Lemma A.1. To remove this condition, we have to re-scale ψ_v, $\widetilde{W}_{\mathrm{o}v}$ and $\mathrm{d}x$ so that $\psi_v(x) = \psi_v^\natural(\delta x)$, $\widetilde{W}_{\mathrm{o}v}(g) = \widetilde{W}_{\mathrm{o}v}^\natural(a(\delta,\delta)g)$ and $\mathrm{d}x = \sqrt{|\delta|_v}\,\mathrm{d}^\natural x$. Applying Lemma A.1, we may transform (A.4) into

$$I_v^\sharp(\gamma) = \frac{|\zeta|_v\overline{\omega}_v(\delta)}{\sqrt{|\delta|_v}} \sum_{\substack{\nu\in F_v^\times/\mathcal{O}_v^\times \\ 1\leqslant|\nu|_v\leqslant|\zeta/\alpha_1\alpha_2|_v \\ |\gamma/\zeta\alpha_1|_v\leqslant|\nu/\delta|_v^2}} \widetilde{W}_{\mathrm{o}v}\begin{pmatrix}\gamma/\nu\zeta & & \\ & \nu/\delta & \\ & & \zeta/\alpha\end{pmatrix}\mathrm{Kl}_v(1,-\gamma/\zeta,\nu), \tag{A.6}$$

where $\mathrm{Kl}_v(\alpha,\beta;\nu)$ is the local Kloosterman sum defined in Definition 4.1.

Finally, our proof is completed by combining (A.2), (A.3) and (A.6).

Part II. Analysis over Archimedean fields

In the following sections, we shall do analysis over a local Archimedean field F_v ($v\,|\,\infty$). For simplicity, we shall suppress v from our notation. Accordingly, F will be an Archimedean local field, and $N = [F:\mathbb{R}]$. Henceforth, x, y will always stand for real variables, while z, u for complex variables; in the complex setting, we shall usually use the polar coordinates $z = xe^{i\phi}$ and $u = ye^{i\theta}$.

7. Stationary phase lemmas

For later use, we collect here some useful stationary phase lemmas in one dimension or two dimensions.

7.1. The one-dimensional case

Consider one-dimensional oscillatory integrals of the form

$$\int_a^b e(f(x))w(x)\,\mathrm{d}x.$$

In practice, the phase function $f(x) = f(x; \lambda, \dots)$ usually contains some (real) parameters. It is convenient to transform the phase into the form $\lambda f(x)$ by change of variables, but clearly this cannot always be done. For instance, one may consider a phase of the form $\lambda^{1/3}x^2 - x^3$ or $x - \lambda \log x$.

Firstly, we record [3, Lemma A.1], which is an improved version of [8, Lemma 8.1].

Lemma 7.1. *Let $w(x)$ be a smooth function with support in (a, b) and $f(x)$ be a real smooth function on $[a, b]$. Suppose that there are parameters $P, Q, R, S, Z > 0$ such that*

$$f^{(i)}(x) \ll_i Z/Q^i, \quad w^{(j)}(x) \ll_j S/P^j,$$

for $i \geqslant 2$ and $j \geqslant 0$, and

$$|f'(x)| \geqslant R.$$

Then for any $A \geqslant 0$ we have

$$\int_a^b e(f(x))w(x)\,\mathrm{d}x \ll_A (b-a)S\left(\frac{Z}{R^2Q^2} + \frac{1}{RQ} + \frac{1}{RP}\right)^A.$$

Secondly, the second derivative test as follows is usually sufficient for our purpose—it is as strong as the stationary phase estimate in most of the cases. See [19, Lemma 5.1.3].

Lemma 7.2. *Let $f(x)$ be a real smooth function on (a, b) with $f''(x) \geqslant \lambda > 0$. Let $w(x)$ be a real smooth function on $[a, b]$, and let V be its total variation plus its maximum modulus. Then*

$$\left|\int_a^b e(f(x))w(x)\,\mathrm{d}x\right| \leqslant \frac{4V}{\sqrt{\pi\lambda}}.$$

Finally, when the phase is of the form $\lambda f(x)$, we record here a generalization of the stationary phase estimate in [60, Theorem 1.1.1] ($X = 1$ in [60]). See [56, §2.4].

Lemma 7.3. *Let $S > 0$ and $\sqrt{\lambda} \geqslant X \geqslant 1$. Let $w(x; \lambda)$ be a smooth function with support in (a, b) for all λ, and $f(x)$ be a real smooth function on $[a, b]$. Suppose that $\lambda^j \partial_x^i \partial_\lambda^j w(x; \lambda) \ll_{i,j} SX^{i+j}$ and that $f(x_0) = f'(x_0) = 0$ at a point $x_0 \in (a, b)$, with $f''(x_0) \neq 0$ and $f'(x) \neq 0$ for all $x \in [a, b] \smallsetminus \{x_0\}$. Then*

$$\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} \int_a^b e(\lambda f(x))w(x; \lambda)\,\mathrm{d}x \ll_j \frac{SX^j}{\lambda^{1/2+j}}.$$

We have deliberately avoided here the use of [68, Lemma 6.3] (or the asymptotic expansion in [8, Proposition 8.2]) with an arbitrary phase function, because it does not currently have a generalization in two dimensions.

7.2. *The two-dimensional case*

Next, we turn to two-dimensional oscillatory integrals of the form

$$\iint_D e(f(x, y))w(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Firstly, we have the two-dimensional generalization of Lemma 7.1 as follows.

Lemma 7.4. *Let $D \subset \mathbb{R}^2$ be a bounded domain. Let $w(x, y)$ be a smooth function with support on D and $f(x, y)$ be a real smooth function on the closure $\overline{D}$. Suppose that there are parameters $P, Q, \Upsilon, \Phi, R, S, Z > 0$ such that*

$$(\partial/\partial x)^i (\partial/\partial y)^j f(x, y) \ll_{i,j} Z/Q^i \Phi^j, \quad (\partial/\partial x)^k (\partial/\partial y)^l w(x, y) \ll_{k,l} S/P^k \Upsilon^l, \tag{7.1}$$

for $i, j, k, l \geqslant 0$ with $i + j \geqslant 2$, and

$$|f'(x, y)|^2 = (\partial f(x, y)/\partial x)^2 + (\partial f(x, y)/\partial y)^2 \geqslant R^2. \tag{7.2}$$

Then

$$\iint_D e(f(x, y)) w(x, y)\, \mathrm{d}x\, \mathrm{d}y \ll_A \operatorname{Area}(D) S \cdot \left\{ \frac{1}{R} \left(\frac{1}{P} + \frac{1}{\Upsilon} + \frac{1}{Q} + \frac{1}{\Phi} \right) + \frac{Z^2}{R^3} \left(\frac{1}{Q^3} + \frac{1}{\Phi^3} \right) \right\}^A \tag{7.3}$$

for any $A \geqslant 0$.

Proof. We start with a useful simple lemma.

Lemma 7.5. *Let $f(x, y)$ be a smooth function. Let $i, j, n \geqslant 0$. Then $\partial_x^i \partial_y^j (f(x, y)^n)$ is a linear combination of products in the form*

$$f(x, y)^{n - \sum k_{\nu\mu}} \prod_{\nu,\mu} (\partial_x^\nu \partial_y^\mu f(x, y))^{k_{\nu\mu}},$$

$$k_{00} = 0, \quad \sum_{\nu,\mu} \nu k_{\nu\mu} = i, \quad \sum_{\nu,\mu} \mu k_{\nu\mu} = j, \quad \sum_{\nu,\mu} k_{\nu\mu} \leqslant n.$$

For brevity, we write $g(x, y) = |f'(x, y)|^2$. By Lemma 7.5 along with (7.1) and the trivial inequalities $|\partial_x f(x, y)|, |\partial_y f(x, y)| \leqslant \sqrt{g(x, y)}$, we infer that

$$\partial_x^i \partial_y^j g(x, y) \ll \left(\left(\frac{Z}{Q} + \frac{Z}{\Phi} \right) \sqrt{g(x, y)} + \left(\frac{Z^2}{Q^2} + \frac{Z^2}{\Phi^2} \right) \right) \frac{1}{Q^i \Phi^j} \tag{7.4}$$

for $i + j \geqslant 1$. Our idea is to repeatedly apply Hörmander's elaborate partial integration (see [17, Theorem 7.7.1]) as follows. Define the differential operator

$$\mathrm{D} = \frac{\partial_x f(x, y)}{g(x, y)} \frac{\partial}{\partial x} + \frac{\partial_y f(x, y)}{g(x, y)} \frac{\partial}{\partial y},$$

so that $\mathrm{D}(e(f(x, y))) = 2\pi i \cdot e(f(x, y))$. Consequently,

$$\mathrm{D}^* = -\frac{1}{2\pi i} \left(\frac{\partial}{\partial x} \frac{\partial_x f(x, y)}{g(x, y)} + \frac{\partial}{\partial y} \frac{\partial_y f(x, y)}{g(x, y)} \right)$$

is the adjoint of $(1/2\pi i) \cdot \mathrm{D}$ and

$$\int_a^b \int_c^d e(f(x, y)) w(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_a^b \int_c^d e(f(x, y)) \mathrm{D}^{*n} w(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

for any integer $n \geqslant 0$. By a straightforward inductive argument, it may be shown that $\mathrm{D}^{*n} w(x, y)$ is a linear combination of all the terms occurring in the product-rule expansions of

$$\partial_x^i \partial_y^j \big\{(\partial_x f(x, y))^i (\partial_y f(x, y))^j g(x, y)^n w(x, y)\big\} / g(x, y)^{2n}, \quad i + j = n.$$

Now let $i_1, i_2 \leqslant i$ and $j_1, j_2 \leqslant j$. It follows from Lemma 7.5 along with (7.1), (7.2), and the trivial inequalities $|\partial_x f(x, y)|, |\partial_y f(x, y)| \leqslant \sqrt{g(x, y)}$ that

$$\partial_x^{i_1} \partial_y^{j_1} \{(\partial_x f(x, y))^i (\partial_y f(x, y))^j\} \ll \left\{1 + \frac{Z}{R}\left(\frac{1}{Q} + \frac{1}{\Phi}\right)\right\}^{i_1 + j_1} \frac{g(x, y)^{(i+j)/2}}{Q^{i_1} \Phi^{j_1}}.$$

Similarly, Lemma 7.5, (7.2), and (7.4) yield

$$\partial_x^{i_2} \partial_y^{j_2} (g(x, y)^n) \ll \left\{1 + \frac{Z^2}{R^2}\left(\frac{1}{Q^2} + \frac{1}{\Phi^2}\right)\right\}^{i_2 + j_2} \frac{g(x, y)^n}{Q^{i_2} \Phi^{j_2}}.$$

Thus

$$\begin{aligned}
&\mathrm{D}^{*n} w(x, y) \\
&\quad \ll \frac{S}{R^n} \sum_{i+j=n} \frac{1}{P^i \Upsilon^j} \sum_{\substack{i_1+i_2 \leqslant i \\ j_1+j_2 \leqslant j}} \frac{P^{i_1+i_2} \Upsilon^{j_1+j_2}}{Q^{i_1+i_2} \Phi^{j_1+j_2}} \left\{1 + \frac{Z}{R}\left(\frac{1}{Q} + \frac{1}{\Phi}\right)\right\}^{i_1+j_1+2i_2+2j_2} \\
&\quad \ll S \left\{\frac{1}{R}\left(\frac{1}{P} + \frac{1}{\Upsilon} + \frac{1}{Q} + \frac{1}{\Phi}\right) + \frac{Z^2}{R^3}\left(\frac{1}{Q^3} + \frac{1}{\Phi^3}\right)\right\}^n,
\end{aligned}$$

as desired. ∎

Secondly, we need a two-dimensional generalization of the second derivative test in Lemma 7.2. A very useful version in the literature is [42, Lemma 4] (see also [62, Lemma 5]), in which it is assumed that

$$\begin{aligned}
&|\partial^2 f / \partial x^2| \geqslant \lambda > 0, \quad |\partial^2 f / \partial y^2| \geqslant \rho > 0, \\
&|\det f''| = |\partial^2 f / \partial x^2 \cdot \partial^2 f / \partial y^2 - (\partial^2 f / \partial x \partial y)^2| \geqslant \lambda \rho,
\end{aligned} \tag{7.5}$$

on the integration domain $D = [a, b] \times [c, d]$. However, their bound $1/\sqrt{\lambda\rho}$ would not be desirable if $(\partial^2 f / \partial x \partial y)^2$ is very large compared to $\partial^2 f / \partial x^2 \cdot \partial^2 f / \partial y^2$ so that the former dominates in $\det f''$. This is because the choice of coordinates is not quite appropriate. In general, it seems that some work is required to find the optimal coordinates. Fortunately, in our application, we shall have $\partial^2 f / \partial x^2 = -\partial^2 f / \partial y^2$ (see (13.24)) and the change of coordinates may be simply chosen to be

$$\sqrt{2}\, x = x' + y', \quad \sqrt{2}\, y = x' - y'.$$

As in [42], we first suppose that $w(x, y) \equiv 1$. Let $f(x, y)$ be a real smooth function on the rectangle $[a, b] \times [c, d]$ such that

$$\partial^2 f / \partial x^2 = -\partial^2 f / \partial y^2 \tag{7.6}$$

with

$$\max\{|\partial^2 f/\partial x^2|, |\partial^2 f/\partial x\partial y|\} \geqslant \lambda > 0. \tag{7.7}$$

We would like to prove

$$\int_a^b \int_c^d e(f(x,y))\,\mathrm{d}x\,\mathrm{d}y \preccurlyeq \frac{1}{\lambda},$$

with an absolute implied constant. Note that $|\det f''| \geqslant \lambda^2$, so this is in essence the expected stationary phase estimate.

For $|\partial^2 f/\partial x^2| \geqslant |\partial^2 f/\partial x\partial y|$, Lemma 4 in [61] (with $D = [a,b]\times[c,d]$) gives us the bound $1/\lambda$ as expected. Now assume that $|\partial^2 f/\partial x^2| < |\partial^2 f/\partial x\partial y|$. Let x', y' be as above. Then $\partial^2 f/\partial x'^2 = -\partial^2 f/\partial y'^2 = \partial^2 f/\partial x\partial y$ and $\partial^2 f/\partial x'\partial y' = \partial^2 f/\partial x^2$. By applying [61, Lemma 4] (with D the rotated rectangle) again to the integral after the change of variables, we also obtain the bound $1/\lambda$.

To extend the result to smooth $w(x,y)$ with support in $(a,b)\times(c,d)$, we apply partial integration once in each variable.

Lemma 7.6. *Suppose that f, w, and λ are as above satisfying (7.6) and (7.7). Let*

$$V = \int_a^b \int_c^d \left|\frac{\partial^2 w(x,y)}{\partial x\partial y}\right| \mathrm{d}x\,\mathrm{d}y.$$

Then

$$\int_a^b \int_c^d e(f(x,y))w(x,y)\,\mathrm{d}x\,\mathrm{d}y \preccurlyeq \frac{V}{\lambda},$$

with an absolute implied constant.

Finally, we remark that the generalization of Lemma 7.3 in two (or higher) dimensions as in [60, Theorem 1.1.4] is not sufficient for our purpose because of the angular argument. We refer the reader to [56, §§2.4, 6.1] for discussions in this regard.

8. Analysis of Bessel integrals

Let $B_\nu(x)$ and $B_\nu(z)$ be the Bessel kernels for $F = \mathbb{R}$ and $F = \mathbb{C}$ as in Definition 3.4, respectively. For $1 \preccurlyeq T^\varepsilon \leqslant M \leqslant T^{1-\varepsilon}$, let $h(\nu)$ be (a local component of) the test function as defined in §6. Let $\mathcal{H}(x)$ and $\mathcal{H}(z)$ be the corresponding Bessel integrals,

$$\mathcal{H}(x) = \int_{-\infty}^{\infty} h(\nu)B_{i\nu}(x)\nu\tanh(\pi\nu)\,\mathrm{d}\nu, \quad \mathcal{H}(z) = \int_{-\infty}^{\infty} h(\nu)B_{i\nu}(z)\nu^2\,\mathrm{d}\nu; \tag{8.1}$$

see (3.15).

8.1. Analytic properties of Bessel integrals

We collect here estimates and integral representations for the Bessel integrals. Our attempt is to have a unified presentation, so several results are not necessarily optimal. For the details, we refer the reader to [33] (and also [30, 68] for the real case).

Lemma 8.1. *We have the following estimates for Bessel integrals of small argument:*

(1) *When F is real, for $|x| \leqslant 1$ we have*
$$\mathcal{H}(x) \ll_{A',\varepsilon} M|x|^{1/2}/T^{2A'-1}. \tag{8.2}$$

(2) *When F is complex, for $|z| \leqslant 1$ we have*
$$\mathcal{H}(z) \ll_{A',\varepsilon} M|z|/T^{4A'-2}. \tag{8.3}$$

Proof. These estimates may be derived from modifying the proofs of [33, Lemma 3.2, A.4, and A.6] by shifting the integral contour far right to $\mathrm{Im}(\nu) = A' + \varepsilon$.[2] In view of Lemma 5.1 (2) and (6.4), the test function $h(\nu)$ is holomorphic for $|\mathrm{Im}(\nu)| \leqslant A' + \frac{9}{32}$, and, along with the bound
$$|J_\nu(z)| \ll \frac{|z^\nu|}{\Gamma\big(\nu + \frac{1}{2}\big)}, \quad |z| \leqslant 4\pi,$$
one may estimate the residues and the integral after the contour shift. To be explicit, one has
$$\mathcal{H}(x) \ll e^{-M^2/T^2}|x|^{1/2}\sum_{k=0}^{A'-1}|x|^k + MT^{1+\varepsilon}(|x|^{1/2}/T)^{2A'+2\varepsilon} \ll M|x|^{1/2}/T^{2A'-1},$$
$$\mathcal{H}(z) \ll e^{-M^2/T^2}|z|\sum_{k=0}^{2A'-1}|z|^k + MT^{2+\varepsilon}(|z|/T^2)^{2A'+2\varepsilon} \ll M|z|/T^{4A'-2}. \qquad \blacksquare$$

Lemma 8.2. *There exists a Schwartz function $g(r)$ satisfying $g^{(j)}(r) \ll_{j,A,A',\varepsilon} (1+|r|)^{-A}$ for any $j, A \geqslant 0$, and such that*

(1) *if F is real, then $\mathcal{H}(x) = \mathcal{H}_+(x) + \mathcal{H}_-(x) + O(T^{-A})$ for $|x| > 1$, with*
$$\mathcal{H}_\pm(x^2) = MT^{1+\varepsilon}\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e(Tr/\pi \mp 2x\cosh r)\,\mathrm{d}r, \tag{8.4}$$
$$\mathcal{H}_\pm(-x^2) = MT^{1+\varepsilon}\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e(Tr/\pi \pm 2x\sinh r)\,\mathrm{d}r, \tag{8.5}$$
for $x > 1$;

(2) *if F is complex, then $\mathcal{H}(z) = \mathcal{H}_+(z) + \mathcal{H}_-(z) + O(T^{-A})$ for $|z| > 1$, with*
$$\mathcal{H}_\pm(x^2e^{2i\phi}) = MT^{2+\varepsilon}\int_0^\pi\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e(2Tr/\pi \mp 4x\,\mathrm{trh}(r,\omega;\phi))\,\mathrm{d}r\,\mathrm{d}\omega, \tag{8.6}$$
for $x > 1$, where $\mathrm{trh}(r,\omega;\phi)$ is the "trigonometric-hyperbolic" function defined by
$$\mathrm{trh}(r,\omega;\phi) = \cosh r\cos\omega\cos\phi - \sinh r\sin\omega\sin\phi. \tag{8.7}$$

Furthermore,

(3) *for real x with $1 < |x| \ll T^2$, we have $\mathcal{H}(x) = O(T^{-A})$;*

(4) *for complex z with $1 < |z| \ll T^2$, we have $\mathcal{H}(z) = O(T^{-A})$.*

[2] In the notation of [33], $t = \nu$, $H_{T,M}(\sqrt{z}) = \mathcal{H}(z)$, and $H^\pm_{T,M}(\sqrt{x}) = \mathcal{H}(\pm x)$.

Proof. See [33, (3.2), (3.3), (A.16), and (A.21)][3] for the integral representations in (1) and (2). The important point is that the Fourier transform of $g(\pi r/N)$ is equal to $e^{-\nu^2}$ (for real ν) up to a harmless factor; although the factor involves T and M, one may easily verify that it is bounded by $\ll_{A',\varepsilon} (1 + |\nu|)^{N(6A'+1)+\varepsilon}$ with the implied constant independent on T and M, and so are its derivatives (see (5.15) and (5.16)).

The statements in (3) and (4) follow from simple applications of (one-dimensional) stationary phase to the integrals in (1) and (2); see [33, Lemmas 3.5, A.5, and A.8].

For the real case, we also refer to [30, §§4, 5] and [68, §7]. ■

Remark 8.3. In the real case, it is easy to prove that $\mathscr{H}(x^2)$ or $\mathscr{H}(-x^2)$ is negligibly small unless $x \gg TM^{1-\varepsilon}$ or $x \asymp T$, respectively. See [30].

Corollary 8.4. *We have uniform estimates for Bessel integrals as follows:*

(1) *When F is real, we have*

$$\mathscr{H}(x) \ll_{A',\varepsilon} \begin{cases} T^{1+\varepsilon} & \text{if } |x| \gg T^2, \\ M|x|^{1/2}/T^{2A'-1} & \text{if } |x| \ll T^2. \end{cases} \tag{8.8}$$

(2) *When F is complex, we have*

$$\mathscr{H}(z) \ll_{A',\varepsilon} \begin{cases} T^{2+\varepsilon} & \text{if } |z| \gg T^2, \\ M|z|/T^{4A'-2} & \text{if } |z| \ll T^2. \end{cases} \tag{8.9}$$

8.2. *Preliminary analysis of the trigonometric-hyperbolic function*

Let $\mathrm{trh}(r,\omega;\phi)$ be the trigonometric-hyperbolic function (8.7). Since $\mathrm{trh}(r,\omega;\phi+\pi) = \mathrm{trh}(r,\omega+\pi;\phi) = -\mathrm{trh}(r,\omega;\phi)$, we may restrict ourselves to $\phi,\omega \in [0,\pi)$. When $(r,\omega) \neq (0,\pi/2)$, $\mathrm{trh}(r,\omega;\phi)$ can be written in a unique way as

$$\mathrm{trh}(r,\omega;\phi) = \rho(r,\omega)\cos(\phi + \theta(r,\omega)), \tag{8.10}$$

where $\rho(r,\omega) > 0$ is defined by

$$\rho(r,\omega) = \sqrt{\sinh^2 r + \cos^2\omega} = \sqrt{\cosh^2 r - \sin^2\omega} = \sqrt{\frac{\cosh 2r + \cos 2\omega}{2}}, \tag{8.11}$$

and $\theta(r,\omega)$ is determined by

$$\cos\theta(r,\omega) = \frac{\cosh r \cos\omega}{\rho(r,\omega)}, \quad \sin\theta(r,\omega) = \frac{\sinh r \sin\omega}{\rho(r,\omega)}. \tag{8.12}$$

By defining $\mathrm{trh}(r,\omega) = \rho(r,\omega)e^{i\theta(r,\omega)}$, the function $x\,\mathrm{trh}(r,\omega;\phi)$ in (8.6) is $\mathrm{Re}(z\,\mathrm{trh}(r,\omega))$ for $z = xe^{i\phi}$.

[3]Strictly speaking, the test function in [33] is like the $k(\nu)$ in (6.1), while our test function $h(\nu)$ has extra factors (see (6.4)). However, these factors do not play an essential role.

Remark 8.5. Since

$$\mathrm{trh}(r,0;0)=\mathrm{trh}(r,0)=\cosh r,\quad \mathrm{trh}(r,\pi/2;\pi/2)=i\,\mathrm{trh}(r,\pi/2)=-\sinh r,$$

the reader should observe the resemblance between the r-integral in (8.6) for $\phi=\omega=0$ or $\phi=\omega=\pi/2$ and the integral in (8.4) or (8.5) respectively.

Lemma 8.6. *Suppose that* $(r,\omega)\neq(0,\pi/2)$ *and* $|r|<1$.

(1) *We have*

$$\frac{\partial\theta(r,\omega)}{\partial r}=\frac{\sin 2\omega}{2\rho(r,\omega)^2},\quad \frac{\partial\theta(r,\omega)}{\partial\omega}=\frac{\sinh 2r}{2\rho(r,\omega)^2}.$$

(2) *We have*

$$\frac{\partial^{i+j}}{\partial r^i\partial\omega^j}\left(\frac{1}{\rho(r,\omega)^2}\right)\ll_{i,j}\frac{1}{\rho(r,\omega)^{i+j+2}}.$$

(3) *Consequently, for* $i+j\geqslant 1$, *we have*

$$\frac{\partial^{i+j}\theta(r,\omega)}{\partial r^i\partial\omega^j}\ll_{i,j}\frac{1}{\rho(r,\omega)^{i+j}}.$$

Proof. By (8.12), we have $\tan\theta(r,\omega)=\tanh r\tan\omega$, so

$$\frac{\partial\theta(r,\omega)}{\partial r}=\frac{\sin\omega\cos\omega}{\cosh^2 r\cos^2\omega+\sinh^2 r\sin^2\omega}=\frac{\sin 2\omega}{\cosh 2r+\cos 2\omega},$$

and similarly

$$\frac{\partial\theta(r,\omega)}{\partial\omega}=\frac{\sinh r\cosh r}{\cosh^2 r\cos^2\omega+\sinh^2 r\sin^2\omega}=\frac{\sinh 2r}{\cosh 2r+\cos 2\omega}.$$

The estimates for $\rho(r,\omega)$ in (2) readily follow from an inductive argument by using the identity obtained from applying the i-th r-derivative and the j-th ω-derivative to

$$\frac{\cosh 2r+\cos 2\omega}{\rho(r,\omega)^2}=2,$$

along with the inequalities

$$\sinh 2r,\sin 2\omega\ll\rho(r,\omega),\quad \cosh 2r,\cos 2\omega\ll 1,$$

where the expression $\rho(r,\omega)=\sqrt{\sinh^2 r+\cos^2\omega}$ is used.

Finally, combining the foregoing results, it is straightforward to bound the derivatives of $\theta(r,\omega)$ as in (3). ∎

9. Remarks on Xiaoqing Li's analysis

We briefly recall several aspects of Xiaoqing Li's analysis in [30], and explain the issues for its generalization to the complex setting or the case when the number field has multiple infinite places.

In the real setting of [30], $T^{3/8+\varepsilon} \leqslant M \leqslant T^{1/2}$ and $TM^{1-\varepsilon} \leqslant x \leqslant T^{3/2+\varepsilon} \leqslant M^4$. By expanding $\cosh r$ in a Taylor series, and disregarding the non-oscillatory factors from the terms of order $\geqslant 4$, the integral $\mathcal{H}_\pm(x^2)$ in (8.4) essentially turns into

$$MT^{1+\varepsilon}e(\mp 2x)\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e(Tr/\pi \mp xr^2)\,\mathrm{d}r.$$

Xiaoqing Li's next step is to complete the square, getting

$$MT^{1+\varepsilon}e\left(\mp 2x \pm \frac{T^2}{4\pi^2 x}\right)\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e\left(\mp x\left(r \pm \frac{T}{2\pi x}\right)^2\right)\mathrm{d}r;$$

by Parseval, the integral is seen to be a non-oscillatory function of x. The secondary exponential factor $e(\pm T^2/4\pi^2 x)$ plays an important role in her second application of Voronoï summation.

