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Exponential rarefaction of maximal real algebraic hypersurfaces

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Abstract. Given an ample real Hermitian holomorphic line bundle L over a real algebraic variety X, the space of real holomorphic sections of $L^{\otimes d}$ inherits a natural Gaussian probability measure. We prove that the probability that the zero locus of a real holomorphic section s of $L^{\otimes d}$ defines a maximal hypersurface tends to 0 exponentially fast as d goes to infinity. This extends to any dimension a result of Gayet and Welschinger (2011) valid for maximal real algebraic curves inside a real algebraic surface.

The starting point is a low degree approximation property which relates the topology of the real vanishing locus of a real holomorphic section of $L^{\otimes d}$ with the topology of the real vanishing locus a real holomorphic section of $L^{\otimes d'}$ for a sufficiently smaller $d' < d$. Such a statement is inspired by the recent work of Diatta and Lerario (2022).

Keywords. Real algebraic varieties, random hypersurfaces, Betti numbers

1. Introduction

1.1. Real algebraic varieties and maximal hypersurfaces

Let (X, c_X) be a real algebraic variety, that is, a complex (smooth, projective and connected) algebraic variety equipped with an antiholomorphic involution $c_X : X \to X$, called the *real structure*. For example, the projective space $\mathbb{C}P^n$ equipped with the standard conjugation *conj* is a real algebraic variety. More generally, the solutions of a system of homogeneous real polynomial equations in $n + 1$ variables define a real algebraic variety X inside $\mathbb{C}P^n$, whose real structure is the restriction of *conj* to X. The real locus $\mathbb{R}X$ of a real algebraic variety is the set of fixed points of the real structure, that is, $\mathbb{R}X = \mathrm{Fix}(c_X)$. The real locus $\mathbb{R}X$ is either empty, or a finite union of n-dimensional $\mathcal{C}^\infty$-manifolds, where n is the (complex) dimension of X. The study of the topology of real algebraic varieties has been a central topic in real algebraic geometry since the works

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of Harnack and Klein on the topology of real algebraic curves [14, 18] and Hilbert's famous sixteenth problem [28]. A fundamental restriction on the topology of the real locus of a real algebraic variety is given by the Smith–Thom inequality [25] which asserts that the total Betti number of the real locus $\mathbb{R}X$ of a real algebraic variety is bounded from above by the total Betti number of the complex locus:

$$\sum_{i=0}^{n} \dim H_i(\mathbb{R}X, \mathbb{Z}/2) \leq \sum_{i=0}^{2n} \dim H_i(X, \mathbb{Z}/2). \tag{1}$$

We will more compactly write $b_*(\mathbb{R}X) \leq b_*(X)$, where b_* denotes the total Betti number (i.e. the sum of the $\mathbb{Z}/2$-Betti numbers). For a real algebraic curve C, inequality (1) is known as Harnack's inequality [14, 18] and reads $b_0(\mathbb{R}C) \leq g + 1$, where g is the genus of C. A real algebraic variety which realizes equality in (1) is called *maximal*.

In this paper, we will study maximal real algebraic hypersurfaces inside a fixed real algebraic variety X. More precisely, we are interested in the following question: given a real linear system of divisors in X, what is the probability of finding a divisor defining a real algebraic maximal hypersurface? The goal of the paper is to show that real algebraic maximal hypersurfaces are exponentially rare inside their linear system.

1.2. Real Hermitian line bundles and Gaussian measures

In order to answer the previous question, let $\pi : (L, c_L) \to (X, c_X)$ be an ample real holomorphic line bundle over X, that is, an ample holomorphic line bundle L over X equipped with a real structure c_L such that $\pi \circ c_L = c_X \circ \pi$. We equip L with a smooth real Hermitian metric h of positive curvature ω (we recall that *real* means $c_L^* h = \bar{h}$). We denote by $\mathbb{R}H^0(X, L^d)$ the space of real global sections of $L^{\otimes d} = L^d$, that is, the space of holomorphic sections $s \in H^0(X, L^d)$ such that $s \circ c_X = c_{L^d} \circ s$. This space is naturally equipped with an $\mathcal{L}^2$-scalar product defined by

$$\langle s_1, s_2 \rangle_{\mathcal{L}^2} = \int_X h^d(s_1, s_2) \frac{\omega^{\wedge n}}{n!} \tag{2}$$

for any pair of real global sections $s_1, s_2 \in \mathbb{R}H^0(X, L^d)$, where h^d is the real Hermitian metric on L^d induced by h. In turn, the $\mathcal{L}^2$-scalar product (2) naturally induces a Gaussian probability measure μ_d defined by

$$\mu_d(A) = \frac{1}{\sqrt{\pi}^{N_d}} \int_{s \in A} e^{-\|s\|^2_{\mathcal{L}^2}} \, \mathrm{d}s \tag{3}$$

for any open set $A \subset \mathbb{R}H^0(X, L^d)$, where N_d is the dimension of $\mathbb{R}H^0(X, L^d)$ and ds the Lebesgue measure induced by the $\mathcal{L}^2$-scalar product (2). The probability space we will consider is then $(\mathbb{R}H^0(X, L^d), \mu_d)$.

Example 1.1. When (X, c_X) is the n-dimensional projective space and (L, c_L, h) is the degree 1 real holomorphic line bundle equipped with the standard Fubini–Study metric, then the vector space $\mathbb{R}H^0(X, L^d)$ is isomorphic to the space $\mathbb{R}_d^{\text{hom}}[X_0, \dots, X_n]$ of degree d homogeneous real polynomials in $n+1$ variables, and the $\mathcal{L}^2$-scalar product is the one that makes the family $\{\sqrt{\binom{(n+d)!}{n!\alpha_0!\cdots\alpha_n!}}X_0^{\alpha_0}\cdots X_n^{\alpha_n}\}_{\alpha_0+\cdots+\alpha_n=d}$ of monomials an orthonormal basis. A random polynomial with respect to the Gaussian probability measure induced by this scalar product is called a *Kostlan polynomial*.

1.3. Statement of the main results

The zero locus of a real global section s_d of L^d is denoted by Z_{s_d} and its real locus by $\mathbb{R}Z_{s_d}$. The total Betti number $b_*(Z_{s_d})$ of Z_{s_d} has the asymptotics $b_*(Z_{s_d}) = v(L)d^n + O(d^{n-1})$ as $d\to\infty$, where $v(L) := \int_X c_1(L)^n$ is called the *volume* of the line bundle L (see, for example, [12, Lemma 3]). If the total Betti number $b_*(\mathbb{R}Z_{s_d})$ of the real locus of a sequence of real algebraic hypersurfaces has the same asymptotics, then the hypersurfaces are called *asymptotically maximal*. The existence of asymptotically maximal hypersurfaces is known in many cases, for example for real algebraic surfaces [10, Theorem 5], for projective spaces [17] and for toric varieties [4, Theorem 1.3]. The first main result of the paper shows that asymptotically maximal hypersurfaces are very rare in their linear system. More precisely:

Theorem 1.2. *Let (X, c_X) be a real algebraic variety of dimension n and (L, c_L) be a real Hermitian line bundle of positive curvature. Then there exists a positive $a_0 < v(L)$ such that, for any $a > a_0$,*

$$\mu_d\{s\in\mathbb{R}H^0(X,L^d), b_*(\mathbb{R}Z_s)\geq ad^n\} = O(d^{-\infty})$$

as $d\to\infty$.

The notation $O(d^{-\infty})$ stands for $O(d^{-k})$ for any $k\in\mathbb{N}$ and the measure μ_d is the Gaussian measure defined in (3). Note that we actually have more than "asymptotically maximal hypersurfaces are very rare". Indeed, asymptotically maximal hypersurfaces correspond to the asymptotics $b_*(\mathbb{R}Z_s) = v(L)d^n + O(d^{n-1})$, while in Theorem 1.2 we consider bigger subsets of $\mathbb{R}H^0(X,L^d)$ of the form $b_*(\mathbb{R}Z_s)\geq ad^n$ for $v(L) > a > a_0$.

For maximal real algebraic hypersurfaces, the rarefaction is even exponential. More precisely:

Theorem 1.3. *Let (X, c_X) be a real algebraic variety of dimension n and (L, c_L) be a real Hermitian line bundle of positive curvature. For any $a > 0$ there exists $c > 0$ such that*

$$\mu_d\{s\in\mathbb{R}H^0(X,L^d),\ b_*(\mathbb{R}Z_s)\geq b_*(Z_s) - ad^{n-1}\} = O(e^{-c\sqrt{d}\log d})$$

as $d\to\infty$. Moreover, if the real Hermitian metric on L is analytic, one has the estimate

$$\mu_d\{s\in\mathbb{R}H^0(X,L^d),\ b_*(\mathbb{R}Z_s)\geq b_*(Z_s) - ad^{n-1}\} = O(e^{-cd}).$$

Remark 1.4. The $\mathcal{L}^2$-scalar product on $\mathbb{R}H^0(X, L^d)$ also induces a Fubini–Study volume on the linear system $P(\mathbb{R}H^0(X, L^d))$. The sets considered in Theorems 1.2

and 1.3 are cones in $\mathbb{R}H^0(X, L^d)$ and the volume (with respect to the Fubini–Study form) of their projectivization coincides with the Gaussian measures estimated in Theorems 1.2 and 1.3.

Theorems 1.3 extends to any dimension a result of Gayet and Welschinger [10], in which the authors prove, using the theory of laminary currents, that maximal real curves are exponentially rare in a real algebraic surface. We stress that our techniques are different from those of [10].

For Kostlan polynomials (see Example 1.1) Theorems 1.2 and 1.3 were proven by Diatta and Lerario [9, Theorem 8] as a corollary of a low degree approximation property. Here we adopt the same strategy: Theorems 1.2 and 1.3 will be consequences of a general "approximation theorem" which states that, for some $b < 1$, with very high probability, the zero locus of a real section of L^d is diffeomorphic to the zero locus of a real section of $L^{\lfloor bd \rfloor}$, where $\lfloor bd \rfloor$ is the greatest integer less than or equal to bd. More precisely:

Theorem 1.5. *Let (X, c_X) be a real algebraic variety and (L, c_L) be a real Hermitian line bundle of positive curvature.*

(i) *There exists a positive $b_0 < 1$ such that for any $b_0 < b < 1$ the following holds: the probability that, for a real section s of L^d, there exists a real section s' of $L^{\lfloor bd \rfloor}$ such that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s'})$ are isotopic is $1 - O(d^{-\infty})$ as $d \to \infty$.*

(ii) *For any $k \in \mathbb{N}$ there exists $c > 0$ such that the following holds: the probability that, for a real section s of L^d, there exists a real section s' of L^{d-k} such that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s'})$ are isotopic is $1 - O(e^{-c\sqrt{d}\log d})$ as $d \to \infty$. If moreover the real Hermitian metric on L is analytic, this probability is $1 - O(e^{-cd})$ as $d \to \infty$.*

Assertion (i) (resp. (ii)) of Theorem 1.5, together with the Smith–Thom inequality (1) and the asymptotics $b_*(Z_{s_d}) = v(L)d^n + O(d^{n-1})$ for $s_d \in \mathbb{R}H^0(X, L^d)$, will imply Theorem 1.2 (resp. Theorem 1.3). It is also worth pointing out that assertions (i) and (ii) of Theorem 1.5 are independent of each other, although their proofs, which will be sketched in Section 4, are similar.

