

Algebraic kk -theory and the KH-isomorphism conjecture

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Abstract. We relate the Davis–Lück homology with coefficients in Weibel’s homotopy K -theory to the equivariant algebraic kk -theory using homotopy theory and adjointness theorems. We express the left hand side of the assembly map for the KH-isomorphism conjecture in terms of equivariant algebraic kk -groups.

1. Introduction

Kasparov KK -theory, introduced in [21], is the major tool in noncommutative topology. To every pair (A, B) of separable C^* -algebras it associates an abelian group $KK(A, B)$ that is a common generalization both of topological K -theory $K_*^{\text{top}}(B) \cong KK_*(\mathbb{C}, B)$ and topological K -homology. Kasparov KK -theory was deeply studied by Cuntz, who gave an alternative description of the groups $KK(A, B)$ and provided a new perspective on the theory [10]. The groups $KK(A, B)$ form the hom-sets of a category KK that is the target of the universal homotopy invariant, C^* -stable and split-exact functor from separable C^* -algebras into an additive category; these universal properties were established by Higson [18]. Kasparov developed in [22] an equivariant version of KK -theory, KK^G , for separable C^* -algebras with an action of a group G by $*$ -automorphisms. Equivariant KK -theory was used in [4] to formulate the Baum–Connes conjecture with coefficients and is one of the basic tools in the proofs of results about it. Cuntz also analyzed KK -theory in algebraic terms and defined a bivariant K -theory for all locally convex algebras [11].

Motivated by the works of Cuntz and Higson, algebraic kk -theory was introduced by Cortiñas and Thom in [9] as a completely algebraic counterpart of Kasparov KK -theory. To every pair (A, B) of algebras over a commutative ring ℓ it associates an abelian group $kk(A, B)$ that generalizes Weibel’s homotopy K -theory KH defined in [31]. By [9, Main Theorem] we have $\text{KH}_*(B) \cong kk_*(\ell, B)$ for every algebra B . The groups $kk(A, B)$ form the hom-sets of a category kk that is the target of the universal (polynomial) homotopy invariant, M_∞ -stable and excisive functor from Alg_ℓ into a triangulated category. An equivariant version of algebraic kk -theory, kk^G , was developed in [15] for ℓ -algebras

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with an action of a group G . By Garkusha’s representability theorems [17], the groups $kk(A, B)$ can be recovered as the homotopy groups of certain spectra $\mathbb{K}(A, B)$. These spectra were constructed in [17] and were later extended in [27] to spectra $\mathbb{K}^G(A, B)$ representing $kk^G(A, B)$.

We may summarize the above into a dictionary between Kasparov KK -theory and algebraic kk -theory:

continuous homotopy invariance	\longleftrightarrow	polynomial homotopy invariance
C^* -stability	\longleftrightarrow	M_∞ -stability
split exactness	\longleftrightarrow	excision
KK -theory of C^* -algebras	\longleftrightarrow	kk -theory of ℓ -algebras
$KK(C, B) \cong K_*^{\text{top}}(B)$	\longleftrightarrow	$kk_*(\ell, B) \cong \text{KH}_*(B)$
KK^G -theory of G - C^* -algebras	\longleftrightarrow	kk^G -theory of G -algebras
formulation of the Baum–Connes conjecture with coefficients using KK^G -theory	\longleftrightarrow	?

This paper is concerned with the last line of this dictionary. The Baum–Connes conjecture with coefficients was originally formulated using KK^G -theory, but no analogue is known on the algebraic side. We shed light on this by exploring the relation between kk^G -theory and the KH-isomorphism conjecture – the latter was introduced in [3] using the homotopical approach to isomorphism conjectures developed in [13]. Our main theorem states that the domain of the KH-assembly map can be described in terms of kk^G -groups in way that is completely analogous to the case of the Baum–Connes assembly map. As explained in Section 2.2.4, a group G and a G -algebra B satisfy the KH-isomorphism conjecture if certain morphism

$$H_*^G(E_{\mathcal{F}in}(G); \mathbf{KH}_B) \rightarrow \text{KH}_*(B \rtimes G),$$

called the *assembly map*, is an isomorphism. Here $E_{\mathcal{F}in}(G)$ is the classifying space of G with respect to its family of finite subgroups and $H_*^G(-; \mathbf{KH}_B)$ is a G -homology theory such that $H_*^G(G/H; \mathbf{KH}_B) \cong \text{KH}_*(B \rtimes H)$ for every subgroup $H \subseteq G$. We prove the following result; see Section 2.1 for the precise definition of $\ell^{(X)}$ and Section 2.2 for the definition of $(G, \mathcal{F}in)$ -complex.

Main Theorem (cf. Theorem 7.11 and Remark 7.12). *Let G be a group such that $|H|$ is invertible in the base ring ℓ for every finite subgroup $H \subseteq G$. Let B be a G -algebra. Then for every $(G, \mathcal{F}in)$ -complex Z there is a natural isomorphism*

$$H_*^G(Z; \mathbf{KH}_B) \cong \text{colim}_{\substack{X \subseteq Z \\ G\text{-finite}}} kk_*^G(\ell^{(X)}, B) \tag{1.1}$$

where $\ell^{(X)}$ denotes the algebra of finitely supported polynomial functions on X .

For $Z = G/H$ with $H \subseteq G$ a finite subgroup, the isomorphism (1.1) is easily understood using the adjointness theorems in kk -theory. Let $H \subseteq G$ be a finite subgroup whose

order is invertible in ℓ . By the Green–Julg theorem [15, Theorem 5.2.1] we have a natural isomorphism

$$kk^H(A^\tau, B) \cong kk(A, B \rtimes H) \tag{1.2}$$

for $A \in \text{Alg}_\ell$ and $B \in H\text{Alg}_\ell$. Here A^τ denotes the algebra A considered as an H -algebra with trivial H -action. By the adjunction between induction and restriction [15, Theorem 6.14] we have a natural isomorphism

$$kk^G(\text{Ind}_H^G(A), B) \cong kk^H(A, \text{Res}_G^H(B)) \tag{1.3}$$

for $A \in H\text{Alg}_\ell$ and $B \in G\text{Alg}_\ell$. Since $\text{Ind}_H^G(A^\tau) = \bigoplus_{G/H} A = A^{(G/H)}$, upon composing (1.2) and (1.3) we get a natural isomorphism

$$kk^G(A^{(G/H)}, B) \cong kk(A, B \rtimes H) \tag{1.4}$$

for $A \in \text{Alg}_\ell$ and $B \in G\text{Alg}_\ell$. As it will turn out, the identification (1.1) is obtained by glueing the isomorphisms (1.4) with homotopy theoretic techniques. In this process we will replace the groups kk and kk^G by the spectra \mathbb{K} and \mathbb{K}^G that respectively represent them; see [17] and Appendix C.

The paper is organized as follows. In Section 2, we recall definitions and preliminaries that are used throughout the article. In Section 3, we give explicit descriptions of the unit and counit of the adjunction (1.4). In Section 4, we define a triangulated functor

$$\mathcal{R}(- \rtimes G/H) : kk^G \rightarrow kk$$

that is naturally isomorphic to the crossed product with H . This allows us to replace the right-hand side of (1.4) by an actual functor on $\text{Or}(G)$; see Definition 2.5. In Section 5, we prove that the isomorphism

$$kk^G(A^{(G/H)}, B) \cong kk(A, \mathcal{R}(B \rtimes G/H)) \tag{1.5}$$

is natural in G/H ; see Theorem 5.8. Section 6 is the technical core of this work and is devoted to lifting the isomorphism (1.5) to a weak equivalence of $\text{Or}(G, \mathcal{F}in)$ -spectra. By Lemma 5.5, we can describe the isomorphism (1.5) as the composite of the morphisms in the zig-zag (6.2). Upon replacing kk by \mathbb{K} and kk^G by \mathbb{K}^G we obtain a zig-zag as follows:

$$\begin{array}{ccc} \mathbb{K}(A, \mathcal{R}(B \rtimes G/H)) & \xrightarrow{(-)^{(G/H)}} & \mathbb{K}^G(A^{(G/H)}, [\mathcal{R}(B \rtimes G/H)]^{(G/H)}) \\ & & \uparrow \wr \\ \mathbb{K}^G(A^{(G/H)}, M_G B) & \xleftarrow{\mathcal{R}(\xi_{G/H})} & \mathbb{K}^G(A^{(G/H)}, \mathcal{R}[(B \rtimes G/H)^{(G/H)}]). \end{array} \tag{1.6}$$

Here the technical difficulties arise:

- (1) How to consider the spectra on the right column as functors $\text{Or}(G, \mathcal{F}in) \rightarrow \text{Sp}$ (or to replace them by ones)?
- (2) Once the previous question has been addressed, are the morphisms in (1.6) natural in G/H ?

An answer to (1) is provided in Sections 6.3 and 6.4. In Section 6.6 we introduce a model category that allows us to build models for certain homotopy coends. An answer to (2) is given in Theorem 6.34. The latter is the key technical result of this paper. In Section 7 we finally prove our Main Theorem (Theorem 7.11). In Section 8 we prove a first result towards obtaining a kk -theoretic description of the KH-assembly map.

2. Notation, conventions and preliminaries

Throughout this text, ℓ denotes a commutative ring with unit and G denotes a group. We write Alg_ℓ for the category of not necessarily unital ℓ -algebras and algebra homomorphisms. The objects of Alg_ℓ are simply called *algebras*. Tensor products over ℓ are denoted by \otimes . A G -algebra is an algebra with a left action of G . We write $G\text{Alg}_\ell$ for the category of G -algebras and G -equivariant algebra homomorphisms. The category of simplicial sets is denoted by \mathbb{S} . A G -simplicial set is a simplicial set endowed with a left action of G . We write \mathbb{S}^G for the category of G -simplicial sets with equivariant morphisms.

2.1. Algebras of polynomial functions on a simplicial set

Let B be an algebra. For $n \geq 0$, the algebra B^{Δ^n} of B -valued polynomial functions on the standard n -simplex is defined as $B^{\Delta^n} := B[t_0, \dots, t_n] / \langle t_0 + \dots + t_n - 1 \rangle$. For a simplicial set X , the algebra B^X of B -valued polynomial functions on X is defined as

$$B^X := \text{Hom}_{\mathbb{S}}(X, B^\Delta)$$

where B^Δ is the simplicial algebra $[n] \mapsto B^{\Delta^n}$. Note that the functor $B^\square : \mathbb{S} \rightarrow \text{Alg}_\ell$ sends colimits to limits. We summarize useful properties of this construction in the following lemma.

Lemma 2.1 ([9, Lemma 3.1.2 and Proposition 3.1.3]). *Let B be an algebra.*

- (1) *If $X \subseteq Y$ is an inclusion of simplicial sets, then $B^Y \rightarrow B^X$ is surjective.*
- (2) *If K is a finite simplicial set, then ℓ^K is free as ℓ -module and there is a natural isomorphism of algebras $B^K \cong B \otimes \ell^K$.*

Let X be a simplicial set. We recall the definition of the algebra $B^{(X)}$ of finitely supported B -valued polynomial functions on X ; see [8, Section 9.3] for details. The support of a polynomial function $\phi \in B^X$ is defined as the simplicial subset of X generated by

$$\bigcup_{n \geq 0} \{\sigma \in X_n : \phi(\sigma) \neq 0\}.$$

Put

$$B^{(X)} := \{\phi \in B^X : \text{supp}(\phi) \text{ is finite}\}.$$

Then $B^{(X)} \subseteq B^X$ is a two-sided ideal. If K is a finite simplicial set, then we have $B^{(K)} = B^K$.

Remark 2.2. The assignment $X \mapsto B^{(X)}$ is natural only for *proper* maps of simplicial sets – i.e. morphisms $f : X \rightarrow Y$ such that $f^{-1}(K)$ is finite for every finite simplicial subset $K \subseteq Y$.

Remark 2.3. If $\{X_i\}_i$ is a family of simplicial sets then we have

$$B(\bigsqcup_i X_i) \cong \bigoplus_i B^{(X_i)}.$$

The behavior of $B^{(-)}$ with respect to general colimits of proper maps is analyzed in [8, Section 9.3, conditions (∂_0) and (∂_1)]. In particular, upon applying $B^{(-)}$ to a pushout square of simplicial sets where all the morphisms are proper we obtain a pullback square of algebras.