In the complex setting, however, the corresponding conditions are $T^{3/4+\varepsilon} \leqslant M \leqslant T$ and $T \leqslant x \leqslant T^{3/2+\varepsilon} \leqslant M^2$. After expanding $\cosh r$ and $\sinh r$ in Taylor series, only the factors of order 0 and 1 are oscillatory, and the integral $\mathcal{H}_\pm(x^2e^{2i\phi})$ in (8.6) is essentially

$$MT^{2+\varepsilon}\int_0^\pi\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e\big(2Tr/\pi \mp 4x(\cos\omega\cos\phi - r\sin\omega\sin\phi)\big)\,\mathrm{d}r\,\mathrm{d}\omega.$$

Hence, we are unable to produce a secondary exponential factor. Even if there were such a factor, the analysis would be conceivably difficult, because $\cos\omega\cos\phi$ would go down to the denominator together with x.

Moreover, when the number field has more than one infinite place, a more serious issue is that the condition $x \leqslant T^{3/2+\varepsilon}$ is not necessarily valid for every infinite place.

At any rate, it is better not to expand $\cosh r$ or $\sinh r$ in Taylor series at this stage, and to allow M be a small power of T.

10. Stationary phase for the Hankel transforms

In this section, we consider certain integrals that will arise from the Hankel transforms over $\mathbb{R}$ and $\mathbb{C}$. For the real case, it is simply a matter of applying the method of stationary phase in one dimension. For the complex case, the double integral has already been investigated in [56, §6.1], but there are certain difficulties in two dimensions—Lemma 7.4 is not applicable for the particular phase function, and [60, Theorem 1.1.4] is not sufficient as we also need to differentiate the angular argument.

10.1. The one-dimensional case

First, in the real setting, we need to consider the integral

$$I(\lambda) = \int_{-\infty}^{\infty} e(\lambda(3x^2 - 2x^3))w(x;\lambda)\,\mathrm{d}x. \tag{10.1}$$

Fix $\Delta > 1$. Let $\rho, S > 0$ and $X \geqslant 1$. Suppose that the function $w(x; \lambda)$ is supported in $\{x : |x| \in [\rho, \Delta^{1/6}\rho]\}$ and its derivatives satisfy

$$x^i \lambda^j \partial_x^i \partial_\lambda^j w(x; \lambda) \ll_{i,j} SX^{i+j}.$$

Define

$$I^\natural(\lambda) = e(-\lambda) I(\lambda). \tag{10.2}$$

Lemma 10.1. *Let A and j be non-negative integers.*

(1) *For either $\rho \geqslant \sqrt{\Delta}$ or $\rho \leqslant 1/\sqrt{\Delta}$, we have*

$$I^\natural(\lambda) \ll_A S\rho \left(\frac{X}{|\lambda| \rho^2 (\rho + 1)} \right)^A.$$

(2) *Assume that $X \leqslant \sqrt{|\lambda|}$. For $1/\Delta \leqslant \rho \leqslant \Delta$, we have*

$$\lambda^j \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} I^\natural(\lambda) \ll_j \frac{SX^j}{\sqrt{|\lambda|}}.$$

Proof. Note that the phase function $3x^2 - 2x^3$ has a unique non-zero stationary point at $x_0 = 1$. The estimates in (1) readily follow from Lemma 7.1; in the case $\rho \geqslant \sqrt{\Delta}$, choose $P = \rho/X$, $Q = \rho$, $Z = |\lambda|\rho^3$, $R = |\lambda|\rho^2$, and in the case $\rho \leqslant 1/\sqrt{\Delta}$, choose $P = \rho/X$, $Q = 1$, $Z = |\lambda|$, $R = |\lambda|\rho$. The estimates in (2) essentially follow from Lemma 7.3. ■

10.2. The two-dimensional case

Second, consider the following double integral that will arise from the complex Hankel transform

$$I(\lambda, \psi) = \int_0^{2\pi} \int_0^\infty e(2\lambda f(x, \phi; \psi)) w(x, \phi; \lambda, \psi)\, \mathrm{d}x\, \mathrm{d}\phi, \tag{10.3}$$

with

$$f(x, \phi; \psi) = 3x^2 \cos(2\phi + \psi) - 2x^3 \cos 3\phi. \tag{10.4}$$

Fix $\Delta > 1$. Let $\rho, S > 0$ and $X \geqslant 1$. Suppose that $w(x, \phi; \lambda, \psi)$ is supported in $\{(x, \phi) : x \in [\rho, \Delta^{1/6}\rho]\}$ and its derivatives satisfy

$$x^i \lambda^k \partial_x^i \partial_\phi^j \partial_\lambda^k \partial_\psi^l w(x, \phi; \lambda, \psi) \ll_{i,j,k,l} SX^{i+j+k+l}.$$

Define

$$I^\natural(\lambda, \psi) = e(-2\lambda \cos 3\psi) I(\lambda, \psi). \tag{10.5}$$

Results from [56, Lemmas 6.1 and 6.3] are quoted in the following lemma with slightly altered notation.

Lemma 10.2. *Let A, k, l be non-negative integers.*

(1) *For either $\rho \geqslant \sqrt{\Delta}$ or $\rho \leqslant 1/\sqrt{\Delta}$, we have*

$$I^\natural(\lambda, \psi) \ll_A S\rho \left(\frac{X}{\lambda \rho^2 (\rho + 1)} \right)^A.$$

(2) *Assume that* $X \leqslant \sqrt{\lambda}$. *For* $1/\Delta \leqslant \rho \leqslant \Delta$, *we have*

$$\lambda^k \frac{\partial^{k+l}}{\partial \lambda^k \partial \psi^l} I^{\natural}(\lambda, \psi) \ll_{k,l} \frac{S X^{k+l}}{\lambda}.$$

11. Analysis of the Hankel transforms, I

Let $w(x)$ be a smooth function supported on $[1, \Delta]$ satisfying $w^{(i)}(x) \ll_i \log^i T$ for all $i \geqslant 0$. For $|\Lambda| \geqslant T^2$, define

$$w(x, \Lambda) = w(|x|)\mathcal{H}(\Lambda x) \tag{11.1}$$

if F is real, and

$$w(z, \Lambda) = w(|z|)\mathcal{H}(\Lambda z) \tag{11.2}$$

if F is complex. Let $\widetilde{w}(y, \Lambda)$ and $\widetilde{w}(u, \Lambda)$ be their Hankel transforms (see Definition 4.4) defined by

$$\widetilde{w}(y, \Lambda) = \int w(x, \Lambda) J_\pi(xy)\, \mathrm{d}x, \quad \widetilde{w}(u, \Lambda) = \iint w(z, \Lambda) J_\pi(zu)\, \mathrm{d}z, \tag{11.3}$$

and modify $\widetilde{w}(y, \Lambda)$ and $\widetilde{w}(u, \Lambda)$ by exponential factors as follows:

$$\widetilde{w}^{\natural}(y, \Lambda) = e(-y/\Lambda)\widetilde{w}(y, \Lambda), \quad \widetilde{w}^{\natural}(u, \Lambda) = e(-2\,\mathrm{Re}(u/\Lambda))\widetilde{w}(u, \Lambda). \tag{11.4}$$

Roughly speaking, our wish is to transform $\widetilde{w}^{\natural}(y, \Lambda)$ and $\widetilde{w}^{\natural}(u, \Lambda)$ into the shape

$$\frac{M T^{1+\varepsilon}}{\sqrt{|y|}} \Phi^{\sigma}(y/\Lambda), \qquad \frac{M T^{2+\varepsilon}}{|u|} \Phi^{\sigma}(u/\Lambda),$$

with $\sigma = 0, -, +, \flat$ in various circumstances. It turns out that the analytic properties of $\Phi^{\sigma}(x)$ or $\Phi^{\sigma}(z)$ depend only mildly on Λ and M, so, for brevity, this dependence will be suppressed from our notation.

11.1. The small-argument case

We first consider the case when the Hankel transforms have relatively small argument. However, this case arises only when there are infinitely many units in the number field.

The following lemma is essentially due to [5, Lemma 7] and [56, Lemma 6.4] (as indicated in Remark 4.3, Blomer has a slightly different normalization).

Lemma 11.1. *For* $w \in C_c^{\infty}(F^{\times})$ *define* $\|w\|_{L^\infty}$ *to be its sup-norm. If* w *is supported in a fixed compact set* $K \subset F^{\times}$, *then its Hankel transform* $\widetilde{w}$ *has the following estimates*:

(1) *when* F *is real,*

$$y^i (\mathrm{d}/\mathrm{d}y)^i \widetilde{w}(y) \ll_{i,K} \|w\|_{L^\infty} \cdot (|y|^{1/3} + 1)^i / |y|^{1/3};$$

(2) *when F is complex,*

$$u^i \overline{u}^j (\partial/\partial u)^i (\partial/\partial \overline{u})^j \widetilde{w}(u) \ll_{i,j,K} \|w\|_{L^\infty} \cdot (|u|^{1/3}+1)^{i+j}/|u|^{2/3}.$$

As a consequence of Corollary 8.4 and Lemma 11.1, for $|y| \leqslant T^\varepsilon$ we have

$$y^i \frac{\mathrm{d}^i \widetilde{w}(y, \Lambda)}{\mathrm{d}y^i} \ll_i \frac{T^{1+(i+1)\varepsilon}}{|y|^{1/3}} \ll \frac{MT^{1+(i+1)\varepsilon}}{\sqrt{|y|}},$$

and in polar coordinates,

$$y^i \frac{\partial^{i+j} \widetilde{w}(ye^{i\theta}, \Lambda)}{\partial y^i \partial \theta^j} \ll_{i,j} \frac{T^{2+(i+j+1)\varepsilon}}{y^{2/3}} \ll \frac{MT^{2+(i+j+1)\varepsilon}}{y}.$$

Corollary 11.2. *Let $|\Lambda| \gg T^2$. Artificially define $\Phi^0(x)$ and $\Phi^0(z)$ by*

$$\widetilde{w}^\natural(y, \Lambda) = \frac{MT^{1+\varepsilon}}{\sqrt{|y|}} \Phi^0(y/\Lambda), \quad \widetilde{w}^\natural(u, \Lambda) = \frac{MT^{2+\varepsilon}}{|u|} \Phi^0(u/\Lambda), \tag{11.5}$$

with $x = y/\Lambda$ and $z = u/\Lambda$.

(1) *When F is real, for $|x| \leqslant T^\varepsilon/|\Lambda|$ we have*

$$x^i \frac{\mathrm{d}^i \Phi^0(x)}{\mathrm{d}x^i} \ll_i T^{i\varepsilon}. \tag{11.6}$$

(2) *When F is complex, for $x \leqslant T^\varepsilon/|\Lambda|$ we have*

$$x^i \frac{\partial^{i+j} \Phi^0(xe^{i\phi})}{\partial x^i \partial \phi^j} \ll T^{(i+j)\varepsilon}. \tag{11.7}$$

11.2. Application of stationary phase

Our next goal is to deduce integral representations of $\widetilde{w}^\natural(y, \Lambda)$ and $\widetilde{w}^\natural(u, \Lambda)$ from those of the GL_2-Bessel integrals $\mathcal{H}(x)$ and $\mathcal{H}(z)$ in Lemma 8.2 (1, 2) along with the asymptotic formulae for the GL_3-Bessel kernels $J_\pi(x)$ and $J_\pi(z)$ in Lemma 4.12.

Proposition 11.3. *Suppose that $|y| > T^\varepsilon$ and $|\Lambda| \gg T^2$. Define $\mathrm{hyp}_\pm(r)$ to be the hyperbolic function*

$$\mathrm{hyp}_+(r) = \cosh r, \quad \mathrm{hyp}_-(r) = -\sinh r. \tag{11.8}$$

There are smooth functions $V_\pm(r; y, \Lambda)$ with support in the region defined by

$$(1/\Delta) \cdot |y|^{1/3}/|\Lambda|^{1/2} \leqslant |\mathrm{hyp}_\pm(r)| \leqslant \Delta \cdot |y|^{1/3}/|\Lambda|^{1/2}, \tag{11.9}$$

satisfying

$$\begin{aligned} (\partial/\partial r)^i V_+(r; y, \Lambda) &\ll_i \log^i T, \\ r^i (\partial/\partial r)^i V_-(r; y, \Lambda) &\ll_i \log^i T, \end{aligned} \tag{11.10}$$

such that

$$\widetilde{w}^{\natural}(y, \Lambda) = \frac{MT^{1+\varepsilon}}{\sqrt{|y|}}(\Phi_{+}(y, \Lambda) + \Phi_{-}(y, \Lambda)) + O(T^{-A}), \tag{11.11}$$

with

$$\Phi_{\pm}(y, \Lambda) = \int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} e(Tr/\pi)g(Mr)e(-y\,\mathrm{hyp}_{\pm}^{\natural}(r)^2/\Lambda)V_{\pm}(r; y, \Lambda)\,\mathrm{d}r, \tag{11.12}$$

in which g is a Schwartz function, and

$$\mathrm{hyp}_{+}^{\natural}(r) = \tanh r, \quad \mathrm{hyp}_{-}^{\natural}(r) = \coth r. \tag{11.13}$$

The reader may find the integral $\Phi_{\pm}(y, \Lambda)$ in [68, (8.16)] and [18, (4.20)]. Our analysis is slightly different, however, as our strategy is to first transform the x-integral into $I(\lambda)$ as in (10.1) with phase $\lambda(2x^3 - 3x^2)$ and then use the stationary phase results in Lemma 10.1, while Young and Binrong Huang directly apply general stationary phase results [68, Lemma 6.3] with phase in an arbitrary form. A technical remark is that, in view of (11.10), $V_{+}(r; y, \Lambda)$ is more than an "inert" function in the sense of Young [68].

Proof of Proposition 11.3. To start with, let us assume $\Lambda > 0$, for $\widetilde{w}^{\natural}(y, -\Lambda) = \widetilde{w}^{\natural}(-y, \Lambda)$.

By Lemma 4.12 (1), the contribution to $\widetilde{w}(y, \Lambda)$ from the leading term in (4.22) is the following integral:

$$I(y^{1/3}, \Lambda) = \int e(3(xy)^{1/3})\mathcal{H}(\Lambda x)w(|x|)\,\frac{\mathrm{d}x}{|xy|^{1/3}};$$

the contributions from lower-order terms are similar and may be handled in the same manner. Also, it follows from Corollary 8.4 (1) that the error term in (4.22) yields an $O(T^{1+\varepsilon}/|y|^{(K+1)/3}) = O(T^{-A})$ for $|y| > T^{\varepsilon}$ if we choose K large, say $K > 3(A+1)/\varepsilon$.

Next, we change the variables x and y to $\pm x^6$ and y^3 so that

$$I(y, \Lambda) = \sum_{\pm} \frac{1}{|y|} \int_0^{\infty} e(\pm 3x^2 y)\mathcal{H}(\pm\Lambda x^6)a(x)\,\mathrm{d}x,$$

where $a(x) = 6x^3 w(x^6)$ is supported on $[1, \Delta^{1/6}]$ and satisfies $a^{(i)}(x) \ll_i \log^i T$. By formulae (8.4) and (8.5) in Lemma 8.2, we infer that $I(y, \Lambda)$ may be written as

$$I(y, \Lambda) = MT^{1+\varepsilon} \sum_{\pm} \int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} e(Tr/\pi)g(Mr)I_{\pm}(r; y, \Lambda)\,\mathrm{d}r + O(T^{-A}),$$

where

$$I_{\pm}(r; y, \Lambda) = \frac{1}{|y|} \int_{-\infty}^{\infty} e(\pm 3x^2 y \mp 2\sqrt{\Lambda}\,x^3\,\mathrm{hyp}_{\pm}(r))a(|x|)\,\mathrm{d}x.$$

On changing x to $xy/\sqrt{\Lambda}\,\mathrm{hyp}_{\pm}(r)$ (which needs $r \neq 0$ to guarantee $\mathrm{hyp}_{-}(r) \neq 0$), the inner integral $I_{\pm}(r; y, \Lambda)$ turns into

$$\frac{1}{\sqrt{\Lambda}\,|\mathrm{hyp}_{\pm}(r)|} \int_{-\infty}^{\infty} e\left(\pm\frac{y^3}{\Lambda\,\mathrm{hyp}_{\pm}(r)^2}(3x^2 - 2x^3)\right)a\left(\frac{|xy|}{\sqrt{\Lambda}\,|\mathrm{hyp}_{\pm}(r)|}\right)\mathrm{d}x,$$

and it is exactly the integral $I(\lambda)$ defined as in (10.1) if one lets $\lambda = \pm y^3/\Lambda\, \mathrm{hyp}_\pm(r)^2$, $\rho = \sqrt{\Lambda}\, |\mathrm{hyp}_\pm(r)|/|y|$ $(= \sqrt{|y/\lambda|})$, and

$$w(x;\lambda) = \sqrt{|\lambda/y^3|}\cdot a(|x|\sqrt{|\lambda/y|}),$$

with

$$x^i\lambda^j\partial_x^i\partial_\lambda^j w(x;\lambda) \ll_{i,j} \sqrt{|\lambda/y^3|}\cdot \log^{i+j} T.$$

Let $I^\natural(\lambda) = e(-\lambda)I(\lambda)$ be as in (10.2). We introduce a smooth function $v(x)$ such that $v(x) \equiv 1$ on $[1/\sqrt{\Delta}, \sqrt{\Delta}]$ and $v(x) \equiv 0$ on $(0, 1/\Delta] \cup [\Delta, \infty)$. According to Lemma 10.1 (1), if $\rho = \sqrt{|y/\lambda|}$ is not in the interval $(1/\sqrt{\Delta}, \sqrt{\Delta})$, then

$$I^\natural(\lambda) \ll \frac{1}{|y|}\left(\frac{\log T}{\sqrt{|y^3/\lambda|} + |y|}\right)^K < \frac{\log^K T}{|y|^{K+1}},$$

and hence $I^\natural(\lambda)(1 - v(\lambda/y))$ only contributes to the error term. By Lemma 10.1 (2),

$$\lambda^j\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\big(I^\natural(\lambda)v(|\lambda/y|)\big) \ll_j \frac{\log^j T}{\sqrt{|y|^3}}.$$

Finally, let[4]

$$V_\pm(r; y^3, \Lambda) = \sqrt{|y|^3}\, I^\natural(\lambda)v(|\lambda/y|), \qquad (\lambda = \pm y^3/(\Lambda\, \mathrm{hyp}_\pm(r)^2)),$$

then, after changing y into $y^{1/3}$, the expression of $\widetilde{w}^\natural(y, \Lambda)$ (defined in (11.4)) given by (11.11) and (11.12) readily follow from the arguments above, along with the identity

$$-\mathrm{hyp}_\pm^\natural(r)^2 = -1 \pm \frac{1}{\mathrm{hyp}_\pm(r)^2},$$

and, to deduce (11.10) one needs the estimates[5]

$$\mathrm{hyp}_\pm(r)\frac{\mathrm{d}^j}{\mathrm{d}r^j}\left(\frac{1}{\mathrm{hyp}_\pm(r)}\right) \ll_j \frac{1}{|\mathrm{hyp}_\pm(r)|^j}$$

for $|r| < 1$. ∎

Proposition 11.4. *Suppose that $|u| > T^\varepsilon$ and $|\Lambda| \gg T^2$. Recall the definition of $\rho(r,\omega)$ in (8.11). There is a smooth function $V(r,\omega;u,\Lambda)$ with support in the region defined by*

$$(1/\Delta)\cdot|u|^{1/3}/|\Lambda|^{1/2} \leqslant \rho(r,\omega) \leqslant \Delta\cdot|u|^{1/3}/|\Lambda|^{1/2}, \tag{11.14}$$

satisfying

$$\rho(r,\omega)^{i+j}(\partial/\partial r)^i(\partial/\partial\omega)^j V(r,\omega;u,\Lambda) \ll_{i,j} \log^{i+j} T, \tag{11.15}$$

[4]This $V_\pm$ function is only from the leading term in (4.22), so, to be strict, one must also include those $V_\pm$ functions constructed from the lower-order terms in (4.22).

[5]We also need Faà di Bruno's formula for higher derivatives of composite functions (see [26]).

such that

$$\widetilde{w}^{\natural}(u,\Lambda) = \frac{MT^{2+\varepsilon}}{|u|}\Phi(u,\Lambda) + O(T^{-A}) \tag{11.16}$$

with

$$\Phi(u,\Lambda) = \int_0^{\pi}\int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} e(2Tr/\pi)g(Mr)e(-2\,\mathrm{Re}(u\,\mathrm{trh}^{\natural}(r,\omega)/\Lambda))V(r,\omega;u,\Lambda)\,\mathrm{d}r\,\mathrm{d}\omega, \tag{11.17}$$

in which g is a Schwartz function, and $\mathrm{trh}^{\natural}(r,\omega) = \rho^{\natural}(r,\omega)e^{i\theta^{\natural}(r,\omega)}$ *is defined by*

$$\rho^{\natural}(r,\omega) = \frac{\cosh 2r - \cos 2\omega}{\cosh 2r + \cos 2\omega}, \quad \tan(\theta^{\natural}(r,\omega)/2) = \frac{\sin 2\omega}{\sinh 2r}. \tag{11.18}$$

It is remarkable that the square-root signs in the formulae of $\rho(r,\omega)$ in (8.11) are no longer in the formula of $\rho^{\natural}(r,\omega)$ in (11.18). This makes our life easier in polar coordinates.

Proof of Proposition 11.4. The first stage of proof will be similar to that of Proposition 11.3.

Let us assume without loss of generality that $\Lambda > 0$ and consider the integral

$$I(u^{1/3},\Lambda) = \sum_{\xi^3=1}\iint e(6\,\mathrm{Re}(\xi(zu)^{1/3}))\mathscr{H}(\Lambda z)w(|z|)\,\frac{\mathrm{d}z}{|zu|^{2/3}},$$

which is the contribution from the three leading terms in (4.23) in Lemma 4.12 (2). Substituting the variables z and u by z^6 and u^3, we have

$$I(u,\Lambda) = \frac{1}{|u|^2}\iint_{\mathbb{C}^{\times}/\{\pm1\}} e(6\,\mathrm{Re}(z^2u))\mathscr{H}(\Lambda z^6)a(|z|)\,\mathrm{d}z/|z|,$$

where $a(x) = 36x^7w(x^6)$ is supported on $[1,\Delta^{1/6}]$ and satisfies $a^{(i)}(x) \ll_i \log^i T$.

Let $z = xe^{i\phi}$ and $u = ye^{i\theta}$. By formula (8.6) in Lemma 8.2, we infer that $I(ye^{i\theta},\Lambda)$ may be written as

$$I(ye^{i\theta},\Lambda) = MT^{2+\varepsilon}\int_0^{\pi}\int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} e(2Tr/\pi)g(Mr)I(r,\omega;ye^{i\theta},\Lambda)\,\mathrm{d}r\,\mathrm{d}\omega + O(T^{-A}),$$

in which

$$I(r,\omega;ye^{i\theta},\Lambda) = \frac{2}{y^2}\int_0^{2\pi}\int_0^{\infty} e\big(6x^2y\cos(2\phi+\theta) - 4\sqrt{\Lambda}x^3\,\mathrm{trh}(r,\omega;3\phi)\big)a(x)\,\mathrm{d}x\,\mathrm{d}\phi.$$

At this point, we assume $(r,\omega) \neq (0,\pi/2)$ and invoke the expression of $\mathrm{trh}(r,\omega;3\phi)$ as in (8.10). On changing the variables x and ϕ to $xy/\sqrt{\Lambda}\,\rho(r,\omega)$ and $\phi - \theta(r,\omega)/3$, respectively, the integral $I(r,\omega;ye^{i\theta},\Lambda)$ turns into

$$\frac{2}{\sqrt{\Lambda}\,\rho(r,\omega)y}\int_0^{2\pi}\int_0^{\infty} e\left(\frac{2y^3}{\Lambda\rho(r,\omega)^2}f(x,\phi;\theta-2\theta(r,\omega)/3)\right)a\left(\frac{xy}{\sqrt{\Lambda}\,\rho(r,\omega)}\right)\mathrm{d}x\,\mathrm{d}\phi.$$

According to the notation in §10.2, the phase function $f(x, \phi; \psi)$ is defined by (10.4), and the integral above is of the form $I(\lambda, \psi)$ as in (10.3) if one lets $\lambda = y^3/\varLambda\rho(r,\omega)^2$, $\psi = \theta - 2\theta(r,\omega)/3$, and $\rho = \sqrt{\varLambda}\,\rho(r,\omega)/y$ $(= \sqrt{y/\lambda})$; the weight function

$$w(x;\lambda) = 2\sqrt{\lambda/y^5}\cdot a(x\sqrt{\lambda/y})$$

has bounds

$$x^i\lambda^k\partial_x^i\partial_\lambda^k w(x;\lambda) \ll_{i,k} \sqrt{\lambda/y^5}\log^{i+k} T.$$

Next, we apply Lemma 10.2 to $I^\natural(\lambda,\psi) = e(-2\lambda\cos 3\psi)I(\lambda,\psi)$ as in (10.5). Let $v(x)$ be a smooth function such that $v(x)\equiv 1$ on $[1/\sqrt{\varDelta}, \sqrt{\varDelta}]$ and $v(x)\equiv 0$ on $(0, 1/\varDelta]\cup[\varDelta,\infty)$. Lemma 10.2 (1) implies that $I^\natural(\lambda,\psi)(1 - v(\lambda/y))$ only contributes an error term, while Lemma 10.2 (2) yields the estimates

$$\lambda^k\frac{\partial^{k+l}}{\partial\lambda^k\partial\psi^l}(I^\natural(\lambda,\psi)v(\lambda/y)) \ll_{k,l} \frac{\log^{k+l} T}{y^3}.$$

Keeping in mind that $\psi = \theta - 2\theta(r,\omega)/3$ and $\lambda = y^3/\varLambda\rho(r,\omega)^2$, the estimates above along with those for $1/\rho(r,\omega)^2$ and $\theta(r,\omega)$ in Lemma 8.6 imply that[6]

$$\rho(r,\omega)^{i+j}\omega(\partial/\partial r)^i(\partial/\partial\omega)^j V(r,\omega; y^3e^{3i\theta},\varLambda) \ll_{i,j} 1,$$

with

$$V(r,\omega; y^3e^{3i\theta},\varLambda) = y^3 I^\natural(\lambda,\psi)v(\lambda/y),$$

supported in the region $y/\varDelta\sqrt{\varLambda} \leqslant \rho(r,\omega) \leqslant \varDelta y/\sqrt{\varLambda}$.

In view of (10.5) and (11.4), we need to compute the exponential factor $e(2\lambda\cos 3\psi - 2y^3\cos 3\theta/\varLambda)$, in which

$$2\lambda\cos 3\psi = \frac{2y^3\cos(3\theta - 2\theta(r,\omega))}{\varLambda\rho(r,\omega)^2}.$$

After reverting $y^3e^{3i\theta}$ to $ye^{i\theta}$, the proof is completed if we can prove (11.18) for $\rho^\natural(r,\omega)$ and $\theta^\natural(r,\omega)$ given by

$$\cos\theta - \frac{\cos(\theta - 2\theta(r,\omega))}{\rho(r,\omega)^2} = \rho^\natural(r,\omega)\cos(\theta + \theta^\natural(r,\omega)).$$

[6] We need to use here the simple fact: For a composite function $f(\lambda(r,\omega),\theta(r,\omega))$ in general, its derivative $\partial_r^i\partial_\omega^j f(\lambda(r,\omega),\theta(r,\omega))$ is a linear combination of

$$\partial_\lambda^k\partial_\theta^l f(\lambda(r,\omega),\theta(r,\omega))\prod_{\nu=1}^{k}\partial_r^{i_\nu}\partial_\omega^{j_\nu}\lambda(r,\omega)\prod_{\mu=1}^{l}\partial_r^{i'_\mu}\partial_\omega^{j'_\mu}\theta(r,\omega),$$
$$\sum i_\nu + \sum i'_\mu = i, \quad \sum j_\nu + \sum j'_\mu = j.$$

This is a two-dimensional Faà di Bruno's formula in a less precise form.