Remark 1.6. Theorem 1.5 implies not only that maximal hypersurfaces are rare, but that "maximal configurations" are. For instance, we will show in Section 5 that in some suitable real algebraic surfaces the probability that a real algebraic curve has a deep nest of ovals is exponentially small (roughly speaking, a nest of ovals means several ovals inside each other); see Theorems 5.3 and 5.7.

As mentioned earlier, when $(X, c_X) = (\mathbb{C}P^n, conj)$, $L = \mathcal{O}(1)$ and the Hermitian metric on L equals the Fubini–Study metric (that is, the case of Kostlan polynomials, see Example 1.1), a low degree approximation property was recently proven by Diatta and Lerario [9] (see also [5]), so that Theorem 1.5 is a natural generalization of their result. Actually, in [9] the authors prove a general low degree approximation property for Kostlan polynomials: for instance, they prove that a degree d Kostlan polynomial can be approximated (in the sense of Theorem 1.5) by a degree $b\sqrt{d\log d}$ Kostlan polynomial, $b>0$, with probability $1-O(d^{-a})$, with $a>0$ depending on b.

We stress that in [9], it is essential that the real algebraic variety considered is the projective space and that the metric on $\mathcal{O}(1)$ is the Fubini–Study metric (and not, for instance, a small perturbation of it). Indeed, in this situation,

- the induced $\mathscr{L}^2$-scalar product on $\mathbb{R}_d^{\mathrm{hom}}[X_0,\dots,X_n]$ is invariant under the action of the orthogonal group $O(n+1)$, which acts on the variables $X_0,\dots,X_n$ (equivalently, the group $O(n+1)$ acts by real holomorphic isometries on $(\mathbb{C}P^n, conj)$);
- there exists a canonical $O(n+1)$-invariant decomposition $\mathbb{R}_d^{\mathrm{hom}}[X_0,\dots,X_n]=\bigoplus_{d-\ell\in 2\mathbb{N}} V_{d,\ell}$, where $V_{d,\ell}$ is the space of homogeneous harmonic polynomials of degree ℓ, thanks to which it is possible to define projections of degree d polynomials to lower degree ones.

These two properties, together with the classification [19] of the $O(n+1)$-invariant scalar products on $\mathbb{R}_d^{\mathrm{hom}}[X_0,\dots,X_n]$, are fundamental for the proof of the results in [9]. This is a very special feature of Kostlan polynomials and the reason why in our general case some of the approximations of [9] cannot be obtained. Indeed,

- on a general real algebraic variety equipped with a Kähler metric ω the group of holomorphic isometries is trivial;
- given a real Hermitian holomorphic line bundle $L\to X$ there is no canonical decomposition of $\mathbb{R}H^0(X,L^d)$.

Hence, in order to obtain an approximation property for sections of line bundles on a general real algebraic variety, we have to use a different strategy, which we will explain in more detail in Section 1.4. In particular, in our proof, in contrast to the case of polynomials [9], the complex locus of the variety X (and not only the real one) plays a fundamental role. Indeed, we will consider real subvarieties of X with empty real loci and we will study the real sections of L^d that vanish along these subvarieties. These real sections are the fundamental tool for our low degree approximation property.

In the proof of Theorem 1.5, we will also need to understand how much we can perturb a real section $s\in\mathbb{R}H^0(X,L^d)$ without changing the topology of its real locus. This leads us to study two quantities related to the discriminant $\mathbb{R}\Delta_d\subset\mathbb{R}H^0(X,L^d)$, that is, the subset of sections which do not vanish transversally along $\mathbb{R}X$. More precisely, we will consider the volume (with respect to the Gaussian measure μ_d) of tubular neighborhoods of $\mathbb{R}\Delta_d$ and the "distance to the discriminant" function. Such quantities have already been used in the case of Kostlan polynomials in [9, Section 4], but in our general framework the lack of symmetries makes their computation more delicate and requires the use of the Bergman kernel's estimates along the diagonal [3, 21, 29].

In particular, denoting by $\mathrm{dist}_{\mathbb{R}\Delta_d}(s)$ the distance (induced by the $\mathscr{L}^2$-scalar product (2)) from a section s to the discriminant $\mathbb{R}\Delta_d$, we obtain the uniform estimate (see Lemma 3.1)

$$\mathrm{dist}_{\mathbb{R}\Delta_d}(s)=\min_{x\in\mathbb{R}X}\left(\frac{\|s(x)\|^2_{h^d}}{d^n}+\frac{\|\nabla s(x)\|^2_{h^d}}{d^{n+1}}\right)^{1/2}(\pi^{n/2}+O(d^{-1/2}))$$

as $d \to \infty$. This estimate is a generalization of a result of Raffalli [23] who proved a similar formula for polynomials (in that case, the formula is exact and not just asymptotic), which was used in [9].

Finally, let us conclude this section by pointing out that, while in the present paper we are interested in some rare events (that is, real algebraic hypersurfaces with rich topology), the expected value $\mathbb{E}[b_i(\mathbb{R}Z_{s_d})]$ of the i-th Betti number of $\mathbb{R}Z_{s_d}$ is bounded from below and from above by respectively $c_{i,n}d^{n/2}$ and $C_{i,n}d^{n/2}$, for some positive constants $0 < c_{i,n} \leq C_{i,n}$ (see [11, 13]). Moreover, in the case of the 0-th Betti number (that is, the number of connected components), it is known that $\lim_{d\to\infty} \frac{1}{d^{n/2}}\mathbb{E}[b_0(\mathbb{R}Z_{s_d})]$ exists and is positive [22].

1.4. Idea of the proof of Theorem 1.5

In this section, we sketch the proof of assertion (ii) of Theorem 1.5; the proof of (i) is similar.

We want to prove that, with very high probability, the real vanishing locus of a real section s of L^d is ambient isotopic to the real vanishing locus of a real section s' of L^{d-k}.

The first fact we will use is the existence of a real section σ of L^k, for some suitable even $k \in 2\mathbb{N}$ large enough, with the properties that σ vanishes transversally and $\mathbb{R}Z_\sigma = \emptyset$ (see Proposition 2.1). In order to obtain such a section σ, we consider an integer m such that L^m is very ample and $\{s_1, \dots, s_N\}$ is a basis of $\mathbb{R}H^0(X, L^m)$. Then σ is any general small perturbation of the section $\sum_{i=1}^N s_i^{\otimes 2}$ of L^{2m}.

Let us define $\mathbb{R}H_{d,\sigma}$ to be the vector space of real global sections $s \in \mathbb{R}H^0(X, L^d)$ such that s vanishes along the vanishing locus Z_σ of σ. We also denote by $\mathbb{R}H_{d,\sigma}^\perp$ the orthogonal complement of $\mathbb{R}H_{d,\sigma}$ with respect to the $\mathcal{L}^2$-scalar product defined in (2). Then, for any section $s \in \mathbb{R}H^0(X, L^d)$, there exists a unique decomposition $s = s_\sigma^\perp + s_\sigma^0$ with $s_\sigma^0 \in \mathbb{R}H_{d,\sigma}$ and $s_\sigma^\perp \in \mathbb{R}H_{d,\sigma}^\perp$.

The fundamental point, which we will prove in Section 2.2 using the theory of partial Bergman kernels [7, 24], is that the "orthogonal component" $s_\sigma^\perp$ of s has a very small $\mathcal{C}^1$-norm along the real locus $\mathbb{R}X$. A geometric reason for this is that the space $\mathbb{R}H_{d,\sigma}^\perp$ is generated by the peak sections [26] which have a peak on Z_σ and, as $Z_\sigma \cap \mathbb{R}X = \emptyset$, the pointwise $\mathcal{C}^1$-norm of these peak sections is very small along $\mathbb{R}X$ (indeed, a peak section has a very small $\mathcal{C}^1$-norm on any compact set disjoint from its peak).

From the fact that $s_\sigma^\perp$ has a very small $\mathcal{C}^1$-norm along the real locus $\mathbb{R}X$, we deduce that s is a "small pertubation" of s_σ^0; Thom's Isotopy Lemma would therefore imply that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s_\sigma^0})$ are isotopic *if* s_σ^0 has a large enough $\mathcal{C}^1$-norm along $\mathbb{R}X$. This last implication can be translated in terms of distance from s_σ^0 to the discriminant $\mathbb{R}\Delta_d$, that is, the space of real sections which do not vanish transversally along $\mathbb{R}X$: if s_σ^0 is far enough from the discriminant, then the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s_\sigma^0})$ are isotopic.

Using the Bergman kernel, we are able to estimate the "distance to the discriminant" function (Lemma 3.1). These estimates, together with an approach similar to [9, Section 4], allow us to prove that, with very high probability, s_σ^0 is far enough from the discriminant, so that, with very high probability, the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s_\sigma^0})$ are isotopic (Lemmas 4.3 and 4.4).

Finally, we prove that the section s_σ^0 can be written as $\sigma \otimes s'$ for some real section s' of L^{d-k}. In particular, as σ does not have any real zero, we have the equality $\mathbb{R}Z_{s_\sigma^0} = \mathbb{R}Z_{s'}$, which proves Theorem 1.5.

1.5. Organization of the paper

The paper is organized as follows.

In Section 2, we prove the existence of a real section σ of L^k with empty real vanishing locus and then we will study the real sections of L^d which vanish along Z_σ. This leads us to consider logarithmic and partial Bergman kernels in Section 2.2.

In Section 3, we study the geometry of the discriminant $\mathbb{R}\Delta_d \subset \mathbb{R}H^0(X, L^d)$ and compute several related quantities: its degree, the volume (with respect to the Gaussian measure μ_d) of tubular neighborhoods and the "distance to the discriminant" function.

