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Liouville-type theorems for double-phase problems involving the Grushin operator

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Abstract. In this paper, we are concerned with the double-phase problem involving the Grushin operator in the whole space $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

$$-\operatorname{div}_G(|\nabla_G u|^{p-2}\nabla_G u + w(z)|\nabla_G u|^{q-2}\nabla_G u) = f(z)|u|^{r-1}u,$$

where ∇_G is the Grushin gradient, Δ_G is the Grushin operator, $q \geq p \geq 2$, $r > q-1$ and $w, f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are two nonnegative functions satisfying some growth conditions at infinity. Our purpose is to establish some Liouville-type theorems for stable weak solutions or for weak solutions which are stable outside a compact set of the equation above.

1. Introduction and main results

The Liouville-type theorems, which are concerned with the nonexistence of nontrivial solutions of PDEs, have been intensively developed in the past few decades since they have emerged as a fundamental tool for many applications to the qualitative properties of solutions of partial differential equations; see, e.g., [5,6,16,27,28]. In this paper, we study Liouville-type theorems for the double-phase problem involving the Grushin operator in the whole space $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

$$-\operatorname{div}_G(|\nabla_G u|^{p-2}\nabla_G u + w(z)|\nabla_G u|^{q-2}\nabla_G u) = f(z)|u|^{r-1}u, \tag{1.1}$$

where $\nabla_G = (\nabla_x, |x|^\gamma \nabla_y)$, $\Delta_G = \operatorname{div}_G \circ \nabla_G = \Delta_x + |x|^{2\gamma}\Delta_y$, $q \geq p \geq 2$, and $r > q-1$. In this paper, we assume that $w, f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are two nonnegative functions verifying the following condition: there exist $R_0, C_1, C_2 > 0$ and $a, b \in \mathbb{R}$ such that

$$w(z) \leq C_1 |z|_G^a \tag{1.2}$$

and

$$f(z) \geq C_2 |z|_G^b \tag{1.3}$$

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for all $|z|_G > R_0$. Here, $z = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and

$$|z|_G = \left[|x|^{2(\gamma+1)} + (\gamma+1)^2|y|^2\right]^{\frac{1}{2(\gamma+1)}}.$$

Note that, there is a group of dilations $\{\delta_t\}_{t>0}$

$$\delta_t : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_t(z) = \delta_t(x, y) = (tx, t^{\gamma+1}y)$$

such that the operator $\Delta_G = \operatorname{div}_G \circ \nabla_G = \Delta_x + |x|^{2\gamma}\Delta_y$ is δ_t-homogeneous of degree two. This operator is elliptic when $|x| \neq 0$ and degenerates on $\{0\} \times \mathbb{R}^{N_2}$.

Let us first recall that, in the case $\gamma = 0$, $w = 0$, $p = 2$, the left-hand side of (1.1) becomes $-\Delta u$. The most well-known Liouville-type result for nonlinear elliptic equations involving the Laplace operator was given in the celebrated article [16], in which the authors proved that the Lane–Emden equation

$$-\Delta u = u^r \quad \text{in } \mathbb{R}^N$$

has no positive solution if and only if $r < \frac{N+2}{N-2}$. About the class of stable solutions of the Lane–Emden equation, Farina [13] obtained an optimal Liouville-type theorem; see also [30].

Next, in the case $\gamma = 0$, $w = 0$, $p \geq 2$, equation (1.1) becomes

$$-\Delta_p u = f(z)|u|^{r-1}u \quad \text{in } \mathbb{R}^N.$$

The reader is referred to [29] for the nonexistence result of positive solutions and to [7, 9] for the nonexistence result of stable solutions of this equation.

Considering (1.1) in the case $\gamma = 0$, $w \geq 0$, $q \geq p \geq 2$, the operator of the left-hand side becomes the double-phase operator, and we have the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u + w(z)|\nabla u|^{q-2}\nabla u) = f(z)|u|^{r-1}u \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

Following the ideas in the papers [9, 14], Phuong Le [24] established the nonexistence of nontrivial stable solutions in $W^{1,H}_{\mathrm{loc}}(\mathbb{R}^N)$ of (1.4) under the condition

$$N < \min\left\{\frac{p(\alpha_0 + r) + b(\alpha_0 + p - 1)}{r - p + 1}, \frac{(q - a)(\alpha_0 + r) + b(\alpha_0 + q - 1)}{r - q + 1}\right\},$$

where

$$\alpha_0 := \frac{2r - q + 1 + 2\sqrt{r(r - q + 1)}}{q - 1}.$$

We now consider the general case $\gamma \geq 0$. In recent years, there has been an increasing interest in Liouville-type theorems for elliptic equations involving the operator $\Delta_G = \operatorname{div}_G \circ \nabla_G = \Delta_x + |x|^{2\gamma}\Delta_y$; see, e.g., [11, 12, 23, 31, 33]. Currently, this operator is usually named Grushin operator. The operators of this kind were first introduced and studied by Franchi and Lanconelli [15]. Recently, they were named by Kogoj and Lanconelli [19]

Δ_λ-Laplacians, under the additional assumption that the operators are homogeneous of degree two with respect to a group of dilations; see also [1,10,20–22,25,31,32]. The operator considered by Grushin [17] is a very particular case of the Δ_λ-Laplacians; Grushin studied this operator by adding lower terms with complex coefficients; see also [3]. For the equation

$$-\Delta_G u = u^r \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

the Liouville-type theorem was proved by Monticelli [26] for nonnegative classical solutions and by Yu [33] for nonnegative weak solutions. The optimal condition on the range of the exponent is

$$r < \frac{Q+2}{Q-2},$$

where

$$Q = N_1 + (\gamma + 1)N_2 \tag{1.5}$$

is called the homogeneous dimension of $\mathbb{R}^N$ associated to the Grushin operator. The Liouville results for the nonlinear elliptic equations involving p-Laplace-type Grushin operator

$$-\operatorname{div}_G(w(z)|\nabla_G u|^{p-2}\nabla_G u) = f(z)g(u) \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

was studied in [31], where $g(u) = e^u$ or $g(u) = -u^{-p}$. However, to our best knowledge, there has not any work treating the double-phase problem (1.1) involving the Grushin operator. The purpose of this paper is to establish some nonexistence results of nontrivial stable solutions of (1.1). Notice that double-phase differential operators and their corresponding energy functionals appear in nonlinear elasticity theory, strongly anisotropic materials, Lavrentiev's phenomenon, and so on; see, e.g., [2,4,8,18,34–36].

We next recall some notations which will be used in the sequel. Let $\Omega \subset \mathbb{R}^N$ be an open domain, and let $H : \Omega \times [0,\infty) \to [0,\infty)$ be the function $(z,t) \mapsto t^p + w(z)t^q$. Put

$$\rho_H(u) = \int_\Omega H(z,|u|) = \int_\Omega (|u|^p + w(z)|u|^q)$$

and

$$L^H(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \rho_H(u) < \infty\},$$

with the norm

$$\|u\|_H = \inf\left\{\tau > 0 \mid \rho_H\left(\frac{u}{\tau}\right) \le 1\right\}.$$

Define

$$W^{1,H}(\Omega) = \{u \in L^H(\Omega) \mid |\nabla_G u| \in L^H(\Omega)\},$$

with the norm

$$\|u\|_{1,H} = \||\nabla_G u|\|_H + \|u\|_H.$$

Denote by $W_0^{1,H}(\Omega)$ the closure of $C_c^1(\Omega)$ with respect to the $\|\cdot\|_{1,H}$ norm and

$$W_{\text{loc}}^{1,H}(\Omega) = \{u : \Omega \to \mathbb{R} \mid u\varphi \in W_0^{1,H}(\Omega) \text{ for all } \varphi \in C_c^1(\Omega)\}.$$

In order to state our results, we need the following definitions.