If X is a G -simplicial set, then B^X and $B^{(X)}$ are G -algebras with the action defined by $(g \cdot \phi)(\sigma) := \phi(g^{-1} \cdot \sigma)$ for $\phi \in B^X$, $n \geq 0$ and $\sigma \in X_n$. If $X \rightarrow Y$ is a proper morphism in \mathbb{S}^G , then $B^{(Y)} \rightarrow B^{(X)}$ is a morphism in $G\text{Alg}_\ell$.

Example 2.4. Let $H \subseteq G$ be a subgroup and let $G/H = \{uH : u \in G\}$ be the set of left cosets of H . We can consider the G -set G/H as a 0-dimensional G -simplicial set. For any algebra A we have

$$\begin{aligned} A^{G/H} &= A^{\bigsqcup_{uH \in G/H} \{uH\}} \cong \prod_{uH \in G/H} A^{\{uH\}} \cong \prod_{G/H} A, \\ A^{(G/H)} &= A^{(\bigsqcup_{uH \in G/H} \{uH\})} \cong \bigoplus_{uH \in G/H} A^{\{uH\}} \cong \bigoplus_{G/H} A. \end{aligned}$$

In particular, $\ell^{(G/H)} \cong \bigoplus_{G/H} \ell$ is a free ℓ -module with base $\{\chi_{uH} : uH \in G/H\}$. When considering χ_{uH} as a function $G/H \rightarrow \ell$ with finite support, then $\chi_{uH}(vH) = \delta_{uH,vH}$ (Kronecker delta). Note that we have $g \cdot \chi_{uH} = \chi_{guH}$ for every $g \in G$. For any algebra A we have

$$A^{(G/H)} \cong \bigoplus_{G/H} A \cong A \otimes \bigoplus_{G/H} \ell \cong A \otimes \ell^{(G/H)}.$$

For $a \in A$ and $uH \in G/H$ we sometimes write $a\chi_{uH} \in A^{(G/H)}$ for the element corresponding to $a \otimes \chi_{uH} \in A \otimes \ell^{(G/H)}$ under the isomorphism above.

2.2. Isomorphism conjectures

In this section, we recall the homotopical approach to the isomorphism conjectures developed in [13]. By [8, Section 2], it is equivalent to work in the topological or in the simplicial setting; we choose to do the latter.

Definition 2.5. Let G be a group. The *orbit category* of G is the category $\text{Or}(G)$ whose objects are the G -sets G/H with $H \subseteq G$ a subgroup and whose morphisms are the G -equivariant functions.

2.2.1. Model structure on G -simplicial sets determined by a family of subgroups.

For $X \in \mathbb{S}^G$ and $H \subseteq G$ a subgroup, let X^H be the simplicial subset of X formed by those simplices that are fixed by H . Note that

$$X^H \cong \text{Hom}_G(G/H, X)$$

so that, for fixed X , $G/H \mapsto X^H$ is a functor on $\text{Or}(G)$. Let \mathcal{F} be a nonempty family of subgroups of G closed under conjugation and under taking subgroups. As explained in [8, Section 2], \mathbb{S}^G admits a model structure where a morphism $f : X \rightarrow Y$ is a weak equivalence (resp. a fibration) if and only if $f^H : X^H \rightarrow Y^H$ is a weak equivalence (resp. a fibration) in \mathbb{S} for every $H \in \mathcal{F}$. Moreover, the cofibrant objects, called (G, \mathcal{F}) -complexes, are those G -simplicial sets X such that the stabilizer of every simplex of X is in \mathcal{F} . These can be alternatively described as those G -simplicial sets X that can be built from cells of the form $G/H \times \Delta^n$ with $H \in \mathcal{F}$; see Appendix A. We write $E_{\mathcal{F}}(G) \rightarrow \text{pt}$ for a cofibrant replacement of the point and call $E_{\mathcal{F}}(G)$ a model for the classifying space of G with respect to \mathcal{F} .

2.2.2. Equivariant homology and assembly map. Let Sp be the category of spectra; see Appendix C for details. An $\text{Or}(G)$ -spectrum is a functor $\text{Or}(G) \rightarrow \text{Sp}$. Fix an $\text{Or}(G)$ -spectrum \mathbf{E} and define $H^G(-; \mathbf{E}) : \mathbb{S}^G \rightarrow \text{Sp}$ as the coend

$$H^G(X; \mathbf{E}) := \int^{G/H} X_+^H \wedge \mathbf{E}(G/H). \tag{2.6}$$

The groups $H_*^G(X; \mathbf{E}) := \pi_* H^G(X; \mathbf{E})$ assemble into a homology theory of G -simplicial sets such that

$$H_*^G(G/H; \mathbf{E}) \cong \pi_* \mathbf{E}(G/H)$$

for any $G/H \in \text{Or}(G)$; see [13, Lemma 6.1]. The assembly map for the triple $(\mathbf{E}, G, \mathcal{F})$ is the morphism

$$H_*^G(E_{\mathcal{F}}(G); \mathbf{E}) \rightarrow H_*^G(\text{pt}; \mathbf{E}) \cong \pi_* \mathbf{E}(G/G) \tag{2.7}$$

induced on homology by the cofibrant replacement $E_{\mathcal{F}}(G) \rightarrow \text{pt}$. We say that the isomorphism conjecture for the triple $(\mathbf{E}, G, \mathcal{F})$ holds if (2.7) is an isomorphism.

Remark 2.8. Let $\text{Or}(G, \mathcal{F})$ be the full subcategory of $\text{Or}(G)$ whose objects are those G/H with $H \in \mathcal{F}$. If the functor \mathbf{E} is only defined on $\text{Or}(G, \mathcal{F})$, the formula (2.6) still makes sense but defines a homology theory in the full subcategory of \mathbb{S}^G whose objects are the (G, \mathcal{F}) -complexes.

Remark 2.9. Any morphism $f : \mathbf{E} \rightarrow \mathbf{F}$ of $\text{Or}(G, \mathcal{F})$ -spectra induces a natural transformation $f_* : H^G(-; \mathbf{E}) \rightarrow H^G(-; \mathbf{F})$ of functors from the category of (G, \mathcal{F}) -complexes into Sp . Upon taking homotopy groups, we get a natural transformation

$$f_* : H_*^G(X; \mathbf{E}) \rightarrow H_*^G(X; \mathbf{F}). \tag{2.10}$$

Moreover, if f is an objectwise weak equivalence, then (2.10) is an isomorphism by [13, Lemma 4.6].

2.2.3. The Baum–Connes conjecture with coefficients. Let G be a countable group and B be a separable G - C^* -algebra. The Baum–Connes conjecture fits into the framework described above by considering the family $\mathcal{F}in$ of finite subgroups and an $\text{Or}(G)$ -spectrum \mathbf{K}_B such that

$$\pi_* \mathbf{K}_B(G/H) \cong K_*^{\text{top}}(B \rtimes_r H)$$

for every subgroup $H \subseteq G$. Here \rtimes_r denotes the reduced crossed product. The equivalence of the corresponding assembly map with the original one in [4] was recently proved independently in [23] and in [6].

2.2.4. The KH-isomorphism conjecture. Let G be a group and let B be a G -algebra. The KH-isomorphism conjecture was introduced in [3, Section 7] using the machinery described above; see [25, Section 15.3] for the status of this conjecture. The conjecture is obtained upon considering the family $\mathcal{F}in$ of finite subgroups of G and an $\text{Or}(G)$ -spectrum \mathbf{KH}_B such that

$$\pi_* \mathbf{KH}_B(G/H) \cong \text{KH}_*(B \rtimes H) \tag{2.11}$$

for every subgroup $H \subseteq G$. The original construction of \mathbf{KH}_B was done in [3]. In Section 7, we provide a different construction using the spectra that represent algebraic kk -theory [17].

2.3. Algebraic kk -theory

Let \mathcal{C} denote either the category Alg_ℓ or $G\text{Alg}_\ell$.

2.3.1. Homotopy invariance. Two morphisms $f, g : A \rightarrow B$ in \mathcal{C} are *elementary homotopic* if there exists a morphism $H : A \rightarrow B[t]$ such that $\text{ev}_0 \circ H = f$ and $\text{ev}_1 \circ H = g$. Here, $B[t]$ is the algebra of polynomials with coefficients in B and $\text{ev}_i : B[t] \rightarrow B$ is the evaluation at i . If B is a G -algebra we let G act trivially on t . The relation of elementary homotopy is easily seen to be reflexive and symmetric but not transitive. We consider the equivalence relation generated by elementary homotopy and call it *homotopy*.

Let \mathcal{C} be a category. A functor $F : \mathcal{C} \rightarrow \mathcal{C}$ is *homotopy invariant* if it sends homotopic morphisms to the same morphism. It is easily verified that F is homotopy invariant if and only if $F(A \subseteq A[t])$ is an isomorphism for every A .

2.3.2. Matrix stability. Let A be an algebra and X be an infinite set. We write $M_X A$ for the algebra of finitely supported matrices with coefficients in A that are indexed over $X \times X$; when $A = \ell$ we just write M_X . Note that $M_X A \cong M_X \otimes A$. The assignment $X \mapsto M_X$ depends covariantly on X with respect to injective functions.

Fix $x \in X$ and let $\iota_x : A \rightarrow M_X A$ be given by $\iota_x(a) = e_{x,x} \otimes a$. Let \mathcal{C} be a category. A functor $F : \mathcal{C} \rightarrow \mathcal{C}$ is *M_X -stable* if $F(\iota_x : A \rightarrow M_X A)$ is an isomorphism for every A . This definition is independent of the choice of x by the following result.

Lemma 2.12 ([7, Proposition 2.2.6]). *Let \mathcal{C} be a category and let $F : G\text{Alg}_\ell \rightarrow \mathcal{C}$ be an M_2 -stable functor. Let $B \subseteq C$ be an inclusion of G -algebras. Suppose that C is unital and let $V \in C$ be an invertible element such that $VB, BV^{-1} \subseteq B$ and $g \cdot V = V$ for all $g \in G$.*

Then the formula $\phi^V(b) = VbV^{-1}$ defines a G -algebra homomorphism $\phi^V : B \rightarrow B$ such that $F(\phi^V) = \text{id}_{F(B)}$.

Proof. It is easily verified that $g \cdot V^{-1} = V^{-1}$ for all $g \in G$ and that ϕ^V defines a G -algebra homomorphism. The rest of the proof of [7, Proposition 2.2.6] carries over verbatim. ■

Let S be a G -set and let $|S|$ be its underlying set. We write M_S for the algebra $M_{|S|}$ endowed with the G -action defined by $g \cdot e_{s,t} = e_{g \cdot s, g \cdot t}$. Let $G_+ = G \sqcup \{+\}$ and let $\iota : \ell \rightarrow M_{G_+}$ (respectively $\iota' : M_G \rightarrow M_{G_+}$) be the morphism induced by the inclusion $\{+\} \subset G_+$ (resp. $G \subset G_+$). Let \mathcal{C} be a category. There is a notion of G -stability for functors $G\text{Alg}_\ell \rightarrow \mathcal{C}$; see [15, Section 3.1] for details. It turns out that any G -stable functor sends the morphisms

$$B \xrightarrow{\iota} M_{G_+} \otimes B \xleftarrow{\iota'} M_G \otimes B \tag{2.13}$$

to isomorphisms, for every $B \in G\text{Alg}_\ell$.

Remark 2.14. For any G -algebra B , we have an isomorphism $R_{S,B} : (M_S \otimes B) \rtimes G \rightarrow M_{|S|} \otimes (B \rtimes G)$ defined by

$$R_{S,B}((e_{s,t} \otimes b) \rtimes g) = e_{s, g^{-1}t} \otimes (b \rtimes g). \tag{2.15}$$

This isomorphism is natural in S with respect to injective morphisms of G -sets and natural in B with respect to morphisms of G -algebras.

2.3.3. Excision. An *extension* of algebras (respectively of G -algebras) is a short exact sequence

$$\mathcal{E} : A \longrightarrow B \begin{array}{c} \longleftarrow \text{---} \\ \longrightarrow \text{---} \end{array} C \tag{2.16}$$

that splits in the category of ℓ -modules (resp. G -modules). Let (\mathcal{T}, Ω) be a triangulated category. A functor $F : \mathcal{C} \rightarrow \mathcal{T}$ is *excisive* [9, Section 6.6] if it associates to every extension (2.16) a morphism $\partial_{\mathcal{E}} : \Omega F(C) \rightarrow F(A)$ that fits into a triangle as follows:

$$\Omega F(C) \xrightarrow{\partial_{\mathcal{E}}} F(A) \longrightarrow F(B) \longrightarrow F(C).$$

These triangles are required to be natural with respect to morphisms of extensions.

