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Standard λ-lattices, rigid C* tensor categories, and (bi)modules

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Abstract. In this article, we construct a 2-shaded rigid C* multitensor category with canonical unitary dual functor directly from a standard λ-lattice. We use the notions of traceless Markov towers and lattices to define the notion of module and bimodule over standard λ-lattice(s), and we explicitly construct the associated module category and bimodule category over the corresponding 2-shaded rigid C* multitensor category.

As an example, we compute the modules and bimodules for Temperley–Lieb–Jones standard λ-lattices in terms of traceless Markov towers and lattices. Translating into the unitary 2-category of bigraded Hilbert spaces, we recover De Commer–Yamashita's classification of $\mathcal{TLJ}$ module categories in terms of edge weighted graphs, and a classification of $\mathcal{TLJ}$ bimodule categories in terms of biunitary connections on square-partite weighted graphs.

As an application, we show that every (infinite depth) subfactor planar algebra embeds into the bipartite graph planar algebra of its principal graph.

1. Introduction

Since Jones' landmark article [22], the modern theory of subfactors has developed deep connections to numerous branches of mathematics, including representation theory, category theory, knot theory, topological quantum field theory, statistical mechanics, conformal field theory, and free probability. The standard invariant of a type II_1 subfactor was first defined as a standard λ-lattice [38]. Since it has been reinterpreted as a planar algebra [24] and a Q-system [30], or unitary Frobenius algebra object, in a rigid C* tensor category [35].

The following theorem is a well-known folklore result. It is for instance mentioned in this form in [1, Rem. 2.1]. A similar result with planar algebras in place of tensor categories was announced in [27]. The folklore proof of this result makes use of Popa's subfactor reconstruction theorem [38, Thm. 3.1]. (Similarly, for a given standard λ-lattice, Jones proved in [24, Thm. 4.2.1] that one can construct a subfactor planar algebra by passing through Popa's subfactor reconstruction theorem [38, Thm. 3.1].) One primary motivation of this paper is to give a direct argument without making a detour via subfactors.

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Theorem (Folklore). *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{array}{l}\text{Standard } \lambda\text{-lattices} \\ A = (A_{i,j})_{0\le i\le j}\end{array}\right\} \cong \left\{\begin{array}{l}\text{Pairs } (\mathcal{A}, X) \text{ with } \mathcal{A} \text{ a 2-shaded rigid C}^* \text{ multiten-} \\ \text{sor category with a generator } X\text{, i.e., } 1_{\mathcal{A}} = 1^+ \oplus 1^-, \\ 1^+, 1^- \text{ are simple and } X = 1^+ \otimes X \otimes 1^-\end{array}\right\}.$$

Equivalence on the left-hand side is unital $*$-isomorphism of standard λ-lattices; equivalence on the right-hand side is unitary equivalence between their Cauchy completions which maps generator to generator.

Given $(\mathcal{A}, X)$, it is well known that one can obtain a standard λ-lattice A by

$$A_{i,j} := \begin{cases} \mathrm{id}_{X^{\mathrm{alt}\otimes 2k}} \otimes \mathrm{End}(X^{\mathrm{alt}\otimes(j-2k)}) & i = 2k, \\ \mathrm{id}_{X^{\mathrm{alt}\otimes(2k+1)}} \otimes \mathrm{End}(\overline{X}^{\mathrm{alt}\otimes(j-2k-1)}) & i = 2k+1, \end{cases}$$

where $\overline{X}$ is a dual of X and

$$X^{\mathrm{alt}\otimes n} := \underbrace{X \otimes \overline{X} \otimes X \otimes \cdots}_{n \text{ tensorands}}$$

and similarly for $\overline{X}^{\mathrm{alt}\otimes n}$. The inclusion $A_{i,j} \subset A_{i,j+1}$ sends x to $x \otimes \mathrm{id}$, the inclusion $A_{i+1,j} \subset A_{i,j}$ sends x to x. The Jones projections are defined using the canonical balanced evaluation and coevaluation for X.

Going the other way directly is harder. Using [7, Def. 3.1], we construct a skeletal (when $d > 1$) W*-category explicitly from A whose objects are $[n, \pm]$ for $n \ge 0$ and whose hom spaces can be identified with the algebras $A_{i,j}$. We endow it with a tensor structure using the 2-shift map in the standard λ-lattice, which is a trace-preserving $*$-isomorphism $S_{i,j} : A_{i,j} \to A_{i+2,j+2}$ [4, Cor. 2.8]. We call this skeletal category a *planar tensor category*, and we provide a string diagram calculus to perform computations. The Cauchy completion of this planar tensor category is the target 2-shaded rigid C* multitensor category.

Given a standard λ-lattice A, an A-module is a Markov tower as a standard A-module. In more detail, let $A = (A_{i,j})_{0\le i\le j<\infty}$ be a standard λ-lattice with Jones projection $\{e_i\}_{i\ge 1}$ and compatible conditional expectations. An A-module is a *Markov tower* of finite dimensional von Neumann algebras $(M_n)_{n\ge 0}$ such that $A_{0,n} \subset M_n$ together with conditional expectations $E_i : M_i \to M_{i-1}$ implemented by the Jones projections, which satisfy the appropriate commuting square conditions.

$$\begin{array}{ccccccccccc} M_0 & \subset & M_1 & \subset & M_2 & \subset & \cdots & \subset & M_n & \subset & \cdots \\ \cup & & \cup & & \cup & & & & \cup & & \\ A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & A_{0,n} & \subset & \cdots \\ & & \cup & & \cup & & & & \cup & & \\ & & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & A_{1,n} & \subset & \cdots \end{array}$$

We refer the reader to Definition 2.3 below for the complete definition.

We warn the reader that our definition is slightly different from the original one from [7, Def. 3.1]; our tower of algebras $(M_n)_{n\geq 0}$ does not necessarily have a Markov trace. An important difference in our construction is that we do *not* use the trace, but rather the commuting square of conditional expectations. In Section 3.3, by using this technique, we are able to discuss arbitrary modules over a standard λ-lattice instead of merely pivotal modules.

We call an A-module *standard* if $[M_i, A_{k,l}] = 0$ for $i \leq k \leq l$. Similar techniques used in our new proof of the Folklore theorem above, we obtain the following theorem.

Theorem A. *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{matrix}\text{Traceless Markov towers } M = \\ (M_i)_{i\geq 0} \text{ with } \dim(M_0) = 1 \text{ as} \\ \text{standard right modules over a} \\ \text{standard } \lambda\text{-lattice } A\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable} \\ \text{semisimple right } \mathcal{A}\text{-module C}^* \text{ category} \\ \text{together with a choice of simple object} \\ Z = Z \lhd 1_{\mathcal{A}}^{+}\end{matrix}\right\}.$$

Equivalence on the left-hand side is $*$-isomorphism of traceless Markov towers as standard A-modules; equivalence on the right-hand side is unitary $\mathcal{A}$-module equivalence on Cauchy completions which maps the simple base object to simple base object.

Tracial Markov towers as standard A-modules correspond to pivotal $\mathcal{A}$-module categories.

In Section 4, we discuss bimodules. Given two standard λ-lattices A and B, we define an A-B bimodule as a *standard Markov lattice*, which consists of a doubly indexed sequence $M = (M_{i,j})_{i,j\geq 0}$ of finite dimensional von Neumann algebras with two sequences of Jones projections $(e_i)_{i\geq 1}$ and $(f_j)_{j\geq 1}$ where the following conditions hold.

(a) $M_{i,j} \subset M_{i,j+1}$ and $M_{i,j} \subset M_{i+1,j}$ are unital inclusions.

(b) $M_{-,j} = (M_{i,j}, E_{i,j}^{M,l}, e_{i+1})_{i\geq 0}$ are Markov towers with the same modulus d_0 and $e_i \in M_{i+1,j}$ for all i; $M_{i,-} = (M_{i,j}, E_{i,j}^{M,r}, f_{j+1})_{j\geq 0}$ are Markov towers with the same modulus d_1 and $f_j \in M_{i,j+1}$ for all j. We call M of modulus (d_0, d_1).

$$\begin{matrix} M_{i+1,j} & \subset & M_{i+1,j+1} \\ \cup & & \cup \\ M_{i,j} & \subset & M_{i,j+1} \end{matrix}$$

(c) The commuting square condition:

$$\begin{tikzcd} M_{i+1,j} \arrow[d, "E_{i+1,j}^{M,l}"'] & M_{i+1,j+1} \arrow[l, "E_{i+1,j+1}^{M,r}"'] \arrow[d, "E_{i+1,j+1}^{M,l}"] \\ M_{i,j} & M_{i,j+1} \arrow[l, "E_{i,j+1}^{M,r}"] \end{tikzcd}$$

is a commuting square, i.e., $E_{i,j+1}^{M,r} \circ E_{i,j}^{M,l} = E_{i,j+1}^{M,l} \circ E_{i+1,j+1}^{M,r}$.

We require $A^{\mathrm{op}}_{i,0} \subset M_{i,0}$ and $B_{0,j} \subset M_{0,j}$ for all i, j with conditional expectations satisfying the appropriate commuting square conditions. Here, we take the *opposite* λ-lattice A^{op} of A, where $A^{\mathrm{op}}_{i,j}$ is the opposite algebra of $A_{i,j}$, so the indices for A and B are transposed.

$$
\begin{array}{ccccccccccccc}
\cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
A_{3,1} & \subset & A_{3,0} & \subset & M_{3,0} & \subset & M_{3,1} & \subset & M_{3,2} & \subset & M_{3,3} & \subset & \\
\cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
A_{2,1} & \subset & A_{2,0} & \subset & M_{2,0} & \subset & M_{2,1} & \subset & M_{2,2} & \subset & M_{2,3} & \subset & \\
\cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
A_{1,1} & \subset & A_{1,0} & \subset & M_{1,0} & \subset & M_{1,1} & \subset & M_{1,2} & \subset & M_{1,3} & \subset & \\
 & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
 & & A_{0,0} & \subset & M_{0,0} & \subset & M_{0,1} & \subset & M_{0,2} & \subset & M_{0,3} & \subset & \\
 & & & & \cup & & \cup & & \cup & & \cup & & \\
 & & & & B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & B_{0,3} & \subset & \\
 & & & & & & \cup & & \cup & & \cup & & \\
 & & & & & & B_{1,1} & \subset & B_{1,2} & \subset & B_{1,3} & \subset &
\end{array}
$$

We call an A-B bimodule *standard* if $[M_{i,j}, A_{p,q}] = 0$ for $i \le q \le p$; $[M_{i,j}, B_{k,l}] = 0$, for $j \le k \le l$. Similar to the proof of the Folklore theorem and Theorem A above, we obtain the following theorem.

Theorem B. *There is a bijective correspondence between equivalence classes of the following:*

$$
\left\{\begin{array}{l}
\text{Traceless Markov lattices } M = \\
(M_{i,j})_{i,j\ge 0} \text{ with } \dim(M_{0,0}) = 1 \\
\text{as standard } A\text{-}B \text{ bimodules over} \\
\text{standard } \lambda\text{-lattices } A, B
\end{array}\right\}
\cong
\left\{\begin{array}{l}
\text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable} \\
\text{semisimple C}^* \ \mathcal{A}\text{-}\mathcal{B} \text{ bimodule category} \\
\text{together with a choice of simple object} \\
Z = 1^+_{\mathcal{A}} \triangleright Z \triangleleft 1^+_{\mathcal{B}}
\end{array}\right\}.
$$

Equivalence on the left-hand side is $*$-isomorphism on the traceless Markov lattice as a standard A-B bimodule; equivalence on the right-hand side is unitary $\mathcal{A}$-$\mathcal{B}$ bimodule equivalence between their Cauchy completions which maps the simple base object to simple base object.

Tracial Markov lattices as standard A-B bimodules correspond to pivotal $\mathcal{A}$-$\mathcal{B}$ bimodule categories.

Examples. As a natural corollary from Theorem A, a Markov tower corresponds to a Temperley–Lieb–Jones ($\mathcal{TLJ}$) module category. This result generalizes the pivotal module case from [7, Thm. A.]. To translate our classification into that of [10] which uses fair and balanced graphs, we obtain an elegant graphical version of a Markov tower using a W* 2-subcategory $\mathcal{C}(\Lambda, \omega)$ of bigraded Hilbert spaces $\mathsf{BigHilb}$ which is built from a fair and balanced graph (Λ, ω). Our approach is inspired by Ocneanu's path algebras [12, 21, 36].

The following diagram shows how these notions are related to each other in Section 5:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Markov tower}\\ M \text{ with modulus } d\end{array}} \arrow[r, "\text{Section 5.6}"] \arrow[d, leftrightarrow, "\text{Section 3}"'] & \boxed{\begin{array}{c}\text{balanced } d\text{-fair}\\ \text{bipartite graph } (\Lambda,\omega)\end{array}} \arrow[d, leftrightarrow, "\text{Section 5.3}"] \\
\boxed{\begin{array}{c}\text{indecomposable semisimple C}^*\\ \mathcal{TLJ}(d)\text{-module category } \mathcal{M}\end{array}} \arrow[r, leftrightarrow, "\text{Section 5.7}"'] & \boxed{\begin{array}{c}\text{2-subcategory } \mathcal{C}(K,\mathrm{ev}_K)\\ \text{of } \mathsf{BigHilb}\end{array}} \arrow[ul, "\text{Section 5.4}"']
\end{tikzcd}$$

As an application, in the unitary pivotal/tracial setting, we obtain the embedding theorem for (infinite depth) subfactor planar algebras (cf. [34]).

Theorem C. *Every (infinite depth) subfactor planar algebra embeds in any bipartite graph planar algebra of its principal graph with respect to a module category. In particular, it embeds in the bipartite graph planar algebra of its (dual) principal graph.*

By Theorem B above, a Markov lattice corresponds to a $\mathcal{TLJ}$-$\mathcal{TLJ}$ bimodule category. By work-in-progress of Penneys–Peters–Snyder, pivotal $\mathcal{TLJ}$-$\mathcal{TLJ}$ bimodule categories correspond to Ocneanu's biunitary connections on associative square-partite graphs with vertex weightings. For the non-pivotal case, the weighting on the square-partite graph is the edge-weighting and we obtain the non-pivotal analog of a biunitary connection. To translate between these classifications, we use the well-known fact that a commuting square of finite dimensional von Neumann algebras gives a biunitary connection [12, 21, 36, 41]. We then introduce a graphical version of a Markov lattice using a W* 2-subcategory $\mathcal{C}(\Phi)$ of $\mathsf{BigHilb}$ obtained from a biunitary connection Φ. It turns out that the biunitary connection Φ corresponds to the bimodule associator of the bimodule category. The following diagram shows how these notions are related to each other in Section 6:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Markov lattice } M\\ \text{with modulus } (d_0,d_1)\end{array}} \arrow[r, "\text{Section 6.4}"] \arrow[d, leftrightarrow, "\text{Section 4}"'] & \boxed{\begin{array}{c}\text{balanced } (d_0,d_1)\text{-fair}\\ \text{square-partite graph } (\Lambda,\omega)\\ \text{with biunitary connection } \Phi\end{array}} \arrow[d, leftrightarrow, "\text{Section 6.2}"] \\
\boxed{\begin{array}{c}\text{indecomposable semisimple C}^*\\ \mathcal{TLJ}(d_0)\text{-}\mathcal{TLJ}(d_1)\\ \text{bimodule category } \mathcal{M}\end{array}} \arrow[r, leftrightarrow, "\text{Section 6.5}"'] & \boxed{\begin{array}{c}\text{2-subcategory } \mathcal{C}(\Phi)\\ \text{of } \mathsf{BigHilb}\end{array}} \arrow[ul, "\text{Section 6.3}"']
\end{tikzcd}$$

2. Standard λ-lattices and tensor category

2.1. Traceless Markov tower and its properties

Definition 2.1. Let $A \subset B$ be a unital inclusion of finite von Neumann algebras. A *conditional expectation* $E : M \to N$ is a positive linear map satisfying the following conditions:

(a) $E(x) = x$ for all $x \in A$,

(b) $E(axb) = aE(x)b$ for all $a, b \in A$, $x \in B$.

Definition 2.2. Let C be a unital C*-algebra. We call a linear functional $\mathrm{tr} : C \to \mathbb{C}$ a *trace* if it satisfies the following conditions:

(a) (tracial) $\mathrm{tr}(xy) = \mathrm{tr}(yx)$, for all $x, y \in C$.

(b) (positive) $\mathrm{tr}(x^*x) \geq 0$, for all $x \in C$.

(c) (faithful) $\mathrm{tr}(x^*x) = 0$ if and only if $x = 0$.

In addition, we call tr *unital* if $\mathrm{tr}(1) = 1$.

Definition 2.3. A *traceless Markov tower* $M = (M_n, E_n, e_{n+1})_{n\geq 0}$ consists of a sequence $(M_n)_{n\geq 0}$ of finite dimensional von Neumann algebras, such that M_n is unitally included in M_{n+1}. For each n, there is a faithful conditional expectation $E_n : M_n \to M_{n-1}$ together with a sequence of *Jones projections* $e_n \in M_{n+1}$ for all $n \geq 1$, such that:

(M1) The projections (e_n) satisfy the Temperley–Lieb–Jones relations:

(TLJ1) $e_n^2 = e_n = e_n^*$ for all n.

(TLJ2) $[e_i, e_j] = 0$ for $|i - j| > 1$.

(TLJ3) There is a fixed constant called the *modulus* $d > 0$ such that $e_n e_{n\pm 1} e_n = d^{-2} e_n$ for all n.

(M2) For all $x \in M_n$, $e_n x e_n = E_n(x) e_n$.

(M3) $E_{n+1}(e_n) = d^{-2} \cdot 1$ for all $n \geq 1$.

(M4) (pull down) $M_{n+1} e_n = M_n e_n$ for all $n \geq 1$.

In the following, all Markov towers are *traceless* unless stated otherwise.

Proposition 2.4. *Some properties of a traceless Markov tower include:*

(1) $[x, e_k] = 0$, *for* $x \in M_n$, $k \geq n + 1$.

(2) *The map* $M_n \ni x \mapsto x e_n \in M_{n+1}$ *is injective.*

(3) *For* $x \in M_{n+1}$, $d^2 E_{n+1}(x e_n)$ *is the unique element* $y \in M_n$ *such that* $x e_n = y e_n$.

(4) *Property* (3) *is equivalent to* (M3).

(5) *If* $x \in M_n$ *and* $[x, e_n] = 0$, *then* $x \in M_{n-1}$. *Together with (1), we have* $M_{n-1} = M_n \cap \{e_n\}'$.

(6) $e_n M_{n+1} e_n = M_{n-1} e_n$.

Proof. (1) For $x \in M_n$ and $k \geq n + 1$, $E_k(x) = x$, $E_k(x^*) = x^*$, then

$$x e_k = E_k(x) e_k = e_k x e_k = (e_k x^* e_k)^* = (E_k(x^*) e_k)^* = (x^* e_k)^* = e_k x.$$

(2) If $x \in M_n$ and $x e_n = 0$, then by (M3),

$$0 = E_{n+1}(x e_n) = x E_{n+1}(e_n) = d^{-2} x.$$

Thus, $x \mapsto x e_n$ is injective.

(3) By (M4) and (2), the existence and uniqueness hold. Then by (M3),

$$E_{n+1}(xe_n) = E_{n+1}(ye_n) = yE_{n+1}(e_n) = d^{-2}y,$$

so $y = d^2 E_{n+1}(xe_n)$.

(4) We first show that (3) implies (2). If $x \in M_n$ and $xe_n = 0$, then $0 = d^2 E_{n+1}(xe_n)$ is the unique element such that $xe_n = 0 = d^2 E_{n+1}(xe_n)e_n$. Therefore, $x = d^2 E_{n+1}(xe_n) = 0$.

Let $x = e_n$, then we have $d^2 E_{n+1}(e_n)e_n = e_n$. Since $d^2 E_{n+1}(e_n)$ and $1 \in M_n$, we have $d^2 E_{n+1}(e_n) = 1$ by (2).

(5) Since $xe_n = e_n x$,

$$E_n(x)e_n = e_n x e_n = x e_n e_n = x e_n.$$

Then by (2), $E_n(x) = x$, which implies $x \in M_{n-1}$.

(6) By (M2) and (M4). ∎

We will explore more properties of traceless Markov towers in Section 5.

Remark 2.5. If there is a faithful tracial state tr_n on each M_n with $\mathrm{tr}_{n+1}|_{M_n} = \mathrm{tr}_n$ and E_n is the canonical trace-preserving conditional expectation for $n = 1, 2, \ldots$, then M is called a *tracial Markov tower*. Thus, tracial Markov towers defined in [7] are also traceless Markov towers.

Example 2.6 (Markov tower without a trace). Let $d > 0$ such that $d^2 > 4$. There is a unique $\lambda \in (0, \frac{1}{2})$ such that $d^{-2} = \lambda(1-\lambda)$. Then $d\lambda + d(1-\lambda) = d$ and $\frac{1}{d\lambda} + \frac{1}{d(1-\lambda)} = d$. Let ε_{ij} denote the matrix units of $M_2(\mathbb{C})$, $i, j = 1, 2$, and $1 = \varepsilon_{11} + \varepsilon_{22} \in M_2(\mathbb{C})$.

Define $E_\lambda : M_2(\mathbb{C}) \to \mathbb{C}$ by $E_\lambda(\varepsilon_{11}) = \lambda$, $E_\lambda(\varepsilon_{22}) = 1-\lambda$ and $E_\lambda(\varepsilon_{12}) = E_\lambda(\varepsilon_{21}) = 0$. It is clear that E_λ is a normal faithful conditional expectation and not tracial.

Define $e_\lambda \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ by

$$e_\lambda = (1-\lambda)\varepsilon_{11} \otimes \varepsilon_{11} + \lambda\varepsilon_{22} \otimes \varepsilon_{22} + \sqrt{\lambda(1-\lambda)}(\varepsilon_{12} \otimes \varepsilon_{12} + \varepsilon_{21} \otimes \varepsilon_{21}),$$

and one can check that:

(a) e_λ is a projection.

(b) $E_\lambda(e_\lambda) = d^{-2}(\varepsilon_{11} + \varepsilon_{22}) = d^{-2} \cdot 1$.

(c) $(e_\lambda \otimes 1)(1 \otimes e_{1-\lambda})(e_\lambda \otimes 1) = d^{-2}(e_\lambda \otimes 1)$ and $(e_{1-\lambda} \otimes 1)(1 \otimes e_\lambda)(e_{1-\lambda} \otimes 1) = d^{-2}(e_{1-\lambda} \otimes 1)$.

Define $\mathrm{id} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ to be the identity map. Let $M_n := M_2(\mathbb{C})^{\otimes n}$. The inclusion $M_n \subset M_{n+1}$ maps x to $x \otimes \mathrm{id}$. Jones projection $e_{2n+1} = 1^{\otimes 2n} \otimes e_{1-\lambda} \in M_{2n+2}$ and $e_{2n+2} = 1^{\otimes 2n+1} \otimes e_\lambda \in M_{2n+3}$, $n = 0, 1, 2, \ldots$. The conditional expectation is defined as follows:

$$E_{2n+1} = \mathrm{id}^{\otimes 2n+1} \otimes E_\lambda, \quad E_{2n+2} = \mathrm{id}^{\otimes 2n+2} \otimes E_{1-\lambda}.$$

Now we build a Markov tower with modulus d and without a trace:

$$1 \xleftarrow{E_\lambda} M_2(\mathbb{C})^{\otimes 2} \xleftarrow{\mathrm{id}\otimes E_{1-\lambda}} M_2(\mathbb{C})^{\otimes 3} \xleftarrow{\mathrm{id}^{\otimes 2}\otimes E_\lambda} M_2(\mathbb{C})^{\otimes 4} \longleftarrow \cdots$$

2.2. Standard λ-lattice and its properties

Definition 2.7 ([38]). Let $A = (A_{i,j})_{0\le i\le j<\infty}$ be a system of finite dimensional C* algebras with $A_{i,i} = \mathbb{C}$ with unital inclusions $A_{i,j} \subset A_{k,l}$, for $k \le i, j \le l$.

$$\begin{array}{ccccccccccc}
A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & A_{0,3} & \subset & A_{0,4} & \subset & \cdots \\
 & & \cup & & \cup & & \cup & & \cup & & \\
 & & A_{1,1} & \subset & A_{1,2} & \subset & A_{1,3} & \subset & A_{1,4} & \subset & \cdots \\
 & & & & \cup & & \cup & & \cup & & \\
 & & & & A_{2,2} & \subset & A_{2,3} & \subset & A_{2,4} & \subset & \cdots \\
 & & & & & & \cup & & \cup & & \\
 & & & & & & A_{3,3} & \subset & A_{3,4} & \subset & \cdots \\
 & & & & & & & & \cup & & \\
 & & & & & & & & A_{4,4} & \subset & \cdots \\
 & & & & & & & & & & \ddots
\end{array}$$

Let $E^r_{i,j} : A_{i,j} \to A_{i,j-1}$ be the (horizontal) faithful conditional expectation, $j = 1, 2, \ldots, i = 0, \ldots, j-1$ and $E^l_{i,j} : A_{i,j} \to A_{i+1,j}$ be the (vertical) faithful normal conditional expectation $i = 0, 1, \ldots, j = i+1, i+2, \ldots$. We also require that

(a1) (commuting square condition)

$$\begin{tikzcd}
A_{i,j} \arrow[d, "E^l_{i,j}"'] & A_{i,j+1} \arrow[l, "E^r_{i,j+1}"'] \arrow[d, "E^l_{i,j+1}"] \\
A_{i+1,j} & A_{i+1,j+1} \arrow[l, "E^r_{i+1,j+1}"]
\end{tikzcd}$$

is a commuting square, i.e., $E^l_{i,j} \circ E^r_{i,j+1} = E^r_{i+1,j+1} \circ E^l_{i,j+1}$.

(a2) (existence of Jones λ-projections)
There exists a sequence of Jones projections $\{e_i\}_{i\ge 1}$ in $\bigcup_n A_{0,n}$ such that

(b1) $e_j \in A_{i-1,k}$, for $1 \le i \le j+1 \le k$.

(b2) The projections satisfy the Temperley–Lieb–Jones relations:

(TLJ1) $e_i^2 = e_i = e_i^*$ for all i.

(TLJ2) $e_i e_j = e_j e_i$ for $|i-j| > 1$.

(TLJ3) There is a fixed constant $d > 0$ called the modulus such that $e_i e_{i\pm 1} e_i = d^{-2} e_i$ for all i.

(b3) $e_j x e_j = E^r_{i,j}(x) e_j$, for $x \in A_{i,j}, i+1 \le j$.

(b4) $e_i x e_i = E^l_{i,j}(x) e_i$, for $x \in A_{i,j}, i+1 \le j$.

(a3) (Markov conditions)

(c1) $\dim A_{i,j} = \dim A_{i,j+1} e_j = \dim A_{i+1,j+1}$, for $i \le j$.

(c2) $E^r_{i,j+1}(e_j) = E^l_{j-1,k}(e_j) = d^{-2} 1$, for $j \ge i+1, k \ge j+1$.

Then $A = (A_{i,j})_{0\le i\le j<\infty}$ is called a *λ-lattice* of commuting squares. If there is a faithful tracial state $\mathrm{tr}_{i,j}$ on $A_{i,j}$ such that $\mathrm{tr}_{i+1,j}|_{A_{i,j}} = \mathrm{tr}_{i,j+1}|_{A_{i,j}} = \mathrm{tr}_{i,j}$ and $E^r_{i,j}, E^l_{i,j}$ are the canonical trace-preserving conditional expectation, then A is called a *tracial λ-lattice*.

Definition 2.8 ([38]). A λ-lattice $(A_{i,j})_{0\le i\le j}$ is called a *standard λ-lattice* if $[A_{i,j}, A_{k,l}] = 0$ for $i \le j \le k \le l$. This condition is called the *standard condition*.

Remark 2.9. In the definition of (standard) λ-lattice, we may not require a trace and the conditional expectations are trace-preserving. In fact, the reader can construct an example of (standard) λ-lattice without a trace from Example 2.6 easily.

Warning. From now on, we will *not* further discuss the traceless standard λ-lattices, though the following statements do not require the trace at all!

Remark 2.10. Each row $A_i = (A_{i,j})_{j\ge i}$ is a Markov tower, $i = 0, 1, 2, \ldots$; each column $A_j = (A_{i,j})^0_{i=j}$ is a Markov tower, $j = 1, 2, \ldots$. From Proposition 2.4, we have

(1) If $x \in A_{i,j}$, $[x, e_k] = 0$ for $k \ge j+1$; $[x, e_l] = 0$ for $1 \le l \le i-1$.

(2) The map $A_{i,j} \ni x \mapsto xe_j \in A_{i,j+1}$ is injective; the map $A_{i,j} \ni x \mapsto xe_i \in A_{i-1,j}$ is injective.

(3) The Markov condition is equivalent to the pull-down condition:

(c1)′ $d^2 E^r_{i,j+1}(xe_j)e_j = xe_j$, for $x \in A_{i,j+1}$, $j \ge i \ge 0$.

(c2)′ $d^2 E^l_{i-1,j}(xe_i)e_i = xe_i$, for $x \in A_{i-1,j}$, $j \ge i \ge 1$.

The following property was proved in [38, Prop. 1.4] by using the trace, here we provide another proof without it.

Proposition 2.11. *Let*

$$\begin{array}{ccccccccc}
A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & A_{0,3} & \subset & \cdots \\
 & & \cup & & \cup & & \cup & & \\
 & & A_{1,1} & \subset & A_{1,2} & \subset & A_{1,3} & \subset & \cdots
\end{array}$$

be a λ-sequence of commuting squares, and define $A_{i,j} := A_{i-1,j} \cap \{e_{i-1}\}' = A_{1,j} \cap \{e_1, \ldots, e_{i-1}\}'$, $2 \le i \le j$. Then $(A_{i,j})_{0\le i\le j<\infty}$ is a λ-lattice of commuting squares.

Proof. We construct $A_{i,j}$ and conditional expectation $E^l_{i-1,j} : A_{i-1,j} \to A_{i,j}$ by induction on i, and show that Jones projections $\{e_{i+1}, \ldots, e_{j-1}\} \subset A_{i,j}$ for $i+2 \le j$. Suppose $A_{i-1,j}$ is constructed (or given) and $\{e_i, \ldots, e_{j-1}\} \subset A_{i-1,j}$, We define $A_{i,j} := A_{i-1,j} \cap \{e_{i-1}\}'$. Then clearly, $\{e_{i+1}, \ldots, e_{j-1}\} \subset A_{i,j}$.

According to Proposition 2.4 (5) and (6), for each $x \in A_{i-1,j} \subset A_{i-2,j}$, there exists a $y \in A_{i,j}$ such that

$$ye_{i-1} = e_{i-1}xe_{i-1}.$$

By Proposition 2.4 (2), $A_{i-1,j} \ni y \mapsto ye_{i-1} \in A_{i-2,j}$ is injective, so y is unique for each given x. This technique is often used in this section. We define $E^l_{i-1,j}(x) := y$. Now we show that $E^l_{i-1,j}$ is a faithful normal conditional expectation:

(a) It is clear that $E^l_{i-1,j}$ is linear, and $E^l_{i-1,j}(1) = 1$.

(b) $E^l_{i-1,j}(x^*) = E^l_{i-1,j}(x)^*$:

$$\begin{aligned} E^l_{i-1,j}(x)^* e_{i-1} &= \big(e_{i-1} E^l_{i-1,j}(x)\big)^* = (e_{i-1} x e_{i-1})^* \\ &= e_{i-1} x^* e_{i-1} = E^l_{i-1,j}(x^*) e_{i-1}. \end{aligned}$$

(c) $E^l_{i-1,j}(axb) = aE^l_{i-1,j}(x)b$ for $a, b \in A_{i,j}$: Note that $[a, e_{i-1}] = [b, e_{i-1}] = 0$, then

$$\begin{aligned} E^l_{i-1,j}(axb)e_{i-1} &= e_{i-1} axb e_{i-1} = a e_{i-1} x e_{i-1} b \\ &= aE^l_{i-1,j}(x) e_{i-1} b = aE^l_{i-1,j}(x) b e_{i-1}. \end{aligned}$$

(d) $E^l_{i-1,j}(x^*x) \geq E^l_{i-1,j}(x)^* E^l_{i-1,j}(x)$, which follows that $E^l_{i-1,j}$ is positive:

$$\begin{aligned} E^l_{i-1,j}(x)^* E^l_{i-1,j}(x) e_{i-1} &= E^l_{i-1,j}(x)^* e_{i-1} x e_{i-1} = e_{i-1} x^* e_{i-1} x e_{i-1} \\ &\leq e_{i-1} x^* x e_{i-1} = E^l_{i-1,j}(x^* x) e_{i-1}, \end{aligned}$$

so $E^l_{i-1,j}(x^*x) \geq E^l_{i-1,j}(x)^* E^l_{i-1,j}(x)$ by applying the inductive hypothesis that $E^l_{i-2,j}$ is a positive conditional expectation and $E^l_{i-2,j}(e_{i-1}) = d^{-2} \cdot 1$.

(e) $E^l_{i-1,j}(x^*x) = 0$ if and only if $x = 0$, i.e., $E^l_{i-1,j}$ is faithful:

$$0 = E^l_{i-1,j}(x^*x) e_{i-1} = e_{i-1} x^* x e_{i-1} = (x e_{i-1})^* (x e_{i-1}),$$

which follows that $x e_{i-1} = 0$. Note that $A_{i-1,j} \ni x \mapsto x e_{i-1} \in A_{i-2,j}$ is an injection, so $x = 0$.

Then define $E^r_{i,j+1} : A_{i,j+1} \to A_{i,j}$ as the restriction of $E^r_{i-1,j+1}$ on $A_{i,j+1}$, which is also a conditional expectation.

Now we prove the commuting square condition $E^l_{i-1,j} \circ E^r_{i-1,j+1} = E^r_{i,j+1} \circ E^l_{i-1,j+1}$: for $x \in A_{i-1,j+1}$,

$$\begin{aligned} E^l_{i-1,j}\big(E^r_{i-1,j+1}(x)\big) e_{i-1} &= e_{i-1} E^r_{i-1,j+1}(x) e_{i-1}, \\ E^r_{i,j+1}\big(E^l_{i-1,j+1}(x)\big) e_{i-1} &= E^r_{i-1,j+1}\big(E^l_{i-1,j+1}(x)\big) e_{i-1} \\ &= E^r_{i-1,j+1}\big(E^l_{i-1,j+1}(x) e_{i-1}\big) \\ &= E^r_{i-1,j+1}(e_{i-1} x e_{i-1}) \\ &= e_{i-1} E^r_{i-1,j+1}(x) e_{i-1}. \end{aligned}$$

Finally, we prove the Markov condition:

(a) $\dim A_{i,j} = \dim A_{i-1,j} \cap \{e_{i-1}\}' = \dim A_{i-1,j} \cap \{e_{j-1}\}' = \dim A_{i-1,j-1}$.

(b) $E^r_{i,j+1}(e_j) = E^r_{i-1,j+1}(e_j) = d^{-2}1$.

(c) $E^l_{i-1,j}(e_i) e_{i-1} = e_{i-1} e_i e_{i-1} = d^{-2} e_{i-1}$, so $E^l_{i-1,j}(e_i) = d^{-2} \cdot 1$. ∎

Corollary 2.12. *Let $(A_{i,j})_{i\le j,i=0,1}$ be a λ-sequence of commuting squares. If $A_{i,j} := \{e_1,\dots,e_{i-1}\}' \cap A_{i,j}$, for all $2 \le i \le j$, then $(A_{i,j})_{0\le i\le j}$ is a standard λ-lattice if and only if $(A_{i,j})_{i\le j,i=0,1}$ satisfies*

$$[A_{0,1}, A_{1,j}] = 0, \quad \forall 1 \le j,$$
$$[A_{0,i}, A_{i,j}] = 0, \quad \forall 2 \le i \le j.$$

Now we define the opposite standard λ-lattice, which will be used in Definition 4.5.

Definition 2.13. $A^{\mathrm{op}} = (A_{i,j})_{0\le j\le i}$ is the *opposite* of λ-lattice A if $A^{\mathrm{op}}_{j,i} = A_{i,j}$ as opposite algebras, $E^{\mathrm{op},l}_{j,i} = E^r_{i,j}$, $E^{\mathrm{op},r}_{j,i} = E^l_{i,j}$ for $i \le j$.

Example 2.14. The Temperley–Lieb–Jones algebra $\mathrm{TLJ}(d)$ forms a standard λ-lattice with the modulus d by letting $A_{i,i} = A_{i,i+1} = \mathbb{C}$ and $A_{i,j} = \langle e_{i+1},\dots,e_{j-1}\rangle$ for $j-i \ge 2$, which is called a Temperley–Lieb–Jones standard λ-lattice.

Example 2.15 ([38]). If $A_0 \subset A_1$ is a unital inclusion of type II_1 subfactors with finite index and $A_0 \subset A_1 \subset A_2 \subset A_3 \subset \cdots$ is the Jones tower from the basic construction, then $A_{i,j} := A_i' \cap A_j$ forms a standard λ-lattice, which is called the *standard invariant* of $A_0 \subset A_1$.

2.3. The 2-shift map

In this section, we discuss an important type of $*$-isomorphism in a standard λ-lattice, so-called the 2-shift map [4]. Here we provide the definition by using conditional expectations and Jones projections instead of tracial states and Pimsner–Popa basis.

For $i, k \ge 0$, define the following element of $A_{l,i+2k}$, $l+1 \le i+2k$:

$$e^i_k := d^{k(k-1)}(e_{k+i}e_{k+i-1}\cdots e_{i+1})(e_{k+i+1}e_{k+i}\cdots e_{n-k+2})\cdots(e_{2k+i-1}e_{2k+i-2}\cdots e_{k+i}).$$

For $i, j, k \ge 0$, define the following element of $A_{l,i+j+2k}$, $l+1 \le i+j+2k$,

$$e^i_{j,k} = d^{jk} e^i_k e^{i+1}_k \cdots e^{i+j}_k.$$

Clearly,

$$e_n = e^{n-1}_1 = e^{n-1}_{0,1},\ e^i_k = e^i_{0,k},\ (e^i_k)^2 = (e^i_k)^* = e^i_k,\ e^i_{j,k}(e^i_{j,k})^* = e^i_{0,k},\ (e^i_{j,k})^* e^i_{j,k} = e^{i+j}_{0,k}.$$

Definition 2.16 (Multi-step condition expectation). Define the k-step horizontal conditional expectation as

$$E^{r,k}_{i,j} = E^r_{i,j+1-k} \circ E^r_{i,j+2-k} \circ \cdots \circ E^r_{i,j} : A_{i,j} \to A_{i,j-k} \quad \text{for } k \le j-i$$

and we have $E^{r,1}_{i,j} = E^r_{i,j}$; the k-step vertical conditional expectation as

$$E^{l,k}_{i,j} = E^l_{i+k-1,j} \circ E^r_{i+k-2,j} \circ \cdots \circ E^l_{i,j} : A_{i,j} \to A_{i+k} \quad \text{for } k \le j-i$$

and we have $E^{l,1}_{i,j} = E^l_{i,j}$.

In particular, the trace is made by the composition of conditional expectations, i.e., $E_{i,j-k}^{l,i-j+k} \circ E_{i,j}^{r,k} = \mathrm{tr} = E_{i+t,j}^{r,j-i-t} \circ E_{i,j}^{l,t}$, for $0 \le k \le j-i, 0 \le t \le j-i$.

Definition 2.17 (2-shift map). Define the 2-*shift* map $S_{i,j} : A_{i,j} \to A_{i+2,j+2}, i \le j$ by

$$S_{i,j}(x) := d^{2j-2i+2} E_{i,j+2}^{l}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j \cdots e_{i+1}).$$

Proposition 2.18. *The followings are the properties of the* 2*-shift map.*

(1) $S_{i,j}$ *is well defined, i.e.,* $S_{i,j}(x) \in A_{i+2,j+2}$ *for* $x \in A_{i,j}$.

(2) $S_{i,j}$ *is a unital* $*$*-isomorphism.*

(3) (*commuting parallelogram*) $S_{i,j-1} \circ E_{i,j}^r = E_{i+2,j+2}^r \circ S_{i,j}$ *and* $S_{i+1,j} \circ E_{i,j}^l = E_{i+2,j+2}^l \circ S_{i,j}$.

(4) $S_{i,j+1}(x) = S_{i,j}(x)$ *for* $x \in A_{i,j}$ *and* $S_{i-1,j}(x) = S_{i,j}(x)$ *for* $x \in A_{i,j}$.

(5) (*shift*) $e_{i+1}e_{i+2}\cdots e_{j+1}x = S_{i,j}(x)e_{i+1}e_{i+2}\cdots e_{j+1}$ *for* $x \in A_{i,j}$. *Taking adjoints,* $xe_{j+1}e_j\cdots e_{i+1} = e_{j+1}e_j\cdots e_{i+1}S_{i,j}(x)$. *In other word,* $e_{j-i,1}^{i}x = S_{i,j}(x)e_{j-i,1}^{i}$.

(6) $S_{i,j}$ *is trace-preserving.*

(7) $S_{i,j}(e_k) = e_{k+2}$, *where* $i+1 \le k \le j-1$.