Definition 1.1. A function $u \in W^{1,H}_{\mathrm{loc}}(\mathbb{R}^N)$ is said to be a weak solution of equation (1.1) if $f(z)|u|^r \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, and for all $\varphi \in C^1_c(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \left(|\nabla_G u|^{p-2}\nabla_G u + w(z)|\nabla_G u|^{q-2}\nabla_G u\right)\cdot\nabla_G\varphi = \int_{\mathbb{R}^N} f(z)|u|^{r-1}u\varphi. \tag{1.6}$$

Moreover, u is called a finite energy solution if

$$\int_{\mathbb{R}^N} \left(|\nabla_G u|^p + w(z)|\nabla_G u|^q + f(z)|u|^{r+1}\right) < \infty.$$

Definition 1.2. We say that a weak solution u of (1.1)

- is stable if for all $\varphi \in C^1_c(\mathbb{R}^N)$ we have

$$\begin{aligned} L_u(\varphi) := &\int_{\mathbb{R}^N} \left[|\nabla_G u|^{p-2}|\nabla_G\varphi|^2 + (p-2)|\nabla_G u|^{p-4}(\nabla_G u\cdot\nabla_G\varphi)^2\right] \\ &+ \int_{\mathbb{R}^N} w(z)\left[|\nabla_G u|^{q-2}|\nabla_G\varphi|^2 + (q-2)|\nabla_G u|^{q-4}(\nabla_G u\cdot\nabla_G\varphi)^2\right] \\ &- r\int_{\mathbb{R}^N} f(z)|u|^{r-1}\varphi^2 \geq 0, \end{aligned} \tag{1.7}$$

- is stable outside a compact set $K \subset \mathbb{R}^N$ if $L_u(\varphi) \geq 0$ only holds for $\varphi \in C^1_c(\mathbb{R}^N \setminus K)$,
- has Morse index equal to $k \geq 1$ if k is the maximal dimension of a subspace $V \subset C^1_c(\mathbb{R}^N)$ such that $L_u(\varphi) < 0$ for all $\varphi \in V \setminus \{0\}$.

Notice that the stability condition (1.7) is nothing but the fact that the second variation at u of the energy functional

$$E(u) = \int_{\mathbb{R}^N} \left(\frac{|\nabla_G u|^p}{p} + \frac{w(z)|\nabla_G u|^q}{q} - \frac{f(z)|u|^{r+1}}{r+1}\right)$$

is nonnegative. Therefore, all the local minima of the functional are stable solutions of (1.1). Besides, we also know that every finite Morse index solution is stable outside a compact set. Indeed, there exist $m_0 \geq 1$ and $V_0 := \mathrm{Span}\{\varphi_1, \ldots, \varphi_{m_0}\} \subset C^1_c(\mathbb{R}^N)$ such that $L_u(\varphi) < 0$ for any $\varphi \in V_0 \setminus \{0\}$. Therefore, $L_u(\varphi) \geq 0$ for every $\varphi \in C^1_c(\mathbb{R}^N \setminus K)$, where $K := \bigcup_{j=1}^{m_0} \mathrm{supp}(\varphi_j)$.

Our first result concerning the classification of stable weak solutions of (1.1) is as follows.

Theorem 1.3. *Let $q \geq p \geq 2, r > q-1$. Suppose that $w, f \in L^1_{\mathrm{loc}}(\mathbb{R}^N; [0,\infty))$ satisfying (1.2) and (1.3). Under the condition*

$$Q < \min\left\{\frac{p(\alpha_0+r)+b(\alpha_0+p-1)}{r-p+1}, \frac{(q-a)(\alpha_0+r)+b(\alpha_0+q-1)}{r-q+1}\right\}, \tag{1.8}$$

where

$$\alpha_0 := \frac{2r-q+1+2\sqrt{r(r-q+1)}}{q-1},$$

equation (1.1) *has only the trivial stable weak solution. Here, the homogeneous dimension* Q *is given in* (1.5).

Next, we consider a special case of f and w in (1.1), i.e.,

$$-\operatorname{div}_G(|\nabla_G u|^{p-2}\nabla_G u + |z|_G^a |\nabla_G u|^{q-2}\nabla_G u) = |z|_G^b |u|^{r-1}u \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \tag{1.9}$$

We next give some conditions ensuring the nonexistence of nontrivial finite energy solutions of (1.9). Our second result is as follows.

Theorem 1.4. *Let* $q > p \geq 2, r > q - 1$. *Suppose that* u *is a finite energy solution of* (1.9) *satisfying* $|\nabla_G u|^{p-2}\nabla_G u + |z|_G^a |\nabla_G u|^{q-2}\nabla_G u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$. *If*

$$\frac{Q+b}{r+1} > \max\left\{\frac{Q-p}{p}, \frac{Q-q+a}{q}\right\},$$

then u *must be trivial, i.e.,* $u = 0$.

Under the same assumptions in Theorem 1.4, we will prove that any weak solution, which is stable outside a compact set, is also a finite energy solution. Therefore, we obtain the following theorem.

Theorem 1.5. *Let* $q > p \geq 2, r > q - 1$. *Suppose that* u *is a weak solution of* (1.9) *which is stable outside a compact set and* $|\nabla_G u|^{p-2}\nabla_G u + |z|_G^a |\nabla_G u|^{q-2}\nabla_G u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$. *If*

$$\frac{Q+b}{r+1} > \max\left\{\frac{Q-p}{p}, \frac{Q-q+a}{q}\right\},$$

then u *must be trivial, i.e.,* $u = 0$.

In particular, our results generalize those in [24] from the Laplace operator to the Grushin operator. Our results are also extensions of those in [31] to the double-phase problem. Inspired by [11, 24, 31], our approach in this paper is also based on the energy method and a Pohozaev identity involving the Grushin operator established below.

This paper is organized as follows. In Section 2, we prove Theorem 1.3. The proof of Theorem 1.4 is given in Section 3. The last section is devoted to the proof of Theorem 1.5.

2. Proof of Theorem 1.3

We begin this section by fixing some notations. Let $\Omega_R = B_1(0, R) \times B_2(0, R^{\gamma+1})$, where $B_1(0, R) \subset \mathbb{R}^{N_1}$ and $B_2(0, R^{\gamma+1}) \subset \mathbb{R}^{N_2}$ are the Euclidean balls.

First, we prove the following a priori estimate for stable weak solutions of (1.1).