2.3.4. Bivariant K -theory categories. Let X be an infinite set. There exists a triangulated category kk (see [9] for $X = \mathbb{N}$ and [27] for general X) endowed with a functor

$$j : \text{Alg}_\ell \rightarrow kk$$

that is homotopy invariant, M_X -stable and excisive. This functor j is moreover universal in the following sense: any functor $\text{Alg}_\ell \rightarrow (\mathcal{T}, \Omega)$ that is homotopy invariant, M_X -stable and excisive factors uniquely through j [9, Theorem 6.6.2]. By a theorem of Cortiñas and Thom [9, Main Theorem], homotopy K -theory is representable in kk since we have

$$kk_*(\ell, B) \cong \text{KH}_*(B)$$

for every $B \in \text{Alg}_\ell$. Here $kk_n(A, B)$ is defined as $kk(j(A), \Omega^n j(B))$. This representability theorem is a major computation of [9].

Let G be a group. There exists a triangulated category kk^G (see [15] for countable G and [27] for general G) endowed with a functor

$$j^G : G\text{Alg}_\ell \rightarrow kk^G$$

that is homotopy invariant, G -stable and excisive. This functor j^G is moreover universal in the following sense: any functor $G\text{Alg}_\ell \rightarrow (\mathcal{T}, \Omega)$ that is homotopy invariant, G -stable and excisive factors uniquely through j^G [15, Theorem 4.1.1].

Throughout this paper, by kk we shall mean the universal $M_{\mathbb{N} \times |G|}$ -stable kk -theory.

Example 2.17. Let B be a G -algebra. By G -stability, the morphisms (2.13) induce an isomorphism $j^G(B) \cong j^G(M_G B)$ in kk^G . To ease notation we often omit j^G and write $B \cong M_G B$. By homotopy invariance, we have isomorphisms $B \cong B^{\Delta^n}$ in kk^G , since B^{Δ^n} is isomorphic to the algebra of polynomials in n variables with coefficients in B .

3. Adjoint theorems revisited

Let G be a group and let $H \subseteq G$ be a finite subgroup whose order n is invertible in ℓ . Recall the notation and conventions from Example 2.4. By [15, Theorem 5.2.1] and [15, Theorem 6.14] we have an adjunction isomorphism

$$kk^G(A^{(G/H)}, B) \cong kk(A, B \rtimes H) \tag{3.1}$$

for any $A \in \text{Alg}_\ell$ and any $B \in G\text{Alg}_\ell$. For $A \in \text{Alg}_\ell$, let $\theta_A : A \rightarrow A^{(G/H)} \rtimes H$ be the algebra homomorphism defined by

$$\theta_A(a) = a\chi_H \rtimes \frac{1}{n} \sum_{h \in H} h.$$

Put $\eta_A = j(\theta_A) \in kk(A, A^{(G/H)} \rtimes H)$. We will show that η_A is a unit for the adjunction (3.1). For $B \in G\text{Alg}_\ell$, let $\rho_B : (B \rtimes H)^{(G/H)} \rightarrow M_G \otimes B$ be the G -algebra homomorphism defined by

$$\rho_B((b \rtimes h)\chi_{wH}) = \sum_{p \in wH} e_{p,ph} \otimes (p \cdot b). \tag{3.2}$$

Let $\varepsilon_B \in kk^G((B \rtimes H)^{(G/H)}, B)$ be the following composite in kk^G , where the isomorphism on the right is given by the zig-zag (2.13):

$$(B \rtimes H)^{(G/H)} \xrightarrow{j^G(\rho_B)} M_G \otimes B \cong B. \tag{3.3}$$

We will show that ε_B is a counit for the adjunction (3.1).

Lemma 3.4. *For any $B \in G\text{Alg}_\ell$ we have $(\varepsilon_B \rtimes H) \circ \eta_{B \rtimes H} = \text{id}_{B \rtimes H}$ in kk .*

Proof. It is easily verified that the following diagram in kk commutes, where all the arrows are isomorphisms:

$$\begin{array}{ccc}
 (M_G \otimes B) \rtimes H & \xrightarrow{(\iota')_*} & (M_{G_+} \otimes B) \rtimes H \xleftarrow{\iota_*} B \rtimes H \\
 R_{G,B} \downarrow (2.15) & & R_{G_+,B} \downarrow (2.15) \\
 M_{|G|} \otimes (B \rtimes H) & \xrightarrow{(\iota')_*} & M_{|G_+|} \otimes (B \rtimes H).
 \end{array}$$

Thus, the isomorphism $(M_G \otimes B) \rtimes H \cong B \rtimes H$ in kk induced by the zig-zag (2.13) equals the following composite:

$$(M_G \otimes B) \rtimes H \xrightarrow[\cong]{j(R_{G,B})} M_{|G|} \otimes (B \rtimes H) \xleftarrow[\cong]{j(e_{1,1} \otimes -)} B \rtimes H.$$

To prove the lemma, it will be enough to show that the composite

$$B \rtimes H \xrightarrow{\theta_{B \rtimes H}} (B \rtimes H)^{(G/H)} \rtimes H \xrightarrow{\rho_{B \rtimes H}} (M_G \otimes B) \rtimes H \xrightarrow{R_{G,B}} M_{|G|} \otimes (B \rtimes H)$$

and the inclusion $e_{1,1} \otimes - : B \rtimes H \rightarrow M_{|G|} \otimes (B \rtimes H)$ induce the same morphism in kk . Let Γ be the algebra of matrices with coefficients in $\tilde{B} \rtimes G$ indexed by $G \times G$ that have only finitely many nonzero coefficients in each column and each row. Notice that Γ is a unital algebra that contains $M_{|G|} \otimes (B \rtimes H)$ as a subalgebra. Put

$$V = \sum_{g \in G} e_{g,g} \otimes (1 \rtimes g) \in \Gamma. \tag{3.5}$$

We have

$$\begin{aligned}
 [R_{G,B} \circ (\rho_{B \rtimes H}) \circ \theta_{B \rtimes H}](b \rtimes h) &= \frac{1}{n} \sum_{p,q \in H} e_{p,q} \otimes ((p \cdot b) \rtimes phq^{-1}) \\
 &= V \left(\frac{1}{n} \sum_{p,q \in H} e_{p,q} \otimes (b \rtimes h) \right) V^{-1}.
 \end{aligned}$$

Moreover, $\frac{1}{n} \sum_{p,q \in H} e_{p,q}$ is a conjugate of $e_{1,1}$ in $M_{|G|}$; see [15, Remark 3.1.11]. By [7, Proposition 2.2.6] we have

$$j[R_{G,B} \circ (\rho_{B \rtimes H}) \circ \theta_{B \rtimes H}] = j(e_{1,1} \otimes -) : B \rtimes H \rightarrow M_{|G|} \otimes (B \rtimes H)$$

as we wanted to prove. ■

Lemma 3.6. *For any $A \in \text{Alg}_\ell$ we have $\varepsilon_{A^{(G/H)}} \circ [(\eta_A)^{(G/H)}] = \text{id}_{A^{(G/H)}}$ in kk^G .*

Proof. It is easily verified that the composite

$$A^{(G/H)} \xrightarrow{(\eta_A)^{(G/H)}} (A^{(G/H)} \rtimes H)^{(G/H)} \xrightarrow{\varepsilon_{A^{(G/H)}}} M_G \otimes A^{(G/H)} \xrightarrow{\iota'} M_{G_+} \otimes A^{(G/H)}$$

is given by

$$a\chi_{wH} \mapsto \frac{1}{n} \sum_{p,q \in H} e_{wp,wq} \otimes a\chi_{wH}. \tag{3.7}$$

To prove the lemma, it suffices to show that the above formula induces the same morphism as $\iota : A^{(G/H)} \rightarrow M_{G_+} \otimes A^{(G/H)}$, $\iota(a\chi_{wH}) = e_{+,+} \otimes a\chi_{wH}$, upon applying $j^G : G\text{Alg}_\ell \rightarrow kk^G$. Let \tilde{A} be the unitalization of A and let $\Gamma_{G_+}(\tilde{A})$ be the set of those matrices with coefficients in \tilde{A} , indexed over $G_+ \times G_+$, that have finitely many nonzero coefficients in each row and each column. Then $\Gamma_{G_+}(\tilde{A})$ is a G -algebra with the usual matrix multiplication and the G -action defined by $(g \cdot a)_{x,y} := a_{g^{-1}x, g^{-1}y}$. Moreover, we have inclusions of G -algebras as follows:

$$M_{G_+} \otimes A^{(G/H)} \subseteq (M_{G_+} \otimes A)^{G/H} \subseteq (\Gamma_{G_+}(\tilde{A}))^{G/H}.$$

The G -action on the right is described by

$$(g \cdot f)(tH) := g \cdot f(g^{-1}tH),$$

where $f : G/H \rightarrow \Gamma_{G_+}(\tilde{A})$ is a function and $g \in G$. We will show that there exists an invertible $V \in (\Gamma_{G_+}(\tilde{A}))^{G/H}$ such that the following diagram commutes, where the horizontal morphism is given by (3.7):

$$\begin{array}{ccc} A^{(G/H)} & \longrightarrow & M_{G_+} \otimes A^{(G/H)} \\ & \searrow \iota & \downarrow \phi^V \\ & & M_{G_+} \otimes A^{(G/H)}. \end{array} \tag{3.8}$$

Once this is done, the result will follow by Lemma 2.12. For $wH \in G/H$, define

$$V_{wH} := \sum_{x \in wH} \left(\frac{n-1}{n} e_{x,x} + e_{+,x} + e_{x,+} \right) - \sum_{x,y \in wH} \frac{1}{n} e_{x,y} + \sum_{g \in G \setminus wH} e_{g,g} \in \Gamma_{G_+}(\tilde{A}). \tag{3.9}$$

Note that $g \cdot V_{wH} = V_{g wH}$ for all $g \in G$. We can picture V_{wH} as a block diagonal matrix having an identity block in the coordinates corresponding to $g \in G \setminus wH$, and the following block in the coordinates corresponding to elements of $wH \cup \{+\}$:

$$wH \left\{ \begin{array}{c} \overbrace{\left(\begin{array}{ccccc|c} \frac{n-1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} & -\frac{1}{n} & 1 \\ -\frac{1}{n} & \frac{n-1}{n} & \dots & -\frac{1}{n} & -\frac{1}{n} & 1 \\ \vdots & & \ddots & & \vdots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \dots & \frac{n-1}{n} & -\frac{1}{n} & 1 \\ -\frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} & \frac{n-1}{n} & 1 \\ \hline 1 & 1 & \dots & 1 & 1 & 0 \end{array} \right)}^{wH} \end{array} \right. .$$

It is easily verified that V_{gH} is invertible with inverse given by

$$V_{wH}^{-1} := \sum_{x \in wH} \left(\frac{n-1}{n} e_{x,x} + \frac{1}{n} e_{+,x} + \frac{1}{n} e_{x,+} \right) - \sum_{x,y \in wH} \frac{1}{n} e_{x,y} + \sum_{g \in G \setminus wH} e_{g,g} \in \Gamma_{G_+}(\tilde{A}).$$

Again, we can think of V_{wH}^{-1} as a block diagonal matrix having an identity block in the coordinates corresponding to $g \in G \setminus wH$, and the following block in the coordinates corresponding to elements of $wH \cup \{+\}$:

$$wH \left\{ \begin{array}{ccccc|c} & \overbrace{\hspace{10em}}^{wH} & & & & \\ \left(\begin{array}{cccc|c} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} & \frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} & \frac{1}{n} \\ \vdots & & \ddots & & \vdots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} & -\frac{1}{n} & \frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{n-1}{n} & \frac{1}{n} \\ \hline \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & 0 \end{array} \right) & & & & & \end{array} \right.$$

Define $V^{\pm 1} : G/H \rightarrow \Gamma_{G_+}(\tilde{A})$ by $V^{\pm 1}(wH) := V_{wH}^{\pm 1}$. Then $V, V^{-1} \in (\Gamma_{G_+}(\tilde{A}))^{G/H}$ are mutual inverses and we have

$$V(M_{G_+} \otimes A^{(G/H)}), (M_{G_+} \otimes A^{(G/H)})V^{-1} \subseteq (M_{G_+} \otimes A^{(G/H)}).$$

Moreover, $g \cdot V = V$ for all $g \in G$. An easy computation shows that the triangle (3.8) commutes. This finishes the proof. ■

Proposition 3.10 (cf. [15, Theorems 5.2.1 and 6.14]). *Let G be a group and let $H \subseteq G$ be a finite subgroup whose order is invertible in ℓ . Then the morphisms*

$$\eta_A \in kk(A, A^{(G/H)} \rtimes H) \quad \text{and} \quad \varepsilon_B \in kk^G((B \rtimes H)^{(G/H)}, B)$$

defined above are respectively the unit and the counit of an adjunction

$$kk^G(A^{(G/H)}, B) \cong kk(A, B \rtimes H). \tag{3.11}$$

Proof. It follows immediately from Lemmas 3.4 and 3.6. ■

4. Crossed product with G/H

Let $\text{Or}(G, \mathcal{F}in)$ be the full subcategory of $\text{Or}(G)$ (see Definition 2.5) whose objects are those G/H with finite H . We want to show that the adjunction (3.11) is natural in G/H . The first thing to do is to replace the right-hand side of (3.11) by an actual functor on $\text{Or}(G, \mathcal{F}in)$.