Proof. (1) Note that $S_{i,j}(x) \in A_{i+1,j+2}$, we shall show that $E_{i+1,j+2}^l(S_{i,j}(x)) = S_{i,j}(x)$. Since $E_{i+1,j+2}^l(S_{i,j}(x)) - S_{i,j}(x) \in A_{i+1,j+2}$ and the map $A_{i+1,j+2} \ni y \mapsto ye_{i+1} \in A_{i,j+2}$ is injective, we shall show that $E_{i+1,j+2}^l(S_{i,j}(x))e_{i+1} = S_{i,j}(x)e_{i+1}$.

$$\begin{aligned}
&E_{i+1,j+2}^l(S_{i,j}(x))e_{i+1} \\
&\quad = e_{i+1}S_{i,j}(x)e_{i+1} \\
&\quad = d^{2j-2i+2}e_{i+1}E_{i,j+2}^l(e_{i+1}e_{i+2}\cdots e_j x e_{j+1}e_j\cdots e_{i+1})e_{i+1} \\
&\quad = d^{2j-2i}e_{i+1}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1}e_j\cdots e_{i+1}) && \text{(pull down)} \\
&\quad = d^{2j-2i}e_{i+1}e_{i+2}\cdots e_j x e_{j+1}e_j\cdots e_{i+1} \\
&\quad = d^{2j-2i+2}E_{i,j+2}^l(e_{i+1}e_{i+2}\cdots e_j x e_{j+1}e_j\cdots e_{i+1})e_{i+1} && \text{(pull down)} \\
&\quad = S_{i,j}(x)e_{i+1}.
\end{aligned}$$

(2) For $x \in A_{i,j}$, we have $[x, e_{j+1}] = 0$. First, we show that $S_{i,j}$ is a homomorphism, i.e., $S_{i,j}(xy) = S_{i,j}(x)S_{i,j}(y)$ for $x, y \in A_{i,j}$. Note that the map $A_{i+2,j+2} \subset A_{i+1,j+2} \ni y \mapsto ye_{i+1} \in A_{i,j+2}$ is injective, we shall prove that $S_{i,j}(xy)e_{i+1} = S_{i,j}(x)S_{i,j}(y)e_{i+1}$.

$$\begin{aligned}
&S_{i,j}(x)S_{i,j}(y)e_{i+1} \\
&\quad = d^{2j-2i+2}S_{i,j}(x)E_{i,j+2}^l(e_{i+1}e_{i+2}\cdots e_j y e_{j+1}e_j\cdots e_{i+1})e_{i+1} \\
&\quad = d^{2j-2i}S_{i,j}(x)e_{i+1}e_{i+2}\cdots e_j y e_{j+1}e_j\cdots e_{i+1} && \text{(pull down)} \\
&\quad = d^{2j-2i}\cdot d^{2j-2i}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1}e_j\cdots e_{i+1})(e_{i+1}e_{i+2}\cdots e_j y e_{j+1}e_j\cdots e_{i+1}) \\
& && \text{(pull down)} \\
&\quad = d^{2j-2i+2}e_{i+1}e_{i+2}\cdots e_j x e_{j+1}e_j y e_{j+1}e_j\cdots e_{i+1} && (e_k e_{k\pm1} e_k = d^{-2}e_k)
\end{aligned}$$

$$= d^{2j-2i+2} e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j e_{j+1} y e_j \cdots e_{i+1} \qquad ([y, e_{j+1}] = 0)$$
$$= d^{2j-2i} e_{i+1}e_{i+2}\cdots e_j x e_{j+1} y e_j \cdots e_{i+1}$$
$$= d^{2j-2i} e_{i+1}e_{i+2}\cdots e_j x y e_{j+1} e_j \cdots e_{i+1}$$
$$= d^{2j-2i+2} E^l_{i,j+2}(e_{i+1}e_{i+2}\cdots e_j x y e_{j+1} e_j \cdots e_{i+1}) e_{i+1} \qquad \text{(pull down)}$$
$$= S_{i,j}(xy) e_{i+1}.$$

Next, $S_{i,j}$ is a $*$-homomorphism. Note that $E^l_{i,j+2}$ is a $*$-homomorphism, we have

$$\begin{aligned} S_{i,j}(x^*) &= d^{2j-2i+2} E^l_{i,j+2}(e_{i+1}e_{i+2}\cdots e_j x^* e_{j+1} e_j \cdots e_{i+1}) \\ &= d^{2j-2i+2} E^l_{i,j+2}\big((e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j \cdots e_{i+1})^*\big) \\ &= d^{2j-2i+2} E^{l,*}_{i,j+2}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j \cdots e_{i+1}) \\ &= S^*_{i,j}(x). \end{aligned}$$

When $x = 1$,

$$\begin{aligned} e_{i+1}e_{i+2}\cdots e_j e_{j+1} e_j \cdots e_{i+1} &= d^{-2} e_{i+1}e_{i+2}\cdots e_{j-1} e_j e_{j-1} \cdots e_{i+1} \\ &= \cdots = d^{2(i-j+2)} e_{i+1}e_{i+2}e_{i+1} = d^{2(i-j)} e_{i+1}. \end{aligned}$$

Thus, $S_{i,j}(1) = d^2 E^l_{i,j+2}(e_{i+1}) = 1$, i.e., $S_{i,j}$ is unital.

In order to prove that $S_{i,j}$ is an isomorphism, we shall show $S_{i,j}$ is injective and surjective.

If $S_{i,j}(x) = 0$, then

$$\begin{aligned} 0 = S_{i,j}(x)e_{i+1} &= d^{2j-2i} e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j \cdots e_{i+1} \\ &= d^{2j-2i} (e_{i+1}e_{i+2}\cdots e_j) x e_{j+1} (e_{i+1}e_{i+2}\cdots e_j)^*, \end{aligned}$$

which follows that $xe_{j+1} = 0$. Since the map $A_{i,j} \ni y \mapsto ye_{j+1} \in A_{i,j+1}$ is injective, we have $x = 0$.

Note that $\dim A_{i,j} = \dim A_{i+1,j+1} = \dim A_{i+2,j+2} < \infty$, so the injectivity implies the surjectivity. Thus, $S_{i,j}$ is a unital $*$-isomorphism.

(3) For $x \in A_{i,j}$, $E^r_{i,j}(x) \in A_{i,j-1}$ and $[E^r_{i,j}(x), e_j] = 0$,

$$\begin{aligned} &S_{i,j-1} \circ E^r_{i,j}(x) \\ &\quad = d^{2j-2i} E^l_{i,j+1}\big(e_{i+1}e_{i+2}\cdots e_j E^r_{i,j}(x) e_{j+1} e_j \cdots e_{i+1}\big) \\ &\quad = d^{2j-2i} E^l_{i,j+1}\big(e_{i+1}e_{i+2}\cdots E^r_{i,j}(x) e_j e_{j+1} e_j \cdots e_{i+1}\big) \\ &\quad = d^{2j-2i+2} E^l_{i,j+1}\big(e_{i+1}e_{i+2}\cdots E^r_{i,j}(x) e_j \cdots e_{i+1}\big) \\ &\quad = d^{2j-2i+2} E^l_{i,j+1}(e_{i+1}e_{i+2}\cdots e_j x e_j \cdots e_{i+1}), \end{aligned}$$

$$\begin{aligned} &E^r_{i+2,j+2} \circ S_{i,j}(x) \\ &\quad = E^r_{i+2,j+2} \circ E^l_{i+1,j+2} \circ S_{i,j}(x) \\ &\quad = E^l_{i+1,j+1} \circ E^r_{i+1,j+2} \circ S_{i,j}(x) \qquad \text{(commuting square)} \end{aligned}$$

$$
\begin{aligned}
&= d^{2j-2i} E^l_{i+1,j+1} \circ E^r_{i+1,j+2} \circ E^l_{i,j+2}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j \cdots e_{i+1}) \\
&= d^{2j-2i} E^l_{i+1,j+1} \circ E^l_{i,j+1} \circ E^r_{i,j+2}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j \cdots e_{i+1}) \\
&= d^{2j-2i} E^l_{i+1,j+1} \circ E^l_{i,j+1}\big(e_{i+1}e_{i+2}\cdots e_j x E^r_{i,j+2}(e_{j+1}) e_j \cdots e_{i+1}\big) \\
&= d^{2j-2i+2} E^l_{i+1,j+1} \circ E^l_{i,j+1}(e_{i+1}e_{i+2}\cdots e_j x e_j \cdots e_{i+1}) \\
&= E^l_{i+1,j+1}\big(S_{i,j-1} \circ E^r_{i,j}(x)\big) \qquad \text{(since } S_{i,j-1} \circ E^r_{i,j}(x) \in A_{i+2,j+1}\text{)} \\
&= S_{i,j-1} \circ E^r_{i,j}(x).
\end{aligned}
$$

Thus, $S_{i,j-1} \circ E^r_{i,j} = E^r_{i+2,j+2} \circ S_{i,j}$.

Note that $\{e_{i+1}, \ldots, e_{j-1}\} \subset A_{i,j}$, we have

$$
E^l_{i,j+2}(e_k x e_n) = e_k E^l_{i,j+2}(x) e_n \quad \text{for all } k, n \in \{i+1, \ldots, j-1\}. \tag{$*$}
$$

In order to prove that $S_{i+1,j} \circ E^l_{i,j} = E^l_{i+2} \circ S_{i,j}$, by Remark 2.10 (2), we shall show that $S_{i+1,j} \circ E^l_{i,j}(x) e_{i+2} = E^l_{i+2,j+2} \circ S_{i,j}(x) e_{i+2}$ for all $x \in A_{i,j}$.

$$
\begin{aligned}
&S_{i+1,j} \circ E^l_{i,j}(x) e_{i+2} \\
&\quad = d^{2j-2i} E^l_{i+1,j+2}\big(e_{i+2}\cdots e_j E^l_{i,j}(x) e_{j+1}\cdots e_{i+2}\big) e_{i+2} \\
&\quad = d^{2j-2i-2} e_{i+2}\cdots e_j E^l_{i,j}(x) e_{j+1}\cdots e_{i+2}, \qquad \text{(pull down)} \\
&E^l_{i+2,j+2} \circ S_{i,j}(x) e_{i+2} \\
&\quad = d^{2j-2i+2} E^l_{i+2,j+2}\big(E^l_{i,j+2}(e_{i+1}\cdots e_j x e_{j+1}\cdots e_{i+1})\big) e_{i+2} \\
&\quad = d^{2j-2i+2} e_{i+2} E^l_{i,j+2}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1}\cdots e_{i+2}e_{i+1}) e_{i+2} \qquad \text{(by ($*$))} \\
&\quad = d^{2j-2i+2} E^l_{i,j+2}(e_{i+2}e_{i+1}e_{i+2}\cdots e_j x e_{j+1}\cdots e_{i+2}e_{i+1}e_{i+2}) \\
&\quad = d^{2j-2i-2} E^l_{i,j+2}(e_{i+2}\cdots e_j x e_{j+1}\cdots e_{i+2}) \\
&\quad = d^{2j-2i-2} e_{i+2}e_{i+1}\cdots e_j E^l_{i,j+2}(x) e_{j+1}\cdots e_{i+1}e_{i+2} \qquad \text{(by ($*$))} \\
&\quad = d^{2j-2i-2} e_{i+2}e_{i+1}\cdots e_j E^l_{i,j}(x) e_{j+1}\cdots e_{i+1}e_{i+2} \qquad \text{(commuting square)} \\
&\quad = S_{i+1,j} \circ E^l_{i,j}(x) e_{i+2}.
\end{aligned}
$$

Thus, $S_{i+1,j} \circ E^l_{i,j} = E^l_{i+2} \circ S_{i,j}$.

(4) This is a particular case of (3) by the property of conditional expectation.

(5) For $x \in A_{i,j}$, $[x, e_{j+1}] = 0$,

$$
\begin{aligned}
&S_{i,j}(x) e_{i+1}e_{i+2}\cdots e_{j+1} \\
&\quad = d^{2j-2i+2} E^l_{i,j+2}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j \cdots e_{i+2}e_{i+1}) e_{i+1}e_{i+2}\cdots e_{j+1} \\
&\quad = d^{2j-2i}(e_{i+1}e_{i+2}\cdots e_j x e_{j+1} e_j \cdots e_{i+2}e_{i+1}) e_{i+2}\cdots e_{j+1} \qquad \text{(pull down)} \\
&\quad = d^{2j-2i-2}(e_{i+1}e_{i+2}\cdots e_j x) e_{j+1}\cdots e_{i+2}\cdots e_{j+1} \\
&\quad = \cdots \qquad (e_t e_{t\pm 1} e_t = d^{-2} e_t) \\
&\quad = e_{i+1}e_{i+2}\cdots e_j x e_{j+1} \\
&\quad = e_{i+1}e_{i+2}\cdots e_j e_{j+1} x.
\end{aligned}
$$

(6) By (3) and Definition 2.16.

(7) Note that the map $A_{i+2,j+2} \subset A_{i+1,j+2} \ni y \mapsto ye_{i+1} \in A_{i,j+2}$ is injective, we shall prove that $S_{i,j}(e_k)e_{i+1} = e_{k+2}e_{i+1}$. For $i+1 \le k \le j-1$,

$$\begin{aligned}
S_{i,j}(e_k)e_{i+1} &= d^{2j-2i+2}E^{l}_{i,j+2}(e_{i+1}e_{i+2}\cdots e_j e_k e_{j+1}e_j\cdots e_{i+1})e_{i+1} \\
&= d^{2j-2i}e_{i+1}e_{i+2}\cdots e_j e_k e_{j+1}e_j\cdots e_{i+1} && \text{(pull down)} \\
&= d^{2j-2i}e_{i+1}\cdots e_{k-1}e_k e_{k+1}e_k e_{k+2}\cdots e_j e_{j+1}e_j\cdots e_{k+2}e_{k+1}\cdots e_{i+1} \\
& && ([e_i,e_j]=0 \text{ for } |i-j|\ge 2) \\
&= d^{2j-2i}e_{i+1}\cdots e_{k-1}(e_k e_{k+1}e_k)(e_{k+2}\cdots e_j e_{j+1}e_j\cdots e_{k+2})e_{k+1}\cdots e_{i+1} \\
&= d^{2k-2i}e_{i+1}\cdots e_{k-1}e_k e_{k+2}e_{k+1}e_k\cdots e_{i+1} && (e_t e_{t\pm1}e_t = d^{-2}e_t) \\
&= d^{2k-2i}e_{k+2}e_{i+1}\cdots e_{k-1}e_k e_{k+1}e_k\cdots e_{i+1} \\
&= e_{k+2}e_{i+1} && (e_t e_{t\pm1}e_t = d^{-2}e_t)
\end{aligned}$$

■

Definition 2.19 (2n-shift map). Define $S^{(n)}_{i,j} : A_{i,j} \to A_{i+2n,j+2n}$ by

$$S^{(n)}_{i,j} = S_{i+2(n-1),j+2(n-1)} \circ S^{(n-1)}_{i,j} = S_{i+2(n-1),j+2(n-1)} \circ S_{i+2(n-2),j+2(n-2)} \circ \cdots \circ S_{i,j}$$

to be the $2n$-*shift map*.

Proposition 2.20. *The followings are the properties of the* $2n$*-shift map.*

(1) $S^{(n)}_{i,j}$ *is a unital* $*$*-isomorphism.*

(2) (*commuting parallelogram*) $S^{(n)}_{i,j-1} \circ E^{r,k}_{i,j} = E^{r,k}_{i+2n,j+2n} \circ S^{(n)}_{i,j}$ *and* $S^{(n)}_{i+1,j} \circ E^{l,k}_{i,j} = E^{l,k}_{i+2n,j+2n} \circ S^{(n)}_{i,j}$.

(3) $S^{(n)}_{i,j+k}(x) = S^{(n)}_{i,j}(x)$ *for* $x \in A_{i,j}$ *and* $S^{(n)}_{i-k,j}(x) = S^{(n)}_{i,j}(x)$ *for* $x \in A_{i,j}$.

(4) (*shift*) *For* $x \in A_{i,j}$, $e^{i}_{j-i,n}x = S^{(n)}_{i,j}(x)e^{i}_{j-i,n}$. *By taking adjoint,* $xe^{i,*}_{j-i,n} = e^{i,*}_{j-i,n}S^{(n)}_{i,j}(x)$.

(5) $S^{(n)}_{i,j}$ *is trace-preserving.*

Proof. (1), (2), (3), (5) follow from Proposition 2.18.

(4) First, we show that

$$e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1}x = S^{n}_{i,j}(x)e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1} \quad \text{for } x \in A_{i,j},$$

$$\begin{aligned}
&S^{n}_{i,j}(x)e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1} \\
&\quad = S_{i+2(n-1),j+2(n-1)}(S^{(n-1)}_{i,j}(x))e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1} \\
&\quad = e^{i+2(n-1)}_{j-i,1}S^{(n-1)}_{i,j}(x)e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1} \\
&\quad = \cdots \\
&\quad = e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1}x.
\end{aligned}$$

Second, $e^i_{j-i,n} = a^i_{j-i,n} e^{i+2(n-1)}_{j-i,1} e^{i+2(n-2)}_{j-i,1} \cdots e^i_{j-i,1} b^i_{j-i,n}$ with $a^i_{j-i,n} \in A_{i,i+2n}$ and $b^i_{j-i,n} \in A_{j,j+2n}$, which will be showed below in Lemma 2.21 and 2.22. Then by the standard condition, since $x \in A_{i,j}$ and $S^{(n)}(x) \in A_{i+2n,j+2n}$, we have

$$[S^{(n)}_{i,j}(x), a^i_{j-i,n}] = 0 \quad \text{and} \quad [x, b^i_{j-i,n}] = 0,$$

which follows that

$$\begin{aligned} S^{(n)}_{i,j}(x) e^i_{j-i,n} &= S^{(n)}_{i,j}(x) a^i_{j-i,n} e^{i+2(n-1)}_{j-i,1} e^{i+2(n-2)}_{j-i,1} \cdots e^i_{j-i,1} b^i_{j-i,n} \\ &= a^i_{j-i,n} S^{(n)}_{i,j}(x) e^{i+2(n-1)}_{j-i,1} e^{i+2(n-2)}_{j-i,1} \cdots e^i_{j-i,1} b^i_{j-i,n} \\ &= a^i_{j-i,n} e^{i+2(n-1)}_{j-i,1} e^{i+2(n-2)}_{j-i,1} \cdots e^i_{j-i,1} x b^i_{j-i,n} \\ &= a^i_{j-i,n} e^{i+2(n-1)}_{j-i,1} e^{i+2(n-2)}_{j-i,1} \cdots e^i_{j-i,1} b^i_{j-i,n} x \\ &= e^i_{j-i,n} x. \end{aligned}$$

■

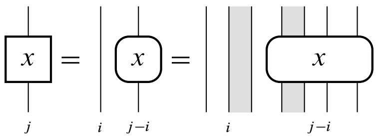
2.4. String diagram explanation

In this section, we use the Temperley–Lieb–Jones (TLJ) string diagram to explain the elements in $A_{i,j}$, horizontal (right) and vertical (left) conditional expectations, the Jones projections, $2n$-shift maps and their properties.

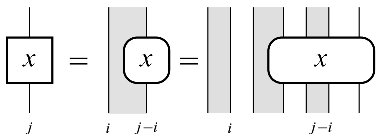
In the following sections, we will use these diagrams to do the algebraic computation and readers may interpret these diagrams directly into the algebraic computations by looking at the dictionary here.

(λ1) Element $x \in A_{i,j}$. $A_{i,j}$ is a (rectangular) box space with j shaded/unshaded strands where the left i strands are straight strands and together with a $j-i$ box space. We set the left part of left most strand to be always unshaded; the shading on the left part of the $j-i$ box space depends on the parity of i:

If $2 \mid i$:

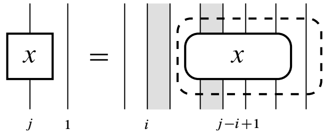


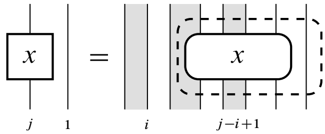
If $2 \nmid i$:



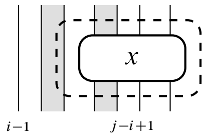
Remark. The reader shall understand the meaning of rectangular box and round box of an element. And the shading type of an element is the shading on the left of the round box.

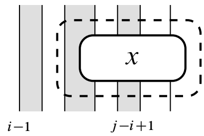
(λ2) Horizontal inclusion $x \in A_{i,j} \subset A_{i,j+1}$. The inclusion $A_{i,j} \subset A_{i,j+1}$ means adding one straight strand on the right and regarding the $j-i$ box space in $A_{i,j}$ as a part of the $j-i+1$ box space in $A_{i,j+1}$ together with the straight strand, which does not change the shading type of the box space:

If $2 \mid i$: 

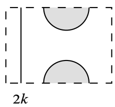
If $2 \nmid i$: 

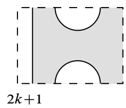
(λ3) Vertical inclusion $x \in A_{i,j} \subset A_{i-1,j}$. The inclusion $A_{i,j} \subset A_{i-1,j}$ means regarding the right most straight strand together with the original $j - i$ box space in $A_{i,j}$ as a part of the $j - i + 1$ box space in $A_{i-1,j}$, which changes the shading type of the box space:

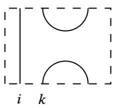
If $2 \mid i$: 

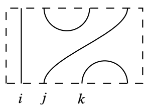
If $2 \mid i$: 

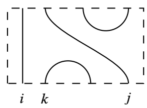
(λ4) Jones projections:

$e_{2k+1} =: d^{-1}$ 

$e_{2k+2} = d^{-1}$ 

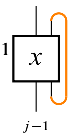
$e^i_k = d^{-k}$ 

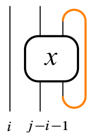
$e^i_{j,k} = d^{-k}$ 

$e^{i,*}_{j,k} = d^{-k}$ 

Remark. See the string diagram calculation of Jones projections in the Temperley–Lieb–Jones algebra.

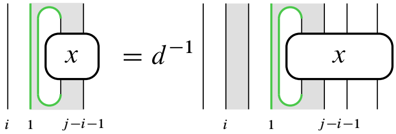
(λ5) Horizontal (right) conditional expectation $E^r_{i,j} : A_{i,j} \to A_{i,j-1}$, $x \in A_{i,j}$:

$E^r_{i,j}(x) = d^{-1}$ 

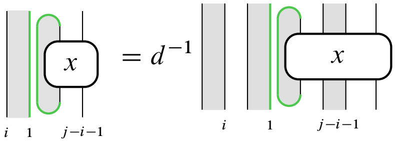
$= d^{-1}$ 

(λ6) Vertical (left) conditional expectation $E^l_{i,j} : A_{i,j} \to A_{i+1,j}$, $x \in A_{i,j}$. The vertical (left) conditional expectation is the left conditional expectation acting on the left of the box space and then adding one straight strand on the left of the box space, which changes the shading type of box space:

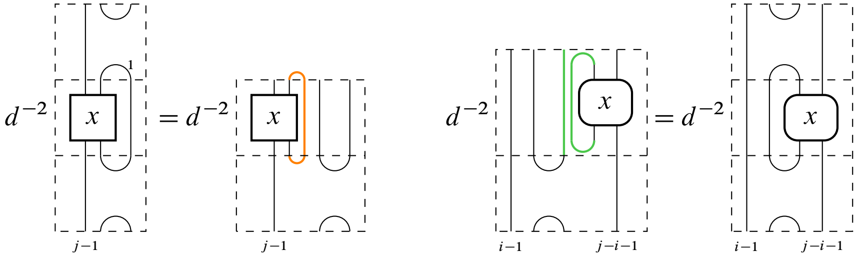
If $2 \mid i$: $E^l_{i,j}(x) = d^{-1}$



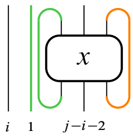
If $2 \nmid i$: $E^l_{i,j}(x) = d^{-1}$



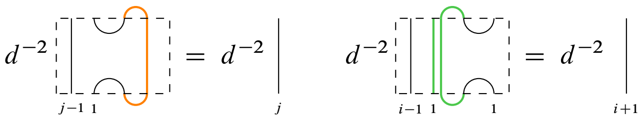
(λ7) $e_j x e_j = E^r_{i,j}(x) e_j$, for $x \in A_{i,j}$, $i+1 \le j$; $e_i x e_i = E^l_{i,j}(x) e_i$, for $x \in A_{i,j}$, $i+1 \le j$:



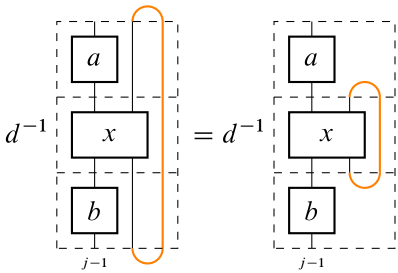
(λ8) Commuting square of conditional expectation: For $x \in A_{i,j}$, $E^l_{i,j} \circ E^r_{i,j+1}(x) = E^r_{i+1,j+1} \circ E^l_{i,j+1}(x)$:

$E^l_{i,j} \circ E^r_{i,j+1}(x) =$  $= E^r_{i+1,j+1} \circ E^l_{i,j+1}(x)$

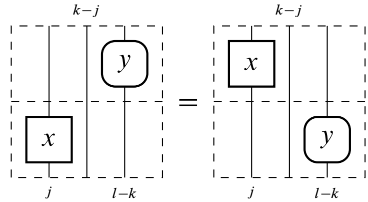
(λ9) $E^r_{i,j+1}(e_j) = E^l_{j-1,k}(e_j) = d^{-2}1$, for $j \ge i+1, k \ge j+1$.



(λ10) Conditional expectation property $E^r_{i,j}(axb) = aE^r_{i,j}(x)b$, for $x \in A_{i,j}$, $a, b \in A_{i,j-1}$; $E^l_{i,j}(axb) = aE^l_{i,j}(x)b$, for $x \in A_{i,j}$, $a, b \in A_{i+1,j}$.



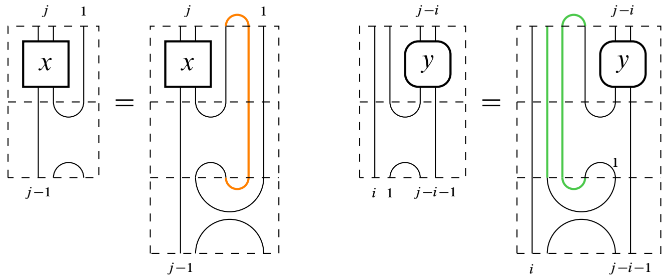
(λ11) Standard condition: For $x \in A_{i,j}$, $y \in A_{k,l}$ with $k \geq j$, then we regard x, y as elements in $A_{i,l}$, $xy = yx$.



(λ12) Pull down condition

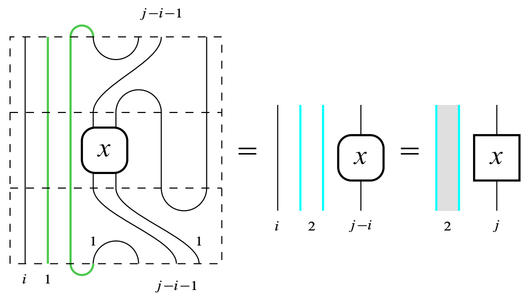
$d^2 E^r_{i,j+1}(xe_j)e_j = xe_j$, for $x \in A_{i,j+1}$, $j \geq i \geq 0$;

$d^2 E^l_{i-1,j}(xe_i)e_i = xe_i$, for $x \in A_{i-1,j}$, $j \geq i \geq 1$:



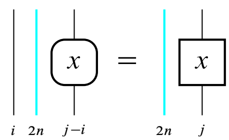
(λ13) 2-shift map $S_{i,j} : A_{i,j} \to A_{i+2,j+2}$: For $x \in A_{i,j}$,

$S_{i,j}(x) =$



$2n$-shift map $S^{(n)}_{i,j} : A_{i,j} \to A_{i+2n,j+2n}$: For $x \in A_{i,j}$,

$S^{(n)}_{i,j}(x) =$

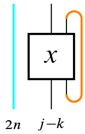


(λ14) Commuting parallelogram:

For $x \in A_{i,j}$, $S^{(n)}_{i,j-1} \circ E^{r,k}_{i,j}(x) = E^{r,k}_{i+2n,j+2n} \circ S^{(n)}_{i,j}(x)$;

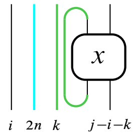
For $x \in A_{i,j}$, $S^{(n)}_{i+1,j} \circ E^{l,k}_{i,j}(x) = E^{l,k}_{i+2n,j+2n} \circ S^{(n)}_{i,j}(x)$.

$$S^{(n)}_{i,j-1} \circ E^{r,k}_{i,j}(x) =$$



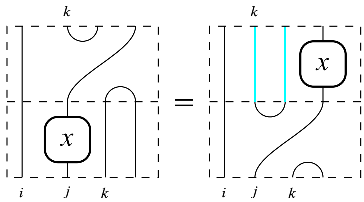
$$= E^{r,k}_{i+2n,j+2n} \circ S^{(n)}_{i,j}(x)$$

$$S^{(n)}_{i+1,j} \circ E^{l,k}_{i,j}(x) =$$



$$= E^{l,k}_{i+2n,j+2n} \circ S^{(n)}_{i,j}(x)$$

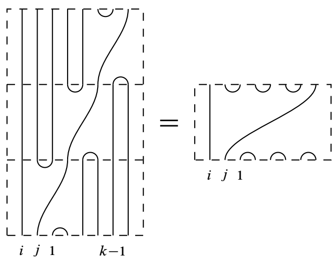
(λ15) Shift property: For $x \in A_{i,j}$, $e^{i}_{j,k}x = S^{k}_{i,j}(x)e^{i}_{j,k}$.

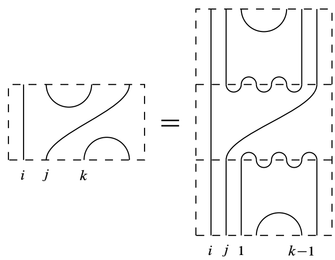


2.5. Some useful lemmas

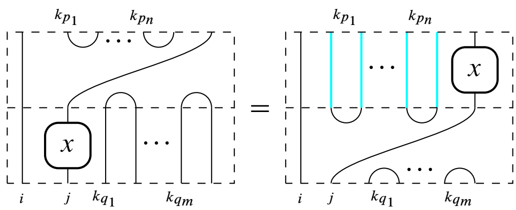
In this section, we are going to show some important lemmas. One can interpret the string diagram computation into algebraic computation by the above dictionary.

Lemma 2.21.





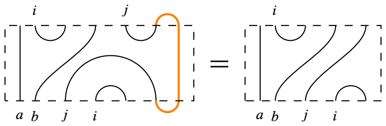
Lemma 2.22. *For* $\sum_{l=1}^{n} k_{p_l} = \sum_{r=1}^{m} k_{q_r}$, $k_{p_l}, k_{q_r} \in \mathbb{Z}_{\geq 0}$, *and* $x \in A_{i,j}$, *we have:*



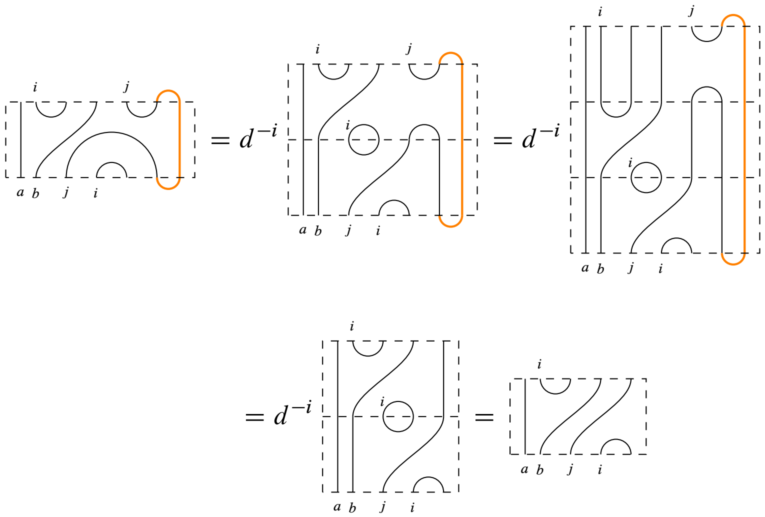
Proof. By the above lemma. ∎

These two lemmas are used in the proof of Proposition 2.20 (4).

Lemma 2.23.

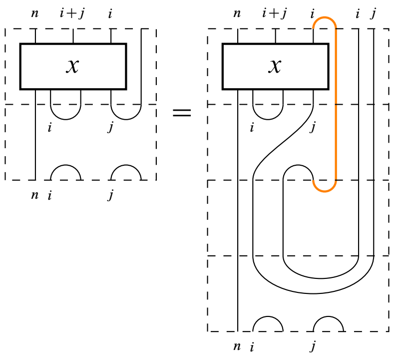


Proof.

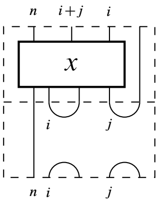


∎

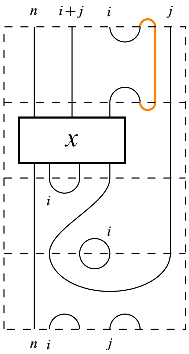
Lemma 2.24 ([7]). *For $x \in A_{m,n+2i+j}$, $m \leq n + 2i + j$, we have:*



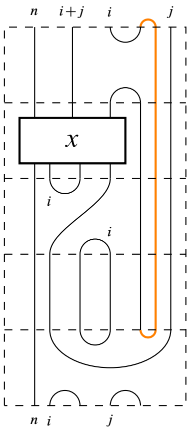
Proof.



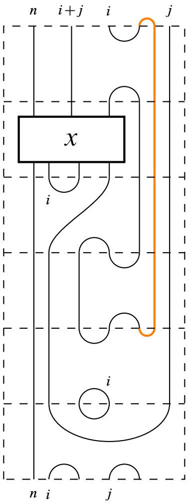
$\overset{(\lambda 9)}{=} d^{-i}$



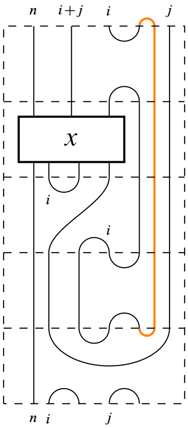
$\overset{(\lambda 10)}{=} d^{-i}$



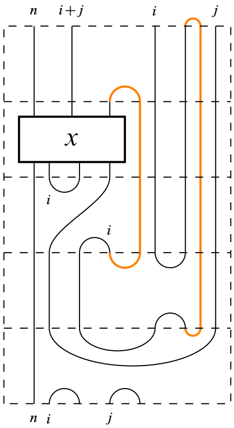
$\overset{(\lambda 9)}{=} d^{-i}$



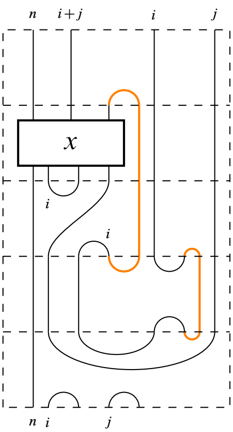
$\overset{(\lambda 10)}{=}$



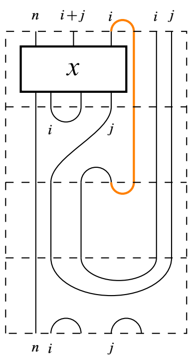
$\overset{(\lambda 7)}{=}$



$\overset{(\lambda 10)}{=}$



$\overset{(\lambda 9)}{=}$



■

2.6. From standard λ-lattice to pivotal planar tensor category

2.6.1. Planar tensor category.

Definition 2.25. A planar tensor category $\mathcal{A}_0$ is a 2-shaded rigid C* multitensor category (see Definition 2.49) with the following properties.

(a) $\mathcal{A}_0$ is a 2-shaded category with objects $[n,+]$, $[n,-]$, $n \in \mathbb{Z}_{\geq 0}$, where $1^+ := [0,+]$, $1^- := [0,-]$ are simple and the tensor unit $1_{\mathcal{A}_0} = 1^+ \oplus 1^-$, which means $\mathcal{A}_0$ is 2-shaded.

(b) $\mathcal{A}_0$ is a strict tensor category. The tensor product of objects are

$[m,?] \otimes [n,?]$	$[2i,+]$	$[2i+1,+]$	$[2i,-]$	$[2i+1,-]$
$[n,+]$	$[2i+n,+]$	0	0	$[2i+1+n,-]$
$[n,-]$	0	$[2i+1+n,+]$	$[2i+n,-]$	0

(c) $\mathcal{A}_0$ is rigid. There is an involution $\overline{(\cdot)}$ such that $\overline{[2i,\pm]} = [2i,\pm]$, $\overline{[2i+1,+]} = [2i+1,-]$ and $\overline{\overline{(\cdot)}} = \mathrm{id}$. For $X \in \mathcal{A}_0$, there exist

(1) $\mathrm{ev}_X : \overline{X} \otimes X \to 1^?$, where $? = +$ if X is unshaded on the right, i.e., $X = 1^+ \otimes X$, $? = -$ if X is shaded on the right, i.e., $X = 1^- \otimes X$;

(2) $\mathrm{coev}_X : 1^? \to X \otimes \overline{X}$, where $? = +$ if X is unshaded on the left, $? = -$ if X is shaded on the left.

such that

- $(\mathrm{id}_X \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X$.
- $(\mathrm{ev}_X \otimes \mathrm{id}_{\overline{X}}) \circ (\mathrm{id}_{\overline{X}} \otimes \mathrm{coev}_X) = \mathrm{id}_{\overline{X}}$.
- $\mathrm{ev}_{\overline{X}} := (\mathrm{coev}_X)^\dagger$ and $\mathrm{coev}_{\overline{X}} = (\mathrm{coev}_X)^\dagger$.

In other words, $\overline{(\cdot)}$ is a unitary dual functor, which will be discussed in Section 2.7.1.

Definition 2.26. We call a planar tensor category $\mathcal{A}_0$ *pivotal*, if the left trace Tr_L and right trace Tr_R defined as follows are faithful normal tracial. For $X = [2k+1,+]$ and $f \in \mathcal{A}_0(X \to X)$, since $\overline{[2k+1,+]} = [2k+1,-]$, we define

$$\mathrm{ev}_X \circ (\mathrm{id}_{\overline{X}} \otimes f) \circ \mathrm{ev}_X^\dagger =: \mathrm{Tr}_L(f)\mathrm{id}_{1^+},$$
$$\mathrm{coev}_X^\dagger \circ (f \otimes \mathrm{id}_{\overline{X}}) \circ \mathrm{coev}_X =: \mathrm{Tr}_R(f)\mathrm{id}_{1^-}.$$

We call $\mathcal{A}_0$ *spherical* if $\mathrm{Tr}_R(f) = \mathrm{Tr}_L(f)$ for all f. Similar for other three cases $[2k,+]$, $[2k,-]$, and $[2k+1,-]$.

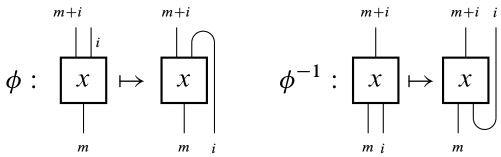
And there exists a $d > 0$ such that $\mathrm{ev}_{\overline{[n,?]}} \circ \mathrm{coev}_{[n,?]} = d^{2n} \cdot 1^?$, $? = +,-$.

Remark 2.27. The traces $\mathrm{Tr}_L, \mathrm{Tr}_R$ are defined in the sense of Definition 2.45.

Definition 2.28. The 2-shaded Temperley–Lieb–Jones multitensor category $\mathcal{TLJ}(d)$ is a planar tensor category with the endomorphism spaces being 2-shaded Temperley–Lieb–Jones algebras with modulus d, namely, $\mathrm{End}([n,+])$ is a 2-shaded Temperley–Lieb–Jones algebra with n points on one side and unshaded on the left; $\mathrm{End}([n,-])$ is a 2-shaded Temperley–Lieb algebra with n points on one side and shaded on the left.

Remark 2.29. The morphisms in $\mathcal{A}_0$ are determined by its representation in endomorphism and its domain and range.

There is a canonical isomorphism $\phi : \mathcal{A}_0([m,+],[m+2i,+]) \to \mathcal{A}_0([m+i,?] \to [m+i,?])$ by Frobenius reciprocity, where $? = +$ if i is even and $? = -$ if i is odd.



For morphism $x \in \mathcal{A}([m,?] \to [n,?])$, we can write a triple $(\phi(x); [m,?],[n,?])$ to represent x, where $\phi(x) \in \mathrm{End}([\frac{m+n}{2},?])$, which is called the *endomorphism representation part* of x. In the following context, we simply write x instead of $\phi(x)$ in the triple $(x; [m,?],[n,?])$.

2.6.2. From standard λ-lattice to pivotal planar tensor category. We regard the elements in algebra $A_{i,j}$ as endomorphisms in the category and the idea in Remark 2.29 gives us the way to represent the morphism by using its corresponding endomorphism, source and target, then we can construct a pivotal planar tensor category from a given standard λ-lattice.

Definition 2.30. Let $A = (A_{i,j})_{0\le i\le j}$ be a standard λ-lattice. We define a planar tensor category $\mathcal{A}_0$ from A as follows.

(a) The objects of $\mathcal{A}_0$ are the symbols $[n,+],[n,-]$ for $n \in \mathbb{Z}_{\ge 0}$.

(b) Given $n \ge 0$, define $\mathcal{A}_0([n,+] \to [n,+]) := A_{0,n}$ and $\mathcal{A}_0([n,-] \to [n,-]) := A_{1,n+1}$. Define $1 := [0,+] \oplus [0,-]$.

(c) The identity morphism in $\mathcal{A}_0([n,+] \to [n,+])$ is $1_{A_{0,n}}$ and in $\mathcal{A}_0([n,-] \to [n,-])$ is $1_{A_{1,n+1}}$.

(d) For $(x; [n,+],[n+2k,+])$ (or $(x; [n+2k,+],[n,+])$), we define the dagger structure as $(x; [n,+],[n+2k,+])^\dagger := (x^*; [n+2k,+],[n,+])$, where $x, x^* \in A_{0,n+k}$; for $(x; [n,-],[n+2k,-])$ (or $(x; [n+2k,-],[n,-])$), we define $(x; [n,-],[n+2k,-])^\dagger := (x^*; [n+2k,-],[n,-])$, where $x, x^* \in A_{1,n+k+1}$.