We have

$$\rho^{\natural}(r,\omega)=\sqrt{\left(1-\frac{\cos 2\theta(r,\omega)}{\rho(r,\omega)^2}\right)^2+\left(\frac{\sin 2\theta(r,\omega)}{\rho(r,\omega)^2}\right)^2},$$

$$\cos\theta^{\natural}(r,\omega)=\frac{1}{\rho^{\natural}(r,\omega)}-\frac{\cos 2\theta(r,\omega)}{\rho^{\natural}(r,\omega)\rho(r,\omega)^2},\quad \sin\theta^{\natural}(r,\omega)=\frac{\sin 2\theta(r,\omega)}{\rho^{\natural}(r,\omega)\rho(r,\omega)^2}.$$

By the definitions of $\rho(r,\omega)$ and $\theta(r,\omega)$ in (8.11) and (8.12), we have

$$\begin{aligned}1-\frac{\cos 2\theta(r,\omega)}{\rho(r,\omega)^2}&=1-\frac{4(\cosh^2 r\cos^2\omega-\sinh^2 r\sin^2\omega)}{(\cosh 2r+\cos 2\omega)^2}\\&=1-\frac{2(\cosh 2r\cos 2\omega+1)}{(\cosh 2r+\cos 2\omega)^2}=\frac{\cosh^2 2r+\cos^2 2\omega-2}{(\cosh 2r+\cos 2\omega)^2}\\&=\frac{\sinh^2 2r-\sin^2 2\omega}{(\cosh 2r+\cos 2\omega)^2},\end{aligned}$$

and similarly

$$\frac{\sin 2\theta(r,\omega)}{\rho(r,\omega)^2}=\frac{2\sinh 2r\cdot\sin 2\omega}{(\cosh 2r+\cos 2\omega)^2}.$$

We conclude that

$$\rho^{\natural}(r,\omega)=\frac{\sinh^2 2r+\sin^2 2\omega}{(\cosh 2r+\cos 2\omega)^2}=\frac{\cosh 2r-\cos 2\omega}{\cosh 2r+\cos 2\omega},$$

$$\cos\theta^{\natural}(r,\omega)=\frac{\sinh^2 2r-\sin^2 2\omega}{\sinh^2 2r+\sin^2 2\omega},\quad \sin\theta^{\natural}(r,\omega)=\frac{2\sinh 2r\cdot\sin 2\omega}{\sinh^2 2r+\sin^2 2\omega},$$

and hence

$$\tan(\theta^{\natural}(r,\omega)/2)=\frac{\sin 2\omega}{\sinh 2r}.$$ ■

Corollary 11.5. *We have* $\widetilde{w}(y,\Lambda)$, $\widetilde{w}(u,\Lambda)=O(T^{-A})$ *for* $|y|$, $|u|\geqslant|\Lambda|^{3/2}$, *respectively.*

Proof. This is clear from (11.9) and (11.14). ■

11.3. Analysis of the new trigonometric-hyperbolic function

For later use, we record here some results concerning the trigonometric-hyperbolic function $\mathrm{trh}^{\natural}(r,\omega)$ that arose in Proposition 11.4. By (11.18), we have

$$\frac{\partial\rho^{\natural}(r,\omega)}{\partial r}=\frac{4\sinh 2r\cos 2\omega}{(\cosh 2r+\cos 2\omega)^2},\qquad \frac{\partial\rho^{\natural}(r,\omega)}{\partial\omega}=\frac{4\cosh 2r\sin 2\omega}{(\cosh 2r+\cos 2\omega)^2}\tag{11.19}$$

$$\frac{\partial\theta^{\natural}(r,\omega)}{\partial r}=-\frac{4\cosh 2r\sin 2\omega}{\cosh^2 2r-\cos^2 2\omega},\qquad \frac{\partial\theta^{\natural}(r,\omega)}{\partial\omega}=\frac{4\sinh 2r\cos 2\omega}{\cosh^2 2r-\cos^2 2\omega}.\tag{11.20}$$

Note that $\sinh^2 2r+\sin^2 2\omega=\cosh^2 2r-\cos^2 2\omega$.

Lemma 11.6. *Define* $\mathrm{trh}^\natural(r,\omega;\phi) = \rho^\natural(r,\omega)\cos(\phi+\theta^\natural(r,\omega))$. *For* $|r|<1$, *we have*

$$\frac{\partial^{i+j}\,\mathrm{trh}^\natural(r,\omega;\phi)}{\partial r^i\,\partial\omega^j} \ll_{i,j} \frac{\rho^\natural(r,\omega)}{(\cosh^2 2r - \cos^2 2\omega)^{(i+j)/2}}.$$

Proof. Set $\psi = \phi + \theta^\natural(r,\omega)$. From (11.19) and (11.20) we deduce that

$$\frac{\partial\,\mathrm{trh}^\natural(r,\omega;\phi)}{\partial r} = \frac{4\sinh 2r\cos 2\omega}{(\cosh 2r+\cos 2\omega)^2}\cos\psi + \frac{4\cosh 2r\sin 2\omega}{(\cosh 2r+\cos 2\omega)^2}\sin\psi,$$
$$\frac{\partial\,\mathrm{trh}^\natural(r,\omega;\phi)}{\partial \omega} = \frac{4\cosh 2r\sin 2\omega}{(\cosh 2r+\cos 2\omega)^2}\cos\psi - \frac{4\sinh 2r\cos 2\omega}{(\cosh 2r+\cos 2\omega)^2}\sin\psi.$$

For $i+j\geqslant 1$, we may prove by induction that $\partial^{i+j}\,\mathrm{trh}^\natural(r,\omega;\phi)/\partial r^i\,\partial\omega^j$ is a linear combination of

$$\frac{\sinh^{i_1} 2r\cosh^{i_2} 2r\sin^{j_1} 2\omega\cos^{j_2} 2\omega}{(\cosh 2r+\cos 2\omega)^{k_1+l+2}(\cosh 2r-\cos 2\omega)^{k_2+l}}\cdot\begin{Bmatrix}\cos\psi\\ \sin\psi\end{Bmatrix},$$

with

$$k_1+k_2+l\leqslant i+j-1,\quad 2(k_1+k_2+l)\leqslant i+j+i_1+j_1-2,$$
$$i_1+i_2+j_1+j_2 = k_1+k_2+2l+2,\quad i_2\leqslant i+1,\quad j_2\leqslant j+1.$$

Such a fraction is bounded by

$$\begin{aligned}&\frac{(\sinh^2 2r+\sin^2 2\omega)^{(i_1+j_1)/2}}{(\cosh 2r+\cos 2\omega)^{k_1+l+2}(\cosh 2r-\cos 2\omega)^{k_2+l}}\\ &\quad= \frac{1}{(\cosh 2r+\cos 2\omega)^{k_1+l-(i_1+j_1)/2+2}(\cosh 2r-\cos 2\omega)^{k_2+l-(i_1+j_1)/2}}\\ &\quad\ll \frac{1}{(\cosh 2r+\cos 2\omega)^{(i+j)/2-k_2+1}(\cosh 2r-\cos 2\omega)^{(i+j)/2-k_1-1}}\\ &\quad\ll \frac{1}{(\cosh 2r+\cos 2\omega)^{(i+j)/2+1}(\cosh 2r-\cos 2\omega)^{(i+j)/2-1}},\end{aligned}$$

as desired. Note that $2(k_1+k_2+l)\leqslant i+j+i_1+j_1-2$ is used here for the first inequality. ∎

By (11.19) and (11.20), we have

$$\frac{\partial\log\rho^\natural(r,\omega)}{\partial r} = \frac{\partial\theta^\natural(r,\omega)}{\partial\omega} = \frac{4\sinh 2r\cos 2\omega}{\sinh^2 2r+\sin^2 2\omega}, \tag{11.21}$$
$$\frac{\partial\theta^\natural(r,\omega)}{\partial r} = -\frac{\partial\log\rho^\natural(r,\omega)}{\partial\omega} = -\frac{4\cosh 2r\sin 2\omega}{\sinh^2 2r+\sin^2 2\omega}. \tag{11.22}$$

Similar to Lemma 11.6, one can establish the following lemma.

Lemma 11.7. *For $|r| < 1$, we have*

$$\frac{\partial^{i+j} \log \rho^{\natural}(r,\omega)}{\partial r^i \partial \omega^j}, \ \frac{\partial^{i+j} \theta^{\natural}(r,\omega)}{\partial r^i \partial \omega^j} \ll_{i,j} \frac{1}{(\sinh^2 2r + \sin^2 2\omega)^{(i+j)/2}}$$

for $i + j \geqslant 1$.

Furthermore, it follows from (11.21) and (11.22) that

$$\frac{\partial^2 \log \rho^{\natural}}{\partial r^2} = \frac{\partial^2 \theta^{\natural}}{\partial r \partial \omega} = -\frac{\partial^2 \log \rho^{\natural}}{\partial \omega^2} = -\frac{8 \cosh 2r \cos 2\omega \, (\sinh^2 2r - \sin^2 2\omega)}{(\sinh^2 2r + \sin^2 2\omega)^2}, \tag{11.23}$$

$$\frac{\partial^2 \theta^{\natural}}{\partial r^2} = -\frac{\partial^2 \log \rho^{\natural}}{\partial r \partial \omega} = -\frac{\partial^2 \theta^{\natural}}{\partial \omega^2} = \frac{8 \sinh 2r \sin 2\omega \, (\cosh^2 2r + \cos^2 2\omega)}{(\sinh^2 2r + \sin^2 2\omega)^2}. \tag{11.24}$$

Notation

For simplicity of exposition, we introduce some non-standard notation.

Notation 11.8. For $\varDelta > 1$ and $X > 0$, let $x \sim_{\varDelta} X$ stand for $x \in [X, \varDelta X]$.

Notation 11.9. Let $\varDelta > 1$. We write $X \approx_{\varDelta} Y$ if $1/c_{\varDelta} \leqslant X/Y \leqslant c_{\varDelta}$ for some $c_{\varDelta} > 1$ such that $c_{\varDelta} \to 1$ as $\varDelta \to 1$. We write $X \lll_{\varDelta} Y$ if $|X| \leqslant \delta_{\varDelta} Y$ for some $\delta_{\varDelta} > 0$ such that $\delta_{\varDelta} \to 0$ as $\varDelta \to 1$.

11.4. Preliminary analysis of the Φ-integrals

For convenience of the further analysis by the Mellin technique in §13, we introduce certain partitions of Φ-integrals.

For the real case, the partition for $\Phi_{+}(y, \varLambda)$ is hidden in the proof of [68, Lemma 8.2], but the case of $\Phi_0^{+}(y/\varLambda)$ seems to be missing there.

Corollary 11.10. *Let $\varDelta > 1$ be fixed. Let $A \geqslant 1$. Let $|y| \sim_{\varDelta} Y$. Suppose that $Y \geqslant T^{\varepsilon}$ and $|\varLambda| \geqslant T^2$. Define $\Phi_{\pm}(y, \varLambda)$ by (11.12) and (11.13). We have*

$$\Phi_{+}(y, \varLambda) = \Phi^{+}(y/\varLambda) + O(T^{-A}) \tag{11.25}$$

for $Y^{2/3} \approx_{\varDelta} |\varLambda|$, where $\Phi^{+}(x)$ is supported on $|x| \geqslant M^{1-\varepsilon} T$ and

$$\Phi^{+}(x) = \begin{cases} \Phi_1^{+}(x) & \text{if } M^{1-\varepsilon} T \leqslant |x| \leqslant T^{2-\varepsilon}, \\ \Phi_0^{+}(x) & \text{if } |x| \geqslant T^{2-\varepsilon}, \end{cases} \tag{11.26}$$

where $\Phi_1^{+}(x)$ is given by

$$\Phi_1^{+}(x) = \int e(Tr/\pi - x \tanh^2 r) V^{+}(r) \, \mathrm{d}r \tag{11.27}$$

with $V^{+}(r)$ supported in $r \approx_{\varDelta} T/2\pi x$ and satisfying $r^i (\mathrm{d}/\mathrm{d}r)^i V^{+}(r) \ll_i \log^i T$, and where $\Phi_0^{+}(x)$ satisfies

$$x^i (\mathrm{d}/\mathrm{d}x)^i \Phi_0^{+}(x) \ll_i T^{(i+1)\varepsilon} / \sqrt{|x|}, \tag{11.28}$$

while

$$\Phi_-(y, \Lambda) = \Phi^-(y/\Lambda) \tag{11.29}$$

for $Y^{2/3} \leqslant |\Lambda|/M^{2-\varepsilon}$, *with*

$$\Phi^-(x) = \int e(Tr/\pi - x\coth^2 r) V^-(r)\,\mathrm{d}r, \tag{11.30}$$

where $V^-(r)$ *is supported in* $|r| \approx_\Delta Y^{1/3}/|\Lambda|^{1/2}$ *with* $r^i(\mathrm{d}/\mathrm{d}r)^i V^-(r) \ll_i \log^i T$.

Proof. The case of $\Phi_-(y, \Lambda)$ is obvious, while $\Phi_+(y, \Lambda)$ requires some discussion. The nature of the integral $\Phi_+(y, \Lambda)$ changes when $x = y/\Lambda$ moves beyond $T^{2-\varepsilon}$. For $|x| \ll T^{2-\varepsilon}$ or $|x| \gg T^{2-\varepsilon}$, respectively, the integral $\Phi_+(y, \Lambda)$ will be turned into $\Phi_1^+(x)$ or $\Phi_0^+(x)$ by smoothly truncating the r-integration near $T/2\pi x$ or at $\pm T^\varepsilon/T$.

Let $x = y/\Lambda$. The phase function $Tr/\pi - x\tanh^2 r$ has a unique stationary point $r_0 \approx_\Delta T/2\pi x$. For $|r_0| \leqslant M^\varepsilon/M$, it is necessary that $|x| \gg M^{1-\varepsilon}T$, because otherwise $\Phi_+(y, \Lambda)$ is negligible by Lemma 7.1 with $Z = |x|$, $Q = 1$, $R = T$, and $P = 1/M$. When $|x| \ll T^{2-\varepsilon}$, we have $|x|/T^2 \ll 1/T^\varepsilon$, and hence Lemma 7.1 (now $P = T/|x|$) implies that only a negligibly small error is lost if we restrict the integration to the interval $r \approx_\Delta T/2\pi x$ via a smooth partition of unity, giving $\Phi_1^+(x)$.

Next assume $|x| \gg T^{2-\varepsilon}$ so that $|r_0| < T^\varepsilon/\Delta T$. On applying Lemma 7.1 again with $R = |x|/T^{1-\varepsilon}$ and $P = T^\varepsilon/T$, we are left to consider the integral $\Phi_0^+(x)$ restricted to $|r| \leqslant T^\varepsilon/T$. The factor $e(Tr/\pi)$ is no longer oscillatory and may be absorbed into the weight function. To prove the estimates in (11.28) for the derivatives of $\Phi_0^+(x)$, we differentiate the integral and then confine the integration to $|r| \leqslant T^\varepsilon/\sqrt{|x|}$; Lemma 7.1 is used for the last time with $R = \sqrt{|x|}\,T^\varepsilon$ and $P = T^\varepsilon/\sqrt{|x|}$. Alternatively, one can also use Lemma 7.2.

Note that the fact that $V_+(r; y, \Lambda)$ has almost bounded derivatives (see (11.10)) is used implicitly to determine the P's. ∎

Corollary 11.11. *Let* $\Delta > 1$ *be fixed. Let* $A \geqslant 1$. *Let* $|u| \sim_\Delta Y$. *Suppose that* $Y \gg T^\varepsilon$ *and* $|\Lambda| \gg T^2$. *Define* $\Phi(u, \Lambda)$ *by* (11.17) *and* (11.18). *We have*

$$\begin{aligned}\Phi(u, \Lambda) = &\sum_{\substack{T^\varepsilon/T < \rho < 1/\sqrt{2}\Delta \\ \rho \approx_\Delta T/2\pi|\Lambda|^{1/2}}} \Phi_\rho^+(u/\Lambda) + \Phi_0^+(u/\Lambda) \\ &+ \sum_{\substack{\rho < 1/\sqrt{2}\Delta \\ \rho \approx_\Delta Y^{1/3}/|\Lambda|^{1/2}}} \Phi_\rho^-(u/\Lambda) + \Phi^\flat(u/\Lambda) + O(T^{-A}),\end{aligned} \tag{11.31}$$

where $\rho = \Delta^{-k/2}$ *for integers* k, $\Phi^\flat(u/\Lambda)$ *exists only when* $|\Lambda| \approx_\Delta T^2/2\pi^2$ *and* $Y \approx_\Delta T^3/8\pi^3$, $\Phi_\rho^+(z)$, $\Phi_\rho^-(z)$, *and* $\Phi^\flat(z)$ *are integrals of the form*

$$\iint e(2Tr/\pi - 2\,\mathrm{Re}(z\,\mathrm{trh}^\natural(r, \omega)))g(Mr)V(r, \omega)\,\mathrm{d}r\,\mathrm{d}\omega \tag{11.32}$$

with weight functions $V = V_\rho^+, V_\rho^-, V^\flat$ *supported in*

$$\cos^2\omega \approx_\Delta Y^{2/3}/|\Lambda|, \quad \sqrt{r^2+\sin^2\omega} \sim_\Delta \rho, \tag{11.33}$$

$$\sqrt{r^2+\cos^2\omega} \sim_\Delta \rho, \tag{11.34}$$

$$|\cos 2\omega| \leqslant \Delta - 1, \tag{11.35}$$

respectively, satisfying

$$\frac{\partial^{i+j} V_\rho^\pm(r,\omega)}{\partial r^i \partial\omega^j} \ll_{i,j} \frac{\log^{i+j} T}{\rho^{i+j}}, \quad \frac{\partial^{i+j} V^\flat(r,\omega)}{\partial r^i \partial\omega^j} \ll_{i,j} \log^{i+j} T, \tag{11.36}$$

and $\Phi_0^+(z)$ *has bounds*

$$x^i \frac{\partial^{i+j}\Phi_0^+(xe^{i\phi})}{\partial x^i \partial\phi^j} \ll_{i,j} \frac{T^{(i+j+1)\varepsilon}}{\max\{T^{2-\varepsilon}, x\}}. \tag{11.37}$$

Clearly, the integral $\Phi^\flat(z)$ has no counterpart in the real case. A similar partition on $\Phi^\flat(z)$ will be needed, but it seems more appropriate to introduce it when we apply the Mellin technique in §13.

Proof of Corollary 11.11. We start by dividing the ω-integral via a smooth partition of $[0,\pi]$ into the union of three regions where the inequalities

$$\sin\omega \leqslant \frac{1}{\sqrt{2\Delta}}, \quad |\cos\omega| \leqslant \frac{1}{\sqrt{2\Delta}}, \quad |\cos 2\omega| \leqslant \Delta - 1,$$

are valid, respectively.

The second integral turns into the sum of $\Phi_\rho^-(u/\Lambda)$ after employing a Δ-adic partition with respect to $\sqrt{r^2+\cos^2\omega}$. Note that (11.14) amounts to the condition $\rho \approx_\Delta Y^{1/3}/|\Lambda|^{1/2}$ in (11.31), and (11.15) is required to deduce the estimates for $V_\rho^-(r,\omega)$ in (11.36).

It remains to analyze the first and the third integrals. Keep in mind that because of (11.14) we necessarily have $\cos^2\omega \approx_\Delta Y^{2/3}/|\Lambda|$ in the first case and $Y^{2/3} \approx_\Delta |\Lambda|/2$ in the third case. Moreover, (11.15) shows that the weight functions have almost bounded derivatives as $\rho(r,\omega) \gg 1$ for both cases.

Let $z = u/\Lambda$ and write $z = xe^{i\phi}$. The phase function in (11.17) or (11.32) is equal to

$$f(r,\omega;x,\phi) = Tr/\pi - x\rho^\natural(r,\omega)\cos(\phi + \theta^\natural(r,\omega)).$$

Set $\psi = \phi + \theta^\natural(r,\omega)$ for brevity. In view of (11.19) and (11.20), we have

$$\partial f/\partial r = T/\pi + x(A\cos\psi + B\sin\psi), \quad \partial f/\partial\omega = x(B\cos\psi - A\sin\psi),$$

where

$$A = \frac{4\sinh 2r\cos 2\omega}{(\cosh 2r + \cos 2\omega)^2}, \quad B = \frac{4\cosh 2r\sin 2\omega}{(\cosh 2r + \cos 2\omega)^2}.$$

It is clear that

$$(\partial f/\partial r)^2 + (\partial f/\partial \omega)^2 \geqslant (T/\pi - x\sqrt{A^2+B^2})^2, \tag{11.38}$$

in which

$$A^2 + B^2 = \frac{16(\cosh 2r - \cos 2\omega)}{(\cosh 2r + \cos 2\omega)^3} = \frac{4(\sinh^2 r + \sin^2 \omega)}{(\cosh^2 r - \sin^2 \omega)^3}. \tag{11.39}$$

In the first case, (11.38) and (11.39) imply that

$$|f'(r,\omega;x,\phi)|^2 \gg T^2 + x^2(r^2 + \sin^2\omega),$$

except for $r^2 + \sin^2\omega \approx_\Delta T^2/4\pi^2|\Lambda|$ (since $\cos^2\omega \approx_\Delta Y^{2/3}/|\Lambda|$ and $x \sim_\Delta Y/|\Lambda|$). By Lemma 11.6, we have

$$\frac{\partial^{i+j} f(r,\omega;x,\theta)}{\partial r^i \partial \omega^j} \ll_{i,j} \frac{x}{(r^2+\sin^2\omega)^{(i+j)/2-1}}$$

for $i + j \geqslant 2$. We truncate smoothly the first integral at $\sqrt{r^2+\sin^2\omega} = T^\varepsilon/T$ and apply a Δ-adic partition of unity with respect to the value of $\sqrt{r^2+\sin^2\omega}$ over $(T^\varepsilon/T, 1/\sqrt{2\Delta})$. In this way, the integral splits into $\sum_\rho \Phi_\rho^+(z) + \Phi_0^+(z)$. On applying Lemma 7.4 with $Q = \Phi = \Upsilon = \rho$, $P = \min\{\rho, 1/M\}$, $Z = x\rho^2$, and $R = T + x\rho$, we infer that the integral $\Phi_\rho^+(z)$ is negligibly small unless $\rho \approx_\Delta T/2\pi\sqrt{|\Lambda|}$ ($\rho T > T^\varepsilon$ is required). When $x \leqslant T^{2-\varepsilon}$, the estimates for $\Phi_0^+(xe^{i\phi})$ in (11.37) follow from trivial estimation. When $x > T^{2-\varepsilon}$, we may further restrict the integration to $\sqrt{r^2+\sin^2\omega} \leqslant T^\varepsilon/\sqrt{x}$. To see this, we absorb $e(2Tr/\pi)$ into the weight function, and apply Lemma 7.4 with $Q = \Phi = P = \Upsilon = T^\varepsilon/\sqrt{x}$, $Z = T^\varepsilon$, and $R = \sqrt{x}\,T^\varepsilon$. Again, (11.37) follows trivially.

For the third case, it remains to prove that the integral restricted to $|\cos 2\omega| \leqslant \Delta - 1$ is negligible unless $x \approx_\Delta T/4\pi$. To this end, observe that $\sqrt{A^2+B^2} \approx_\Delta 4$, and it follows from (11.38) that

$$|f'(r,\omega;x,\phi)|^2 \gg T^2 + x^2$$

unless $x \approx_\Delta T/4\pi$. By Lemma 11.6, we have

$$\frac{\partial^{i+j} f(r,\omega;x,\theta)}{\partial r^i \partial \omega^j} \ll_{i,j} x$$

for $i + j \geqslant 2$. The proof is completed by applying Lemma 7.4 with $Q = \Phi = 1$, $P = 1/M$, $\Upsilon = 1$, $Z = x$, and $R = T + x$. ∎

12. Stationary phase for the Mellin transforms

In this section, we fix a smooth function $v(x)$ such that $v(x) \equiv 1$ on $[1/2, 2]$ and $v(x) \equiv 0$ on $(0, 1/3] \cup [3, \infty)$.

12.1. The real case

As in §2, let $\widehat{\mathfrak{a}} = \mathbb{R} \times \{0, 1\}$ and define $\chi_{iv,m}(x) = |x|^{iv}(x/|x|)^m$ for $(v, m) \in \widehat{\mathfrak{a}}$. The Mellin transform of $f \in C_c^\infty(\mathbb{R}^\times)$ is defined by

$$\breve{f}(v, m) = \int_{\mathbb{R}^\times} f(x)\chi_{iv,m}(x)\,\mathrm{d}^\times x,$$

and the Mellin inversion reads

$$f(x) = \frac{1}{4\pi} \iint_{\widehat{\mathfrak{a}}} \breve{f}(v, m)\overline{\chi_{iv,m}(x)}\,\mathrm{d}\mu(v, m).$$

Lemma 12.1. *Let $R, S > 0$ and $X \geqslant 1$. Suppose that $w(x)$ is smooth and $x^i w^{(i)}(x) \ll_i SX^i$ for $|x| \in [R/3, 3R]$. We have*

$$w(x) = \iint_{\widehat{\mathfrak{a}}} \xi(v, m)\overline{\chi_{iv,m}(x)}\,\mathrm{d}\mu(v, m)$$

whenever $|x| \in [R/2, 2R]$, with the function $\xi(v, m)$ satisfying $\xi(v, m) \ll S$ and $\xi(v, m) = O(RST^{-A})$ if $|v| > T^\varepsilon X$.

Proof. Let $\xi(v, m)$ be the Mellin transform of $4\pi v(|x|/R)w(x)$. The first estimate for $\xi(v, m)$ is trivial. The second is an easy consequence of Lemma 7.1 with phase function $v(\log|x|)/2\pi$. ■

Lemma 12.2. *Let $R > 1$. We have*

$$e(x) = \iint_{\widehat{\mathfrak{a}}} \xi_R(v, m)\overline{\chi_{iv,m}(x)}\,\mathrm{d}\mu(v, m)$$

whenever $|x| \in [R/2, 2R]$, where $\xi_R(v, m) = O((R + |v|)^{-A})$ unless $|v| \asymp R$, in which case $\xi_R(v, m) \ll 1/\sqrt{R}$.

Proof. Let $\xi_R(v, m)$ be the Mellin transform of $4\pi v(|x|/R)e(x)$. To derive the estimates for $\xi_R(v, m)$, apply Lemmas 7.1 and 7.2 (the second derivative test) with phase function $x + (v \log|x|)/2\pi$. ■

12.2. The complex case

As in §2, let $\widehat{\mathfrak{a}} = \mathbb{R} \times \mathbb{Z}$ and define $\chi_{iv,m}(z) = |z|^{2iv}(z/|z|)^m$ for $(v, m) \in \widehat{\mathfrak{a}}$. The Mellin transform of $f \in C_c^\infty(\mathbb{C}^\times)$ is defined by

$$\breve{f}(v, m) = \int_{\mathbb{C}^\times} f(z)\chi_{iv,m}(z)\,\mathrm{d}^\times z,$$

and the Mellin inversion reads

$$f(z) = \frac{1}{4\pi^2} \iint_{\widehat{\mathfrak{a}}} \breve{f}(v, m)\overline{\chi_{iv,m}(z)}\,\mathrm{d}\mu(v, m).$$

In polar coordinates,

$$\breve{f}(\nu, m) = 2\int_0^\infty \int_0^{2\pi} f(xe^{i\phi})x^{2i\nu}e^{im\phi}\,\frac{\mathrm{d}\phi\,\mathrm{d}x}{x}.$$

Lemma 12.3. *Let $R, S > 0$ and $X \geqslant 1$. Let $w(z)$ be smooth with $x^i\partial_x^i\partial_\phi^j w(xe^{i\phi}) \ll_{i,j} SX^{i+j}$ for $x \in [R/3, 3R]$. We have*

$$w(z) = \iint_{\widehat{\mathfrak{a}}} \xi(2\nu, m)\overline{\chi_{i\nu,m}(z)}\,\mathrm{d}\mu(\nu, m),$$

whenever $|z| \in [R/2, 2R]$, with the function $\xi(\nu, m)$ satisfying $\xi(\nu, m) \ll S$ and $\xi(\nu, m) = O(R^2ST^{-A})$ if $\sqrt{\nu^2 + m^2} > T^\varepsilon X$.