In Section 4, we prove our main results, namely Theorems 1.2, 1.3 and 1.5.

Finally, in Section 5, we study the depth of the nests of real algebraic curves inside some real algebraic surfaces, namely Hirzebruch surfaces (Theorem 5.3) and del Pezzo surfaces (Theorem 5.7).

2. Sections vanishing along a fixed hypersurface

In this section, we prove the existence of a real global section σ of L^k with empty real vanishing locus and we study the real sections of L^d which vanish along Z_σ.

2.1. A global section with empty real vanishing locus

Let (L, c_L) be an ample real holomorphic line bundle over a real algebraic variety (X, c_X). A *real section* of L is a global holomorphic section s of L such that $s \circ c_X = c_L \circ s$.

Proposition 2.1. *There exists an even positive integer k_0 such that for any even $k \geq k_0$ there exists a real section σ of L^k with the following properties:* (i) *σ vanishes transversally and* (ii) *$\mathbb{R}Z_\sigma$ is empty.*

Proof. Let m_0 be the smallest integer such that L^{m_0} is very ample and set $k_0 = 2m_0$. For any integer $m \geq m_0$, fix a basis $s_1, \dots, s_{N_m}$ of $\mathbb{R}H^0(X, L^m)$ and consider the real section $s = \sum_{i=1}^{N_m} s_i^{\otimes 2}$, which is a real section of L^{2m} whose real vanishing locus is empty. Note that any small perturbation of s inside $\mathbb{R}H^0(X, L^{2m})$ will have empty real vanishing locus, and the discriminant (i.e. the sections which do not vanish transversally) is an algebraic hypersurface of $\mathbb{R}H^0(X, L^{2m})$. We can then find a small perturbation σ of s which has the desired properties, namely σ is a real global section of L^{2m} vanishing transversally and whose real vanishing locus is empty. ∎

Definition 2.2. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, where k is a fixed large enough integer, and let Z_σ be its vanishing locus. For any integer ℓ, we will write σ^ℓ for the section $\sigma^{\otimes\ell}$ of $L^{k\ell}$. For any integers d and ℓ, let H_{d,σ^ℓ} be the subspace of $H^0(X, L^d)$ consisting of the sections which vanish along Z_σ to order at least ℓ. Similarly, $\mathbb{R}H_{d,\sigma^\ell}$ is the subspace of $\mathbb{R}H^0(X, L^d)$ of real sections which vanish along Z_σ to order at least ℓ.

Proposition 2.3. *Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1 for some fixed k large enough. For any integers d and ℓ, the space H_{d,σ^ℓ} coincides with the space of sections $s \in H^0(X, L^d)$ such that $s = \sigma^\ell \otimes s'$ for some $s' \in H^0(X, L^{d-k\ell})$. Similarly, $\mathbb{R}H_{d,\sigma^\ell}$ coincides with the space of real sections $s \in \mathbb{R}H^0(X, L^d)$ such that $s = \sigma^\ell \otimes s'$ for some $s' \in \mathbb{R}H^0(X, L^{d-k\ell})$.*

Proof. If a section s of $H^0(X, L^d)$ is of the form $s = \sigma^\ell \otimes s'$ for some $s' \in \mathbb{R}H^0(X, L^{d-k\ell})$, then $s \in H_{d,\sigma^\ell}$ (that is, s vanishes along Z_σ to order at least ℓ). Moreover, if such a section is real then $s \in \mathbb{R}H_{d,\sigma^\ell}$. Let us now prove the other inclusion. Let $s \in H^0(X, L^d)$ vanish along Z_σ to order at least ℓ. We want to prove that there exists $s' \in H^0(X, L^{d-k})$ such that $s = \sigma^\ell \otimes s'$. For this, let $\{U_i\}_i$ be a cover of X by open subsets such that the line bundle L is obtained by gluing together the local models $\{U_i \times \mathbb{C}\}_i$ using the maps $f_{ij} : (U_i \cap U_j) \ni (x, v) \times \mathbb{C} \mapsto (x, g_{ij}(x)v) \in (U_i \cap U_j) \times \mathbb{C}$, where $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ are holomorphic maps. Observe that, for any integer d, the line bundle L^d is obtained by gluing together the same local models $\{U_i \times \mathbb{C}\}_i$ using the maps $f_{ij}^d : (U_i \cap U_j) \ni (x, v) \times \mathbb{C} \mapsto (x, g_{ij}^d(x)v) \in (U_i \cap U_j) \times \mathbb{C}$. Using these trivializations, a global section s of L^d is equivalent to the data of local holomorphic functions s_i on U_i such that $s_i = g_{ij}^d s_j$ on $U_i \cap U_j$. We can then locally define the section s' we are looking for by setting $s'_i = s_i/\sigma_i^\ell$. Indeed, these are holomorphic functions because s_i vanishes along the zero set of σ_i to order at least ℓ. With this definition, it is straightforward to check that the family $\{s'_i\}_i$ glues together and defines a global section s' of $L^{d-k\ell}$ with $s = \sigma^\ell \otimes s'$.

The proof for the real case follows from the complex case and from that fact that if s and σ are real sections, then s' is also real. ∎

2.2. $\mathscr{L}^2$-orthogonal complement to $\mathbb{R}H_{d,\sigma^\ell}$ and $\mathscr{C}^1$-estimates

In this section, we equip L with a real Hermitian metric h with positive curvature and consider the induced $\mathscr{L}^2$-scalar product on $\mathbb{R}H^0(X, L^d)$ given by (2). The main goal is to study the real sections of L^d which are $\mathscr{L}^2$-orthogonal to the space $\mathbb{R}H_{d,\sigma^\ell}$ defined in Definition 2.2.

Definition 2.4 ($\mathscr{C}^1$-norm). Let $K \subset X$ be a compact set. We define the $\mathscr{C}^1(K)$-norm of a global section s of L^d to be $\|s\|_{\mathscr{C}^1(K)} = \max_{x\in K} \|s(x)\|_{h^d} + \max_{x\in K} \|\nabla s(x)\|_{h^d}$, where $\|\cdot\|_{h^d}$ is the norm induced by the Hermitian metric h^d, ∇ is the Chern connection on L^d induced by h and $\|\nabla s(x)\|_{h^d}^2 = \sum_{i=1}^n \|\nabla_{v_i} s(x)\|_{h^d}^2$, with $\{v_1, \dots, v_n\}$ an orthonormal basis of $T_x X$.

Remark 2.5. Recall that $\mathbb{R}H_{d,\sigma^\ell} = H_{d,\sigma^\ell} \cap \mathbb{R}H^0(X, L^d)$. The orthogonal complement (with respect to the $\mathcal{L}^2$-scalar product given by (2)) of $\mathbb{R}H_{d,\sigma^\ell}$ inside $\mathbb{R}H^0(X, L^d)$ coincides with $H^\perp_{d,\sigma^\ell} \cap \mathbb{R}H^0(X, L^d)$. Here, $H^\perp_{d,\sigma^\ell}$ is the orthogonal complement of H_{d,σ^ℓ} with respect to the $\mathcal{L}^2$-Hermitian product defined by

$$\langle s_1, s_2\rangle_{\mathcal{L}^2} = \int_X h^d(s_1, s_2)\frac{\omega^{\wedge n}}{n!} \tag{4}$$

for any global sections $s_1, s_2 \in H^0(X, L^d)$.

Proposition 2.6. *Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1 for some fixed k large enough. There exists a positive real number t_0 such that, for any $t \in (0, t_0)$, we have the uniform estimate $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} = O(d^{-\infty})$ as $d \to \infty$ for any real section $\tau \in \mathbb{R}H^\perp_{d,\sigma^{\lfloor td\rfloor}}$ with $\|\tau\|_{\mathcal{L}^2} = 1$. Here, $\mathbb{R}H_{d,\sigma^{\lfloor td\rfloor}}$ is as in Definition 2.2.*

Proof. Let $\tau_1, \dots, \tau_{m_d}$ be an orthonormal basis of $H^\perp_{d,\sigma^{\lfloor td\rfloor}}$ and $s_1, \dots, s_{N_d-m_d}$ be an orthonormal basis of $H_{d,\sigma^{\lfloor td\rfloor}}$ (the Hermitian products are the ones induced by (4)). We set

$$P_d^\perp(x) = \sum_{i=1}^{m_d}\|\tau_i(x)\|^2_{h^d} \quad\text{and}\quad P_d^0(x) = \sum_{i=1}^{N_d-m_d}\|s_i(x)\|^2_{h^d},$$

so that $P_d^\perp + P_d^0$ equals the Bergman function P_d (i.e. the value of the Bergman kernel on the diagonal). The function P_d^0 equals the *partial Bergman kernel* of order $\lfloor td\rfloor$ associated with the subvariety Z_σ (see [7]). By [7, Theorem 1.3], for any compact subset K of X which is disjoint from Z_σ and for any $r \in \mathbb{N}$, there exists a positive real number $t_0(K)$ such that, for any $t < t_0(K)$, one has $\|P_d - P_d^0\|_{\mathcal{C}^r(K)} = O(d^{-\infty})$. Now, we have $P_d^\perp = P_d - P_d^0$, so that, by choosing $K = \mathbb{R}X$, which is disjoint from Z_σ by construction of σ, we obtain $\|P_d^\perp\|_{\mathcal{C}^r(\mathbb{R}X)} = O(d^{-\infty})$ as long as $t < t_0 = t_0(\mathbb{R}X)$. This implies that for any $t < t_0$ and any $\tau \in H^\perp_{d,\sigma^{\lfloor td\rfloor}}$ with $\|\tau\|_{\mathcal{L}^2} = 1$, one has $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} = O(d^{-\infty})$, and in particular this happens if $\tau \in \mathbb{R}H^\perp_{d,\sigma^{\lfloor td\rfloor}}$. ■

Proposition 2.7. *Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1 for some fixed k large enough. There exists $c > 0$ (depending on k) such that we have the uniform estimate $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} = O(e^{-c\sqrt{d}\log d})$ for any real section $\tau \in \mathbb{R}H^\perp_{d,\sigma}$ with $\|\tau\|_{\mathcal{L}^2} = 1$. If the real Hermitian metric on L is analytic, then we have the uniform estimate $\|\tau\|_{\mathcal{C}^1(\mathbb{R}X)} = O(e^{-cd})$ for any real section $\tau \in \mathbb{R}H^\perp_{d,\sigma}$ with $\|\tau\|_{\mathcal{L}^2} = 1$.*