(e) We define composition in six cases.

(C1) $(y; [n+2i,+],[n+2i+2j,+]) \circ (x; [n,+],[n+2i,+])$
$= (d^i E^{r,i}_{0,n+2i+j}(yxe^n_{j,i}); [n,+],[n+2i+2j,+])$,
where $x \in A_{0,n+i}$, $y \in A_{0,n+2i+j}$ and $d^i E^{r,i}_{0,n+2i+j}(yxe^n_{j,i}) \in A_{0,n+i+j}$.

(C2) $(y; [n+2i+2j,+],[n+2i,+]) \circ (x; [n,+],[n+2i+2j,+])$
$= (d^i E^{r,i+j}_{0,n+2i+j}(yxe^{n,*}_{j,i}); [n,+],[n+2i,+])$,
where $x \in A_{0,n+i+j}$, $y \in A_{0,n+2i+j}$ and $d^i E^{r,i+j}_{0,n+2i+j}(yxe^{n,*}_{j,i}) \in A_{0,n+i}$.

(C3) $(y; [n,+], [n+2i+2j,+]) \circ (x; [n+2i,+], [n,+])$
$= (d^i y e_{j,i}^{n,*} x; [n+2i,+], [n+2i+2j,+])$,
where $x \in A_{0,n+i}$, $y \in A_{0,n+i+j}$ and $d^i y e_{j,i}^{n,*} x \in A_{0,n+2i+j}$.

(C4) $(y; [n+2i,-], [n+2i+2j,-]) \circ (x; [n,-], [n+2i,-])$
$= (d^i E_{1,n+2i+j+1}^{r,i}(yxe_{j,i}^{n+1}); [n,+], [n+2i+2j,+])$,
where $x \in A_{1,n+i+1}$, $y \in A_{1,n+2i+j+1}$ and $d^i E_{1,n+2i+j+1}^{r,i}(yxe_{j,i}^{n+1}) \in A_{1,n+i+j+1}$.

(C5) $(y; [n+2i+2j,-], [n+2i,-]) \circ (x; [n,-], [n+2i+2j,-])$
$= (d^i E_{1,n+2i+j+1}^{r,i+j}(yxe_{j,i}^{n+1,*}); [n,-], [n+2i,-])$,
where $x \in A_{1,n+i+j+1}$, $y \in A_{1,n+2i+j+1}$ and $d^i E_{1,n+2i+j+1}^{r,i+j}(yxe_{j,i}^{n+1,*}) \in A_{1,n+i+1}$.

(C6) $(y; [n,-], [n+2i+2j,-]) \circ (x; [n+2i,-], [n,-])$
$= (d^i y e_{j,i}^{n+1,*} x; [n+2i,-], [n+2i+2j,-])$,
where $x \in A_{1,n+i+1}$, $y \in A_{1,n+i+j+1}$ and $d^i y e_{j,i}^{n+1,*} x \in A_{1,n+2i+j+1}$.

If $x \in \mathcal{A}_0([n+2i,-] \to [n,-])$ and $y \in \mathcal{A}_0([n,-] \to [n+2i+2j,-])$, we define

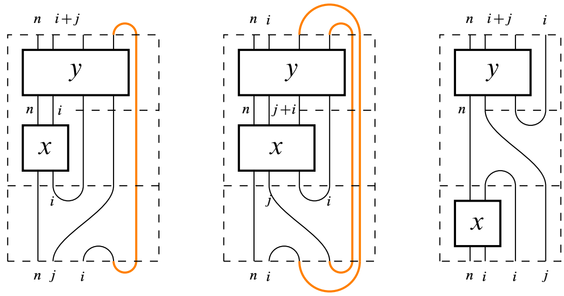
$$y \circ x := d^i y e_{j,i}^{n+1,*} x \in A_{1,n+2i+j+1} = \mathcal{A}_0\big([n+2i,-] \to [n+2i+2j,-]\big).$$

We define the composition $x^\dagger \circ y^\dagger := (y \circ x)^\dagger$, which defines composition

$$\mathcal{A}_0\big([n+2i+2j,-] \to [n,-]\big) \otimes \mathcal{A}_0\big([n,-] \to [n+2i,-]\big) \\ \to \mathcal{A}_0\big([n+2i+2j,-] \to [n+2i,-]\big).$$

According to [7, §3.4], the composition and dagger structure are well defined as Markov tower, and $\mathcal{A}_0$ is a C* category.

Before we define the tensor product of morphisms, we use string diagrams to explain the composition. The box space in the following diagram is always the endomorphism representation of the corresponding morphism.



(C1) (C2) (C3)

The string diagram of case (C4) comes from the string diagram of case (C1) by adding a straight strand on the leftmost of the diagram and changing the shading. In the same way, we obtain (C5) from (C2) and (C6) from (C3).

Now we define the tensor product of morphisms.

x	$x \otimes 1_j$
$(x;[m,+],[m+2i,+]), i \le j$	$(xe^{m}_{j-i,i};[m+j,+],[m+2i+j,+])$
$(x;[m,+],[m+2i,+]), i > j$	$(xe^{m,*}_{i-j,j};[m+j,+],[m+2i+j,+])$
$(x;[m,-],[m+2i,-]), i \le j$	$(xe^{m+1}_{j-i,i};[m+j,-],[m+2i+j,-])$
$(x;[m,-],[m+2i,-]), i > j$	$(xe^{m+1,*}_{i-j,j};[m+j,-],[m+2i+j,-])$

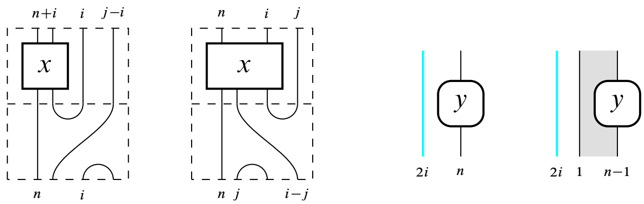
Definition 2.31. $x \otimes 1$ and $1 \otimes y$, $x, y \in \mathrm{Hom}(\mathcal{A}_0)$:

First, we define $x \otimes 1$ as

x	$x \otimes 1_j$
$(x;[m,+],[m+2i,+]), i \le j$	$(xe^{m}_{j-i,i};[m+j,+],[m+2i+j,+])$
$(x;[m,+],[m+2i,+]), i > j$	$(xe^{m,*}_{i-j,j};[m+j,+],[m+2i+j,+])$
$(x;[m,-],[m+2i,-]), i \le j$	$(xe^{m+1}_{j-i,i};[m+j,-],[m+2i+j,-])$
$(x;[m,-],[m+2i,-]), i > j$	$(xe^{m+1,*}_{i-j,j};[m+j,-],[m+2i+j,-])$

Because of the shading, we define $1 \otimes y$ as:

y	$1_{2i} \otimes y$	$1_{2i+1} \otimes y$
$(y;[n,+],[n \pm 2j,+])$	$(S^{(i)}_{0,n\pm j}(y);[n+2i,+],$ $[n+2i \pm 2j,+])$	0
$(y;[n,-],[n \pm 2j,-])$	0	$(S^{(i)}_{1,n+1\pm j}(y);[n+2i,-],$ $[n+2i \pm 2j,-])$



$i \le j$ $\quad$ $i > j$

Proposition 2.32. *For $x, y \in \mathrm{Hom}(\mathcal{A}_0)$, $(x \otimes 1) \circ (1 \otimes y) = (1 \otimes y) \circ (x \otimes 1)$.*

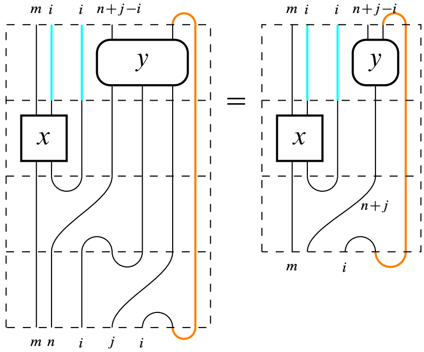
Proof. Here, we check the case $(x;[m,+],[m+2i,+])$ and $(y;[n,+],[n+2j,+])$, where $2 \mid m$ (or $(y;[n,-],[n+2j,-])$ if $2 \nmid m$) and $n + j \ge i$. We shall prove that

$$\begin{aligned}
&\big((x;[m,+],[m+2i,+]) \otimes (1;[n+2j,+],[n+2j,+])\big)\\
&\quad\circ \big((1;[m,+],[m,+]) \otimes (y;[n,+],[n+2j,+])\big)\\
&= \big((1;[m+2i,+],[m+2i,+]) \otimes (y;[n,+],[n+2j,+])\big)\\
&\quad\circ \big((x;[m,+],[m+2i,+]) \otimes (1;[n,+];[n,+])\big).
\end{aligned}$$

First, they both in $\mathcal{A}_0([m+n,+] \to [m+n+2i+2j,+])$.

The right-hand side:

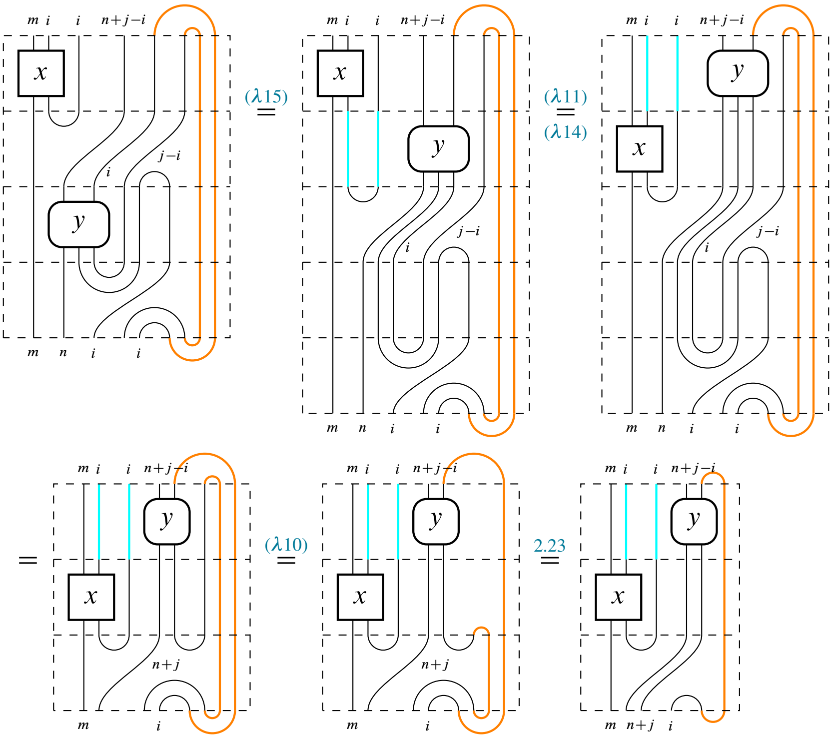
$$\begin{gathered}\big((1;[m+2i,+],[m+2i,+])\otimes\big(y;[n,+],[n+2j,+]\big)\big)\\ \circ\big(\big(x;[m,+],[m+2i,+]\big)\otimes\big(1;[n,+];[n,+]\big)\big):\end{gathered}$$



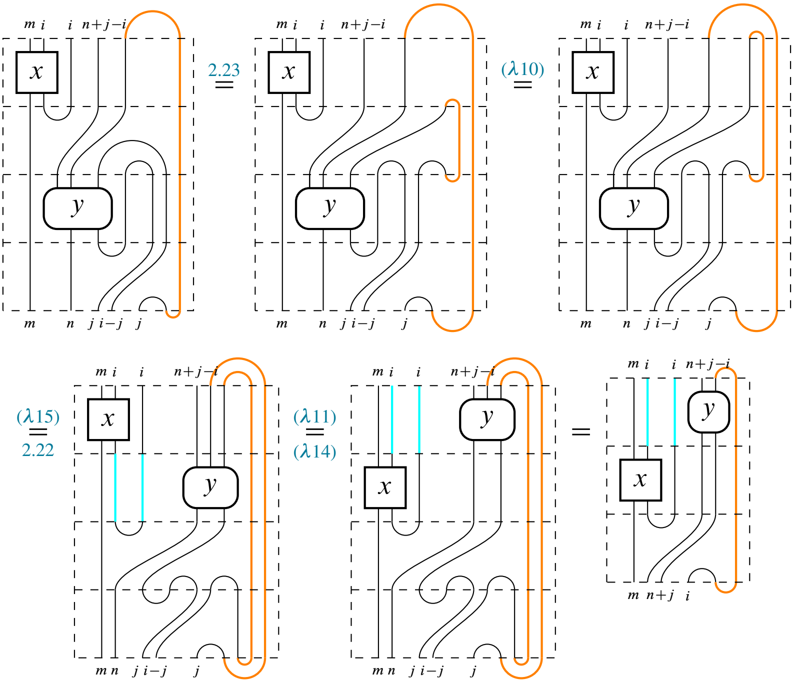
The left-hand side:

$$\begin{gathered}\big(\big(x;[m,+],[m+2i,+]\big)\otimes\big(1;[n+2j,+],[n+2j,+]\big)\big)\\ \circ\big(\big(1;[m,+],[m,+]\big)\otimes\big(y;[n,+],[n+2j,+]\big)\big):\end{gathered}$$

(1) If $i \le j$,



(2) If $i > j$,



Therefore, $(x \otimes 1) \circ (1 \otimes y) = (1 \otimes y) \circ (x \otimes 1)$ in this case. The remaining cases are left to the reader. ∎

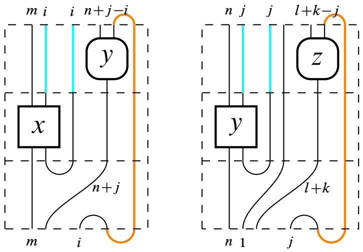
Definition 2.33 (Tensor product of morphisms). Define $x \otimes y := (x \otimes 1) \circ (1 \otimes y)$.

We need to prove that the tensor product defined above is functorial and associative.

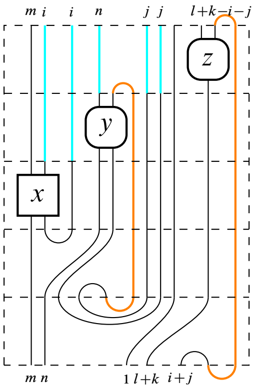
Proposition 2.34. *The tensor product is associative and strict, i.e., for* $x, y, z \in \mathrm{Hom}(\mathcal{A}_0)$, $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.

Proof. Here,we check the case $(x; [m,+], [m+2i,+])$, $(y; [n,+], [n+2j,+])$ and $(z; [l,-], [l+2k,-])$, where $2 \mid m, 2 \nmid n$ and $n + j \geq i, l + k \geq i + j$. Then $(x \otimes y) \otimes z$, $x \otimes (y \otimes z) \in \mathcal{A}_0([m+n+l,+] \to [m+n+l+2i+2j+2k,+])$.

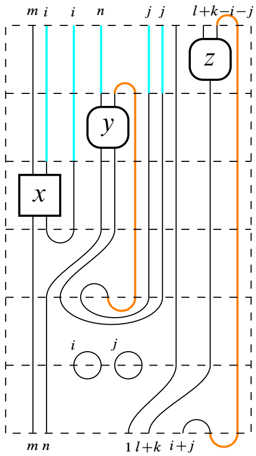
By Proposition 2.32, the endomorphism representation parts of $x \otimes y$ and $y \otimes z$ are defined in this way:



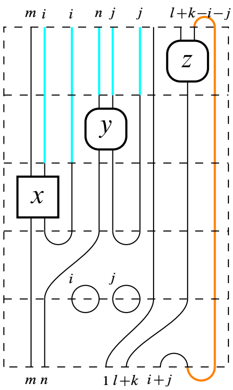
Then $(x \otimes y) \otimes z$:



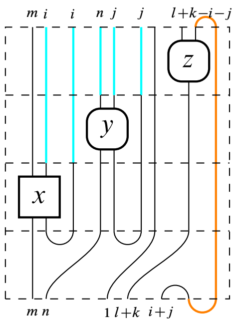
$= d^{-i-j}$



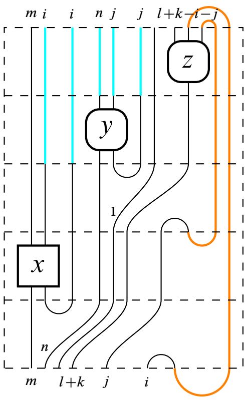
$\overset{2.24}{=} d^{-i-j}$



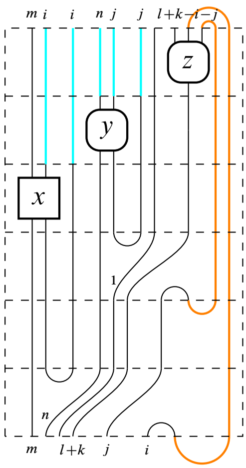
$=$

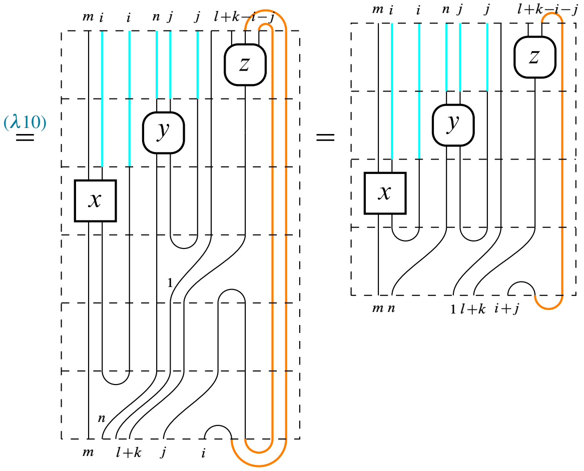


And $x \otimes (y \otimes z)$:



$\overset{(\lambda 10)}{=}$





Therefore, $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ in this case. Readers can check the rest of the cases by using the string diagram dictionary and the lemmas. ■

Proposition 2.35. *For* $x, y \in \mathrm{Hom}(\mathcal{A}_0)$,

$$(x \circ y) \otimes 1 = (x \otimes 1) \circ (y \otimes 1) \quad \textit{and} \quad 1 \otimes (x \circ y) = (1 \otimes x) \circ (1 \otimes y).$$

Proof. By our construction,

$$1 \otimes (x \circ y) = (1 \otimes x) \circ (1 \otimes y)$$

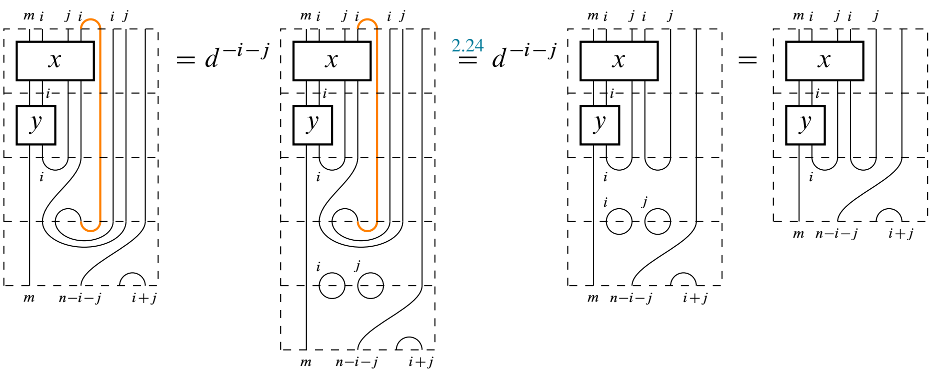
only uses the fact that the shift map is a $*$-homomorphism.

As for $(x \circ y) \otimes 1 = (x \otimes 1) \circ (y \otimes 1)$, we check the case $(x; [m, +], [m + 2i, +])$ and $(y; [m + 2i], [m + 2i + 2j, +])$, where $n \geq i + j$. Then

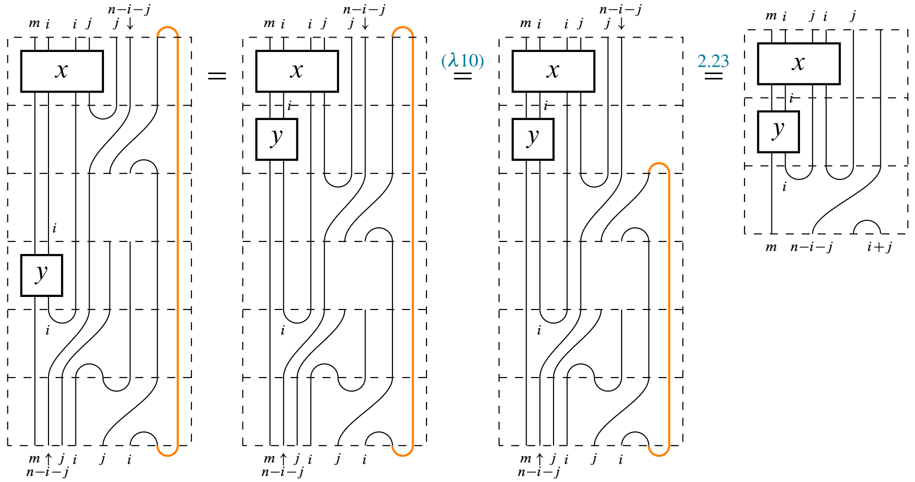
$$(x \circ y) \otimes 1_n, \quad (x \otimes 1_n) \circ (y \otimes 1_n) \in \mathcal{A}_0\big([m + n, +] \to [m + n + 2i + 2j, +]\big).$$

Next, let us compare their endomorphism representation parts.

$(x \circ y) \otimes 1_n$:



$(x \otimes 1_n) \circ (y \otimes 1_n)$:



Only the straight strands are allowed in the blank.

Therefore, $(x \circ y) \otimes 1 = (x \otimes 1) \circ (y \otimes 1)$ in this case. Readers can check the rest of the cases by using the string diagram dictionary and the lemmas. ∎

Corollary 2.36. *The tensor product is functorial, i.e., for $x, y, z, w \in \mathrm{Hom}(\mathcal{A}_0)$,*

$$(x \circ y) \otimes (z \circ w) = (x \otimes z) \circ (y \otimes w).$$

Therefore, the tensor product in Definition 2.33 is well defined.

Next, we show that $\mathcal{A}_0$ has a pivotal structure.

Definition 2.37 (ev and coev). Note that $[n, \pm] \otimes \overline{[n, \pm]} = [2n; \pm]$; $\overline{[n,+]} \otimes [n,+] = [2n,+]$ if $2 \mid n$ and $[2n,-]$ if $2 \nmid n$; $\overline{[n,-]} \otimes [n,-] = [2n,-]$ if $2 \mid n$ and $[2n,+]$ if $2 \nmid n$.

Define

$$\begin{aligned}
&\mathrm{coev}_{[n,+]} : 1^+ \to [2n,+] = [n,+] \otimes \overline{[n,+]} && \text{as } \mathrm{coev}_{[n,+]} = \big(d^n; [0,+], [2n,+]\big),\\
&\mathrm{ev}_{[n,+]} : \overline{[n,+]} \otimes [n,+] = [2n,?] \to 1^? && \text{as } \mathrm{ev}_{[n,+]} = \big(d^n; [2n,?], [0,?]\big),\ ? = +,\ \text{if } 2 \mid n,\\
&\mathrm{coev}_{[n,-]} : 1^- \to [2n,-] = [n,-] \otimes \overline{[n,-]} && \text{as } \mathrm{coev}_{[n,-]} = \big(d^n; [0,-], [2n,-]\big),\\
&\mathrm{ev}_{[n,-]} : \overline{[n,-]} \otimes [n,-] = [2n,?] \to 1^? && \text{as } \mathrm{ev}_{[n,-]} = \big(d^n; [2n,?], [0,?]\big),\ ? = -,\ \text{if } 2 \mid n.
\end{aligned}$$

Proposition 2.38. *$\mathcal{A}_0$ is rigid.*

Proof. First, we prove that

$$(\mathrm{id}_{[n,+]} \otimes \mathrm{ev}_{[n,+]}) \circ (\mathrm{coev}_{[n,+]} \otimes \mathrm{id}_{[n,+]}) = \mathrm{id}_{[n,+]}.$$

Note that

$$\mathrm{id}_{[n,+]} \otimes \mathrm{ev}_{[n,+]} = \big(S^{(n)}(d^n); [2n+n,+], [0+n,+]\big) = \big(d^n; [3n,+], [n,+]\big)$$

and

$$\begin{aligned}\mathrm{coev}_{[n,+]} \otimes \mathrm{id}_{[n,+]} &= \big(d^n e^0_{(n-n),n}; [0+n,+], [2n+n,+]\big)\\ &= \big(d^n e^0_{0,n}; [n,+], [3n,+]\big).\end{aligned}$$

Then by the composition case (C2), where $i=0$, $j=n$,

$$\begin{aligned}&(\mathrm{id}_{[n,+]} \otimes \mathrm{ev}_{[n,+]}) \circ (\mathrm{coev}_{[n,+]} \otimes \mathrm{id}_{[n,+]})\\ &\quad= \big(d^n; [3n,+], [n,+]\big) \circ \big(d^n e^0_{0,n}; [n,+], [3n,+]\big)\\ &\quad= \big(d^0 E^{r,0+n}_{0,n+2n}(d^{2n} e^0_{0,n} e^{n,*}_{j,i}); [n,+], [n+2i,+]\big)\\ &\quad= \big(d^{2n} E^{r,n}_{0,3n}(e^0_{0,n}); [n,+], [n,+]\big)\\ &\quad= \big(1; [n,+], [n,+]\big) = \mathrm{id}_{[n,+]}.\end{aligned}$$

The other three cases are left to the reader. Therefore, $\mathcal{A}_0$ is rigid. ■

Proposition 2.39. $\mathcal{A}_0$ *is pivotal and spherical.*

Proof. First, we prove that the right trace Tr_R is a normal faithful trace. Let $X=[n,+]$. Given $(f; [n,+], [n,+])$, $f \otimes \mathrm{id}_{\overline{[n,+]}} = (f; [2n,+], [2n,+])$, then

$$\begin{aligned}\mathrm{Tr}_R(f) &= \mathrm{coev}^\dagger_{[n,+]} \circ (f \otimes \mathrm{id}_{\overline{[n,+]}}) \circ \mathrm{coev}_{[n,+]}\\ &= \big(d^n; [2n,+], [0,+]\big) \circ \big(f; [2n,+], [2n,+]\big) \circ \big(d^n; [0,+], [2n,+]\big)\\ &= \big(d^n; [2n,+], [0,+]\big) \circ \big(d^n E^{r,n}_{0,2n}(f \cdot d^n e^0_{0,n}); [0,+], [2n,+]\big)\\ &= \big(d^n; [2n,+], [0,+]\big) \circ \big(f; [0,+], [2n,+]\big)\\ &= \big(d^0 E^{r,n}_{0,n}(f e^{0,*}_{n,0}); [0,+]; [0,+]\big)\\ &= \big(\mathrm{tr}(f); [0,+], [0,+]\big).\end{aligned}$$

The third equality uses Definition 2.30(e)(C1), where $n=0$, $i=n$, $j=0$; the forth equality uses (λ9); the fifth equality uses (C2), where $n=i=0$, $j=n$.

The case $X=[n,-]$ is left to the reader.

Next, we prove that the left trace Tr_L is a normal faithful trace. Let $X=[2n,+]$. Given $(f; [2n,+], [2n,+])$, $\mathrm{id}_{\overline{[2n,+]}} \otimes f = (S^{(n)}_{0,2n}(f); [4n,+], [4n,+])$, then

$$\begin{aligned}\mathrm{Tr}_L(f) &= \mathrm{ev}_{[2n,+]} \circ (\mathrm{id}_{\overline{[2n,+]}} \otimes f) \circ \mathrm{ev}^\dagger_{[2n,+]}\\ &= \big(d^{2n}; [4n,+], [0,+]\big) \circ \big(S^{(n)}_{0,2n}(f); [4n,+], [4n,+]\big) \circ \big(d^{2n}; [0,+], [4n,+]\big)\\ &= \big(d^{2n}; [4n,+], [0,+]\big) \circ \big(d^{2n} E^{r,2n}_{0,4n}(S^{(n)}_{0,2n}(f) \cdot d^{2n} e^0_{0,2n}); [0,+], [4n,+]\big)\\ &= \big(d^{4n} \cdot d^0 E^{r,2n}_{0,2n}(E^{r,2n}_{0,4n}(S^{(n)}_{0,2n}(f) e^0_{0,2n}) e^{0,*}_{0,2n}); [0,+], [0,+]\big)\\ &= \big(\mathrm{tr}(f); [0,+], [0,+]\big).\end{aligned}$$

The last equality: note that $e^{0,*}_{0,2n} = 1$ and $E^{r,2n}_{0,2n} \circ E^{r,2n}_{0,4n} = \mathrm{tr} = E^{r,2n}_{2n,4n} \circ E^{l,2n}_{0,4n}$, $S^{(n)}_{0,2n}(f) \in A_{2n,4n}$ and $S^{(n)}_{0,2n}$ is trace-preserving, then

$$\begin{aligned}
&d^{4n} \cdot d^0 E^{r,2n}_{0,2n}\big(E^{r,2n}_{0,4n}(S^{(n)}_{0,2n}(f)e^0_{0,2n})e^{0,*}_{0,2n}\big)\\
&\quad = d^{4n}\,\mathrm{tr}\,\big(S^{(n)}_{0,2n}(f)e^0_{0,2n}\big)\\
&\quad = d^{4n} E^{r,2n}_{2n,4n} \circ E^{l,2n}_{0,4n}\big(S^{(n)}_{0,2n}(f)e^0_{0,2n}\big)\\
&\quad = E^{r,2n}_{2n,4n}\big(S^{(n)}_{0,2n}(f)\big) \quad \big(\text{by Proposition 2.20 (2)}\big)\\
&\quad = E^{r,2n}_{0,2n}(f) = \mathrm{tr}(f).
\end{aligned}$$

The cases $X = [2n+1,+], [n,-]$ are left to the reader.

Therefore, $\mathrm{Tr}_R = \mathrm{Tr}_L$ is the trace, so $\mathcal{A}_0$ has a pivotal structure.

Moreover, by the composition case (C2), where $i = n = 0$, $j = n$,

$$\begin{aligned}
\mathrm{ev}_{[n,+]} \circ \mathrm{coev}_{[n,+]} &= \big(d^n; [2n,+], [0,+]\big) \circ \big(d^n; [0,+], [2n,+]\big)\\
&= \big(d^0 E^{r,n}_{0,2n}(d^{2n} e^{0,*}_{n,0}); [0,+], [0,+]\big)\\
&= \big(d^{2n}; [0,+], [0,+]\big) = d^{2n} \cdot 1^+.
\end{aligned}$$

Similarly, $\mathrm{ev}_{\overline{[n,-]}} \circ \mathrm{coev}_{[n,-]} = d^{2n} \cdot 1^-$. ∎

Combining the above propositions, $\mathcal{A}_0$ constructed from a standard λ-lattice is a pivotal planar tensor category.

2.7. From 2-shaded rigid C* multitensor category to standard λ-lattice

In this section, we show the relation between the 2-shaded rigid C* multitensor category and planar tensor category, and give the construction from the category to standard λ-lattice.

2.7.1. Rigid C* multitensor category. In this subsection, we are going to briefly review the unitary dual functors in a rigid C* (multi)tensor category $\mathcal{C}$ [37].

Definition 2.40. [44, 45] Recall that every object $c \in \mathcal{C}$ is *dualizable*, i.e., there is an object $\bar{c} \in \mathcal{C}$ together with morphisms $\mathrm{ev}_c \in \mathcal{C}(\bar{c} \otimes c \to 1_{\mathcal{C}})$ and $\mathrm{coev}_c \in \mathcal{C}(1_{\mathcal{C}} \to c \otimes \bar{c})$ satisfying the zigzag condition:

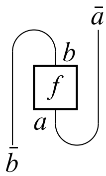
$$(\mathrm{id}_c \otimes \mathrm{ev}_c) \circ (\mathrm{coev}_c \otimes \mathrm{id}_c) = \mathrm{id}_c,$$

$$(\mathrm{ev}_c \otimes \mathrm{id}_{\bar{c}}) \circ (\mathrm{id}_{\bar{c}} \otimes \mathrm{coev}_c) = \mathrm{id}_{\bar{c}}.$$

We also require that every object $c \in \mathcal{C}$ admits a predual object $\underline{c}$ such that $\overline{(\underline{c})} \cong c$.

Definition 2.41. A choice of dual for every object in $\mathcal{C}$ assembles into a *dual functor* $\overline{(\cdot)} : \mathcal{C} \to \mathcal{C}^{\mathrm{mop}}$, which is a tensor functor with a canonical tensorator $\nu_{a,b}$. To be precise, for a morphism $f \in \mathcal{C}(a \to b)$, define

$$\bar{f} := (\mathrm{ev}_b \otimes \mathrm{id}_{\bar{a}}) \circ (\mathrm{id}_{\bar{b}} \otimes f \otimes \mathrm{id}_{\bar{a}}) \circ (\mathrm{id}_{\bar{b}} \otimes \mathrm{coev}_a) : \bar{b} \to \bar{a}.$$

$\bar{f} :=$ 

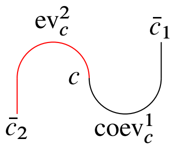
The tensorator $\nu_{a,b} : \bar{a} \otimes \bar{b} \to \overline{b \otimes a}$ is defined as

$$\nu_{a,b} := (\mathrm{ev}_a \otimes \mathrm{id}_{\overline{b\otimes a}}) \circ (\mathrm{id}_{\bar{a}} \otimes \mathrm{ev}_b \otimes \mathrm{id}_a \otimes \mathrm{id}_{\overline{b\otimes a}}) \circ (\mathrm{id}_{\bar{a}\otimes\bar{b}} \otimes \mathrm{coev}_{b\otimes a}).$$

Note that ν is completely determined by ev and coev.

Proposition 2.42. *Any two dual functors $\overline{(\cdot)}_1$ and $\overline{(\cdot)}_2$ are equivalent up to a unique natural isomorphism. Define $\zeta : \overline{(\cdot)}_2 \to \overline{(\cdot)}_1$ as follows: for $c \in \mathcal{C}$,*

$$\zeta_c := (\mathrm{ev}^2_c \otimes \mathrm{id}_{\bar{c}_1}) \circ (\mathrm{id}_{\bar{c}_2} \otimes \mathrm{coev}^1_c).$$

$\zeta_c =$ 

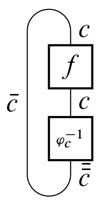
Then we have $\zeta(\bar{f}_2) = \zeta_a \circ \bar{f}_2 \circ \zeta_b^{-1} = \overline{\zeta(f)}_1$ for all $f \in \mathcal{C}(a \to b)$.

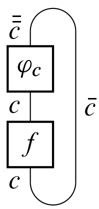
Definition 2.43. [11] A *pivotal structure* on a rigid monoidal category $\mathcal{C}$ is a pair $(\overline{(\cdot)}, \varphi)$, where $\overline{(\cdot)}$ is a dual functor and $\varphi : \mathrm{id} \Rightarrow \overline{\overline{(\cdot)}}$ is a monoidal natural isomorphism. To be precise, for all $a, b \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
a \otimes b \arrow[r, "\varphi_a \otimes \varphi_b"] \arrow[d, "\varphi_{a\otimes b}"'] & \overline{\overline{a}} \otimes \overline{\overline{b}} \arrow[d, "\overline{\nu_{\bar{b},\bar{a}}}"] \\
\overline{\overline{a \otimes b}} & \overline{\bar{b} \otimes \bar{a}} \arrow[l, phantom]
\end{tikzcd}$$

Definition 2.44 (Pivotal trace). Let $1_{\mathcal{C}} = \bigoplus_{i=1}^r 1_i$ be a decomposition into simples. For $c \in \mathcal{C}$ and $f \in \mathcal{C}(c \to c)$, define the left/right *pivotal traces* tr^{φ}_L and $\mathrm{tr}^{\varphi}_R : \mathcal{C}(c \to c) \to \mathcal{C}(1_{\mathcal{C}} \to 1_{\mathcal{C}}) \cong M_r(\mathbb{C})$ by

$$\mathrm{tr}^{\varphi}_L(f) := \mathrm{ev}_c \circ (\mathrm{id}_{\bar{c}} \otimes f) \circ (\mathrm{id}_{\bar{c}} \otimes \varphi_c^{-1}) \circ \mathrm{coev}_{\bar{c}}$$
$$\mathrm{tr}^{\varphi}_R(f) := \mathrm{ev}_{\bar{c}} \circ (\varphi_c \otimes \mathrm{id}_{\bar{c}}) \circ (f \otimes \mathrm{id}_{\bar{c}}) \circ \mathrm{coev}_c.$$

$\mathrm{tr}^{\varphi}_L(f) =$ 

$\mathrm{tr}^{\varphi}_R(f) =$ 

The traces are tracial and non-degenerate.

Definition 2.45. Let $p_i \in \mathcal{C}(1_\mathcal{C} \to 1_\mathcal{C})$ be the projection onto 1_i, $i = 1, 2, \dots, r$. We define the $M_r(\mathbb{C})$-valued traces Tr_L^φ and Tr_R^φ by the formulas:

$$\big(\mathrm{Tr}_L^\varphi(f)\big)_{i,j} \mathrm{id}_{1_j} := \mathrm{tr}_L^\varphi(p_i \otimes f \otimes p_j),$$
$$\big(\mathrm{Tr}_R^\varphi(f)\big)_{i,j} \mathrm{id}_{1_i} := \mathrm{tr}_R^\varphi(p_i \otimes f \otimes p_j).$$

Note that Tr_L^φ and Tr_R^φ are tracial, and $\mathrm{Tr}_L^\varphi(\bar{f}) = \mathrm{Tr}_R^\varphi(f)^T$ for all $f \in \mathcal{C}(c \to c)$.

We call the pivotal structure $(\overline{(\cdot)}, \varphi)$ *spherical*, if $\mathrm{Tr}_L^\varphi(f) = \mathrm{Tr}_R^\varphi(f)$, for all $c \in \mathcal{C}$, $f \in \mathcal{C}(c \to c)$.

Definition 2.46. For each $c \in C$, define $\mathrm{Dim}_L^\varphi, \mathrm{Dim}_R^\varphi \in M_r(\mathbb{C})$ by

$$\mathrm{Dim}_L^\varphi(c) := \mathrm{Tr}_L^\varphi(\mathrm{id}_c), \quad \mathrm{Dim}_R^\varphi(c) := \mathrm{Tr}_R^\varphi(\mathrm{id}_c).$$

If c is simple, then $\mathrm{Dim}_L^\varphi(c), \mathrm{Dim}_R^\varphi(c)$ have only one non-zero entry, which we denote $\dim_L^\varphi(c), \dim_R^\varphi(c)$ respectively.

If the pivotal structure $(\overline{(\cdot)}, \varphi)$ is spherical, $\mathrm{Dim}_L^\varphi(c) = \mathrm{Dim}_R^\varphi(c) := \mathrm{Dim}(c)$ for all object c.

Definition 2.47. A *dagger structure* on a $\mathbb{C}$-linear category is a collection of anti-linear maps $\dagger : \mathcal{C}(c \to d) \to \mathcal{C}(d \to c)$ for all $c, d \in \mathcal{C}$ such that $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$ and $(f^\dagger)^\dagger = f$. A morphism $f : \mathcal{C}(a \to b)$ is called unitary if $f^\dagger = f^{-1}$.

A dagger (multi)tensor category is a (multi)tensor category equipped with a dagger structure so that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms f, g, and all associator and unitors are unitary.

Definition 2.48. A functor between dagger categories $F : \mathcal{C} \to \mathcal{D}$ is called a *dagger functor* if $F(f^\dagger) = F(f)^\dagger$ for all $f \in \mathrm{Hom}(\mathcal{C})$.

Definition 2.49 (Rigid C* (multi)tensor category). A C* category is a dagger category which is Cauchy complete and each endomorphism algebra is a C*-algebra, where the dagger structure is compatible with the $*$-structure.

A C* (multi)tensor category is a dagger (multi)tensor category whose underlying dagger category is C*.

A rigid C* (multi)tensor category is a C* (multi)tensor category equipped with a dual functor. It is known that a rigid C* multitensor category is Cauchy complete if and only if it is semisimple [31].

Proposition 2.50 (Unitary dual functor). *Fix a dual functor $\overline{(\cdot)}$ on a rigid* C* *(multi)tensor category $\mathcal{C}$, the followings are equivalent:*

(1) *$\overline{(\cdot)}$ is a unitary dual functor, i.e., for all $a, b \in \mathcal{C}$, $f \in \mathcal{C}(a \to b)$, the tensorator $\nu_{a,b}$ is unitary and $\bar{f}^\dagger = \overline{f^\dagger}$.*

(2) *Defining $\varphi_c := (\mathrm{coev}_c^\dagger \otimes \mathrm{id}_{\bar{\bar{c}}}) \circ (\mathrm{id}_c \otimes \mathrm{coev}_{\bar{c}})$ is a pivotal structure $\varphi : \mathrm{id} \Rightarrow \overline{\overline{(\cdot)}}$.*

Proof. [42], see also [37, Prop. 3.9]. ∎

Definition 2.51. Two unitary dual functors are called *unitary equivalent*, if the canonical natural transformation ζ from Proposition 2.42 is unitary, i.e., ζ_c is unitary for all $c \in \mathcal{C}$.