Proof. Let $\xi(2\nu, m)$ be the Mellin transform of $4\pi^2 v(|z|/R)w(z)$. In polar coordinates, apply Lemma 7.1 to the x- or ϕ-integral with phase function $(\nu\log x)/2\pi$ or $m\phi/2\pi$, respectively. ∎

The complex analogue of Lemma 12.2 is as follows. However, its proof requires considerably more work.

Lemma 12.4. *Let $R \geqslant 1$. We have*

$$e(2\,\mathrm{Re}(z)) = \iint_{\widehat{\mathfrak{a}}} \xi_R(2\nu, m)\overline{\chi_{i\nu,m}(z)}\,\mathrm{d}\mu(\nu, m)$$

whenever $|z| \in [R/2, 2R]$, where $\xi_R(\nu, m) = O((R + |\nu| + |m|)^{-A})$ unless $\sqrt{\nu^2 + m^2} \asymp R$, in which case $\xi_R(\nu, m) \ll (\log R)/R$.

Let $\xi_R(2\nu, m) = \xi(2\nu, m)$ be the Mellin transform of $4\pi^2 v(|z|/R)e(2\,\mathrm{Re}(z))$. Write

$$\xi(\nu, m) = 8\pi^2 R^{i\nu}\int_0^\infty \int_0^{2\pi} v(x)e(f(x, \phi; \nu, m))\,\frac{\mathrm{d}\phi\,\mathrm{d}x}{x}$$

with

$$f(x, \phi) = f(x, \phi; \nu, m) = 2Rx\cos\phi + (\nu\log x + m\phi)/2\pi.$$

We have

$$f'(x, \phi) = (2R\cos\phi + \nu/2\pi x, -2Rx\sin\phi + m/2\pi), \tag{12.1}$$

and hence there is a unique stationary point (x_0, ϕ_0) given by

$$x_0 = \frac{\sqrt{\nu^2 + m^2}}{4\pi R}, \quad \cos\phi_0 = -\frac{\nu}{\sqrt{\nu^2 + m^2}}, \quad \sin\phi_0 = \frac{m}{\sqrt{\nu^2 + m^2}}.$$

12.2.1. Applying Hörmander's partial integration. First, we prove $\xi(\nu, m) = O((R + |\nu| + |m|)^{-A})$ for any $A \geqslant 1$ unless $x_0 \in [1/4, 4]$, say. The arguments below are similar to those in [56, §6.1]. Our idea is to modify Hörmander's elaborate partial integration. To this end, we introduce

$$g(x,\phi) = x(\partial_x f(x,\phi))^2 + (1/x)(\partial_x f(x,\phi))^2$$
$$= \left(2R\sqrt{x} - \frac{\sqrt{\nu^2+m^2}}{2\pi\sqrt{x}}\right)^2 + \frac{2R}{\pi}(\sqrt{\nu^2+m^2} + \nu\cos\phi - m\sin\phi). \quad (12.2)$$

It is clear that

$$g(x,\phi) \gg \begin{cases} R^2 x & \text{if } x \geqslant 4x_0/3, \\ (\nu^2+m^2)/x & \text{if } x \leqslant 3x_0/4. \end{cases} \quad (12.3)$$

Define the differential operator

$$\mathrm{D} = \frac{x\partial_x f(x,\phi)}{g(x,\phi)}\frac{\partial}{\partial x} + \frac{\partial_\phi f(x,\phi)}{xg(x,\phi)}\frac{\partial}{\partial\phi}$$

so that $\mathrm{D}(e(f(x,\phi))) = 2\pi i \cdot e(f(x,\phi))$; its adjoint operator is given by

$$\mathrm{D}^* = -\frac{1}{2\pi i}\left(\frac{\partial}{\partial x}\frac{x\partial_x f(x,\phi)}{g(x,\phi)} + \frac{\partial}{\partial\phi}\frac{\partial_\phi f(x,\phi)}{xg(x,\phi)}\right),$$

and

$$\xi(\nu,m) = \frac{8\pi^2 R^{i\nu}}{(2\pi i)^n}\int_0^\infty\int_0^{2\pi}\mathrm{D}^{*n}(v(x)/x)e(f(x,\phi))\,\mathrm{d}\phi\,\mathrm{d}x.$$

For integer $n \geqslant 0$, $\mathrm{D}^{*n}(v(x)/x)$ is a linear combination of all the terms occurring in the expansions of

$$\partial_x^i\partial_\phi^j\{(x\partial_x f(x,\phi))^i(\partial_\phi f(x,\phi)/x)^j g(x,\phi)^n(v(x)/x)\}/g(x,\phi)^{2n}, \quad i+j=n.$$

Moreover, we have

$$x\partial_x f(x,\phi) \ll Rx + |\nu|, \quad \partial_\phi^{j+1}(x\partial_x f(x,\phi)) \ll Rx, \quad \partial_x\partial_\phi^j(x\partial_x f(x,\phi)) \ll R,$$
$$\partial_\phi f(x,\phi)/x \ll R + |m|/x, \ \partial_x^{i+1}(\partial_\phi f(x,\phi)/x) \ll |m|/x^{i+2}, \ \partial_\phi^{j+1}(\partial_\phi f(x,\phi)/x) \ll R,$$
$$x^2\partial_x g(x,\phi) \ll R^2x^2 + \nu^2 + m^2, \ x^{i+3}\partial_x^{i+2}g(x,\phi) \ll \nu^2 + m^2, \ \partial_\phi^{j+1}g(x,\phi) \ll \sqrt{\nu^2+m^2},$$
$$\partial_x^2(x\partial_x f(x,\phi)) = 0, \quad \partial_x\partial_\phi(\partial_\phi f(x,\phi)/x) = 0, \quad \partial_x\partial_\phi g(x,\phi) = 0,$$

for $i, j \geqslant 0$. Now assume that $x \in [1/3, 3]$ and $x_0 \notin [1/4, 4]$. Then (12.3) yields

$$g(x,\phi) \gg R^2 + \nu^2 + m^2.$$

Let $i_1, i_2 \leqslant i$ and $j_1, j_2 \leqslant j$. From the estimates above, it is straightforward to prove that

$$\partial_x^{i_1}\partial_\phi^{j_1}\left\{(x\partial_x f(x,\phi))^i(\partial_\phi f(x,\phi)/x)^j\right\} \ll (R+|\nu|)^i(R+|m|)^j,$$

and

$$\frac{\partial_x^{i_2}\partial_\phi^{j_2}g(x,\phi)^n}{g(x,\phi)^{2n}} \ll \sum_{k_1+2k_2\leqslant i_2}\sum_{l\leqslant j_2}\frac{(R^2+\nu^2+m^2)^{k_1}(\nu^2+m^2)^{k_2+l/2}}{g(x,\phi)^{n+k_1+k_2+l}}.$$

Combining these, we conclude that

$$\begin{aligned}\xi(\nu, m) &\ll \sum_{k_1+2k_2+l\leqslant n} \int_{1/3}^{3} \int_0^{2\pi} \frac{(R+|\nu|+|m|)^{n+2(k_1+k_2)+l}}{g(x,\phi)^{n+k_1+k_2+l}} \,\mathrm{d}\phi\,\mathrm{d}x \\ &\ll \sum_{l\leqslant n} \frac{1}{(R+|\nu|+|m|)^{n+l}} \\ &\ll \frac{1}{(R+|\nu|+|m|)^{n}},\end{aligned}$$

as desired.

12.2.2. Applying Olver's uniform asymptotic formula. Next, we need to prove the bound $\xi(\nu, m) \ll (\log R)/R$ when $x_0 \in [1/4, 4]$. This may be easily deduced from the same bound for the unweighted integrals as follows.

Lemma 12.5. *Suppose that $x_0 \in [1/4, 4]$. For $b > a > 0$, define*

$$I(a,b) = \int_a^b \int_0^{2\pi} e(2Rx\cos\phi) e^{im\phi} x^{i\nu-1}\,\mathrm{d}\phi\,\mathrm{d}x.$$

Then for any $b > a \geqslant 1/8$, we have $I(a,b) \ll (\log R)/R$, where the implied constant is absolute.

Firstly, we write

$$I(a,b) = \int_a^b \int_0^{2\pi} e(f(x,\phi)) \frac{\mathrm{d}\phi\,\mathrm{d}x}{x},$$

and apply Hörmander's elaborate partial integration once, obtaining

$$\begin{aligned}&\frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{\partial_x f(b,\phi)}{g(b,\phi)} e(f(b,\phi)) - \frac{\partial_x f(a,\phi)}{g(a,\phi)} e(f(a,\phi))\right)\mathrm{d}\phi \\ &-\frac{1}{2\pi i}\int_a^b\int_0^{2\pi}\left(\frac{\partial}{\partial x}\left(\frac{\partial_x f(x,\phi)}{g(x,\phi)}\right) + \frac{1}{x^2}\frac{\partial}{\partial\phi}\left(\frac{\partial_\phi f(x,\phi)}{g(x,\phi)}\right)\right) e(f(x,\phi))\,\mathrm{d}x\,\mathrm{d}\phi.\end{aligned}$$

For $a \geqslant 2x_0$ ($\geqslant 1/2$), one uses (12.1), (12.2), and the first lower bound in (12.3) to bound this by $1/aR \ll 1/R$. The case when $b \leqslant x_0/2$ is similar: we use the second lower bound in (12.3).

The problem is thus reduced to the case when $2x_0 \geqslant b > a \geqslant x_0/2$. Assume $m \geqslant 0$ for simplicity. We invoke Bessel's integral representation for $J_m(z)$ (see [65, 2.2 (1)]):

$$J_m(z) = \frac{1}{2\pi i^m}\int_0^{2\pi} e^{iz\cos\phi + im\phi}\,\mathrm{d}\phi,$$

and hence

$$I(a,b) = 2\pi i^m \int_a^b J_m(4\pi Rx) x^{i\nu-1}\,\mathrm{d}x. \tag{12.4}$$

According to [50, §7.13.1],

$$J_m(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi m}{2} - \frac{\pi}{4}\right) + O\left(\frac{m^2+1}{x^{3/2}}\right), \quad x \geqslant m^2 + 1.$$

It follows that if $m \leqslant R^{1/4}$, then $I(a,b) = O(1/\sqrt{R|\nu|}) = O(1/R)$ by Lemma 7.2.

For $m > R^{1/4}$, we employ Olver's uniform asymptotic formula, in particular, Lemma B.1 in Appendix B.

Recall that $x_0 = \sqrt{\nu^2 + m^2}/4\pi R$. For brevity, set $c = 4\pi R/m$, $x_0' = m/2\pi R$, and $x_0^+ = (m + m^{1/3})/4\pi R$.

When $R^{1/4} < m \leqslant \pi R x_0$, so that $2 \leqslant cx \ll m^3$ for all $x \in [x_0/2, 2x_0]$, by Lemma B.1 (4), the integral in (12.4) turns into

$$I(a,b) = \sum_{\pm} \int_a^b e(f_\pm(x)/2\pi) w_\pm(x)\, \mathrm{d}x + O(1/R)$$

with

$$f_\pm(x) = \pm m\gamma(cx) + \nu \log x,$$
$$w_\pm(x) = 2\sqrt{2}\pi i^m \frac{W_\pm(m\gamma(cx))}{m^{1/2}((cx)^2-1)^{1/4} x},$$

in which $\gamma(x) = \sqrt{x^2-1} - \operatorname{arc\,sec} x$. We have

$$\gamma'(x) = \frac{\sqrt{x^2-1}}{x}, \quad \gamma''(x) = \frac{1}{x^2\sqrt{x^2-1}}.$$

Therefore

$$f_\pm'(x) = \frac{1}{x}(\pm m\sqrt{(cx)^2-1} + \nu), \quad f_\pm''(x) = \frac{1}{x^2}\left(\pm \frac{m}{\sqrt{(cx)^2-1}} - \nu\right).$$

Moreover,

$$w_\pm(x),\ w_\pm'(x) \ll \frac{1}{\sqrt{R}};$$

note that $\gamma(x) = x + O(1)$ for $x \geqslant 2$ (see (B.6)). If $\pm\nu > 0$, then $|f_\pm'(x)| \gg m + |\nu| \gg R$, and the integral is $O(1/R^{3/2})$ by partial integration (the first derivative test). If $\pm\nu < 0$, then $|f_\pm''(x)| \gg m + |\nu| \gg R$, and the integral is $O(1/R)$ by the second derivative test in Lemma 7.2.

Suppose now $x_0/2 < x_0'$ so that necessarily $m \asymp R$. When $\max\{x_0/2, x_0^+\} \leqslant a < b \leqslant x_0'$, we apply Lemma B.1 (3) and then divide the integral in (12.4) by a dyadic partition with respect to $cx - 1$; the error term is $O(1/m) = O(1/R)$, and the resulting integrals can be treated in a manner similar to the above. We just need to notice that $m/\sqrt{(cx)^2-1}$ would dominate ν in $f_\pm''(x)$ when $cx - 1$ is small, in which case only Lemma 7.2 is applied. However, by doing the dyadic partition, we might lose a $\log R$.

Finally, assume that $x_0/2 < x_0^+$, and consider the case when $x_0/2 \leqslant a < b \leqslant x_0^+$. We use Lemma B.1 (1, 2) to bound the integral $I(a,b)$ as in (12.4) by

$$\begin{aligned} &\ll \frac{1}{m^{1/3}} \int_{1/c}^{x_0^+} \mathrm{d}x + \frac{1}{m^{1/3}} \int_{1/2c}^{1/c} \exp\big(-\tfrac{1}{3} m (2-2cx)^{3/2}\big)\, \mathrm{d}x \\ &\ll \frac{1}{m^{1/3}} \int_{0}^{1/m^{2/3}} \mathrm{d}y + \frac{1}{m^{1/3}} \int_{0}^{1} \exp\big(-\tfrac{1}{3} m y^{3/2}\big)\, \mathrm{d}y \\ &\ll \frac{1}{m}. \end{aligned}$$

Remark 12.6. The log R in Lemma 12.4 or 12.5 could be removed on applying the stationary phase method ([19, Lemma 5.5.6] for example) instead of the second derivative test, as revealed by the formula

$$|I(0,\infty)| = \frac{2\pi}{\sqrt{m^2+\nu^2}}$$

for $m \neq 0$; this may be seen from

$$I(0,\infty) = \int_0^\infty \int_0^{2\pi} e(2Rx\cos\phi) e^{im\phi} x^{i\nu-1}\, \mathrm{d}\phi\, \mathrm{d}x = \frac{\pi i^{|m|}\Gamma\big(\frac{1}{2}(|m|+i\nu)\big)}{(2\pi R)^{i\nu}\Gamma\big(\frac{1}{2}(|m|-i\nu)+1\big)},$$

which is a consequence of Weber's integral formula in [65, 13.24 (1)].

13. Analysis of the Hankel transforms, II

In this final analytic section, our primary object is to use the Mellin technique and the stationary phase method to analyze the Φ-integrals in §11.4. We remind the reader that the expressions of these Φ-integrals depend only mildly on M and Λ.

Definition 13.1. Let $U \gg 1$ and $(\kappa, n) \in \hat{\mathfrak{a}}$. Define

$$\begin{aligned} \hat{\mathfrak{a}}(U) &= \{(\nu,m) \in \hat{\mathfrak{a}} : \sqrt{\nu^2+m^2} \leqslant U\}, \\ \hat{\mathfrak{a}}'(U) &= \{(\nu,m) \in \hat{\mathfrak{a}} : \sqrt{\nu^2+m^2} \asymp U\}, \end{aligned} \tag{13.1}$$

$$\hat{\mathfrak{a}}_{\kappa,n}(U) = \{(\nu,m) \in \hat{\mathfrak{a}} : \sqrt{(\nu-\kappa)^2+(m-n)^2} \leqslant U\}. \tag{13.2}$$

For convenience, we shall not distinguish $\xi_R(\nu,m)$ and $\xi_U(\nu,m)$ when $R \asymp U$; see Lemmas 12.2 and 12.4.

13.1. The real case

Lemma 13.2. *Fix a constant $\Delta > 1$ with $\log \Delta$ small. Let $|\Lambda| \gg T^2$. Let $|x| \sim_\Delta X$. Let $\Phi^0(x)$, $\Phi^-(x)$ and $\Phi^+(x)$ be given as in Corollaries 11.2 and 11.10. For $\sigma = 0, -, +$, we have*

$$\Phi^\sigma(x) = \frac{T^\varepsilon}{\sqrt{A^\sigma}} \iint_{\hat{\mathfrak{a}}(U^\sigma)} \lambda^\sigma(\nu,m)\chi_{i\nu,m}(x)\, \mathrm{d}\mu(\nu,m) + O(T^{-A}) \tag{13.3}$$

for

$$\begin{aligned} &X \lll T^{\varepsilon}/|\varLambda| && \textit{if } \sigma = 0, \\ &T^{\varepsilon}/|\varLambda| \lll X \lll \sqrt{|\varLambda|}/M^{3-\varepsilon} && \textit{if } \sigma = -, \\ &X \approx_{\Delta} \sqrt{|\varLambda|} \ggg M^{1-\varepsilon}T && \textit{if } \sigma = +, \end{aligned} \tag{13.4}$$

with

$$U^{\sigma} = \begin{cases} T^{\varepsilon}, \\ |X\varLambda|^{1/3}, \\ \max\{T^2/X, T^{\varepsilon}\}, \end{cases} \qquad A^{\sigma} = \begin{cases} 1, & \textit{if } \sigma = 0, \\ |\varLambda| & \textit{if } \sigma = -, \\ \max\{T^{2-\varepsilon}, X\}, & \textit{if } \sigma = +, \end{cases} \tag{13.5}$$

and $\lambda^{\sigma}(\nu, m) \lll 1$ *for all* σ.

Firstly, note that, according to Corollaries 11.2 and 11.10, $\Phi^{\sigma}(x)$ vanishes unless X satisfies (13.4) in various cases.

For $\Phi^0(x)$ and $\Phi^+(x) = \Phi_0^+(x)$, it is easy to establish (13.3) by Lemma 12.1, along with (11.6) and (11.28). This settles the case $\sigma = 0$ and partially the case $\sigma = +$ for $X \ggg T^{2-\varepsilon}$.

Next, we consider the integral $\Phi_1^+(x)$ as defined in (11.27) for $M^{1-\varepsilon}T \lll |x| \lll T^{2-\varepsilon}$. Since $|x \tanh^2 r| \approx_{\Delta} T^2/4\pi^2 X$ for $|r| \approx_{\Delta} T/2\pi X$, up to a negligible error, we can rewrite $\Phi_1^+(x)$ using Lemma 12.2 as follows:

$$\iint_{\hat{\mathfrak{a}}'(T^2/X)} \overline{\xi_{T^2/X}(\nu, m)} \chi_{i\nu,m}(x) \int e(Tr/\pi) \chi_{i\nu,m}(\tanh^2 r) V^+(r)\, \mathrm{d}r\, \mathrm{d}\mu(\nu, m)$$

with $\xi_{T^2/X}(\nu, m) = O(\sqrt{X}/T)$. Write the inner integral as an exponential integral with phase $f_+(r) = (Tr + \nu \log|\tanh r|)/\pi$. Note that $f_+''(r) = \nu(\tanh^2 r - \coth^2 r))/\pi$ is of size $|\nu|/r^2 \asymp X$. By the second derivative test (Lemma 7.2), the r-integral is $O((\log T)/\sqrt{X})$, and $\xi_{T^2/X}(\nu, m)/\sqrt{X} = O(1/T)$, leading to $\sqrt{A^+} = T$ for $X \lll T^{2-\varepsilon}$ as claimed.

Finally, let $\Phi^-(x)$ be as defined in (11.30). Note that $|r| \approx_{\Delta} Y^{1/3}/|\varLambda|^{1/2}$ there amounts to $|r| \approx_{\Delta} X^{1/3}/|\varLambda|^{1/6}$ for $X = Y/|\varLambda|$, and hence $|x \coth^2 r| \approx_{\Delta} |X\varLambda|^{1/3}$. By Lemma 12.2, up to a negligible error, the integral $\Phi^-(x)$ can be rewritten as

$$\iint_{\hat{\mathfrak{a}}'(|X\varLambda|^{1/3})} \overline{\xi_{|X\varLambda|^{1/3}}(\nu, m)} \chi_{i\nu,m}(x) \int e(Tr/\pi) \chi_{i\nu,m}(\coth^2 r) V^-(r)\, \mathrm{d}r\, \mathrm{d}\mu(\nu, m).$$

with $\xi_{|X\varLambda|^{1/3}}(\nu, m) = O(1/|X\varLambda|^{1/6})$. Now the phase function of the inner integral is $f_-(r) = (Tr + \nu \log|\coth r|)/\pi$. Since $f_-''(r)\ (= -f_+''(r))$ is of size $|\nu|/r^2 \asymp |\varLambda^2/X|^{1/3}$, by the second derivative test the r-integral is $O(X^{1/6}(\log T)/|\varLambda|^{1/3})$, and $\xi_{|X\varLambda|^{1/3}}(\nu, m) \cdot X^{1/6}/|\varLambda|^{1/3} = O(1/\sqrt{|\varLambda|})$, as desired.

Remark 13.3. In the case $\sigma = -$, $f_-(r)$ has a stationary point at $|\nu|/T \asymp |X\varLambda|^{1/3}/T$, while $V^-(r)$ is supported on $|r| \asymp X^{1/3}/|\varLambda|^{1/6}$, so a consistency check shows that $|\varLambda| \asymp T^2$. However, this would have been implied at an early stage when analyzing the Bessel integral $\mathcal{H}(-x^2)$ (see Remark 8.3).

13.2. The complex case

Lemma 13.4. *Fix a constant $\Delta > 1$ with $\log \Delta$ small. Let $|\Lambda| \gg T^2$. Let $|z| \sim_\Delta X$. Let $\Phi^0(z)$, $\Phi^-_\rho(z)$, $\Phi^+_\rho(z)$, and $\Phi^+_0(z)$ be as in Corollaries 11.2 and 11.11. Set*

$$\Phi^+(z) = \sum_{\substack{T^\varepsilon/T<\rho<1/\sqrt{2}\Delta \\ \rho\approx_\Delta T/2\pi|\Lambda|^{1/2}}} \Phi^+_\rho(z) + \Phi^+_0(z), \quad \Phi^-(z) = \sum_{\substack{\rho<1/\sqrt{2}\Delta \\ \rho\approx_\Delta X^{1/3}/|\Lambda|^{1/6}}} \Phi^-_\rho(z).$$

For $\sigma = 0, -, +$, we have

$$\Phi^\sigma(z) = \frac{T^\varepsilon}{A^\sigma} \iint_{\hat{\mathfrak{a}}(U^\sigma)} \lambda^\sigma(\nu, m)\chi_{i\nu,m}(z)\,\mathrm{d}\mu(\nu, m) + O(T^{-A}), \tag{13.6}$$

for

$$\begin{aligned} &X \ll T^\varepsilon/|\Lambda| && \text{if } \sigma = 0,\\ &T^\varepsilon/|\Lambda| \ll X \ll \sqrt{|\Lambda|} && \text{if } \sigma = -,\\ &X \asymp \sqrt{|\Lambda|} && \text{if } \sigma = +, \end{aligned} \tag{13.7}$$

with

$$U^\sigma = \begin{cases} T^\varepsilon, \\ |X\Lambda|^{1/3}, \\ \max\{T^2/X, T^\varepsilon\}, \end{cases} \quad A^\sigma = \begin{cases} 1, & \text{if } \sigma = 0,\\ |\Lambda|, & \text{if } \sigma = -,\\ \max\{T^{2-\varepsilon}, X\}, & \text{if } \sigma = +, \end{cases} \tag{13.8}$$

and $\lambda^\sigma(\nu, m) \ll 1$ for all σ.

Lemma 13.5. *Fix a constant $\Delta > 1$ with $\log \Delta$ small. Let $|\Lambda| \approx_\Delta T^2/2\pi^2$ and $X \approx_\Delta T/4\pi$. For $|z| \sim_\Delta X$, let $\Phi^\flat(z)$ be as in Corollary 11.11. We have*

$$\Phi^\flat(z) = \sum_{T^\varepsilon/K^\flat<\rho<\Delta-1} \Phi^\flat_\rho(z) + \Phi^\flat_0(z) + O(T^{-A})$$

with $K^\flat = \min\{(T/M)^{1/2}, T^{1/4}\}$,

$$\Phi^\flat_\rho(z) = \frac{T^\varepsilon}{A^\flat_\rho} \iint_{\hat{\mathfrak{a}}_{0,\lfloor T\rfloor}(U^\flat_\rho)\cup\hat{\mathfrak{a}}_{0,\lceil -T\rceil}(U^\flat_\rho)} \lambda^\flat_\rho(\nu, m)\chi_{i\nu,m}(z)\,\mathrm{d}\mu(\nu, m), \tag{13.9}$$

where $\rho = \Delta^{-k/2}$ or 0,

$$U^\flat_\rho = T\rho^2, \quad A^\flat_\rho = T^2\rho, \tag{13.10}$$

$$U^\flat_0 = \begin{cases} T^{1/2+\varepsilon}, \\ T^{1/2+\varepsilon}, \\ MT^\varepsilon, \end{cases} \quad A^\flat_0 = \begin{cases} T^{5/3}, & \text{if } T^\varepsilon \leqslant M \leqslant T^{1/3},\\ M^{1/2}T^{3/2}, & \text{if } T^{1/3} < M \leqslant T^{1/2},\\ M^{1/2}T^{3/2}, & \text{if } T^{1/2} < M \leqslant T^{1-\varepsilon}, \end{cases} \tag{13.11}$$

and $\lambda^\flat_\rho(\nu, m), \lambda^\flat_0(\nu, m) \ll 1$.

Remark 13.6. It is important that r and ω play symmetric roles in the arguments below. Note that the restriction $|r| \leqslant M^{\varepsilon}/M$ does not apply to the ω-variable. Nevertheless, we could let $M = T^{\varepsilon}$ so that not much symmetry is lost. This symmetry seems unique for the first moment of $\mathrm{GL}_3 \times \mathrm{GL}_2$ or the cubic moment of GL_2—it does not occur, for example, in the case of the second moment of GL_2.

For $\Phi^0(z)$ and $\Phi_0^+(z)$, it is easy to establish (13.6) by Lemma 12.3, along with (11.7) and (11.37). This settles the case $\sigma = 0$ and partially the case $\sigma = +$ for $X \gg T^{2-\varepsilon}$.