Proposition 2.1. *Let* u *be a stable weak solution of* (1.1). *Suppose that* $\alpha \geq 1$, $r > \frac{(q-1)(\alpha+1)^2}{4\alpha}$ *and* $m \geq 1$ *so that*

$$\min\left\{\frac{(mq-p)(\alpha+r)}{\alpha+p-1}, \frac{(mq-q)(\alpha+r)}{\alpha+q-1}\right\} \geq mq. \tag{2.1}$$

Then, for all $\eta \in C_c^1(\mathbb{R}^N;[0,1])$ *and* $\nabla_G \eta = 0$ *in* Ω_{R_0}*, there is a positive constant* C *depending on* p, q, r, α, m *such that*

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla_G u|^p |u|^{\alpha-1} + w(z)|\nabla_G u|^q |u|^{\alpha-1} + f(z)|u|^{\alpha+r})\eta^{qm} \\ \leq C \int_{\mathbb{R}^N\setminus\Omega_{R_0}} f(z)^{-\frac{\alpha+p-1}{r-p+1}} |\nabla_G \eta|^{\frac{p(\alpha+r)}{r-p+1}} \\ + C \int_{\mathbb{R}^N\setminus\Omega_{R_0}} w(z)^{\frac{\alpha+r}{r-q+1}} f(z)^{-\frac{\alpha+q-1}{r-q+1}} |\nabla_G \eta|^{\frac{q(\alpha+r)}{r-q+1}}.\end{aligned}$$

Proof. For each $k \in \mathbb{N}$, we define

$$a_k(t) = \begin{cases} |t|^{\frac{\alpha-1}{2}} t, & |t| < k, \\ k^{\frac{\alpha-1}{2}} t, & |t| \geq k, \end{cases} \quad \text{and} \quad b_k(t) = \begin{cases} |t|^{\alpha-1} t, & |t| < k, \\ k^{\alpha-1} t, & |t| \geq k. \end{cases}$$

By direct calculations, we have

$$a_k'(t) = \begin{cases} \frac{\alpha+1}{2} |t|^{\frac{\alpha-1}{2}}, & |t| < k, \\ k^{\frac{\alpha-1}{2}}, & |t| \geq k, \end{cases} \quad \text{and} \quad b_k'(t) = \begin{cases} \alpha |t|^{\alpha-1}, & |t| < k, \\ k^{\alpha-1}, & |t| \geq k. \end{cases}$$

It follows that

$$\begin{aligned} a_k(t)^2 = t b_k(t), \quad a_k'(t)^2 \leq \frac{(\alpha+1)^2}{4\alpha} b_k'(t), \\ |a_k(t)|^s a_k'(t)^{2-s} + |b_k(t)|^s b_k'(t)^{1-s} \leq C_{\alpha,s} |t|^{\alpha+s-1} \end{aligned} \tag{2.2}$$

for all $t \in \mathbb{R}$ and $s \geq 2$.

By density arguments, we observe that (1.6) holds for all $\varphi \in W_0^{1,H}(\mathbb{R}^N)$. Moreover, one may easily check that if $u \in W_{\text{loc}}^{1,H}(\mathbb{R}^N)$, then $a_k(u), b_k(u) \in W_{\text{loc}}^{1,H}(\mathbb{R}^N)$ for any $k \in \mathbb{N}$. This implies that $a_k(u)\psi^{\frac{\beta}{2}} \in W_0^{1,H}(\mathbb{R}^N)$ and $b_k(u)\psi^\beta \in W_0^{1,H}(\mathbb{R}^N)$ for all $\psi \in C_c^1(\mathbb{R}^N)$, $k \in \mathbb{N}$, and $\beta \geq q$.

Let $\beta \geq q$, $\varepsilon \in (0,1)$, and $\psi \in C_c^1(\mathbb{R}^N)$ satisfying $0 \leq \psi \leq 1$. Using

$$\varphi = b_k(u)\psi^\beta$$

as a test function in (1.6) gives us

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla_G u|^p b_k'(u)\psi^\beta + \beta \int_{\mathbb{R}^N} |\nabla_G u|^{p-2} b_k(u)\psi^{\beta-1} \nabla_G u \cdot \nabla_G \psi \\ &+ \int_{\mathbb{R}^N} w(z)|\nabla_G u|^q b_k'(u)\psi^\beta + \beta \int_{\mathbb{R}^N} w(z)|\nabla_G u|^{q-2} b_k(u)\psi^{\beta-1} \nabla_G u \cdot \nabla_G \psi \\ &= \int_{\mathbb{R}^N} f(z)|u|^{r-1} u b_k(u)\psi^\beta. \end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}
&-\beta \int_{\mathbb{R}^N} w(z)|\nabla_G u|^{q-2} b_k(u)\psi^{\beta-1}\nabla_G u\cdot\nabla_G\psi \\
&\le \beta \int_{\mathbb{R}^N} w(z)|\nabla_G u|^{q-1}|b_k(u)|\psi^{\beta-1}|\nabla_G\psi| \\
&\le \int_{\mathbb{R}^N} \Big\{\varepsilon\Big(w(z)^{\frac{q-1}{q}}|\nabla_G u|^{q-1} b_k'(u)^{\frac{q-1}{q}}\psi^{\frac{(q-1)\beta}{q}}\Big)^{\frac{q}{q-1}} \\
&\qquad + C_\varepsilon\Big(w(z)^{\frac{1}{q}}|b_k(u)|b_k'(u)^{\frac{1-q}{q}}\psi^{\frac{\beta-q}{q}}|\nabla_G\psi|\Big)^q\Big\} \\
&= \varepsilon \int_{\mathbb{R}^N} w(z)|\nabla_G u|^q b_k'(u)\psi^\beta \\
&\quad + C_\varepsilon \int_{\mathbb{R}^N} w(z)|b_k(u)|^q b_k'(u)^{1-q}\psi^{\beta-q}|\nabla_G\psi|^q.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&-\beta \int_{\mathbb{R}^N} |\nabla_G u|^{p-2} b_k(u)\psi^{\beta-1}\nabla_G u\cdot\nabla_G\psi \\
&\le \varepsilon \int_{\mathbb{R}^N} |\nabla_G u|^p b_k'(u)\psi^\beta + C_\varepsilon \int_{\mathbb{R}^N} |b_k(u)|^p b_k'(u)^{1-p}\psi^{\beta-p}|\nabla_G\psi|^p.
\end{aligned}$$

Employing the two estimates above and (2.2), we get that

$$\begin{aligned}
&(1-\varepsilon)\int_{\mathbb{R}^N} \big(|\nabla_G u|^p + w(z)|\nabla_G u|^q\big) b_k'(u)\psi^\beta \\
&\le \int_{\mathbb{R}^N} f(z)|u|^{r-1}u b_k(u)\psi^\beta \\
&\quad + C_\varepsilon \int_{\mathbb{R}^N} |b_k(u)|^p b_k'(u)^{1-p}\psi^{\beta-p}|\nabla_G\psi|^p \\
&\quad + C_\varepsilon \int_{\mathbb{R}^N} w(z)|b_k(u)|^q b_k'(u)^{1-q}\psi^{\beta-q}|\nabla_G\psi|^q \\
&\le \int_{\mathbb{R}^N} f(z)|u|^{r-1}u b_k(u)\psi^\beta + C_\varepsilon \int_{\mathbb{R}^N} |u|^{\alpha+p-1}\psi^{\beta-p}|\nabla_G\psi|^p \\
&\quad + C_\varepsilon \int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1}\psi^{\beta-q}|\nabla_G\psi|^q.
\end{aligned} \tag{2.3}$$

Applying Schwartz's inequality for the stability condition (1.7), we obtain

$$\begin{aligned}
&(p-1)\int_{\mathbb{R}^N} |\nabla_G u|^{p-2}|\nabla_G\varphi|^2 + (q-1)\int_{\mathbb{R}^N} w(z)|\nabla_G u|^{q-2}|\nabla_G\varphi|^2 \\
&\ge r\int_{\mathbb{R}^N} f(z)|u|^{r-1}\varphi^2
\end{aligned} \tag{2.4}$$

for all $\varphi \in C_c^1(\mathbb{R}^N)$. By density arguments, we have that (2.4) holds true for all $\varphi \in W_0^{1,H}(\mathbb{R}^N)$. Next, using $\varphi = a_k(u)\psi^{\frac{\beta}{2}} \in W_0^{1,H}(\mathbb{R}^N)$ as a test function in (2.4), we obtain

$$\begin{aligned}
&(p-1)\int_{\mathbb{R}^N} |\nabla_G u|^{p-2}\left|a_k'(u)\psi^{\frac{\beta}{2}}\nabla_G u + \frac{\beta}{2}a_k(u)\psi^{\frac{\beta-2}{2}}\nabla_G\psi\right|^2\\
&+(q-1)\int_{\mathbb{R}^N} w(z)|\nabla_G u|^{q-2}\left|a_k'(u)\psi^{\frac{\beta}{2}}\nabla_G u + \frac{\beta}{2}a_k(u)\psi^{\frac{\beta-2}{2}}\nabla_G\psi\right|^2\\
&\quad\ge r\int_{\mathbb{R}^N} f(z)|u|^{r-1}a_k(u)^2\psi^\beta.
\end{aligned}$$