Proposition 2.52. *For a unitary dual functor $\overline{(\cdot)}$, the left/right pivotal traces have alternate formulas:*

$$\mathrm{tr}_L^{\varphi}(f) = \mathrm{ev}_c \circ (\mathrm{id}_{\bar{c}} \otimes f) \circ \mathrm{ev}_c^{\dagger},$$
$$\mathrm{tr}_R^{\varphi}(f) = \mathrm{coev}_c^{\dagger} \circ (f \otimes \mathrm{id}_{\bar{c}}) \circ \mathrm{coev}_c.$$

Theorem 2.53 ([2], [37, Prop. 3.24]). *For a rigid* C* *(multi)tensor category $\mathcal{C}$, there exists a unique unitary dual functor whose induced pivotal structure is spherical up to unitary equivalence. In other words, the pivotal structure can be trivial, so that $\mathrm{ev}_{\bar{c}} = \mathrm{coev}_c^{\dagger}$ and $\mathrm{coev}_{\bar{c}} = \mathrm{ev}_c^{\dagger}$ for all $c \in \mathcal{C}$.*

2.7.2. 2-shaded rigid C* multitensor category with a choice of the generator and planar tensor category. Let $\mathcal{A}$ be a 2-shaded rigid C* multitensor category together with $1 = 1^+ \oplus 1^-$, where $1^+, 1^-$ are simple, and a generator $X = 1^+ \otimes X \otimes 1^-$. Here, the generating means for any simple object P, it is a direct summand of $X^{\mathrm{alt}\otimes n}$ or $\overline{X}^{\mathrm{alt}\otimes n}$ (defined below) for some $n \in \mathbb{Z}_{\geq 0}$.

Let $\overline{(\cdot)}$ be a unitary dual functor that induced a spherical pivotal structure φ. Note that only $(+,-)$ entry of $\mathrm{Dim}(X)$ is non-zero and we denote this number as d_X to be the modulus of category $\mathcal{C}$.

Construction 2.54. We construct a planar tensor category $\mathcal{A}_0$ from $(\mathcal{A}, X)$. By MacLane's coherence theorem, $\mathcal{A}$ is unitary equivalent to a strict tensor category with the above properties and the dual functor is strict, without loss of generality, we also denote it as $\mathcal{A}$. Construct the pivotal planar tensor category $\mathcal{A}_0$ as follows:

(a) Objects: Define $[0,+] := 1^+$, $[0,-] := 1^-$, and

$$[n,+] := [n-1,+] \otimes X^? = \underbrace{\big(\cdots\big((X \otimes \overline{X}) \otimes X\big) \otimes \cdots\big) \otimes X^?}_{n \text{ tensorands}} =: X^{\mathrm{alt}\otimes n},$$

where $X^? = \overline{X}$ if n is even and X if n is odd, and

$$[n,-] := [n-1,-] \otimes X^? = \underbrace{\big(\cdots\big((\overline{X} \otimes X) \otimes \overline{X}\big) \otimes \cdots\big) \otimes X^?}_{n \text{ tensorands}} =: \overline{X}^{\mathrm{alt}\otimes n},$$

where $X^? = X$ if n is even and $\overline{X}$ if n is odd, for $n \in \mathbb{Z}_{\geq 0}$.

(b) Morphisms: $\mathcal{A}_0$ is the full subcategory of $\mathcal{A}$ with above objects.

(c) Duality: The dual functor is unitary as a dual functor on the subcategory, which also induces a spherical pivotal structure on the subcategory.

Given $\mathcal{A}_0$ to be a pivotal planar tensor category, then its Cauchy completion $\widehat{\mathcal{A}_0}$ is a Cauchy completed 2-shaded rigid C* multitensor category with a generator $[1,+]$ and a canonical unitary dual functor $\overline{(\cdot)}_1$.

Proposition 2.55. *Suppose $\mathcal{A}_0$ is a pivotal planar tensor category constructed from $(\mathcal{A}, X)$, then there is a unitary equivalence between $(\widehat{\mathcal{A}_0}, [1,+])$ and the Cauchy completion of $(\mathcal{A}, X)$ with respect to their unitary dual functors.*

Remark 2.56. Suppose $\mathcal{A}, \mathcal{B}$ are two 2-shaded rigid C* multitensor categories with generator X and Y respectively and $\mathcal{A}_0, \mathcal{B}_0$ are corresponding pivotal planar tensor categories. Then $\mathcal{A}_0$ and $\mathcal{B}_0$ are unitary equivalent if and only if the Cauchy completions of $\mathcal{A}$ and $\mathcal{B}$ are unitary equivalent which maps generator to generator.

Remark 2.57. The planar tensor category $\mathcal{A}_0$ is not Cauchy complete, i.e., additive complete and idempotent complete. In fact, as for skeletalness, strictness, and Cauchy complete, most tensor categories can require at most two of them. $\mathsf{Hilb}(G)$ is an exception.

2.7.3. From planar tensor category to standard λ-lattice.

Construction 2.58. Let $\mathcal{A}_0$ be a pivotal planar tensor category with modulus d. Define $A_{0,j} = \mathrm{End}([j,+])$, $A_{1,j} = \mathrm{id}_{[1,+]} \otimes \mathrm{End}([j-1,-])$, $j \in \mathbb{Z}_{\geq 0}$, so that $A_{0,0} = A_{1,1} = \mathbb{C}$. In general, for $i \leq j$, define

$$A_{i,j} = \begin{cases} \mathrm{id}_{[i,+]} \otimes \mathrm{End}\left([j-i,+]\right) & 2 \mid i, \\ \mathrm{id}_{[i,+]} \otimes \mathrm{End}\left([j-i,-]\right) & 2 \nmid i. \end{cases}$$

Then we check $A = (A_{i,j})_{i,j\geq 0}$ to be a standard λ-lattice.

(a) The vertical inclusion $A_{i+1,j} \subset A_{i,j}$ is clear. The right inclusion: the right inclusion send $x \in A_{i,j}$ to $x \otimes \mathrm{id}_{[1,?]} \in A_{i,j+1}$, where $? = +$ if $2 \mid j$ and $? = -$ if $2 \nmid j$.

(b) Horizontal conditional expectation: Define $E^r_{i,j} : A_{i,j} \to A_{i,j-1}$ by

$$E^r_{i,2k}(x) = d^{-1}(\mathrm{id}_{[2k-1,+]} \otimes \mathrm{ev}_{\overline{[1,+]}}) \circ \left(x \otimes [1,+]\right) \circ (\mathrm{id}_{[2k-1,+]} \otimes \mathrm{coev}_{[1,+]}),$$
$$E^r_{i,2k+1}(x) = d^{-1}(\mathrm{id}_{[2k,+]} \otimes \mathrm{ev}_{\overline{[1,-]}}) \circ \left(x \otimes [1,-]\right) \circ (\mathrm{id}_{[2k,+]} \otimes \mathrm{coev}_{[1,-]}).$$

(c) Vertical conditional expectation: Define $E^l_{i,j} : A_{i,j} \to A_{i+1,j}$ by

$$E^l_{2k,j} = d^{-1}(\mathrm{id}_{[2k+2,+]} \otimes \mathrm{ev}_{\overline{[1,+]}}) \circ (\mathrm{id}_{[2,+]} \otimes x) \circ (\mathrm{id}_{[2k+2,+]} \otimes \mathrm{coev}_{[1,+]}),$$
$$E^l_{2k+1,j} = d^{-1}(\mathrm{id}_{[2k+3,+]} \otimes \mathrm{ev}_{\overline{[1,-]}}) \circ (\mathrm{id}_{[2,+]} \otimes x) \circ (\mathrm{id}_{[2k+3,+]} \otimes \mathrm{coev}_{[1,-]}).$$

(d) Jones projection: the nth Jones projection is defined as

$$e_{2k+1} = d^{-1} \cdot \mathrm{id}_{[2k,+]} \otimes (\mathrm{coev}_{[1,+]} \circ \mathrm{ev}_{\overline{[1,+]}}) \in A_{i,2k+2},$$
$$e_{2k+2} = d^{-1} \cdot \mathrm{id}_{[2k+1,+]} \otimes (\mathrm{coev}_{[1,-]} \circ \mathrm{ev}_{\overline{[1,-]}}) \in A_{i,2k+3}.$$

The check that $A = (A_{i,j})_{j\geq i\geq 0}$ satisfies Definition 2.7 (a), (b), (c) and the standard condition is left to the reader. In particular, $e_n e_{n\pm 1} e_n = d^{-2} e_n$, $E^r_{i,j+1}(e_j) = E^l_{j-1,k}(e_j) = d^{-2}1$.

Note that the dual functor is unitary and we divide the loop parameter, the composition of these conditional expectations is actually a unital trace on A.

Remark 2.59. The idea of drawing the string diagram explanation in Section 2.4 comes from here.

In this section, the class of unitary equivalent pairs $(\mathcal{A}, X)$ with $\mathcal{A}$ a 2-shaded rigid C* multitensor category and X a generator induces the class of isomorphic pivotal planar tensor categories; in Section 2.6, the class of isomorphic pivotal planar tensor categories is one to one corresponding to the class of isomorphic standard λ-lattices.

Combining above discussion, we can deduce the equivalence between standard λ-lattice A and pair 2-shaded rigid C* multitensor category with a generator $(\mathcal{A}, X)$.

Theorem 2.60. *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{array}{l}\text{Standard } \lambda\text{-lattices}\\ A = (A_{i,j})_{0\le i\le j}\end{array}\right\} \cong \left\{\begin{array}{l}\text{Pairs } (\mathcal{A}, X) \text{ with } \mathcal{A} \text{ a 2-shaded rigid C}^* \text{ multiten-}\\ \text{sor category with a generator } X\text{, i.e., } 1_{\mathcal{A}} = 1^+ \oplus 1^-,\\ 1^+, 1^- \text{ are simple and } X = 1^+ \otimes X \otimes 1^-\end{array}\right\}.$$

Equivalence on the left-hand side is unital $*$-isomorphism of standard λ-lattices; equivalence on the right-hand side is the unitary equivalence between their Cauchy completions which maps generator to generator.

3. Markov towers as standard right module over standard λ-lattice and module categories

Now we move to the module case. One motivation that regards a Markov tower as a right module over a standard λ-lattice is to answer the question in [7, Rem. 3.34].

3.1. Markov tower as a standard right module over standard λ-lattice

Definition 3.1.

$$\begin{array}{ccccccccccc}
M_0 & \subset & M_1 & \subset & M_2 & \subset & \cdots & \subset & M_n & \subset & \cdots\\
\cup & & \cup & & \cup & & & & \cup & & \\
A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & A_{0,n} & \subset & \cdots\\
 & & \cup & & \cup & & & & \cup & & \\
 & & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & A_{1,n} & \subset & \cdots
\end{array}$$

Let $A = (A_{i,j})_{0\le i\le j<\infty}$ be a standard λ-lattice with Jones projection $\{e_i\}_{i\ge 1}$ and compatible conditional expectations. Let $M = (M_n, e_n)_{n\ge 0}$ be a Markov tower with conditional expectation $E_i : M_i \to M_{i-1}$, $i \ge 1$. (M and A share the same Jones projections.) We call a Markov tower M a *standard right A-module* if it satisfies the following three conditions.

(a) $A_{0,i} \subset M_i$ is a unital inclusion, $i = 0, 1, 2, \ldots$.

(b) $E_i|_{A_{0,i}} = E^r_{0,i}$, $i = 1, 2, \ldots$.

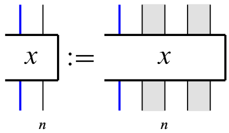
(c) (standard condition) $[M_i, A_{k,l}] = 0$ for $i \le k \le l$.

In the rest of this section, we only consider the Markov tower with $\dim(M_0) = 1$ unless stated.

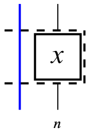
3.2. String diagram explanation

We now introduce the diagrammatic explanation of the element, conditional expectation, Jones projection and their relations in a Markov tower with the same spirit in Section 2.4.

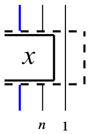
(MT1) Element $x \in M_n$:



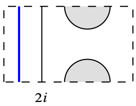
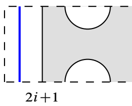
(MT2) Vertical inclusion $x \in A_{0,n} \subset M_n$:



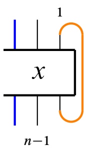
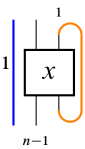
(MT3) Horizontal inclusion $x \in M_n \subset M_{n+1}$:



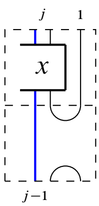
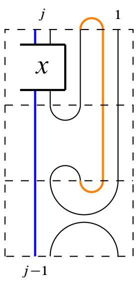
(MT4) Jones projections:

$e_{2i+1} = d^{-1}$  $\in M_{2i+2}$ $\quad e_{2i+2} = d^{-1}$  $\in M_{2i+3}$

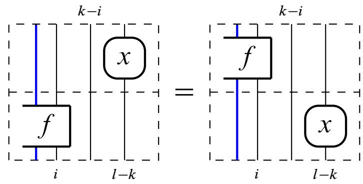
(MT5) Conditional expectation $E_n : M_n \to M_{n-1}$ and $E_n|_{A_{0,n}} = E^r_{0,n}$:

$E_n(x) = d^{-1}$  $, x \in M_n \quad E_n(x) = E^r_{0,n}(x) = d^{-1}$  $, x \in A_{0,n}$

(MT6) Pull down condition: For $x \in M_{n+1}$, $xe_n = dE_{n+1}(xe_n)e_n$.

 $=$ 

(MT7) Standard condition: For $f \in M_i$, $x \in A_{k,l}$ with $k \geq i$, then we regard ϕ, x as elements in M_l, $fx = xf$.



3.3. From Markov tower as a standard module to planar module category

3.3.1. Planar module category over planar tensor category.

Definition 3.2. Let $\mathcal{A}_0$ be a planar tensor category defined in Definition 2.25. Let $\mathcal{M}_0$ be an indecomposable semisimple C* right $\mathcal{A}_0$-module category with following properties:

(a) Object: The objects of $\mathcal{M}_0$ are $[n] = [n]_{\mathcal{M}_0}$, $n \in \mathbb{Z}_{\geq 0}$, where $[0]$ is simple.

(b) The tensor product of objects are

$$[m]_{\mathcal{M}_0} \lhd [n,+]_{\mathcal{A}_0} = \begin{cases} [m+n]_{\mathcal{M}_0} & \text{if } 2 \mid m, \\ 0 & \text{if } 2 \nmid m, \end{cases}$$

$$[m]_{\mathcal{M}_0} \lhd [n,-]_{\mathcal{A}_0} = \begin{cases} 0 & \text{if } 2 \mid m, \\ [m+n]_{\mathcal{M}_0} & \text{if } 2 \nmid m. \end{cases}$$

(c) Only $\mathcal{M}_0([n] \to [n \pm 2i])$ is non-zero, $n, i \in \mathbb{Z}_{\geq 0}$. The module product of morphisms in $\mathrm{Hom}(\mathcal{M}_0)$ and $\mathrm{Hom}(\mathcal{A}_0)$ should match the shading types.

(d) $\mathcal{M}_0$ is a strict right $\mathcal{A}_0$-module category, i.e., the module associator is identity. For $x_1, x_2 \in \mathcal{A}_0$ and $f \in \mathcal{M}_0$,

$$(f \lhd x_1) \lhd x_2 = f \lhd (x_1 \otimes x_2).$$

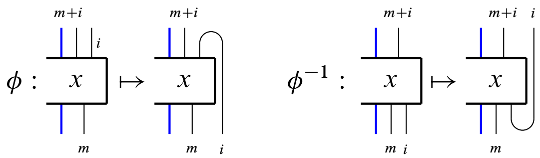
(e) $\mathcal{M}_0$ is a C* category with a natural dagger structure such that $\lhd$ is a dagger functor, i.e., for $x \in \mathrm{Hom}(\mathcal{A}_0)$ and $f \in \mathrm{Hom}(\mathcal{M}_0)$,

$$(f \lhd x)^\dagger = f^\dagger \lhd x^\dagger.$$

Such module category is called a *planar module category*.

Remark 3.3. Similar to Remark 2.29, every morphism in $\mathcal{M}_0$ is determined by its representation as an endomorphism and its domain and range.

There is a canonical isomorphism $\phi : \mathcal{M}_0([m] \to [m+2i]) \to \mathcal{M}_0([m+i] \to [m+i])$ by using the rigid structure on $\mathcal{A}_0$.



For morphism $x \in \mathcal{M}_0([m],[n])$, we can write a triple $(\phi(x);[m],[n])$ to represent x, where $\phi(x) \in \mathrm{End}([\frac{m+n}{2}])$, which is called the *endomorphism representation part* of x. In the following context, we simply write x instead of $\phi(x)$ in the triple $(x;[m],[n])$.

3.3.2. From Markov tower as a standard module to planar module category. Define the multi-step conditional expectation $E_n^m = E_{n-m+1} \circ \cdots \circ E_n$, for $m \leq n$. Similar to Definition 2.30, we may regard the elements in M_n as endomorphisms in the category, we can construct a planar module category from a given Markov tower as a standard module over a standard λ-lattice.

Definition 3.4. Let $M = (M_n)_{n\geq 0}$ be a Markov tower as a standard right module over standard λ-lattice $A = (A_{i,j})$ with $\dim(M_0) = 1$. We define a planar module category $\mathcal{M}_0$ from M as follows.

(a) The objects of $\mathcal{M}_0$ are the symbols $[n]$ for $n \in \mathbb{Z}_{\geq 0}$.

(b) Given $n \geq 0$, define $\mathcal{M}_0([n] \to [n]) := M_n$.

(c) The identity morphism in $\mathcal{M}_0([n] \to [n])$ is 1_{M_n}.

(d) For $(f;[m],[n])$ with $2 \mid m+n$, we define $(f;[m],[n])^\dagger := (f^*;[n],[m])$, where $f, f^* \in M_{\frac{m+n}{2}}$.

(e) We define composition in three cases.

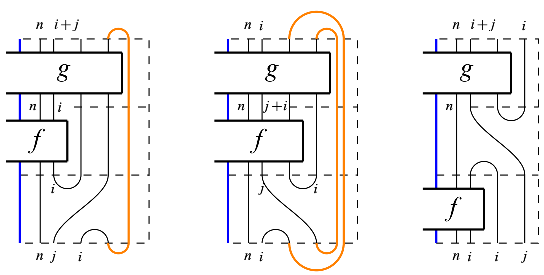
(C1) $(g;[n+2i],[n+2i+2j]) \circ (f;[n],[n+2i])$
$= (d^i E^i_{n+2i+j}(gfe^n_{j,i});[n],[n+2i+2j])$,
where $f \in M_{n+i}, g \in M_{n+2i+j}$ and $d^i E^i_{n+2i+j}(gfe^n_{j,i}) \in M_{n+i+j}$.

(C2) $(g;[n+2i+2j],[n+2i]) \circ (f;[n],[n+2i+2j])$
$= (d^i E^{i+j}_{n+2i+j}(gfe^{n,*}_{j,i});[n],[n+2i])$,
where $f \in M_{n+i+j}, g \in M_{n+2i+j}$ and $d^i E^{i+j}_{n+2i+j}(gfe^{n,*}_{j,i}) \in M_{n+i}$.

(C3) $(g;[n],[n+2i+2j]) \circ (f;[n+2i],[n])$
$= (d^i ge^{n,*}_{j,i} f;[n+2i],[n+2i+2j])$,
where $f \in M_{n+i}, g \in M_{n+i+j}$ and $d^i ge^{n,*}_{j,i} f \in M_{n+2i+j}$.

For the other cases, we can use the dagger structure $f^\dagger \circ g^\dagger := (g \circ f)^\dagger$ to define.

Similarly, the composition and the dagger structure are well defined, and $\mathcal{M}_0$ is C* according to [7, §3.4].



(C1) (C2) (C3)

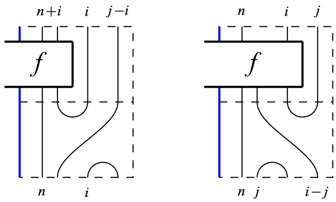
Remark 3.5. Readers can observe the similarity between the diagrammatic explanation of elements in M_n and $A_{i,n}$, the difference only appears on the leftmost. Moreover, a similar version of Lemma 2.23 and Lemma 2.24 holds for the Markov tower case.

Now we define the module action of morphisms.

Definition 3.6. $f \lhd 1$ and $1 \lhd x$, $f \in \mathrm{Hom}(\mathcal{M}_0)$ and $x \in \mathrm{Hom}(\mathcal{A}_0)$. The idea is the same as in Definition 2.31.

First, we define $f \lhd 1$ as

f	$f \lhd 1_j$
$(f; [m], [m+2i]), i \leq j$	$(fe^{m}_{j-i,i}; [m+j], [m+2i+j])$
$(f; [m], [m+2i]), i > j$	$(fe^{m,*}_{i-j,j}; [m+j], [m+2i+j])$



$i \leq j$ $\quad$ $i > j$

The definition of $1 \lhd x$ will be the same as $1 \otimes x$ by using the 2-shift maps in Definition 2.31.

The proof of the following propositions is the same as in Propositions 2.32, 2.34, and 2.35.

Proposition 3.7. *For $f \in \mathrm{Hom}(\mathcal{M}_0)$, $x \in \mathrm{Hom}(\mathcal{A}_0)$, $(f \lhd 1) \circ (1 \lhd x) = (1 \lhd x) \circ (f \lhd 1)$.*

Definition 3.8. Define $f \lhd x := (f \lhd 1) \circ (1 \lhd x)$.

The following propositions guarantee the module action defined above is well defined.

Proposition 3.9. *For $f \in \mathrm{Hom}(\mathcal{M}_0)$, $x, y \in \mathrm{Hom}(\mathcal{A}_0)$, $(f \lhd x) \lhd y = f \lhd (x \otimes y)$.*

Proposition 3.10. *For $f, g \in \mathrm{Hom}(\mathcal{M}_0)$, $(f \circ g) \lhd 1 = (f \lhd 1) \circ (g \lhd 1)$ and $1 \lhd (x \otimes y) = (1 \lhd x) \circ (1 \lhd y)$.*

3.4. Indecomposable semisimple C* $\mathcal{A}$-module categories and planar $\mathcal{A}_0$-module categories

3.4.1. Indecomposable semisimple C* $\mathcal{A}$-module category. Let $\mathcal{A}$ be a 2-shaded rigid C* multitensor category with a generator $X = 1^+ \otimes X \otimes 1^-$ with a canonical unitary dual functor $\overline{(\cdot)}$. Let $\mathcal{M}$ be a Cauchy complete indecomposable semisimple C* $\mathcal{A}$-module category. Note that there is a natural dagger structure on $\mathcal{M}$, and the module action $\lhd$ is a dagger functor, namely, for morphism $f \in \mathrm{Hom}(\mathcal{M})$ and $x \in \mathrm{Hom}(\mathcal{A})$,

$$(f \lhd x)^\dagger = f^\dagger \lhd x^\dagger.$$

We call a module category $\mathcal{M}$ *indecomposable* if for any two simple objects $P, Q \in \mathcal{M}$, Q is a direct summand of $P \lhd X^{\mathrm{alt}\otimes n}$ if $P = P \lhd 1^+$ ($P \lhd \overline{X}^{\mathrm{alt}\otimes n}$ if $P = P \lhd 1^-$) for some $n \in \mathbb{Z}_{\geq 0}$.

Construction 3.11. Let $\mathcal{A}_0$ be a planar tensor category obtained from $(\mathcal{A}, X)$ via the construction in Section 2.7.2. By MacLane's coherence theorem, $\mathcal{M}_{\mathcal{A}}$ is unitary equivalent to a strict one, i.e., $\mathcal{M}$ and $\mathcal{A}$ are strict and the right module associator is trivial. Then $\mathcal{M}$ is also a strict right $\mathcal{A}_0$-module category.

We construct the planar $\mathcal{A}_0$-module category $\mathcal{M}_0$ as follows:

(a) Objects: Pick a simple object $Z = Z \lhd 1^+ \in \mathcal{M}$, define $[0] := Z$, and

$$[n+1] := [n] \lhd [1, ?],$$

where $[1, ?] = [1, +]$ if $2 \mid n$ and $[1, ?] = [1, -]$ if $2 \nmid n$.

(b) Morphisms: $\mathcal{M}_0$ is a full subcategory of $\mathcal{M}$ with above objects.

Given $\mathcal{M}_0$ to be a planar $\mathcal{A}_0$-module category, then its Cauchy completion $\widehat{\mathcal{M}_0}$ is an $\widehat{\mathcal{A}_0}$-module, compatible with the dagger structure. The proof is left to the reader as an exercise.

Remark 3.12. Suppose $\mathcal{M}_0$ is a planar $\mathcal{A}_0$-module category constructed from $(\mathcal{M}, Z)$ over $(\mathcal{A}, X)$, then there is a unitary equivalence between $\mathcal{M}$ as $\mathcal{A}$-module and $\widehat{\mathcal{M}_0}$ as $\widehat{\mathcal{A}_0}$-module, which sends base object to base object.

3.4.2. From planar module category to Markov tower as a standard module over a standard λ-lattice.

Construction 3.13. Let $\mathcal{M}_0$ be a planar $\mathcal{A}_0$-module category with modulus d and A is a standard λ-lattice constructed from $\mathcal{A}_0$ as in Section 2.7.3. Define $M_j = \mathrm{End}([j])$, $j \in \mathbb{Z}_{\geq 0}$. Then we check $M = (M_j)_{j \geq 0}$ to be a Markov tower as a standard A-module.

(a) The horizontal inclusion $M_j \subset M_{j+1}$ sends $x \in M_j$ to $x \lhd \mathrm{id}_{[1,?]} \in M_{j+1}$, where $? = +$ if $2 \mid j$ and $? = -$ if $2 \nmid j$. The vertical inclusion $A_{0,j} \subset M_j$ sends $x \in A_{0,j}$ to $\mathrm{id}_{[0]} \lhd x \in M_j$.

(b) Conditional expectation: Define $E_j^M : M_j \to M_{j-1}$ by

$$E_{2k}^M(x) = d^{-1}(\mathrm{id}_{[2k-1]} \otimes \mathrm{ev}_{\overline{[1,+]}}) \circ \big(x \lhd [1,+]\big) \circ (\mathrm{id}_{[2k-1]} \otimes \mathrm{coev}_{[1,+]}),$$
$$E_{2k+1}^M(x) = d^{-1}(\mathrm{id}_{[2k]} \otimes \mathrm{ev}_{\overline{[1,-]}}) \circ \big(x \lhd [1,-]\big) \circ (\mathrm{id}_{[2k]} \otimes \mathrm{coev}_{[1,-]}).$$

(c) Jones projections: the same Jones projections in A and identify $e_n \in A_{0,n+1}$ with $1 \lhd e_n \in M_{n+1}$.

The check that M is a Markov tower and a standard A-module is left to the reader. In particular, we have $E_{n+1}(e_n) = d^{-2} \cdot 1$.

In this section, we show that the class of unitary equivalent pairs $(\mathcal{M}, Z)$ with $\mathcal{M}$ an indecomposable right $\mathcal{A}$-module category and Z a simple base point induces the equivalent class of planar module categories; according to Section 3.3.2, the class of equivalent planar module categories is one to one corresponding to the class of isomorphic Markov towers as standard module over the isomorphic standard λ-lattices.

Combining above discussion, we can deduce the equivalence between $(\mathcal{M}, Z)$ as $\mathcal{A}$-module category and Markov tower M as standard A-module.

Theorem 3.14. *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{array}{l}\text{Traceless Markov tower } M = \\ (M_i)_{i\geq 0} \text{ with } \dim(M_0) = 1 \text{ as} \\ \text{a standard right module over a} \\ \text{standard } \lambda\text{-lattice } A\end{array}\right\} \cong \left\{\begin{array}{l}\text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable} \\ \text{semisimple } \mathrm{C}^* \text{ right } \mathcal{A}\text{-module category} \\ \text{together with a choice of simple object} \\ Z = Z \lhd 1_{\mathcal{A}}^{+}\end{array}\right\}.$$

Equivalence on the left-hand side is $*$-isomorphism of traceless Markov towers as standard A-modules; equivalence on the right-hand side is unitary $\mathcal{A}$-module category equivalence on their Cauchy completions which maps the simple base object to the simple base object.

Corollary 3.15. *Any Markov tower M with modulus d and* $\dim(M_0) = 1$ *is naturally a standard right* $\mathrm{TLJ}(d)$*-module, where* $\mathrm{TLJ}(d)$ *is a Temperley–Lieb–Jones standard λ-lattice as in Example 2.14, which corresponds to an indecomposable semisimple* C^* *right $\mathcal{TLJ}(d)$-module category with a simple base object.*

Remark 3.16. The tracial case will be discussed in Section 7.1.

4. Markov lattices as standard bimodule over two standard λ-lattices and bimodule categories

In this section, we extend the discussion into the bimodule case. We give the notion of Markov lattices and Markov lattices as bimodule over two standard λ-lattices, by using a similar method, which corresponds to bimodule categories.

4.1. Markov lattice and basic properties

Definition 4.1 (Markov lattice). A tuple $M = (M_{i,j}, E_{i,j}^{M,l}, E_{i,j}^{M,r}, e_i, f_j)_{i,j\geq 0}$ is called a Markov lattice if the following conditions hold.

$$\begin{array}{ccc} M_{i+1,j} & \subset & M_{i+1,j+1} \\ \cup & & \cup \\ M_{i,j} & \subset & M_{i,j+1} \end{array}$$

(a) $M_{i,j} \subset M_{i,j+1}$ and $M_{i,j} \subset M_{i+1,j}$ are unital inclusions.

(b) $M_{-,j} = (M_{i,j}, E_{i,j}^{M,l}, e_i)_{i\geq 0}$ are Markov towers with the same modulus d_0 and $e_i \in M_{i+1,j}$ for all j; $M_{i,-} = (M_{i,j}, E_{i,j}^{M,r}, f_j)_{j\geq 0}$ are Markov towers with the same modulus d_1 and $f_j \in M_{i,j+1}$ for all i. We call M of modulus (d_0, d_1).

(c) The commuting square condition:

$$\begin{tikzcd} M_{i+1,j} \arrow[d, "E^{M,l}_{i+1,j}"'] & M_{i+1,j+1} \arrow[l, "E^{M,r}_{i+1,j+1}"'] \arrow[d, "E^{M,l}_{i+1,j+1}"] \\ M_{i,j} & M_{i,j+1} \arrow[l, "E^{M,r}_{i,j+1}"] \end{tikzcd}$$

is a commuting square, i.e., $E^{M,r}_{i,j+1} \circ E^{M,l}_{i,j} = E^{M,l}_{i,j+1} \circ E^{M,r}_{i+1,j+1}$.

Here are some properties of Markov lattice.

Proposition 4.2. *Let $M = (M_{i,j}, E^{M,l}_{i,j}, E^{M,r}_{i,j}, e_i, f_j)_{i,j\geq 0}$ be a Markov lattice.*

(1) $E^{M,r}_{i+1,j+1}(e_i) = e_i$ *and* $E^{M,l}_{i+1,j+1}(f_j) = f_j$ *for each* $i, j = 1, 2, \ldots$.

(2) $[f_j, e_i] = 0$ *for each* $i, j = 1, 2, 3, \ldots$.

Proof. (1) Note that $e_i \in M_{i+1,j} \subset M_{i+1,j+1}$ and $E^r_{i+1,j+1} : M_{i+1,j+1} \to M_{i+1,j}$ is a conditional expectation, we have $E^r_{i+1,j+1}(e_i) = e_i$. Similarly, $E^{M,l}_{i+1,j+1}(f_j) = f_j$.

(2) By Proposition 2.4 (1). ■

Remark 4.3. If there is a faithful tracial state $\mathrm{tr}^M_{i,j}$ on $M_{i,j}$ such that $\mathrm{tr}^M_{i+1,j}|_{M_{i,j}} = \mathrm{tr}^M_{i,j+1}|_{M_{i,j}} = \mathrm{tr}^M_{i,j}$ and $E^{M,r}_{i,j}, E^{M,l}_{i,j}$ are the canonical faithful trace-preserving conditional expectations for $i, j = 0, 1, 2, \ldots$, then M is called a *tracial Markov lattice*.

Remark 4.4. It is worth mentioning that a single commuting square of finite dimensional C*-algebras is a particular version of a Markov lattice. For a more detailed discussion, see Section 6.6.

In the rest of this section, we only consider the traceless Markov lattice with $\dim(M_{0,0}) = 1$ unless stated.

4.2. Markov lattice as a standard bimodule over two standard λ-lattices

Definition 4.5 (Markov lattice as a standard bimodule over two standard λ-lattices).

$$\begin{array}{ccccccccccccc}
\cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
A_{3,1} & \subset & A_{3,0} & \subset & M_{3,0} & \subset & M_{3,1} & \subset & M_{3,2} & \subset & M_{3,3} & \subset & \\
\cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
A_{2,1} & \subset & A_{2,0} & \subset & M_{2,0} & \subset & M_{2,1} & \subset & M_{2,2} & \subset & M_{2,3} & \subset & \\
\cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
A_{1,1} & \subset & A_{1,0} & \subset & M_{1,0} & \subset & M_{1,1} & \subset & M_{1,2} & \subset & M_{1,3} & \subset & \\
 & & \cup & & \cup & & \cup & & \cup & & \cup & & \\
 & & A_{0,0} & \subset & M_{0,0} & \subset & M_{0,1} & \subset & M_{0,2} & \subset & M_{0,3} & \subset & \\
 & & & & \cup & & \cup & & \cup & & \cup & & \\
 & & & & B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & B_{0,3} & \subset & \\
 & & & & & & \cup & & \cup & & \cup & & \\
 & & & & & & B_{1,1} & \subset & B_{1,2} & \subset & B_{1,3} & \subset &
\end{array}$$

Let $A^{\mathrm{op}} = (A_{i,j})_{0\le j\le i<\infty}$ $B = (B_{i,j})_{0\le i\le j<\infty}$ be two standard λ-lattices with Jones projection $e_i \in A_{i+1,j}$, $f_j \in B_{i,j+1}$ respectively and compatible conditional expectations. Here, A and M share the same Jones projections e_i; B and M share the same Jones projections f_j. (Warning: here we use the opposite λ-lattice A^{op}, see Definition 2.13)

Let $M = (M_{i,j}, e_i, f_j)_{i,j\ge 0}$ be a Markov lattice with conditional expectation $E^{M,r}$, $E^{M,l}$. We call a Markov lattice M a standard A-B bimodule where the left action is the opposite action if it satisfies the following three conditions.

(a) $A_{i,0} \subset M_{i,0}$, $B_{0,j} \subset M_{0,j}$ are unital inclusions, $i, j = 0, 1, 2, \ldots$.

(b) $E^{M,l}_{i,0}|_{A_{i,0}} = E^{A,l}_{i,0}$, $E^{M,r}_{0,j}|_{B_{0,j}} = E^{B,r}_{0,j}$ $i = 1, 2, \ldots$.

(c) (standard condition) $[M_{i,j}, A_{p,q}] = 0$ for $i \le q \le p$; $[M_{i,j}, B_{k,l}] = 0$, for $j \le k \le l$.

Remark 4.6. The standard condition implies that

$$[A_{p,q}, B_{k,l}] = 0 \quad \text{for all } q \le p, k \le l$$

since $A_{p,q} \subset A_{p,0} \subset M_{p,0}$ and $B_{k,l} \subset B_{0,l} \subset M_{0,l}$. Moreover,

$$E^{M,r}_{i,j}|_{A_{k,l}} = \mathrm{id}, \quad E^{M,l}_{i,j}|_{B_{k,l}} = \mathrm{id}.$$

In particular, we have

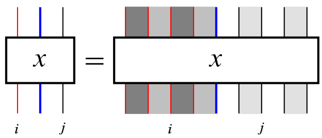
$$E^{M,r}_{i,j}(e_k) = e_k, \quad E^{M,l}_{i,j}(f_l) = f_l$$

for Jones projections.

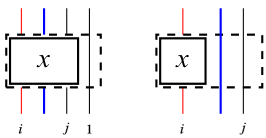
4.3. String diagram explanation

We now provide the string diagram explanation of the element, conditional expectation, Jones projection and their relations in a Markov lattice with the same spirit in Section 3.2.

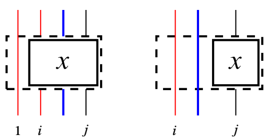
(ML1) Element $x \in M_{i,j}$:



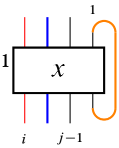
(ML2) Horizontal inclusion $x \in M_{i,j} \subset M_{i,j+1}$ and $x \in A_{i,0} \subset M_{i,j}$:



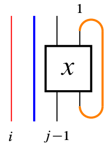
(ML3) Vertical inclusion $x \in M_{i,j} \subset M_{i+1,j}$ and $x \in B_{0,j} \subset M_{i,j}$:



(ML4) Horizontal conditional expectation $E^{M,r}_{i,j} : M_{i,j} \to M_{i,j-1}$ and $E^{M,r}_{i,j}|_{B_{0,j}} = E^{B,r}_{0,j}$:

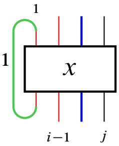
$$E^{M,r}_{i,j}(x) = d_1^{-1}$$ 

$, \quad x \in M_{i,j},$

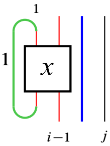
$$E^{M,r}_{i,j}(x) = E^{B,r}_{0,j}(x) = d_1^{-1}$$ 

$, \quad x \in B_{0,j}$

(ML5) Vertical conditional expectation $E^{M,l}_{i,j} : M_{i,j} \to M_{i-1,j}$ and $E^{M,l}_{i,j}|_{A_{i,0}} = E^{A,l}_{i,0}$:

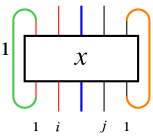
$$E^{M,l}_{i,j}(x) = d_0^{-1}$$ 

$, \quad x \in M_{i,j},$

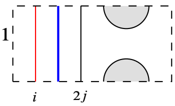
$$E^{M,l}_{i,j}(x) = E^{A,l}_{i,0}(x) = d_0^{-1}$$ 

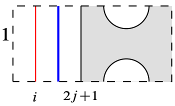
$, \quad x \in A_{i,0}$

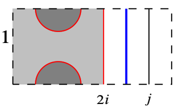
(ML6) Commuting square of conditional expectation
$E^{M,r}_{i,j+1} \circ E^{M,l}_{i,j} = E^{M,l}_{i,j+1} \circ E^{M,r}_{i+1,j+1} : M_{i+1,j+1} \to M_{i,j}$, $x \in M_{i+1,j+1}$:

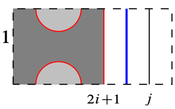
$$E^{M,r}_{i,j+1} \circ E^{M,l}_{i,j}(x) = E^{M,l}_{i,j+1} \circ E^{M,r}_{i+1,j+1}(x) = d_0^{-1} d_1^{-1}$$ 

(ML7) Horizontal Jones projections $f_j \in M_{i,j+1}$ and vertical Jones projections $e_i \in M_{i+1,j}$:

$$f_{2j+1} = d_1^{-1}$$ 

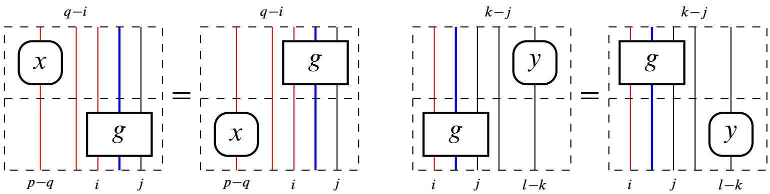
$$f_{2j+2} = d_1^{-1}$$ 

$$e_{2i+1} = d_0^{-1}$$ 

$$e_{2i+2} = d_0^{-1}$$ 

(ML8) Standard condition:

- $[M_{i,j}, A_{p,q}] = 0$, for $i \le q \le p$. For $g \in M_{i,j}, x \in A_{p,q}$, regard them as elements in $M_{p,j}$, then $gx = xg$;
- $[M_{i,j}, B_{k,l}] = 0$, for $j \le k \le l$. For $g \in M_{i,j}, y \in B_{k,l}$, regard them as elements in $M_{i,l}$, then $gy = yg$:



4.4. From Markov lattice as standard bimodule to planar bimodule category

4.4.1. Planar bimodule category. Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be planar tensor categories. Let $\mathcal{M}_0$ be a C* $\mathcal{A}_0$-$\mathcal{B}_0$ bimodule category with following properties:

(a) Object: The objects of $\mathcal{M}_0$ are $[m,n] = [m,n]_{\mathcal{M}_0}, m,n \in \mathbb{Z}_{\ge 0}$, where $[0,0] := 1_{\mathcal{M}_0}$ is simple.

(b) The module tensor product of objects are

$$[i,+]_{\mathcal{A}_0} \rhd [m,n]_{\mathcal{M}_0} = [m+i,n]_{\mathcal{M}_0}, \quad [i,-]_{\mathcal{A}_0} \rhd [m,n]_{\mathcal{M}_0} = 0,$$
$$[m,n]_{\mathcal{M}_0} \lhd [j,+]_{\mathcal{B}_0} = [i,n+j]_{\mathcal{M}_0}, \quad [m,n]_{\mathcal{M}_0} \lhd [j,-]_{\mathcal{B}_0} = 0,$$
$$\big([i,+]_{\mathcal{A}_0} \rhd [m,n]_{\mathcal{M}_0}\big) \lhd [j,+]_{\mathcal{B}_0}$$
$$= [m+i,n+j]_{\mathcal{M}_0} = [i,+]_{\mathcal{A}_0} \rhd \big([m,n]_{\mathcal{M}_0} \lhd [j,+]_{\mathcal{B}_0}\big).$$

(c) Only $\mathcal{M}_0([m,n] \to [m \pm 2i, n \pm 2j])$ is non-zero, $m,n,i,j \in \mathbb{Z}_{\ge 0}$. The module tensor product of morphisms in $\mathrm{Hom}(\mathcal{A}_0)$, $\mathrm{Hom}(\mathcal{M}_0)$ and $\mathrm{Hom}(\mathcal{M}_0)$ should match the shading types.