In view of (11.18),

$$\rho^{\natural}(r,\omega) = \frac{\sinh^2 r + \sin^2 \omega}{\cosh^2 r - \sin^2 \omega} = \frac{\cosh^2 r - \cos^2 \omega}{\sinh^2 r + \cos^2 \omega} = \frac{\cosh 2r - \cos 2\omega}{\cosh 2r + \cos 2\omega}.$$

From (11.33)–(11.35) and the conditions for the ρ-sums in (11.31), we deduce that

$$x\rho^{\natural}(r,\omega) \approx_{\Delta} \begin{cases} |X\Lambda|^{1/3}\rho^2 \asymp T^2/X & \text{if } \sigma = +, \\ X(1-\rho^2)/\rho^2 \asymp |X\Lambda|^{1/3} & \text{if } \sigma = -, \\ X \approx_{\Delta} T/4\pi & \text{if } \sigma = \flat. \end{cases}$$

Applying Lemma 12.4 to the exponential factor $e(-2\,\mathrm{Re}(z\, \mathrm{trh}^{\natural}(r,\omega)))$ in the integral (11.32), we have

$$\Phi_{\rho}^{\pm}(xe^{i\phi}) = \iint_{\hat{\mathfrak{a}}'(U^{\pm})} \overline{\xi_{U^{\pm}}(2\nu,m)} I_{\rho}^{\pm}(2\nu,m)\chi_{i\nu,m}(xe^{i\phi})\, \mathrm{d}\mu(\nu,m) + O(T^{-A}), \tag{13.12}$$

$$\Phi^{\flat}(xe^{i\phi}) = \iint_{\hat{\mathfrak{a}}'(T)} \overline{\xi_{T}(2\nu,m)} I^{\flat}(2\nu,m)\chi_{i\nu,m}(xe^{i\phi})\, \mathrm{d}\mu(\nu,m) + O(T^{-A}), \tag{13.13}$$

where $\xi_{U^{\pm}}(\nu,m) = O((\log T)/U^{\pm})$, $\xi_T(\nu,m) = O((\log T)/T)$, and

$$I_{\rho}^{\pm}(\nu,m) = \iint e(f(r,\omega;\nu,m)/2\pi)g(Mr)V_{\rho}^{\pm}(r,\omega)\, \mathrm{d}r\, \mathrm{d}\omega, \tag{13.14}$$

$$I^{\flat}(\nu,m) = \iint e(f(r,\omega;\nu,m)/2\pi)g(Mr)V^{\flat}(r,\omega)\, \mathrm{d}r\, \mathrm{d}\omega, \tag{13.15}$$

$$f(r,\omega;\nu,m) = 4Tr + \nu\log\rho^{\natural}(r,\omega) + m\theta^{\natural}(r,\omega). \tag{13.16}$$

Analysis of $f(r,\omega;\nu,m)$. By (11.21) and (11.22), we have

$$\partial f/\partial r = 4(T + \nu A_1 - mB_1), \quad \partial f/\partial\omega = 4(mA_1 + \nu B_1), \tag{13.17}$$

with

$$A_1 = \frac{\sinh 2r \cos 2\omega}{\sinh^2 2r + \sin^2 2\omega}, \quad B_1 = \frac{\cosh 2r \sin 2\omega}{\sinh^2 2r + \sin^2 2\omega}. \tag{13.18}$$

Since

$$\sinh^2 2r \cos^2 2\omega + \cosh^2 2r \sin^2 2\omega = \sinh^2 2r + \sin^2 2\omega,$$

the stationary point (r_0, ω_0) is given by the equations

$$\sinh 2r_0 \cos 2\omega_0 = \nu/T, \quad \cosh 2r_0 \sin 2\omega_0 = -m/T. \tag{13.19}$$

Also note that

$$A_1^2 + B_1^2 = \frac{1}{\sinh^2 2r + \sin^2 2\omega}. \tag{13.20}$$

It follows from (13.17) that

$$(\partial f/\partial r)^2 + (\partial f/\partial \omega)^2 = 16\big(T - \sqrt{(\nu^2+m^2)(A_1^2+B_1^2)}\big)^2 + 32T\big(\sqrt{(\nu^2+m^2)(A_1^2+B_1^2)} + \nu A_1 - mB_1\big).$$

From this, it is easy to prove the following lemma.

Lemma 13.7. *We have*

$$|f'(r,\omega;\nu,m)|^2 \gg T^2 + \frac{\nu^2+m^2}{\sinh^2 2r + \sin^2 2\omega} \tag{13.21}$$

unless

$$|\sinh 2r \cos 2\omega \cdot T - \nu|, |\cosh 2r \sin 2\omega \cdot T + m| \ll_\Delta \sqrt{\nu^2+m^2}. \tag{13.22}$$

Note that the conditions in (13.22) imply

$$\frac{\nu^2+m^2}{\sinh^2 2r + \sin^2 2\omega} \approx_\Delta T^2, \tag{13.23}$$

and they describe a small neighborhood of (r_0, ω_0).

By (11.23) and (11.24), we have

$$f'' = -8\begin{pmatrix} \nu A_2 - mB_2 & mA_2 + \nu B_2 \\ mA_2 + \nu B_2 & -\nu A_2 + mB_2 \end{pmatrix} \tag{13.24}$$

with

$$A_2 = \frac{\cosh 2r \cos 2\omega\,(\sinh^2 2r - \sin^2 2\omega)}{(\sinh^2 2r + \sin^2 2\omega)^2}, \quad B_2 = \frac{\sinh 2r \sin 2\omega\,(\cosh^2 2r + \cos^2 2\omega)}{(\sinh^2 2r + \sin^2 2\omega)^2}. \tag{13.25}$$

It is clear that for $|r| < 1$ we have

$$A_2, B_2 \ll \frac{\sqrt{\sinh^2 2r + \cos^2 2\omega}}{\sinh^2 2r + \sin^2 2\omega}. \tag{13.26}$$

Some computations show that

$$f''(r_0,\omega_0;\nu,m) = -\frac{8T}{\sinh^2 2r_0 + \sin^2 2\omega_0}\begin{pmatrix} \sinh 2r_0 \cosh 2r_0 & \sin 2\omega_0 \cos 2\omega_0 \\ \sin 2\omega_0 \cos 2\omega_0 & -\sinh 2r_0 \cosh 2r_0 \end{pmatrix}.$$

In light of this, we have the following lemma.

Lemma 13.8. *For any* (r, ω) *satisfying* (13.22), *we have*

$$\nu A_2 - m B_2 - \frac{T \sinh 2r \cosh 2r}{\sinh^2 2r + \sin^2 2\omega} \lll_\Delta \frac{\sqrt{\nu^2 + m^2}\sqrt{\sinh^2 2r + \cos^2 2\omega}}{\sinh^2 2r + \sin^2 2\omega}, \tag{13.27}$$

$$m A_2 + \nu B_2 - \frac{T \sin 2\omega \cos 2\omega}{\sinh^2 2r + \sin^2 2\omega} \lll_\Delta \frac{\sqrt{\nu^2 + m^2}\sqrt{\sinh^2 2r + \cos^2 2\omega}}{\sinh^2 2r + \sin^2 2\omega}. \tag{13.28}$$

Proof. (13.27) is a consequence of (13.22) and (13.26), since its left hand side may be written as

$$(\nu - \sinh 2r \cos 2\omega \cdot T) A_2 - (m + \cosh 2r \sin 2\omega \cdot T) B_2,$$

and similarly for (13.28). ■

Moreover, by Lemma 11.7,

$$\frac{\partial^{i+j} f(r, \omega; \nu, m)}{\partial r^i \partial \omega^j} \ll_{i,j} \frac{\sqrt{\nu^2 + m^2}}{(\sinh^2 2r + \sin^2 2\omega)^{(i+j)/2}} \tag{13.29}$$

for $i + j \geqslant 2$.

The case $\sigma = \pm$. For Lemma 13.4 it remains to prove the bounds

$$I_\rho^+(\nu, m) \ll (\log^2 T)/X, \quad I_\rho^-(\nu, m) \ll X^{1/3}(\log^2 T)/|\Lambda|^{2/3}.$$

Recall from (11.33) and (11.34) that $V_\rho^\sigma(r, \omega)$ is supported on

$$\sqrt{r^2 + \sin^2 \omega} \sim_\Delta \rho \quad \text{or} \quad \sqrt{r^2 + \cos^2 \omega} \sim_\Delta \rho \tag{13.30}$$

according as $\sigma = +$ or $-$, and from (11.36) we have

$$\partial_r^i \partial_\omega^j V_\rho^\sigma(r, \omega) \ll_{i,j} ((\log T)/\rho)^{i+j}. \tag{13.31}$$

In view of Lemma 13.7, (13.23), and (13.30), together with the identity

$$\sinh^2 2r + \sin^2 2\omega = 4(\sinh^2 r + \sin^2 \omega)(\sinh^2 r + \cos^2 \omega),$$

one would expect the integral $I_\rho^\sigma(\nu, m)$ to be negligibly small unless

$$(1 - \rho^2)\rho^2 \approx_\Delta \frac{\nu^2 + m^2}{4T^2}. \tag{13.32}$$

To see this, we apply Lemma 7.4 with $Q = \Phi = \Upsilon = \rho$, $P = \min\{\rho, 1/M\}$, $Z = \sqrt{\nu^2 + m^2}$, and $R = T + \sqrt{\nu^2 + m^2}/\rho$, which are determined by (13.21), (13.29), and (13.31). For $\sigma = +$, the condition $\rho > T^\varepsilon/T$ is required here. For $\sigma = -$, it is slightly easier because $R \asymp \sqrt{\nu^2 + m^2}/\rho$ in view of $\sqrt{\nu^2 + m^2}/\rho \asymp \sqrt{|\Lambda|} \gg T$.

Next, some remarks on (13.32) are in order. For $\sigma = +$, it is pleasant to check that (13.32) is consistent with $\rho \approx_\Delta T/2\pi|\Lambda|^{1/2}$, $\sqrt{\nu^2 + m^2} \asymp T^2/X$, and $X \asymp \sqrt{|\Lambda|}$. For $\sigma = -$, since $\rho \approx_\Delta X^{1/3}/|\Lambda|^{1/6}$ and $\sqrt{\nu^2 + m^2} \asymp |X\Lambda|^{1/3}$, (13.32) would imply that

$|\varLambda| \asymp T^2$. We remark that the condition $|\varLambda| \asymp T^2$ for $\sigma = -$ also arose in the real case (see Remark 13.3).

Consider rectangular regions of the form

$$|\pm 2r(1-2\rho^2)\cdot T - v|, |\sin 2\omega \cdot T + m| \ll_\Delta \sqrt{v^2+m^2}. \tag{13.33}$$

Given (13.30) (and $|r| \leqslant M^\varepsilon/M$), the regions defined by (13.22) and (13.33) contain one another if we choose the implied constants suitably. By Lemma 7.4, we infer that only a negligibly small error is lost if the integral is restricted to the region (13.33). Next, Lemma 13.8, along with (13.30) and (13.32), implies that, when $\log \Delta > 0$ is a small constant (now we use Notation 11.9), $|vA_2 - mB_2| \gg T/\rho$ if $|r| \gg |\sin 2\omega|$ and $|mA_2 + vB_2| \gg T/\rho$ if $|\sin 2\omega| \gg |r|$. Now we exploit the second derivative test as in Lemma 7.6 to deduce

$$I_\rho^{\pm}(v,m) \ll \rho(\log^2 T)/T.$$

Finally, since

$$\rho/T \ll \begin{cases} 1/|\varLambda|^{1/2} \ll 1/X & \text{if } \sigma = +, \\ X^{1/3}/T|\varLambda|^{1/6} \ll X^{1/3}/|\varLambda|^{2/3} & \text{if } \sigma = -, \end{cases}$$

we arrive at the desired estimates. Recall here that $X \asymp \sqrt{|\varLambda|}$ if $\sigma = +$ and $T \asymp \sqrt{|\varLambda|}$ if $\sigma = -$.

The case $\sigma = \flat$. Set $K = \min\{(T/M)^{1/2}, T^{1/3}\}$. We introduce a smooth partition to the integral $I^\flat(v,m)$ in (13.15) according to the value of $\sqrt{r^2+\cos^2 2\omega}$,

$$I^\flat(v,m) = \sum_{T^\varepsilon/K<\rho<\Delta^{-1}} I_\rho^\flat(v,m) + I_0^\flat(v,m)$$

with

$$I_\rho^\flat(v,m) = \iint e(f(r,\omega;v,m)/2\pi) g(Mr) V_\rho^\flat(r,\omega)\, \mathrm{d}r\, \mathrm{d}\omega,$$

where $\rho = \Delta^{-k/2}$ or 0, $V_\rho^\flat(r,\omega)$ or $V_0^\flat(r,\omega)$ is supported on

$$\sqrt{r^2+\cos^2 2\omega} \sim_\Delta \rho, \qquad \sqrt{r^2+\cos^2 2\omega} \ll T^\varepsilon/K,$$

respectively, and

$$\partial_r^i \partial_\omega^j V_\rho^\flat(r,\omega) \ll_{i,j} ((\log T)/\rho)^{i+j}, \qquad \partial_r^i \partial_\omega^j V_0^\flat(r,\omega) \ll_{i,j} (K/T^\varepsilon)^{i+j}.$$

Consequently, we have a partition of $\Phi^\flat(xe^{i\phi})$ (see (13.13)) in the same fashion:

$$\Phi^\flat(v,m) = \sum_{T^\varepsilon/K<\rho<\Delta^{-1}} \Phi_\rho^\flat(v,m) + \Phi_0^\flat(v,m), \tag{13.34}$$

with

$$\Phi_\rho^\flat(xe^{i\phi}) = \iint_{\widehat{\mathfrak{a}}'(T)} \overline{\xi_T(2v,m)} I_\rho^\flat(2v,m) \chi_{iv,m}(xe^{i\phi})\, \mathrm{d}\mu(v,m). \tag{13.35}$$

An obvious distinction between the cases $\sigma = \pm$ and $\flat$ is the scale of $\sinh^2 2r + \sin^2 2\omega$ which arose ubiquitously in the denominators of the derivatives of $f(r, \omega; \nu, m)$. Its scale grows from ρ^2 to 1 when σ changes from $\pm$ to $\flat$. As a consequence, we lose $1/\rho^2$ in the stationary-phase bound for $I^{\flat}_{\rho}(\nu, m)$ (it could be even worse than the trivial bound if ρ were very close to 0). Fortunately, however, we shall be able to recover the loss by shrinking the integral domain $\widehat{\mathfrak{a}}'(T)$ to the union of $\widehat{\mathfrak{a}}_{0,\lfloor T \rfloor}(U^{\flat}_{\rho})$ and $\widehat{\mathfrak{a}}_{0,\lceil -T \rceil}(U^{\flat}_{\rho})$.

Now we return to the analysis of the integral $I^{\flat}_{\rho}(\nu, m)$.

Similar to the case $\sigma = \pm$, we deduce from the second derivative test (Lemma 7.6) that

$$I^{\flat}_{\rho}(\nu, m) \ll (\log^2 T)/T\rho.$$

On the other hand, we have the trivial bound

$$I^{\flat}_{0}(\nu, m) \ll T^{\varepsilon}/\max\{(MT)^{1/2}, T^{2/3}\}.$$

Recall that $\sqrt{\nu^2 + m^2} \asymp T$. Note that $\sinh 2r, \cos 2\omega \ll \sqrt{r^2 + \cos^2 2\omega} \ll \rho$ and $\sinh^2 2r + \sin^2 2\omega = 1 + O(\rho^2)$. It follows from (13.18) and (13.20) that $A_1, B_1 \pm 1 = O(\rho^2)$. Consequently, in view of (13.17) and (13.18),

$$\partial f/\partial r = 4(T \mp m) + O(T\rho^2), \quad \partial f/\partial \omega = \pm 4\nu + O(T\rho^2).$$

Moreover, (13.29) now reads

$$\partial^{i+j} f/\partial r^i \partial \omega^j \ll_{i,j} T$$

for $i + j \geqslant 2$. Set $U = \max\{T\rho^2, T^{1/2+\varepsilon}\} = \max\{T\rho^2, T^{1/2+\varepsilon}, MT^{\varepsilon}\}$. Then $|\partial f/\partial \omega| \gg U$ for $|\nu| \gg U$, and $|\partial f/\partial r| \gg U$ for $|T \mp m| \gg U$. On applying Lemma 7.1 to the ω- or r-integral, with $P = \rho$ or $\min\{\rho, 1/M\}$, $Q = 1$, $Z = T$, and $R = U$, we find that $I^{\flat}_{\rho}(\nu, m)$ is negligibly small for such ν or m (it is important here that $T/U^2, M/U \leqslant 1/T^{\varepsilon}$). Similarly, if we put $U_0 = \max\{T^{1/2+\varepsilon}, MT^{\varepsilon}\}$, then $I^{\flat}_{0}(\nu, m)$ is negligibly small unless $\nu = O(U_0)$ and $m = \pm T + O(U_0)$.

Lemma 13.5 follows if we truncate the ρ-sum in (13.34) at $\rho = T^{\varepsilon}/T^{1/4}$ in the case $M < T^{1/2}$ and absorb the sum of $\Phi^{\flat}_{\rho}(\nu, m)$ over smaller ρ's into $\Phi^{\flat}_{0}(\nu, m)$.

Appendix B. Olver's uniform asymptotic formula for Bessel functions

In this appendix, we recollect Olver's uniform asymptotic formula for Bessel functions of large order and prove some of its implications that will be useful in §12.2.2 for our study of certain Mellin integrals over $\mathbb{C}$. For our purpose, we only consider here $J_m(mx)$ with large integer order m and positive real variable x.

According to the works of Olver [48, 49], we have

$$J_m(mx) = \left(\frac{4\zeta}{1 - x^2}\right)^{1/4} \left\{ \frac{\mathrm{Ai}(m^{2/3}\zeta)}{m^{1/3}} \sum_{s=0}^{k} \frac{A_s(\zeta)}{m^{2s}} + \frac{\mathrm{Ai}'(m^{2/3}\zeta)}{m^{5/3}} \sum_{s=0}^{k-1} \frac{B_s(\zeta)}{m^{2s}} + O\left(\frac{\left|\exp(-\frac{2}{3} m \zeta^{3/2})\right|}{m^{2k+1}(1 + m^{1/6}|\zeta|^{1/4})} \right) \right\}, \quad \text{(B.1)}$$

where $\mathrm{Ai}(y)$ is the Airy function,

$$\begin{aligned} \frac{2}{3}\zeta^{3/2} &= \log\frac{1+\sqrt{1-x^2}}{x} - \sqrt{1-x^2}, && 0 < x \leqslant 1, \\ \frac{2}{3}(-\zeta)^{3/2} &= \sqrt{x^2-1} - \operatorname{arcsec} x, && x > 1, \end{aligned} \tag{B.2}$$

and

$$A_s(\zeta) = \sum_{j=0}^{2s} b_j \zeta^{-3j/2} U_{2s-j}(v), \quad \zeta^{1/2} B_s(\zeta) = -\sum_{j=0}^{2s+1} a_j \zeta^{-3j/2} U_{2s-j+1}(v), \tag{B.3}$$

in which $a_0 = b_0 = 1$,

$$a_s = \frac{1}{3^{2s}(2s)!}\frac{\Gamma\big(3s+\frac{1}{2}\big)}{\Gamma\big(\frac{1}{2}\big)}, \quad b_s = -\frac{6s+1}{6s-1} a_s,$$

and $U_s(v)$ are polynomials in $v = 1/\sqrt{1-x^2}$, with the first three found to be

$$U_0 = 1, \quad U_1 = (3v - 5v^3)/24, \quad U_2 = (81v^2 - 462v^4 + 385v^6)/1152; \tag{B.4}$$

see [48, §4] and [49, Theorem B] for the expansion and the error term as in (B.1), and [48, §6] for the coefficients $A_s(\zeta)$ and $B_s(\zeta)$ as in (B.3). By [48, (4.13), (4.14)],

$$x = 1 - \frac{\zeta}{2^{1/3}} + \frac{3\zeta^2}{10\cdot 2^{2/3}} + O(\zeta^3), \quad |\zeta| \lll 1, \tag{B.5}$$

$$x = \tfrac{2}{3}(-\zeta)^{3/2} + O(1), \qquad -\zeta \ggg 1. \tag{B.6}$$

As for the Airy function, if we set $\gamma = \frac{2}{3}y^{3/2}$ for $y > 0$, then it is well-known (see [1, (10.4.14)–(10.4.17)]) that

$$\mathrm{Ai}(y) = \frac{\sqrt{y}}{\sqrt{3}\pi} K_{1/3}(\gamma), \qquad \mathrm{Ai}(-y) = \frac{\sqrt{y}}{3}\big(J_{1/3}(\gamma) + J_{-1/3}(\gamma)\big),$$

$$\mathrm{Ai}'(y) = -\frac{y}{\sqrt{3}\pi} K_{2/3}(\gamma), \quad \mathrm{Ai}'(-y) = \frac{y}{3}\big(J_{2/3}(\gamma) - J_{-2/3}(\gamma)\big).$$

For $|y| \lll 1$, we have $\mathrm{Ai}(y) = O(1)$ (see [1, (10.4.2)]). For $y \ggg 1$, it follows from [65, 7.21 (1, 3), 7.23 (1), §7.3] that

$$\mathrm{Ai}(y) = O\bigg(\frac{\exp(-\gamma)}{y^{1/4}}\bigg), \qquad \mathrm{Ai}'(y) = O(y^{1/4}\exp(-\gamma)), \tag{B.7}$$

$$\mathrm{Ai}(-y) = \sum_{\pm} \frac{\exp(\pm i\gamma)}{y^{1/4}} W_{\pm}(\gamma), \quad \mathrm{Ai}'(-y) = O(y^{1/4}), \tag{B.8}$$

with $\gamma^i W_{\pm}^{(i)}(\gamma) \ll_i 1$.

Lemma B.1. *Let $m \ggg 1$ and $x > 0$. For $x > 1$, define $\gamma(x) = \sqrt{x^2-1} - \operatorname{arcsec} x$.*

(1) *For $|x-1| \leqslant 1/m^{2/3}$, we have $J_m(mx) = O(1/m^{1/3})$.*

(2) *For $\frac{1}{2} \leqslant x \leqslant 1 - 1/m^{2/3}$, we have*

$$J_m(mx) = O\bigg(\frac{\exp\big(-\frac{1}{3}m(2-2x)^{3/2}\big)}{m^{1/2}(1-x)^{1/4}}\bigg). \tag{B.9}$$

(3) *For* $1 + 1/m^{2/3} \leqslant x \leqslant 2$, *we have*

$$J_m(mx) = \sqrt{2}\sum_{\pm}\frac{\exp(\pm i m\gamma(x))}{m^{1/2}(x^2-1)^{1/4}}W_{\pm}(m\gamma(x)) + O\bigg(\frac{1}{m^{7/6}(x-1)^{1/4}}\bigg), \quad \text{(B.10)}$$

in which $\gamma^i W_{\pm}^{(i)}(\gamma) \ll_i 1$ *for* $\gamma \gg 1$.

(4) *For* $2 \leqslant x \ll m^{13/3}$, *the asymptotic formula* (B.10) *holds with an error term* $O(1/mx)$.

Proof. First let $k = 0$ in (B.1). The estimate in (1) is clear. For $0 < x \leqslant 1$, it is easy to prove (compare with (B.5))

$$\log\frac{1+\sqrt{1-x^2}}{x} - \sqrt{1-x^2} \geqslant \tfrac{1}{3}(2-2x)^{3/2}.$$

Then (B.9) is a direct consequence of (B.1), (B.5), and (B.7).[7] The asymptotic formula (B.10) in (3) is obvious in view of (B.1), (B.5), and (B.8). As for (4), we let $k = 1$ in (B.1). For $x \geqslant 2$, it follows from (B.3), (B.4), and (B.6) that $B_0(\zeta) = O(1/\zeta^2)$ and $A_1(\zeta) = O(1/\zeta^3)$. By (B.6) and (B.8), the two lower-order main terms and the error term in (B.1) are $O(1/(mx)^{3/2})$, $O(1/(mx)^{5/2})$, and $O(1/m^{19/6}x^{1/2})$, respectively; all of these are $O(1/mx)$ provided $x \ll m^{13/3}$. ∎

Part III. Proof of Theorem 1.1

14. Setup

We start by introducing the spectral mean of L-values

$$\sum_{f\in\mathcal{B}}\omega_f k^{\natural}(\nu_f)L\big(\tfrac{1}{2},\pi\otimes f\big) + \frac{c_0}{4\pi}\int_{-\infty}^{\infty}\omega(t)k^{\natural}(t)\big|L\big(\tfrac{1}{2}+it,\pi\big)\big|^2\,\mathrm{d}t,$$

in which $k^{\natural}(\nu)$ is the test function defined in §6. Recall that $k^{\natural}(\nu) > 0$ for $\nu \in \mathfrak{a}$, and that $k^{\natural}(\nu) \gg 1$ if $|\nu_v - T_v| \leqslant M_v$ for all $v\,|\,\infty$. When $f \in \mathcal{B}$ is exceptional in the sense that $\nu_{f,v}$ is not real for some $v\,|\,\infty$, the weight $k^{\natural}(\nu_f)$ would be negligibly small (although not necessarily positive), for at this place v we have $k^{\natural}(\nu_{f,v}) = o(e^{-T_v^2/M_v^2})$ and $T_v \geqslant \mathrm{N}(T)^{\varepsilon}$ by assumption. Thus, in view of (3.20) in Lemma 3.6, along with the non-negativity of the L-values, Theorem 1.1 follows if we are able to prove that the spectral mean is bounded by $\mathrm{N}^{\natural}(M)\mathrm{N}(T)^{5/4+\varepsilon}$.

Applying the approximate functional equations (5.9) and (5.12), the above spectral mean may be written as

$$2\sum\sum_{\mathfrak{n}_1,\mathfrak{n}_2\subset\mathcal{O}}\frac{A(\mathfrak{n}_1,\mathfrak{n}_2)}{\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2)^{1/2}}\bigg\{\sum_{f\in\mathcal{B}}\omega_f k^{\natural}(\nu_f)\lambda_f(\mathfrak{n}_2)V\big(\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2\mathfrak{D}^{-3});\nu_f\big) + \frac{c_0}{4\pi}\int_{-\infty}^{\infty}\omega(t)k^{\natural}(t)\tau_{it}(\mathfrak{n}_2)V\big(\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2\mathfrak{D}^{-3});t\big)\,\mathrm{d}t\bigg\}.$$

[7] Note that (B.9) would also follow from Nicholson's asymptotic formula in [37, §3.14.3].