4.1. Non-unital linear categories

Definition 4.1. A non-unital ℓ -linear category \mathcal{C} consists of:

- (1) a set of objects $\text{ob } \mathcal{C}$,
- (2) an ℓ -module $\mathcal{C}(x, y)$ for every $x, y \in \text{ob } \mathcal{C}$, and
- (3) ℓ -module homomorphisms

$$\circ : \mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \rightarrow \mathcal{C}(x, z) \tag{4.2}$$

for every $x, y, z \in \text{ob } \mathcal{C}$, that are associative in the obvious way.

Non-unital ℓ -linear categories with only one object can be identified with (non-unital) algebras. In the sequel, we will refer to non-unital ℓ -linear categories simply as *linear categories*.

Definition 4.3. Let \mathcal{C} and \mathcal{D} be linear categories. A *linear functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of a function $F : \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{D}$ together with ℓ -module homomorphisms

$$F_{x,y} : \mathcal{C}(x, y) \rightarrow \mathcal{D}(F(x), F(y)) \tag{4.4}$$

that are compatible with the composition.

We will write Cat_ℓ for the category whose objects are linear categories and whose morphisms are linear functors.

Example 4.5. Let B be a G -algebra and let $H \subseteq G$ be a subgroup. We proceed to define a linear category $B \rtimes G/H$. The objects of $B \rtimes G/H$ are the elements of G/H and the hom-modules are given by

$$(B \rtimes G/H)(uH, vH) := B \otimes \ell[vHu^{-1}].$$

The composition law in $B \rtimes G/H$ is defined as follows:

$$\begin{aligned} (B \rtimes G/H)(vH, wH) \otimes (B \rtimes G/H)(uH, vH) &\xrightarrow{\circ} (B \rtimes G/H)(uH, wH) \\ (b \otimes g) \otimes (\tilde{b} \otimes \tilde{g}) &\longmapsto b(g \cdot \tilde{b}) \otimes g\tilde{g}. \end{aligned}$$

In the terminology of [8], $B \rtimes G/H$ is the crossed product of B with the transport groupoid of G/H . It is easily verified that $B \rtimes G/H$ depends covariantly on B and on G/H .

Let \mathcal{C} and \mathcal{D} be linear categories. The *tensor product* $\mathcal{C} \otimes \mathcal{D}$ is the linear category with objects $\text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{D})$ and such that

$$(\mathcal{C} \otimes \mathcal{D})((c, d), (\tilde{c}, \tilde{d})) = \mathcal{C}(c, \tilde{c}) \otimes \mathcal{D}(d, \tilde{d}).$$

Composition is defined in the usual way, using the composition laws in \mathcal{C} and \mathcal{D} and the commutativity of the tensor product of ℓ -modules.

We proceed to recall some definitions from [8, Section 3]. Let \mathcal{C} be a linear category. Put

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{x,y \in \text{ob } \mathcal{C}} \mathcal{C}(x, y).$$

If $f \in \mathcal{A}(\mathcal{C})$, write $f_{y,x}$ for its component in $\mathcal{C}(x, y)$. Then $\mathcal{A}(\mathcal{C})$ is an algebra with multiplication given by

$$(gf)_{y,x} = \sum_{z \in \text{ob } \mathcal{C}} g_{y,z} \circ f_{z,x}.$$

Example 4.6. Let \mathcal{C} and \mathcal{D} be linear categories. It is easily verified that

$$\mathcal{A}(\mathcal{C} \otimes \mathcal{D}) \cong \mathcal{A}(\mathcal{C}) \otimes \mathcal{A}(\mathcal{D}).$$

Example 4.7. Let B be a G -algebra and $H \subseteq G$ be a subgroup. We can view $\mathcal{A}(B \rtimes G/H)$ as a subalgebra of $M_{|G/H|}(B \rtimes G)$ via the inclusion that sends $b \otimes g \in (B \rtimes G/H)(uH, vH)$ to $e_{vH, uH} \otimes (b \rtimes g)$.

A disadvantage of $\mathcal{A}(\mathcal{C})$ is that it is not natural with respect to all linear functors, but only with respect to those that are injective on objects; see [8, p. 1231]. To fix this, one defines the algebra $\mathcal{R}(\mathcal{C})$ [8, Section 3.4]. If M is an ℓ -module, write $T(M) = \bigoplus_{n \geq 1} M^{\otimes n}$ for the unaugmented tensor algebra. Put

$$\mathcal{R}(\mathcal{C}) := T(\mathcal{A}(\mathcal{C})) / \langle \{g \otimes f - g \circ f : f \in \mathcal{C}(x, y), g \in \mathcal{C}(y, z), x, y, z \in \text{ob } \mathcal{C}\} \rangle.$$

This defines a functor $\mathcal{R} : \text{Cat}_\ell \rightarrow \text{Alg}_\ell$.

Remark 4.8. Let G be a group. One can define a G -category as a linear category \mathcal{C} such that the hom-modules $\mathcal{C}(x, y)$ are G -modules and the composition law (4.2) is G -equivariant, endowing $\mathcal{C}(y, z) \otimes \mathcal{C}(x, y)$ with the diagonal G -action. If \mathcal{C} and \mathcal{D} are G -categories, a G -functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a linear functor such that the morphisms (4.4) are G -equivariant. These definitions give rise to a category $G\text{Cat}_\ell$ whose objects are G -categories and whose morphisms are G -functors.

If \mathcal{C} is a G -category, then $\mathcal{A}(\mathcal{C})$ and $\mathcal{R}(\mathcal{C})$ are G -algebras in a natural way. Thus, we have a functor $\mathcal{R} : G\text{Cat}_\ell \rightarrow G\text{Alg}_\ell$.

Example 4.9. Let \mathcal{C} be a linear category and let D be an algebra. We claim that there is a natural morphism

$$\mathcal{R}(\mathcal{C} \otimes D) \rightarrow \mathcal{R}(\mathcal{C}) \otimes D.$$

To see this, first note that there is a linear functor $\mathcal{C} \rightarrow \mathcal{R}(\mathcal{C})$ that takes $f \in \mathcal{C}(x, y)$ to the class in $\mathcal{R}(\mathcal{C})$ of $f \in \mathcal{C}(x, y) \subseteq \mathcal{A}(\mathcal{C}) \subseteq T(\mathcal{A}(\mathcal{C}))$. Upon tensoring this functor with D and then applying $\mathcal{R}(-)$, we get the desired morphism

$$\mathcal{R}(\mathcal{C} \otimes D) \rightarrow \mathcal{R}(\mathcal{R}(\mathcal{C}) \otimes D) = \mathcal{R}(\mathcal{C}) \otimes D.$$

If \mathcal{C} is a G -category and D is a G -algebra, then this is a morphism of G -algebras.

For a linear category \mathcal{C} , there is a morphism $p : \mathcal{R}(\mathcal{C}) \rightarrow \mathcal{A}(\mathcal{C})$ induced by multiplication in $\mathcal{A}(\mathcal{C})$.

Lemma 4.10 ([8, Lemma 3.4.3]). *Let \mathcal{C} be a linear category (respectively a G -category). Then the morphism*

$$p : \mathcal{R}(\mathcal{C}) \rightarrow \mathcal{A}(\mathcal{C})$$

induces an isomorphism in kk (resp. in kk^G).

Proof. The proof of [8, Lemma 3.4.3] carries on verbatim in this setting. ■

Corollary 4.11. *Let $\mathcal{C} \in \text{Cat}_\ell$ and $D \in \text{Alg}_\ell$ (resp. $\mathcal{C} \in G\text{Cat}_\ell$ and $D \in G\text{Alg}_\ell$). Then the morphism*

$$\mathcal{R}(\mathcal{C} \otimes D) \rightarrow \mathcal{R}(\mathcal{C}) \otimes D$$

of Example 4.9 is an isomorphism in kk (resp. in kk^G).

Proof. It is easily verified that the following diagram commutes in Alg_ℓ (resp. in $G\text{Alg}_\ell$):

$$\begin{array}{ccc} \mathcal{R}(\mathcal{C} \otimes D) & \longrightarrow & \mathcal{R}(\mathcal{C}) \otimes D \\ p \downarrow & & \downarrow p \otimes D \\ \mathcal{A}(\mathcal{C} \otimes D) & \xrightarrow{\cong} & \mathcal{A}(\mathcal{C}) \otimes D. \end{array}$$

Indeed, it suffices to check commutativity on the generators $f \otimes d$, and this is immediate. The result follows from Lemma 4.10. ■

4.2. Crossed product with G/H

Fix $G/H \in \text{Or}(G)$ and recall the definition of $B \rtimes G/H$ from Example 4.5. We claim that the composite functor

$$G\text{Alg}_\ell \xrightarrow{- \rtimes G/H} \text{Cat}_\ell \xrightarrow{\mathcal{R}} \text{Alg}_\ell \xrightarrow{j} kk \tag{4.12}$$

factors through $j^G : G\text{Alg}_\ell \rightarrow kk^G$. To prove this, it suffices to show that it is excisive, homotopy invariant and G -stable [15, Theorem 4.1.1]. Homotopy invariance and G -stability follow easily from the following.

Lemma 4.13. *Let G be a group, let $H \subseteq G$ be a subgroup and let $uH \in G/H$. Then there is a natural isomorphism*

$$\nu_{uH} : j(- \rtimes uHu^{-1}) \xrightarrow{\cong} j\mathcal{R}(- \rtimes G/H) \tag{4.14}$$

of functors $G\text{Alg}_\ell \rightarrow kk$.

Proof. Let $B \in G\text{Alg}_\ell$. Consider $B \rtimes uHu^{-1} \subseteq B \rtimes G/H$ as the full subcategory whose only object is uH . Upon applying \mathcal{R} to this inclusion, we get an algebra homomorphism

$$B \rtimes uHu^{-1} = \mathcal{R}(B \rtimes uHu^{-1}) \longrightarrow \mathcal{R}(B \rtimes G/H).$$

Let v_{uH} be the image of this morphism in kk . Clearly, v_{uH} is a natural transformation $j(- \rtimes uHu^{-1}) \rightarrow j\mathcal{R}(- \rtimes G/H)$. To finish the proof, we will show that v_{uH} is an isomorphism. We claim that there is an isomorphism α that fits in the following commutative diagram in kk :

$$\begin{array}{ccc}
 B \rtimes uHu^{-1} & \xrightarrow{v_{uH}} & \mathcal{R}(B \rtimes G/H) \\
 & \searrow^{e_{uH, uH} \otimes -} & \cong \downarrow p \\
 & & \mathcal{A}(B \rtimes G/H) \\
 & & \cong \downarrow \alpha \\
 & & M_{|G/H|}(B \rtimes uHu^{-1}).
 \end{array}$$

Here, the bent arrow is induced by the inclusion into the (uH, uH) -coefficient and it is an isomorphism by matrix invariance. It follows that v_{uH} is an isomorphism too. The isomorphism α is constructed as in the proof of [8, Lemma 3.2.6]. More precisely, let $s : G/H \rightarrow G$ be a section of the projection such that $s(uH) = u$. Write $\hat{g} = s(gH)$ for $g \in G$. For $b \otimes g \in (B \rtimes G/H)(sH, tH)$ put

$$\alpha(b \otimes g) := e_{tH, sH} \otimes ((u\hat{t}^{-1}) \cdot b) \rtimes u\hat{t}^{-1}g\hat{s}u^{-1}.$$

It is straightforward to verify that this formula defines an isomorphism of algebras $\alpha : \mathcal{A}(B \rtimes G/H) \rightarrow M_{|G/H|}(B \rtimes uHu^{-1})$. ■

Corollary 4.15 (cf. [15, Proposition 5.1.2 and Section 6]). *Let G be a group and let $H \subseteq G$ be a subgroup. Then the functor*

$$j\mathcal{R}(- \rtimes G/H) : G\text{Alg}_\ell \rightarrow kk$$

is homotopy invariant and G -stable.