Applying the inequality

$$|z_1+z_2|^2 \le (1+\delta)|z_1|^2 + C_\delta|z_2|^2 \quad \text{for } z_1, z_2 \in \mathbb{R}^N,\ \delta>0,$$

we arrive at

$$\begin{aligned}
&r\int_{\mathbb{R}^N} f(z)|u|^{r-1}a_k(u)^2\psi^\beta\\
&\le \left(p-1+\frac{\varepsilon}{2}\right)\int_{\mathbb{R}^N}|\nabla_G u|^p a_k'(u)^2\psi^\beta + A_\varepsilon\int_{\mathbb{R}^N}|\nabla_G u|^{p-2}a_k(u)^2\psi^{\beta-2}|\nabla_G\psi|^2\\
&\quad+\left(q-1+\frac{\varepsilon}{2}\right)\int_{\mathbb{R}^N} w(z)|\nabla_G u|^q a_k'(u)^2\psi^\beta\\
&\quad+B_\varepsilon\int_{\mathbb{R}^N} w(z)|\nabla_G u|^{q-2}a_k(u)^2\psi^{\beta-2}|\nabla_G\psi|^2. \qquad (2.5)
\end{aligned}$$

In the case $q>2$, employing Young's inequality, we get that

$$\begin{aligned}
&B_\varepsilon\int_{\mathbb{R}^N} w(z)|\nabla_G u|^{q-2}a_k(u)^2\psi^{\beta-2}|\nabla_G\psi|^2\\
&\le \int_{\mathbb{R}^N}\left\{\frac{\varepsilon}{2}\left(w(z)^{\frac{q-2}{q}}|\nabla_G u|^{q-2}a_k'(u)^{\frac{2(q-2)}{q}}\psi^{\frac{(q-2)\beta}{q}}\right)^{\frac{q}{q-2}}\right.\\
&\qquad\left.+C_\varepsilon\left(w(z)^{\frac{2}{q}}a_k(u)^2a_k'(u)^{\frac{2(2-q)}{q}}\psi^{\frac{2(\beta-q)}{q}}|\nabla_G\psi|^2\right)^{\frac{q}{2}}\right\}\\
&=\frac{\varepsilon}{2}\int_{\mathbb{R}^N} w(z)|\nabla_G u|^q a_k'(u)^2\psi^\beta + C_\varepsilon\int_{\mathbb{R}^N} w(z)|a_k(u)|^q a_k'(u)^{2-q}\psi^{\beta-q}|\nabla_G\psi|^q.
\end{aligned}$$

We see that this inequality is also true in the case $q=2$. By the same argument, we have

$$\begin{aligned}
&A_\varepsilon\int_{\mathbb{R}^N}|\nabla_G u|^{p-2}a_k(u)^2\psi^{\beta-2}|\nabla_G\psi|^2\\
&\le\frac{\varepsilon}{2}\int_{\mathbb{R}^N}|\nabla_G u|^p a_k'(u)^2\psi^\beta + C_\varepsilon\int_{\mathbb{R}^N}|a_k(u)|^p a_k'(u)^{2-p}\psi^{\beta-p}|\nabla_G\psi|^p.
\end{aligned}$$

Putting these two estimates back into (2.5) gives

$$
\begin{aligned}
& r \int_{\mathbb{R}^N} f(z)|u|^{r-1} a_k(u)^2 \psi^\beta \\
& \leq (p-1+\varepsilon) \int_{\mathbb{R}^N} |\nabla_G u|^p a_k'(u)^2 \psi^\beta + (q-1+\varepsilon) \int_{\mathbb{R}^N} w(z)|\nabla_G u|^q a_k'(u)^2 \psi^\beta \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} |a_k(u)|^p a_k'(u)^{2-p} \psi^{\beta-p} |\nabla_G \psi|^p \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} w(z)|a_k(u)|^q a_k'(u)^{2-q} \psi^{\beta-q} |\nabla_G \psi|^q.
\end{aligned}
$$

Using the fact that $q \geq p$ and (2.2), we derive

$$
\begin{aligned}
& r \int_{\mathbb{R}^N} f(z)|u|^{r-1} a_k(u)^2 \psi^\beta \\
& \leq (q-1+\varepsilon) \int_{\mathbb{R}^N} (|\nabla_G u|^p + w(z)|\nabla_G u|^q) a_k'(u)^2 \psi^\beta \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} |u|^{\alpha+p-1} \psi^{\beta-p} |\nabla_G \psi|^p + C_\varepsilon \int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1} \psi^{\beta-q} |\nabla_G \psi|^q \\
& \leq \frac{(q-1+\varepsilon)(\alpha+1)^2}{4\alpha} \int_{\mathbb{R}^N} (|\nabla_G u|^p + w(z)|\nabla_G u|^q) b_k'(u) \psi^\beta \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} |u|^{\alpha+p-1} \psi^{\beta-p} |\nabla_G \psi|^p + C_\varepsilon \int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1} \psi^{\beta-q} |\nabla_G \psi|^q. \qquad (2.6)
\end{aligned}
$$

Combining (2.3) and (2.6), we obtain

$$
\begin{aligned}
& r \int_{\mathbb{R}^N} f(z)|u|^{r-1} a_k(u)^2 \psi^\beta \\
& \leq \frac{(q-1+\varepsilon)(\alpha+1)^2}{4\alpha(1-\varepsilon)} \int_{\mathbb{R}^N} f(z)|u|^{r-1} u b_k(u) \psi^\beta \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} |u|^{\alpha+p-1} \psi^{\beta-p} |\nabla_G \psi|^p + C_\varepsilon \int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1} \psi^{\beta-q} |\nabla_G \psi|^q.
\end{aligned}
$$

Using (2.2) gives us

$$
\begin{aligned}
& r \int_{\mathbb{R}^N} f(z)|u|^{r-1} a_k(u)^2 \psi^\beta \\
& \leq \frac{(q-1+\varepsilon)(\alpha+1)^2}{4\alpha(1-\varepsilon)} \int_{\mathbb{R}^N} f(z)|u|^{r-1} a_k(u)^2 \psi^\beta \\
& \quad + C_\varepsilon \int_{\mathbb{R}^N} |u|^{\alpha+p-1} \psi^{\beta-p} |\nabla_G \psi|^p + C_\varepsilon \int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1} \psi^{\beta-q} |\nabla_G \psi|^q.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& D_\varepsilon \int_{\mathbb{R}^N} f(z)|u|^{r-1} a_k(u)^2 \psi^\beta \\
& \leq C_\varepsilon \int_{\mathbb{R}^N} |u|^{\alpha+p-1} \psi^{\beta-p} |\nabla_G \psi|^p + C_\varepsilon \int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1} \psi^{\beta-q} |\nabla_G \psi|^q,
\end{aligned}
$$

where

$$D_\varepsilon := r - \frac{(q-1+\varepsilon)(\alpha+1)^2}{4\alpha(1-\varepsilon)}.$$