By (5.13) in Lemma 5.1 (1), we may truncate the sum over $\mathfrak{n}_1, \mathfrak{n}_2$ at $\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2) \leqslant \mathrm{N}(T)^{3+\varepsilon}$. We then apply (5.14) in Lemma 5.1 (1), in which we choose $U = \log \mathrm{N}(T)$. The error term is again negligible, and we need to prove

$$\sum\sum_{\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2)\leqslant \mathrm{N}(T)^{3+\varepsilon}} \frac{A(\mathfrak{n}_1,\mathfrak{n}_2)}{\mathrm{N}(\mathfrak{n}_1^2\mathfrak{n}_2)^{1/2+u}} \Bigg\{ \sum_{f\in\mathscr{B}} \omega_f h(\nu_f)\lambda_f(\mathfrak{n}_2) + \frac{c_0}{4\pi}\int_{-\infty}^{\infty} \omega(t)h(t)\tau_{it}(\mathfrak{n}_2)\,\mathrm{d}t \Bigg\} \ll \mathrm{N}^{\natural}(M)\mathrm{N}(T)^{5/4+\varepsilon}.$$

uniformly in $u \in [\varepsilon - i\log \mathrm{N}(T), \varepsilon + i\log\mathrm{N}(T)]$. By the Hecke relation (4.8), the left-hand side is equal to

$$\sum\sum\sum_{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}_1^2\mathfrak{n}_2)\leqslant \mathrm{N}(T)^{3+\varepsilon}} \frac{\mu(\mathfrak{d})A(\mathfrak{n}_1,1)A(1,\mathfrak{n}_2)}{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}_1^2\mathfrak{n}_2)^{1/2+u}} \Bigg\{ \sum_{f\in\mathscr{B}} \omega_f h(\nu_f)\lambda_f(\mathfrak{d}\mathfrak{n}_2) + \frac{c_0}{4\pi}\int_{-\infty}^{\infty} \omega(t)h(t)\tau_{it}(\mathfrak{d}\mathfrak{n}_2)\,\mathrm{d}t \Bigg\}.$$

We now apply the Kuznetsov trace formula (3.17) of Proposition 3.5, with $\mathfrak{m}_1 = \mathfrak{d}\mathfrak{n}_2$ and $\mathfrak{m}_2 = \mathfrak{O}$, obtaining a diagonal sum

$$c_1\mathscr{H} \sum_{\mathrm{N}(\mathfrak{n})\leqslant \mathrm{N}(T)^{3/2+\varepsilon}} \frac{A(\mathfrak{n},1)}{\mathrm{N}(\mathfrak{n})^{1+2u}}, \tag{14.1}$$

and an off-diagonal sum

$$c_2 \sum\sum_{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)\leqslant \mathrm{N}(T)^{3+\varepsilon}} \sum_{\mathfrak{c}\in\widetilde{C}_F} \sum_{\epsilon\in\mathfrak{O}^\times/\mathfrak{O}^{\times 2}} \sum_{\substack{\gamma\in\mathfrak{a}^{-1}/\mathfrak{O}^\times \\ \mathrm{N}(\gamma)\leqslant \mathrm{N}(T)^{3+\varepsilon}/\mathrm{N}(\mathfrak{a}\mathfrak{d}^3\mathfrak{n}^2)}} \frac{\mu(\mathfrak{d})A(\mathfrak{n},1)A(1,\gamma\mathfrak{a})}{\mathrm{N}(\gamma\mathfrak{a}\mathfrak{d}^3\mathfrak{n}^2)^{1/2+u}} \cdot \sum_{c\in\mathfrak{c}^{-1}} \frac{\mathrm{KS}(\epsilon\gamma, \mathfrak{a}\mathfrak{d}\mathfrak{D}^{-1}; 1/\beta_{\mathfrak{d}}, \mathfrak{D}^{-1}; c, \mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} \mathscr{H}\left(\frac{\epsilon\gamma}{\beta_{\mathfrak{d}}c^2}\right), \tag{14.2}$$

in which $\mathfrak{a} \in \widetilde{C}_F$ is determined by $\mathfrak{a} \sim (\mathfrak{c}\mathfrak{D})^2\mathfrak{d}^{-1}$, and $\beta_{\mathfrak{d}} = \beta_{\mathfrak{c},\mathfrak{a}\mathfrak{d}\mathfrak{D}^{-1}\cdot\mathfrak{D}^{-1}} = [(\mathfrak{c}\mathfrak{D})^2(\mathfrak{a}\mathfrak{d})^{-1}]$.

Lemma 14.1. *For the test function $h(\nu)$ defined as in (6.1)–(6.4) (see also (5.8), (5.11)), we have the following estimate for $\mathscr{H}$ (defined by (3.1), (3.15)):*

$$\mathscr{H} \ll \mathrm{N}^{\natural}(M)\mathrm{N}(T)^{1+\varepsilon}. \tag{14.3}$$

Proof. In view of (3.1) and (3.15), the integral $\mathscr{H}$ splits into the product of

$$\int_{-\infty}^{\infty} h_v(\nu)\tanh(\pi\nu)\nu\,\mathrm{d}\nu$$

if v is real, and

$$\int_{-\infty}^{\infty} h_v(\nu)\nu^2\,\mathrm{d}\nu$$

if v is complex, which, in view of (6.5), are bounded by

$$\int_0^\infty e^{-(\nu-T_v)^2/M_v^2}\nu^{1+\varepsilon}\,d\nu + T_v^{-A} \ll M_v T_v^{1+\varepsilon},$$

and

$$\int_0^\infty e^{-(\nu-T_v)^2/M_v^2}\nu^{2+\varepsilon}\,d\nu + T_v^{-A} \ll M_v T_v^{2+\varepsilon},$$

respectively. Then (14.3) follows immediately. ■

It follows from Cauchy–Schwarz, (4.15), and (14.3) that the diagonal sum in (14.1) is bounded by $\mathrm{N}^\natural(M)\mathrm{N}(T)^{1+\varepsilon}$, as expected.

For the off-diagonal sum, our aim is to execute Voronoï summation in the γ-variable, so we must unfold the $\epsilon\gamma$-sum from a sum over $\mathfrak{a}^{-1}/\mathfrak{O}^{\times 2}$ to a sum over $\mathfrak{a}^{-1}$. For this, we set $\mathfrak{q} = c\mathfrak{c}$ and fold the c-sum into a $\mathfrak{q}$-sum over ideals. Thus (14.2) is rewritten as

$$2c_2 \sum\sum_{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)\leqslant \mathrm{N}(T)^{3+\varepsilon}} \sum_{\mathfrak{c}\in\widetilde{C}_F} \frac{\mu(\mathfrak{d})A(\mathfrak{n},1)}{\mathrm{N}(\mathfrak{a}\mathfrak{d}^3\mathfrak{n}^2)^{1/2+u}} \cdot \sum_{\mathfrak{q}\sim\mathfrak{c}} \frac{1}{\mathrm{N}(\mathfrak{q})} \sum_{\substack{\gamma\in\mathfrak{a}^{-1}\\ \mathrm{N}(\gamma)\leqslant \mathrm{N}(T)^{3+\varepsilon}/\mathrm{N}(\mathfrak{a}\mathfrak{d}^3\mathfrak{n}^2)}} \frac{A(1,\gamma\mathfrak{a})}{\mathrm{N}(\gamma)^{1/2+u}} \mathrm{KS}(\gamma, \mathfrak{a}\mathfrak{d}\mathfrak{D}^{-1}; 1/\beta_\mathfrak{d}, \mathfrak{D}^{-1}; c_\mathfrak{q}, \mathfrak{c})\mathscr{H}\bigg(\frac{\gamma}{\beta_\mathfrak{d} c_\mathfrak{q}^2}\bigg), \tag{14.4}$$

where $c_\mathfrak{q} = [\mathfrak{c}^{-1}\mathfrak{q}]$, and $\mathfrak{a}$, $\beta_\mathfrak{d}$ are defined after (14.2). In view of (2.1), it will be convenient to introduce $V(\mathfrak{b}) \in \mathfrak{a}_+$ for every non-zero ideal $\mathfrak{b}$ with

$$V(\mathfrak{b})_v = \mathrm{N}(\mathfrak{b})^{\theta_v}, \quad \theta_v = \log T_v/\log \mathrm{N}(T), \tag{14.5}$$

so that

$$1/|\beta_\mathfrak{d}|_v \asymp V(\mathfrak{d})_v^{N_v}, \quad |c_\mathfrak{q}|_v \asymp V(\mathfrak{q})_v^{N_v}, \tag{14.6}$$

for each $v\,|\,\infty$. The main actors are $\mathfrak{q}$ and γ, so we shall be concerned with the last two summations in the second line of (14.4).

15. First reductions

Next, we need to do a smooth Δ-adic partition of unity in $|\gamma|_v$ for each $v\,|\,\infty$, where $\Delta > 1$ is a fixed constant with $\log\Delta$ small. However, when F is neither rational nor imaginary quadratic, an issue with the infinitude of units is that one has $|\gamma|_v \to 0$ when γ ranges in $\mathfrak{a}^{-1} \smallsetminus \{0\}$. This may be addressed by proving that if $|\gamma|_v V(\mathfrak{d})_v^{N_v}/V(\mathfrak{q})_v^{2N_v} \leqslant T_v^{2N_v}$ (so that $|\gamma/\beta_\mathfrak{d} c_\mathfrak{q}^2|_v \ll T_v^{2N_v}$ by (14.6)) for any given $v\,|\,\infty$, then the contribution is negligibly small; critical are the second estimates for Bessel integrals in Corollary 8.4 and the assumption that T_v is large ($T_v \geqslant \mathrm{N}(T)^\varepsilon$) for every $v\,|\,\infty$. To this end, we use Weil's bound for Kloosterman sums (3.14) and the estimates for Bessel integrals in Corollary 8.4 to bound the contribution by

$$\sum_{S \subsetneq S_\infty} \frac{\mathrm{N}^\natural(M)\mathrm{N}(T)^{1+\varepsilon}}{|T|^{2A'}_{S_\infty \smallsetminus S}} \sum\sum_{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2) \leqslant \mathrm{N}(T)^{3+\varepsilon}} \sum_{\mathfrak{c} \in \widetilde{C}_F} \frac{|A(\mathfrak{n},1)|}{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)^{1/2+\varepsilon}} \cdot \sum_{\mathfrak{q} \sim \mathfrak{c}} \frac{1}{\mathrm{N}(\mathfrak{q})^{1/2-\varepsilon}|V(\mathfrak{d}^{-1}\mathfrak{q}^2)|^{1/2}_{S_\infty \smallsetminus S}} \sum_{\substack{\gamma \in F^S_\infty(T^2 V(\mathfrak{d}^{-1}\mathfrak{q}^2)) \\ \mathrm{N}(\gamma) \leqslant \mathrm{N}(T)^{3+\varepsilon}/\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)}} \frac{|A(1,\gamma\mathfrak{a})|}{|\mathrm{N}(\gamma)|^{\varepsilon}|\gamma|^{1/2}_S},$$

where $F^S_\infty(V) \subset F_\infty$ is defined in (4.17). Because of the occurrence of $|T|^{2A'}_{S_\infty \smallsetminus S}$, this sum is negligibly small on choosing A' to be large. Note that if (4.18) in Lemma 4.10 is applied to bound the γ-sum by

$$\frac{\mathrm{N}(T)^{9/4+\varepsilon}}{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)^{3/4}} \sum_{\gamma \in F^S_\infty(T^2 V(\mathfrak{d}^{-1}\mathfrak{q}^2))} \frac{|A(1,\gamma\mathfrak{a})|}{|\mathrm{N}(\gamma)|^{3/4+\varepsilon}|\gamma|^{1/2}_S} \ll \frac{\mathrm{N}(T)^{11/4+\varepsilon}\mathrm{N}(\mathfrak{q})^{1/2-\varepsilon}}{|T|_S \mathrm{N}(\mathfrak{d})^{5/2}\mathrm{N}(\mathfrak{n})^{3/2}|V(\mathfrak{d}^{-1}\mathfrak{q}^2)|^{1/2}_S},$$

then $|V(\mathfrak{d}^{-1}\mathfrak{q}^2)|^{1/2}_{S_\infty \smallsetminus S}$ and $|V(\mathfrak{d}^{-1}\mathfrak{q}^2)|^{1/2}_S$ are combined into $\mathrm{N}(\mathfrak{d}^{-1}\mathfrak{q}^2)^{1/2}$, and the $\mathfrak{q}$-sum is convergent.

We may therefore impose the condition $|\gamma|_v V(\mathfrak{d})^{N_v}_v / V(\mathfrak{q})^{2N_v}_v > T^{2N_v}_v$ for all $v \,|\, \infty$. Note that necessarily $|\gamma|_v > T^{(1-\varepsilon)N_v}_v$ for all $v \,|\, \infty$, since $\mathrm{N}(\mathfrak{d}) \leqslant \mathrm{N}(T)^{1+\varepsilon}$. By a smooth partition of unity on the γ-sum, the problem can be reduced to proving the following proposition.

Proposition 15.1. *Let $\mathfrak{d}$ be a square-free integral ideal with $\mathrm{N}(\mathfrak{d}) \leqslant \mathrm{N}(T)^{1+\varepsilon}$. Let $\mathfrak{a}, \mathfrak{c} \in \widetilde{C}_F$ satisfy $\mathfrak{a}\mathfrak{d} \sim (\mathfrak{c}\mathfrak{D})^2$. Set $\beta_\mathfrak{d} = [(\mathfrak{c}\mathfrak{D})^2(\mathfrak{a}\mathfrak{d})^{-1}]$. Let $R \in \mathfrak{a}_+$ be such that*

$$\mathrm{N}(R) \leqslant \mathrm{N}(T)^{3+\varepsilon}/\mathrm{N}(\mathfrak{d})^3. \tag{15.1}$$

Fix $\Delta > 1$ with $\log \Delta$ sufficiently small. For each $v \,|\, \infty$, let $f_v(r)$ be a smooth function supported on $[R_v, \Delta R_v]$ satisfying $f^{(i)}_v(r) \ll_i (\log \mathrm{N}(T)/R_v)^i$ for all $i \geqslant 0$. Suppose that $\mathcal{H}(x)$ is the Bessel transform of $h(\nu)$ given in (3.15), with $h(\nu)$ defined as in (6.1)–(6.3). Define

$$\mathcal{S}_\mathfrak{d}(T,R) = \sum_{\mathfrak{q} \sim \mathfrak{c}} \frac{1}{\mathrm{N}(\mathfrak{q})} \sum_{\gamma \in \mathfrak{a}^{-1}} A(1,\gamma\mathfrak{a})\mathrm{KS}(\gamma, \mathfrak{a}\mathfrak{d}\mathfrak{D}^{-1}; 1/\beta_\mathfrak{d}, \mathfrak{D}^{-1}; c_\mathfrak{q}, \mathfrak{c}) f\left(\gamma, \frac{1}{\beta_\mathfrak{d} c^2_\mathfrak{q}}\right), \tag{15.2}$$

where $c_\mathfrak{q} = [\mathfrak{c}^{-1}\mathfrak{q}]$, the $\mathfrak{q}$-sum is finite, subject to the conditions

$$V(\mathfrak{d}^{-1}\mathfrak{q}^2)_v \ll R_v/T^2_v, \quad v \,|\, \infty, \tag{15.3}$$

with $V(\mathfrak{d}^{-1}\mathfrak{q}^2) \in \mathfrak{a}_+$ *defined as in* (14.5)*, and* $f(x, 1/\beta_\mathfrak{d} c_\mathfrak{q}^2)$ *is the product of*

$$f_v\left(x_v, \frac{1}{\beta_\mathfrak{d} c_\mathfrak{q}^2}\right) = f_v(|x_v|)\mathcal{H}_v\left(\frac{x_v}{\beta_\mathfrak{d} c_\mathfrak{q}^2}\right), \quad v \,|\, \infty. \tag{15.4}$$

Then

$$\mathcal{S}_\mathfrak{d}(T, R) \ll_{\varepsilon,\pi,F} \mathrm{N}^\natural(M)\mathrm{N}(T)^{1/2+\varepsilon}(\mathrm{N}(R)\mathrm{N}(\mathfrak{d}))^{3/4} + \frac{\mathrm{N}^\natural(M)\mathrm{N}(R)\mathrm{N}(\mathfrak{d})}{\mathrm{N}(T)^{1/3-\varepsilon}}. \tag{15.5}$$

To deduce Theorem 1.1 from Proposition 15.1, we use (15.5) to bound the sum in (14.4) by the sum of

$$\mathrm{N}^\natural(M)\mathrm{N}(T)^{1/2+\varepsilon} \sum\sum_{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)\leqslant \mathrm{N}(T)^{3+\varepsilon}} \frac{|A(\mathfrak{n},1)|}{\mathrm{N}(\mathfrak{d})^{3/4}\mathrm{N}(\mathfrak{n})}\left(\frac{\mathrm{N}(T)^{3+\varepsilon}}{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)}\right)^{1/4} \ll \mathrm{N}^\natural(M)\mathrm{N}(T)^{5/4+\varepsilon}$$

and

$$\frac{\mathrm{N}^\natural(M)\mathrm{N}(T)^\varepsilon}{\mathrm{N}(T)^{1/3}} \sum\sum_{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)\leqslant \mathrm{N}(T)^{3+\varepsilon}} \frac{|A(\mathfrak{n},1)|}{\mathrm{N}(\mathfrak{d})^{1/2}\mathrm{N}(\mathfrak{n})}\left(\frac{\mathrm{N}(T)^{3+\varepsilon}}{\mathrm{N}(\mathfrak{d}^3\mathfrak{n}^2)}\right)^{1/2} \ll \mathrm{N}^\natural(M)\mathrm{N}(T)^{7/6+\varepsilon}.$$

16. Application of the Voronoï summation

By Definition 3.2, we open the Kloosterman sum as follows:

$$\mathrm{KS}(\gamma, \mathfrak{a}\mathfrak{d}\mathfrak{D}^{-1}; 1/\beta_\mathfrak{d}, \mathfrak{D}^{-1}; c_\mathfrak{q}, \mathfrak{c}) = \sum_{x\in(\mathfrak{a}\mathfrak{d}(\mathfrak{c}\mathfrak{D})^{-1}/\mathfrak{a}\mathfrak{d}(\mathfrak{c}\mathfrak{D})^{-1}\mathfrak{q})^\times} \psi_\infty\left(\frac{\gamma x}{c_\mathfrak{q}} + \frac{x^{-1}}{\beta_\mathfrak{d} c_\mathfrak{q}}\right),$$

where $x^{-1} \in ((\mathfrak{a}\mathfrak{d})^{-1}\mathfrak{c}\mathfrak{D}/(\mathfrak{a}\mathfrak{d})^{-1}\mathfrak{c}\mathfrak{D}\mathfrak{q})^\times$ is as defined in Definition 3.1. On applying the Voronoï summation formula in Proposition 4.8, up to the constant $\mathrm{N}(\mathfrak{a})/\mathrm{N}(\mathfrak{D})^{3/2}$, the γ-sum in $\mathcal{S}_\mathfrak{d}(T, R)$ is transformed into

$$\sum_{\mathfrak{b}\subset\mathfrak{q}_1\subset\mathcal{O}} \frac{1}{\mathrm{N}(\mathfrak{b}\mathfrak{q}_1)} \sum_{\gamma\in\mathfrak{a}(\mathfrak{b}\mathfrak{q}_1^2\mathfrak{D}^3)^{-1}\smallsetminus\{0\}} A(\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{q}_1^2\mathfrak{D}^3\gamma, \mathfrak{b}\mathfrak{q}_1^{-1}) T_\mathfrak{b}(\gamma; \mathfrak{q}, \mathfrak{q}_1)\widetilde{f}\left(\gamma, \frac{1}{\beta_\mathfrak{d} c_\mathfrak{q}^2}\right), \tag{16.1}$$

where $\mathfrak{b} = (\mathfrak{d}, \mathfrak{q})^{-1}\mathfrak{q}$, the function $\widetilde{f}(y, 1/\beta_\mathfrak{d} c_\mathfrak{q}^2)$ is the Hankel transform of $f(x, 1/\beta_\mathfrak{d} c_\mathfrak{q}^2)$ $(x, y \in F_\infty^\times)$ as in Definition 4.4, with

$$\widetilde{f}_v(y_v, 1/\beta_\mathfrak{d} c_\mathfrak{q}^2) = \int_{F_v^\times} f_v(x_v, 1/\beta_\mathfrak{d} c_\mathfrak{q}^2) J_{\pi_v}(x_v y_v)\, \mathrm{d}x_v, \tag{16.2}$$

and the exponential sum

$$T_\mathfrak{b}(\gamma; \mathfrak{q}, \mathfrak{q}_1) = \sum_{x\in(\mathfrak{a}\mathfrak{d}(\mathfrak{c}\mathfrak{D})^{-1}/\mathfrak{a}\mathfrak{d}(\mathfrak{c}\mathfrak{D})^{-1}\mathfrak{q})^\times} \psi_\infty\left(\frac{x^{-1}}{\beta_\mathfrak{d} c_\mathfrak{q}}\right) \mathrm{Kl}_\mathfrak{b}(1, -\gamma c_\mathfrak{q}/x; \mathfrak{q}_1). \tag{16.3}$$

To see $\mathfrak{b} = (\mathfrak{d}, \mathfrak{q})^{-1}\mathfrak{q}$, use $x/c_{\mathfrak{q}} \in (\mathfrak{a}\mathfrak{D}^{-1}\mathfrak{d}\mathfrak{q}^{-1}/\mathfrak{a}\mathfrak{D}^{-1}\mathfrak{d})^{\times}$ to deduce $R = \{v : \mathrm{ord}_v(\mathfrak{d}\mathfrak{q}^{-1}) < 0\} = \{v : \mathfrak{p}_v | (\mathfrak{d}, \mathfrak{q})^{-1}\mathfrak{q}\}$ and $\mathrm{ord}_v((c_{\mathfrak{q}}/x)\mathfrak{a}\mathfrak{D}^{-1}) = \mathrm{ord}_v(\mathfrak{d}^{-1}\mathfrak{q}) = \mathrm{ord}_v((\mathfrak{d}, \mathfrak{q})^{-1}\mathfrak{q})$ for every $v \in R$.

We conclude that, up to a constant,

$$\mathcal{S}_{\mathfrak{d}}(T, R) = \sum_{\substack{\mathfrak{q}_1 | \mathfrak{b},\, \mathfrak{b} | \mathfrak{q},\, \mathfrak{q} \sim \mathfrak{c} \\ (\mathfrak{b}, \mathfrak{d}\mathfrak{b}\mathfrak{q}^{-1}) = (1)}} \sum\sum \frac{1}{\mathrm{N}(\mathfrak{b}\mathfrak{q}_1\mathfrak{q})} \sum_{\gamma \in \mathfrak{a}(\mathfrak{b}\mathfrak{q}_1^2\mathfrak{D}^3)^{-1}} A(\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{q}_1^2\mathfrak{D}^3\gamma, \mathfrak{b}\mathfrak{q}_1^{-1}) T_{\mathfrak{d}}(\gamma; \mathfrak{q}, \mathfrak{q}_1) \tilde{f}\left(\gamma, \frac{1}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right), \tag{16.4}$$

where the $\mathfrak{q}$-sum is subject to the conditions in (15.3), and so are the $\mathfrak{q}_1$- and $\mathfrak{b}$-sums.

17. Transformation of exponential sums

Next, we need to compute the exponential sum $T_{\mathfrak{d}}(\gamma; \mathfrak{q}, \mathfrak{q}_1)$ as in (16.3).

17.1. The special case $F = \mathbb{Q}$

For purely expository purposes, we first compute the exponential sum in the case when $F = \mathbb{Q}$. For this, Nunes [47] quoted a result of Blomer [5] for the corresponding character sum and then set the character $\chi = 1$. However, when $\chi = 1$, some of Blomer's manipulations become unnecessary, so it is easier to just compute in a direct manner.

More precisely, in the notation of [5,47], let $\mathfrak{a} = \mathfrak{c} = \mathfrak{c}_1 = \mathfrak{D} = (1)$, $\mathfrak{d} = (\delta)$, $\beta_{\mathfrak{d}} = 1/\delta$, $\mathfrak{q} = (c)$, $\mathfrak{b} = (c_1)$ $(= (c/(c,\delta)))$, $\mathfrak{q}_1 = (c_1/n_1)$ $(n_1 \mid c_1)$, $c_{\mathfrak{q}} = c$, $c_{\mathfrak{q}_1} = c_1/n_1$, and $\gamma = n_1^2 n_2/c_1^3$. After suitable changes, the exponential sum in (16.3) turns into

$$\sum_{d \,(\mathrm{mod}\, c)}^{\star} e\left(\frac{d}{c}\right) S(d, \bar{\delta}_1 n_2; c_1/n_1), \tag{17.1}$$

where $\delta_1 = \delta/(c,\delta)$, and c or n_2 could have signs. For simplicity, set $f_1 = c_1/n_1$. Opening the Kloosterman sum, we obtain

$$\sum_{d \,(\mathrm{mod}\, c)}^{\star} e\left(\frac{d}{c}\right) \sum_{a \,(\mathrm{mod}\, f_1)}^{\star} e\left(\frac{ad}{f_1} + \frac{\bar{\delta}_1 \bar{a} n_2}{f_1}\right) = \sum_{a \,(\mathrm{mod}\, f_1)}^{\star} e\left(\frac{\bar{\delta}_1 \bar{a} n_2}{f_1}\right) \sum_{d \,(\mathrm{mod}\, c)}^{\star} e\left(\frac{d(ac/f_1 + 1)}{c}\right).$$

The d-sum is a Ramanujan sum, and it may be evaluated with the aid of Möbius inversion. We then arrive at

$$\sum_{c_2 | c} c_2 \mu(c/c_2) \sum_{\substack{a \,(\mathrm{mod}\, f_1) \\ ac/f_1 \equiv -1 \,(\mathrm{mod}\, c_2)}}^{\star} e\left(\frac{\bar{\delta}_1 \bar{a} n_2}{f_1}\right).$$

We necessarily have $(c_2, c/f_1) = 1$, and hence $c_2 \mid f_1$. Moreover, we may assume that c/c_2 is square-free. By introducing the new variable $b = (\bar{a} + c/f_1)/c_2$, the sum above is transformed into

$$e\left(-\frac{\overline{\delta}_1(c/f_1)n_2}{f_1}\right)\sum_{\substack{c_2|f_1\\(c_2,c/f_1)=1}} c_2\mu(c/c_2)\sum_{\substack{b\,(\mathrm{mod}\,f_1/c_2)\\(bc_2-c/f_1,\,f_1)=1}} e\left(\frac{\overline{\delta}_1 bn_2}{f_1/c_2}\right).$$

Finally, Möbius inversion turns the innermost sum into

$$\sum_{f_2|f_1}\mu(f_2)\sum_{\substack{b\,(\mathrm{mod}\,f_1/c_2)\\bc_2\equiv c/f_1\,(\mathrm{mod}\,f_2)}} e\left(\frac{\overline{\delta}_1 bn_2}{f_1/c_2}\right).$$

As $bc_2\equiv c/f_1(\mathrm{mod}\ f_2)$, it is easy to see that if $p\mid f_2$, then $p\nmid c_2$ and $p\parallel c$ (recall that $(c_2,c/f_1)=1$ and c/c_2 is square-free). Let $\breve{f}_1$ denote the square-free part of f_1. Then $f_2\mid\breve{f}_1$ and $(f_2,c_2)=1$; in particular, f_2 divides f_1/c_2. Consequently, the sum above is equal to

$$\sum_{\substack{f_2|\breve{f}_1\\(f_2,c_2)=1\\(f_1/c_2f_2)|n_2}}\frac{f_1}{c_2f_2}\mu(f_2)e\left(\frac{\overline{\delta}_1\overline{c}_2(c/f_1)n_2}{f_1/c_2}\right),$$

in which $\overline{c}_2c_2\equiv 1\ (\mathrm{mod}\ f_2)$. We conclude that the exponential sum in (17.1) is equal to

$$e\left(-\frac{\overline{\delta}_1(c/f_1)n_2}{f_1}\right)\sum\sum_{\substack{c_2|f_1,\ f_2|\breve{f}_1\\(c_2,c/f_1)=(c_2,f_2)=1\\c_2f_2n_2=f_1n_2'}}\frac{f_1}{f_2}\mu(c/c_2)\mu(f_2)e\left(\frac{\overline{\delta}_1\overline{c}_2(c/f_1)n_2'}{f_2}\right). \tag{17.2}$$

17.2. The general case

By Lemma 4.7, the Kloosterman sum in (16.3) is

$$\mathrm{Kl}_{\mathfrak{b}}(1,-\gamma c_{\mathfrak{q}}/x;\mathfrak{q}_1)=\varphi(\mathfrak{q}_1)\int_{\pi_{(\mathfrak{q}_1\mathfrak{D})^{-1}}\widehat{\mathbb{O}}_{\mathfrak{b}}^{\times}}\psi_{\mathfrak{b}}\left(y-\frac{c_{\mathfrak{q}}\gamma}{xy}\right)\mathrm{d}^{\times}y.$$

Let $\widehat{\mathbb{O}}^{\times}=\prod_{v\nmid\infty}\mathbb{O}_v^{\times}$. We may also transform the x-sum in (16.3) into an integral over $\pi_{\mathfrak{a}\mathfrak{b}(\mathfrak{c}\mathfrak{D})^{-1}}\widehat{\mathbb{O}}^{\times}$. More precisely,

$$T_{\mathfrak{b}}(\gamma;\mathfrak{q},\mathfrak{q}_1)=\varphi(\mathfrak{q})\varphi(\mathfrak{q}_1)I_{\mathfrak{b}}(\gamma;\mathfrak{q},\mathfrak{q}_1), \tag{17.3}$$

where

$$I_{\mathfrak{b}}(\gamma;\mathfrak{q},\mathfrak{q}_1)=\int_{\pi_{\mathfrak{a}\mathfrak{b}(\mathfrak{c}\mathfrak{D})^{-1}}\widehat{\mathbb{O}}^{\times}}\int_{\pi_{(\mathfrak{q}_1\mathfrak{D})^{-1}}\widehat{\mathbb{O}}_{\mathfrak{b}}^{\times}}\psi_f\left(-\frac{1}{\beta_{\mathfrak{b}}c_{\mathfrak{q}}x}\right)\psi_{\mathfrak{b}}\left(y-\frac{c_{\mathfrak{q}}\gamma}{xy}\right)\mathrm{d}^{\times}y\,\mathrm{d}^{\times}x.$$

On changing y into $-1/\beta_{\mathfrak{b}}xy$ and then x into $-1/\beta_{\mathfrak{b}}x$, we obtain

$$I_{\mathfrak{b}}(\gamma;\mathfrak{q},\mathfrak{q}_1)=\int_{\pi_{\mathfrak{c}^{-1}\mathfrak{q}_1}\widehat{\mathbb{O}}_{\mathfrak{b}}^{\times}}\psi_{\mathfrak{b}}(\beta_{\mathfrak{b}}c_{\mathfrak{q}}\gamma y)\int_{\pi_{(\mathfrak{c}\mathfrak{D})^{-1}}\widehat{\mathbb{O}}^{\times}}\psi_f\left(\frac{x}{c_{\mathfrak{q}}}\right)\psi_{\mathfrak{b}}\left(\frac{x}{y}\right)\mathrm{d}^{\times}x\,\mathrm{d}^{\times}y.$$

It is clear that $I_{\mathfrak{d}}(\gamma;\mathfrak{q},\mathfrak{q}_1)$ may be factored into a product of local integrals $I_v(\gamma;\mathfrak{q}_v,\mathfrak{q}_{1v})$ (the $\mathfrak{d}$ is suppressed from the subscript for brevity). For non-Archimedean v, define $d_v = \mathrm{ord}_v(\mathfrak{D})$, $r_v = \mathrm{ord}_v(\mathfrak{c})$, $s_v = \mathrm{ord}_v(\mathfrak{q})$, and $s_{1v} = \mathrm{ord}_v(\mathfrak{q}_1)$. For $\mathfrak{p}_v \nmid \mathfrak{b}$, the local integral is

$$I_v(\gamma;\mathfrak{q}_v,\mathcal{O}_v) = \int_{v(x)=-r_v-d_v} \psi_v\left(\frac{x}{c_{\mathfrak{q}}}\right) \mathrm{d}^\times x. \tag{17.4}$$

For $\mathfrak{p}_v \mid \mathfrak{b}$, the local integral is

$$I_v(\gamma;\mathfrak{q}_v,\mathfrak{q}_{1v}) = \int_{v(y)=s_{1v}-r_v} \psi_v(\beta_{\mathfrak{d}} c_{\mathfrak{q}}\gamma y) \int_{v(x)=-r_v-d_v} \psi_v\left(x\left(\frac{1}{y}+\frac{1}{c_{\mathfrak{q}}}\right)\right) \mathrm{d}^\times x\, \mathrm{d}^\times y. \tag{17.5}$$

The following lemma is standard.