Proof. The proof follows the same idea as that of Proposition 2.6. Let $\tau_1, \dots, \tau_{m_d}$ be an orthonormal basis of $H^\perp_{d,\sigma}$ and $s_1, \dots, s_{N_d-m_d}$ be an orthonormal basis of $H_{d,\sigma}$ (the Hermitian products are the ones induced by (4)). We set $P_d^\perp(x) = \sum_{i=1}^{m_d}\|\tau_i(x)\|^2_{h^d}$ and $P_d^0(x) = \sum_{i=1}^{N_d-m_d}\|s_i(x)\|^2_{h^d}$, so that $P_d^\perp + P_d^0$ equals the Bergman function P_d. The function P_d^0 is called the *logarithmic Bergman kernel* associated with the subvariety Z_σ (see [24]). For this kernel, we have, for any $r \in \mathbb{N}$, the estimates

$$\|P_d - P_d^0\|_{\mathcal{C}^r(K)} = O(e^{-c\sqrt{d}\log d}) \tag{5}$$

and, in the analytic case,

$$\|P_d - P_d^0\|_{\mathscr{C}^r(K)} = O(e^{-cd}), \tag{6}$$

where K is a fixed compact set disjoint from Z_σ. For a proof of these estimates, one can follow line for line the proof of [24, Theorem 3.4] (stating that for any sequence of compact sets K_d whose distance from Z_σ is greater than $(\log d)/\sqrt{d}$, one has $\|P_d - P_d^0\|_{\mathscr{C}^r(K_d)} = O(d^{-\infty})$) and just replace there:

- the sequence of compact sets K_d by a fixed compact set K;
- the $\mathscr{C}^r$-norm at a point $z_d \in K_d$ (depending on d) of a peak section [26] peaking at Z_σ (which is $O(d^{-\infty})$ if $\operatorname{dist}(K_d, Z_\sigma) > (\log d)/\sqrt{d}$) by the $\mathscr{C}^r$-norm at a fixed point $z \in K$ (not depending on d) of a peak section peaking at Z_σ. In this case, the latter $\mathscr{C}^r$-norm is $O(e^{-c\sqrt{d}\log d})$ (and even $O(e^{-cd})$ if the metric on L is analytic). This last fact is a standard and direct consequence of the exponential decay of the Bergman kernel, which can be found for instance in [15, Theorem 1.1 and Corollary 1.4].

Let us continue the proof of the proposition. By definition, we have $P_d^\perp = P_d - P_d^0$, so that, by the estimate (5) for $K = \mathbb{R}X$, which is disjoint from Z_σ by construction of σ, we obtain $\|P_d^\perp\|_{\mathscr{C}^r(\mathbb{R}X)} = O(e^{-c\sqrt{d}\log d})$. This implies that for any $\tau \in H_{d,\sigma}^\perp$, with $\|\tau\|_{\mathscr{L}^2} = 1$, one has $\|\tau\|_{\mathscr{C}^1(\mathbb{R}X)} = O(e^{-c\sqrt{d}\log d})$, and in particular this happens if $\tau \in \mathbb{R}H_{d,\sigma}^\perp$. Similarly, using the estimate (6) for the analytic case, we have $\|\tau\|_{\mathscr{C}^1(\mathbb{R}X)} = O(e^{-cd})$ for any $\tau \in \mathbb{R}H_{d,\sigma}^\perp$ with $\|\tau\|_{\mathscr{L}^2} = 1$. ∎

3. Geometry of the discriminant: distance, degree and volume

Let $\mathbb{R}\Delta_d \subset \mathbb{R}H^0(X, L^d)$ denote the discriminant, that is, the subset of sections which do not vanish transversally along $\mathbb{R}X$. In this section, we will study and compute several quantities related to the discriminant: its degree, the volume (with respect to the Gaussian measure μ_d) of tubular conical neighborhoods (see Definition 3.3) and the "distance to the discriminant" function.

As in the previous section, we equip L with a real Hermitian metric h with positive curvature and consider the induced $\mathscr{L}^2$-scalar product on $\mathbb{R}H^0(X, L^d)$ given by (2). Given $s \in \mathbb{R}H^0(X, L^d)$, the next lemma gives us an explicit formula for the distance from s to the discriminant $\mathbb{R}\Delta_d$. This distance is computed with respect to the $\mathscr{L}^2$-scalar product, that is,

$$\operatorname{dist}_{\mathbb{R}\Delta_d}(s) := \min_{\tau \in \mathbb{R}\Delta_d} \|s - \tau\|_{\mathscr{L}^2}.$$

In the case of polynomials, the distance has already been computed by Raffalli [23] and used in [9].

Lemma 3.1 (Distance to the discriminant). *Let (L, c_L) be a real Hermitian ample line bundle over a real algebraic variety X of dimension n. Denote by $\|\cdot\|_{h^d}$ the Hermitian* *metric on L^d induced by a Hermitian metric h on L with positive curvature. Then, as $d \to \infty$, we have the uniform estimate*

$$\mathrm{dist}_{\mathbb{R}\Delta_d}(s) = \min_{x \in \mathbb{R}X} \left(\frac{\|s(x)\|^2_{h^d}}{d^n} + \frac{\|\nabla s(x)\|^2_{h^d}}{d^{n+1}} \right)^{1/2} (\pi^{n/2} + O(d^{-1/2})).$$

Proof. We will follow the approach of [23, Section 3] which we combine with Bergman kernel estimates along the diagonal [3, 8, 21, 29]. Let us define $\mathbb{R}\Delta_{d,x}$ to be the linear subspace of real sections which do not vanish transversally to x, that is,

$$\mathbb{R}\Delta_{d,x} = \{s \in \mathbb{R}H^0(X, L^d) : s(x) = 0 \text{ and } \nabla s(x) = 0\}.$$

Let us fix some notations:

- We fix once for all an orthonormal basis $s_1, \dots, s_{N_d}$ of $\mathbb{R}H^0(X, L^d)$ (with respect to the $\mathcal{L}^2$-scalar product (2)). We denote by $S(x)$ the vector $(s_1(x), \dots, s_{N_d}(x)) \in (\mathbb{R}L^d_x)^{N_d}$.
- We will also fix an orthonormal basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ of $T_x\mathbb{R}X$ (the Riemannian metric is the one induced by the curvature form ω of the real Hermitian line bundle (L, h)).
- Let ∇ be a connection on L^d which is metric, real and such that, if we read the connection ∇ using real normal coordinates around x and the real normal trivialization of L^d over a neighborhood of x, then we obtain the trivial connection (see [20, Section 3.1] for the definition of real normal coordinates and real normal trivialization and [20, Section 3.3] for the existence of such a connection; in the following, we will not use explicitly the properties of these normal coordinates or of the normal trivializations).
- We denote by ∇_j the covariant derivative along $\frac{\partial}{\partial x_j}$ on L^d with respect to the connection ∇, that is, $\nabla_j = \nabla_{\partial/\partial x_j}$.
- For a real section s, we will write $s(x)^* \in (\mathbb{R}L^d_x)^*$ to denote the linear function on $\mathbb{R}L^d_x$ defined by $\mathbb{R}L^d_x \ni e \mapsto h^d_x(s(x), e)$.
- We will denote by M the $(n+1) \times N_d$ matrix with entries in $\mathbb{R}L^d_x$ whose $n+1$ lines are $S(x), \nabla_1 S(x), \dots, \nabla_n S(x)$, where $\nabla_j S(x) = (\nabla_j s_1(x), \dots, \nabla_j s_{N_d}(x)) \in (\mathbb{R}L^d_x)^{N_d}$. Note that, for d large enough, the matrix M has maximal rank. Indeed, for d large enough and for any $v \in T_x\mathbb{R}X$, one can find a real section τ_1 such that $\tau_1(x) \neq 0$ and a real section τ_2 such that $\nabla_v \tau_2(x) \neq 0$ (such sections always exist for large d by positivity of L).
- Finally, we denote by M^t the $N_d \times (n+1)$ matrix with entries in $(\mathbb{R}L^d_x)^*$ whose $n+1$ columns are $S^*(x), \nabla_1 S^*(x), \dots, \nabla_n S^*(x)$, where

$$\nabla_j S^*(x) = ((\nabla_j s_1(x))^*, \dots, (\nabla_j s_{N_d}(x))^*).$$

Using these notations, a real holomorphic section $s = \sum_i a_i s_i$ lies in $\mathbb{R}\Delta_{d,x}$ if and only if $s(x) = 0$ and $\nabla_j s(x) = 0$ for any $j \in \{1, \dots, n\}$. Equivalently, it lies in $\mathbb{R}\Delta_{d,x}$ if and only $Ma = 0$ where $a = (a_1, \dots, a_{N_d})^t \in \mathbb{R}^{N_d}$ and the multiplication is the standard matrix multiplication.

We now compute $\mathrm{dist}(s, \mathbb{R}\Delta_{d,x})$ for any $s \in \mathbb{R}H^0(X, L^d)$. By definition, the distance $\mathrm{dist}(s, \mathbb{R}\Delta_{d,x})$ equals $\min_{s+\tau\in\mathbb{R}\Delta_{d,x}} \|\tau\|_{\mathscr{L}^2}$, that is, it equals the $\mathscr{L}^2$-norm of a section τ which is orthogonal to $\mathbb{R}\Delta_{d,x}$ and such that $s+\tau \in \mathbb{R}\Delta_{d,x}$. Writing $s = \sum_i a_i s_i$ and $\tau = \sum_i b_i s_i$, the last condition reads $M(a+b) = 0$, where $a = (a_1, \dots, a_{N_d})^t$, $b = (b_1, \dots, b_{N_d})^t$ and the multiplication is the standard matrix multiplication. The condition "τ is orthogonal to $\mathbb{R}\Delta_{d,x}$" can be written as "there exists $\alpha \in (\mathbb{R}L_x^d)^{n+1}$ such that $b = M^t\alpha$". Putting together these two conditions, we obtain $M(a + M^t\alpha) = Ma + MM^t\alpha = 0$. Now, let $A = MM^t$, which is an $(n+1)\times(n+1)$ matrix with entries in $\mathbb{R}L_x^d \otimes (\mathbb{R}L_x^d)^* = \mathrm{End}(\mathbb{R}L_x^d)$. Recall that, for d large enough, the matrix M has maximal rank and so A is invertible, and A^{-1} is the $(n+1)\times(n+1)$ matrix with entries in $\mathrm{End}((\mathbb{R}L_x^d)^*)$ such that $AA^{-1} = A^{-1}A = \mathrm{Diag}(\mathrm{Id}_{\mathbb{R}L_x^d})$, the diagonal matrix with $\mathrm{Id}_{\mathbb{R}L_x^d}$ along the diagonal.