(d) $\mathcal{M}_0$ is a strict $\mathcal{A}_0$-$\mathcal{B}_0$ bimodule category, i.e., the left/right module associator and bimodule associator are trivial. For $x, x_1, x_2 \in \mathrm{Hom}(\mathcal{A}_0)$, $g \in \mathrm{Hom}(\mathcal{M}_0)$ and $y, y_1, y_2 \in \mathrm{Hom}(\mathcal{B}_0)$,

$$x_2 \rhd (x_1 \rhd g) = (x_2 \otimes x_1) \rhd g,$$
$$(g \lhd y_1) \lhd y_2 = g \lhd (y_1 \otimes y_2),$$
$$(x \rhd g) \lhd y = x \rhd (g \lhd y).$$

(e) $\mathcal{M}_0$ is a C* category with a natural dagger structure such that $\lhd$ and $\rhd$ are dagger functors, i.e., for $x \in \mathrm{Hom}(\mathcal{A}_0)$, $g \in \mathrm{Hom}(\mathcal{M}_0)$ and $y \in \mathrm{Hom}(\mathcal{B}_0)$,

$$(x \rhd g \lhd y)^\dagger = x^\dagger \rhd g^\dagger \lhd y^\dagger.$$

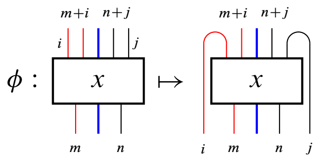
Such a bimodule category is called a *planar bimodule category*.

Remark 4.7. As in Remark 3.3, the morphisms in $\mathcal{M}_0$ is determined by its representation as an endomorphism and its domain and range.

There is a canonical isomorphism

$$\phi : \mathcal{M}_0\big([m,n] \to [m+2i, n+2j]\big) \to \mathcal{M}_0\big([m+i, n+j] \to [m+i, n+j]\big)$$

by using the rigid structure on $\mathcal{A}_0$ and $\mathcal{B}_0$.



Remark 4.8. Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be planar bimodule categories over the same planar tensor category. If they are unitary monoidal equivalent, then they are unitary isomorphic.

4.4.2. From Markov lattice as standard bimodule to planar bimodule category. Use a similar notion as we define the planar module category in Definition 3.4.

Define the multi-step conditional expectations $E^{l,i}_{m,n} := E^{M,l}_{m-i+1,n} \circ \cdots \circ E^{M,l}_{m,n}$ and $E^{r,k}_{m,n} := E^{M,r}_{m,n-k+1} \circ \cdots \circ E^{M,r}_{m,n}$.

Definition 4.9. Let A, B be standard λ-lattices and $M = (M_{m,n})_{m,n\geq 0}$ be a Markov lattice as a standard A-B bimodule with $\dim(M_{0,0}) = 1$. We define a planar bimodule category $\mathcal{M}_0$ from M as follows.

(a) The objects of $\mathcal{M}_0$ are the symbols $[m,n]$ for $m,n \in \mathbb{Z}_{\geq 0}$.

(b) Given $m,n \geq 0$, define $\mathcal{M}_0([m,n] \to [m,n]) := M_{m,n}$.

(c) The identity morphism in $\mathcal{M}_0([m,n] \to [m,n])$ is $1_{M_{m,n}}$.

(d) For $(f; [m_1,n_1],[m_2,n_2])$ with $2 \mid m_1+m_2$ and $2 \mid n_1+n_2$, define

$$\big(f; [m_1,n_1],[m_2,n_2]\big)^\dagger := \big(f^*; [m_2,n_2],[m_1,n_1]\big),$$

where $f, f^* \in M_{\frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}}$.

(e) Define the composition in nine cases.

(C11) $(h; [m+2i, n+2k], [m+2i+2j, n+2k+2t]) \circ (g; [m,n], [m+2i, n+2k])$
$= (d_0^i d_1^k E^{l,i}_{m+2i+j,n+k+t}(E^{r,k}_{m+2i+j,n+2k+t}(hgf^n_{t,k}e^m_{j,i})); [m,n],$
$[m+2i+2j, n+2k+2t])$, where $g \in M_{m+i,n+k}$, $h \in M_{m+2i+j,n+2k+t}$ and $d_0^i d_1^k E^{l,i}_{m+2i+j,n+k+t}(E^{r,k}_{m+i+j,n+2k+t}(hgf^n_{t,k}e^m_{j,i})) \in M_{m+i+j,n+k+t}$.

(C12) $(h; [m+2i, n+2k+2t], [m+2i+2j, n+2k]) \circ (g; [m,n],$
$[m+2i, n+2k+2t])$
$= (d_0^i d_1^k E^{l,i}_{m+2i+j,n+k}(E^{r,k+t}_{m+2i+j,n+2k+t}(hgf^{n,*}_{t,k}e^m_{j,i})); [m,n],$
$[m+2i+2j, n+2k])$, where $g \in M_{m+i,n+k+t}$, $h \in M_{m+2i+j,n+2k+t}$ and $d_0^i d_1^k E^{l,i}_{m+2i+j,n+k}(E^{r,k+t}_{m+2i+j,n+2k+t}(hgf^{n,*}_{t,k}e^m_{j,i})) \in M_{m+i+j,n+k}$.

(C13) $(h; [m+2i, n][m+2i+2j, n+2k+2t]) \circ (g; [m, n+2k], [m+2i, n])$
$= (d_0^i d_1^k E^{l,i}_{m+2i+j,n+2k+l}(hf^{n,*}_{l,k} g e^m_{j,i}); [m, n+2k], [m+2i+2j,$
$n+2k+2t])$, where $g \in M_{m+i,n+k}$, $h \in M_{m+2i+j,n+k+t}$
and $d_0^i d_1^k E^{l,i}_{m+2i+j,n+2k+l}(hf^{n,*}_{l,k} g e^m_{j,i}) \in M_{m+i+j,n+2k+t}$.

(C21) $(h; [m+2i+2j, n+2k], [m+2i, n+2k+2t]) \circ (g; [m, n],$
$[m+2i+2j, n+2k])$
$= (d_0^i d_1^k E^{l,i+j}_{m+2i+j,n+k+t}(E^{r,k}_{m+2i+j,n+2k+t}(hgf^n_{t,k} e^{m,*}_{j,i})); [m, n],$
$[m+2i, n+2k+2t])$, where $g \in M_{m+i+j,n+k}$, $h \in M_{m+2i+j,n+2k+t}$
and $d_0^i d_1^k E^{l,i+j}_{m+2i+j,n+k+t}(E^{r,k}_{m+2i+j,n+2k+t}(hgf^n_{t,k} e^{m,*}_{j,i})) \in M_{m+i,n+2k+t}$.

(C22) $(h; [m+2i+2j, n+2k+2t], [m+2i, n+2k]) \circ (g; [m, n],$
$[m+2i+2j, n+2k+2t])$
$= (d_0^i d_1^k E^{l,i+j}_{m+2i+j,n+k}(E^{r,k+t}_{m+2i+j,n+2k+t}(hgf^{n,*}_{t,k} e^{m,*}_{j,i})); [m, n],$
$[m+2i, n+2k])$, where $g \in M_{m+i+j,n+k+t}$, $h \in M_{m+2i+j,n+2k+t}$
and $d_0^i d_1^k E^{l,i+j}_{m+2i+j,n+k}(E^{r,k+t}_{m+2i+j,n+2k+t}(hgf^{n,*}_{t,k} e^{m,*}_{j,i})) \in M_{m+i,n+k}$.

(C23) $(h; [m+2i+2j, n], [m+2i, n+2k+2t]) \circ (g; [m, n+2k],$
$[m+2i+2j, n])$
$= (d_0^i f_2^k E^{l,i+j}_{m+2i+j,n+2k+t}(hf^{n,*}_{t,k} g e^{m,*}_{j,i}); [m, n+2k],$
$[m+2i, n+2k+2t])$, where $g \in M_{m+i+j,n+k}$, $h \in M_{m+2i+j,n+k+t}$
and $d_0^i f_2^k E^{l,i+j}_{m+2i+j,n+2k+t}(hf^{n,*}_{t,k} g e^{m,*}_{j,i}) \in M_{m+i,n+2k+t}$.

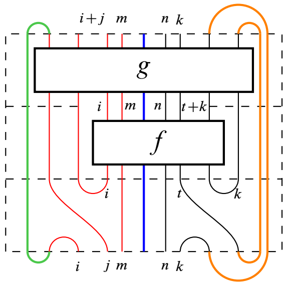
(C31) $(h; [m, n+2k], [m+2i+2j, n+2k+2t]) \circ (g; [m+2i, n],$
$[m, n+2k])$
$= (d_0^i d_1^k E^{r,k}_{m+2i+j,n+2k+t}(he^{m,*}_{j,i} g f^n_{t,k}); [m+2i, n],$
$[m+2i+2j, n+2k+2t])$, where $g \in M_{m+i,n+k}$, $h \in M_{m+i+j,n+2k+t}$
and $d_0^i d_1^k E^{r,k}_{m+2i+j,n+2k+t}(he^{m,*}_{j,i} g f^n_{t,k}) \in M_{m+2i+j,n+k+t}$.

(C32) $(h; [m, n+2k+2t], [m+2i+2j, n+2k]) \circ (g; [m+2i, n],$
$[m, n+2k+2t])$
$= (d_0^i d_1^k E^{r,k+t}_{m+2i+j,n+2k+t}(he^{m,*}_{j,i} g f^{n,*}_{t,k}); [m+2i, n],$
$[m+2i+2j, n+2k])$, where $g \in M_{m+i,n+k+t}$, $h \in M_{m+i+j,n+2k+t}$
and $d_0^i d_1^k E^{r,k+t}_{m+2i+j,n+2k+t}(he^{m,*}_{j,i} g f^{n,*}_{t,k}) \in M_{m+2i+j,n+k}$.

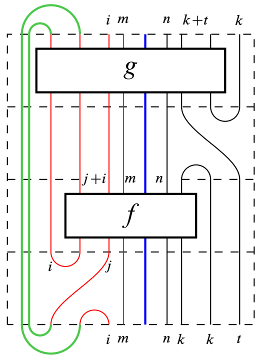
(C33) $(h; [m, n], [m+2i+2j, n+2k+2t]) \circ (g; [m+2i, n+2k], [m, n])$
$= (d_0^i d_1^k hf^{n,*}_{t,k} e^{m,*}_{j,i} g; [m+2i, n+2k], [m+2i+2j, n+2k+2t])$,
where $g \in M_{m+i,n+k}$, $h \in M_{m+i+j,n+k+t}$ and
$d_0^i d_1^k hf^{n,*}_{t,k} e^{m,*}_{j,i} g \in M_{m+2i+j,n+2k+t}$.

For the other cases, we can use the dagger structure $g^\dagger \circ h^\dagger := (h \circ g)^\dagger$ to define.

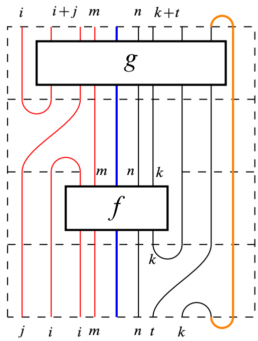
Similarly, we use the string diagrams to explain the composition.



(C12)



(C23)



(C31)

The composition is well defined and $\mathcal{M}_0$ is a C* category as before.

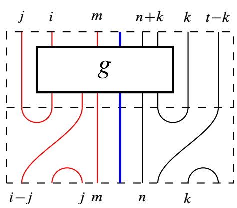
Remark 4.10. The composition is well defined, because of the commuting square of left/right conditional expectation condition and Proposition 4.2.

Definition 4.11. $1 \rhd g \lhd 1$, $x \rhd 1$ and $1 \lhd y$, $g \in \mathrm{Hom}(\mathcal{M}_0)$, $x \in \mathrm{Hom}(\mathcal{A}_0)$ and $y \in \mathrm{Hom}(\mathcal{B}_0)$.

The idea is the same as in Definition 3.6. First, we define $1 \rhd g \lhd 1$ as

g	$1_j \rhd g \lhd 1_t$
$(g; [m,n], [m+2i, n+2k]),\ i \le j, k \le t$	$(ge^m_{j-i,i} f^n_{t-k,k}; [m+j, n+t], [m+2i+j, m+2k+t])$
$(g; [m,n], [m+2i, n+2k]),\ i > j, k \le t$	$(ge^{m,*}_{i-j,j} f^n_{t-k,k}; [m+j, n+t], [m+2i+j, m+2k+t])$
$(g; [m,n], [m+2i, n+2k]),\ i \le j, k > t$	$(ge^m_{j-i,i} f^{n,*}_{k-t,t}; [m+j, n+t], [m+2i+j, m+2k+t])$
$(g; [m,n], [m+2i, n+2k]),\ i > j, k > t$	$(ge^{m,*}_{i-j,j} f^{n,*}_{k-t,t}; [m+j, n+t], [m+2i+j, m+2k+t])$

Note that here we use the fact that the Jones projection $[e_i, f_k] = 0$ for all $i, k \ge 1$ and hence $(1 \rhd g) \lhd 1 = 1 \rhd (g \lhd 1) =: 1 \rhd g \lhd 1$.



$i \ge j, k \le t$

The definitions of $x \rhd 1$ and $1 \lhd y$ will be the same as $x \otimes 1$ and $1 \otimes y$ in Definition 2.31 by using the shift maps.

The proof of the following propositions is the same as in the Markov tower case with the fact in Remark 4.6. To be precise, the diagrammatic proof can be split as left-hand side and right-hand side independently, and the proof on each side is the same as the Markov tower case.

Proposition 4.12. *$\mathcal{M}_0$ is a left $\mathcal{A}_0$-module. That is,*

(1) *For $g \in \mathrm{Hom}(\mathcal{M}_0)$, $x \in \mathrm{Hom}(\mathcal{A}_0)$, $(1 \lhd g) \circ (x \lhd 1) = (x \lhd 1) \circ (1 \lhd g)$.*

(2) *For $g \in \mathrm{Hom}(\mathcal{M}_0)$, $x_1, x_2 \in \mathrm{Hom}(\mathcal{A}_0)$, $x_2 \rhd (x_1 \rhd g) = (x_2 \otimes x_1) \rhd g$.*

(3) *For $g_1, g_2 \in \mathrm{Hom}(\mathcal{M}_0)$, $x_1, x_2 \in \mathrm{Hom}(\mathcal{A}_0)$, $1 \rhd (g_1 \circ g_2) = (1 \rhd g_1) \circ (1 \rhd g_2)$ and $(x_1 \circ x_2) \rhd 1 = (x_1 \rhd 1) \circ (x_2 \rhd 1)$.*

Proposition 4.13. *Similarly, $\mathcal{M}_0$ is a right $\mathcal{B}_0$-module. That is,*

(1) *For $g \in \mathrm{Hom}(\mathcal{M}_0)$, $y \in \mathrm{Hom}(\mathcal{B}_0)$, $(g \lhd 1) \circ (1 \lhd y) = (1 \lhd y) \circ (g \lhd 1)$.*

(2) *For $g \in \mathrm{Hom}(\mathcal{M}_0)$, $y_1, y_2 \in \mathrm{Hom}(\mathcal{B}_0)$, $(g \lhd y_1) \lhd y_2 = g \lhd (y_1 \otimes y_2)$.*

(3) *For $g_1, g_2 \in \mathrm{Hom}(\mathcal{M}_0)$, $y_1, y_2 \in \mathrm{Hom}(\mathcal{B}_0)$, $(g_1 \circ g_2) \lhd 1 = (g_1 \lhd 1) \circ (g_2 \lhd 1)$ and $1 \lhd (x_1 \circ x_2) = (1 \lhd x_1) \circ (1 \lhd x_2)$.*

Proposition 4.14. *$\mathcal{M}_0$ is a $\mathcal{A}_0$-$\mathcal{B}_0$ bimodule. That is, for $g \in \mathrm{Hom}(\mathcal{M}_0)$, $x \in \mathrm{Hom}(\mathcal{A}_0)$, $y \in \mathrm{Hom}(\mathcal{B}_0)$, $(x \rhd 1) \circ (1 \lhd y) \circ (1 \rhd g \lhd 1) = (1 \lhd y) \circ (x \rhd 1) \circ (1 \rhd g \lhd 1)$.*

Proof. By Remark 4.6. ■

Definition 4.15. Define $x \rhd g \lhd y := (x \rhd 1) \circ (1 \lhd y) \circ (1 \rhd g \lhd 1)$.

4.5. Indecomposable semisimple C* $\mathcal{A}$-$\mathcal{B}$ bimodules and planar $\mathcal{A}_0$-$\mathcal{B}_0$ bimodule categories

4.5.1. Indecomposable semisimple C* $\mathcal{A}$-$\mathcal{B}$ bimodule category. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-shaded rigid C* multitensor categories with generators $X = 1_{\mathcal{A}}^+ \otimes X \otimes 1_{\mathcal{A}}^-$ and $Y = 1_{\mathcal{B}}^+ \otimes Y \otimes 1_{\mathcal{B}}^-$. Let $\mathcal{M}$ be a Cauchy complete indecomposable semisimple C* $\mathcal{A}$-$\mathcal{B}$ bimodule category. Note that there is a natural dagger structure on $\mathcal{M}$, and the left/right module actions are dagger functors, i.e., for morphism $g \in \mathrm{Hom}(\mathcal{M})$, $x \in \mathrm{Hom}(\mathcal{A})$ and $y \in \mathrm{Hom}(\mathcal{B})$,

$$(x \rhd g)^\dagger = x^\dagger \rhd g^\dagger, \quad (f \lhd y)^\dagger = f^\dagger \lhd y^\dagger.$$

We call $\mathcal{M}$ *indecomposable* if for any two simple objects $P, Q \in \mathcal{M}$ (without loss of generality, $P = 1_{\mathcal{A}}^+ \rhd P \lhd 1_{\mathcal{B}}^+$), Q is a direct summand of $(X^{\mathrm{alt}\otimes m} \rhd P) \lhd Y^{\mathrm{alt}\otimes n}$ for some $m, n \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{A}_0, \mathcal{B}_0$ be planar tensor categories constructed from $(\mathcal{A}, X)$ and $(\mathcal{B}, Y)$ respectively. By MacLane's coherence theorem, ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is unitary equivalent to a strict one, i.e., $\mathcal{A}, \mathcal{B}$ are strict, the right/left module associators and the bimodule associator are trivial. This strict category is also a strict $\mathcal{A}_0$-$\mathcal{B}_0$ bimodule category. Without loss of generality, we also denote it as $\mathcal{M}$.

Pick a simple object $Z = 1_{\mathcal{A}}^+ \rhd Z \lhd 1_{\mathcal{B}}^+ \in \mathcal{M}$, then we construct a planar $\mathcal{A}_0$-$\mathcal{B}_0$ bimodule category $\mathcal{M}_0$ as follows:

(a) Objects: Define $[0, 0] := Z$, and

$$[m+1, 0] := [1, ?]_{\mathcal{A}_0} \rhd [m, 0], \quad [m, n+1] := [m, n] \lhd [1, ?]_{\mathcal{B}_0},$$

where $[1, ?]_{\mathcal{A}_0} = [1, +]_{\mathcal{A}_0}$ if $2 \mid m$ and $[1, ?]_{\mathcal{A}_0} = [1, -]_{\mathcal{A}_0}$ if $2 \nmid m$; $[1, ?]_{\mathcal{B}_0} = [1, +]_{\mathcal{B}_0}$ if $2 \nmid n$ and $[1, ?]_{\mathcal{B}_0} = [1, -]_{\mathcal{B}_0}$ if $2 \mid n$.

(b) $\mathcal{M}_0$ is a full subcategory of $\mathcal{M}$ with above objects.

Given $\mathcal{M}_0$ to be a planar $\mathcal{A}_0$-$\mathcal{B}_0$ bimodule category, for a similar reason, its Cauchy completion $\widehat{\mathcal{M}_0}$ is a $\widehat{\mathcal{A}_0}$-$\widehat{\mathcal{B}_0}$ bimodule category, compatible with the dagger structure.

Remark 4.16. Suppose $\mathcal{M}_0$ is a planar $\mathcal{A}_0$-$\mathcal{B}_0$ bimodule category constructed from $\mathcal{M}$ over $(\mathcal{A}, X)$ and $(\mathcal{B}, Y)$, then there is a unitary equivalence between $\mathcal{M}$ as $\mathcal{A}$-$\mathcal{B}$ bimodule category and $\widehat{\mathcal{M}_0}$ as $\widehat{\mathcal{A}_0}$-$\widehat{\mathcal{B}_0}$ bimodule category, which maps base object to base object.

4.5.2. From planar bimodule to Markov lattice as standard bimodule.

Construction 4.17. Now let $M_{i,j} = \mathrm{End}([i,j])$, $i,j \in \mathbb{Z}_{\geq 0}$. After identifying $f \in M_{i,j}$ with $\mathrm{id}_{[1,?]} \rhd f \in M_{i+1,j}$ and $f \lhd \mathrm{id}_{[1,?]} \in M_{i,j+1}$ and identifying $x \in A_{i,0} = \mathrm{End}([i,+]_{\mathcal{A}_0})$ with $x \lhd \mathrm{id}_{[0,j]} \in M_{i,j}$ and $y \in B_{0,j} = \mathrm{End}([j,+]_{\mathcal{B}_0})$ with $\mathrm{id}_{[i,0]} \lhd y \in M_{i,j}$. It is easy to show that $M = (M_{i,j})_{i,j\geq 0}$ is a Markov lattice as a standard A-B bimodule with modulus (d_0, d_1).

Similar to the module case, combining the above discussion, we have the following theorem.

Theorem 4.18. *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{array}{l}\text{Traceless Markov lattice } M = \\ (M_{i,j})_{i,j\geq 0} \text{ with } \dim(M_{0,0}) = \\ 1 \text{ as a standard } A\text{-}B \text{ bimodule} \\ \text{over standard } \lambda\text{-lattices } A, B\end{array}\right\} \cong \left\{\begin{array}{l}\text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable} \\ \text{semisimple C}^* \ \mathcal{A}\text{-}\mathcal{B} \text{ bimodule category} \\ \text{together with a choice of simple object} \\ Z = 1_{\mathcal{A}}^{+} \rhd Z \lhd 1_{\mathcal{B}}^{+}\end{array}\right\}.$$

Equivalence on the left-hand side is the $*$-isomorphism on the traceless Markov lattice as standard A-B bimodule; the equivalence on the right-hand side is the unitary $\mathcal{A}$-$\mathcal{B}$ bimodule category equivalence between their Cauchy completions which maps the simple base object to simple base object.

Corollary 4.19. *Any Markov lattice M with modulus (d_0, d_1) and* $\dim(M_{00}) = 1$ *is naturally a standard* $\mathrm{TLJ}(d_0) - \mathrm{TLJ}(d_1)$ *bimodule, which corresponds to an indecomposable semisimple* C* $\mathcal{TLJ}(d_0)$-$\mathcal{TLJ}(d_1)$ *bimodule category with a simple base object.*

Remark 4.20. The tracial case will be discussed in Section 7.3.

5. Markov towers, bigraded Hilbert spaces, and balanced fair graphs

In this section, as an application, we are going to classify all indecomposable semisimple $\mathcal{TLJ}$-modules (see Corollary 3.15) to get Markov tower, which are also the same as balanced d-fair bipartite graphs [10]. We will explain exactly how these two classifications agree by directly constructing the correspondence passing through the 2-category BigHilb [13]. Although this is known [10, 13], we explain in detail here so that we are able to do the bimodules in Section 6 below.

5.1. Balanced d-fair bipartite graph

In [10], the authors classify unshaded unoriented $\mathcal{TLJ}(d)$-modules in terms of the combinatorial data of fair and balanced graphs. This classification was generalized to $\mathcal{TLJ}(\Gamma)$-modules in [13], where $\mathcal{TLJ}(\Gamma)$ is a generalized Temperley–Lieb–Jones category associated to a weighted bidirected graph Γ. We will be interested in the special case of 2-shaded $\mathcal{TLJ}(d)$-modules.

Notation 5.1. Let Λ be a graph where $V(\Lambda)$ is the set of vertices and $E(\Lambda)$ is the set of edges. Let $s, t : E(\Lambda) \to V(\Lambda)$ be the source and target functions respectively.

Definition 5.2. Let Λ be a bipartite graph with vertices

$$V(\Lambda) = V_0 \sqcup V_1 \quad \text{and} \quad \left\{e | s(e),\ t(e) \in V_i\right\} = \varnothing,\ i = 0, 1.$$

Let $\omega : E(\Lambda) \to (0, \infty)$ be the weighting on the edges of graph [13].

We call (Λ, ω) a d-*fair* graph if for each $P \in V_0$, $Q \in V_1$

$$\sum_{\{e|s(e)=P\}} w(e) = \sum_{\{e|s(e)=Q\}} w(e) = d.$$

We call (Λ, ω) a *balanced* graph if there exists an involution $\overline{(\cdot)}$ on $E(\Lambda)$ that switches sources and targets for each $e \in E(\Lambda)$ and

$$\omega(e)\omega(\bar{e}) = 1.$$

Proposition 5.3 (cf. [10, Prop. 3.1]). *Suppose (Λ, ω) is a balanced d-fair bipartite graph. Then the graph is locally finite, i.e., the number of edges coming in or out of any vertex is uniformly bounded:*

$$\#\left\{e : s(e) = P\right\} = \#\left\{e : t(e) = P\right\} \le d^2 \quad \textit{for any vertex } P.$$

Proof. Suppose P has N edges, then there exists an edge $e_0 : P \to Q$ such that $\omega(e_0) \le \frac{d}{N}$ and hence $\omega(\overline{e_0}) = \frac{1}{\omega(e_0)} \ge \frac{N}{d}$. Note that

$$d = \sum_{\{e|s(e)=Q\}} \omega(e) \ge \omega(\overline{e_0}) \ge \frac{N}{d},$$

which follows that $N \le d^2 < \infty$. ∎

Definition 5.4. We call $\theta : (\Lambda, \omega) \to (\Lambda', \omega')$ an isomorphism of edge-weighted graphs if θ is a graph isomorphism and $\omega'(\theta(e)) = \omega(e)$ for each $e \in E(\Lambda)$.

5.2. BigHilb and 2-subcategory $\mathfrak{C}(K, \mathrm{ev}_K)$

We refer the reader to [15] and [6, Def. 2.2] for the full definition of W*-category and W* 2-category.

Definition 5.5. Let U, V be countable sets. Define a W*-category $\mathsf{Hilb}_f^{U\times V}$ as follows:

(a) Object: $U \times V$-bigraded Hilbert spaces

$$H = \bigoplus_{\substack{u\in U\\ v\in V}} H_{uv},$$

where H_{uv} is finite dimensional for each pair (u, v), and only finite many H_{uv} is non-trivial for each fixed $u \in U$ or each fixed $v \in V$.

(b) Morphism: The morphisms are defined as uniformly bounded operators

$$f = \bigoplus_{\substack{u\in U\\ v\in V}} f_{uv} : H \to G,$$

where $f_{uv} : H_{uv} \to G_{uv}$ are morphisms in Hilb_f, the category of finitely dimensional Hilbert spaces. Here the direct sum is taken as von Neumann algebra direct sum. Uniformly boundedness means

$$\sup_{\substack{u\in U\\ v\in V}} \|f_{uv}\| < \infty.$$

(c) The composition: For morphisms f, g, define the composition entry-wisely as

$$g \circ f := \bigoplus_{\substack{u\in U\\ v\in V}} g_{uv} \circ f_{uv}.$$

(d) The identity morphism: Define the identity morphism $\mathrm{id}_H : H \to H$ as

$$\mathrm{id}_H := \bigoplus_{\substack{u\in U\\ v\in V}} \mathrm{id}_{H_{uv}},$$

where $\mathrm{id}_{H,uv} = \mathrm{id}_{H_{uv}}$ is the identity map on H_{uv}.

Definition 5.6. Let $\mathsf{BigHilb}$ be a W* 2-category defined as follows:

(a) Object: Countable sets.

(b) For objects U, V, $\mathrm{Hom}(U, V) = \mathsf{Hilb}_f^{U\times V}$.

(c) The composition of 1-morphisms: For 1-morphisms $H : U \to V$, $G : V \to W$, the composition of U, V denoted by $\otimes$ is defined as

$$G \circ H = H \otimes G := \bigoplus_{\substack{u\in U\\ w\in W}} \bigoplus_{v\in V} H_{uv} \otimes G_{vw} : U \to W,$$

where the $\otimes$ on the right-hand side is the tensor product of Hilbert spaces. The operator is analogous to matrix multiplication, the product is replaced by tensor product and the sum is replaced by direct sum. Clearly, $(H \otimes G) \otimes L = H \otimes (G \otimes L)$.

(d) The identity 1-morphism: For an object U, the identity 1-morphism

$$\mathbb{C}^{|U|} \in \mathrm{Hom}(U, U)$$

is defined as

$$\mathbb{C}^{|U|} := \bigoplus_{u,v \in U} \delta_{u=v} \cdot \mathbb{C}.$$

(e) The dual 1-morphism: For 1-morphism

$$H = \bigoplus_{\substack{u \in U \\ v \in V}} H_{uv} : U \to V,$$

define its dual as

$$\bar{H} := \bigoplus_{\substack{v \in V \\ u \in U}} \bar{H}_{vu} : V \to U,$$

where $\bar{H}_{vu} := \overline{H_{uv}}$ and $\overline{H_{uv}}$ is the complex conjugate Hilbert space of H_{uv}.

(f) Tensor product of 2-morphisms. Let $H_1, H_2 : U \to V$, $G_1, G_2 : V \to W$, and $f : H_1 \to H_2$, $g : G_1 \to G_2$, define $f \otimes g$ as

$$(f \otimes g)_{uw} := \bigoplus_{v \in V} f_{uv} \otimes g_{vw} : \bigoplus_{v \in V} H_{1,uv} \otimes G_{1,vw} \to \bigoplus_{v \in V} H_{2,uv} \otimes G_{2,vw}.$$

Clearly, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

(g) Dagger structure: For a 2-morphism $f = \bigoplus_{u,v} f_{uv} : H \to G$, define its adjoint

$$f^\dagger := \bigoplus_{u,v} f_{uv}^* : G \to H,$$

where f_{uv}^* is the adjoint of f_{uv} as a bounded linear map. Clearly, $(f^\dagger)^\dagger = f$.

Note that for each hom space $\mathrm{Hom}(H \to K)$ as a infinite direct sum of finite dimensional spaces has a predual, and for countable sets T, U, V, K and 1-morphisms $H_1, H_2 : U \to V$, $G : V \to W$, $K : T \to U$, the following maps are weak* continuous:

- $\mathrm{id}_K \otimes - : \mathrm{Hom}(H_1 \to H_2) \to \mathrm{Hom}(K \otimes H_1 \to K \otimes H_2)$ given by $f \mapsto \mathrm{id}_K \otimes f$.
- $- \otimes \mathrm{id}_G : \mathrm{Hom}(H_1 \to H_2) \to \mathrm{Hom}(H_1 \otimes G \to H_2 \otimes G)$ given by $f \mapsto f \otimes \mathrm{id}_G$.

According to [6, Def. 2.2, Prop. 2.4], BigHilb is a W* 2-category.

Definition 5.7. We call a 1-morphism $H : U \to V$ *dualizable*, if there exist evaluation and coevaluation 2-morphisms $\mathrm{ev}_H : \bar{H} \otimes H \to \mathbb{C}^{|V|}$ and $\mathrm{coev}_H : \mathbb{C}^{|U|} \to H \otimes \bar{H}$ meeting the *zigzag condition*:

$$(\mathrm{id}_H \otimes \mathrm{ev}_H) \circ (\mathrm{coev}_H \otimes \mathrm{id}_H) = \mathrm{id}_H,$$
$$(\mathrm{ev}_H \otimes \mathrm{id}_{\bar{H}}) \circ (\mathrm{id}_{\bar{H}} \otimes \mathrm{coev}_H) = \mathrm{id}_{\bar{H}}.$$

We are going to discuss the evaluation and coevaluation ev_H and coev_H in more detail.

Definition 5.8. Note that

$$\mathrm{ev}_{H,uv} : \bigoplus_{w} \bar{H}_{uw} \otimes H_{wv} = (\bar{H} \otimes H)_{uv} \to (\mathbb{C}^{|V|})_{uv} = \delta_{u=v} \cdot \mathbb{C},$$

only $\mathrm{ev}_{H,vv}$ is non-zero for $v \in V$. Let $C_{H,vu} : \bar{H}_{vu} \otimes H_{uv} = \overline{H_{uv}} \otimes H_{uv} \to \mathbb{C}$ such that $\mathrm{ev}_{H,vv} = \bigoplus_{u \in U} C_{H,vu}$. Similarly, only $\mathrm{coev}_{H,uu} : \mathbb{C} \to (H \otimes \bar{H})_{uu} = \bigoplus_{v \in V} H_{uv} \otimes \bar{H}_{vu}$ is non-zero for $u \in U$. Let $D_{H,uv} : \mathbb{C} \to H_{uv} \otimes \bar{H}_{vu} = H_{uv} \otimes \overline{H_{uv}}$ such that $\mathrm{coev}_{H,uu} = \bigoplus_{v \in V} D_{H,uv}$.

Then

$$\begin{aligned}
\mathrm{id}_{H,uv} &= \big((\mathrm{id}_H \otimes \mathrm{ev}_H) \circ (\mathrm{coev}_H \otimes \mathrm{id}_H)\big)_{uv} \\
&= (\mathrm{id}_H \otimes \mathrm{ev}_H)_{uv} \circ (\mathrm{coev}_H \otimes \mathrm{id}_H)_{uv} \\
&= \Big(\bigoplus_{w \in V} \mathrm{id}_{H,uw} \otimes \mathrm{ev}_{H,wv}\Big) \circ \Big(\bigoplus_{t \in U} \mathrm{coev}_{H,ut} \otimes \mathrm{id}_{H,tv}\Big) \\
&= (\mathrm{id}_{H,uv} \otimes \mathrm{ev}_{H,vv}) \circ (\mathrm{coev}_{H,uu} \otimes \mathrm{id}_{H,uv}) \\
&= (\mathrm{id}_{H,uv} \otimes C_{H,vu}) \circ (D_{H,uv} \otimes \mathrm{id}_{H,uv}),
\end{aligned}$$

for $u \in U$, $v \in V$. Similarly,

$$\begin{aligned}
\mathrm{id}_{\bar{H},vu} &= (\mathrm{ev}_{H,vv} \otimes \mathrm{id}_{\bar{H},vu}) \circ (\mathrm{id}_{\bar{H},vu} \otimes \mathrm{coev}_{H,uu}) \\
&= (C_{H,vu} \otimes \mathrm{id}_{\bar{H},vu}) \circ (\mathrm{id}_{\bar{H},vu} \otimes D_{H,uv}),
\end{aligned}$$

for $v \in V$, $u \in U$.

Remark 5.9. ev_H and coev_H are completely determined by $C_{H,uv}$ and $D_{H,uv}$.

Definition 5.10. Let $\mathcal{C}(K, \mathrm{ev}_K) = \mathcal{C}(K, \mathrm{ev}_K, \mathrm{coev}_K)$ be a 2-subcategory of BigHilb with a 1-morphism generator $K : V_0 \to V_1$ and distinguished 2-morphisms evaluation and coevaluation $\mathrm{ev}_K, \mathrm{coev}_K$. We require that

(a) K is dualizable.

(b) The evaluation and coevaluation for the dual $\bar{K}$:

$$\mathrm{ev}_{\bar{K}} := (\mathrm{coev}_K)^\dagger \quad \text{and} \quad \mathrm{coev}_{\bar{K}} := (\mathrm{ev}_K)^\dagger.$$

(c) They satisfy the *d-fairness condition*, namely,

$$\mathrm{ev}_{\bar{K}} \circ \mathrm{coev}_K = d \cdot \mathrm{id}_{\mathbb{C}^{|V_0|}}, \quad \mathrm{ev}_K \circ \mathrm{coev}_{\bar{K}} = d \cdot \mathrm{id}_{\mathbb{C}^{|V_1|}}.$$

In other words,

$$C_{\bar{K},uv} = (D_{K,uv})^\dagger, \quad D_{\bar{K},vu} = (C_{K,vu})^\dagger,$$

and

$$\text{For each } P \in V_0, \quad \sum_{Q \in V_1} C_{\bar{K},PQ} \circ D_{K,PQ} = d \cdot \mathrm{id}_{\mathbb{C}},$$

$$\text{For each } Q \in V_1, \quad \sum_{P \in V_0} C_{K,QP} \circ D_{\bar{K},QP} = d \cdot \mathrm{id}_{\mathbb{C}},$$

Here, the 1-morphism generator means all the 1-morphism is Cauchy generated by K and $\overline{K}$.

Remark 5.11. Note that K satisfies the d-fairness condition, the maps $\mathrm{id}_K \otimes -$ and $- \otimes \mathrm{id}_K$ (also for $\overline{K}$) are bounded by d and hence weak* continuous. Therefore, $\mathcal{C}(K, \mathrm{ev}_K)$ is a rigid W* 2-subcategory of BigHilb.

Remark 5.12. ev_K, coev_K, $\mathrm{ev}_{\overline{K}}$ and $\mathrm{coev}_{\overline{K}}$ are determined by one of them in $\mathcal{C}(K, \mathrm{ev}_K)$.

Proposition 5.13. *The followings are some properties of $\mathcal{C}(K, \mathrm{ev}_K)$.*

(1) *Let $V = V_0 \sqcup V_1$, then all the 1-morphisms in $\mathcal{C}(K, \mathrm{ev}_K)$, including $K, \overline{K}$, can be regarded as $V \times V$-bigraded Hilbert spaces. So we can regard $\mathcal{C}(K, \mathrm{ev}_K)$ as a 2-category with one object V. Then all the 2-morphisms can be regarded as $V \times V$-bigraded uniformly bounded operators.*
If $(P, Q) \notin V_0 \times V_1$, then $K_{PQ} = \overline{K}_{QP} = 0$, which follows that

$$C_{K,QP} = D_{K,PQ} = 0.$$

The zigzag condition between them still hold.

(2) *All the 1-morphisms in $\mathcal{C}(K, \mathrm{ev}_K)$ are dualizable.*

(3) $\sup_{P \in V_0, Q \in V_1} \dim(K_{PQ}) < \infty$. *In fact, we will see* $\sup_{P \in V_0, Q \in V_1} \dim(K_{PQ}) \leq d^2$ *in the next section (Section 5.3) together with Proposition 5.3.*

(4) *There exist standard spherical evaluation and coevaluation in 2-morphisms:*

$$\mathrm{ev}_K^{\mathrm{st}} : \overline{K} \otimes K \to \mathbb{C}^{|V_1|}, \quad \mathrm{coev}_K^{\mathrm{st}} : \mathbb{C}^{|V_0|} \to K \otimes \overline{K},$$
$$\mathrm{ev}_{\overline{K}}^{\mathrm{st}} := (\mathrm{coev}_K)^\dagger, \quad \mathrm{coev}_{\overline{K}}^{\mathrm{st}} := (\mathrm{ev}_K)^\dagger.$$

In more details, Let $\{\varepsilon_i\}_{i=1}^k$ be the orthonormal basis (ONB) of K_{PQ} and $\{\epsilon_i^*\}$ be the dual basis of $\overline{K_{PQ}}$, $P \in V_0$, $Q \in V_1$ then

$$C_{K,QP}^{\mathrm{st}} : \overline{K}_{QP} \otimes K_{PQ} = \overline{K_{PQ}} \otimes K_{PQ} \to \mathbb{C},$$
$$D_{K,ab}^{\mathrm{st}} : \mathbb{C} \to K_{PQ} \otimes \overline{K}_{QP} = K_{PQ} \otimes \overline{K_{PQ}},$$
$$C_{\overline{K},PQ}^{\mathrm{st}} := (D_{K,PQ}^{\mathrm{st}})^\dagger, \quad D_{\overline{K},QP}^{\mathrm{st}} := (C_{K,QP}^{\mathrm{st}})^\dagger$$

are defined as

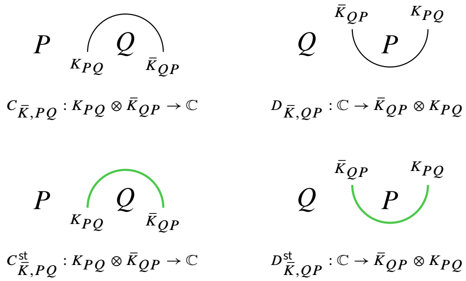
$$C_{K,QP}^{\mathrm{st}} : \varepsilon_i^* \otimes \varepsilon_j \mapsto \delta_{i=j}, \quad D_{K,PQ}^{\mathrm{st}} : 1 \mapsto \sum_{i=1}^k \varepsilon_i \otimes \varepsilon_i^*.$$

Note that $\mathrm{ev}_K^{\mathrm{st}}$ and $\mathrm{coev}_K^{\mathrm{st}}$ are well defined 2-morphisms because of (3), and the definitions of $\mathrm{ev}_K^{\mathrm{st}}$ and $\mathrm{coev}_K^{\mathrm{st}}$ do not depend on the choice of ONB on each K_{PQ} and they also meet the zigzag condition.