Lemma 17.1. *We have*

$$(\mathrm{N}(\mathfrak{p}_v)-1)\int_{v(x)=-r-d_v} \psi_v(ax)\,\mathrm{d}^\times x = \begin{cases} \mathrm{N}(\mathfrak{p}_v)-1 & \textit{if } v(a)\geqslant r, \\ -1 & \textit{if } v(a)=r-1, \\ 0 & \textit{if otherwise.} \end{cases}$$

For $\mathfrak{p}_v \nmid \mathfrak{b}$, Lemma 17.1 implies that the integral in (17.4) is just $\mu(\mathfrak{q}_v)/\varphi(\mathfrak{q}_v)$.

For $\mathfrak{p}_v \mid \mathfrak{b}$, we first observe that

$$v(\beta_{\mathfrak{d}} c_{\mathfrak{q}}\gamma) \geqslant r_v - 2s_{1v} - d_v,$$

for $(\beta_{\mathfrak{d}}) = (\mathfrak{c}\mathfrak{D})^2(\mathfrak{a}\mathfrak{d})^{-1}$, $(c_{\mathfrak{q}}) = \mathfrak{c}^{-1}\mathfrak{q}$, $\gamma \in \mathfrak{a}(\mathfrak{b}\mathfrak{q}_1^2\mathfrak{D}^3)^{-1}$, and $\mathfrak{b} = (\mathfrak{d},\mathfrak{q})^{-1}\mathfrak{q}$. Hence the integral in (17.5) is reduced to that in (17.4) if $s_{1v}=0$, and one may henceforth assume $s_{1v}\geqslant 1$.

Keep in mind that $v(c_{\mathfrak{q}}) = s_v - r_v$ and $v(y) = s_{1v}-r_v$. On applying Lemma 17.1 to the x-integral in (17.5), we obtain

$$I_v(\gamma;\mathfrak{q}_v,\mathfrak{q}_{1v}) = \sum_{\nu=0,1} \frac{(-1)^\nu \mathrm{N}(\mathfrak{p}_v)^{1-\nu}}{\mathrm{N}(\mathfrak{p}_v)-1} I_v^\nu(\gamma;\mathfrak{q}_v,\mathfrak{q}_{1v}) \tag{17.6}$$

with

$$I_v^\nu(\gamma;\mathfrak{q}_v,\mathfrak{q}_{1v}) = \int_{\substack{v(y)=s_{1v}-r_v \\ v(y+c_{\mathfrak{q}})\geqslant s_v+s_{1v}-r_v-\nu}} \psi_v(\beta_{\mathfrak{d}} c_{\mathfrak{q}}\gamma y)\,\mathrm{d}^\times y \quad (\nu=0,1). \tag{17.7}$$

First, consider the case when $s_v>\nu$. For $s_{1v}<s_v$, we have $v(y+c_{\mathfrak{q}}) = v(y) = s_{1v}-r_v < s_v+s_{1v}-r_v-\nu$, and hence $I_v^\nu(\gamma;\mathfrak{q}_v,\mathfrak{q}_{1v}) = 0$ as the integration is on an empty set. For $s_{1v}=s_v$, we introduce the new variable $w = y+c_{\mathfrak{q}}$ so that the resulting w-integral is on $\mathfrak{p}_v^{2s_v-r_v-\nu}$ (since $v(w)>v(y)$ by the condition $s_v>\nu$):

$$I_v^\nu(\gamma;\mathfrak{q}_v,\mathfrak{q}_v) = \frac{\mathrm{N}(\mathfrak{D}_v)^{1/2}\mathrm{N}(\mathfrak{p}_v)^{s_v-r_v+1}}{\mathrm{N}(\mathfrak{p}_v)-1}\cdot \psi_v(-\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2\gamma)\int_{\mathfrak{p}_v^{2s_v-r_v-\nu}} \psi_v(\beta_{\mathfrak{d}} c_{\mathfrak{q}}\gamma w)\,\mathrm{d}w.$$

Recall that ψ_v has conductor $\mathfrak{D}_v^{-1}$ and $\mathfrak{p}_v^{2s_v-r_v-\nu}$ has measure $\mathrm{N}(\mathfrak{D}_v)^{-1/2}\mathrm{N}(\mathfrak{p}_v)^{r_v+\nu-2s_v}$. It follows that $I_v^\nu(\gamma;\mathfrak{q}_v,\mathfrak{q}_v)=0$ unless $v(\beta_{\mathfrak{d}}c_{\mathfrak{q}}\gamma)\geqslant r_v-2s_v-d_v+\nu$, in which case

$$I_v^\nu(\gamma;\mathfrak{q}_v,\mathfrak{q}_v)=\psi_v(-\beta_{\mathfrak{d}}c_{\mathfrak{q}}^2\gamma)\frac{\mathrm{N}(\mathfrak{p}_v)^\nu}{\varphi(\mathfrak{q}_v)}.$$

Next, we consider the remaining case when $s_v=s_{1v}=\nu=1$. Then the second condition in the integration domain in (17.7) reads $v(y+c_{\mathfrak{q}})\geqslant 1-r_v$, and it can be dropped because it is implied by the first condition $v(y)=1-r_v$, along with $v(c_{\mathfrak{q}})=1-r_v$. Thus

$$I_v^1(\gamma;\mathfrak{p}_v,\mathfrak{p}_v)=\int_{v(y)=1-r_v}\psi_v(\beta_{\mathfrak{d}}c_{\mathfrak{q}}\gamma y)\,\mathrm{d}^\times y.$$

By Lemma 17.1,

$$I_v^1(\gamma;\mathfrak{p}_v,\mathfrak{p}_v)=\sum_{\substack{\mu=0,1\\ v(\beta_{\mathfrak{d}}c_{\mathfrak{q}}\gamma)\geqslant r_v-1-d_v-\mu}}\frac{(-1)^\mu\mathrm{N}(\mathfrak{p}_v)^{1-\mu}}{\mathrm{N}(\mathfrak{p}_v)-1}.$$

Lemma 17.2. *We have the following formulae for $I_v(\gamma;\mathfrak{q}_v,\mathfrak{q}_{1v})$:*

(1) $I_v(\gamma;\mathfrak{q}_v,\mathfrak{O}_v)=\mu(\mathfrak{q}_v)/\varphi(\mathfrak{q}_v)$.

(2) *For* $\mathrm{ord}_v(\mathfrak{q})>1$, *we have* $I_v(\gamma;\mathfrak{q}_v,\mathfrak{q}_{1v})=0$ *if* $\mathrm{ord}_v(\mathfrak{q})>\mathrm{ord}_v(\mathfrak{q}_1)$, *and*

$$I_v(\gamma;\mathfrak{q}_v,\mathfrak{q}_v)=\psi_v(-\beta_{\mathfrak{d}}c_{\mathfrak{q}}^2\gamma)\frac{\mathrm{N}(\mathfrak{p}_v)}{\mathrm{N}(\mathfrak{p}_v)-1}\frac{1}{\varphi(\mathfrak{q}_v)}\sum_{\substack{\nu=0,1\\ v(\beta_{\mathfrak{d}}c_{\mathfrak{q}}^2\gamma\mathfrak{D})\geqslant\nu-v(\mathfrak{q})}}(-1)^\nu.$$

(3) *We have*

$$I_v(\gamma;\mathfrak{p}_v,\mathfrak{p}_v)=\psi_v(-\beta_{\mathfrak{d}}c_{\mathfrak{q}}^2\gamma)\frac{\mathrm{N}(\mathfrak{p}_v)}{(\mathrm{N}(\mathfrak{p}_v)-1)^2}\sum\sum_{\substack{0\leqslant\mu\leqslant\nu\leqslant 1\\ v(\beta_{\mathfrak{d}}c_{\mathfrak{q}}^2\gamma\mathfrak{D})\geqslant\nu-\mu-1}}\frac{(-1)^{\nu+\mu}}{\mathrm{N}(\mathfrak{p}_v)^\mu}\psi_v(\beta_{\mathfrak{d}}c_{\mathfrak{q}}^2\gamma)^\mu.$$

As a consequence of (17.3) and Lemma 17.2, it is straightforward to deduce the following formula for $T_{\mathfrak{b}}(\gamma;\mathfrak{q},\mathfrak{q}_1)$. The reader may compare (17.8) with (17.2).

Corollary 17.3. *Let $\breve{\mathfrak{q}}_1$ denote the square-free part of $\mathfrak{q}_1$. We have $T_{\mathfrak{b}}(\gamma;\mathfrak{q},\mathfrak{q}_1)=0$ unless $(\mathfrak{q}_1,\mathfrak{q}\mathfrak{q}_1^{-1})=(1)$, in which case*

$$T_{\mathfrak{b}}(\gamma;\mathfrak{q},\mathfrak{q}_1)$$
$$=\psi_{\mathfrak{b}}(-\beta_{\mathfrak{d}}c_{\mathfrak{q}}^2\gamma)\mu(\mathfrak{q}\mathfrak{q}_1^{-1})\mathrm{N}(\mathfrak{q}_1)\sum\sum_{\substack{\mathfrak{q}_2|\mathfrak{q}_1,\ \mathfrak{f}|\breve{\mathfrak{q}}_1\\ (\mathfrak{q}_2,\mathfrak{q}\mathfrak{q}_1^{-1})=(\mathfrak{q}_2,\mathfrak{f})=(1)\\ \gamma\in\mathfrak{a}(\mathfrak{b}\mathfrak{q}_1\mathfrak{q}_2\mathfrak{f}\mathfrak{D}^3)^{-1}}}\frac{\mu(\mathfrak{q}_1\mathfrak{q}_2^{-1})\mu(\mathfrak{f})}{\mathrm{N}(\mathfrak{f})}\psi_{\mathfrak{f}}(\beta_{\mathfrak{d}}c_{\mathfrak{q}}^2\gamma),\tag{17.8}$$

where $\psi_{\mathfrak{b}}$ and $\psi_{\mathfrak{f}}$ are defined in Definition 4.5.

18. Further reductions

Set $\mathfrak{r} = (\mathfrak{d}, \mathfrak{q})^{-1}\mathfrak{d}$. Note that $v(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma \mathfrak{D}) \geqslant 0$ and hence $\psi_v(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma) = 1$ if $\mathfrak{p}_v \nmid \mathfrak{r}\mathfrak{b}$. It follows that the $\psi_{\mathfrak{b}}(-\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma)$ in (17.8) can be written as

$$\psi_{\mathfrak{b}}(-\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma) = \psi_\infty(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma)\psi_{\mathfrak{r}}(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma), \tag{18.1}$$

where we have used the fact that ψ is trivial on F.

Next, in view of (15.4), (16.2), and (11.1)–(11.3), we have

$$f_v\left(x_v, \frac{1}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right) = w_v\left(\frac{x_v}{R_v}, \frac{R_v}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right), \quad \widetilde{f}_v\left(y_v, \frac{1}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right) = R_v \widetilde{w}_v\left(R_v y_v, \frac{R_v}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right).$$

We combine the $\psi_\infty(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 y)$ that occurred in (18.1) with $\widetilde{f}(y, 1/\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2)$ to form

$$\widetilde{f}^{\natural}\left(y, \frac{1}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right) = \frac{1}{\mathrm{N}(R)}\psi_\infty(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma)\widetilde{f}\left(y, \frac{1}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right), \tag{18.2}$$

so that, according to (11.4),

$$\widetilde{f}_v^{\natural}\left(y_v, \frac{1}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right) = \widetilde{w}_v^{\natural}\left(R_v y_v, \frac{R_v}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right). \tag{18.3}$$

In light of our analysis in §11, it is very natural to introduce $\Phi(x)$ such that

$$\Phi(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 y) = \frac{\sqrt{\mathrm{N}(y)\mathrm{N}(R)}}{\mathrm{N}^{\natural}(M)\mathrm{N}(T)^{1+\varepsilon}}\widetilde{f}^{\natural}\left(y, \frac{1}{\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2}\right). \tag{18.4}$$

Set $\mathfrak{d}_0 = (\mathfrak{d}, \mathfrak{q})$, $\mathfrak{b} = \mathfrak{b}_1\mathfrak{q}_1$, and $\mathfrak{q}_1 = \mathfrak{f}_1\mathfrak{q}_2$. By (17.8), (18.1), (18.2), and (18.4), we can now reorganize the sum $\mathcal{S}_{\mathfrak{d}}(T, R)$ in (16.4) as follows:

$$\mathrm{N}^{\natural}(M)\mathrm{N}(T)^{1+\varepsilon}\sqrt{\mathrm{N}(R)} \sum_{\mathfrak{d}=\mathfrak{d}_0\mathfrak{r}} \frac{1}{\mathrm{N}(\mathfrak{d}_0)} \sum\sum_{\substack{(\mathfrak{b}_1,\mathfrak{f}_1)=(1)\\ (\mathfrak{b}_1\mathfrak{f}_1,\mathfrak{d})=(1)}} \frac{\mu(\mathfrak{d}_0\mathfrak{b}_1)\mu(\mathfrak{f}_1)}{\mathrm{N}(\mathfrak{b}_1\mathfrak{f}_1)^2} \sum_{\mathfrak{f}|\mathfrak{f}_1} \frac{\mu(\mathfrak{f})}{\mathrm{N}(\mathfrak{f})} \mathcal{S}^{\mathfrak{d},\mathfrak{r}}_{\mathfrak{b}_1,\mathfrak{f}_1,\mathfrak{f}}(T, R) \tag{18.5}$$

with

$$\mathcal{S}^{\mathfrak{d},\mathfrak{r}}_{\mathfrak{b}_1,\mathfrak{f}_1,\mathfrak{f}}(T, R) = \sum_{\substack{(\mathfrak{q}_2,\mathfrak{d}\mathfrak{b}_1\mathfrak{f})=(1)\\ \mathfrak{q}=\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1\mathfrak{q}_2\sim\mathfrak{c}}} \sum_{\gamma\in\mathfrak{a}(\mathfrak{b}_1\mathfrak{f}\mathfrak{f}_1^2\mathfrak{q}_2^3\mathfrak{D}^3)^{-1}} \frac{A(\mathfrak{a}^{-1}\mathfrak{b}_1(\mathfrak{f}_1\mathfrak{q}_2\mathfrak{D})^3\gamma, \mathfrak{b}_1)}{\mathrm{N}(\mathfrak{q}_2)^2\sqrt{\mathrm{N}(\gamma)}} \psi_{\mathfrak{r}\mathfrak{f}}(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma)\Phi(\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 \gamma), \tag{18.6}$$

where the sums over $\mathfrak{b}_1$, $\mathfrak{f}_1$, and $\mathfrak{q}_2$ must be subject to the conditions (see (15.3))

$$V(\mathfrak{b}_1\mathfrak{f}_1\mathfrak{q}_2)_v \lll \sqrt{R_v V(\mathfrak{d})_v}/T_v V(\mathfrak{d}_0)_v, \quad v \mid \infty. \tag{18.7}$$

Let $\mathfrak{c}_2 \in \widetilde{C}_F$ be such that $\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1 \cdot \mathfrak{c}_2 \sim \mathfrak{c}$. Set $c_{\mathfrak{q}_2} = [\mathfrak{c}_2^{-1}\mathfrak{q}_2]$ and $\mathfrak{b}_2 = (\mathfrak{d}_0\mathfrak{b}_1)^{-1}\mathfrak{r}\mathfrak{f}_1\mathfrak{c}_2\mathfrak{D}$. By (2.1) and (14.5), we have

$$|c_{\mathfrak{q}_2}|_v \asymp V(\mathfrak{q}_2)_v^{N_v}, \quad v \,|\, \infty. \tag{18.8}$$

Substituting γ by $\gamma/\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2 c_{\mathfrak{q}_2}$, up to a harmless factor, we have

$$\mathcal{S}_{\mathfrak{b}_1,\mathfrak{f}_1,\mathfrak{f}}^{\mathfrak{d},\mathfrak{r}}(T,R) = \frac{\mathrm{N}(\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1)}{\sqrt{\mathrm{N}(\mathfrak{d})}} \sum_{\substack{\mathfrak{q}_2\sim\mathfrak{c}_2 \\ (\mathfrak{q}_2,\mathfrak{d}\mathfrak{b}_1\mathfrak{f})=(1)}} \sum_{\gamma\mathfrak{b}_2\subset\mathfrak{f}_1\mathfrak{f}^{-1}} \frac{A(\gamma\mathfrak{b}_2,\mathfrak{b}_1)}{\sqrt{\mathrm{N}(\gamma\mathfrak{q}_2)}} \psi_{\mathfrak{r}\mathfrak{f}}(\gamma/c_{\mathfrak{q}_2})\Phi(\gamma/c_{\mathfrak{q}_2}). \tag{18.9}$$

By Corollary 11.5, (14.6), and (18.8), for each $v \,|\, \infty$ we have $\Phi_v(\gamma/c_{\mathfrak{q}_2}) = O(T_v^{-A})$ when $|\gamma|_v^{1/N_v} > \sqrt{R_v V(\mathfrak{d})_v}/V(\mathfrak{q}\mathfrak{q}_2^{-1})_v$. Arguing as in §15, we can impose the condition $\gamma \in F_\infty^{\emptyset}(\sqrt{RV(\mathfrak{d})}/V(\mathfrak{q}\mathfrak{q}_2^{-1}))$ (see (4.17)) with a negligible error. Since we also have $\mathrm{N}(\gamma) \geqslant \mathrm{N}(\mathfrak{d}_0\mathfrak{b}_1)/\mathrm{N}(\mathfrak{r}\mathfrak{f}) \geqslant 1/\mathrm{N}(\mathfrak{d}\mathfrak{q}) \geqslant 1/\mathrm{N}(T)^{1/2+\varepsilon}$, due to (15.1) and (15.3), $|\gamma|_v \geqslant T_v^{-A}$ for each γ, so there is no issue with a Δ-adic partition in the γ-sum as we had in §15. Again, the assumption that $T_v \geqslant \mathrm{N}(T)^\varepsilon$ for all $v \,|\, \infty$ is required here.

Definition 18.1. For $V \in \mathfrak{a}_+$ and $\Delta > 1$, define

$$F_\infty^\Delta(V) = \{x \in F_\infty : |x_v| \in [V_v, \sqrt{\Delta}\, V_v) \text{ for all } v \in S_\infty\}. \tag{18.10}$$

Let $\sigma \in \{0,-,+,\flat\}^{|S_\infty|}$ ($\sigma_v = \flat$ only if v is complex). As we have seen in §§11 and 13, each local Φ_v equals (the sum of) Φ^{σ_v} for $\sigma_v = 0,-,+,\flat$ in various circumstances; with abuse of notation, $\Phi^\flat = \Phi^\flat_\rho$ or $\Phi^\flat_0$. The product of such $\Phi^{\sigma_v}(x_v)$ will be denoted by $\Phi^\sigma(x)$.

Lemma 18.2. *Let notation be as above. Fix $\Delta > 1$ as in §13. Let $C, \Gamma \in \mathfrak{a}_+$ satisfy*

$$1 \ll \mathrm{N}(C) \ll \frac{\sqrt{\mathrm{N}(R)\mathrm{N}(\mathfrak{d})}}{\mathrm{N}(T)\mathrm{N}(\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1)}, \quad \frac{\mathrm{N}(\mathfrak{d}_0\mathfrak{b}_1)}{\mathrm{N}(\mathfrak{r}\mathfrak{f})} \ll \mathrm{N}(\Gamma) \ll \frac{\sqrt{\mathrm{N}(R)\mathrm{N}(\mathfrak{d})}}{\mathrm{N}(\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1)}, \tag{18.11}$$

$$1 \ll C_v \ll \frac{\sqrt{R_v V(\mathfrak{d})_v}}{T_v V(\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1)_v}, \quad \frac{1}{T_v^A} \ll \Gamma_v \ll \frac{\sqrt{R_v V(\mathfrak{d})_v}}{V(\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1)_v}. \tag{18.12}$$

Define

$$\mathcal{S}^\sigma(T,R;\Gamma,C) = \sum_{\substack{\mathfrak{q}_2\sim\mathfrak{c}_2 \\ (\mathfrak{q}_2,\mathfrak{d}\mathfrak{b}_1\mathfrak{f})=(1) \\ c_{\mathfrak{q}_2}\in F_\infty^\Delta(C)}} \sum_{\substack{\gamma\mathfrak{b}_2\subset\mathfrak{f}_1\mathfrak{f}^{-1} \\ \gamma\in F_\infty^\Delta(\Gamma)}} \frac{A(\gamma\mathfrak{b}_2,\mathfrak{b}_1)}{\sqrt{\mathrm{N}(\gamma\mathfrak{q}_2)}} \psi_{\mathfrak{r}\mathfrak{f}}(\gamma/c_{\mathfrak{q}_2})\Phi^\sigma(\gamma/c_{\mathfrak{q}_2}). \tag{18.13}$$

Then

$$\mathcal{S}^\sigma(T,R;\Gamma,C) \ll \frac{\mathrm{N}(\mathfrak{d})^{5/4}\mathrm{N}(\mathfrak{f})^{57/64}}{\mathrm{N}(\mathfrak{d}_0)^2\mathrm{N}(\mathfrak{b}_1)^{57/64}\mathrm{N}(\mathfrak{f}_1)^{25/64}} \frac{\mathrm{N}(T)^\varepsilon\mathrm{N}(R)^{1/4}}{\mathrm{N}(T)^{1/2}} + \frac{\mathrm{N}(\mathfrak{d})^{3/2}\mathrm{N}(\mathfrak{f})^{57/64}}{\mathrm{N}(\mathfrak{d}_0)^{5/2}\mathrm{N}(\mathfrak{b}_1)^{89/64}\mathrm{N}(\mathfrak{f}_1)^{57/64}} \frac{\mathrm{N}(T)^\varepsilon\mathrm{N}(R)^{1/2}}{\mathrm{N}(T)^{4/3}}. \tag{18.14}$$

Granted that Lemma 18.2 holds, we can now finish the proof of Proposition 15.1. By the discussions above Lemma 18.2, the bound in (18.14) applies to the double sum in (18.9), and hence

$$\mathcal{S}^{\mathfrak{d},\mathfrak{r}}_{\mathfrak{b}_1,\mathfrak{f}_1,\mathfrak{f}}(T,R) \ll \frac{\mathrm{N}(\mathfrak{d})^{3/4}\mathrm{N}(\mathfrak{b}_1)^{7/64}\mathrm{N}(\mathfrak{f}_1)^{39/64}\mathrm{N}(\mathfrak{f})^{57/64}}{\mathrm{N}(\mathfrak{d}_0)} \frac{\mathrm{N}(T)^{\varepsilon}\mathrm{N}(R)^{1/4}}{\mathrm{N}(T)^{1/2}} + \frac{\mathrm{N}(\mathfrak{d})\mathrm{N}(\mathfrak{f}_1)^{7/64}\mathrm{N}(\mathfrak{f})^{57/64}}{\mathrm{N}(\mathfrak{d}_0)^{3/2}\mathrm{N}(\mathfrak{b}_1)^{25/64}} \frac{\mathrm{N}(T)^{\varepsilon}\mathrm{N}(R)^{1/2}}{\mathrm{N}(T)^{4/3}}.$$

Applying this to (18.5) leads to the estimate in (15.5).