Then, for d large enough, we have $\alpha = -A^{-1}Ma$, so that $b = -M^tA^{-1}Ma$. We then obtain

$$\begin{aligned} \mathrm{dist}^2(s, \mathbb{R}\Delta_{d,x}) &= \|\tau\|^2_{\mathscr{L}^2} = \|b\|^2 = \|M^tA^{-1}Ma\|^2 \\ &= a^tM^t(A^{-1})^tMM^tA^{-1}Ma \\ &= a^tM^t(A^{-1})^tMa. \end{aligned} \tag{7}$$

Observe now that A is the following $(n+1)\times(n+1)$ matrix:

$$\begin{bmatrix} \langle S(x), S^*(x)\rangle & \langle \nabla_1 S(x), S^*(x)\rangle & \cdots & \langle \nabla_n S(x), S^*(x)\rangle \\ \langle S(x), \nabla_1 S^*(x)\rangle & \langle \nabla_1 S(x), \nabla_1 S^*(x)\rangle & \cdots & \langle \nabla_n S(x), \nabla_1 S^*(x)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle S(x), \nabla_n S^*(x)\rangle & \langle \nabla_1 S(x), \nabla_n S^*(x)\rangle & \cdots & \langle \nabla_n S(x), \nabla_n S^*(x)\rangle \end{bmatrix}$$

where the notation $\langle S(x), S^*(x)\rangle$ stands for $\sum_{i=1}^{N_d} s_i(x)\otimes s_i(x)^* \in \mathrm{End}(\mathbb{R}L_x^d)$, the notation $\langle S(x), \nabla_j S^*(x)\rangle$ stands for $\sum_{i=1}^{N_d} s_i(x)\otimes(\nabla_j s_i(x))^* \in \mathrm{End}(\mathbb{R}L_x^d)$ and so on for the other terms.

Note that the quantities $\langle S(x), S^*(x)\rangle$, $\langle \nabla_i S(x), S^*(x)\rangle$, $\langle S(x), \nabla_i S^*(x)\rangle$ and $\langle \nabla_i S(x), \nabla_j S^*(x)\rangle$ equal the Bergman kernel and its first derivatives at (x,x). The asymptotics of these quantities are well-known (see, for instance, [3, 8, 21, 29]). In our case, we are in the same setting as in [20, Corollary 3.8], so we have

$$\begin{aligned} \langle S(x), S^*(x)\rangle &= \frac{d^n}{\pi^n}(\mathrm{Id}_{\mathbb{R}L_x^d} + O(d^{-1})), \\ \langle \nabla_i S(x), S^*(x)\rangle &= O(d^{n-1/2}), \\ \langle S(x), \nabla_i S^*(x)\rangle &= O(d^{n-1/2}), \\ \langle \nabla_i S(x), \nabla_j S^*(x)\rangle &= O(d^n) \quad \text{for } i \neq j, \\ \langle \nabla_i S(x), \nabla_i S^*(x)\rangle &= \frac{d^{n+1}}{\pi^n}(\mathrm{Id}_{\mathbb{R}L_x^d} + O(d^n)), \end{aligned}$$

as $d\to\infty$. As a consequence, as $d\to\infty$, the matrix A equals

$$\begin{bmatrix} \frac{d^n}{\pi^n}\mathrm{Id}_{\mathbb{R}L_x^d} + O(d^{n-1}) & O(d^{n-1/2}) & \cdots & O(d^{n-1/2}) \\ O(d^{n-1/2}) & \frac{d^{n+1}}{\pi^n}\mathrm{Id}_{\mathbb{R}L_x^d} + O(d^n) & \cdots & O(d^n) \\ \vdots & \vdots & \ddots & \vdots \\ O(d^{n-1/2}) & O(d^n) & \cdots & \frac{d^{n+1}}{\pi^n}\mathrm{Id}_{\mathbb{R}L_x^d} + O(d^n) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{d^n}{\pi^n}\mathrm{Id}_{\mathbb{R}L_x^d} & 0 & \cdots & 0 \\ 0 & \frac{d^{n+1}}{\pi^n}\mathrm{Id}_{\mathbb{R}L_x^d} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{d^{n+1}}{\pi^n}\mathrm{Id}_{\mathbb{R}L_x^d} \end{bmatrix} (\mathrm{Id} + O(d^{-1/2})),$$

so that A^{-1} equals

$$A^{-1} = \begin{bmatrix} d^{-n}\mathrm{Id}_{\mathbb{R}L_x^d} & 0 & \cdots & 0 \\ 0 & d^{-n-1}\mathrm{Id}_{\mathbb{R}L_x^d} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d^{-n-1}\mathrm{Id}_{\mathbb{R}L_x^d} \end{bmatrix} (\pi^n\mathrm{Id} + O(d^{-1/2})).$$

In particular, recalling the definition of the matrix M and the vector a given above, we find that (7) equals

$$\left(\frac{\|s(x)\|_{h^d}^2}{d^n} + \frac{\|\nabla s(x)\|_{h^d}^2}{d^{n+1}}\right)(\pi^n + O(d^{-1/2})).$$

Taking the minimum over $x \in \mathbb{R}X$, we obtain the result. ■

Lemma 3.2 (Degree of the discriminant). *Let (L, c_L) be a real ample line bundle over a real algebraic variety X of dimension n and denote by $\mathbb{R}\Delta_d$ the discriminant in $\mathbb{R}H^0(X, L^d)$. Then there exists a homogeneous real polynomial Q_d vanishing on $\mathbb{R}\Delta_d$ and such that* $\deg(Q_d) = (n+1)\int_X c_1(X)^n d^n + O(d^{n-1})$.

Proof. Let $\Delta_d \subset H^0(X, L^d)$ be the (complex) discriminant. First, note that if $s \in \mathbb{R}\Delta_d$ then $s \in \Delta_d$, so that if we find a real polynomial vanishing on Δ_d, then it will vanish also on $\mathbb{R}\Delta_d$. We will now find a polynomial vanishing along Δ_d and estimate its degree. Since Δ_d is a cone in $H^0(X, L^d)$, the degree of Δ_d equals the number of intersection points of a generic line γ in $\mathbb{P}(H^0(X, L^d))$ with $\mathbb{P}(\Delta_d)$. We remark that a line γ in $\mathbb{P}(H^0(X, L^d))$ induces a Lefschetz pencil $u : X \dashrightarrow \mathbb{C}P^1$ sending $x \in X$ to $[s_1(x) : s_2(x)] \in \mathbb{C}P^1$, where s_1 and s_2 are two distinct sections lying on γ. Now, the cardinality of $\gamma \cap \mathbb{P}(\Delta_d)$ corresponds exactly to the number of singular fibers of the Lefschetz pencil u. By [1, Proposition 2.3], we have $\deg(\mathbb{P}(\Delta_d)) = (n+1)\int_X c_1(X)^n d^n + O(d^{n-1})$. In particular, there exists a homogeneous polynomial Q_d of degree $\deg(\mathbb{P}(\Delta_d))$ vanishing on Δ_d. Also, if $s \in \Delta_d$, then $c_L \circ s \circ c_X \in \Delta_d$, so that Δ_d is a real algebraic hypersurface in $H^0(X, L^d)$ (with respect to the natural real structure $s \mapsto c_{L^d} \circ s \circ c_X$). It is then possible to choose the polynomial Q_d to be real. ■

Definition 3.3. A *tubular conical neighborhood* of $\mathbb{R}\Delta_d$ is a tubular neighborhood of $\mathbb{R}\Delta_d$ in $\mathbb{R}H^0(X, L^d)$ which is also a cone (that is, if s is in the neighborhood, then λs is also in the neighborhood, for any $\lambda \in \mathbb{R}^*$).

To define a tubular conical neighborhood of $\mathbb{R}\Delta_d$, it is enough to define its trace on the unit sphere $S_d := \{s \in \mathbb{R}H^0(X, L^d) : \|s\|_{\mathcal{L}^2} = 1\}$. We denote by $S\Delta_d$ the trace of the discriminant on S_d, that is, $S\Delta_d = S_d \cap \mathbb{R}\Delta_d$. The next lemma estimates the volume of *very small* tubular conical neighborhoods of the discriminant $\mathbb{R}\Delta_d$, namely tubular conical neighborhoods whose trace on S_d is of the form $\{s \in S_d : \operatorname{dist}(s, S\Delta_d) < r_d\}$ for r_d a sequence of real numbers which goes to zero *at least* as d^{-2n}. For the case of Kostlan polynomials, that volume has already been estimated by Diatta and Lerario [9, Proposition 4].

Lemma 3.4 (Volume of tubular conical neighborhoods). *Let (L, c_L) be a real Hermitian ample line bundle over a real algebraic variety (X, c_X) of dimension n. Then there exists a positive constant c (not depending on d) such that, for any sequence r_d satisfying $r_d \leq c d^{-2n}$, one has*

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d) : \operatorname{dist}_{\mathbb{R}\Delta_d}(s) \leq r_d \|s\|_{\mathcal{L}^2}\} = O(r_d d^{2n}).$$

Here, μ_d is the Gaussian probability measure defined in (3).

Proof. We start with a standard remark about Gaussian measures on Euclidean spaces: if S_d is the unit sphere in $\mathbb{R}H^0(X, L^d)$, and ν_d the probability measure induced by its volume form (i.e. for any $A \subset S_d$, $\nu_d(A) = \operatorname{Vol}(A)\operatorname{Vol}(S_d)^{-1}$), then the Gaussian measure of every cone C_d in $\mathbb{R}H^0(X, L^d)$ equals $\nu_d(C_d \cap S_d)$. We apply this remark to the tubular conical neighborhood $C_d = \{\operatorname{dist}_{\mathbb{R}\Delta_d}(s) \leq r_d \|s\|_{\mathcal{L}^2}\}$ and find that its Gaussian measure equals $\nu_d\{s \in S_d : \operatorname{dist}(s, S\Delta_d) \leq r_d\}$, where $S\Delta_d$ denote the trace of the discriminant on S_d. By Lemma 3.2, for d large enough, there exists a polynomial of degree bounded by $2(n+1)\int_X c_1(X)^n d^n$ whose zero locus contains $S\Delta_d$. We are thus under the hypothesis of [6, Theorem 21.1], which gives us

$$\nu_d\{s \in S_d : \operatorname{dist}(s, S\Delta_d) \leq r_d\} \leq c N_d d^n r_d \tag{8}$$

for some constant $c > 0$, where $N_d = \dim \mathbb{R}H^0(X, L^d)$. By the Riemann–Roch Theorem, we have $N_d = O(d^n)$, so that the right-hand side of (8) is $O(r_d d^{2n})$. The lemma then follows from the fact that the ν_d-measure of the set $\{s \in S_d : \operatorname{dist}(s, S\Delta_d) \leq r_d\}$ appearing in (8) equals the Gaussian measure of its cone. This implies

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d) : \operatorname{dist}_{\mathbb{R}\Delta_d}(s) \leq r_d \|s\|_{\mathcal{L}^2}\} = O(r_d d^{2n}),$$

which concludes the proof. ∎

4. Proof of the main results

In this section, we prove our main results, Theorems 1.2, 1.3 and 1.5. We will use the notations of the previous sections, in particular we consider an ample Hermitian real holomorphic line bundle (L, c_L, h) over a real algebraic variety (X, c_X) and we denote by μ_d the Gaussian measure on $\mathbb{R}H^0(X, L^d)$ defined in (3).