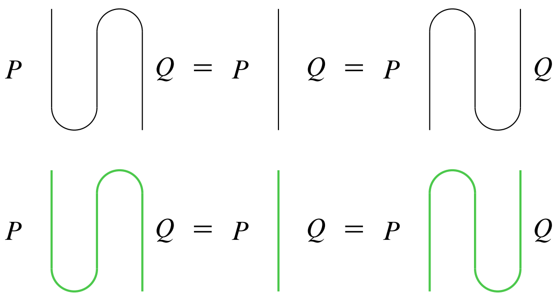
Notation 5.14. Now, we use graphic calculus to describe $\mathcal{C}(K, \mathrm{ev}_K)$. The idea is from the graphical calculus for 2Hilb [39]. However, in their paper, they only care about the case when $\mathrm{ev} = \mathrm{ev}^{\mathrm{st}}$ and $\mathrm{coev} = \mathrm{coev}^{\mathrm{st}}$, which is not necessarily true in our context.

First, we provide the single object version:

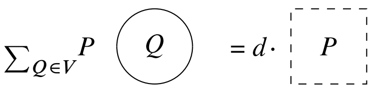
(1) For $P \in V_0$, $Q \in V_1$, $C_{\bar{K},PQ}$, $D_{\bar{K},QP}$, $C^{\mathrm{st}}_{\bar{K},PQ}$ and $D^{\mathrm{st}}_{\bar{K},QP}$.



(2) Rigidity:

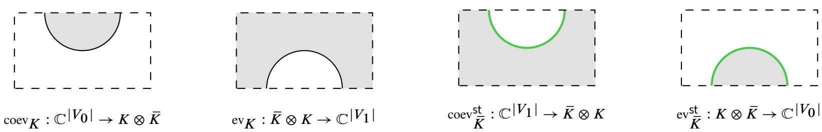


(3) d-fairness. For $P \in V$,

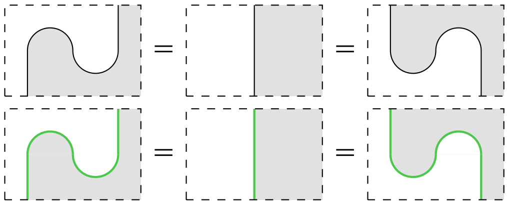


Then the graphical calculus version: In the n-category setting, n-morphisms are used to label codimension n cells of an n-manifold. So here, 0-morphisms in BigHilb label regions of the plane, 1-morphisms label strings from left to right, and 2-morphisms label tickets (including ev and coev) from bottom to top. Shading is just shorthand for the labeling. The unshaded region indicates the object V_0 and the shaded region indicates V_1.

(1) coev_K, ev_K, $\mathrm{coev}^{\mathrm{st}}_{\bar{K}}$ and $\mathrm{ev}^{\mathrm{st}}_{\bar{K}}$.

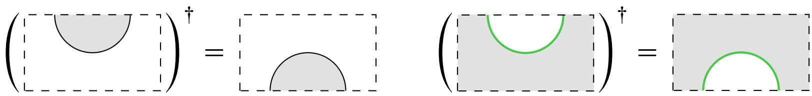


(2) Rigidity:



(3) d-fairness:

(4) Dagger structure on ev and $\mathrm{ev}^{\mathrm{st}}$.



5.3. The 2-subcategory of BigHilb generated by a balanced d-fair bipartite graph

In this section, we show the relation between 2-categories $\mathcal{C}(K, \mathrm{ev}_K)$ and d-fair bipartite graphs (Λ, ω). Then we may regard the generator K as a Hilb-enriched graph and the edge-weighting ω gives the interesting dual pair.

Construction 5.15. First, we construct a W* 2-subcategory $\mathcal{C}(\Lambda, \omega)$ of BigHilb from a balanced d-fair bipartite graph (Λ, ω) as follows:

(a) Object is $V = V(\Lambda) = V_0 \sqcup V_1$, which is a countable set.

(b) The 1-morphism generator $K = K_\Lambda$: At $(P, Q) \in V_0 \times V_1$, K_{PQ} is the Hilbert space with ONB $\{|e\rangle : e \in E(\Lambda),\ s(e) = P,\ t(e) = Q\}$ and other entries are 0. The uniform boundedness condition follows from Proposition 5.3.
As for the dual 1-morphism $\overline{K}$, at entry $(Q, P) \in V_1 \times V_0$, $\overline{K}_{QP}$ is the Hilbert space with ONB

$$\big\{|e\rangle : e \in E(\Lambda),\ s(e) = Q,\ t(e) = P\big\} = \big\{|\bar{e}\rangle : e \in E(\Lambda),\ s(e) = P,\ t(e) = Q\big\},$$

where $\overline{(\cdot)}$ is the involution of edge.
So we may regard K as a Hilb-enriched graph.

(c) All the 1-morphisms are Cauchy generated by K and $\overline{K}$.

(d) 2-morphisms are $V \times V$-bigraded uniformly bounded operators between those 1-morphisms.

(e) The edge-weighting gives the distinguished evaluation and coevaluation ev and coev. Note that K_{PQ} is a Hilbert space with orthonormal basis

$$\big\{|e\rangle : e \in E(\Lambda),\ s(e) = P,\ t(e) = Q\big\},$$

then $\{|\bar{e}\rangle : e \in E(\Lambda),\ s(e) = P,\ t(e) = Q\}$ is an orthonormal basis for $\overline{K}_{QP}$.

Define

$$
\begin{aligned}
C_{\bar{K},PQ} &: K_{PQ} \otimes \bar{K}_{QP} \to \mathbb{C} \\
&\quad \text{by } |e\rangle \otimes |\overline{e'}\rangle \mapsto \delta_{e=e'} w(e)^{\frac{1}{2}},\ e : P \to Q, \\
D_{K,PQ} &: \mathbb{C} \to K_{PQ} \otimes \bar{K}_{QP} \\
&\quad \text{by } 1 \mapsto \sum_{e:P\to Q} w(e)^{\frac{1}{2}} |e\rangle \otimes |\bar{e}\rangle = \sum_{e:Q\to P} w(\bar{e})^{\frac{1}{2}} |\bar{e}\rangle \otimes |e\rangle, \\
C_{K,QP} &: \bar{K}_{QP} \otimes K_{PQ} \to \mathbb{C} \\
&\quad \text{by } |e\rangle \otimes |\overline{e'}\rangle \mapsto \delta_{e=e'} w(e)^{\frac{1}{2}},\ e : Q \to P, \\
D_{\bar{K},QP} &: \mathbb{C} \to \bar{K}_{QP} \otimes K_{PQ} \\
&\quad \text{by } 1 \mapsto \sum_{e:Q\to P} w(e)^{\frac{1}{2}} |e\rangle \otimes |\bar{e}\rangle = \sum_{e:P\to Q} w(\bar{e})^{\frac{1}{2}} |\bar{e}\rangle \otimes |e\rangle.
\end{aligned}
$$

Proposition 5.16. *$\mathcal{C}(\Lambda, \omega)$ satisfies the condition in Definition 5.10.*

Proof. We shall prove that $C(\Lambda, \omega)$ is rigid and d-fair.

(a) Rigidity: For each $P, Q \in V$, $e : P \to Q$,

$$
\begin{aligned}
&(C_{\bar{K},PQ} \otimes \mathrm{id}_{K,PQ}) \circ (\mathrm{id}_{K,PQ} \otimes D_{\bar{K},QP})(|e\rangle \otimes 1) \\
&\qquad = (C_{\bar{K},PQ} \otimes \mathrm{id}_{K,PQ})\Big(|e\rangle \otimes \sum_{e:P\to Q} w(\bar{e})^{\frac{1}{2}} |\bar{e}\rangle \otimes |e\rangle\Big) \\
&\qquad = w(e)^{\frac{1}{2}} w(\bar{e})^{\frac{1}{2}} |e\rangle = |e\rangle, \\
&(\mathrm{id}_{K,PQ} \otimes C_{K,QP}) \circ (D_{K,PQ} \otimes \mathrm{id}_{K,QP})(1 \otimes |e\rangle) \\
&\qquad = (\mathrm{id}_{K,PQ} \otimes C_{K,QP})\Big(\sum_{e:P\to Q} w(e)^{\frac{1}{2}} |e\rangle \otimes |\bar{e}\rangle \otimes |e\rangle\Big) \\
&\qquad = w(e)^{\frac{1}{2}} w(\bar{e})^{\frac{1}{2}} |e\rangle = |e\rangle.
\end{aligned}
$$

(b) d-fairness:

$$
\begin{aligned}
\sum_{Q\in V_1} C_{\bar{K},PQ} \circ D_{K,PQ}(1) &= \sum_{Q\in V} C_{\bar{K},PQ}\Big(\sum_{e:P\to Q} w(e)^{\frac{1}{2}} |e\rangle \otimes |\bar{e}\rangle\Big) \\
&= \sum_{\{e|s(e)=P\}} w(e)^{\frac{1}{2}} w(e)^{\frac{1}{2}} = d, \\
\sum_{P\in V_0} C_{K,QP} \circ D_{\bar{K},QP}(1) &= \sum_{a\in V} C_{K,QP}\Big(\sum_{e:Q\to P} w(e)^{\frac{1}{2}} |e\rangle \otimes |\bar{e}\rangle\Big) \\
&= \sum_{\{e|s(e)=Q\}} w(e)^{\frac{1}{2}} w(e)^{\frac{1}{2}} = d.
\end{aligned}
$$

∎

Remark 5.17. Suppose $\theta : (\Lambda, \omega) \to (\Lambda', \omega')$ is an isomorphism of edge-weighted graphs (see Definition 5.4). We construct a unitary equivalence between $\mathcal{C}(\Lambda, \omega)$ and $\mathcal{C}(\Lambda', \omega')$. For the 1-morphism generators K_Λ and $K_{\Lambda'}$, we have

$$K_{\Lambda,PQ} \cong K_{\Lambda',\theta(P)\theta(Q)}$$

as finite dimensional Hilbert spaces, via the bijection of ONBs given by $|e\rangle \mapsto |\theta(e)\rangle$. Denote by $u_\theta : K_\Lambda \to K_{\Lambda'}$ this unitary isomorphism.

As for the evaluation ev_{K_Λ} and $\mathrm{ev}_{K_{\Lambda'}}$, we look at $C_{K_\Lambda,PQ}$ and $C_{K_{\Lambda'},\theta(P)\theta(Q)}$ (see Definition 5.8). Note that $C_{K_{\Lambda'},\theta(P)\theta(Q)} : \overline{K}_{\Lambda',\theta(Q)\theta(P)} \otimes K_{\Lambda',\theta(P)\theta(Q)} \to \mathbb{C}$ by

$$\big|\theta(e)\big\rangle \otimes \big|\theta(e')\big\rangle \mapsto \delta_{\theta(e)=\theta(e')}\omega'\big(\theta(e)\big) = \delta_{e=e'}\omega(e), \quad \forall\, e : Q \to P \in E(\Lambda).$$

We have

$$C_{K_{\Lambda'},\theta(P)\theta(Q)} = C_{K_\Lambda,PQ} \circ (\overline{u_\theta}^\dagger_{QP} \otimes u_\theta{}^\dagger_{PQ}).$$

In other words,

$$\mathrm{ev}_{K_{\Lambda'}} = \mathrm{ev}_{K_\Lambda} \circ (\overline{u_\theta}^\dagger \otimes u_\theta^\dagger).$$

Therefore, $\mathcal{C}(\Lambda, \omega)$ and $\mathcal{C}(\Lambda', \omega')$ are unitary equivalent up to the unitary 2-morphism u_θ.

Next, start with a 2-category $\mathcal{C}(K, \mathrm{ev}_K)$, we construct a balanced d-fair bipartite graph (Λ, ω).

Definition 5.18. For $P \in V_0$, $Q \in V_1$, let $v_{PQ} : K_{PQ} \to \overline{K_{PQ}} = \overline{K}_{QP}$ be the canonical dual map that $\xi \mapsto \xi^*$ and $v_{PQ}^\dagger : \overline{K}_{QP} \to K_{PQ}$ defined by $\xi^* \to \xi^{**} = \xi$. Then

$$v_{PQ}^\dagger \circ v_{PQ} = \mathrm{id}_{K,PQ} \quad \text{and} \quad v_{PQ} \circ v_{PQ}^\dagger = \mathrm{id}_{\overline{K},QP}.$$

Define

$$\begin{aligned}
\varphi_{K,PQ} &: \overline{K}_{QP} \to K_{PQ} \quad \text{by } \varphi_{K,PQ} = (\mathrm{id}_{K,PQ} \otimes C^{\mathrm{st}}_{K,QP}) \circ (D_{K,PQ} \otimes v_{PQ}^\dagger),\\
\varphi_{\overline{K},QP} &: K_{PQ} \to \overline{K}_{QP} \quad \text{by } \varphi_{\overline{K},QP} = (\mathrm{id}_{\overline{K},QP} \otimes C^{\mathrm{st}}_{\overline{K},PQ}) \circ (D_{\overline{K},QP} \otimes v_{PQ}^\dagger).
\end{aligned}$$

Proposition 5.19. *Here are some properties for φ_K and $\varphi_{\overline{K}}$.*

(1) $\varphi_{K,PQ} \circ \varphi_{\overline{K},QP} = \mathrm{id}_{K,PQ}$.

(2) $\sum_{Q \in V_1} \mathrm{Tr}(\varphi_{K,PQ}^\dagger \circ \varphi_{K,PQ}) = \sum_{P \in V_0} \mathrm{Tr}(\varphi_{\overline{K},QP}^\dagger \circ \varphi_{\overline{K},QP}) = d$.

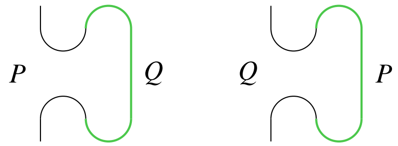
Proof. See [10, Prop. 1.8], [13, Prop. 3.10]. ∎

Construction 5.20. Define the graph Λ to be $V(\Lambda) := V$ and the number of edges from $P \in V_0$ to $Q \in V_1$ to be $\dim K_{PQ}$. Define edge-weighting function $\omega : E(\Lambda) \to (0, \infty)$ as the multiset

$$\begin{aligned}
\big\{\omega(e)\big\}_{e:P\to Q} &:= \big\{\text{eigenvalues of } \varphi_{K,PQ} \circ \varphi_{K,PQ}^\dagger\big\},\\
\big\{\omega(e)\big\}_{e:Q\to P} &:= \big\{\text{eigenvalues of } \varphi_{\overline{K},QP} \circ \varphi_{\overline{K},QP}^\dagger\big\}.
\end{aligned}$$

From Proposition 5.19 above, (Λ,ω) is a d-fair and balanced bipartite graph. To be precise, (1) gives the balance condition, and (2) gives the d-fairness. In fact,

$$\begin{aligned}\varphi_{K,PQ}\circ\varphi_{K,PQ}^{\dagger} &= (\mathrm{id}_{K,PQ}\otimes C^{\mathrm{st}}_{K,QP})\circ(D_{K,PQ}\otimes \mathrm{id}_{K,PQ})\\ &\qquad\circ(C_{\bar{K},PQ}\otimes\mathrm{id}_{K,PQ})\circ(\mathrm{id}_{K,PQ}\otimes D^{\mathrm{st}}_{\bar{K},QP}),\\ \varphi_{\bar{K},QP}\circ\varphi_{\bar{K},QP}^{\dagger} &= (\mathrm{id}_{\bar{K},QP}\otimes C^{\mathrm{st}}_{\bar{K},PQ})\circ(D_{\bar{K},QP}\otimes \mathrm{id}_{\bar{K},QP})\\ &\qquad\circ(C_{K,QP}\otimes\mathrm{id}_{\bar{K},QP})\circ(\mathrm{id}_{\bar{K},QP}\otimes D^{\mathrm{st}}_{K,PQ}).\end{aligned}$$



Remark 5.21. For a given 2-category $\mathcal{C}(K,\mathrm{ev}_K)$, let (Λ,ω) be the balanced d-fair bipartite graph obtained from Construction 5.20. When we construct the 1-morphism generator $K=K_\Lambda$ in $\mathcal{C}(\Lambda,\omega)$ from the bipartite graph Λ, we secretly make a choice of ONB for each $(K_\Lambda)_{PQ}$, so there is a unitary 2-morphism $\alpha: K\to K_\Lambda$ such that $\mathrm{ev}_K=\mathrm{ev}_{K_\Lambda}\circ(\bar{\alpha}\otimes\alpha)$. Therefore, $\mathcal{C}(K,\mathrm{ev}_K)$ and $\mathcal{C}(\Lambda,\omega)$ are unitary equivalent up to a unitary 2-morphism α.

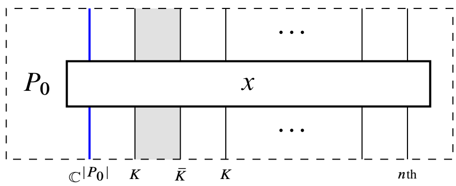
5.4. From $\mathcal{C}(K,\mathrm{ev}_K)$ to Markov tower

Construction 5.22. Here, we are going to build a tower of algebra from the 2-category $\mathcal{C}(K,\mathrm{ev}_K)$ discussed above with a chosen point, say $P_0\in V_0$. Let $\mathbb{C}^{|P_0|}$ be a 1-morphism with all the entry being 0 except $(\mathbb{C}^{|P_0|})_{P_0P_0}=\mathbb{C}$.

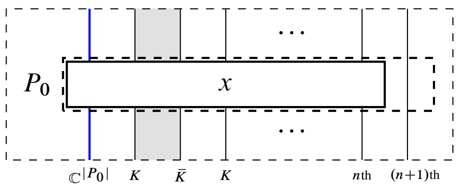
Note that $\mathbb{C}^{|P_0|}\otimes K^{\mathrm{alt}\otimes n}$ is a 1-morphism for each $n\in\mathbb{Z}_{\geq 0}$.

Let $M_n=\mathrm{End}(\mathbb{C}^{|P_0|}\otimes K^{\mathrm{alt}\otimes n})$ and identify $M_n\ni x$ with $x\otimes\mathrm{id}_{K^?}\in M_{n+1}$, where $K^?=K$ if $2\mid n$, $K^?=\bar{K}$ if $2\nmid n$. We use the graphical calculus to show $M=(M_n)_{n\geq 0}$ is a Markov tower.

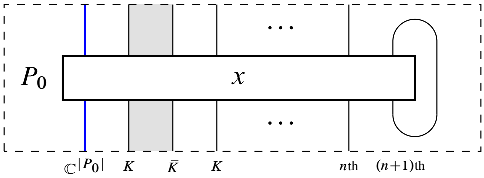
(1) Element $x\in M_n$:



(2) Inclusion $x\in M_n\subset M_{n+1}$:

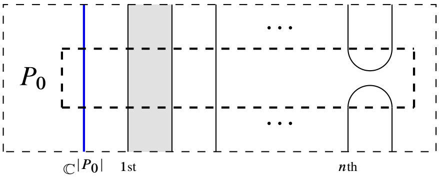


(3) Conditional expectation $E_{n+1} : M_{n+1} \to M_n$, $x \in M_n$:

$E_{n+1}(x) = d^{-1}$ 

Here, the choice of the duality pair $(\mathrm{coev}_K, (\mathrm{coev}_K)^\dagger)$ or $(\mathrm{ev}_K, (\mathrm{ev}_K)^\dagger)$ depends on the shading.

(4) Jones projection $e_n \in M_{n+1}$:

$e_n = d^{-1}$ 

(5) The pull down property is true automatically in this setting. See diagram (MT6) in Section 3.2.

5.5. More properties of Markov tower

Here, we are going to explore more properties of Markov tower. The tracial version has been proved in [17, Thm. 4.1.4, Thm. 4.6.3] and [7, Prop. 3.4]. For convenience, here we will prove those properties for the traceless case.

Lemma 5.23. *Suppose $A \subset B$ is a unital inclusion of finite dimensional* C^**-algebras and $E : B \to A$ is a faithful conditional expectation. Then there is an orthonormal basis $\{u_i\}_{i\in I}$ such that $\sum_{i\in I} u_i E(u_i^* x) = x$ for all $x \in B$, where $|I| < \infty$.*

Proof. Regard B as a right A-module equipped with an A-valued inner product $\langle x|y\rangle_A := E(x^*y)$. Note that A and B are finite dimensional, so B is a finitely generated projective Hilbert A-module. By [14, Thm. 4.1] and [26, Lemma. 1.7], there exists an orthonormal basis $\{u_i\}_{i\in I} \subset B$ such that $x = \sum_{i\in I} u_i \langle u_i|x\rangle_A = \sum_{i\in I} u_i E(u_i^* x)$ for all $x \in B$ and $|I| < \infty$. ∎

Proposition 5.24. *The tracial version has been proved in* [17, Thm. 4.1.4, Thm. 4.6.3] *and* [7, Prop. 3.4].

(1) *$X_{n+1} := M_n e_n M_n$ is a 2-sided ideal of M_{n+1} and hence M_{n+1} splits as a direct sum of von Neumann algebras $X_{n+1} \oplus Y_{n+1}$. We also define $Y_0 = M_0$, $Y_1 = M_1$ so that $X_0 = X_1 = 0$. X_{n+1} is called the old stuff and Y_{n+1} is called the new stuff.*

(2) *X_{n+1} is isomorphic to $M_n \otimes_{M_{n-1}} M_n$, which is the basic construction from $E_n : M_n \to M_{n-1}$. Denote this isomorphism as ϕ. Here, $M_n \otimes_{M_{n-1}} M_n$ is a $*$-algebra with multiplication $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 E_n(y_1 x_2) \otimes y_2$ and adjoint $(x \otimes y)^* = y^* \otimes x^*$.*

(3) *If $y \in Y_{n+1}$ and $x \in X_n$, then $yx = 0$ in M_{n+1}. Hence $E_{n+1}(Y_{n+1}) \subset Y_n$, which means the new stuff comes from the old new stuff.*

(4) *If $Y_n = 0$, then $Y_k = 0$ for all $k \geq n$.*

Proof. (1) Note that $M_{n+1}e_n = M_n e_n$, then $M_{n+1}M_n e_n M_n \subset M_{n+1}e_n M_n = M_n e_n M_n$ and $M_n e_n M_n M_{n+1} = (M_{n+1}M_n e_n M_n)^* \subset (M_n e_n M_n)^* = M_n e_n M_n$.

(2) See Watatani index theory [43, §1] with Lemma 5.23.

(3) Note that as a finite dimensional von Neumann algebra, $M_{n+1} = \bigoplus_i M_{n+1}p_i$, where p_i are the minimum central projections. So if $y \in Y_{n+1}$, then $y = \sum_j m_j p_j$, where $[p_j, e_n] = 0$.

For $ae_{n-1}b \in X_n$ and $m_j p_j \in Y_{n+1}$, by Jones projection property,

$$m_j p_j a e_{n-1} b = d^{-2} m_j p_j a e_{n-1} e_n e_{n-1} b = d^{-2} m_j a e_{n-1} p_j e_n e_{n-1} b = 0,$$

so $yx = 0$ for any $x \in X_n$, $y \in Y_{n+1}$.

Let $X_n = \bigoplus_k M_n q_k$, where q_k are the minimum central projections. For any $y \in Y_{n+1}$,

$$q_k E_{n+1}(y) = E_{n+1}(q_k y) = 0 \quad \text{for all } k,$$

which implies that $E_{n+1}(y) \in Y_n$.

(4) By (3) and faithfulness of E_n. ∎

5.6. From Markov tower to $\mathscr{C}(\Lambda, \omega)$

Now we are able to extract the so-called principal graph data from the Markov tower, which is similar to the classical tracial Markov tower [36], [21, §4.2].

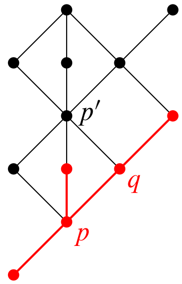
If A is a finite dimensional C*-algebra, we write $\pi(A)$ to be the set of minimal central projections of A. If $A \subset B$ is a unital inclusion of finite dimensional C*-algebras, then the inclusion matrix is the $\pi(A) \times \pi(B)$ matrix, with (p, q)th entry being $(\dim_{\mathbb{C}}(pqA'pq \cap pqBpq))^{\frac{1}{2}}$. If $A \subset B \subset B_1$ is a basic construction, then the inclusion matrix of $B \subset B_1$ is the transpose of the inclusion matrix of $A \subset B$ [17, §2], [21].

The inclusion matrix of $A \subset B$ can be described as the *Bratteli diagram* of $A \subset B$, whose vertices are the minimal central projections and the number of edges between p and q is the (p, q)th entry.

The Bratteli diagram Δ of the Markov tower $M = (M_n)_{n \geq 0}$ contains all the Bratteli diagram Δ_n of $M_n \subset M_{n+1}$. Then by the property of inclusion matrix of basic construction and Proposition 5.24 (2), the Bratteli diagram for $M_n \subset M_{n+1}$ contains the reflection of the Bratteli diagram of $M_{n-1} \subset M_n$ and new part, which is called the *principal part*. A vertex in the new part is called a *new vertex*, otherwise, called an *old vertex*. The reflected vertex from a new vertex is called a *new old vertex*. Moreover, for a new vertex $p \in Y_n$, denote p' to be the new old vertex of p in M_{n+2}.

The *principal graph* Λ contains the new part in the Bratteli diagram Δ, so its vertices are new vertices. To be precise, $V(\Lambda)$ contains all the minimal central projections p in the new stuff. By Proposition 5.24 (4), the new stuff comes from the old new stuff, then for $p, q \in \Lambda$, $E(\Lambda)$ contains all the edges between p and q.

It is clear that both the Bratteli diagram and the principal graph are bipartite. We can also use the principal graph to construct the Bratteli diagram by doing the reflection at each level.



The red part is principal part

p' is the new old vertex of p

p, q are new vertices

Let us then compute the edge weighting $w : E(\Lambda) \to (0, \infty)$. Before that, we first give a lemma:

Lemma 5.25. *The followings are some properties for the relative commutant in* BigHilb*:*

(1) *Let $H_1, H_2, \ldots, H_n, G_1, G_2, \ldots, G_n$ be finite dimensional Hilbert spaces. We identify $B(H_i)$ with $B(H_i) \otimes \mathrm{id}_{G_i}$ and $B(G_i)$ with $\mathrm{id}_{H_i} \otimes B(G_i)$ as subalgebras in $B(\bigoplus_{i=1}^n H_i \otimes G_i)$ for each $i = 1, \ldots, n$, then the relative commutant*

$$\bigcap_{i=1}^{n} \left(B(H_i)' \cap B\left(\bigoplus_{i=1}^{n} H_i \otimes G_i \right) \right) = \bigoplus_{i=1}^{n} B(G_i). \qquad (*)$$

(2) *Let H be a 1-morphism in* BigHilb*, then the center $Z(\mathrm{End}(H))$ is the linear span of all the direct summands of id_H.*

(3) *Let G be another 1-morphism in* BigHilb *such that $H \otimes G$ is nondegenerate, i.e., for each non-zero H_{pq}, there is a non-zero G_{qr} and vice versa. We identify $\mathrm{End}(H)$ with $\mathrm{End}(H) \otimes \mathrm{id}_G$ and $\mathrm{End}(G)$ with $\mathrm{id}_H \otimes \mathrm{End}(G)$ as subalgebras in $\mathrm{End}(H \otimes G)$. Then the relative commutant*

$$\mathrm{End}(H)' \cap \mathrm{End}(H \otimes G) = Z(\mathrm{End}(H)) \otimes \mathrm{End}(G).$$

(4) *Moreover, if H_{pq} is non-zero only when $p = p_0 \in V$, then the relative commutant can be represented as*

$$\mathrm{End}(H)' \cap \mathrm{End}(H \otimes G) = \mathrm{id}_H \otimes \mathrm{End}(G).$$

Reminder. *The tensor product in (1) is the tensor product of Hilbert spaces and bounded operators; the tensor product in (3) and (4) is the tensor product of 1-morphisms/2-morphisms in* BigHilb*, see Definition 5.6.*

Proof. (1) $\supset$ is clear. We show $\subset$.

For $f \in B(\bigoplus_{i=1}^n H_i \otimes G_i)$, $f = \bigoplus_{i,j=1}^n f_{i,j}$, where $f_{i,j} \in B(H_i \otimes G_i, H_j \otimes G_j)$. We shall prove that $f_{i,j} = 0$ for $i \neq j$ and $f_{i,i} \in \mathrm{id}_{H_i} \otimes B(G_i)$ if $f \in$LHS of equation $(*)$.

Let $x_i \in B(H_i)$, then

$$f(x_i \otimes \mathrm{id}_{G_i}) = \bigoplus_{j=1}^{n} f_{i,j}(x_i \otimes \mathrm{id}_{G_i}) = \bigoplus_{k=1}^{n}(x_i \otimes \mathrm{id}_{G_i}) f_{k,i} = (x_i \otimes \mathrm{id}_{G_i}) f,$$

which implies that

$$f_{i,j}(x_i \otimes \mathrm{id}_{G_i}) = (x_i \otimes \mathrm{id}_{G_i}) f_{k,i} = 0 \quad \text{for } k \neq i,\ j \neq i$$

and

$$f_{i,i}(x_i \otimes \mathrm{id}_{G_i}) = (x_i \otimes \mathrm{id}_{G_i}) f_{i,i}.$$

From the first half, if we choose $x_i = \mathrm{id}_{H_i}$, we obtain $f_{i,j} = f_{k,i} = 0$, $j \neq i, k \neq i$; from the second half, from a well-known statement that $B(H_i)' \cap B(H_i \otimes G_i) = B(G_i)$, so that $f_{i,i} \in \mathrm{id}_{H_i} \otimes G_i$.

(2) Clear, see Definition 5.5(d).

(3) $\supset$ is clear. We show $\subset$.

For $f \in \mathrm{End}(H)' \cap \mathrm{End}(H \otimes G)$, we shall prove that $f_{pq} \in \bigoplus_{r\in V} \mathrm{id}_{H_{pr}} \otimes B(H_{rq})$. Note that

$$\big(\mathrm{End}(H \otimes G)\big)_{pq} = \mathrm{End}\big((H \otimes G)_{pq}\big) = B\Big(\bigoplus_{r\in V} H_{pr} \otimes G_{rq}\Big).$$

For $f \in \mathrm{End}(H)' \cap \mathrm{End}(H \otimes G)$, f_{pq} commute with $B(H_{pr}) \otimes \mathrm{id}_{G_{rq}}$ for all $r \in V$. By (1), we have $f_{pq} \in \bigoplus_{r\in V} \mathrm{id}_{H_{pr}} \otimes B(H_{rq})$. Together with (2), we prove this statement.

(4) From (3), for $f \in \mathrm{End}(H)' \cap \mathrm{End}(H \otimes G)$,

$$f = \bigoplus_{q\in V} \mathrm{id}_{H_{p_0 q}} \otimes g^{(q)},$$

where $g^{(q)} \in \mathrm{End}(G)$.

Now we define $g \in \mathrm{End}(G)$ by $g_{ij} := g_{ij}^{(i)}$. Then $f = \mathrm{id}_H \otimes g$. ■

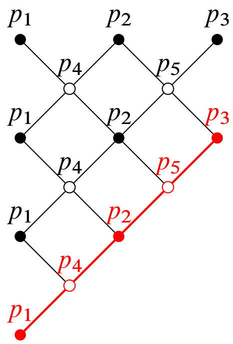
By Section 5.3, we are able to construct a W* 2-subcategory $\mathcal{C}(\Lambda)$ without providing the distinguished evaluation and coevaluation given by the edge weighting, though we still have the canonical evaluation and coevaluation denoted by $\mathrm{ev}^{\mathrm{st}}$ and $\mathrm{coev}^{\mathrm{st}}$, which are drawn in green below. We denote the generators by $K = K_\Lambda$ and $\bar{K}$. From Construction 5.22, let $N_n := \mathrm{End}(\mathbb{C}^{|p_0|} \otimes K^{\mathrm{alt}\otimes n})$.

Notation and observation 5.26. Denote Λ_n to be the subgraph of Λ with vertices depth $\leq n$ and the corresponding Hilb-enriched graph to be $K_n := K_{\Lambda_n}$ and $\bar{K}_n$ the dual space in the sense of Construction 5.15. As a convention, p_0 is of depth 0. Observe that

$$N_n = \mathrm{End}(K_1 \otimes \bar{K}_2 \otimes K_3 \otimes \bar{K}_4 \otimes \cdots \otimes K_n^?).$$

where $K_n^? = K_n$ if $2 \nmid n$, $K_n^? = \bar{K}_n$ if $2 \mid n$.

Example 5.27. Let us take A_5 graph for example. We label the vertices as follows.



Then

$$K_1 = \begin{bmatrix} 0 & 0 & 0 & \mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{K}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & \mathbb{C} & 0 \\ 0 & 0 & 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{K}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ 0 & \mathbb{C} & \mathbb{C} & 0 & 0 \end{bmatrix} = \bar{K}_{4+2k}, \quad K_5 = \begin{bmatrix} 0 & 0 & 0 & \mathbb{C} & 0 \\ 0 & 0 & 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 & 0 & \mathbb{C} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = K_{5+2k}, \ k = 0, 1, 2, \ldots$$

$$K_1 \otimes \bar{K}_2 \otimes K_3 = \begin{bmatrix} 0 & 0 & 0 & \mathbb{C}^2 & \mathbb{C} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_1 \otimes \bar{K}_2 \otimes K_3 \otimes \overline{K_4} = \begin{bmatrix} \mathbb{C}^2 & \mathbb{C}^3 & \mathbb{C} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For this example, observe that $\mathrm{End}(K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_n^?)$ is the semisimple quotient of $\mathrm{TLJ}_n(\sqrt{3})$.

One can regard Λ_n as the subgraph of the Bratteli diagram between depth $n-1$ and n, and K_n is the Hilb-enriched graph of Λ_n. The entry (i, j) in $K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_n^?$ indicates the number of paths from the vertex p_i at depth 0 to the vertex p_j at depth n. Note that the base point is a single vertex p_1, so entry only at $(1, j)$ can be non-zero.

The idea is to transport the Jones projections from the Markov tower (M_n) to the endomorphism algebras (N_n) in order to obtain the edge weighting ω. Let $\psi_n : M_n \to N_n$ be a $*$-algebra isomorphism for each $n \geq 0$ with $\psi_{n+1}|_{M_n} = \psi_n$.

Let us consider the image of Jones projection $\psi(e_n) \in N_{n+1}$. Note that

$$e_n \in M'_{n-1} \cap M_{n+1},$$

so $\psi(e_n) \in N'_{n-1} \cap N_{n+1}$.

Proposition 5.28.

$$N_{n-1}' \cap N_{n+1} = \begin{cases} \mathrm{id}_{K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_{2k-1}} \otimes \mathrm{End}(\bar{K}_{2k} \otimes K_{2k+1}) & n = 2k \\ \mathrm{id}_{K_1 \otimes \bar{K}_2 \otimes \cdots \otimes \bar{K}_{2k}} \otimes \mathrm{End}(K_{2k+1} \otimes \bar{K}_{2k+2}) & n = 2k+1. \end{cases}$$

Proof. Note that

$$K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_n^?$$

satisfies the condition in Lemma 5.25 (3) and (4). ■

Proposition 5.29. *Without loss of generality, let $n = 2k$. There exists a projection $\varepsilon_{2k} \in \mathrm{End}(\bar{K}_{2k} \otimes K_{2k+1})$ such that $\psi(e_{2k}) = \mathrm{id}_{K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_{2k-1}} \otimes \varepsilon_{2k}$.*

Proof. By proposition 5.28, there exists $\varepsilon_{2k} \in \mathrm{End}(\bar{K}_{2k} \otimes K_{2k+1})$ such that

$$\psi(e_{2k}) = \mathrm{id}_{K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_{2k-1}} \otimes \varepsilon_{2k}.$$

Note that e_{2k} is a projection, so is ε_{2k}. ■

Lemma 5.30. *Let H be a Hilbert space and $p \neq 0$ be a projection on H. Suppose $pfp \in \mathbb{C}p$ for all $f \in B(H)$, then $p = r^*r$, where $r : H \to \mathbb{C}$ and $rr^* = 1$.*

Similarly, let H be a 1-morphism in BigHilb *and $p \neq 0$ be a projection on H. Suppose $pfp \in \mathbb{C}p$ for all $f \in \mathrm{End}(H)$, then $p = r^*r$, where $r : H \to \mathbb{C}^{|V|}$ and $rr^* = \mathbb{C}^{|V|}$.*

Proof. For the Hilbert space case: Note that $\mathrm{Im}(fp)$ can be any subspace of H and $\mathrm{Im}(p(fp)) = \mathrm{Im}(p)$, so $\mathrm{Im}(p)$ does not depend on the input, i.e., p facts through $\mathbb{C}$. Let $r : H \to \mathbb{C}$ and $p = r^*r$ with $rr^* = 1$, since $p^* = p = p^*p$.

The similar argument on 1-morphisms in BigHilb. ■

As we see the construction of the Jones projection in Construction 5.22 (4), we shall prove that the Jones projection splits into two pieces.

By Proposition 2.4 (6), $e_n M_{n+1} e_n = M_{n-1} e_n$, so

$$\psi(e_n) N_{n+1} \psi(e_n) = N_{n-1} e_n.$$

Without loss of generality, let $n = 2k$. For each $f \in \mathrm{End}(\bar{K}_{2k} \otimes K_{2k+1})$, $\mathrm{id}_{K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_{2k-1}} \otimes f \in N_{2k+1}$, there exists $x \in N_{2k-1}$ such that

$$\mathrm{id}_{K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_{2k-1}} \otimes (\varepsilon_{2k} f \varepsilon_{2k}) = (x \otimes \mathrm{id}_{\bar{K}_{2k} \otimes K_{2k+1}})(\mathrm{id}_{K_1 \otimes \bar{K}_2 \otimes \cdots \otimes K_{2k-1}} \otimes \varepsilon_{2k}) = x \otimes \varepsilon_{2k},$$

which follows that $\varepsilon_{2k} f \varepsilon_{2k} \in \mathbb{C}\varepsilon_{2k}$.

By Lemma 5.30, there exists $r_{2k} : \bar{K}_{2k} \otimes K_{2k+1} \to \mathbb{C}^{|V_{1,2k-1}|}$ such that

$$\varepsilon_{2k} = r_{2k}^\dagger r_{2k} \quad \text{and} \quad r_{2k} r_{2k}^\dagger = \mathbb{C}^{|V_{1,2k-1}|},$$

where $V_{1,2k+1}$ contains all the simple objects in Λ_{2k+1} with odd depth.

Similarly, we can define $\varepsilon_{2k+1} \in \mathrm{End}(K \otimes \bar{K})$ corresponding to Jones projection e_{2k+1} and there exists

$$r_{2k+1} : K_{2k+1} \otimes \bar{K}_{2k+2} \to \mathbb{C}^{|V_{0,2k}|}$$

such that

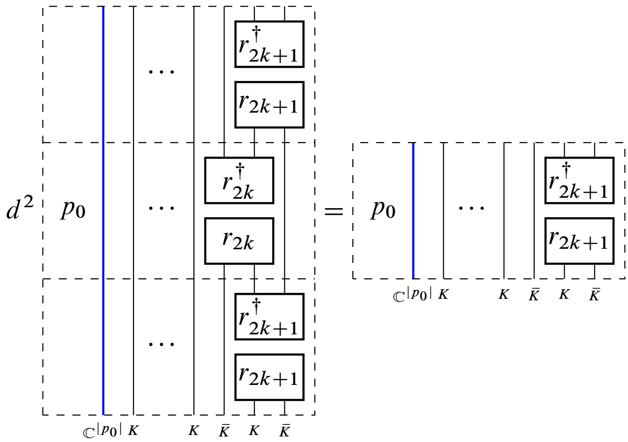
$$\varepsilon_{2k+1} = r_{2k+1}^{\dagger} r_{2k+1} \quad \text{and} \quad r_{2k+1} r_{2k+1}^{\dagger} = \mathbb{C}^{|V_{0,2k}|},$$

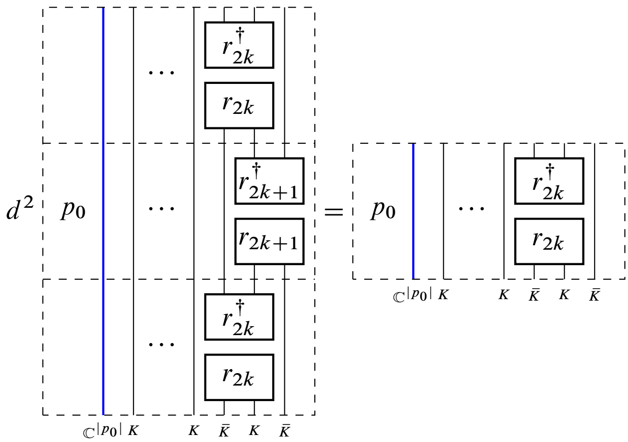
where $V_{0,2k}$ contains all the simple objects in Λ_{2k} with even depth.

Now consider

$$u_{2k} := d(\mathrm{id}_{\bar{K}} \otimes r_{2k+1}) \circ (r_{2k}^{\dagger} \otimes \mathrm{id}_K) \in \mathrm{End}(\bar{K}).$$

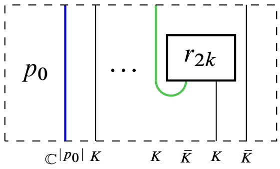
Note that $e_{2k} e_{2k+1} e_{2k} = d^{-2} e_{2k}$ and $e_{2k+1} e_{2k} e_{2k+1} = d^{-2} e_{2k+1}$, we have $u_{2k}^{\dagger} u_{2k} = \mathrm{id}_{\bar{K}_{2k}}$ and $u_{2k} u_{2k}^{\dagger} = \mathrm{id}_{\bar{K}_{2k+2}}$, so u_{2k} is a unitary.

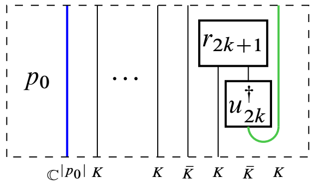




For adjacent simple objects $p, q \in \Lambda$ with p at depth n and q at depth $n+1$, we shall compute the edge weighting on the edges $e : p \to q$ and $e : q \to p$. Without loss of generality, $n = 2k$.