Proof. Write $F : G\text{Alg}_\ell \rightarrow kk$ for the functor $F = j(- \rtimes H)$. By Lemma 4.13 it suffices to show that F is homotopy invariant and G -stable. The functor $- \rtimes H : G\text{Alg}_\ell \rightarrow \text{Alg}_\ell$ is easily seen to send homotopic morphisms in $G\text{Alg}_\ell$ to homotopic morphisms in Alg_ℓ . It follows that F is homotopy invariant. Recall the definition of G -stable functor from [15, Section 3.1]. Let (\mathcal{W}_1, B_1) and (\mathcal{W}_2, B_2) be G -modules by locally finite automorphisms such that $\text{card}(B_i) \leq \text{card}(\mathbb{N} \times G)$ for $i = 1, 2$ and let A be a G -algebra. Then the inclusion

$$(A \otimes \text{End}_\ell^F(\mathcal{W}_1)) \rtimes H \xrightarrow{\tilde{\tau}} (A \otimes \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2)) \rtimes H$$

is identified with

$$(A \rtimes H) \otimes \text{End}_\ell^F(\mathcal{W}_1) \xrightarrow{\tilde{\tau}} (A \rtimes H) \otimes \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \quad \text{by [15, Proposition 5.1.1].}$$

The latter becomes an isomorphism upon applying j by $M_{\mathbb{N} \times |G|}$ -stability of kk . It follows that F is G -stable. ■

We still have to show that the composite of the functors in (4.12) is excisive.

Construction 4.16. Let $\mathcal{E} : A \rightarrow B \rightarrow C$ be an extension in $G\text{Alg}_\ell$ (that splits in $G\text{Mod}_\ell$). We will to construct a triangle in kk as follows, that is natural with respect to morphisms of extensions:

$$\Omega(\mathcal{R}(C \rtimes G/H)) \xrightarrow{\partial_{\mathcal{E} \rtimes G/H}} \mathcal{R}(A \rtimes G/H) \longrightarrow \mathcal{R}(B \rtimes G/H) \longrightarrow \mathcal{R}(C \rtimes G/H).$$

Let $v_H : j(- \rtimes H) \rightarrow j\mathcal{R}(- \rtimes G/H)$ be the natural isomorphism (4.14). Note that $j(- \rtimes H)$ is excisive, since $- \rtimes H : G\text{Alg}_\ell \rightarrow \text{Alg}_\ell$ preserves extensions. To simplify notation, we omit explicit mention to j for the rest of the proof. We have the following commutative diagram of solid arrows in kk , where the top row is a triangle:

$$\begin{array}{ccccccc} \Omega(C \rtimes H) & \xrightarrow{\partial_{\mathcal{E} \rtimes H}} & A \rtimes H & \longrightarrow & B \rtimes H & \longrightarrow & C \rtimes H \\ \cong \downarrow \Omega(v_H) & & \cong \downarrow v_H & & \cong \downarrow v_H & & \cong \downarrow v_H \\ \Omega(\mathcal{R}(C \rtimes G/H)) & \dashrightarrow & \mathcal{R}(A \rtimes G/H) & \longrightarrow & \mathcal{R}(B \rtimes G/H) & \longrightarrow & \mathcal{R}(C \rtimes G/H). \end{array}$$

Define $\partial_{\mathcal{E} \rtimes G/H} : \Omega(\mathcal{R}(C \rtimes G/H)) \rightarrow \mathcal{R}(A \rtimes G/H)$ to be the dashed arrow that makes the left square commute. Then the bottom row becomes a triangle too. This triangle is clearly natural with respect to the extension \mathcal{E} . These morphisms $\partial_{\mathcal{E} \rtimes G/H}$ make $j\mathcal{R}(- \rtimes G/H) : G\text{Alg}_\ell \rightarrow kk$ into an excisive homology theory.

Proposition 4.17. *Let G be a group and let $H \subseteq G$ be a subgroup. Then there exists a unique triangulated functor $\overline{- \rtimes G/H} : kk^G \rightarrow kk$ making the following diagram commute:*

$$\begin{array}{ccc} G\text{Alg}_\ell & \xrightarrow{\mathcal{R}(- \rtimes G/H)} & \text{Alg}_\ell \\ j^G \downarrow & & \downarrow j \\ kk^G & \xrightarrow{\overline{- \rtimes G/H}} & kk. \end{array}$$

Moreover, for every extension $\mathcal{E} : A \rightarrow B \rightarrow C$ in $G\text{Alg}_\ell$ and every $uH \in G/H$, the following square in kk commutes:

$$\begin{array}{ccc} \Omega(C \rtimes uHu^{-1}) & \xrightarrow{\partial_{\mathcal{E} \rtimes uHu^{-1}}} & A \rtimes uHu^{-1} \\ \Omega(v_{uH}) \downarrow \cong_{(4.14)} & & \cong_{(4.14)} \downarrow v_{uH} \\ \Omega(\mathcal{R}(C \rtimes G/H)) & \xrightarrow{\partial_{\mathcal{E} \rtimes G/H}} & \mathcal{R}(A \rtimes G/H). \end{array} \tag{4.18}$$

Proof. The functor

$$j\mathcal{R}(- \rtimes G/H) : G\text{Alg}_\ell \rightarrow kk$$

is homotopy invariant and G -stable by Corollary 4.15. Moreover, endowed with the morphisms $\partial_{\mathcal{E} \rtimes G/H}$ defined in Construction 4.16, it becomes an excisive homology theory. Then the existence of $\overline{- \rtimes G/H} : kk^G \rightarrow kk$ follows from [15, Theorem 4.1.1].

To prove the assertion about (4.18), consider the following diagram in kk :

$$\begin{array}{ccc}
 \Omega(C \rtimes uHu^{-1}) & \xrightarrow{\partial_{\mathcal{E} \rtimes uHu^{-1}}} & A \rtimes uHu^{-1} \\
 \Omega(v_{uH}) \downarrow \cong & & \cong \downarrow v_{uH} \\
 \Omega(\mathcal{R}(C \rtimes G/H)) & \xrightarrow{\partial_{\mathcal{E} \rtimes G/H}} & \mathcal{R}(A \rtimes G/H) \\
 \Omega(v_H)^{-1} \downarrow \cong & & \cong \downarrow (v_H)^{-1} \\
 \Omega(C \rtimes H) & \xrightarrow{\partial_{\mathcal{E} \rtimes H}} & A \rtimes H.
 \end{array} \tag{4.19}$$

The bottom square commutes by definition of $\partial_{\mathcal{E} \rtimes G/H}$; see Construction 4.16. Thus, the commutativity of (4.18) is equivalent to that of the outer square in (4.19). We will see that the latter commutes since it is induced by a morphism of extensions in Alg_ℓ . For $D \in G\text{Alg}_\ell$, let $\varphi_u : D \rtimes uHu^{-1} \rightarrow D \rtimes H$ be the algebra homomorphism defined by $\varphi_u(a \rtimes g) = (u^{-1} \cdot a) \rtimes u^{-1}gu$. We have a morphism of extensions

$$\begin{array}{ccccc}
 A \rtimes uHu^{-1} & \longrightarrow & B \rtimes uHu^{-1} & \longrightarrow & C \rtimes uHu^{-1} \\
 \downarrow \varphi_u & & \downarrow \varphi_u & & \downarrow \varphi_u \\
 A \rtimes H & \longrightarrow & B \rtimes H & \longrightarrow & C \rtimes H
 \end{array}$$

that induces a morphism of triangles in kk . In particular, there is a commutative square in kk as follows:

$$\begin{array}{ccc}
 \Omega(C \rtimes uHu^{-1}) & \xrightarrow{\partial_{\mathcal{E} \rtimes uHu^{-1}}} & A \rtimes uHu^{-1} \\
 \Omega j(\varphi_u) \downarrow & & \downarrow j(\varphi_u) \\
 \Omega(C \rtimes H) & \xrightarrow{\partial_{\mathcal{E} \rtimes H}} & A \rtimes H.
 \end{array}$$

This square turns out to be the outer square of (4.19). To see this, it suffices to show that $j(\varphi_u) = (v_H)^{-1} \circ v_{uH}$ or, equivalently, that

$$j(p) \circ v_H \circ j(\varphi_u) = j(p) \circ v_{uH} \tag{4.20}$$

where $p : \mathcal{R}(D \rtimes G/H) \rightarrow \mathcal{A}(D \rtimes G/H)$ is the morphism of Lemma 4.10. Each side of (4.20) is induced by an algebra homomorphism $D \rtimes uHu^{-1} \rightarrow \mathcal{A}(D \rtimes G/H)$. We will show that both morphisms are conjugate in $M_{|G/H|}(D \rtimes G)$, regarding $\mathcal{A}(D \rtimes G/H)$ as a subalgebra of $M_{|G/H|}(D \rtimes G)$ with the inclusion of Example 4.7. A straightforward verification shows that the left- and the right-hand sides of (4.20) are induced, respectively, by the algebra homomorphisms λ and ρ defined by

$$\begin{aligned}
 \lambda(d \rtimes g) &:= e_{H,H} \otimes (u^{-1} \cdot d) \rtimes u^{-1}gu, \\
 \rho(d \rtimes g) &:= e_{uH,uH} \otimes d \rtimes g.
 \end{aligned}$$

Now put

$$V := \sum_{vH \in G/H} e_{uvH,vH} \otimes 1 \rtimes u \in \Gamma_{|G/H|}(\tilde{D} \rtimes G) \supset M_{|G/H|}(D \rtimes G).$$

$j \circ \mathcal{R} : \text{Cat}_\ell \rightarrow kk$ the vertical arrows become isomorphisms and we get

$$f_* = v_{uK} \circ j(\text{incl}) \circ (v_H)^{-1} : \mathcal{R}(A \rtimes G/H) \longrightarrow \mathcal{R}(A \rtimes G/K).$$

Thus, the commutativity of (4.22) is equivalent to that of the outer square in the following diagram in kk :

$$\begin{array}{ccc} \Omega(\mathcal{R}(C \rtimes G/H)) & \xrightarrow{\partial_{\mathcal{E} \rtimes G/H}} & \mathcal{R}(A \rtimes G/H) \\ \Omega(v_H)^{-1} \downarrow \cong & & \cong \downarrow (v_H)^{-1} \\ \Omega(C \rtimes H) & \xrightarrow{\partial_{\mathcal{E} \rtimes H}} & A \rtimes H \\ \Omega j(\text{incl}) \downarrow & & \downarrow j(\text{incl}) \\ \Omega(C \rtimes uKu^{-1}) & \xrightarrow{\partial_{\mathcal{E} \rtimes uKu^{-1}}} & A \rtimes uKu^{-1} \\ \Omega(v_{uK}) \downarrow \cong & & \cong \downarrow v_{uK} \\ \Omega(\mathcal{R}(C \rtimes G/K)) & \xrightarrow{\partial_{\mathcal{E} \rtimes G/K}} & \mathcal{R}(A \rtimes G/K). \end{array}$$

Here, the bottom and top squares commute by Proposition 4.17. The middle square commutes because it fits into the morphism of triangles induced by $H \subseteq uKu^{-1}$. ■