Notation 4.1. Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1, for some fixed k large enough. Let d and ℓ be two positive integers and let $\mathbb{R}H_{d,\sigma^\ell}$ be as in Definition 2.2. For any real section $s \in \mathbb{R}H^0(X, L^d)$ there exists a unique decomposition $s = s^{\perp}_{\sigma^\ell} + s^0_{\sigma^\ell}$ with $s^0_{\sigma^\ell} \in \mathbb{R}H_{d,\sigma^\ell}$ and $s^{\perp}_{\sigma^\ell} \in \mathbb{R}H^{\perp}_{d,\sigma^\ell}$ (the orthogonal is with respect to the $\mathscr{L}^2$-scalar product defined in (2)).

Proposition 4.2. *Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1 for some fixed k large enough.*

(i) *There exists a positive real number t_0 such that for any $t \in (0, t_0)$ the following holds. Let $C > 0$ and $r \in \mathbb{N}$. For any $w_d \geq Cd^{-r}$, there exists $d_0 \in \mathbb{N}$ such that for any $d \geq d_0$ and any $s \in \mathbb{R}H^0(X, L^d)$, we have*

$$\|s^{\perp}_{\sigma^{\lfloor td \rfloor}}\|_{\mathscr{C}^1(\mathbb{R}X)} < w_d \|s\|_{\mathscr{L}^2}.$$

(ii) *There exist positive constants c_1 and c_2 such that, for any sequence $w_d \geq c_1 e^{-c_2\sqrt{d}\log d}$ and any $s \in \mathbb{R}H^0(X, L^d)$, we have*

$$\|s^{\perp}_{\sigma}\|_{\mathscr{C}^1(\mathbb{R}X)} < w_d \|s\|_{\mathscr{L}^2}.$$

If the real Hermitian metric on L is analytic, then the last estimate is true for any sequence $w_d \geq c_1 e^{-c_2 d}$.

Here, $s^{\perp}_{\sigma}$ and $s^{\perp}_{\sigma^{\lfloor td \rfloor}}$ are given by Notation 4.1.

Proof. Let us prove (i). Fix t_0 given by Proposition 2.6 and take $t < t_0$. Fix a sequence w_d with $w_d \geq Cd^{-r}$ for some fixed $C > 0$ and $r \in \mathbb{N}$.

Let $s \in \mathbb{R}H^0(X, L^d)$. The $\mathscr{L}^2$-norm of the section $(\|s\|_{\mathscr{L}^2})^{-1} s^{\perp}_{\sigma^{\lfloor td \rfloor}}$ is smaller than 1, so, by Proposition 2.6 there exists a constant $c_r > 0$ (not depending on d) such that

$$(\|s\|_{\mathscr{L}^2})^{-1} \|s^{\perp}_{\sigma^{\lfloor td \rfloor}}\|_{\mathscr{C}^1(\mathbb{R}X)} \leq c_r d^{-r-1},$$

which is strictly smaller than w_d for d large enough. This proves (i). The proof of (ii) follows the same lines, using Proposition 2.7 instead of Proposition 2.6. ∎

Following [9, Proposition 3], we now estimate how much we can perturb a real section without changing the topology of its zero locus.

Lemma 4.3. *There exists a positive integer d_0 such that for any $d \geq d_0$ and any real section $s \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_d$, the following holds. For any real global section $s' \in \mathbb{R}H^0(X, L^d)$ such that*

$$\|s - s'\|_{\mathscr{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}} \operatorname{dist}_{\mathbb{R}\Delta_d}(s),$$

the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s'})$ are isotopic.

Proof. By Lemma 3.1, there exists d_0 such that for any $d \geq d_0$ one has

$$\mathrm{dist}_{\mathbb{R}\Delta_d}(s) < 2\pi^{n/2} \min_{x\in\mathbb{R}X}\left(\frac{\|s(x)\|^2_{h^d}}{d^n} + \frac{\|\nabla s(x)\|^2_{h^d}}{d^{n+1}}\right)^{1/2}.$$

In particular, for any $d \geq d_0$, the inequality $\|s - s'\|_{\mathscr{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}}\mathrm{dist}_{\mathbb{R}\Delta_d}(s)$ implies

$$\|s - s'\|_{\mathscr{C}^1(\mathbb{R}X)} < \frac{1}{2}\min_{x\in\mathbb{R}X}\left(\|s(x)\|^2_{h^d} + \frac{\|\nabla s(x)\|^2_{h^d}}{d}\right)^{1/2}. \tag{9}$$

Now, denoting $\delta(s) := \min_{x\in\mathbb{R}X}(\|s(x)\|^2_{h^d} + \|\nabla s(x)\|^2_{h^d}/d)^{1/2}$ and following the lines of [9, Proposition 3], we find that inequality (9) implies that the pairs $(\mathbb{R}X, \mathbb{R}Z_s)$ and $(\mathbb{R}X, \mathbb{R}Z_{s'})$ are isotopic. ∎

Lemma 4.4. *Let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1 for some fixed k large enough. Then we have the following estimates as $d \to \infty$:*

(i) *There exists $t_0 > 0$ such that, for any $t \in (0, t_0)$, we have*

$$\mu_d\left\{s \in \mathbb{R}H^0(X, L^d) : \|s^{\perp}_{\sigma^{\lfloor td\rfloor}}\|_{\mathscr{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}}\mathrm{dist}_{\mathbb{R}\Delta_d}(s)\right\} = 1 - O(d^{-\infty}).$$

(ii) *There exists a positive $c > 0$ such that*

$$\mu_d\left\{s \in \mathbb{R}H^0(X, L^d) : \|s^{\perp}_{\sigma}\|_{\mathscr{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}}\mathrm{dist}_{\mathbb{R}\Delta_d}(s)\right\} = 1 - O(e^{-c\sqrt{d}\log d}).$$

Moreover, if the real Hermitian metric on L is analytic, then the last measure is even $1 - O(e^{-cd})$.

Proof. First, note that, by Proposition 3.4, for any $m \in \mathbb{N}$, setting $r_d = d^{-2n-m}$, one has

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d) : \mathrm{dist}_{\mathbb{R}\Delta_d}(s) > r_d\|s\|_{\mathscr{L}^2}\} = 1 - O(d^{-m}). \tag{10}$$

Also, by Proposition 4.2 (i), there exists a positive t_0 such that, for any $t < t_0$, any integer r, any sequence w_d of the form $C_2 d^{-r}$ and any d large enough, we have

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d) : \|s^{\perp}_{\sigma^{\lfloor td\rfloor}}\|_{\mathscr{C}^1(\mathbb{R}X)} < w_d\|s\|_{\mathscr{L}^2}\} = 1. \tag{11}$$

Putting together (10) and (11), we see that, for any such sequences r_d and w_d,

$$\mu_d\left\{s \in \mathbb{R}H^0(X, L^d) : \|s^{\perp}_{\sigma^{\lfloor td\rfloor}}\|_{\mathscr{C}^1(\mathbb{R}X)} < \frac{w_d}{r_d}\mathrm{dist}_{\mathbb{R}\Delta_d}(s)\right\} = 1 - O(d^{-m}). \tag{12}$$

By choosing $w_d = \frac{d^{\lfloor n/2\rfloor}}{4\pi^{n/2}}r_d$, we then find that for any $m \in \mathbb{N}$,

$$\mu_d\left\{s \in \mathbb{R}H^0(X, L^d) : \|s^{\perp}_{\sigma^{\lfloor td\rfloor}}\|_{\mathscr{C}^1(\mathbb{R}X)} < \frac{d^{\lfloor n/2\rfloor}}{4\pi^{n/2}}\mathrm{dist}_{\mathbb{R}\Delta_d}(s)\right\} = 1 - O(d^{-m}) \tag{13}$$

which implies (i). The proof of (ii) follows the same lines, using Proposition 4.2 (ii) and setting $r_d = d^{-2n}e^{-c_2\sqrt{d}\log d}$, where c_2 is given by Proposition 4.2 (ii). ∎

We can now prove Theorems 1.5, 1.2 and 1.3.