Since $\lim_{\varepsilon\to 0^+} D_\varepsilon = r - \frac{(q-1)(\alpha+1)^2}{4\alpha} > 0$, we can fix some ε sufficiently close to zero such that $D_\varepsilon > 0$. We also fix $\beta = q$. Hence,

$$\int_{\mathbb{R}^N} f(z)|u|^{r-1}a_k(u)^2\psi^q \le C\int_{\mathbb{R}^N} |u|^{\alpha+p-1}\psi^{q-p}|\nabla_G\psi|^p + C\int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1}|\nabla_G\psi|^q.$$

Combining this with (2.3) and using (2.2), we can also bound gradient terms as follows:

$$\begin{aligned}\int_{\mathbb{R}^N}&(|\nabla_G u|^p + w(z)|\nabla_G u|^q)b_k'(u)\psi^q + \int_{\mathbb{R}^N} f(z)|u|^{r-1}a_k(u)^2\psi^q\\ &\le C\int_{\mathbb{R}^N}|u|^{\alpha+p-1}\psi^{q-p}|\nabla_G\psi|^p + C\int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1}|\nabla_G\psi|^q.\end{aligned}$$

Using Fatou's lemma when letting $k\to\infty$, we get that

$$\begin{aligned}\int_{\mathbb{R}^N}&\left(|\nabla_G u|^p|u|^{\alpha-1} + w(z)|\nabla_G u|^q|u|^{\alpha-1} + f(z)|u|^{\alpha+r}\right)\psi^q\\ &\le C\int_{\mathbb{R}^N}|u|^{\alpha+p-1}\psi^{q-p}|\nabla_G\psi|^p + C\int_{\mathbb{R}^N} w(z)|u|^{\alpha+q-1}|\nabla_G\psi|^q. \end{aligned}\tag{2.7}$$

Next, take $\psi = \eta^m$ in (2.7) and apply the Young inequality to find that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(|\nabla_G u|^p|u|^{\alpha-1} + w(z)|\nabla_G u|^q|u|^{\alpha-1} + f(z)|u|^{\alpha+r}\right)\eta^{qm}\\
&\le C\int_{\mathbb{R}^N\setminus\Omega_{R_0}}|u|^{\alpha+p-1}|\nabla_G\eta|^p\eta^{mq-p} + C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w(z)|u|^{\alpha+q-1}|\nabla_G\eta|^q\eta^{q(m-1)}\\
&\le \int_{\mathbb{R}^N\setminus\Omega_{R_0}}\left\{\frac{1}{4}\left(f(z)^{\frac{\alpha+p-1}{\alpha+r}}|u|^{\alpha+p-1}\eta^{mq-p}\right)^{\frac{\alpha+r}{\alpha+p-1}} + C\left(f(z)^{-\frac{\alpha+p-1}{\alpha+r}}|\nabla_G\eta|^p\right)^{\frac{\alpha+r}{r-p+1}}\right\}\\
&\quad+\int_{\mathbb{R}^N\setminus\Omega_{R_0}}\left\{\frac{1}{4}\left(f(z)^{\frac{\alpha+q-1}{\alpha+r}}|u|^{\alpha+q-1}\eta^{q(m-1)}\right)^{\frac{\alpha+r}{\alpha+q-1}}\right.\\
&\qquad\qquad\left.+ C\left(w(z)f(z)^{-\frac{\alpha+q-1}{\alpha+r}}|\nabla_G\eta|^q\right)^{\frac{\alpha+r}{r-q+1}}\right\}\\
&\le \frac{1}{4}\int_{\mathbb{R}^N\setminus\Omega_{R_0}} f(z)|u|^{\alpha+r}\eta^{qm} + C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} f(z)^{-\frac{\alpha+p-1}{r-p+1}}|\nabla_G\eta|^{\frac{p(\alpha+r)}{r-p+1}}\\
&\quad+\frac{1}{4}\int_{\mathbb{R}^N\setminus\Omega_{R_0}} f(z)|u|^{\alpha+r}\eta^{qm} + C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} w(z)^{\frac{\alpha+r}{r-q+1}}f(z)^{-\frac{\alpha+q-1}{r-q+1}}|\nabla_G\eta|^{\frac{q(\alpha+r)}{r-q+1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^N} & \left(|\nabla_G u|^p |u|^{\alpha-1} + w(z)|\nabla_G u|^q |u|^{\alpha-1} + f(z)|u|^{\alpha+r}\right)\eta^{qm} \\ & \leq C \int_{\mathbb{R}^N\setminus\Omega_{R_0}} f(z)^{-\frac{\alpha+p-1}{r-p+1}} |\nabla_G \eta|^{\frac{p(\alpha+r)}{r-p+1}} \\ & \quad + C \int_{\mathbb{R}^N\setminus\Omega_{R_0}} w(z)^{\frac{\alpha+r}{r-q+1}} f(z)^{-\frac{\alpha+q-1}{r-q+1}} |\nabla_G \eta|^{\frac{q(\alpha+r)}{r-q+1}}.\end{aligned}$$

The proof is complete. ∎

Let $R > 0$, $\Omega_{2R} = B_1(0, 2R) \times B_2(0, 2R^{\gamma+1})$, where

$$B_1(0, 2R) \subset \mathbb{R}^{N_1}, \quad B_2(0, 2R^{\gamma+1}) \subset \mathbb{R}^{N_2}$$

are the Euclidean balls. Define

$$\eta_{1,R}(x) = \eta_1\left(\frac{|x|}{R}\right), \quad \eta_{2,R}(y) = \eta_2\left(\frac{|y|}{R^{\gamma+1}}\right),$$

where $\eta_1, \eta_2 \in C_c^\infty([0, +\infty))$, $0 \leq \eta_1, \eta_2 \leq 1$,

$$\eta_i(t) = \begin{cases} 1 & \text{in } [0, 1], \\ 0 & \text{in } [2, +\infty), \end{cases}$$

and for some constant $C > 0$ and $\eta_{1,R}, \eta_{2,R}$ satisfy

$$|\nabla_x \eta_{1,R}| \leq \frac{C}{R}, \quad |\nabla_y \eta_{2,R}| \leq \frac{C}{R^{\gamma+1}}.$$

Lemma 2.2. *The following assertions hold true.*

(i) *There exists a constant $C > 0$ independent of R such that*

$$|\nabla_G \eta_R| \leq \frac{C}{R}, \quad \forall z \in \Omega_{2R},$$

where $\eta_R(z) = \eta_{1,R}(x)\eta_{2,R}(y)$.

(ii) *There exists a constant $C > 0$ independent of R such that if $z \in \Omega_{2R}$, then*

$$|z|_G \leq CR.$$

(iii) *If $z \notin \Omega_R$, then $|z|_G > R$.*

Proof. Proof of (i). We have

$$\nabla_G \eta_R = (\nabla_x \eta_R, |x|^\gamma \nabla_y \eta_R) = (\eta_{2,R} \nabla_x \eta_{1,R}, |x|^\gamma \eta_{1,R} \nabla_y \eta_{2,R}).$$

For any $z = (x, y) \in \Omega_{2R}$, we have $x \in B_1(0, 2R)$. This implies

$$|x| \le 2R.$$

Combining this with the hypothesis about functions $\eta_{i,R}$, $i = 1, 2$, there exists a constant $C > 0$ independent of R such that

$$|\nabla_G \eta_R|^2 = \eta_{2,R}^2 |\nabla_x \eta_{1,R}|^2 + |x|^{2\gamma} \eta_{1,R}^2 |\nabla_y \eta_{2,R}|^2 \le \frac{C}{R^2}, \quad \forall z \in \Omega_{2R}.$$

Therefore, there is a constant $C > 0$ independent of R such that

$$|\nabla_G \eta_R| \le \frac{C}{R}, \forall z \in \Omega_{2R}.$$

Proof of (ii). For any $z = (x, y) \in \Omega_{2R}$, we have $x \in B_1(0, 2R)$, $y \in B_2(0, 2R^{\gamma+1})$. This implies

$$|x| \le 2R \quad \text{and} \quad |y| \le 2R^{\gamma+1}.$$

Then, we get

$$\begin{aligned} |z|_G &= \left[|x|^{2(\gamma+1)} + (\gamma+1)^2 |y|^2\right]^{\frac{1}{2(\gamma+1)}} \\ &\le \left[(2R)^{2(\gamma+1)} + (\gamma+1)^2 (2R^{\gamma+1})^2\right]^{\frac{1}{2(\gamma+1)}}. \end{aligned}$$

By direct calculation, we obtain

$$|z|_G \le CR,$$

where C is a positive constant independent of R.