Define φ_{2k} and φ_{2k+1} as follows:

$\varphi_{2k} = d^{\frac{1}{2}}$ 

$\varphi_{2k+1} = d^{\frac{1}{2}}$ 

and we have the following properties:

(a) $\varphi_{2k+1} \circ \varphi_{2k}^{\dagger} = \mathrm{id}$.

(b) $\mathrm{Tr}(\varphi_{2k}^{\dagger} \circ \varphi_{2k}) = d\,\mathrm{Tr}(r_{2k}^{\dagger} r_{2k}) = d\,\mathrm{Tr}(r_{2k} r_{2k}^{\dagger}) = d$.

(c) $\mathrm{Tr}(\varphi_{2k+1}^{\dagger} \circ \varphi_{2k+1}) = d\,\mathrm{Tr}(u_{2k} r_{2k+1}^{\dagger} r_{2k+1} u_{2k}^{\dagger}) = d\,\mathrm{Tr}(r_{2k+1} r_{2k+1}^{\dagger}) = d$.

Definition 5.31. Define the edge-weighting function ω as the multiset:

$$\{\omega(e)\}_{e:p\to q} := \{\text{eigenvalues of } (\varphi_{2k}^{\dagger} \circ \varphi_{2k})_{pq}\},$$
$$\{\omega(e)\}_{e:q\to p} := \{\text{eigenvalues of } (\varphi_{2k+1}^{\dagger} \circ \varphi_{2k+1})_{pq}\}.$$

Combining Construction 5.20 and our definition with properties for $\varphi_{2k}, \varphi_{2k+1}$, the edge weighting ω we obtained for bipartite graph Λ is d-fair and balanced.

5.7. $\mathcal{C}(K, \mathrm{ev}_K)$ and $\mathrm{End}_0^{\dagger}(\mathcal{M}, F)$

In this section, $\mathcal{TLJ}(d)$ means the 2-shaded pivotal rigid C* multitensor category from Definition 2.28 with endomorphism spaces the Temperley–Lieb algebras and simple generator $X = 1^+ \otimes X \otimes 1^-$.

We have already seen the ways to construct a Markov tower from $\mathcal{C}(K, \mathrm{ev}_K)$ in this section or from $\mathcal{M}$ in Section 3 with a simple base point Z, where $\mathcal{M}$ is an indecomposable semisimple C* $\mathcal{TLJ}(d)$-module category. In this section, we will show their relation to each other.

Definition 5.32 (Endofunctor monoidal category). Define $\mathrm{End}^{\dagger}(\mathcal{M})$ to be a W* tensor category as follows:

(a) Objects: The objects are all the dagger endofunctors of $\mathcal{M}$.

(b) Morphisms: The morphisms are the uniformly bounded natural transformations between these dagger endofunctors which compatible with the dagger structure.

(c) Tensor structure: The tensor product is given by the composition of endofunctors, i.e., $F_1 \otimes F_2 := F_2 \circ F_1$ for endofunctors F_1, F_2.

Definition 5.33. Define $F := - \lhd X$, $\bar{F} := - \lhd \bar{X}$, which are endofunctors of $\mathcal{M}$. Note that F and $\bar{F}$ are adjoint functors, with unit ev_F and counit coev_F induced by ev_X and coev_X.

Define $\mathrm{End}_0^{\dagger}(\mathcal{M}, F)$ to be the full category Cauchy generated by F and $\bar{F}$. Since the generators are dualizable, the category is rigid.

We warn the reader that $\mathrm{End}_0^\dagger(\mathcal{M}, F)$ will only be multitensor $(\dim(\mathrm{End}(\mathrm{id}_{\mathcal{M}})) < \infty)$ when $\mathcal{M}$ is finitely semisimple. Moreover, the dual functor on $\mathrm{End}_0^\dagger(\mathcal{M}, F)$ given by ev_F and coev_F is not a unitary dual functor.

We can give an alternative description of $\mathrm{End}_0^\dagger(\mathcal{M}, F)$ using the following proposition.

Proposition 5.34. *Let $\mathcal{A}$ be a 2-shaded rigid C^* multitensor category with generator X. The follows are equivalent* [18]*:*

(1) *$\mathcal{M}$ is an indecomposable semisimple C^* right $\mathcal{A}$-module category;*

(2) *there is a faithful dagger tensor functor $\phi : \mathcal{A} \to \mathrm{End}^\dagger(\mathcal{M})$, where $\mathrm{End}^\dagger(\mathcal{M})$ is a tensor category with all the dagger endofunctors being objects and uniformly bounded natural transformations being morphisms.*

We see that under this equivalence, $\mathrm{End}_0^\dagger(\mathcal{M}, F) := \phi(\mathcal{A})$ is the W^* category Cauchy tensor generated by the image of the tensor functor $\mathcal{TLJ} \to \mathrm{End}^\dagger(\mathcal{M})$, where $F = - \lhd X$. Then $\mathrm{End}_0^\dagger(\mathcal{M}, F)$ is clearly a rigid C^* tensor category.

At the end of this section, we are going to show that the tensor category $\mathrm{End}_0^\dagger(\mathcal{M}, F)$ (viewed as a 2-category with one object) and 2-category $\mathcal{C}(K, \mathrm{ev}_K)$ are unitarily equivalent.

Construction 5.35. We construct $\mathcal{C}(K, \mathrm{ev}_K)$ from $\mathrm{End}_0^\dagger(\mathcal{M}, F)$ functorially.

(a) Object: Let V_0 be a set of representatives of all isomorphism classes of simple objects $P \in \mathcal{M}$ such that $P = P \lhd 1^+$ and V_1 a set of representatives of all isomorphism classes of simple objects $Q \in \mathcal{M}$ such that $Q = Q \lhd 1^-$. Then the object is the set $V = V_0 \sqcup V_1$.

(b) 1-morphism: Let $G \in \mathrm{End}_0^\dagger(\mathcal{M}, F)$ be an object with adjoint $\overline{G}$. Define the $V \times V$-bigraded Hilbert space H_G by

$$H_{G,PQ} := \mathrm{Hom}\big(Q, G(P)\big),$$

with inner product $\langle f|g\rangle_{G,PQ}$ for $f, g \in \mathrm{Hom}(Q, G(P))$ defined by

$$f^\dagger \circ g = \langle f|g\rangle_{G,PQ} \cdot \mathrm{id}_Q,$$

since Q is simple and $f^\dagger \circ g \in \mathrm{End}(Q) \cong \mathbb{C} \cdot \mathrm{id}_Q$. Note that $\mathrm{Hom}(Q, G(P)) \cong \mathrm{Hom}(\overline{G}(Q), P)$ is a natural isomorphism, so $H_{\overline{G},QP}$ and $H_{G,PQ}$ are dual Hilbert spaces.

(c) Composition of 1-morphisms:

Proposition 5.36. *For $G_1, G_2 \in \mathrm{End}_0^\dagger(\mathcal{M}, F)$, we have $H_{G_1 \circ G_2} \cong H_{G_1} \circ H_{G_2}$ as $V \times V$-bigraded Hilbert spaces, i.e.,*

$$H_{G_1 \circ G_2, PQ} \cong (H_{G_1} \circ H_{G_2})_{PQ} = (H_{G_2} \otimes H_{G_1})_{PQ} = \bigoplus_R H_{G_2,PR} \otimes H_{G_1,RQ}.$$

is a unitary isomorphism between Hilbert spaces for each pair $(P, Q) \in V \times V$.

Proof. Note that the direct sum contains finite many components. For each nonzero component with respect to R, define $\theta_R : H_{G_2,PR} \otimes H_{G_1,RP} \to H_{G_1 \circ G_2,PQ}$ by

$$\theta_R(f_2 \otimes f_1) := G_1(f_2) \circ f_1.$$

First, we prove that θ_R is an isometry, i.e.,

$$\begin{aligned}\big\langle \theta(f_2 \otimes f_1) | \theta(g_2 \otimes g_1) \big\rangle_{G_1 \circ G_2,PQ} &= \langle f_2 \otimes f_1 | g_2 \otimes g_1 \rangle \\ &= \langle f_2 | g_2 \rangle_{G_2,PR} \cdot \langle f_1 | g_1 \rangle_{G_1,RQ}\end{aligned}$$

for $f_2, g_2 \in H_{G_2,PR}$, $f_1, g_1 \in H_{G_1,RQ}$.

$$\begin{aligned}\text{LHS} &= \big\langle G_1(f_2) \circ f_1 | G_1(g_2) \circ g_1 \big\rangle_{G_1 \circ G_2,PQ} \\ &= \big(G_1(f_2) \circ f_1\big)^\dagger \circ \big(G_1(g_2) \circ g_1\big) \\ &= f_1^\dagger \circ G_1(f_2^\dagger \circ g_2) \circ g_1 \qquad (G_1 \text{ is a dagger functor}) \\ &= f_1^\dagger \circ G_1(\langle f_2 | g_2 \rangle_{G_2,PR} \cdot \mathrm{id}_R) \circ g_1 \\ &= \langle f_2 | g_2 \rangle_{G_2,PR} \cdot f_1^\dagger \circ \mathrm{id}_{G_1(R)} \circ g_1 \qquad (G_1 \text{ is a functor}) \\ &= \langle f_2 | g_2 \rangle_{G_2,PR} \cdot f_1^\dagger \circ g_1 \\ &= \text{RHS}.\end{aligned}$$

It follows that $\bigoplus_R \theta_R : \bigoplus_R H_{G_2,PR} \otimes H_{G_1,RQ} \to H_{G_1 \circ G_2,PQ}$ is an isometry. Note that for a semisimple rigid C* category,

$$\begin{aligned}\dim H_{G_1 \circ G_2,PQ} &= \dim \mathrm{Hom}\big(Q, G_1 \circ G_2(P)\big) \\ &= \dim \mathrm{Hom}\big(\overline{G_1}(Q), G_2(P)\big) \\ &= \dim \bigoplus_R \mathrm{Hom}\big(\overline{G_1}(Q), R\big) \otimes \mathrm{Hom}\big(R, G_2(P)\big) \\ &= \dim \bigoplus_R \mathrm{Hom}\big(Q, G_1(R)\big) \otimes \mathrm{Hom}\big(R, G_2(P)\big) \\ &= \dim \bigoplus_R H_{G_1,RQ} \otimes H_{G_2,PR} \\ &= \dim \bigoplus_R H_{G_2,PR} \otimes H_{G_1,RQ}.\end{aligned}$$

Note that $\bigoplus_R \theta_R$ is an isometry and hence injective, so $\bigoplus_R \theta_R : \bigoplus_R H_{G_2,PR} \otimes H_{G_1,RQ} \to H_{G_1 \circ G_2,PQ}$ is a bijection and hence a unitary. ∎

It follows that

$$H_{G_1 \circ G_2} \circ H_{G_3} \cong H_{G_1 \circ G_2 \circ G_3} \cong H_{G_1} \circ H_{G_2 \circ G_3}$$

as $V \times V$-bigraded Hilbert space.

(d) 1-morphism generator: Define $K := H_F$ and $\bar{K} := H_{\bar{F}}$. It is clear that $\mathbb{C}^{|V_0|} = H_{I^+}$ and $\mathbb{C}^{|V_1|} = H_{I^-}$.

(e) 2-morphism: The 2-morphism of $\mathcal{C}(K)$ is the morphism of $\mathrm{End}_0^\dagger(\mathcal{M}, F)$. Let $\alpha : G_1 \to G_2$ be a uniformly bounded natural transformation. Then $\alpha(P) : G_1(P) \to G_2(P)$ and hence

$$\alpha_{PQ} := \alpha_P \circ - : H_{G_1,PQ} = \mathrm{Hom}\big(Q, G_1(P)\big) \to \mathrm{Hom}\big(Q, G_2(P)\big) = H_{G_2,PQ}$$

is a uniformly bounded linear map.

(f) Composition of 2-morphisms: Let $\alpha_1 : G_1 \to G_2$, $\alpha_2 : G_2 \to G_3$ be uniformly bounded natural transformations. Then $G_1(P) \xrightarrow{\alpha_1(P)} G_2(P) \xrightarrow{\alpha_2} G_3(P)$, then

$$\begin{aligned}(\alpha_2 \circ \alpha_1)_{PQ} &= (\alpha_2 \circ \alpha_1)_P \circ - = \alpha_{2,P} \circ \alpha_{1,P} \circ - \\ &= \alpha_{2,PQ} \circ \alpha_{1,PQ} : H_{G_1,PQ} \to H_{G_2,PQ} \to H_{G_3,PQ}.\end{aligned}$$

(g) Tensor product of 2-morphisms: Let $\alpha_1 : G_1 \to G_2$, $\alpha_2 : G_3 \to G_4$ be uniformly bounded natural transformation. Then $\alpha_1 \otimes \alpha_2 : G_3 \circ G_1 = G_1 \otimes G_3 \to G_2 \otimes G_4 = G_4 \circ G_2$ defined as

$$\begin{tikzcd} G_3 \circ G_1 \arrow[r] \arrow[d] & G_3 \circ G_2 \arrow[d] \\ G_4 \circ G_1 \arrow[r] & G_4 \circ G_2 \end{tikzcd} \Longrightarrow \begin{tikzcd} H_{G_1} \otimes H_{G_3} \arrow[r] \arrow[d] & H_{G_2} \otimes H_{G_3} \arrow[d] \\ H_{G_1} \otimes H_{G_4} \arrow[r] & H_{G_2} \otimes H_{G_4} \end{tikzcd}$$

Clearly, the tensor product is strict.

(h) ev_K and coev_K: Define ev_K to be the unit of adjoint pair $(F, \bar{F})$ and coev_K to be the counit of $(F, \bar{F})$. Note that the duality is a property, not an extra structure. The dual functor is generated by the duality of generator, which is not necessarily a unitary dual functor.

Definition 5.37 ([11, Def. 7.2.1]). Let $\mathcal{M}$ and $\mathcal{N}$ be two semisimple C* module category categories over a semisimple rigid C* (multi)tensor category $\mathcal{C}$. A $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ consists of a functor $\psi : \mathcal{M} \to \mathcal{N}$ and a natural isomorphism $s_{X,M} : \psi(M \lhd X) \to \psi(M) \lhd X$ for all $X \in \mathcal{C}$, $M \in \mathcal{M}$ which satisfies the pentagon equation.

We call that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{C}$-module equivalent if ψ is an equivalence of categories.

Let $\mathcal{C} = \mathcal{TLJ}(d)$. Now we discuss the relation between the equivalence on $\mathcal{TLJ}(d)$-module category and the equivalence on $\mathrm{End}_0^\dagger(\mathcal{M}, F)$, where $F = - \lhd X$, and the corresponding 2-category $\mathcal{C}(K, \mathrm{ev}_K)$.

Remark 5.38. Let $\mathcal{M}$ be an indecomposable semisimple $\mathcal{TLJ}(d)$-module C* categories and $(\psi, s) : \mathcal{M} \to \mathcal{M}$ is an $\mathcal{TLJ}(d)$-module equivalence. Then $\psi \in \mathrm{End}^\dagger(M)$ is an object. Since $\mathcal{TLJ}(d)$ is generated by X, $s_{-,-}$ in above Definition 5.37 is determined by $s_{X,-}$. Note that

$$s_{X,-} : \psi\big(F(-)\big) = \psi(- \lhd X) \to \psi(-) \lhd X = F\big(\psi(-)\big)$$

is a unitary natural isomorphism. Note that as an equivalence, ψ maps simple objects in $\mathcal{M}$ to simple objects. Then we have

$$\begin{aligned} H_{F,\psi(P)\psi(Q)} &= \operatorname{Hom}\big(\psi(Q), F\big(\psi(P)\big)\big) \xrightarrow[-\circ s^{-1}]{\sim} \operatorname{Hom}\big(\psi(Q), \psi(F(P))\big) \\ &\cong \operatorname{Hom}\big(Q, F(P)\big) = H_{F,PQ}. \end{aligned}$$

It follows that the 1-morphism generator $K = H_F$ indexed by V and H_F indexed by $\psi(V)$ are unitary equivalent.

Comparing the discussion here with Remark 5.17, the $\mathcal{TLJ}(d)$-module equivalence corresponds to the unitary equivalence on $\mathcal{C}(K, \mathrm{ev}_K)$, which corresponds to isomorphism of edge-weighted graphs (Λ, ω).

Theorem 5.39 (cf. [9, Thm. 6.4], [10, Thm. 2.4], [13, Thm. 4.15]). *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{matrix}\text{Indecomposable semisimple C}^*\\ \mathcal{TLJ}(d)\text{-module categories } \mathcal{M}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{W}^* \text{ 2-subcategories } \mathcal{C}(\Lambda,\omega) \text{ of } \mathsf{BigHilb},\\ \text{where } \Lambda \text{ is a balanced } d\text{-fair bipartite}\\ \text{graph with edge-weighting } \omega\end{matrix}\right\}.$$

Equivalence on the left-hand side is unitary equivalence; equivalence on the right-hand side is the isomorphism of edge-weighted graphs.

Proof. We can prove this correspondence for the version with base point by passing through the Markov tower. According to Construction 5.35, the correspondence holds without fixing the base point. As for the equivalence, see Remark 5.38. ∎

Remark 5.40. Given a semisimple C* category $\mathcal{C}$, similar to Construction 5.35, we get a dagger tensor functor from $\mathrm{End}^\dagger(\mathcal{C})$ to the tensor category $\mathsf{Hilb}_f^{\mathrm{Irr}(\mathcal{C})\times\mathrm{Irr}(\mathcal{C})}$, which is the endomorphism tensor category of the object $\mathrm{Irr}(\mathcal{C})$ in $\mathsf{BigHilb}$. One should view this as a concrete version of $\mathrm{End}^\dagger(\mathcal{C})$. Note that dualizable endofunctors always map to dualizable 1-morphisms.

6. Markov lattices and biunitary connections

6.1. Balanced (d_0, d_1)-fair square-partite graph

Definition 6.1. Let Γ be an oriented square-partite graph with vertices $V(\Gamma) = V_{00} \sqcup V_{01} \sqcup V_{10} \sqcup V_{11}$.

$$\begin{tikzcd} V_{10} \arrow[r, dash, "\Omega_0"] \arrow[d, dash, "\Lambda_0"'] & V_{11} \arrow[d, dash, "\Lambda_1"] \\ V_{00} \arrow[r, dash, "\Omega_1"'] & V_{01} \end{tikzcd}$$

We call that Γ *associative* if for any two vertices on opposite corners of Γ, there are the same number of length 2 paths going either way around Γ. In more detail,

- for any $P \in V_{00}$ and $R \in V_{11}$, there are the same number of length 2 paths from P to R (or R to P) through vertices $Q \in V_{01}$ and through vertices $S \in V_{01}$;
- for any $Q \in V_{01}$ and $S \in V_{10}$, there are the same number of length 2 paths from Q to S (or S to Q) through vertices $P \in V_{00}$ and through vertices $R \in V_{11}$.

Let $\omega : E(\Gamma) \to (0, \infty)$ be a weighting on the edges of graph Γ.

Let Λ_i denote the full subgraph of Γ on $V_{0i} \sqcup V_{1i}$, $i = 0, 1$; let Ω_i denote the full subgraph of Γ on $V_{i0} \sqcup V_{i1}$, $i = 0, 1$. Then $\Lambda_1, \Lambda_2, \Omega_1, \Omega_2$ are oriented bipartite graphs.

We call (Γ, ω) a *balanced* (d_0, d_1)*-fair* square-partite graph if Λ_0, Λ_1 are balanced d_0-fair bipartite graphs and Ω_0, Ω_1 are balanced d_1-fair bipartite graphs.

Remark 6.2. We can define the edge-weighting preserving graph isomorphism literally the same as in Definition 5.4 for balanced (d_0, d_1)-fair square partite graph.

6.2. 2-subcategory $\mathcal{C}(K_0, K_1, L_0, L_1, \mathrm{ev})$ of BigHilb and biunitary connection Φ

Definition 6.3. Let $\mathcal{C}(K_0, K_1, L_0, L_1, \mathrm{ev})$ be a W* 2-subcategory of BigHilb with four 1-morphism generators $K_i : V_{0i} \to V_{1i}$, $L_i : V_{i0} \to V_{i1}$, $i = 0, 1$ and a chosen evaluation and coevaluation for each generator. We require that

(a) K_i, L_i are dualizable, $i = 0, 1$.

(b) The evaluation and coevaluation for the dual:

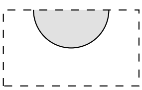
$$\mathrm{ev}_{\overline{?}} := (\mathrm{coev}_?)^\dagger \quad \text{and} \quad \mathrm{coev}_{\overline{?}} := (\mathrm{ev}_?)^\dagger,$$

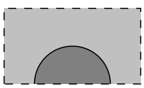
where $? = K_i, L_i, i = 0, 1$.

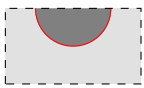
(c) They satisfy the (d_0, d_1)—fairness condition, namely,

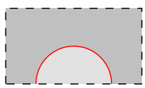
$$\begin{aligned}
\mathrm{ev}_{\overline{K_0}} \circ \mathrm{coev}_{K_0} &= d_0 \cdot \mathrm{id}_{\mathbb{C}^{|V_{00}|}}, & \mathrm{ev}_{K_0} \circ \mathrm{coev}_{\overline{K_0}} &= d_0 \cdot \mathrm{id}_{\mathbb{C}^{|V_{10}|}}, \\
\mathrm{ev}_{\overline{K_1}} \circ \mathrm{coev}_{K_1} &= d_0 \cdot \mathrm{id}_{\mathbb{C}^{|V_{01}|}}, & \mathrm{ev}_{K_1} \circ \mathrm{coev}_{\overline{K_1}} &= d_0 \cdot \mathrm{id}_{\mathbb{C}^{|V_{11}|}}, \\
\mathrm{ev}_{\overline{L_0}} \circ \mathrm{coev}_{L_0} &= d_1 \cdot \mathrm{id}_{\mathbb{C}^{|V_{00}|}}, & \mathrm{ev}_{L_0} \circ \mathrm{coev}_{\overline{L_0}} &= d_1 \cdot \mathrm{id}_{\mathbb{C}^{|V_{01}|}}, \\
\mathrm{ev}_{\overline{L_1}} \circ \mathrm{coev}_{L_1} &= d_1 \cdot \mathrm{id}_{\mathbb{C}^{|V_{10}|}}, & \mathrm{ev}_{L_1} \circ \mathrm{coev}_{\overline{L_1}} &= d_1 \cdot \mathrm{id}_{\mathbb{C}^{|V_{11}|}}.
\end{aligned}$$

Notation 6.4. Now, we provide the graphical calculus to describe $\mathcal{C}(K_0, K_1, L_0, L_1, \mathrm{ev})$. The white region indicates the object V_{00}, the lightest gray for V_{10}, the medium gray for V_{11} and the darkest gray for V_{01}; the black edge indicates K_0, K_1 and red for L_0, L_1, so white and medium gray, lightest gray and darkest gray will not be adjacent.


$\mathrm{coev}_{K_0} : \mathbb{C}^{|V_{00}|} \to K_0 \otimes \overline{K_0}$


$\mathrm{ev}_{K_1} : \overline{K_1} \otimes K_1 \to \mathbb{C}^{|V_{11}|}$


$\mathrm{coev}_{\overline{L_0}} : \mathbb{C}^{|V_{01}|} \to \overline{L_0} \otimes L_0$


$\mathrm{ev}_{\overline{L_1}} : L_1 \otimes \overline{L_1} \to \mathbb{C}^{|V_{10}|}$

Remark 6.5. Similar to the discussion in Section 5.3, from a given balanced (d_0, d_1)-fair square-partite graph (Γ, ω), we can construct a 2-subcategory $\mathcal{C}(\Gamma, \omega)$ of BigHilb; on the other hand, if we start with $\mathcal{C}(K_0, K_1, L_0, L_1, \mathrm{ev})$, we can obtain the (Γ, ω). Moreover, $\mathcal{C}(K_0, K_1, L_0, L_1, \mathrm{ev})$ and $\mathcal{C}(\Gamma, \omega)$ are unitary equivalent.

Similar to the discussion in Remark 5.17, the edge-weighting preserving graph automorphism will result in the unitary equivalence on $\mathcal{C}(\Gamma, \omega)$.

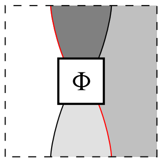
In the rest of this section, we define a special 2-morphism Φ in $\mathcal{C}(K_0, K_1, L_0, L_1, \mathrm{ev})$, called *biunitary connection*.

Definition 6.6 (Biunitary connection). A *biunitary connection*

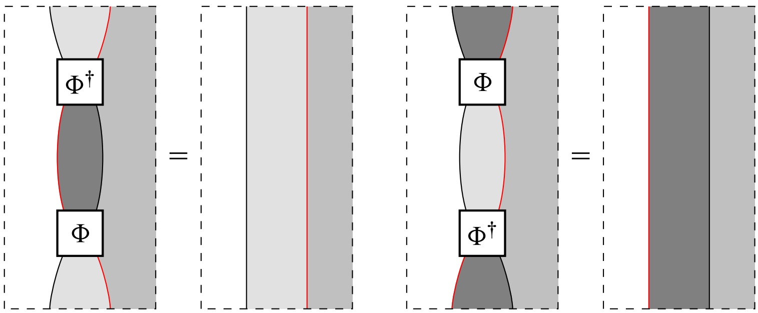
$$\Phi : K_0 \otimes L_1 \to L_0 \otimes K_1$$

is a 2-morphism which is a vertical unitary and a horizontal unitary, as defined as follows. Here is the graphical calculus of Φ.

(1) The biunitary connection Φ:

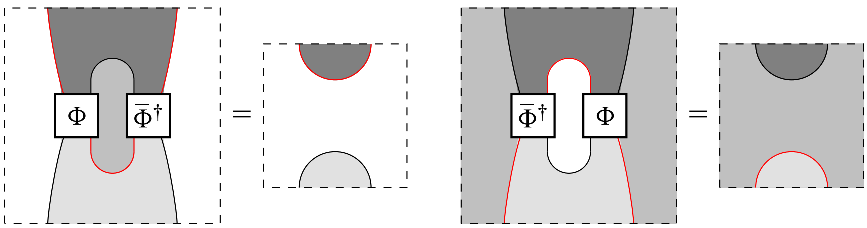


(2) Vertical unitary: $\Phi^\dagger \circ \Phi = \mathrm{id}_{K_0} \otimes \mathrm{id}_{L_1}$ and $\Phi \circ \Phi^\dagger = \mathrm{id}_{L_0} \otimes \mathrm{id}_{K_1}$.



(3) Horizontal unitary:

$$(\mathrm{id}_{L_0} \otimes \mathrm{ev}_{\overline{K_1}} \otimes \mathrm{id}_{\overline{L_0}}) \circ (\Phi \otimes \overline{\Phi}^\dagger) \circ (\mathrm{id}_{K_0} \otimes \mathrm{coev}_{L_1} \otimes \mathrm{id}_{\overline{K_0}}) = \mathrm{coev}_{L_0} \circ \mathrm{ev}_{\overline{K_0}},$$
$$(\mathrm{id}_{\overline{K_1}} \otimes \mathrm{ev}_{L_0} \otimes \mathrm{id}_{K_1}) \circ (\overline{\Phi}^\dagger \otimes \Phi) \circ (\mathrm{id}_{\overline{L_1}} \otimes \mathrm{coev}_{\overline{K_0}} \otimes \mathrm{id}_{L_1}) = \mathrm{coev}_{\overline{K_1}} \circ \mathrm{ev}_{L_1}.$$



Here $\overline{\Phi}$ is defined as the dual of Φ in the sense of Definition 2.41.

Definition 6.7. $\mathcal{C}(K_0, K_1, L_0, L_1, \mathrm{ev})$ equipped with a biunitary connection Φ is written as $\mathcal{C}(K_0, K_1, L_0, L_1, \mathrm{ev}; \Phi)$ or simply $\mathcal{C}(\Phi)$.

Remark 6.8. The existence of Φ implies that

$$\dim(K_0 \otimes L_1)_{uv} = \dim(L_0 \otimes K_1)_{uv},$$
$$\dim(\overline{K_0} \otimes L_0)_{uv} = \dim(L_1 \otimes \overline{K_1})_{uv},$$

for each pair $(u, v) \in V \times V$. In other words, the corresponding square-partite graph is associative.

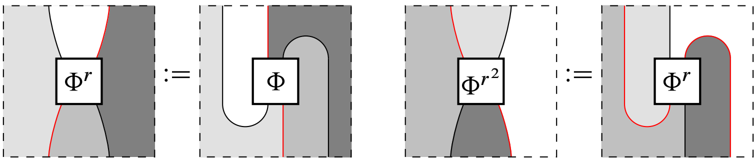
We are going to discuss some properties of biunitary connections.

Definition 6.9 (Rotation by 90°). Define the rotation by 90° to be

$$\Phi^r := (\mathrm{id}_{\overline{K_0}} \otimes \mathrm{id}_{L_0} \otimes \mathrm{ev}_{\overline{K_1}}) \circ (\mathrm{id}_{\overline{K_0}} \otimes \Phi \otimes \mathrm{id}_{\overline{K_1}}) \circ (\mathrm{coev}_{\overline{K_0}} \otimes \mathrm{id}_{L_1} \otimes \mathrm{id}_{\overline{K_1}}).$$

Similarly,

$$\Phi^{r^2} := (\mathrm{id}_{\overline{L_1}} \otimes \mathrm{id}_{\overline{K_0}} \otimes \mathrm{ev}_{\overline{L_0}}) \circ (\mathrm{id}_{\overline{L_1}} \otimes \Phi^r \otimes \mathrm{id}_{\overline{L_0}}) \circ (\mathrm{coev}_{\overline{L_1}} \otimes \mathrm{id}_{\overline{K_1}} \otimes \mathrm{id}_{\overline{L_0}}) = \overline{\Phi}.$$

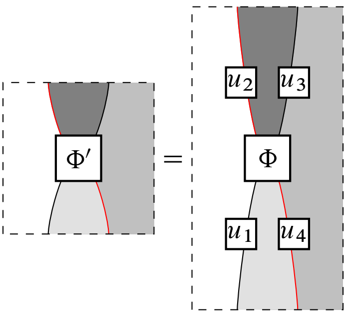


Remark 6.10. Here are some properties for biunitary connections and rotation.

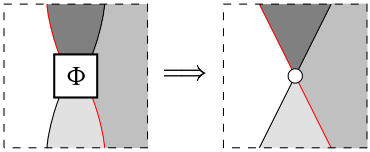
(1) The group $\langle r, \dagger \rangle = \langle r, \dagger | r^4 = \dagger^2 = \mathrm{id}, r\dagger = \dagger r^3 \rangle$ for the biunitary connection is isomorphic to the dihedral group D_4.

(2) Φ is a biunitary connection if and only if Φ^g is both vertical unitary and horizontal unitary, where $g \in \langle r, \dagger \rangle$.

Definition 6.11 ([39, §4]). We call biunitary connections $\Phi : K_0 \otimes L_1 \to L_0 \otimes K_1$ and $\Phi' : K_0' \otimes L_1' \to L_0' \otimes K_1'$ *gauge equivalent*, if there exist unitaries $u_1 : K_0' \to K_0$, $u_2 : L_0 \to L_0'$, $u_3 : K_1 \to K_1'$ and $u_4 : L_1' \to L_1$ such that $\Phi_2 = (u_2 \otimes u_3) \circ \Phi_1 \circ (u_1 \otimes u_4)$.

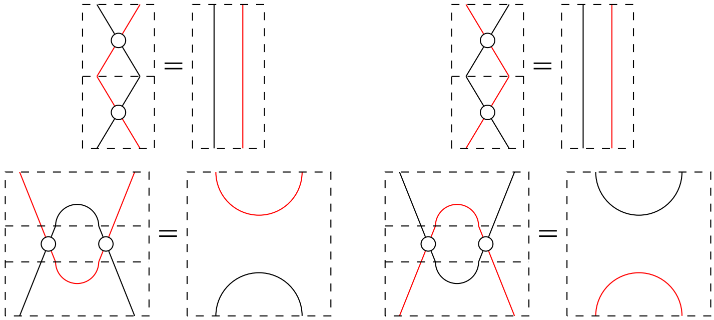


Notation and observation 6.12. Observe that once we know the color of the region and the color of the edge, the biunitary connection in the circle is determined. So we can simplify the graphical calculus of biunitary connection as follows.



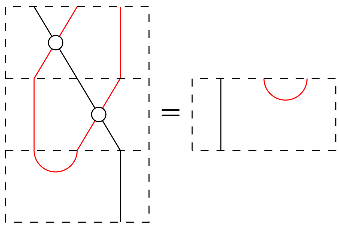
Moreover, if the color of the leftmost region and the color of each edge is determined, then the color of the rest of the regions will be determined. The 4 colors on the leftmost region and 2 colors on the edge (8 cases) can represent all Φ^g, $g \in \langle r, \dagger \rangle$.

Here is the simplified graphical calculus of vertical unitarity and horizontal unitarity. In the following context, We require that the leftmost regions in the uncolored equality have the same color.

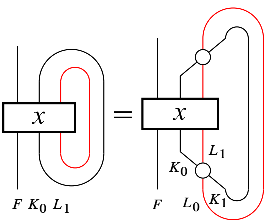


Proposition 6.13. *Here are some properties that will be used in the next section and the proof is left to the reader.*

(1)



(2) *For 2-morphism $x \in \mathrm{End}(F \otimes K_0 \otimes L_1)$, where F is a proper 2-morphism, we have*



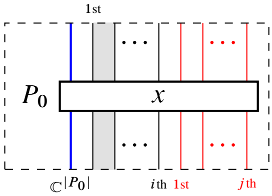
6.3. From $\mathcal{C}(\Phi)$ to Markov lattice

Construction 6.14. Here we are going to construct a Markov lattice from the 2-category $\mathcal{C}(\Phi)$ discussed above with a chosen point, say $P_0 \in V_{00}$. Let $\mathbb{C}^{|P_0|}$ be a 1-morphism with all the entry being 0 except $(\mathbb{C}^{|P_0|})_{P_0 P_0} = \mathbb{C}$.

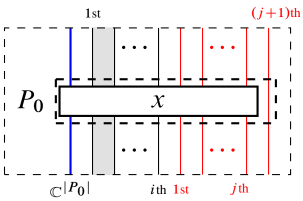
Note that $\mathbb{C}^{|P_0|} \otimes K_0^{\text{alt}\otimes i} \otimes L_?^{\text{alt}\otimes j}$ is a 1-morphism for each $i, j \in \mathbb{Z}_{\geq 0}$.

Let $M_{i,j} = \text{End}(\mathbb{C}^{|P_0|} \otimes K_0^{\text{alt}\otimes i} \otimes L_?^{\text{alt}\otimes j})$, where $L_? = L_0$ if $2 \mid i$ and $L_? = L_1$ if $2 \nmid j$. We use the graphical calculus to show $M = (M_{i,j})_{i,j\geq 0}$ is a Markov lattice.

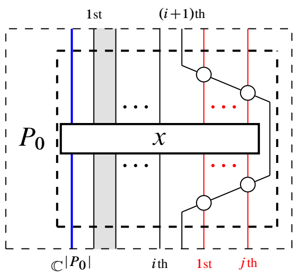
(1) Element $x \in M_{i,j}$:



(2) Horizontal inclusion $x \in M_{i,j} \subset M_{i,j+1}$:

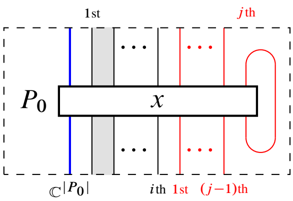


(3) Vertical inclusion $x \in M_{i,j} \subset M_{i+1,j}$:

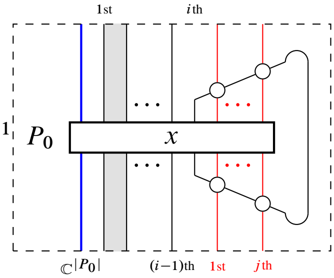


(4) Horizontal conditional expectation $E_{i,j}^{M,r} : M_{i,j} \to M_{i,j-1}, , x \in M_{i,j}$:

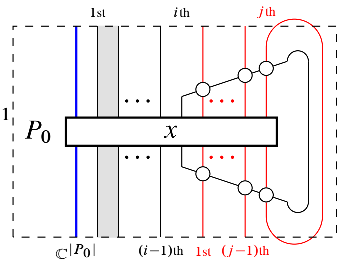
$$E_{i,j}^{M,r}(x) = d_1^{-1}$$

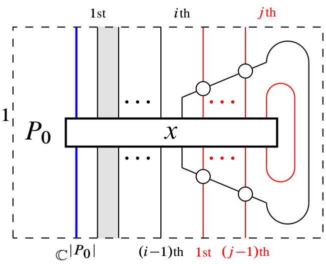


(5) Vertical conditional expectation $E^{M,l}_{i,j} : M_{i,j} \to M_{i-1,j}$, $x \in M_{i,j}$:

$$E^{M,l}_{i,j}(x) = d_0^{-1}$$ 

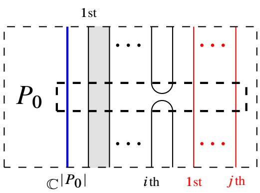
(6) Commuting square of conditional expectations $E^{M,r}_{i-1,j} \circ E^{M,l}_{i-1,j-1} = E^{M,l}_{i-1,j} \circ E^{M,r}_{i,j}$: $M_{i,j} \to M_{i-1,j-1}$, $x \in M_{i,j}$:

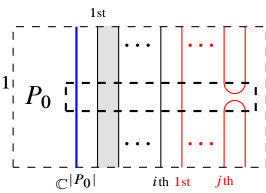
$$E^{M,r}_{i-1,j} \circ E^{M,l}_{i-1,j-1}(x) = d_0^{-1} d_1^{-1}$$ 

$$= d_0^{-1} d_1^{-1}$$ 

$$= E^{M,l}_{i-1,j} \circ E^{M,r}_{i,j}(x)$$

(7) Vertical Jones projections $e_i \in M_{i+1,j}$ and horizontal Jones projection $f_j \in M_{i,j+1}$:

$$e_i = d_0^{-1}$$ 

$$f_j = d_1^{-1}$$ 

(8) It is clear that $M_j = (M_{i,j}, E^{M,l}_{i,j}, e_i)_{i \geq 0}$ are Markov towers with the same modulus d_0 and $e_i \in M_{i+1,j}$ for all i, $i, j = 0, 1, 2, \ldots$; $M_i = (M_{i,j}, E^{M,r}_{i,j}, f_j)_{j \geq 0}$ are Markov towers with the same modulus d_1 and $f_j \in M_{i,j+1}$ for all j.

Remark 6.15. A gauge equivalence $\Phi \sim \Phi'$ will result in an isomorphism of the corresponding Markov lattices.

6.4. From Markov lattice to $\mathscr{C}(\Gamma, \omega; \Phi)$

First, we are going to explore more properties of Markov lattice.

Proposition 6.16. *One can show the following properties for Markov lattice similar to Proposition 5.24.*

(a) $X_{i+1,j+1} := \langle e_i, f_j \rangle$ *is a 2-sided ideal of* $M_{i+1,j+1}$ *and hence* $M_{i+1,j+1}$ *can split as a direct sum of von Neumann algebras* $X_{i+1,j+1} \oplus Y_{i+1,j+1}$. *We also define* $Y_{0,0} = M_{0,0}, Y_{1,0} = M_{1,0}, Y_{0,1} = M_{0,1}, Y_{1,1} = M_{1,1}$ *so that*

$$X_{0,0} = X_{1,0} = X_{0,1} = X_{1,1} = 0.$$

$X_{i+1,j+1}$ *is called the old stuff and* $Y_{i+1,j+1}$ *is called the new stuff.*

(b) *If* $y \in Y_{i+1,j+1}$ *and* $x \in X_{i+1,j}$ *or* $x \in X_{i,j+1}$, *then* $yx = 0$ *in* $M_{i+1,j+1}$. *Hence* $E^r_{i+1,j+1}(Y_{i+1,j+1}) \subset Y_{i+1,j}$ *and* $E^l_{i+1,j+1}(Y_{i+1,j+1}) \subset Y_{i,j+1}$, *which means the new stuff comes from the old new stuff.*

(c) *If* $Y_{i,j} = 0$, *then* $Y_{k,l} = 0$ *for all* $k \geq i, l \geq j$.

Now we are going to construct $\mathscr{C}(\Gamma, \omega; \Phi)$ from a given Markov lattice M.

Construction 6.17. The square partite graph and the edge weighting (Γ, ω):

From Markov lattice M, since each row and column is a Markov tower, we can obtain a Bratteli diagram Δ as in Section 5.6 (which can be viewed as a "lattice-partite" graph). After taking only the new vertices in $\Delta \cap Y_{i,j}$ and the edges between them, we obtain the principal graph Γ_0 because of Proposition 6.16(2). Here, Γ_0 is not necessarily a square-partite graph, so we have to do some identification.

For the new vertices $p_1 \in \Gamma_0 \cap Y_{i,j}$ and $p_2 \in \Gamma_0 \cap Y_{i+2,j-2}$, as in Section 5.6, let p_1' be the new old vertex of p_1 in $M_{i+2,j}$ and p_2' be the new old vertex of p_2 in $M_{i+2,j}$. We identify p_1 with p_2 if $p_2' \in M_{i+2,j} p_1'$ (or equivalently $p_1' \in M_{i+2,j} p_2'$).