Proof of Theorem 1.5. To prove (i), let $\sigma \in \mathbb{R}H^0(X, L^k)$ be a section given by Proposition 2.1 for some fixed k large enough. We want to prove that there exists $b_0 < 1$ such that for any $b > b_0$, the measure

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d) : \exists s' \in \mathbb{R}H^0(X, L^{\lfloor bd \rfloor}) \text{ such that } (\mathbb{R}X, \mathbb{R}Z_s) \sim (\mathbb{R}X, \mathbb{R}Z_{s'})\} \tag{14}$$

is $1 - O(d^{-\infty})$, as $d \to \infty$, where $\sim$ means isotopy. Let $t_0 > 0$ be given by Proposition 4.4 (i) and set $b_0 = 1 - kt_0$. Then, for any $b > b_0$ there exists $t < t_0$ such that $\lfloor bd \rfloor \geq d - k\lfloor td \rfloor$. Fix such b and t. For any $s \in \mathbb{R}H^0(X, L^d)$, let $s = s^{\perp}_{\sigma^{\lfloor td \rfloor}} + s^0_{\sigma^{\lfloor td \rfloor}}$ be the orthogonal decomposition given by Notation 4.1. By Lemma 4.3, if the $\mathscr{C}^1(\mathbb{R}X)$-norm of $s_{\sigma^{\lfloor td \rfloor}}$ is smaller than $\frac{d^{n/2}}{4\pi^{n/2}} \operatorname{dist}_{\mathbb{R}\Delta_d}(s)$, then $(\mathbb{R}X, \mathbb{R}Z_s) \sim (\mathbb{R}X, \mathbb{R}Z_{s^0_{\sigma^{\lfloor td \rfloor}}})$. This implies that the measure (14) is greater than the Gaussian measure of the set

$$\left\{ s \in \mathbb{R}H^0(X, L^d) : \|s^{\perp}_{\sigma^{\lfloor td \rfloor}}\|_{\mathscr{C}^1(\mathbb{R}X)} < \frac{d^{n/2}}{4\pi^{n/2}} \operatorname{dist}_{\mathbb{R}\Delta_d}(s) \right\}, \tag{15}$$

which in turn, by Proposition 4.4 (i), is $1 - O(d^{-\infty})$. We have thus proven that $(\mathbb{R}X, \mathbb{R}Z_s) \sim (\mathbb{R}X, \mathbb{R}Z_{s^0_{\sigma^{\lfloor td \rfloor}}})$ with probability $1 - O(d^{-\infty})$. Now, $s^0_{\sigma^{\lfloor td \rfloor}} \in \mathbb{R}H_{d,\sigma^{\lfloor td \rfloor}}$ which implies, by Proposition 2.3, that there exists $s' \in \mathbb{R}H^0(X, L^{d-k\lfloor td \rfloor})$ such that $s^0_{\sigma^{\lfloor td \rfloor}} = \sigma^{\lfloor td \rfloor} \otimes s'$. Assertion (i) then follows from the fact that the real zero locus of $s^0_{\sigma^{\lfloor td \rfloor}}$ coincides with the real zero locus of s'. Indeed, the latter equals $\mathbb{R}Z_\sigma \cup \mathbb{R}Z_{s'}$, and this is equal to $\mathbb{R}Z_{s'}$ because $\mathbb{R}Z_\sigma = \emptyset$.

The proof of (ii) follows the same lines, using the orthogonal decomposition $s = s^{\perp}_{\sigma} + s^0_{\sigma}$ and Proposition 4.4 (ii). ∎

Proof of Theorem 1.2. Recall that we want to prove that there exists $a_0 < v(L)$ such that, for any $a > a_0$,

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d) : b_*(\mathbb{R}Z_s) < ad^n\} = 1 - O(d^{-\infty})$$

as $d \to \infty$. Let $a_0 := \int_X c_1(L)^n b_0^n$, where b_0 is given by Theorem 1.5 (i), and let $a > a_0$.

By Theorem 1.5 (i), for any $b > b_0$, the real zero locus of a global section s is diffeomorphic to the real zero locus of a global section s' of $L^{\lfloor bd \rfloor}$ with probability $1 - O(d^{-\infty})$. Now, by the Smith–Thom inequality (1), the total Betti number $b_*(\mathbb{R}Z_{s'})$ of the real zero locus a generic section s' of $L^{\lfloor bd \rfloor}$ is smaller than or equal to $b_*(Z_{s'})$, which, by [12, Lemma 3], has the asymptotics $b_*(Z_{s'}) = \int_X c_1(L)^n(\lfloor bd \rfloor)^n + O(d^{n-1})$. In particular, with probability $1 - O(d^{-\infty})$, the total Betti number $b_*(\mathbb{R}Z_s)$ of the real zero locus of a section s of L^d is smaller than $\int_X c_1(L)^n(\lfloor bd \rfloor)^n + \epsilon d^n$ for any $\epsilon > 0$ as $d \to \infty$. Choosing ϵ small enough, we can find $1 > b > b_0$ such that $\int_X c_1(L)^n b^n + \epsilon < a$, which implies the result. ∎

Proof of Theorem 1.3. We want to prove that, for any $a > 0$, there exists $c > 0$ such that

$$\mu_d\{s \in \mathbb{R}H^0(X, L^d) : b_*(\mathbb{R}Z_s) < b_*(Z_s) - ad^{n-1}\} = 1 - O(e^{-c\sqrt{d}\log d})$$

as $d \to \infty$ (and also the similar estimate with $1 - O(e^{-cd})$ on the right-hand side if the real Hermitian metric on L is analytic).

The total Betti number of the zero locus Z_s of a generic section s of L^d has the asymptotics

$$b_*(Z_s) = \int_X c_1(L)^n d^n - \int_X c_1(L)^{n-1} \wedge c_1(X) d^{n-1} + O(d^{n-2}) \tag{16}$$

as $d \to \infty$, where $c_1(X)$ is the first Chern class of X. (This is a classical estimate which follows from the adjunction formula and the Lefschetz hyperplane section theorem. The proof follows the lines of [12, Lemma 3], by just taking care of the second term of the asymptotics which can be derived using [12, Lemma 2].)

Similarly, for any $k \in \mathbb{N}$, the zero locus of a generic section s' of L^{d-k} satisfies

$$b_*(Z_{s'}) = \int_X c_1(L)^n d^n - \Big(\int_X c_1(L)^{n-1} \wedge c_1(X) + nk \int_X c_1(L)^n\Big) d^{n-1} + O(d^{n-2}) \tag{17}$$

as $d \to \infty$. Choose once and for all an integer k large enough such that $nk \int_X c_1(L)^n > a$. In particular, by (16) and (17), the zero loci of generic sections s and s' of L^d and L^{d-k} satisfy

$$b_*(Z_{s'}) < b_*(Z_s) - a d^{n-1} \tag{18}$$

as $d \to \infty$. Now, by Theorem 1.5 (ii), the probability that the real zero locus of a section s of L^d is diffeomorphic to the real zero locus of a section s' of L^{d-k} equals $1 - O(e^{-c\sqrt{d}\log d})$ (or $1 - O(e^{-cd})$ if the real Hermitian metric on L is analytic). By the Smith–Thom inequality and (18), this implies that the probability of the event "the total Betti number of the real zero locus of a section s of L^d is smaller than $b_*(Z_s) - a d^{n-1}$" is $1 - O(e^{-c\sqrt{d}\log d})$ (or $1 - O(e^{-cd})$ if the real Hermitian metric on L is analytic), which proves the theorem. ∎

5. Real curves in algebraic surfaces and depth of nests

In this section, we study the nests defined by a random real algebraic curve inside a real algebraic surface. We will focus on two examples: Hirzebruch surfaces (Section 5.1) and del Pezzo surfaces (Section 5.2). The main results of the section (Theorems 5.3 and 5.7) say that, within these families of surfaces, real algebraic curves with deep nests are rare, in the spirit of Theorems 1.2 and 1.3. In order to define what "deep nests" means in this context, we need to prove some bounds on the depth of nests (Propositions 5.2 and 5.5) which are well known to experts. These bounds play the same role as the Smith–Thom inequality (1) plays in Theorems 1.2 and 1.3. Theorems 5.3 and 5.7 will also follow from our low degree approximation (Theorem 1.5).

Definition 5.1. Let S be a real algebraic surface and let M be a connected component of the real part $\mathbb{R}S$.

- An *oval* in M is an embedded circle $\mathbb{S}^1 \hookrightarrow M$ which bounds a disk.
- Suppose that M is not diffeomorphic to the sphere $\mathbb{S}^2$. Then two ovals form an *injective pair* if one of them is contained in the disk bounded by the other. A *nest* N in M is a collection of ovals such such that each pair of ovals in the collection is an injective pair.

- Suppose that M is diffeomorphic to $\mathbb{S}^2$. Then a *nest* N in M is a collection of ovals such that each connected component of $M \setminus N$ is either a disk or an annulus.
- The number of ovals which form a nest is called the *depth* of the nest.

5.1. Nests in Hirzebruch surfaces

A *Hirzebruch surface* S is a compact complex surface which admits a holomorphic fibration over $\mathbb{C}P^1$ with fiber $\mathbb{C}P^1$. Given a Hirzebruch surface S, there exists a positive integer n such that S is isomorphic to the projective bundle $P(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathcal{O}_{\mathbb{C}P^1})$ over $\mathbb{C}P^1$ (see for example [2]). We will denote such a Hirzebruch surface by Σ_n. Note that each Hirzebruch surface is simply connected, Σ_0 is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, and Σ_1 to $\mathbb{C}P^2$ blown-up at a point.

The Hirzebruch surface Σ_n admits a natural real structure c_n, which we fix from now on, such that $\mathbb{R}\Sigma_n$ is diffeomorphic to a torus if n is even and to a Klein bottle if n is odd (see, for instance, [27, p. 7]).

We denote by F_n any fiber of the natural fibration $\Sigma_n \to \mathbb{C}P^1$ and by B_n the section $P(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \{0\})$ of this fibration. Note that B_n is a real algebraic curve and F_n can be chosen to be a real algebraic curve. The homology classes $[F_n]$ and $[B_n]$ form a basis of $H_2(\Sigma_n, \mathbb{Z})$, with $[F_n] \cdot [F_n] = 0$, $[F_n] \cdot [B_n] = 1$ and $[B_n] \cdot [B_n] = n$, where "$\cdot$" denotes the intersection product of $H_2(\Sigma_n, \mathbb{Z})$.

The second homology group $H_2(\Sigma_n, \mathbb{Z})$ can be identified with the Picard group of Σ_n. We say an algebraic curve in Σ_n *realizes the class* (a, b) (or *is of bidegree* (a, b)) if its homology class equals $a[F_n] + b[B_n]$. If $a, b > 0$, then the divisor associated with the class (a, b) is an ample divisor.

Proposition 5.2. *Let (Σ_n, c_n) be a real Hirzebruch surface and let a, b be two positive integers. Then the depth of a nest of a real algebraic curve in the class (a, b) is no greater than $b/2$.*

Proof. Let C be a real curve of bidegree (a, b) with $a, b > 0$. Let N be a nest in $\mathbb{R}C$ of depth l and choose a point p inside the innermost oval and a point q outside the outer oval of the nest. We can choose p and q so that they lie in the same fiber F_n of the fibration $\Sigma_n \to \mathbb{C}P^1$ and such that the real locus of $\mathbb{R}F_n$ intersects the nest transversally. The real locus of the fiber $\mathbb{R}F_n$ intersects each of the l ovals of the nest in at least two points, so that $\mathbb{R}F_n \cap N \geq 2l$. On the other hand, by Bézout's theorem, we have

$$\mathbb{R}F_n \cap N \leq [F_n] \cdot [C] \leq b,$$

hence the result. ∎

Theorem 5.3. *Let (Σ_n, c_n) be a real Hirzebruch surface and a and b be two positive integers.*

(i) *There exists a positive $\beta_0 < \lfloor b/2 \rfloor$ such that for any $\beta \in (\beta_0, \lfloor b/2 \rfloor)$, the probability that a real algebraic curve of bidegree (da, db) has a nest of depth $\geq \beta d$ is $O(d^{-\infty})$.*

(ii) *For any $k \in \mathbb{N}$ there exists a positive constant c such that the probability that a real algebraic curve of bidegree (da, db) has a nest of depth $\geq \lfloor db/2 \rfloor - k$ equals $O(e^{-c\sqrt{d}\log d})$.*

Proof. Notice that as a probability space, we take the linear system associated with the divisor of bidegree (a, b), rather than the space of real sections of the associated line bundle. As noticed in Remark 1.4, these two probability spaces are equivalent for our purpose.