Proof of (iii). For any $z = (x, y) \notin \Omega_R$, we have $x \notin B_1(0, R)$, $y \notin B_2(0, R^{\gamma+1})$. This implies

$$|x| > R \quad \text{and} \quad |y| > R^{\gamma+1}.$$

We deduce that

$$|z|_G > \left[R^{2(\gamma+1)} + (\gamma+1)^2 (R^{\gamma+1})^2\right]^{\frac{1}{2(\gamma+1)}}.$$

By direct calculation, we have $|z|_G > R$. ∎

Proof Theorem 1.3. Observe that

$$\alpha \ge 1 \quad \text{and} \quad r > \frac{(q-1)(\alpha+1)^2}{4\alpha}$$

are equivalent to $1 \le \alpha < \alpha_0$. Besides, it follows from the assumption $r > q - 1 \ge p - 1$ that there is $m \ge 1$ satisfying (2.1). Clearly, the standard cut-off function η_R is chosen as in Lemma 2.2 satisfying $\eta_R \in C_c^1(\mathbb{R}^N)$, $0 \le \eta_R \le 1$ in $\mathbb{R}^N$ and

$$\begin{cases} \eta_R = 1 & \text{in } \Omega_R, \\ \eta_R = 0 & \text{in } \mathbb{R}^N \setminus \Omega_{2R}, \\ |\nabla_G \eta_R| \le \frac{C}{R} & \text{in } \Omega_{2R} \setminus \Omega_R, \end{cases}$$

where $R > R_0$. For all $R > R_0$, applying the Proposition 2.1 with $\alpha \in [1, \alpha_0)$ and the function η_R as in Lemma 2.2, there is a constant C independent of R such that

$$\begin{aligned}\int_{\Omega_R} &\left(|\nabla_G u|^p |u|^{\alpha-1} + w(z)|\nabla_G u|^q |u|^{\alpha-1} + f(z)|u|^{\alpha+r}\right) \\ &\leq C \int_{\Omega_{2R}\setminus\Omega_R} f(z)^{-\frac{\alpha+p-1}{r-p+1}} |\nabla_G \eta_R|^{\frac{p(\alpha+r)}{r-p+1}} \\ &\quad + C \int_{\Omega_{2R}\setminus\Omega_R} w(z)^{\frac{\alpha+r}{r-q+1}} f(z)^{-\frac{\alpha+q-1}{r-q+1}} |\nabla_G \eta_R|^{\frac{q(\alpha+r)}{r-q+1}}.\end{aligned}$$

Since $|z|_G > R > R_0$, $\forall z \notin \Omega_R$, and using the hypothesis about $w(z)$, $f(z)$, η_R, we obtain

$$\begin{aligned}\int_{\Omega_R} &\left(|\nabla_G u|^p |u|^{\alpha-1} + w(z)|\nabla_G u|^q |u|^{\alpha-1} + f(z)|u|^{\alpha+r}\right) \\ &\leq C \int_{\Omega_{2R}\setminus\Omega_R} |z|_G^{-\frac{b(\alpha+p-1)}{r-p+1}} R^{-\frac{p(\alpha+r)}{r-p+1}} + C \int_{\Omega_{2R}\setminus\Omega_R} |z|_G^{\frac{a(\alpha+r)}{r-q+1}} |z|_G^{-\frac{b(\alpha+q-1)}{r-q+1}} R^{-\frac{q(\alpha+r)}{r-q+1}}.\end{aligned}$$

Combining this with $|z|_G > R$, $\forall z \notin \Omega_R$ and $|z|_G \leq CR$, $\forall z \in \Omega_{2R}$, we have

$$\int_{\Omega_R} \left(|\nabla_G u|^p |u|^{\alpha-1} + w(z)|\nabla_G u|^q |u|^{\alpha-1} + f(z)|u|^{\alpha+r}\right) \leq CR^{\theta}, \tag{2.8}$$

where

$$\theta = Q - \min\left\{\frac{p(\alpha+r) + b(\alpha+p-1)}{r-p+1}, \frac{(q-a)(\alpha+r) + b(\alpha+q-1)}{r-q+1}\right\}.$$

By assumption (1.8), we may choose α sufficiently close to α_0 such that $\theta < 0$. Then, by letting $R \to \infty$ in (2.8), we get the conclusion of the theorem. ∎

3. Proof of Theorem 1.4

We begin this section with the following Pohozaev-type identity involving the Grushin operator.

Proposition 3.1. *Suppose that u is a finite energy solution of equation (1.9) and*

$$|\nabla_G u|^{p-2}\nabla_G u + |z|_G^a |\nabla_G u|^{q-2}\nabla_G u \in W^{1,2}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N).$$

Then, we have

$$\frac{Q-p}{p}\int_{\mathbb{R}^N} |\nabla_G u|^p + \frac{Q-q+a}{q}\int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q = \frac{Q+b}{r+1}\int_{\mathbb{R}^N} |z|_G^b |u|^{r+1}.$$

Proof. By density arguments and the regularity provided, we may use $\nu_R = \eta_R z^* \cdot \nabla_G u$ as a test function in (1.6), where η_R is chosen as in Lemma 2.2 and $z^* = \left(x, \frac{\gamma+1}{|x|^\gamma} y\right)$, we have

$$\int_{\mathbb{R}^N} \left(|\nabla_G u|^{p-2}\nabla_G u \cdot \nabla_G \nu_R + |z|_G^a |\nabla_G u|^{q-2}\nabla_G u \cdot \nabla_G \nu_R\right) = \int_{\mathbb{R}^N} |z|_G^b |u|^{r-1} u \nu_R. \tag{3.1}$$

Using the divergence theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla_G u|^{p-2}\nabla_G u \cdot \nabla_G \nu_R \\
&= \int_{\mathbb{R}^N} |\nabla_G u|^{p-2}(\nabla_G u \cdot \nabla_G \eta_R)(z^* \cdot \nabla_G u) \\
&\quad + \int_{\mathbb{R}^N} |\nabla_G u|^p \eta_R + \int_{\mathbb{R}^N} \eta_R z^* \cdot \nabla_G \left(\frac{|\nabla_G u|^p}{p}\right) \\
&= \int_{\mathbb{R}^N} |\nabla_G u|^{p-2}(\nabla_G u \cdot \nabla_G \eta_R)(z^* \cdot \nabla_G u) \\
&\quad + \int_{\mathbb{R}^N} |\nabla_G u|^p \eta_R - \int_{\mathbb{R}^N} \nabla_G \eta_R \cdot z^* \frac{|\nabla_G u|^p}{p} - \int_{\mathbb{R}^N} \eta_R (\nabla_G \cdot z^*) \frac{|\nabla_G u|^p}{p} \\
&= \int_{\mathbb{R}^N} |\nabla_G u|^{p-2}(\nabla_G u \cdot \nabla_G \eta_R)(z^* \cdot \nabla_G u) \\
&\quad + \int_{\mathbb{R}^N} |\nabla_G u|^p \eta_R - \int_{\mathbb{R}^N} \nabla_G \eta_R \cdot z^* \frac{|\nabla_G u|^p}{p} - Q \int_{\mathbb{R}^N} \eta_R \frac{|\nabla_G u|^p}{p}.
\end{aligned}$$