5. A natural adjunction

Let G be a group and let H be a finite subgroup of G . Recall from Lemma 4.13 that there is a natural isomorphism $v_H : - \rtimes H \rightarrow \overline{- \rtimes G/H}$ of functors $kk^G \rightarrow kk$. If $|H|$ is invertible in ℓ , we have isomorphisms

$$kk^G(A^{(G/H)}, B) \cong kk(A, B \rtimes H) \cong kk(A, \mathcal{R}(B \rtimes G/H)), \tag{5.1}$$

where the isomorphism on the right is induced by v_H and the one on the left is that of Proposition 3.10. In other words, there is an adjunction as follows:

$$(-)^{(G/H)} : kk \rightleftarrows kk^G : \overline{- \rtimes G/H}. \tag{5.2}$$

We will show that this adjunction is natural in G/H . For the rest of this paper, we assume that G satisfies the following property:

$$|H| \text{ is invertible in } \ell \text{ for every } H \in \mathcal{F}in. \tag{5.3}$$

We will prove that for every morphism $f : G/H \rightarrow G/K \in \text{Or}(G, \mathcal{F}in)$ the following square commutes:

$$\begin{array}{ccc} kk(A, \mathcal{R}(B \rtimes G/H)) & \xrightarrow{\cong} & kk^G(A^{(G/H)}, B) \\ f_* \downarrow & & \downarrow f_* \\ kk(A, \mathcal{R}(B \rtimes G/K)) & \xrightarrow{\cong} & kk^G(A^{(G/K)}, B). \end{array} \tag{5.4}$$

The commutativity of this square is not obvious a priori since the middle term of (5.1) is not a functor on $\text{Or}(G, \mathcal{F}in)$. Let $\alpha \in kk(A, \mathcal{R}(B \rtimes G/H))$ and $f : G/H \rightarrow G/K$ in $\text{Or}(G, \mathcal{F}in)$. The commutativity of (5.4) is equivalent to that of the outer square in the following diagram in kk , where $\epsilon_{G/H}$ is the counit of (5.2):

$$\begin{array}{ccccc}
 A^{(G/H)} & \xrightarrow{\alpha^{(G/H)}} & [\mathcal{R}(B \rtimes G/H)]^{(G/H)} & \xrightarrow{\epsilon_{G/H}} & B \\
 \uparrow A^{(f)} & & \uparrow f^* & & \uparrow \epsilon_{G/K} \\
 A^{(G/K)} & \xrightarrow{\alpha^{(G/K)}} & [\mathcal{R}(B \rtimes G/H)]^{(G/K)} & \xrightarrow{f_*} & [\mathcal{R}(B \rtimes G/K)]^{(G/K)}.
 \end{array}$$

The square on the left clearly commutes. The commutativity of the square on the right follows easily from the following result.

Lemma 5.5. *Let G be a group satisfying (5.3). Then:*

- (1) *For every $G/H \in \text{Or}(G, \mathcal{F}in)$ there is a G -functor*

$$\zeta_{G/H} : (B \rtimes G/H)^{(G/H)} \rightarrow M_G \otimes B \tag{5.6}$$

that sends $(b \otimes g)\chi_{sH} \in \text{Hom}(uH, vH) = B \otimes \ell[vHu^{-1}]^{(G/H)}$ to

$$\sum_{p \in sHv^{-1}} e_{p,pg} \otimes (p \cdot b).$$

- (2) *For any $f : G/H \rightarrow G/K$ in $\text{Or}(G, \mathcal{F}in)$, the following square in GCat_ℓ commutes:*

$$\begin{array}{ccc}
 (B \rtimes G/H)^{(G/H)} & \xrightarrow{\zeta_{G/H}} & M_G \otimes B \\
 \uparrow f^* & & \uparrow \zeta_{G/K} \\
 (B \rtimes G/H)^{(G/K)} & \xrightarrow{f_*} & (B \rtimes G/K)^{(G/K)}.
 \end{array} \tag{5.7}$$

- (3) *The counit $\epsilon_{G/H}$ of the adjunction (5.2) fits into the following commutative diagram in kk^G :*

$$\begin{array}{ccc}
 [\mathcal{R}(B \rtimes G/H)]^{(G/H)} & \xleftarrow[\cong]{\text{Corollary 4.11}} & \mathcal{R}[(B \rtimes G/H)^{(G/H)}] \\
 & \searrow \epsilon_{G/H} & \downarrow \mathcal{R}(\zeta_{G/H}) \\
 & & M_G \otimes B \\
 & & \cong \downarrow (2.13) \\
 & & B.
 \end{array}$$

Proof. Let us verify that $\zeta_{G/H}$ as defined in (1) is compatible with composition. Let $f_1 = (b_1 \otimes g_1)\chi_{sH} \in \text{Hom}(uH, vH)$ and $f_2 = (b_2 \otimes g_2)\chi_{tH} \in \text{Hom}(vH, wH)$. We have

$$f_2 \circ f_1 = \delta_{sH,tH}(b_2(g_2 \cdot b_1) \otimes g_2g_1)\chi_{tH}$$

where $\delta_{sH,tH}$ is Kronecker's delta. Then

$$\zeta_{G/H}(f_2 \circ f_1) = \delta_{sH,tH} \sum_{p \in tHw^{-1}} e_{p,pg_2g_1} \otimes p \cdot (b_2(g_2 \cdot b_1)).$$

On the other hand, we have

$$\begin{aligned} \zeta_{G/H}(f_2)\zeta_{G/H}(f_1) &= \left(\sum_{q \in tHw^{-1}} e_{q,qg_2} \otimes (q \cdot b_2) \right) \left(\sum_{p \in sHv^{-1}} e_{p,pg_1} \otimes (p \cdot b_1) \right) \\ &= \sum_{\substack{p \in sHv^{-1} \\ q \in tHw^{-1}}} e_{q,qg_2} e_{p,pg_1} \otimes (q \cdot b_2)(p \cdot b_1) \\ &= \delta_{sH,tH} \sum_{q \in tHw^{-1}} e_{q,qg_2g_1} \otimes (q \cdot b_2)((qg_2) \cdot b_1). \end{aligned}$$

The appearance of Kronecker's delta in the last line is explained as follows: $qg_2 \in tHv^{-1}$ and $p \in sHv^{-1}$ can be equal if and only if $sH = tH$. This shows that $\zeta_{G/H}$ is indeed a well-defined functor.

Let us prove (2). The square commutes on objects since $M_G \otimes B$ has only one object. A morphism $f : G/H \rightarrow G/K$ is determined by $f(H) = xK$ with x such that $H \subseteq xKx^{-1}$. Let $(b \otimes g)\chi_{sK} \in \text{Hom}(uH, vH)$. We have

$$\begin{aligned} \zeta_{G/K}(f_*((b \otimes g)\chi_{sK})) &= \zeta_{G/K}((b \otimes g)\chi_{sK}) \\ &= \sum_{p \in sK(vx)^{-1}} e_{p,pg} \otimes (p \cdot b). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \zeta_{G/H}(f^*((b \otimes g)\chi_{sK})) &= \zeta_{G/H} \left(\sum_{tH \in f^{-1}(sK)} (b \otimes g)\chi_{tH} \right) \\ &= \sum_{tH \in f^{-1}(sK)} \sum_{p \in tHv^{-1}} e_{p,pg} \otimes (p \cdot b) \\ &= \sum_{p \in sKx^{-1}v^{-1}} e_{p,pg} \otimes (p \cdot b). \end{aligned}$$

Here, the last equality follows from the fact that sKx^{-1} is the disjoint union of $f^{-1}(sK)$. This finishes the proof of (2).

Let us prove (3). It follows from (5.1) that the counit $\epsilon_{G/H}$ of the adjunction (5.2) equals $\epsilon_H \circ [(v_H)^{-1}]^{(G/H)}$, where ϵ_H is the counit of (3.11). Recall the definition of ϵ_H from (3.3). Then $\epsilon_{G/H}$ equals the following composite in kk :

$$[\mathcal{R}(B \rtimes G/H)]^{(G/H)} \xrightarrow{[(v_H)^{-1}]^{(G/H)}} (B \rtimes H)^{(G/H)} \xrightarrow[(3.2)]{\rho_B} M_G \otimes B \underset{(2.13)}{\cong} B.$$

The statement in (3) now follows from the commutativity of the following diagram of G -algebras:

$$\begin{array}{ccc}
 [\mathcal{R}(B \rtimes G/H)]^{(G/H)} & \xleftarrow{\text{Corollary 4.11}} & \mathcal{R}[(B \rtimes G/H)^{(G/H)}] \\
 \uparrow (v_H)^{(G/H)} & \nearrow & \downarrow \mathcal{R}(\zeta_{G/H}) \\
 (B \rtimes H)^{(G/H)} & \xrightarrow{\rho_B} & M_G \otimes B.
 \end{array}$$

Here the diagonal morphism is the one induced by the inclusion

$$(B \rtimes H)^{(G/H)} \subseteq (B \rtimes G/H)^{(G/H)}$$

as the full subcategory on the object H . This finishes the proof. ■

We collect the main results of this section in the next theorem.

Theorem 5.8. *Let G be a group satisfying (5.3) and let $H \subseteq G$ be a finite subgroup. Then we have an adjunction as follows:*

$$(-)^{(G/H)} : kk \rightleftarrows kk^G : \overline{- \rtimes G/H}.$$

Moreover, the adjunction isomorphism

$$kk^G(A^{(G/H)}, B) \cong kk(A, \mathcal{R}(B \rtimes G/H))$$

is natural in $G/H \in \text{Or}(G, \mathcal{F}in)$ and the counit of the adjunction is described by Lemma 5.5.

6. Lifting the adjunction to spectra

6.1. The primitive zig-zag

Let $A \in \text{Alg}_\ell$, $B \in \text{GAlg}_\ell$ and $G/N \in \text{Or}(G, \mathcal{F}in)$. By Theorem 5.8 we have an adjunction isomorphism

$$kk^G(A^{(G/N)}, B) \cong kk(A, \mathcal{R}(B \rtimes G/N)) \tag{6.1}$$

that is natural in G/N . Let \mathbb{K} and \mathbb{K}^G be, respectively, the spectra representing kk -theory and kk^G -theory. These were defined in [17] and in [27]; see Section C.2 for details. We would like to lift the isomorphism (6.1) to a natural weak equivalence of spectra

$$\mathbb{K}(A, \mathcal{R}(B \rtimes G/N)) \dashrightarrow \mathbb{K}^G(A^{(G/N)}, B).$$

Here, we want the dashed arrow to represent a zig-zag of $\text{Or}(G, \mathcal{F}in)$ -spectra inducing the isomorphism (6.1) upon taking homotopy groups. As a starting point, let us recall how to obtain this adjunction. By Lemma 5.5 (3), the isomorphism (6.1) equals the following

composite:

$$\begin{CD}
 kk(A, \mathcal{R}(B \rtimes G/N)) @>(-)^{(G/N)}>> kk^G(A^{(G/N)}, [\mathcal{R}(B \rtimes G/N)]^{(G/N)}) \\
 @. @VV\text{Corollary 4.11}\uparrow\mathbb{R}V \\
 kk^G(A^{(G/N)}, M_G B) @<<_{(5.6)}^{\mathcal{R}(\xi_{G/N})}<< kk^G(A^{(G/N)}, \mathcal{R}[(B \rtimes G/N)^{(G/N)}]) \\
 @VV\cong\downarrow V @. \\
 kk^G(A^{(G/N)}, M_{G_+} B) @<<_{\cong}<< kk^G(A^{(G/N)}, B).
 \end{CD} \tag{6.2}$$

The last two morphisms are easily lifted to spectra. Indeed, the G -stability zig-zag (2.13) induces the following zig-zag of weak equivalences that is clearly natural in G/N :

$$\mathbb{K}^G(A^{(G/N)}, M_G B) \xrightarrow{\sim} \mathbb{K}^G(A^{(G/N)}, M_{G_+} B) \xleftarrow{\sim} \mathbb{K}^G(A^{(G/N)}, B).$$

Lifting the rest of (6.2) is somewhat more delicate. If we simply replace groups by spectra, we get the following diagram:

$$\begin{CD}
 \mathbb{K}(A, \mathcal{R}(B \rtimes G/N)) @>(-)^{(G/N)}>> \mathbb{K}^G(A^{(G/N)}, [\mathcal{R}(B \rtimes G/N)]^{(G/N)}) \\
 @. @VV\text{Corollary 4.11}\uparrow\wr V \\
 \mathbb{K}^G(A^{(G/N)}, M_G B) @<<_{(5.6)}^{\mathcal{R}(\xi_{G/N})}<< \mathbb{K}^G(A^{(G/N)}, \mathcal{R}[(B \rtimes G/N)^{(G/N)}]).
 \end{CD} \tag{6.3}$$

This is what we call the *primitive zig-zag*. While the spectra on the left are covariant functors on $\text{Or}(G, \mathcal{F}in)$, this is not the case for those on the right. We should start by replacing the latter by covariant functors on $\text{Or}(G, \mathcal{F}in)$ if we expect a zig-zag that is natural in G/N . In the following sections – taking the primitive zig-zag as our model – we proceed to construct a zig-zag of spectra that depends covariantly on G/N and that induces the isomorphism (6.1) upon taking homotopy groups.