For the pairs of new vertices $p_1 \in \Gamma_0 \cap Y_{i,j}$ and $q_1 \in \Gamma_0 \cap Y_{i+1,j}$, and the pairs of new vertices $p_2 \in \Gamma_0 \cap Y_{i+2,j-2}$ and $q_2 \in \Gamma_0 \cap Y_{i+3,j-2}$, suppose p_1 and p_2 are identified in $M_{i+2,j}$, q_1 and q_2 are identified in $M_{i+3,j}$ on above sense, then the numbers of edges between p_1, q_1 and p_2, q_2 are equal, since they both equal to

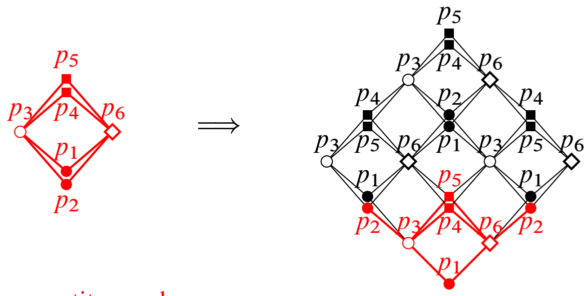
$$\left(\dim_{\mathbb{C}}(p_1'q_1' M_{i+2,j}' p_1'q_1' \cap p_1'q_1' M_{i+3,j} p_1'q_1')\right)^{\frac{1}{2}},$$

see the discussion in Section 5.6. Then we can also identify the edges between p_1, q_1 and p_2, q_2. Similar statement for $p_1 \in \Gamma_0 \cap Y_{i,j}$ and $r_1 \in \Gamma_0 \cap Y_{i,j+1}$, and the pairs of new vertices $p_2 \in \Gamma_0 \cap Y_{i+2,j-2}$ and $r_2 \in \Gamma_0 \cap Y_{i+2,j-1}$. After the above identification as well as the edges between those identified vertices (see the following example), we obtain a graph Γ, which is a square-partite graph.

Then $V_{ij} \subset V(\Gamma)$ contains all the vertices in $V(\Gamma_0) \cap M_{i+2m,j+2n}$, $i, j = 0, 1$, $m, n \in \mathbb{Z}_{\geq 0}$.

The edge-weighting ω can be obtained the same way as in Section 5.6.

Example 6.18. Here we provide an example to see the difference between the square-partite graph and the principal graph of a Markov lattice. In the diagram below, if p_1 is at depth zero, then p_2 is at depth 2 of the principal graph. Therefore, as a new vertex, p_2 will appear in two places $M_{0,2}$ and $M_{2,0}$, but their reflections/new old vertices coincide in $M_{2,2}$.



square-partite graph

principal graph with base point p_1
and Bratteli diagram

Remark 6.19. Suppose vertex $q \in V_{00}$ is at depth $2n$ of the principal graph, then q will first appear in $M_{2i,2n-2i}$, $i = 0, 1, \ldots, n$; if $q \in V_{10}$ is at depth $2n + 1$, then q will first appear in $M_{2i+1,2n-2i}$, $i = 0, 1, \ldots, n$; if $q \in V_{01}$ is at depth $2n + 1$, then q will first appear in $M_{2i,2n+1-2i}$, $i = 0, 1, \ldots, n$; if $q \in V_{11}$ is at depth $2n + 2$, then q will first appear in $M_{2i+1,2n+1-2i}$, $i = 0, 1, \ldots, n$.

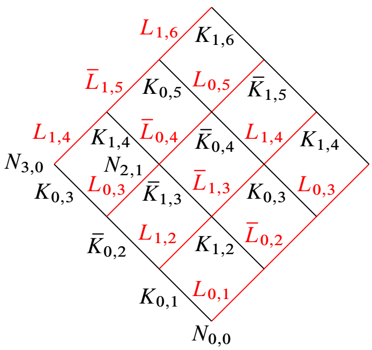
Next, we compute the biunitary connection Φ.

Notation and observation 6.20. We choose $p_0 \in V_{00}$ as the base point, which is at depth 0. Similar to Observation 5.26, denote $\Lambda_{0,n}$ to be the subgraph of Λ_0 with vertices depth $\leq n$, similar definition for $\Omega_{0,n}$, $\Lambda_{1,n}$ and $\Omega_{1,n}$, see Definition 6.1. The corresponding Hilb-enriched graphs are $K_{i,n} := K_{\Lambda_{i,n}}$, $L_{i,n} := L_{\Omega_{i,n}}$. From Construction 6.14, $N_{i,j} := \mathrm{End}(\mathbb{C}^{|p_0|} \otimes K_0^{\mathrm{alt}\otimes i} \otimes L_?^{\mathrm{alt}\otimes j})$. Without loss of generality, let $2 \nmid i$. Observe that

$$N_{i,j} = \mathrm{End}(K_{0,1} \otimes \overline{K}_{0,2} \otimes \cdots \overline{K}_{0,i} \otimes L_{1,i+1} \otimes \overline{L}_{1,i+2} \otimes \cdots \otimes L^?_{1,i+j}),$$

where $L^?_{1,j} = L_{1,j}$ if $2 \nmid j$, $L^?_{1,j} = \overline{L}_{1,j}$ if $2 \mid j$.

Example 6.21. Following Example 6.18,



we have

$$K_{0,1} = \begin{bmatrix} 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{K}_{0,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{0,3} = \begin{bmatrix} 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_{1,4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C} & \mathbb{C} & 0 \end{bmatrix}$$

$$L_{0,1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathbb{C} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{L}_{1,3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{0,1} \otimes \bar{K}_{0,2} \otimes L_{0,3} \cong \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathbb{C}^6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cong K_{0,1} \otimes L_{1,2} \otimes \bar{K}_{1,3}$$
$$\cong L_{0,1} \otimes K_{1,2} \otimes \bar{K}_{1,3}.$$

Similar to Example 5.27, the entry (i, j) in $N_{m,n}$ indicates number of paths from the vertex p_i at depth 0 to the vertex p_j at depth $m + n$. Note that the base point is a single vertex p_1, so only at entry $(1, j)$ can be non-zero.

Remark 6.22. Any automorphism of $M_n(\mathbb{C})$ is inner. To be precise, if $\alpha \in \mathrm{Aut}(M_n(\mathbb{C}))$, then there exists a unitary $u \in M_n(\mathbb{C})$, such that $\alpha(x) = uxu^* = \mathrm{Ad}(u)(x)$, for any $x \in M_n(\mathbb{C})$. Moreover, this unitary u is unique up to a unit scalar. Indeed, if $uxu^* = u_1xu_1^*$ for all $x \in M_n(\mathbb{C})$, then $x(u^*u_1) = (u^*u_1)x$, which implies that u^*u_1 is in the center of $M_n(\mathbb{C})$. Thus, $u^*u_1 = a \in \mathbb{C}$ with $|a| = 1$ and hence $u_1 = au$.

As a corollary, for 1-morphisms H, G, if $\alpha : \mathrm{End}(H) \cong \mathrm{End}(G)$ is a $*$-isomorphism, then there exists a unitary 2-morphism $u : H \to G$ such that $\alpha = \mathrm{Ad}(u)$.

Warning. The unitary u is obtained by taking a unitary $u_{i,j}$ in each entry. Thus any two choices of implementing unitary $u = (u_{i,j})$ and $v = (v_{i,j})$ differ by a matrix of scalars $(a_{i,j})$ which may be distinct. Hence the unitary u is unique up to a matrix of scalars.

Construction 6.23. The biunitary connection Φ: The construction (for the tracial case) has been written in [21, §5.5] in the language of path algebras. For convenience, we will construct it here using our language.

From Construction 6.17 and Remark 6.5, the 2-category $\mathcal{C}(\Gamma, \omega)$ can be constructed.

In order to obtain the biunitary connection Φ, we shall compute it componentwise, which is similar to the idea to compute the edge-weighting in Section 5.6. The goal is to compute

$$\Phi_{pr} : (K_0 \otimes L_1)_{pr} = \bigoplus_{q \in V_{10}} K_{0,pq} \otimes L_{1,qr} \to \bigoplus_{s \in V_{01}} L_{0,ps} \otimes K_{1,sr} = (L_0 \otimes K_1)_{pr}$$

for each pair $(p, r) \in V_{00} \times V_{11}$.

Suppose p is at depth $2n$ of the principal graph and r is at depth $2n+2$. By Remark 6.19, p first appear in $M_{0,2n}$ and r first appears in $M_{1,2n+1}$.

Consider two path models $M_{0,0} \subset M_{0,1} \subset \cdots \subset M_{0,2n} \subset M_{0,2n+1} \subset M_{1,2n+1}$ and $M_{0,0} \subset M_{0,1} \subset \cdots \subset M_{0,2n} \subset M_{1,2n} \subset M_{1,2n+1}$.

Similar to Proposition 5.28, we have

$$N'_{0,2n} \cap N_{1,2n+1} = \mathrm{id}_{K_{0,1} \otimes \bar{K}_{0,2} \otimes \cdots \otimes \bar{K}_{0,2n}} \otimes \mathrm{End}(K_{0,2n+1} \otimes L_{1,2n+1}) \text{ for the first model,}$$
$$N'_{0,2n} \cap N_{1,2n+1} = \mathrm{id}_{K_{0,1} \otimes \bar{K}_{0,2} \otimes \cdots \otimes K_{0,2n-1}} \otimes \mathrm{End}(L_{0,2n} \otimes K_{1,2n+1}) \text{ for the second model.}$$

Let $\psi : M_{1,2n+1} \to N_{1,2n+1}$ denote the $*$-isomorphism onto the first model and $\psi' : M_{1,2n+1} \to N_{1,2n+1}$ denote the $*$-isomorphism onto the second model, then

$$\begin{aligned} \psi &: M'_{0,2n} \cap M_{1,2n+1} \to N'_{0,2n} \cap N_{1,2n+1} \cong \mathrm{End}(K_{0,2n+1} \otimes L_{1,2n+1}), \\ \psi' &: M'_{0,2n} \cap M_{1,2n+1} \to N'_{0,2n} \cap N_{1,2n+1} \cong \mathrm{End}(L_{0,2n} \otimes K_{1,2n+1}) \end{aligned}$$

are $*$-isomorphisms. Then $\psi' \circ \psi^{-1} : \mathrm{End}(K_{0,2n+1} \otimes L_{1,2n+1}) \to \mathrm{End}(L_{0,2n} \otimes K_{1,2n+1})$ is a $*$ isomorphism between two 1-morphisms. By Remark 6.22, their exists a unique unitary u up to a matrix of scalars such that $\psi' \circ \psi^{-1} = \mathrm{Ad}(u)$. We define $\Phi_{pr} := u_{pr}$.

Similar to Remark 5.21, we secretly make a choice of ONB when we construct the generators K_i, L_j from the square-partite graph Γ, $i, j = 0, 1$. Different choice results in multiplying a unitary on each generator. Combining Definition 6.11 of gauge equivalence and above discussion, the biunitary connection Φ we construct here is unique up to gauge equivalence.

6.5. $\mathcal{C}(\Phi)$ and $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$

We have already seen the method to construct a Markov lattice from $\mathcal{C}(\Phi)$ above or from $\mathcal{M}$ in Section 4 with a simple base point, where $\mathcal{M}$ is an indecomposable semisimple C* $\mathcal{A}$-$\mathcal{B}$ bimodule category. In this section, by using a similar technique as in Section 5.7, we will show their relation to each other.

Definition 6.24. Suppose $\mathcal{M}$ is an indecomposable semisimple C* $\mathcal{TLJ}(d_0)$-$\mathcal{TLJ}(d_1)$ bimodule category, where $X = 1^+ \otimes X \otimes 1^-$, $Y = 1^+ \otimes Y \otimes 1^-$ are the generators of

$\mathcal{TLJ}(d_0)$ and $\mathcal{TLJ}(d_1)$ respectively. Define $F = X \triangleright -$, $\overline{F} = \overline{X} \triangleright -$, $G = - \triangleleft Y$, $\overline{G} = - \triangleleft \overline{Y}$, which are endofunctors on $\mathcal{M}$. Note that $(F, \overline{F})$ and $(G, \overline{G})$ are adjoint pairs, with unit $\mathrm{ev}_F, \mathrm{ev}_G$ induced by $\mathrm{ev}_X, \mathrm{ev}_Y$ and counit $\mathrm{coev}_F, \mathrm{coev}_G$ induced by $\mathrm{coev}_{\overline{X}}, \mathrm{coev}_{\overline{Y}}$.

Define $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$ to be the full subcategory of $\mathrm{End}^\dagger(\mathcal{M})$ Cauchy tensor generated by $F, \overline{F}, G, \overline{G}$, so it is a rigid W* tensor category.

We warn the reader that $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$ will only be multitensor ($\dim(\mathrm{End}(\mathrm{id}_{\mathcal{M}})) < \infty$) when $\mathcal{M}$ is finitely semisimple.

Definition 6.25 (Biunitary connection in $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$). Note that the bimodule associator $\alpha_{X,-,Y} : (X \triangleright -) \triangleleft Y \to X \triangleright (- \triangleleft Y)$ is a unitary, which induces a natural isomorphism $\Phi_{F,G} : F \otimes G \to G \otimes F$, where $F \otimes G := G \circ F$. Then

$$\Phi_{G,\overline{F}} : G \otimes \overline{F} \to \overline{F} \otimes G$$

is equal to the 90° rotation $\Phi^r_{F,G}$ defined as follows:

$$\Phi^r_{F,G} := (\mathrm{id}_{\overline{F}} \otimes \mathrm{id}_G \otimes \mathrm{ev}_F) \circ (\mathrm{id}_{\overline{F}} \otimes \Phi_{F,G} \otimes \mathrm{id}_{\overline{F}}) \circ (\mathrm{coev}_F \otimes \mathrm{id}_G \otimes \mathrm{id}_{\overline{F}}).$$

It is easy to show that $\Phi_{F,G}$ is vertical and horizontal unitary and so is $\Phi_{G,\overline{F}}$.

Similar to Section 5.7, we will show that the tensor category $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$ and 2-category $\mathcal{C}(\Phi)$ are unitarily equivalent.

Construction 6.26. We construct $\mathcal{C}(\Phi)$ from $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$ functorially.

(a) Let V_{00} be a set of representatives of all simple objects $P \in \mathcal{M}$ such that $P = 1^+ \triangleright P \triangleleft 1^+$; V_{10} be the set of representatives of all simple objects $Q \in \mathcal{M}$ such that $Q = 1^- \triangleright Q \triangleleft 1^+$; V_{11} be the set of representatives of all simple objects $R \in \mathcal{M}$ such that $R = 1^- \triangleright R \triangleleft 1^-$; V_{01} be the set of representatives of all simple objects $S \in \mathcal{M}$ such that $S = 1^+ \triangleright S \triangleleft 1^-$. Then the objects are the sets $V_{i,j}$, $i, j = 0, 1$ and their union $V = V_{00} \sqcup V_{01} \sqcup V_{11} \sqcup V_{10}$.

(b) 1-morphism: The 1-morphism of $\mathcal{C}(\Phi)$ is the object of $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$. The way to construct the corresponding $V \times V$-bigraded Hilbert space from an endofunctor is the same as in Construction 5.35. The same for the dual 1-morphism and tensor structure/composition.

(c) 2-morphism: The 2-morphism of $\mathcal{C}(\Phi)$ is the morphism of $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$.

(d) 1-morphism generator: Define

$$K_0 := H_{J^+} \otimes H_F,\ \overline{K_0} = H_{J^+} \otimes H_{\overline{F}},\ K_1 := H_{J^-} \otimes H_F,\ \overline{K_1} = H_{J^-} \otimes H_{\overline{F}},$$
$$L_0 := H_{I^+} \otimes H_G,\ \overline{L_0} = H_{I^+} \otimes H_{\overline{G}},\ L_1 := H_{I^-} \otimes H_G,\ \overline{L_0} = H_{I^-} \otimes H_{\overline{G}}.$$

(e) ev and coev. The same as in Construction 5.35 (h).

(f) Biunitary connection: $\Phi : K_0 \otimes L_1 \to L_0 \otimes K_1$ is defined as

$$\Phi_{F,G} : F \otimes G \to G \otimes F.$$

The check that Φ is vertical and horizontal unitary is left to the reader.

Construction 6.27. For the convenience to the reader, we also provide the construction from $\mathcal{C}(\Phi)$ to $\mathrm{End}_0^\dagger(\mathcal{M}, F, G)$:

(a) Object: The object are the 1-morphisms in $\mathcal{C}(\Phi)$. In particular, the generator $F = K_0 \oplus K_1$, $\overline{F} = \overline{K_0} \oplus \overline{K_1}$, $G = L_0 \oplus L_1$ and $\overline{G} = \overline{L_0} \oplus \overline{L_1}$; the unit $I^+ = 1^+ \rhd - = \mathbb{C}^{|V_{00} \sqcup V_{01}|}$, $I^- = 1^- \rhd - = \mathbb{C}^{|V_{10} \sqcup V_{11}|}$, $J^+ = - \lhd 1^+ = \mathbb{C}^{|V_{00} \sqcup V_{10}|}$ and $J^- = - \lhd 1^- = \mathbb{C}^{|V_{01} \sqcup V_{11}|}$.

(b) Morphism: The morphisms are the 2-morphisms in $\mathcal{C}(\Phi)$.

(c) The associator: Note that $F \otimes G = (K_0 \oplus K_1) \otimes (L_0 \oplus L_1) = K_0 \otimes L_1$ and $G \otimes F = (L_0 \oplus L_1) \otimes (K_0 \oplus K_1) = L_0 \otimes K_1$, the associator $\Phi_{F,G} : F \otimes G \to G \otimes F$ is defined as the biunitary connection $\Phi : K_0 \otimes L_1 \to L_0 \otimes K_1$. All the 8 cases of associators are defined as Φ^g, where $g \in \langle r, \dagger \rangle$.

Theorem 6.28. *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{array}{l}\text{Indecomposable semisimple C}^* \\ \mathcal{TLJ}(d_0)\text{-}\mathcal{TLJ}(d_1) \ \text{ bimodule} \\ \text{categories } \mathcal{M}\end{array}\right\} \cong \left\{\begin{array}{l}\text{W}^* \text{ 2-subcategories } \mathcal{C}(\Gamma, \omega; \Phi) \text{ of } \mathsf{BigHilb}, \\ \text{where } \Gamma \text{ is a balanced } (d_0, d_1)\text{-fair square} \\ \text{partite graph with edge-weighting } \omega \text{ and } \Phi \\ \text{a biunitary connection}\end{array}\right\}.$$

Equivalence on the left-hand side is unitary equivalence; equivalence on the right-hand side is the isomorphism on the edge-weighted square-partite graph and gauge equivalence on biunitary connection.

Proof. We can prove this correspondence for the version with base point by using the Markov lattice. According to Construction 6.26, the correspondence holds without fixing the base point.

As for the equivalence, combining Remark 6.5, Definition 6.11 and the last paragraph in Construction 6.23, the isomorphism on the edge-weighted graph (Γ, ω) and gauge equivalence on Φ corresponds to the unitary equivalence on $\mathcal{C}(\Phi)$, which corresponds to the unitary equivalence on $\mathcal{TLJ}(d_0)$-$\mathcal{TLJ}(d_1)$ bimodule category $\mathcal{M}$ based on Construction 6.23 and Remark 5.38. ∎

6.6. Commuting square of finite dimensional C*-algebras

Suppose the following is a commuting square of finite dimensional C*-algebras with conditional expectations:

$$\begin{tikzcd} M_{1,0} \arrow[d, "E_{1,0}^{M,l}"'] & M_{1,1} \arrow[l, "E_{1,1}^{M,r}"'] \arrow[d, "E_{1,1}^{M,l}"] \\ M_{0,0} & M_{0,1} \arrow[l, "E_{0,1}^{M,r}"] \end{tikzcd}$$

Without loss of generality, we assume this commuting square is indecomposable. We can do the basic construction for each horizontal and vertical inclusion, and obtain a Markov lattice. Therefore, by Theorem 6.28, commuting squares of finite dimensional C*-algebras corresponds to semisimple C* $\mathcal{TLJ}$-$\mathcal{TLJ}$ bimodule categories with a base point $Z = 1^+ \triangleright Z \triangleleft 1^+$ such that $\mathrm{End}(Z) \cong M_{0,0}$.

Note that the commuting square may not be Markov, i.e., there is new stuff appearing in the basic construction, the Bratteli diagram of the original commuting square is not necessarily the principal graph of the square-partite graph (See Example 6.18).

7. The tracial case

In this section, we finally discuss the tracial/pivotal case for (bi)module categories. As an application, we prove the module embedding theorem for (infinite depth) graph planar algebra.

7.1. Tracial Markov towers and pivotal module categories

Definition 7.1. [40] Let $\mathcal{C}$ be a rigid C* (multi)tensor category with the canonical spherical unitary dual functor. We call $\mathcal{M}$ a semisimple pivotal C* $\mathcal{C}$-module category, if there exists a pivotal trace $\mathrm{tr}^{\mathcal{M}}$ compatible with the spherical structure on $\mathcal{C}$, i.e.,

$$\mathrm{tr}^{\mathcal{M}}_{m \triangleleft c}(f) = \mathrm{tr}^{\mathcal{M}}_{m}\left((\mathrm{id}_m \triangleleft \mathrm{coev}_c^{\dagger}) \circ (f \triangleleft \mathrm{id}_{\bar{c}}) \circ (\mathrm{id}_m \triangleleft \mathrm{coev}_c)\right),$$

for all $f \in \mathrm{End}(m \triangleleft c)$, where $m \in \mathcal{M}$, $c \in \mathcal{C}$.

Remark 7.2. If $f \in \mathrm{End}(c)$, $c \in \mathcal{C}$ and $m \in \mathcal{M}$,

$$\begin{aligned}\mathrm{tr}^{\mathcal{M}}_{m \triangleleft c}(\mathrm{id}_m \triangleleft f) &= \mathrm{tr}^{\mathcal{M}}_{m}\left((\mathrm{id}_m \triangleleft \mathrm{coev}_c^{\dagger}) \circ \left((\mathrm{id}_m \triangleleft f) \triangleleft \mathrm{id}_{\bar{c}}\right) \circ (\mathrm{id}_m \triangleleft \mathrm{coev}_c)\right) \\ &= \mathrm{tr}^{\mathcal{M}}_{m}\left(\mathrm{id}_m \triangleleft \left(\mathrm{coev}_c^{\dagger} \circ (f \triangleleft \mathrm{id}_{\bar{c}} \circ \mathrm{coev}_c)\right)\right) \\ &= \mathrm{tr}^{\mathcal{M}}_{m}\left(\mathrm{id}_m \triangleleft \mathrm{tr}^{\mathcal{A}}_{c}(f)\right) \\ &= \mathrm{tr}^{\mathcal{M}}_{m}(\mathrm{id}_m) \cdot \mathrm{tr}^{\mathcal{A}}_{c}(f).\end{aligned}$$

Here we call $\mathrm{tr}^{\mathcal{M}}_{m}(\mathrm{id}_m)$ the dimension of object m.

Remark 7.3. [40, §4.1] If $\mathcal{C}$ is fusion and $\mathcal{M}$ is indecomposable, then the pivotal trace $\mathrm{tr}^{\mathcal{M}}$ is unique up to scalar.

Definition 7.4. Let M be a tracial Markov tower (see Remark 2.5). We call M a tracial standard A-module, where A is a standard λ-lattice, if $\mathrm{tr}^{M}|_{A} = \mathrm{tr}^{A}$ and M is a standard A-module, see Definition 3.1.

Let A be a standard λ-lattice. If we start with a tracial standard A-module M, combining the construction in Section 3.3 and the proof in proposition 2.39, we are able to construct a pivotal planar $\mathcal{A}_0$-module category. Furthermore, from this pivotal planar $\mathcal{A}_0$-module category, we can construct an indecomposable semisimple pivotal C* $\mathcal{A}$-module category with a choice of the simple base object. The following is the theorem.

Theorem 7.5. *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{array}{l}\text{Tracial Markov towers } M=(M_i)_{i\geq 0}\\ \text{with } \dim(M_0)=1 \text{ as standard right}\\ \text{modules over a standard } \lambda\text{-lattice } A\end{array}\right\} \cong \left\{\begin{array}{l}\text{Pairs } (\mathcal{M},Z) \text{ with } \mathcal{M} \text{ an indecomposable}\\ \text{semisimple pivotal C}^* \text{ right } \mathcal{A}\text{-module}\\ \text{category together with a choice of simple}\\ \text{base object } Z=Z\lhd 1_{\mathcal{A}}^{+}\end{array}\right\}.$$

Equivalence on the left-hand side is trace-preserving $*$-isomorphism on the tracial Markov tower as standard A-module; equivalence on the right-hand side is the pivotal unitary $\mathcal{A}$-module category equivalence on their Cauchy completions which maps simple base object to simple base object.

Let us look at the balanced d-fair bipartite graph (Λ,ω) from the tracial Markov tower M. Since the evaluation and coevaluation are compatible with the trace, the edge-weighting comes from a vertex-weighting, see [20, Rem. 6.10]. To be precise,

Definition 7.6 (Vertex weighting). Let Λ be a bipartite graph. Let $\nu : V(\Lambda)\to(0,\infty)$ be a weighting on the vertices of Λ which satisfies the Frobenius-Perron condition: for each $P\in V(\Lambda)$,

$$\sum_{\{Q\in V(\Lambda):P,Q \text{ adjacent}\}} \nu(Q)=d\cdot\nu(P).$$

In the sum on the left-hand side, $\nu(Q)$ has a number of edges between $P\to Q$ copies.

From an undirected bipartite graph, one can obtain a directed graph with involution [19, Def. 2.20]. Then for $e:P\to Q$, define $w(e):=\frac{\nu(t(e))}{\nu(s(e))}=\frac{\nu(Q)}{\nu(P)}$. The d-fairness and balance condition in Definition 5.2 follows automatically.

Remark 7.7. Suppose $\mathcal{M}$ is an indecomposable semisimple C* pivotal $\mathcal{A}$-module category with principal graph Λ whose vertices are simple objects of $\mathcal{M}$. We can define the vertex weighting for simple object P as $\nu(P):=\mathrm{Tr}_P(\mathrm{id}_P)$.

Remark 7.8. Note that $\mathcal{M}$ being a pivotal $\mathcal{A}$-module is equivalent to the dagger tensor functor $\mathcal{A}\to\mathrm{End}^{\dagger}(\mathcal{M})$ being pivotal [18, Thm. 3.70], so that its essential image $\mathrm{End}_0^{\dagger}(\mathcal{M},F)$ has a unitary pivotal structure from the pivotal structure in $\mathcal{A}$, where $F=-\lhd X$ is the generator. We also denote the corresponding 2-subcategory of $\mathsf{BigHilb}$ as $\mathcal{C}(K,\phi)$ or $\mathcal{C}(\Lambda,\nu)$.

7.2. The module embedding theorem

Jones' planar algebra, as a form of standard invariant, is a method to construct and classify finite index type II_1 subfactors. The module embedding theorem has been known to Vaughan Jones since he first defined the graph planar algebra [23]. The proof for finite depth case appears in [7, 18, 25]. Many non-trivial subfactor classification results are inspired by this theorem [28, 29], including the extended Haagerup subfactor and its relatives [3, 18].

In this section, our goal is to prove Theorem 7.9, the infinite depth module embedding theorem. We refer the reader to [23] for the full definition of graph planar algebra.

Let (Λ, ν) be a balanced d-fair bipartite graph with vertex weighting described in Definition 7.6. According to the discussion in Section 5.3, $\mathcal{C}(K, \mathrm{ev}_K)$ is a pivotal W* 2-subcategory of $\mathsf{BigHilb}$, where the tensor generator $K = K_\Lambda$ is the Hilb-enriched diagram of Λ, the dual and the pivotal structure are given by the vertex weighting. Following [18, §3], we can define a W* shaded planar algebra $\mathcal{P}(\Lambda)$ with $\mathcal{P}_{n,+} := \mathrm{End}_{\mathsf{BigHilb}}(K^{\mathrm{alt}\otimes n})$ and $\mathcal{P}_{n,-} := \mathrm{End}_{\mathsf{BigHilb}}(\overline{K}^{\mathrm{alt}\otimes n})$.

Let $V = V_0 \cup V_1$ be the set of vertices of Λ. Recall from the definition of $\mathsf{BigHilb}$, we have

$$K_{u,v}^{\mathrm{alt}\otimes n} = \bigoplus_{w_1, w_2, \ldots, w_{n-1} \in V} K_{u,w_1} \otimes \overline{K}_{w_1,w_2} \otimes \cdots \otimes K^{?}_{w_{n-1},v}.$$

Since the degree of each vertex is uniformly bounded, $K_{u,v}^{\mathrm{alt}\otimes n}$ is a finite dimensional Hilbert space. Note that every 2-morphism $f \in \mathrm{End}_{\mathsf{BigHilb}}(K^{\mathrm{alt}\otimes n})$ is determined by its component maps

$$\Bigg\{ f_{u,v} : \bigoplus_{w_1, w_2, \ldots, w_{n-1} \in V} K_{u,w_1} \otimes \overline{K}_{w_1,w_2} \otimes \cdots \otimes K^{?}_{w_{n-1},v}$$
$$\to \bigoplus_{r_1, r_2, \ldots, r_{n-1} \in V} K_{u,r_1} \otimes \overline{K}_{r_1,r_2} \otimes \cdots \otimes K^{?}_{r_{n-1},v} \Bigg\}_{u,v \in V}.$$

Now fix an ONB $\{\varepsilon_{u,v}^k\}$ for each $K_{u,v}$ ($\{\bar{\varepsilon}_{v,u}^k\}$ for $\overline{K}_{v,u}$), and for each pair of paths of length n on Λ from u to v,

$$p = \varepsilon_{u,w_1}^{k_1} \otimes \bar{\varepsilon}_{w_1,w_2}^{k_2} \otimes \cdots \otimes \varepsilon_{w_{n-1},v}^{?,k_n} \quad \text{and} \quad q = \varepsilon_{u,r_1}^{l_1} \otimes \bar{\varepsilon}_{r_1,r_2}^{l_2} \otimes \cdots \otimes \varepsilon_{r_{n-1},v}^{?,l_n}.$$

Let $F_{p\to q}^{u,v} \in \mathrm{End}_{\mathsf{BigHilb}}(K^{\mathrm{alt}\otimes n})$ be the unique (u, v)-component map sending path p to path q and all other paths p' to zero. Then we can see

$$\mathcal{P}_{n,+} = \mathrm{End}_{\mathsf{BigHilb}}(K^{\mathrm{alt}\otimes n}) = \bigoplus_{i,l} \mathrm{span}_{\mathbb{C}} \left\{ F_{p\to q}^{i,l} \right\}_{p,q \text{ paths } u \text{ to } v}.$$

Therefore, each 2-morphism can be represented as a linear combination of these bases.

Following [25, Ex. 3.27], we show this is a bipartite graph planar algebra. In particular, we verify case (1) and the rest of the cases is left to the readers.

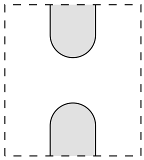
According to the discussion in Definition 5.8, evaluations $\mathrm{ev}_K, \mathrm{ev}_{\overline{K}}$ and coevaluations $\mathrm{coev}_K, \mathrm{coev}_{\overline{K}}$ are completely determined by $C_{\overline{K},uv} : K_{uv} \otimes \overline{K}_{vu} \to \mathbb{C}$. Note that now the edge-weighting comes from the vertex-weighting, so by Construction 5.15, $C_{\overline{K},uv}$ is defined as

$$|e\rangle \otimes |\overline{e'}\rangle \mapsto \delta_{e=e'} w(e)^{\frac{1}{2}} = \delta_{e=e'} \left(\frac{\nu(v)}{\nu(u)} \right)^{\frac{1}{2}} = \delta_{e=e'} \left(\frac{\nu\big(t(e)\big)}{\nu\big(s(e)\big)} \right)^{\frac{1}{2}} \quad \text{for } e : u \to v.$$

And $D_{K,uv} = C_{\overline{K},uv}^{\dagger}$ is defined as

$$1 \mapsto \sum_{e:u\to v} w(e)^{\frac{1}{2}} |e\rangle \otimes |\bar{e}\rangle = \sum_{e:u\to v} \left(\frac{\nu(v)}{\nu(u)} \right)^{\frac{1}{2}} |e\rangle \otimes |\bar{e}\rangle = \sum_{e:u\to v} \left(\frac{\nu\big(t(e)\big)}{\nu\big(s(e)\big)} \right)^{\frac{1}{2}} |e\rangle \otimes |\bar{e}\rangle.$$

Therefore,



$$\in \mathrm{End}_{\mathsf{BigHilb}}(K \otimes \overline{K}) = \mathcal{P}_{2,+} \tag{7.1}$$

is non-zero on each $K_{uv} \otimes \overline{K}_{vu} \to K_{uw} \otimes \overline{K}_{wu}$ component. Once we fix an ONB $\{\varepsilon^k_{u,v}\}$ for $K_{u,v}$, the map on this component is given by

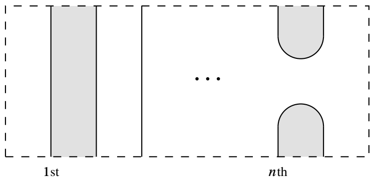
$$\left(\frac{\nu(v)\nu(w)}{\nu(u)\nu(u)}\right)^{\frac{1}{2}} = \left(\frac{\nu\big(t(\varepsilon^k_{u,v})\big)\nu\big(t(\varepsilon^l_{u,w})\big)}{\nu\big(s(\varepsilon^k_{u,v})\big)^2}\right)^{\frac{1}{2}} : \varepsilon^k_{u,v} \otimes \overline{\varepsilon}^k_{v,u} \to \varepsilon^l_{u,w} \otimes \overline{\varepsilon}^l_{w,u}.$$

When we identify it with the loop $[\varepsilon^k_{u,v}\overline{\varepsilon}^k_{v,u}\varepsilon^l_{u,w}\overline{\varepsilon}^l_{w,u}]$, we can regard 2-morphism (7.1) as

$$\sum \left(\frac{\nu\big(t(\varepsilon^k_{u,v})\big)\nu\big(t(\varepsilon^l_{u,w})\big)}{\nu\big(s(\varepsilon^k_{u,v})\big)^2}\right)^{\frac{1}{2}} [\varepsilon^k_{u,v}\overline{\varepsilon}^k_{v,u}\varepsilon^l_{u,w}\overline{\varepsilon}^l_{w,u}],$$

where the sum is taken over all valid loops in $\mathcal{P}_{2,+}$.

For $n \in \mathbb{N}$ is odd,



$$\in \mathrm{End}_{\mathsf{BigHilb}}(K^{\mathrm{alt}\otimes(n+1)}) = \mathcal{P}_{n+1,+}.$$

Note that the first $n-1$ strings are identities, this is equal to

$$\sum \left(\frac{\nu\big(t(\varepsilon^{k_n}_{v_{n-1},v_n})\big)\nu\big(t(\varepsilon^{l_{n+1}}_{w_n,u})\big)}{\nu\big(s(\varepsilon^{k_n}_{v_{n-1},v_n})\big)^2}\right)^{\frac{1}{2}}$$
$$\times \left[\varepsilon^{k_1}_{u,v_1}\overline{\varepsilon}^{k_2}_{v_1,v_2}\cdots\overline{\varepsilon}^{k_{n-1}}_{v_{n-2},v_{n-1}}\varepsilon^{k_n}_{u,v_n}\overline{\varepsilon}^{k_n}_{v_n,u}\varepsilon^{l_n}_{u,w_n}\overline{\varepsilon}^{l_n}_{w_n,u}\varepsilon^{l_{n-1}}_{u,w_{n-1}}\cdots\varepsilon^{l_2}_{w_2,w_1}\overline{\varepsilon}^{l_1}_{w_1,u}\right],$$

where the sum is taken over all valid loops in $\mathcal{P}_{n+1,+}$.

When the bipartite graph is finite depth, this is exactly the Jones graph planar algebra. If the bipartite graph is infinite, this is the analytic version of infinite bipartite graph planar algebra due to Burnstein [5] generalizing Jones' bipartite graph planar algebra [23].

Note that there is a well-know correspondence between [8, 16, 18, 37]:

$$\left\{\begin{matrix}\text{Subfactor planar}\\ \text{algebras } \mathcal{P}_\bullet\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Pairs } (\mathcal{A}, X) \text{ with } \mathcal{A} \text{ a 2-shaded rigid C}^*\\ \text{multitensor category with a generator } X,\\ \text{i.e., } 1_{\mathcal{A}} = 1^+ \oplus 1^-,\ 1^+, 1^- \text{ are simple}\\ \text{and } X = 1^+ \otimes X \otimes 1^-\end{matrix}\right\}.$$

Finally, according to [18, Thm. 3.77], a pivotal dagger tensor functor $\mathcal{C} \to \mathrm{End}^\dagger(\mathcal{M})$ which preserves the dual structure and maps the generator to the generator corresponds to an embedding of shaded planar algebra $\mathcal{P}_\bullet \hookrightarrow \mathcal{G}_\bullet$, where $\mathcal{G}_\bullet$ is the bipartite graph planar algebra of the principal graph of $\mathcal{M}$ as a $\mathcal{A}$-module generated by $- \lhd X$ and X is the generator of $\mathcal{A}$.

Now we choose $\mathcal{M} := \mathcal{A}$ to be a $\mathcal{A}$-module category and pick the tensor functor $\phi : \mathcal{A} \to \mathrm{End}^\dagger(\mathcal{A})$ by $c \mapsto - \lhd c$. Based on the discussion in Section 5.7 and Remark 7.8, the tensor category $\mathrm{End}_0^\dagger(\mathcal{M}, F) := \phi(\mathcal{A})$ is equivalent to the 2-category $\mathcal{C}(K_\Lambda, \mathrm{ev}_{K_\Lambda})$, where Λ is the principal bipartite graph generated by $- \lhd X$ with generator $X = 1^+ \otimes X \otimes 1^- \in \mathcal{A}$, we obtain the module embedding theorem:

Theorem 7.9. *Every subfactor planar algebra $\mathcal{P}_\bullet$ embeds into the graph planar algebra of its principal graph.*

7.3. Tracial Markov lattices and pivotal bimodule categories

Definition 7.10. Let $\mathcal{C}, \mathcal{D}$ be rigid C* (multi)tensor categories with canonical unitary dual functors respectively. We call M a semisimple pivotal C* $\mathcal{C}$-$\mathcal{D}$ bimodule category, if there exists a pivotal trace $\mathrm{tr}^{\mathcal{M}}$ compatible with the spherical structures in $\mathcal{C}$ and $\mathcal{D}$, i.e.,

$$\begin{aligned} \mathrm{tr}^{\mathcal{M}}_{a \rhd m}(f) &= \mathrm{tr}^{\mathcal{M}}_m \big((\mathrm{ev}_a^\dagger \rhd \mathrm{id}_m) \circ (\mathrm{id}_{\bar{a}} \rhd f) \circ (\mathrm{ev}_a \rhd \mathrm{id}_m)\big), \\ \mathrm{tr}^{\mathcal{M}}_{m \lhd b}(f) &= \mathrm{tr}^{\mathcal{M}}_m \big((\mathrm{id}_m \lhd \mathrm{coev}_b^\dagger) \circ (f \lhd \mathrm{id}_{\bar{b}}) \circ (\mathrm{id}_m \lhd \mathrm{coev}_b)\big), \end{aligned}$$

for $f \in \mathrm{End}(a \rhd m \lhd b)$, where $m \in \mathcal{M}, a \in \mathcal{C}, b \in \mathcal{D}$.

Definition 7.11. Let M be a tracial Markov lattice (see Remark 4.3). We call M a tracial standard A-B bimodule, where A, B are standard λ-lattices, if $\mathrm{tr}^M|_A = \mathrm{tr}^A$, $\mathrm{tr}^M|_B = \mathrm{tr}^B$ and M is a standard A-B bimodule, see Definition 4.5.

Similar to Theorem 7.5, we have the following theorem:

Theorem 7.12. *There is a bijective correspondence between equivalence classes of the following:*

$$\left\{\begin{array}{l}\text{Tracial Markov lattice } M = \\ (M_{i,j})_{i,j\geq 0} \text{ with } \dim(M_{0,0}) = \\ 1 \text{ as a standard } A\text{-}B \text{ bimodule} \\ \text{over standard } \lambda\text{-lattices } A, B\end{array}\right\} \cong \left\{\begin{array}{l}\text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable} \\ \text{semisimple C}^* \text{ pivotal } \mathcal{A}\text{-}\mathcal{B} \text{ bimodule} \\ \text{category together with a choice of simple} \\ \text{base object } Z = 1^+_{\mathcal{A}} \rhd Z \lhd 1^+_{\mathcal{B}}\end{array}\right\}.$$

Equivalence on the left-hand side is the trace-preserving $*$-isomorphism on the tracial Markov lattice as standard A-B bimodule; equivalence on the right-hand side is the pivotal unitary $\mathcal{A}$-$\mathcal{B}$ bimodule equivalence between their Cauchy completions which maps the simple base object to simple base object.

Let us look at the balanced (d_0, d_1)-fair square-partite graph (Λ, ω) from the tracial Markov lattice M. Similar to the tracial Markov tower case, edge-weighting comes from vertex-weighting. To be precise,

$$\text{For } P \in V_{00} \sqcup V_{01}, \qquad \sum_{\{e:P\to Q:Q\in V_{10}\sqcup V_{11}\}} \nu(Q) = d_0 \cdot \nu(P),$$

$$\text{For } P \in V_{00} \sqcup V_{01}, \qquad \sum_{\{e:P\to Q:Q\in V_{01}\sqcup V_{11}\}} \nu(Q) = d_1 \cdot \nu(P).$$

Remark 7.13. As for the biunitary connection, the computation does not change at all. In fact, the biunitary connection is independent of the pivotal structure, see Proposition 6.13 (2) and Section 6.5. This now agrees with the usual definition of biunitary connection for the tracial/pivotal case discussed in [12, 21, 32, 33].

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