It follows from $\int_{\mathbb{R}^N} |\nabla_G u|^p < \infty$ and the dominated convergence theorem that

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} |\nabla_G u|^{p-2}\nabla_G u \cdot \nabla_G \nu_R = -\frac{Q-p}{p} \int_{\mathbb{R}^N} |\nabla_G u|^p. \tag{3.2}$$

Similarly,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^{q-2}\nabla_G u \cdot \nabla_G \nu_R \\
&= \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^{q-2}(\nabla_G u \cdot \nabla_G \eta_R)(z^* \cdot \nabla_G u) \\
&\quad + \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q \eta_R + \int_{\mathbb{R}^N} |z|_G^a \eta_R z^* \cdot \nabla_G \left(\frac{|\nabla_G u|^q}{q}\right) \\
&= \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^{q-2}(\nabla_G u \cdot \nabla_G \eta_R)(z^* \cdot \nabla_G u) + \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q \eta_R \\
&\quad - a \int_{\mathbb{R}^N} |z|_G^a \eta_R \frac{|\nabla_G u|^q}{q} - \int_{\mathbb{R}^N} |z|_G^a \nabla_G \eta_R \cdot z^* \frac{|\nabla_G u|^q}{q} - Q \int_{\mathbb{R}^N} |z|_G^a \eta_R \frac{|\nabla_G u|^q}{q}.
\end{aligned}$$

It follows from $\int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q < \infty$ and the dominated convergence theorem that

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^{q-2}\nabla_G u \cdot \nabla_G \nu_R = -\frac{Q-q+a}{q} \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q. \tag{3.3}$$

Now, we take $v_R = \eta_R z^* \cdot \nabla_G u$ on the right-hand side of (3.1) and employing the divergence theorem to find

$$\begin{aligned}
&\int_{\mathbb{R}^N} |z|_G^b |u|^{r-1} u v_R \\
&\quad = \int_{\mathbb{R}^N} |z|_G^b |u|^{r-1} u \eta_R z^* \cdot \nabla_G u \\
&\quad = \int_{\mathbb{R}^N} |z|_G^b \eta_R z^* \cdot \nabla_G \left(\frac{|u|^{r+1}}{r+1} \right) \\
&\quad = -\int_{\mathbb{R}^N} \nabla_G(|z|_G^b) \cdot z^* \eta_R \frac{|u|^{r+1}}{r+1} - \int_{\mathbb{R}^N} |z|_G^b \nabla_G \eta_R \cdot z^* \frac{|u|^{r+1}}{r+1} \\
&\qquad - \int_{\mathbb{R}^N} |z|_G^b \eta_R (\nabla_G \cdot z^*) \frac{|u|^{r+1}}{r+1} \\
&\quad = -b \int_{\mathbb{R}^N} |z|_G^b \eta_R \frac{|u|^{r+1}}{r+1} - \int_{\mathbb{R}^N} |z|_G^b \nabla_G \eta_R \cdot z^* \frac{|u|^{r+1}}{r+1} - Q \int_{\mathbb{R}^N} |z|_G^b \eta_R \frac{|u|^{r+1}}{r+1}.
\end{aligned}$$

Since $\int_{\mathbb{R}^N} |z|_G^b |u|^{r+1} < \infty$, we deduce from the dominated convergence theorem that

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} |z|_G^b |u|^{r-1} u v_R = -\frac{Q+b}{r+1} \int_{\mathbb{R}^N} |z|_G^b |u|^{r+1}. \tag{3.4}$$

The conclusion of the proposition follows from (3.1)–(3.4). ∎

Proof of Theorem 1.4. To prove Theorem 1.4, first, we prove the following identity:

$$\int_{\mathbb{R}^N} |\nabla_G u|^p + |z|_G^a |\nabla_G u|^q = \int_{\mathbb{R}^N} |z|_G^b |u|^{r+1}.$$

Indeed, using $\varphi = u\eta_R$ as a test function in (1.6), where η_R is chosen as in Lemma 2.2, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla_G u|^p \eta_R + \int_{\mathbb{R}^N} |\nabla_G u|^{p-2} u \nabla_G u \cdot \nabla_G \eta_R \\
&+ \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q \eta_R + \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^{q-2} u \nabla_G u \cdot \nabla_G \eta_R \\
&\quad = \int_{\mathbb{R}^N} |z|_G^b |u|^{r+1} \eta_R.
\end{aligned} \tag{3.5}$$

By means of Holder's inequality, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} |\nabla_G u|^{p-2} u \nabla_G u \cdot \nabla_G \eta_R \right| \\
&\quad \le \int_{\mathbb{R}^N} |\nabla_G u|^{p-1} |u| |\nabla_G \eta_R| \\
&\quad \le \left(\int_{\mathbb{R}^N} |\nabla_G u|^p \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |z|_G^b |u|^{r+1} \right)^{\frac{1}{r+1}} \left(\int_{\mathbb{R}^N} |z|_G^{-\frac{bp}{r-p+1}} |\nabla_G \eta_R|^{\frac{p(r+1)}{r-p+1}} \right)^{\frac{r-p+1}{p(r+1)}}.
\end{aligned}$$

Using the hypothesis about η_R and Lemma 2.2, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} |\nabla_G u|^{p-2} u \nabla_G u \cdot \nabla_G \eta_R \right| \\
&\leq C \left(\int_{\mathbb{R}^N} |\nabla_G u|^p \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |z|_G^b |u|^{r+1} \right)^{\frac{1}{r+1}} \left(\int_{\Omega_{2R} \setminus \Omega_R} |z|_G^{-\frac{bp}{r-p+1}} R^{-\frac{p(r+1)}{r-p+1}} \right)^{\frac{r-p+1}{p(r+1)}} \\
&\leq C R^{\frac{Q(r-p+1)}{p(r+1)} - \frac{r+b+1}{r+1}} \left(\int_{\mathbb{R}^N} |\nabla_G u|^p \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |z|_G^b |u|^{r+1} \right)^{\frac{1}{r+1}}.
\end{aligned}$$

Using the condition $\frac{Q+b}{r+1} > \frac{Q-p}{p}$, we see that $\frac{Q(r-p+1)}{p(r+1)} - \frac{r+b+1}{r+1} < 0$. Therefore, by letting $R \to \infty$, we get that

$$\lim_{R \to \infty} \int_{\mathbb{R}^N} |\nabla_G u|^{p-2} u \nabla_G u \cdot \nabla_G \eta_R = 0. \tag{3.6}$$

In the same way, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^{q-2} u \nabla_G u \cdot \nabla_G \eta_R \right| \\
&\qquad \leq \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^{q-1} |u| |\nabla_G \eta_R| \\
&\qquad \leq \left(\int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^N} |z|_G^b |u|^{r+1} \right)^{\frac{1}{r+1}} \\
&\qquad\quad \times \left(\int_{\mathbb{R}^N} |z|_G^{\frac{a(r+1)}{r-q+1}} |z|_G^{-\frac{bq}{r-q+1}} |\nabla_G \eta_R|^{\frac{q(r+1)}{r-q+1}} \right)^{\frac{r-q+1}{q(r+1)}} \\
&\qquad \leq C R^{\frac{Q(r-q+1)}{q(r+1)} + \frac{a}{q} - \frac{r+b+1}{r+1}} \left(\int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^N} |z|_G^b |u|^{r+1} \right)^{\frac{1}{r+1}}.
\end{aligned}$$