6.2. Notation and preliminary definitions

To ease notation, the category $\text{Or}(G, \mathcal{F}in)$ will be denoted by \mathcal{O} for the rest of this section. Its objects – the orbit spaces corresponding to finite subgroups of G – will be denoted by letters r, s and t .

Let \mathcal{C} be a category and Γ be a small category. We will write $B(\Gamma, \mathcal{C})$ for the category $\mathcal{C}^{\Gamma^{op} \times \Gamma}$ of bifunctors $\Gamma^{op} \times \Gamma \rightarrow \mathcal{C}$. Let $f : \Gamma \rightarrow \Lambda$ be a functor between small categories. Then we can restrict along f either of the variables of a bifunctor $\Lambda^{op} \times \Lambda \rightarrow \mathcal{C}$, or both of them, as shown by the following commutative diagram of categories:

$$\begin{CD}
 B(\Lambda, \mathcal{C}) @>f^*>> \mathcal{C}^{\Lambda^{op} \times \Gamma} \\
 @Vf^*VV @V\searrow f^*VV \\
 \mathcal{C}^{\Gamma^{op} \times \Lambda} @>f^*>> B(\Gamma, \mathcal{C}).
 \end{CD}$$

Here, f^* denotes restriction of one of the variables (either the covariant or the contravariant one) and f^\star denotes restriction of both variables.

Define functors $J \in \text{Sp}^\mathcal{O}$ and $M, L \in B(\mathcal{O}, \text{Sp})^\mathcal{O}$ by

$$\begin{aligned} J(t) &:= \mathbb{K}(A, \mathcal{R}(B \rtimes t)), \\ L_t(s, r) &:= \mathbb{K}^G(A^{(t)}, \mathcal{R}[(B \rtimes r)^{(s)}]), \\ M_t(s, r) &:= \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(s)}) \end{aligned} \tag{6.4}$$

for $r, s, t \in \mathcal{O}$. The kk^G -equivalence $\mathcal{R}[(B \rtimes r)^{(s)}] \rightarrow [\mathcal{R}(B \rtimes r)]^{(s)}$ of Corollary 4.11 induces, upon applying $\mathbb{K}^G(A^{(t)}, -)$, a natural transformation $\psi : L \rightarrow M$ that is an objectwise weak equivalence of spectra. With this notation, the primitive zig-zag (6.3) becomes

$$J(t) \xrightarrow{(-)^{(t)}} M_t(t, t) \xleftarrow{\sim} L_t(t, t) \xrightarrow{\mathcal{R}(\xi_t)} \mathbb{K}^G(A^{(t)}, M_G B). \tag{6.5}$$

6.3. Coends enter the game

Fix $t \in \mathcal{O}$. The commutativity of (5.7) suggests that the morphism induced by $\mathcal{R}(\xi_t)$ in (6.5) could be replaced by a morphism from a certain coend, as we proceed to explain. If $f : r \rightarrow s$ is a morphism in \mathcal{O} , the following square commutes by Lemma 5.5 (2):

$$\begin{array}{ccc} (B \rtimes r)^{(r)} & \xrightarrow{\xi_r} & M_G \otimes B \\ \uparrow f^* & & \uparrow \xi_s \\ (B \rtimes r)^{(s)} & \xrightarrow{f_*} & (B \rtimes s)^{(s)}. \end{array}$$

Upon applying $\mathbb{K}^G(A^{(t)}, \mathcal{R}(-))$, we get a commutative diagram as follows:

$$\begin{array}{ccc} L_t(r, r) & \xrightarrow{\xi_r} & \mathbb{K}^G(A^{(t)}, M_G B) \\ \uparrow f^* & & \uparrow \xi_s \\ L_t(s, r) & \xrightarrow{f_*} & L_t(s, s). \end{array} \tag{6.6}$$

For reasons that will become clear later on (see Remark 6.16 and Lemma 6.29) we will take coends over the slice category $\mathcal{O}_{/t}$ of orbit spaces over t . We will denote by u_t the forgetful functor $\mathcal{O}_{/t} \rightarrow \mathcal{O}$. Let now $f : \alpha \rightarrow \beta$ be a morphism in $\mathcal{O}_{/t}$, where $\alpha : r \rightarrow t$ and $\beta : s \rightarrow t$. Then (6.6) equals the following square:

$$\begin{array}{ccc} [(u_t)^\star L_t](\alpha, \alpha) & \xrightarrow{\xi_r} & \mathbb{K}^G(A^{(t)}, M_G B) \\ \uparrow f^* & & \uparrow \xi_s \\ [(u_t)^\star L_t](\beta, \alpha) & \xrightarrow{f_*} & [(u_t)^\star L_t](\beta, \beta). \end{array}$$

By the universal property of the coend, there is a unique morphism ζ making the following triangle commute for all objects $\alpha : r \rightarrow t$ of \mathcal{O}/t :

$$\begin{array}{ccc}
 \int^{\mathcal{O}/t} (u_t)^* L_t & \xrightarrow{\zeta} & \mathbb{K}^G(A^{(t)}, M_G B). \\
 \text{can}_\alpha \uparrow & & \nearrow \\
 [(u_t)^* L_t](\alpha, \alpha) & & \zeta_r
 \end{array} \tag{6.7}$$

Here the vertical morphism is the structural morphism into the coend corresponding to α . In the next section we will prove that ζ depends covariantly on $t \in \mathcal{O}$.

6.4. Defining \mathcal{O} -spectra as objectwise coends

Let us show that the morphisms ζ of (6.7) assemble, for varying t , into a morphism of \mathcal{O} -spectra. We first prove some preliminary lemmas.

Lemma 6.8. *Let \mathcal{C} be a cocomplete category, $f : \Gamma \rightarrow \Lambda$ be a functor between small categories and $T \in B(\Lambda, \mathcal{C})$. Then there is a unique morphism*

$$\int^\Gamma f^* T \longrightarrow \int^\Lambda T \tag{6.9}$$

making the following square in \mathcal{C} commute, for every object γ of Γ :

$$\begin{array}{ccc}
 \int^\Gamma f^* T & \longrightarrow & \int^\Lambda T \\
 \text{can}_\gamma \uparrow & & \uparrow \text{can}_{f(\gamma)} \\
 (f^* T)(\gamma, \gamma) & \equiv & T(f(\gamma), f(\gamma)).
 \end{array}$$

Here the vertical arrows are the structural morphisms into the coends. Moreover, (6.9) is natural in T .

Proof. This is immediate from the universal property of coends. ■

Remark 6.10. For composable functors $\Gamma \xrightarrow{f} \Lambda \xrightarrow{g} \Sigma$ and $T \in B(\Sigma, \mathcal{C})$, the morphisms (6.9) clearly fit into the following commutative triangle:

$$\begin{array}{ccc}
 \int^\Gamma (g \circ f)^* T = \int^\Gamma f^* g^* T & \longrightarrow & \int^\Lambda g^* T \\
 & \searrow & \downarrow \\
 & & \int^\Sigma T.
 \end{array}$$

Lemma 6.11. *Let \mathcal{C} be a cocomplete category and Γ be a small category. Then there is a functor $\mathcal{C} : B(\Gamma, \mathcal{C})^\Gamma \rightarrow \mathcal{C}^\Gamma$ described as follows.*

(1) Let V be an object of $B(\Gamma, \mathcal{C})^\Gamma$, $t \in \Gamma \mapsto V_t \in B(\Gamma, \mathcal{C})$. For $t \in \Gamma$, we have

$$\mathcal{C}(V)(t) = \int^{\Gamma/t} (u_t)^\star V_t.$$

For a morphism $f : t \rightarrow t'$ in Γ , $\mathcal{C}(V)(f)$ equals the composite

$$\int^{\Gamma/t} (u_t)^\star V_t \xrightarrow{V_f} \int^{\Gamma/t} (u_t)^\star V_{t'} = \int^{\Gamma/t} f^\star(u_{t'})^\star V_{t'} \xrightarrow{(6.9)} \int^{\Gamma/t'} (u_{t'})^\star V_{t'}.$$

(2) For a morphism $h : V \rightarrow W$ in $B(\Gamma, \mathcal{C})^\Gamma$, the natural transformation $\mathcal{C}(h)$ has the following components:

$$\int^{\Gamma/t} (u_t)^\star V_t \xrightarrow{\int^{\Gamma/t} (u_t)^\star h_t} \int^{\Gamma/t} (u_t)^\star W_t.$$

Proof. The fact that the equalities in (1) indeed define a functor $\mathcal{C}(V) \in \mathcal{C}^\Gamma$ boils down to the naturality of (6.9) and Remark 6.10. The fact that $\mathcal{C}(h)$ is indeed a natural transformation follows as well from the naturality of (6.9). ■

Lemma 6.12. *The morphism ζ defined by (6.7) is a morphism of \mathcal{O} -spectra.*

Proof. The codomain of ζ is clearly an \mathcal{O} -spectrum. Its domain is an \mathcal{O} -spectrum as well; indeed, it is $\mathcal{C}(t \mapsto L_t)$ where \mathcal{C} is the functor of Lemma 6.11. Let $f : t \rightarrow t'$ be a morphism in \mathcal{O} . We claim that following square commutes:

$$\begin{array}{ccc} \int^{\mathcal{O}/t} (u_t)^\star L_t & \xrightarrow{\zeta} & \mathbb{K}^G(A^{(t)}, M_G B) \\ f \downarrow & & \downarrow f \\ \int^{\mathcal{O}/t'} (u_{t'})^\star L_{t'} & \xrightarrow{\zeta} & \mathbb{K}^G(A^{(t')}, M_G B). \end{array}$$

Indeed, by the universal property of the coend, it suffices to show that the square commutes when precomposed with the structural morphisms

$$[(u_t)^\star L_t](\alpha, \alpha) \rightarrow \int^{\mathcal{O}/t} (u_t)^\star L_t$$

for every object $\alpha : r \rightarrow t$ of \mathcal{O}/t . Upon precomposing with the latter we get the following square, that clearly commutes:

$$\begin{array}{ccc} \mathbb{K}^G(A^{(t)}, \mathcal{R}[(B \rtimes r)^{(r)}]) & \xrightarrow{\zeta_r} & \mathbb{K}^G(A^{(t)}, M_G B) \\ f \downarrow & & \downarrow f \\ \mathbb{K}^G(A^{(t')}, \mathcal{R}[(B \rtimes r)^{(r)}]) & \xrightarrow{\zeta_r} & \mathbb{K}^G(A^{(t')}, M_G B). \end{array}$$