By Theorem 1.5 (i), the real locus of a real algebraic curve of bidegree (da, db) is ambient isotopic to the real locus of a real algebraic curve of bidegree $(\lfloor dta \rfloor, \lfloor dtb \rfloor)$ with probability $1 - O(d^{-\infty})$, where $t_0 < t < 1$ for some t_0. From this, using Proposition 5.2, we find that with probability $1 - O(d^{-\infty})$, the depth of a nest of a real algebraic curve of bidegree (da, db) is at most $\lfloor dtb/2 \rfloor$. Setting $\beta_0 = \lfloor t_0 b/2 \rfloor$, we obtain (i).

The proof of (ii) follows the same lines, by using Theorem 1.5 (i). ∎

5.2. Nests in del Pezzo surfaces

An algebraic surface S is called a *del Pezzo surface* if the anticanonical bundle K_S^* is ample. We will say that a real algebraic curve inside S is *of class d* if it belongs to the real linear system defined by the real divisor $-dK_S$, where $d \in \mathbb{N}$.

The *degree* of a del Pezzo surface is the self-intersection of the canonical class K_S.

Notation 5.4. We denote by m_S the smallest value of $-H \cdot K_S$, where H is a real divisor such that $-H \cdot K_S - 1 \geq 2$.

When S has degree ≥ 3, choosing $H = -K_S$ we can see that m_S is no greater than the degree of S.

Proposition 5.5. *Let (S, c_S) be a real del Pezzo surface of degree ≥ 3, with non-empty real locus. Then the depth of a nest in a real algebraic curve of class d is at most $dm_S/2$.*

Proof. Let m_S be the smallest value of $-H \cdot K_S$, where H is a real divisor such that $-H \cdot K_S - 1 \geq 2$. Fix such an H.

Let Z be a real algebraic curve of class d and let N be a nest in $\mathbb{R}Z$ of depth l. Choose generic points p and q such that p is inside the innermost oval and q is outside the outer oval of the nest (if the connected component M where the nest is located is a sphere, then $M \setminus N$ is exactly two disks, and one of the ovals bordering one of these disks is the innermost oval and the other is the outer one). By [16], there exists a real rational curve C of class H whose real locus passes through p and q. This curve intersects each oval of the nest in at least two points, so that $N \cap \mathbb{R}C \geq 2l$. On the other hand, by Bézout's theorem, we have $\mathbb{R}C \cap N \leq -dK_s \cdot H = dm_S$, hence the result. ∎

Remark 5.6. The same statement and proof also work for many degree 2 del Pezzo surfaces (always using a real rational curve passing through two points, whose existence is ensured by the positivity of certain Welschinger invariants [16]). However, there exist degree 2 del Pezzo surfaces for which such curves may not exist, as explained in [16, Section 2.3]. To avoid explaining these differences, we have preferred to restrict ourselves to del Pezzo surfaces of degree ≥ 3.

Theorem 5.7. *Let (S, c_S) be a del Pezzo surface of degree ≥ 3 with non-empty real locus.*

(i) *There exists a positive $\alpha_0 < \lfloor m_S/2 \rfloor$ such that for any $\alpha \in (\alpha_0, \lfloor m_S/2 \rfloor)$, the probability that a real curve of class d has a nest of depth $\geq \alpha d$ is $O(d^{-\infty})$.*

(ii) *For any $k \in \mathbb{N}$ there exists a positive constant c such that the probability that a real curve of class d has a nest of depth $\geq \lfloor d m_S/2 \rfloor - k$ is $O(e^{-c\sqrt{d}\log d})$.*

Remark 5.8. The degree of a del Pezzo surface is at least 1 and at most 9. The only degree 9 del Pezzo surface is the projective plane $\mathbb{C}P^2$, and in this particular case Theorem 5.7 follows from [9, Theorem 9].

Proof of Theorem 5.7. The proof follows the lines of the proof of Theorem 5.3, using Proposition 5.5 instead of Proposition 5.2. ∎

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References

[1] Ancona, M.: Expected number and distribution of critical points of real Lefschetz pencils. Ann. Inst. Fourier (Grenoble) **70**, 1085–1113 (2020) Zbl 1457.14125 MR 4117057

[2] Beauville, A.: Complex Algebraic Surfaces. 2nd ed., London Math. Soc. Student Texts 34, Cambridge Univ. Press, Cambridge (1996) Zbl 0849.14014 MR 1406314

[3] Berman, R., Berndtsson, B., Sjöstrand, J.: A direct approach to Bergman kernel asymptotics for positive line bundles. Ark. Mat. **46**, 197–217 (2008) MR 2430724

[4] Bertrand, B.: Asymptotically maximal families of hypersurfaces in toric varieties. Geom. Dedicata **118**, 49–70 (2006) Zbl 1103.14033 MR 2239448

[5] Breiding, P., Keneshlou, H., Lerario, A.: Quantitative singularity theory for random polynomials. Int. Math. Res. Notices **2022**, 5685–5719 Zbl 1494.14058 MR 4406118

[6] Bürgisser, P., Cucker, F.: Condition: The Geometry of Numerical Algorithms. Grundlehren Math. Wiss. 349, Springer, Heidelberg (2013) Zbl 1280.65041 MR 3098452

[7] Coman, D., Marinescu, G.: On the first order asymptotics of partial Bergman kernels. Ann. Fac. Sci. Toulouse Math. (6) **26**, 1193–1210 (2017) Zbl 1394.32015 MR 3746626

[8] Dai, X., Liu, K., Ma, X.: On the asymptotic expansion of Bergman kernel. J. Differential Geom. **72**, 1–41 (2006) Zbl 1099.32003 MR 2215454

[9] Diatta, D. N., Lerario, A.: Low-degree approximation of random polynomials. Found. Comput. Math. **22**, 77–97 (2022) MR 4376589

[10] Gayet, D., Welschinger, J.-Y.: Exponential rarefaction of real curves with many components. Publ. Math. Inst. Hautes Études Sci. **113**, 69–96 (2011) Zbl 1227.32028 MR 2805598

[11] Gayet, D., Welschinger, J.-Y.: Lower estimates for the expected Betti numbers of random real hypersurfaces. J. London Math. Soc. (2) **90**, 105–120 (2014) Zbl 1326.14139 MR 3245138

[12] Gayet, D., Welschinger, J.-Y.: What is the total Betti number of a random real hypersurface? J. Reine Angew. Math. **689**, 137–168 (2014) Zbl 1348.14138 MR 3187930

[13] Gayet, D., Welschinger, J.-Y.: Betti numbers of random real hypersurfaces and determinants of random symmetric matrices. J. Eur. Math. Soc. **18**, 733–772 (2016) Zbl 1408.14187 MR 3474455

[14] Harnack, A.: Ueber die Vieltheiligkeit der ebenen algebraischen Curven. Math. Ann. **10**, 189–198 (1876) JFM 08.0438.01 MR 1509883

[15] Hezari, H., Xu, H.: Quantitative upper bounds for Bergman kernels associated to smooth Kähler potentials. Math. Res. Lett. **27**, 743–787 (2020) Zbl 1462.32024 MR 4216567

[16] Itenberg, I., Kharlamov, V., Shustin, E.: Welschinger invariants of real del Pezzo surfaces of degree ≥ 2. Internat. J. Math. **26**, art. 1550060, 63 pp. (2015) Zbl 1351.14035 MR 3372187

[17] Itenberg, I., Viro, O.: Asymptotically maximal real algebraic hypersurfaces of projective space. In: Proceedings of Gökova Geometry-Topology Conference 2006, Gökova Geometry/Topology Conference (GGT), Gökova, 91–105 (2007) Zbl 1180.14055 MR 2404951

[18] Klein, F.: Ueber den Verlauf der Abel'schen Integrale bei den Curven vierten Grades. Math. Ann. **10**, 365–397 (1876) JFM 08.0302.01 MR 1509892

[19] Kostlan, E.: On the distribution of roots of random polynomials. In: From Topology to Computation: Proceedings of the Smalefest (Berkeley, CA, 1990), Springer, New York, 419–431 (1993) Zbl 0788.60069 MR 1246137

[20] Letendre, T.: Variance of the volume of random real algebraic submanifolds. Trans. Amer. Math. Soc. **371**, 4129–4192 (2019) Zbl 1412.53079 MR 3917219

[21] Ma, X., Marinescu, G.: Holomorphic Morse Inequalities and Bergman Kernels. Progr. Math. 254, Birkhäuser, Basel (2007) Zbl 1135.32001 MR 2339952

[22] Nazarov, F., Sodin, M.: Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. J. Math. Phys. Anal. Geom. **12**, 205–278 (2016) Zbl 1358.60057 MR 3522141

[23] Raffalli, C.: Distance to the discriminant. arXiv:1404.7253 (2014)

[24] Sun, J.: Logarithmic Bergman kernel and conditional expectation of Gaussian holomorphic fields. J. Geom. Anal. **31**, 8520–8538 (2021) Zbl 1480.32005 MR 4293939

[25] Thom, R.: Sur l'homologie des variétés algébriques réelles. In: Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Univ. Press, Princeton, NJ, 255–265 (1965) Zbl 0137.42503 MR 0200942

[26] Tian, G.: On a set of polarized Kähler metrics on algebraic manifolds. J. Differential Geom. **32**, 99–130 (1990) MR 1064867

[27] Welschinger, J.-Y.: Courbes algébriques réelles et courbes flexibles sur les surfaces réglées. Prépublication de l'Institut de Recherche Mathématique Avancée 2000/43, Institut de Recherche Mathématique Avancée, Strasbourg (2000) MR 2111893

[28] Wilson, G.: Hilbert's sixteenth problem. Topology **17**, 53–73 (1978) Zbl 0394.57001 MR 498591

[29] Zelditch, S.: Szegő kernels and a theorem of Tian. Int. Math. Res. Notices **1998**, 317–331 Zbl 0922.58082 MR 1616718