By $\frac{Q+b}{r+1} > \frac{Q-q+a}{q}$, there holds $\frac{Q(r-q+1)}{q(r+1)} + \frac{a}{q} - \frac{r+b+1}{r+1} < 0$. Therefore, letting $R \to \infty$, we get that

$$\lim_{R \to \infty} \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^{q-2} u \nabla_G u \cdot \nabla_G \eta_R = 0. \tag{3.7}$$

Combining (3.5)–(3.7) leads to

$$\int_{\mathbb{R}^N} (|\nabla_G u|^p + |z|_G^a |\nabla_G u|^q) = \int_{\mathbb{R}^N} |z|_G^b |u|^{r+1}.$$

Substituting this identity into Proposition 3.1, we obtain

$$\left(\frac{Q+b}{r+1} - \frac{Q-p}{p} \right) \int_{\mathbb{R}^N} |\nabla_G u|^p + \left(\frac{Q+b}{r+1} - \frac{Q-q+a}{q} \right) \int_{\mathbb{R}^N} |z|_G^a |\nabla_G u|^q = 0.$$

Since $\frac{Q+b}{r+1} > \max\{\frac{Q-p}{p}, \frac{Q-q+a}{q}\}$, then it follows that u is constant and hence must be zero. ■

4. Proof of Theorem 1.5

The following a priori estimate holds for weak solutions which are stable outside a compact set.

Proposition 4.1. *Let u be a weak solution of (1.1) which is stable outside a compact set $K \subset \mathbb{R}^N$. Suppose that $\alpha \geq 1$, $r > \frac{(q-1)(\alpha+1)^2}{4\alpha}$, and $m \geq 1$ verifying*

$$\min\left\{\frac{(mq-p)(\alpha+r)}{\alpha+p-1}, \frac{(mq-q)(\alpha+r)}{\alpha+q-1}\right\} \geq mq.$$

Then, for all $\eta \in C_c^1(\mathbb{R}^N \setminus (K \cup \Omega_{R_0}); [0,1])$ and $\nabla_G \eta = 0$ in Ω_{R_0}, there is a constant C depending on p, q, r, α, m such that

$$\begin{aligned}\int_{\mathbb{R}^N} &\left(|\nabla_G u|^p |u|^{\alpha-1} + w(z)|\nabla_G u|^q |u|^{\alpha-1} + f(z)|u|^{\alpha+r}\right)\eta^{qm} \\ &\leq C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} f(z)^{-\frac{\alpha+p-1}{r-p+1}} |\nabla_G \eta|^{\frac{p(\alpha+r)}{r-p+1}} \\ &\quad + C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} w(z)^{\frac{\alpha+r}{r-q+1}} f(z)^{-\frac{\alpha+q-1}{r-q+1}} |\nabla_G \eta|^{\frac{q(\alpha+r)}{r-q+1}}.\end{aligned}$$

The proof of Proposition 4.1 follows closely to that of Proposition 2.1, and so, it will be omitted.

Proof of Theorem 1.5. Let u be a weak solution of (1.9) which is stable outside a compact set K. We fix $R_1 > R_0$ so that $K \subset \Omega_{R_1}$. For any $R > R_1 + 1$, we consider $\xi_R \in C_c^1(\mathbb{R}^N)$ such that $0 \leq \xi_R \leq 1$ in $\mathbb{R}^N$ and

$$\xi_R = \begin{cases} 0 & \text{in } \Omega_{R_1}, \\ 1 & \text{in } \Omega_R \setminus \Omega_{R_1+1}, \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega_{2R}. \end{cases}$$

We may assume furthermore that $|\nabla_G \xi_R| \leq C$ in $\Omega_{R_1+1} \setminus \Omega_{R_1}$ and $|\nabla_G \xi_R| \leq \frac{C}{R}$ in $\Omega_{2R} \setminus \Omega_R$, where C is independent of R. Then, applying Proposition 4.1 with $\alpha = 1$, $w(z) = |z|_G^a$, $f(z) = |z|_G^b$, and $\eta = \xi_R$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} &\left(|\nabla_G u|^p + |z|_G^a |\nabla_G u|^q + |z|_G^b |u|^{r+1}\right)\xi_R^{qm} \\ &\leq C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} |z|_G^{-\frac{bp}{r-p+1}} |\nabla_G \xi_R|^{\frac{p(r+1)}{r-p+1}} \\ &\quad + C \int_{\mathbb{R}^N \setminus \Omega_{R_0}} |z|_G^{\frac{a(r+1)}{r-q+1}} |z|_G^{-\frac{bq}{r-q+1}} |\nabla_G \xi_R|^{\frac{q(r+1)}{r-q+1}}.\end{aligned} \tag{4.1}$$

Using the hypothesis about ξ_R, we have

$$\int_{\Omega_R\setminus\Omega_{R_1+1}} \big(|\nabla_G u|^p + |z|_G^a|\nabla_G u|^q + |z|_G^b|u|^{r+1}\big) \\ \le \int_{\mathbb{R}^N} \big(|\nabla_G u|^p + |z|_G^a|\nabla_G u|^q + |z|_G^b|u|^{r+1}\big)\xi_R^{qm}.$$

We also obtain

$$C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} |z|_G^{-\frac{bp}{r-p+1}}|\nabla_G\xi_R|^{\frac{p(r+1)}{r-p+1}} \\ = C\int_{\Omega_{R_1+1}\setminus\Omega_{R_1}} |z|_G^{-\frac{bp}{r-p+1}}|\nabla_G\xi_R|^{\frac{p(r+1)}{r-p+1}} + C\int_{\Omega_{2R}\setminus\Omega_R} |z|_G^{-\frac{bp}{r-p+1}}|\nabla_G\xi_R|^{\frac{p(r+1)}{r-p+1}} \\ \le C_0 + C\int_{\Omega_{2R}\setminus\Omega_R} |z|_G^{-\frac{bp}{r-p+1}}|\nabla_G\xi_R|^{\frac{p(r+1)}{r-p+1}} \\ \le C_0 + C_1 R^{Q-\frac{p(r+b+1)}{r-p+1}}.$$

Similarly,

$$C\int_{\mathbb{R}^N\setminus\Omega_{R_0}} |z|_G^{\frac{a(r+1)}{r-q+1}}|z|_G^{-\frac{bq}{r-q+1}}|\nabla_G\xi_R|^{\frac{q(r+1)}{r-q+1}} \le C_0 + C_2 R^{Q-\frac{q(r+b+1)-a(r+1)}{r-q+1}}.$$

Substituting these three estimates into (4.1), we have

$$\int_{\Omega_R\setminus\Omega_{R_1+1}} \big(|\nabla_G u|^p + |z|_G^a|\nabla_G u|^q + |z|_G^b|u|^{r+1}\big) \\ \le C_0 + C_1 R^{Q-\frac{p(r+b+1)}{r-p+1}} + C_2 R^{Q-\frac{q(r+b+1)-a(r+1)}{r-q+1}}. \tag{4.2}$$

The condition

$$\frac{Q+b}{r+1} > \max\left\{\frac{Q-p}{p}, \frac{Q-q+a}{q}\right\}$$

implies that

$$Q < \frac{p(r+b+1)}{r-p+1}$$

and

$$Q < \frac{q(r+b+1)-a(r+1)}{r-q+1}.$$

Therefore, letting $R\to\infty$ in (4.2), we receive

$$|\nabla_G u|^p + |z|_G^a|\nabla_G u|^q + |z|_G^b|u|^{r+1} \in L^1(\mathbb{R}^N).$$

This and Theorem 1.4 complete the proof. ∎

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