19. Completion: Proof of Lemma 18.2

For each $\sigma_v \in \{0,-,+,\flat\}$, we first apply Lemmas 13.2, 13.4, and 13.5 to the local $\Phi^{\sigma_v}(\gamma/c_{\mathfrak{q}_2})$, so that the sum $\mathcal{S}^{\sigma}(T,R;\Gamma,C)$ is expressed in a form which is ready for the hybrid large sieve in Corollary C.5. Then, after applying Corollary C.5, we infer that

$$\mathcal{S}^{\sigma}(T,R;\Gamma,C) \ll \sqrt{\mathcal{T}(T,R)}\sqrt{\mathcal{T}^{\sigma}(T,R;\Gamma,C)} \tag{19.1}$$

with

$$\mathcal{T}(T,R) = \mathrm{N}(T)^{\varepsilon}\frac{\mathrm{N}(\mathfrak{r}\mathfrak{f})}{\mathrm{N}(\Gamma)} \sum_{\substack{\gamma\mathfrak{b}_2\subset\mathfrak{f}_1\mathfrak{f}^{-1}\\ \gamma\in F^{\Delta}_{\infty}(\Gamma)}} |A(\gamma\mathfrak{b}_2,\mathfrak{b}_1)|^2, \tag{19.2}$$

$$\mathcal{T}^{\sigma}(T,R;\Gamma,C) = \frac{1}{\mathrm{N}(A^{\sigma})}\left(\mathrm{N}(U^{\sigma}) + \frac{\mathrm{N}(C)}{\mathrm{N}(\mathfrak{r}\mathfrak{f})}\right)\left(\mathrm{N}(U^{\sigma}) + \frac{\mathrm{N}(\Gamma)}{\mathrm{N}(\mathfrak{d}_0\mathfrak{b}_1)}\right). \tag{19.3}$$

Recall that ρ or 0 is suppressed from subscripts when $\sigma = \flat$. By Lemma 4.11, with the Kim–Sarnak exponent $\theta = \frac{7}{32}$ as in (4.24),

$$\mathcal{T}(T,R) \ll \mathrm{N}(T)^{\varepsilon}\frac{\mathrm{N}(\mathfrak{r})^2\mathrm{N}(\mathfrak{f}_1)^{7/32}\mathrm{N}(\mathfrak{f})^{57/32}}{\mathrm{N}(\mathfrak{d}_0)\mathrm{N}(\mathfrak{b}_1)^{25/32}}. \tag{19.4}$$

Set $\sqrt{\Delta}\,X = \Gamma/C$. We have

$$\mathcal{T}^{\sigma}(T,R;\Gamma,C) \ll \frac{\mathrm{N}(U^{\sigma})^2}{\mathrm{N}(A^{\sigma})} + \frac{\mathrm{N}(CU^{\sigma})}{\mathrm{N}(A^{\sigma})}(1+\mathrm{N}(X)) + \frac{\mathrm{N}(C^2X)}{\mathrm{N}(A^{\sigma})}. \tag{19.5}$$

For the three summands in (19.5), we claim that

$$\frac{\mathrm{N}(U^{\sigma})^2}{\mathrm{N}(A^{\sigma})} \ll \mathrm{N}(T)^{\varepsilon}, \tag{19.6}$$

$$\frac{\mathrm{N}(CU^{\sigma})}{\mathrm{N}(A^{\sigma})}(1+\mathrm{N}(X)) \ll \frac{\mathrm{N}(T)^{\varepsilon}\sqrt{\mathrm{N}(R)\mathrm{N}(\mathfrak{d})}}{\mathrm{N}(T)\mathrm{N}(\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1)}, \tag{19.7}$$

$$\frac{\mathrm{N}(C^2X)}{\mathrm{N}(A^{\sigma})} \ll \frac{\mathrm{N}(R)\mathrm{N}(\mathfrak{d})}{\mathrm{N}(T)^{8/3}\mathrm{N}(\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1)^2}. \tag{19.8}$$

Lemma 18.2 readily follows from (19.1), (19.4)–(19.8), with the observation that (19.6) can be absorbed into (19.7).

Clearly, (19.6)–(19.8) follow from the local inequalities

$$\frac{U^{\sigma_v}}{\sqrt{A^{\sigma_v}}} \preccurlyeq T_v^{\varepsilon}, \quad \frac{U^{\sigma_v}}{A^{\sigma_v}}(1+X_v) \preccurlyeq T_v^{\varepsilon}, \quad \frac{C_v^2 X_v}{A^{\sigma_v}} \preccurlyeq \frac{R_v Z_v^2}{T_v^{8/3}}, \tag{19.9}$$

in which $Z_v = \sqrt{V(\mathfrak{d})_v}/V(\mathfrak{d}_0\mathfrak{b}_1\mathfrak{f}_1)_v$ so that the range of C_v in (18.12) reads

$$1 \preccurlyeq C_v \preccurlyeq \sqrt{R_v}\, Z_v/T_v. \tag{19.10}$$

Note that the third inequality in (19.9) can be deduced from

$$\frac{X_v}{A^{\sigma_v}} \preccurlyeq \frac{1}{T_v^{2/3}}. \tag{19.11}$$

Moreover, if we define $\Lambda_v = R_v/\beta_{\mathfrak{d}} c_{\mathfrak{q}}^2$ (see (18.3)), then

$$|\Lambda_v| \asymp R_v Z_v^2/C_v^2. \tag{19.12}$$

It remains to verify (19.9) for various cases (alternatively, (19.11) for all the cases other than $\sigma = +$).

19.1. *The case* $\sigma = 0$

In this case, we have $U^0 = T_v^{\varepsilon}$, $A^0 = 1$, and $X_v \leqslant T_v^{\varepsilon}/|\Lambda_v| \preccurlyeq T_v^{\varepsilon}/T_v^2$. Thus (19.9) and (19.11) are obvious.

19.2. *The case* $\sigma = -$

In this case, we have $U^- = |X_v\Lambda_v|^{1/3}$, $A^- = |\Lambda_v|$, and $X_v \preccurlyeq \sqrt{|\Lambda_v|}$. Therefore

$$\frac{U^-}{\sqrt{A^-}} = \frac{X_v^{1/3}}{|\Lambda_v|^{1/6}} \preccurlyeq 1, \quad \frac{U^-}{A^-}(1+X_v) \preccurlyeq \frac{X_v^{1/3}}{|\Lambda_v|^{1/6}} \preccurlyeq 1, \quad \frac{X_v}{A^-} \preccurlyeq \frac{1}{\sqrt{|\Lambda_v|}} \preccurlyeq \frac{1}{T_v},$$

as desired.

19.3. *The case* $\sigma = +$

In this case, we always have $X_v \asymp \sqrt{|\Lambda_v|}$, and, by (19.12),

$$C_v X_v \asymp \sqrt{R_v}\, Z_v. \tag{19.13}$$

For $T_v \preccurlyeq X_v \preccurlyeq T_v^{2-\varepsilon}$, we have $U^+ = T_v^2/X_v$, $A^+ = T_v^{2-\varepsilon}$. Consequently,

$$\frac{U^+}{\sqrt{A^+}} = \frac{T_v^{1+\varepsilon}}{X_v} \preccurlyeq T_v^{\varepsilon}, \quad \frac{U^+}{A^+}(1+X_v) \preccurlyeq T_v^{\varepsilon},$$

and it follows from (19.13) that

$$\frac{C_v^2 X_v}{A^+} = \frac{T_v^\varepsilon C_v^2 X_v}{T_v^2} \asymp \frac{T_v^\varepsilon R_v Z_v^2}{T_v^2 X_v} \leqslant \frac{T_v^\varepsilon R_v Z_v^2}{T_v^3}.$$

For $X_v \geqslant T_v^{2-\varepsilon}$, we have $U^+ = T_v^\varepsilon$, $A^+ = X_v$. Consequently,

$$\frac{U^+}{\sqrt{A^+}} = \frac{T_v^\varepsilon}{\sqrt{X_v}} \leqslant \frac{T_v^\varepsilon}{T_v}, \quad \frac{U^+}{A^+}(1+X_v) \leqslant T_v^\varepsilon,$$

and it follows from (19.13) that

$$\frac{C_v^2 X_v}{A^+} = C_v^2 \asymp \frac{R_v Z_v^2}{X_v^2} \leqslant \frac{T_v^\varepsilon R_v Z_v^2}{T_v^4}.$$

19.4. The case $\sigma = \flat$

In this case, we always have $|\Lambda_v| \asymp T_v^2$ and $X_v \asymp T_v$.

Firstly, for $T_v^\varepsilon / \min\{(T_v/M_v)^{1/2}, T_v^{1/4}\} < \rho \leqslant 1$, we have $A_\rho^\flat = T_v^2 \rho$ and $U_\rho^\flat = T_v \rho^2$. Therefore

$$\frac{U_\rho^\flat}{\sqrt{A_\rho^\flat}} = \rho^{3/2} \leqslant 1, \quad \frac{U_\rho^\flat}{A_\rho^\flat}(1+X_v) \leqslant \rho \leqslant 1, \quad \frac{X_v}{A_\rho^\flat} \leqslant \frac{1}{T_v \rho} < \frac{1}{T_v^{3/4+\varepsilon}}.$$

Secondly, it follows from (13.11) that

$$\frac{U_0^\flat}{\sqrt{A_0^\flat}} = \frac{T_v^\varepsilon}{T_v^{1/3}}, \quad \frac{U_0^\flat}{A_0^\flat}(1+X_v) \leqslant \frac{T_v^\varepsilon}{T_v^{1/6}}, \quad \frac{X_v}{A_0^\flat} \leqslant \frac{1}{T_v^{2/3}}$$

if $T_v^\varepsilon \leqslant M \leqslant T_v^{1/3}$;

$$\frac{U_0^\flat}{\sqrt{A_0^\flat}} = \frac{T_v^\varepsilon}{(M_v T_v)^{1/4}}, \quad \frac{U_0^\flat}{A_0^\flat}(1+X_v) \leqslant \frac{T_v^\varepsilon}{M_v^{1/2}}, \quad \frac{X_v}{A_0^\flat} \leqslant \frac{1}{(M_v T_v)^{1/2}} < \frac{1}{T_v^{2/3}}$$

if $T_v^{1/3} < M_v \leqslant T_v^{1/2}$; and

$$\frac{U_0^\flat}{\sqrt{A_0^\flat}} = \frac{M_v^{3/4} T_v^\varepsilon}{T_v^{3/4}}, \quad \frac{U_0^\flat}{A_0^\flat}(1+X_v) \leqslant \frac{M_v^{1/2} T_v^\varepsilon}{T_v^{1/2}}, \quad \frac{X_v}{A_0^\flat} \leqslant \frac{1}{(M_v T_v)^{1/2}} < \frac{1}{T_v^{3/4}}$$

if $T_v^{1/2} < M_v \leqslant T_v^{1-\varepsilon}$, all of which are satisfactory.

Appendix C. Gallagher's hybrid large sieve over number fields

In this appendix, we establish Gallagher's hybrid large sieve [13, §1] over number fields.

Let notation be as in §2. Recall that $\hat{\mathfrak{a}} = \prod_{v|\infty} \hat{\mathfrak{a}}_v$ is the unitary dual of $F_\infty^\times$: here $\hat{\mathfrak{a}}_v = \mathbb{R} \times \{0, 1\}$ if v is real and $\hat{\mathfrak{a}}_v = \mathbb{R} \times \mathbb{Z}$ if v is complex.

Let $\mathfrak{a} = \mathbb{R}^{|S_\infty|}$ and $\mathfrak{a}_+ = \mathbb{R}_+^{|S_\infty|}$. For $U \in \mathfrak{a}_+$, define

$$\widehat{\mathfrak{a}}(U) = \{(\nu, m) \in \widehat{\mathfrak{a}} : |\nu_v|, |m_v| \leqslant U_v \text{ for all } v \,|\, \infty\}. \tag{C.1}$$

For $y \in F_\infty^\times$ and $\delta \in \mathfrak{a}_+$, with $\delta_v < \pi$ for each $v \,|\, \infty$, define

$$F_\infty^\times(y;\delta) = \left\{x \in F_\infty^\times : N_v \big|\log|x|_v - \log|y|_v\big|, |\arg(x_v) - \arg(y_v)| \leqslant \delta_v \text{ for all } v \,|\, \infty\right\}, \tag{C.2}$$

where $\arg(x_v)$ lies on the circle $\mathbb{R}/2\pi\mathbb{Z}$ ($\arg(x_v) = 0$ or π if v is real).

Let $\mathfrak{a}$ be a fractional ideal of F. Consider an absolutely convergent series

$$S_\mathfrak{a}(\nu, m) = \sum_{\gamma \in \mathfrak{a} \smallsetminus \{0\}} a_\gamma \chi_{i\nu, m}(\gamma), \quad a_\gamma \in \mathbb{C}. \tag{C.3}$$

Proposition C.1. *Let $U \in \mathfrak{a}_+$ be such that $U_v \geqslant 1$ for all $v \,|\, \infty$. We have*

$$\iint_{\widehat{\mathfrak{a}}(U)} |S_\mathfrak{a}(\nu, m)|^2 \,\mathrm{d}\mu(\nu, m) \ll_F \mathrm{N}(U)^2 \int_{F_\infty^\times} \Big| \sum_{\gamma \in F_\infty^\times(y;1/U) \cap \mathfrak{a}} a_\gamma \Big|^2 \mathrm{d}^\times y. \tag{C.4}$$

Proof. Set $\delta = 1/U$. The right-hand side of (C.4) may be written as

$$\int_{F_\infty^\times} |C_\mathfrak{a}^\delta(y)|^2 \,\mathrm{d}^\times y, \quad C_\mathfrak{a}^\delta(y) = \frac{1}{\mathrm{N}(\delta)} \sum_{\gamma \in F_\infty^\times(y;\delta) \cap \mathfrak{a}} a_\gamma.$$

Put $F_\delta(x) = 1/\mathrm{N}(\delta)$ or 0 according as $x \in F_\infty^\times(1;\delta)$ or not. Then

$$C_\mathfrak{a}^\delta(y) = \sum_{\gamma \in \mathfrak{a} \smallsetminus \{0\}} a_\gamma F_\delta(y/\gamma).$$

Taking the Mellin transform (see (2.2)), we get $\breve{C}_\mathfrak{a}^\delta = S_\mathfrak{a} \cdot \breve{F}_\delta$. Since the series (C.3) converges absolutely, $C_\mathfrak{a}^\delta$ is a bounded integrable function, and hence is square-integrable. By Plancherel's theorem (see (2.4)),

$$\int_{F_\infty^\times} |C_\mathfrak{a}^\delta(y)|^2 \,\mathrm{d}^\times y = \frac{1}{c} \iint_{\widehat{\mathfrak{a}}} |S_\mathfrak{a}(\nu, m) \breve{F}_\delta(\nu, m)|^2 \,\mathrm{d}\mu(\nu, m),$$

for a certain constant c (explicitly, $c = 2^{2r_1 + 2r_2} \pi^{r_1 + 2r_2}$). Since $\breve{F}_\delta(\nu, m)$ is the product of

$$\begin{cases} \dfrac{2\sin(\delta_v \nu_v)}{\delta_v \nu_v} & \text{if } v \text{ is real,} \\[2ex] \dfrac{2\sin(\delta_v \nu_v)}{\delta_v \nu_v} \dfrac{2\sin(\delta_v m_v)}{\delta_v m_v} & \text{if } v \text{ is complex,} \end{cases}$$

we have $\breve{F}_\delta(\nu, m) \gg 1$ for $(\nu, m) \in \widehat{\mathfrak{a}}(1/\delta)$, and the result follows. ∎

We shall apply (C.4) to sums of the form

$$S_\mathfrak{a}(\chi; \nu, m) = \sum_{\gamma \in \mathfrak{a} \smallsetminus \{0\}} a_\gamma \chi(\gamma) \chi_{i\nu, m}(\gamma). \tag{C.5}$$

where $\chi \in \widehat{(\mathfrak{a}/\mathfrak{a}\mathfrak{n})^\times}$ is induced from a character $\chi : (\mathcal{O}/\mathfrak{n})^\times \to \mathbb{C}^\times$ via a (fixed) isomorphism $(\mathfrak{a}/\mathfrak{a}\mathfrak{n})^\times \to (\mathcal{O}/\mathfrak{n})^\times$ (see Definition 3.1), and $\chi(\gamma) = \chi(\gamma + \mathfrak{a}\mathfrak{n})$ or 0 according as $\gamma + \mathfrak{a}\mathfrak{n} \in (\mathfrak{a}/\mathfrak{a}\mathfrak{n})^\times$ or not.

Lemma C.2. *We have*

$$\sum_{\chi\in\widehat{(\mathfrak{a}/\mathfrak{a}\mathfrak{n})^\times}}\Big|\sum_{\gamma\in F_\infty^\times(y;\delta)\cap\mathfrak{a}} a_\gamma\chi(\gamma)\Big|^2 \ll_F \left(\mathrm{N}(\mathfrak{n})+\frac{\mathrm{N}(y)\mathrm{E}(\delta)}{\mathrm{N}(\mathfrak{a})}\right)\sum_{\gamma\in F_\infty^\times(y;\delta)\cap\mathfrak{a}}|a_\gamma|^2, \quad \text{(C.6)}$$

where

$$\mathrm{E}(\delta)=\prod_{v\mid\infty}(e^{\delta_v}-e^{-\delta_v})(2\delta_v)^{N_v-1}$$

so that $\mathrm{N}(y)\mathrm{E}(\delta)$ *is the area of* $F_\infty^\times(y;\delta)$.

Proof. By the orthogonality relations, the left-hand side of (C.6) is equal to

$$\varphi(\mathfrak{n})\sum_{x\in(\mathfrak{a}/\mathfrak{a}\mathfrak{n})^\times}\Big|\sum_{\gamma\in F_\infty^\times(y;\delta)\cap(x+\mathfrak{a}\mathfrak{n})}a_\gamma\Big|^2,$$

and, by the Cauchy–Schwarz inequality, this is bounded by

$$\varphi(\mathfrak{n})\sum_{x\in(\mathfrak{a}/\mathfrak{a}\mathfrak{n})^\times}\left(\frac{\mathrm{N}(y)\mathrm{E}(\delta)}{\mathrm{N}(\mathfrak{a}\mathfrak{n})}+1\right)\sum_{\gamma\in F_\infty^\times(y;\delta)\cap(x+\mathfrak{a}\mathfrak{n})}|a_\gamma|^2. \qquad \blacksquare$$

Proposition C.3. *Let* $U\in\mathfrak{a}_+$ *be such that* $U_v\geqslant 1$ *for all* $v\mid\infty$. *We have*

$$\sum_{\chi\in\widehat{(\mathfrak{a}/\mathfrak{a}\mathfrak{n})^\times}}\iint_{\hat{\mathfrak{a}}(U)}|S_{\mathfrak{a}}(\chi;\nu,m)|^2\,\mathrm{d}\mu(\nu,m)\ll_F\sum_{\gamma\in\mathfrak{a}\smallsetminus\{0\}}(\mathrm{N}(U)\mathrm{N}(\mathfrak{n})+\mathrm{N}(\gamma\mathfrak{a}^{-1}))|a_\gamma|^2. \quad \text{(C.7)}$$

Proof. Using (C.4) and (C.6), the left-hand side of (C.7) is bounded by

$$\begin{aligned}&\mathrm{N}(U)^2\int_{F_\infty^\times}\sum_{\chi\in\widehat{(\mathfrak{a}/\mathfrak{a}\mathfrak{n})^\times}}\Big|\sum_{\gamma\in F_\infty^\times(y;1/U)\cap\mathfrak{a}}a_\gamma\chi(\gamma)\Big|^2\,\mathrm{d}^\times y\\ &\qquad\ll\mathrm{N}(U)^2\int_{F_\infty^\times}\left(\mathrm{N}(\mathfrak{n})+\frac{\mathrm{N}(y)\mathrm{E}(1/U)}{\mathrm{N}(\mathfrak{a})}\right)\sum_{\gamma\in F_\infty^\times(y;1/U)\cap\mathfrak{a}}|a_\gamma|^2\,\mathrm{d}^\times y.\end{aligned}$$

The coefficient of $|a_\gamma|^2$ here is

$$\begin{aligned}&\mathrm{N}(U)^2\mathrm{N}(\mathfrak{n})\iint_{F_\infty^\times(\gamma;1/U)}\mathrm{d}^\times y+\frac{\mathrm{N}(U)^2\mathrm{E}(1/U)}{\mathrm{N}(\mathfrak{a})}\iint_{F_\infty^\times(\gamma;1/U)}\mathrm{d}y\\ &\qquad=\mathrm{N}(2U)\mathrm{N}(\mathfrak{n})+\frac{\mathrm{N}(U)^2\mathrm{E}(1/U)^2\mathrm{N}(\gamma)}{\mathrm{N}(\mathfrak{a})}\ll\mathrm{N}(U)\mathrm{N}(\mathfrak{n})+\mathrm{N}(\gamma\mathfrak{a}^{-1}). \qquad\blacksquare\end{aligned}$$

C.1. A corollary of Gallagher's large sieve

Definition C.4. Let $U\in\mathfrak{a}_+$ and $(\kappa,n)\in\hat{\mathfrak{a}}$. Define

$$\hat{\mathfrak{a}}_{\kappa,n}(U)=\{(\nu,m)\in\hat{\mathfrak{a}}:\sqrt{(\nu_v-\kappa_v)^2+(m_v-n_v)^2}\leqslant U_v\text{ for all }v\mid\infty\}. \quad \text{(C.8)}$$

Note that $\hat{\mathfrak{a}}_{0,0}(U) = \hat{\mathfrak{a}}(U)$ if we slightly modify the definition of $\hat{\mathfrak{a}}(U)$ in (C.1). See also Definition 13.1.

Corollary C.5. *Let $\mathfrak{c} \in \widetilde{C}_F$. For $\mathfrak{q} \sim \mathfrak{c}$, define $c_{\mathfrak{q}} = [\mathfrak{c}^{-1}\mathfrak{q}]$. Let $\mathfrak{a}$ and $\mathfrak{n}$ be ideals with $\mathfrak{n} \subset \mathfrak{O}$ and $(\mathfrak{a}\mathfrak{c}\mathfrak{D})_{\mathfrak{n}} = \mathfrak{n}^{-1}$ (see Definition 4.5). Let $C, \Gamma \in \mathfrak{a}_+$ and $F^{\Delta}_{\infty}(C)$, $F^{\Delta}_{\infty}(\Gamma)$ be defined as in Definition 18.1. Let a_γ and $b_{\mathfrak{q}}$ be sequences of complex numbers for $\gamma \in F^{\Delta}_{\infty}(\Gamma)$ and $\mathfrak{q} \sim \mathfrak{c}$ with $c_{\mathfrak{q}} \in F^{\Delta}_{\infty}(C)$. Define*

$$S_{\mathfrak{n}}(C, \Gamma; \nu, m; a, b) = \sum_{\substack{\mathfrak{q}\sim\mathfrak{c} \\ (\mathfrak{q},\mathfrak{n})=(1) \\ c_{\mathfrak{q}} \in F^{\Delta}_{\infty}(C)}} \sum_{\substack{\gamma\in\mathfrak{a} \\ \gamma\in F^{\Delta}_{\infty}(\Gamma)}} a_\gamma b_{\mathfrak{q}} \psi_{\mathfrak{n}}(\gamma/c_{\mathfrak{q}}) \chi_{i\nu,m}(\gamma/c_{\mathfrak{q}}). \tag{C.9}$$

Let $U \in \mathfrak{a}_+$ be such that $U_v \geqslant 1$ for all $v\,|\,\infty$. For $\kappa, n \in \mathfrak{a}$, define $\hat{\mathfrak{a}}_{\kappa,n}(U)$ as in (C.8). Then

$$\iint_{\hat{\mathfrak{a}}_{\kappa,n}(U)} |S_{\mathfrak{n}}(C, \Gamma; \nu, m; a, b)|\, \mathrm{d}\mu(\nu, m)$$
$$\ll \mathrm{N}(\mathfrak{n})^{1/2+\varepsilon}(\mathrm{N}(U) + \mathrm{N}(C)/\mathrm{N}(\mathfrak{n}))^{1/2}(\mathrm{N}(U) + \mathrm{N}(\Gamma)/\mathrm{N}(\mathfrak{a}\mathfrak{n}))^{1/2}\|a\|_2\|b\|_2, \tag{C.10}$$

where $\|a\|_2^2 = \sum_\gamma |a_\gamma|^2$ and $\|b\|_2^2 = \sum_{\mathfrak{q}} |b_{\mathfrak{q}}|^2$.

Proof. By changing a_γ and $b_{\mathfrak{q}}$ to $a_\gamma \overline{\chi_{i\kappa,n}(\gamma)}$ and $b_{\mathfrak{q}}\chi_{i\kappa,n}(c_{\mathfrak{q}})$ if necessary, we may assume with no loss of generality that $\hat{\mathfrak{a}}_{\kappa,n}(U) = \hat{\mathfrak{a}}(U)$. Moreover, we set $a_\gamma = b_{\mathfrak{q}} = 0$ if $\gamma \notin F^{\Delta}_{\infty}(\Gamma)$ or $c_{\mathfrak{q}} \notin F^{\Delta}_{\infty}(C)$. It will be convenient to view the $\mathfrak{q}$-sum as a sum over $\{c_{\mathfrak{q}} : \mathfrak{q} \sim \mathfrak{c}\} \subset \mathfrak{c}^{-1}$.

Next, we reformulate $S_{\mathfrak{n}}(C, \Gamma; \nu, m; a, b)$ in (C.9) as

$$\sum_{\mathfrak{m}|\mathfrak{n}} \sum_{\substack{\mathfrak{q}\sim\mathfrak{c} \\ (\mathfrak{q},\mathfrak{n})=(1)}} \sum_{\substack{\gamma\in\mathfrak{a}\mathfrak{n}\mathfrak{m}^{-1} \\ (\gamma\mathfrak{c}\mathfrak{D})_{\mathfrak{m}}=\mathfrak{m}^{-1}}} a_\gamma b_{\mathfrak{q}} \psi_{\mathfrak{m}}(\gamma/c_{\mathfrak{q}}) \chi_{i\nu,m}(\gamma/c_{\mathfrak{q}}).$$

For $\chi \in \widehat{((\mathfrak{m}\mathfrak{D}_{\mathfrak{m}})^{-1}/\mathfrak{D}_{\mathfrak{m}}^{-1})^\times}$, define the Gauss sum

$$\tau(\chi) = \sum_{x\in((\mathfrak{m}\mathfrak{D}_{\mathfrak{m}})^{-1}/\mathfrak{D}_{\mathfrak{m}}^{-1})^\times} \chi(x)\psi_{\mathfrak{m}}(x).$$

It is well-known that $|\tau(\chi)| \leqslant \sqrt{\mathrm{N}(\mathfrak{m})}$. By the orthogonality relation,

$$\psi_{\mathfrak{m}}(\gamma/c_{\mathfrak{q}}) = \frac{1}{\varphi(\mathfrak{m})} \sum_{\chi\in\widehat{((\mathfrak{m}\mathfrak{D}_{\mathfrak{m}})^{-1}/\mathfrak{D}_{\mathfrak{m}}^{-1})^\times}} \chi(\gamma/c_{\mathfrak{q}})\tau(\overline{\chi}),$$

for $(\mathfrak{q}, \mathfrak{m}) = (1)$ and $(\gamma\mathfrak{c}\mathfrak{D})_{\mathfrak{m}} = \mathfrak{m}^{-1}$. From these, we deduce that the left-hand side of (C.10) is bounded by

$$\sum_{\mathfrak{m}|\mathfrak{n}} \frac{\sqrt{\mathrm{N}(\mathfrak{m})}}{\varphi(\mathfrak{m})} \sum_{\chi\in\widehat{((\mathfrak{m}\mathfrak{D}_{\mathfrak{m}})^{-1}/\mathfrak{D}_{\mathfrak{m}}^{-1})^\times}} \iint_{\hat{\mathfrak{a}}(U)} \left|S_{\mathfrak{a}\mathfrak{n}\mathfrak{m}^{-1}}(\chi; \nu, m)\overline{S_{\mathfrak{c}^{-1}}(\chi; \nu, m)}\right| \mathrm{d}\mu(\nu, m),$$

where

$$S_{\mathfrak{a}\mathfrak{n}\mathfrak{m}^{-1}}(\chi;\nu,m) = \sum_{\substack{\gamma\in\mathfrak{a}\mathfrak{n}\mathfrak{m}^{-1} \\ (\gamma\mathfrak{c}\mathfrak{D})_{\mathfrak{m}}=\mathfrak{m}^{-1}}} a_\gamma \chi(\gamma)\chi_{i\nu,m}(\gamma),$$

$$S_{\mathfrak{c}^{-1}}(\chi;\nu,m) = \sum_{\substack{\mathfrak{q}\sim\mathfrak{c} \\ (\mathfrak{q},\mathfrak{n})=(1)}} \bar{b}_{\mathfrak{q}} \chi(c_{\mathfrak{q}})\chi_{i\nu,m}(c_{\mathfrak{q}}).$$

Finally, the bound in (C.10) readily follows from Cauchy–Schwarz and Proposition C.3. ∎

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