This proves the lemma. ■

6.5. The morphism φ

In this section, we define a morphism φ that will be part of the natural zig-zag of Theorem 6.34. Let $\delta : \text{Sp}^{\mathcal{O}} \rightarrow B(\mathcal{O}, \text{Sp})$ be the functor that adds a constant contravariant variable, defined by

$$\delta_F(s, r) = F(r) \tag{6.13}$$

for $F \in \text{Sp}^{\mathcal{O}}$ and $r, s \in \mathcal{O}$. Let $(u_t)^* : B(\mathcal{O}, \text{Sp}) \rightarrow \text{Sp}^{(\mathcal{O}/t)^{\text{op}} \times \mathcal{O}}$ be the restriction of the contravariant variable along the forgetful functor. Recall the definitions of $J \in \text{Sp}^{\mathcal{O}}$ and $M_t \in B(\mathcal{O}, \text{Sp})$ from (6.4). Define a morphism

$$\varphi^{\sharp} : (u_t)^* \delta_J \rightarrow (u_t)^* M_t \tag{6.14}$$

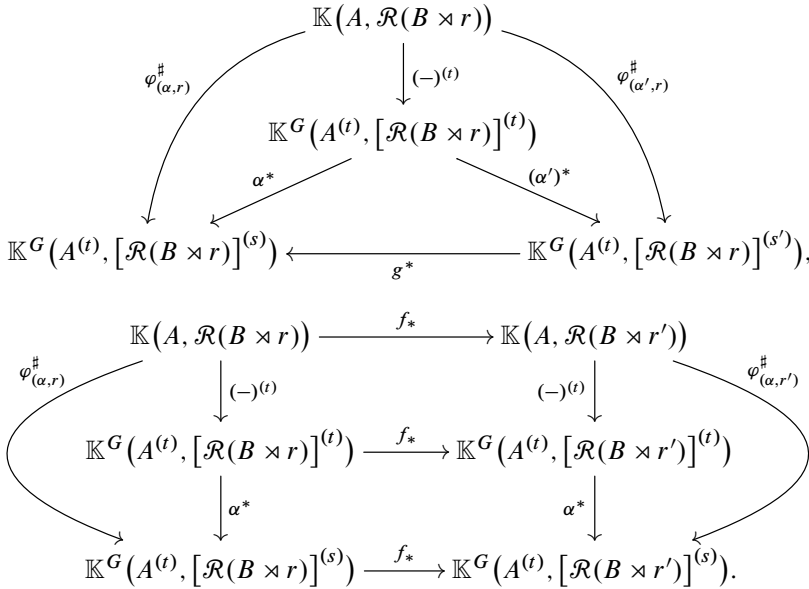
as follows. For objects r of \mathcal{O} and $\alpha : s \rightarrow t$ of \mathcal{O}/t , let

$$\varphi_{(\alpha, r)}^{\sharp} : [(u_t)^* \delta_J](\alpha, r) \rightarrow [(u_t)^* M_t](\alpha, r) \tag{6.15}$$

be the following composition:

$$\mathbb{K}(A, \mathcal{R}(B \rtimes r)) \xrightarrow{(-)^{(t)}} \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(t)}) \xrightarrow{\alpha^*} \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(s)}).$$

It is easily verified that φ^{\sharp} is a natural transformation of bifunctors. Indeed, for morphisms $f : r \rightarrow r'$ in \mathcal{O} and $g : \alpha \rightarrow \alpha'$ in \mathcal{O}/t , the following diagrams commute:



Remark 6.16. The definition of the components of φ^{\sharp} (6.15) makes use of the structural morphism $\alpha : s \rightarrow t$ and it is not clear how to define φ^{\sharp} as a morphism $\delta_J \rightarrow M_t$ in $B(\mathcal{O}, \text{Sp})$. This is one of the reasons that motivated our choice of \mathcal{O}/t as the indexing category for coends.

Construction 6.17. By Remark 6.24 we have, for each $t \in \mathcal{O}$, a pair of adjoint functors

$$(u_t)_! : \mathrm{Sp}^{(\mathcal{O}_t)^{op} \times \mathcal{O}} \rightleftarrows B(\mathcal{O}, \mathrm{Sp}) : (u_t)^*. \quad (6.18)$$

Let $f : t \rightarrow t'$ be a morphism in \mathcal{O} and write $f^* : \mathrm{Sp}^{(\mathcal{O}_{t'})^{op} \times \mathcal{O}} \rightarrow \mathrm{Sp}^{(\mathcal{O}_t)^{op} \times \mathcal{O}}$ for the restriction of the contravariant variable along $f : \mathcal{O}_{t'} \rightarrow \mathcal{O}_t$. Note that $f^* \circ (u_{t'})^* = (u_t)^*$ and consider the following diagram of solid arrows for $F \in \mathrm{Sp}^{\mathcal{O}}$ and $H \in B(\mathcal{O}, \mathrm{Sp})$:

$$\begin{array}{ccc} B(\mathcal{O}, \mathrm{Sp})((u_{t'})_!(u_{t'})^* \delta_F, H) & \xrightarrow{\cong} & \mathrm{Sp}^{(\mathcal{O}_{t'})^{op} \times \mathcal{O}}((u_{t'})^* \delta_F, (u_{t'})^* H) \\ \downarrow & & \downarrow f^* \\ B(\mathcal{O}, \mathrm{Sp})((u_t)_!(u_t)^* \delta_F, H) & \xrightarrow{\cong} & \mathrm{Sp}^{(\mathcal{O}_t)^{op} \times \mathcal{O}}((u_t)^* \delta_F, (u_t)^* H). \end{array}$$

Let the dashed arrow complete the diagram to a commutative square. By the Yoneda lemma, the dashed arrow is induced by precomposition with a unique morphism

$$(u_t)_!(u_t)^* \delta_F \rightarrow (u_{t'})_!(u_{t'})^* \delta_F.$$

The latter, for varying f , assemble into a functor $\mathcal{O} \rightarrow B(\mathcal{O}, \mathrm{Sp})$, $t \mapsto (u_t)_!(u_t)^* \delta_F$. This construction is, moreover, clearly natural in F , so that we get a functor as follows:

$$\begin{array}{ccc} \mathrm{Sp}^{\mathcal{O}} & \longrightarrow & B(\mathcal{O}, \mathrm{Sp})^{\mathcal{O}} \\ F & \longmapsto & (t \mapsto (u_t)_!(u_t)^* \delta_F). \end{array}$$

Remark 6.19. Let $F \in \mathrm{Sp}^{\mathcal{O}}$. Later on, it will be useful to have an explicit description of the functor $\mathcal{O} \rightarrow B(\mathcal{O}, \mathrm{Sp})$, $t \mapsto (u_t)_!(u_t)^* \delta_F$ mentioned in Construction 6.17. Let us first describe the bifunctor $(u_t)_!(u_t)^* \delta_F \in B(\mathcal{O}, \mathrm{Sp})$ for fixed $t \in \mathcal{O}$. For $r, s \in \mathcal{O}$ we have

$$[(u_t)_!(u_t)^* \delta_F](s, r) = \coprod_{\alpha \in \mathcal{O}(s, t)} [(u_t)^* \delta_F](\alpha, r) = \coprod_{\mathcal{O}(s, t)} F(r).$$

For a morphism $f : r \rightarrow r'$ in \mathcal{O} , the induced morphism $f_* = [(u_t)_!(u_t)^* \delta_F](s, f)$ equals the morphism

$$\coprod_{\mathcal{O}(s, t)} F(f) : \coprod_{\mathcal{O}(s, t)} F(r) \longrightarrow \coprod_{\mathcal{O}(s, t)} F(r').$$

For a morphism $g : s' \rightarrow s$ in \mathcal{O} , the induced morphism $g^* = [(u_t)_!(u_t)^* \delta_F](g, r)$ is the unique morphism making the following triangle commute for all $\beta \in \mathcal{O}(s, t)$:

$$\begin{array}{ccc} F(r) & \xrightarrow{\mathrm{can}_\beta} & \coprod_{\mathcal{O}(s, t)} F(r) \\ & \searrow \mathrm{can}_{\beta \circ g} & \downarrow g^* \\ & & \coprod_{\mathcal{O}(s', t)} F(r). \end{array}$$

Now let $h : t \rightarrow t'$ be a morphism in \mathcal{O} . Then the components of the induced natural transformation $h_* : (u_t)_!(u_t)^*\delta_F \rightarrow (u_{t'})_!(u_{t'})^*\delta_F$ are the unique morphisms making the following triangle commute for all $\beta \in \mathcal{O}(s, t)$:

$$\begin{array}{ccc}
 F(r) & \xrightarrow{\text{can}_\beta} & \coprod_{\mathcal{O}(s,t)} F(r) \\
 & \searrow^{\text{can}_{h \circ \beta}} & \downarrow h_* \\
 & & \coprod_{\mathcal{O}(s,t')} F(r).
 \end{array}$$

Lemma 6.20. Fix $t \in \mathcal{O}$ and let $\varphi^\# : (u_t)_!\delta_J \rightarrow (u_t)^*M_t$ be the morphism in $\text{Sp}^{(\mathcal{O}_t)^{op} \times \mathcal{O}}$ defined in (6.14). Under the adjunction (6.18), $\varphi^\#$ corresponds to a morphism

$$\varphi : (u_t)_!(u_t)^*\delta_J \rightarrow M_t$$

in $B(\mathcal{O}, \text{Sp})$. Explicitly, for $r, s \in \mathcal{O}$, the component $\varphi_{(s,r)} : [(u_t)_!(u_t)^*\delta_J](s, r) \rightarrow M_t(s, r)$ is the unique morphism making the following triangle commute for every $\alpha \in \mathcal{O}(s, t)$:

$$\begin{array}{ccc}
 \coprod_{\mathcal{O}(s,t)} J(r) & \xrightarrow{\varphi_{(s,r)}} & M_t(s, r). \\
 \text{can}_\alpha \uparrow & \nearrow \varphi_{(\alpha,r)}^\# & \\
 J(r) & &
 \end{array}$$

Then the latter, for varying t , assemble into a morphism in $B(\mathcal{O}, \text{Sp})^\mathcal{O}$ – where the domain of φ is considered as an object of $B(\mathcal{O}, \text{Sp})^\mathcal{O}$ as explained in Construction 6.17.

Proof. Let $f : t \rightarrow t'$ be a morphism in \mathcal{O} . We have to show that the following square in $B(\mathcal{O}, \text{Sp})$ commutes:

$$\begin{array}{ccc}
 (u_t)_!(u_t)^*\delta_J & \xrightarrow{\varphi} & M_t \\
 f_* \downarrow & & \downarrow f_* \\
 (u_{t'})_!(u_{t'})^*\delta_J & \xrightarrow{\varphi} & M_{t'}.
 \end{array}$$

By Remark 6.19, for $r, s \in \mathcal{O}$, we have $[(u_t)_!(u_t)^*\delta_J](s, r) = \coprod_{\mathcal{O}(s,t)} J(r)$. Thus, it suffices to show that the following square commutes for $r, s \in \mathcal{O}$:

$$\begin{array}{ccc}
 \coprod_{\mathcal{O}(s,t)} J(r) & \xrightarrow{\varphi} & M_t(s, r) \\
 f_* \downarrow & & \downarrow f_* \\
 \coprod_{\mathcal{O}(s,t')} J(r) & \xrightarrow{\varphi} & M_{t'}(s, r).
 \end{array}$$

By the universal property of the coproduct it suffices to show that this square commutes when precomposed with every structural morphism into the coproduct in the upper left

corner. Let $\alpha \in \mathcal{O}(s, t)$. Upon precomposing with $\text{can}_\alpha : J(r) \rightarrow \coprod_{\mathcal{O}(s,t)} J(r)$ we get

$$\begin{array}{ccc} J(r) & \xrightarrow{\varphi_{(\alpha,r)}^\#} & M_t(s, r) \\ & \searrow \varphi_{(f\circ\alpha,r)}^\# & \downarrow f_* \\ & & M_{t'}(s, r). \end{array}$$

Unraveling the definition of $\varphi^\#$, this triangle becomes the following diagram:

$$\begin{array}{ccccc} \mathbb{K}(A, \mathcal{R}(B \rtimes r)) & \xrightarrow{(-)^{(t)}} & \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(t)}) & \xrightarrow{\alpha^*} & \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(s)}) \\ (-)^{(t')} \downarrow & & \downarrow f_* & & \downarrow f_* \\ \mathbb{K}^G(A^{(t')}, [\mathcal{R}(B \rtimes r)]^{(t')}) & \xrightarrow{f^*} & \mathbb{K}^G(A^{(t')}, [\mathcal{R}(B \rtimes r)]^{(t)}) & \xrightarrow{\alpha^*} & \mathbb{K}^G(A^{(t')}, [\mathcal{R}(B \rtimes r)]^{(s)}). \end{array}$$

The square on the right clearly commutes. The one on the left commutes by Lemma C.6. ■

6.6. Model structure on categories of bifunctors

Let \mathcal{C} be a model category and let Γ be a small category. In this section, we endow $B(\Gamma, \mathcal{C})$ with a model structure that will allow us to build models for the *homotopy coends* of certain morphisms of bifunctors. We will need these later on to define the morphisms going backwards in the zig-zag of \mathcal{O} -spectra of Theorem 6.34.

Proposition 6.21 ([26, Proposition A.2.8.2]). *Let \mathcal{C} be a combinatorial model category and let Γ be a small category. Then there exist two combinatorial model structures on \mathcal{C}^Γ :*

- *The injective model structure, denoted $\mathcal{C}_{\text{inj}}^\Gamma$, where weak equivalences and cofibrations are defined objectwise.*
- *The projective model structure, denoted $\mathcal{C}_{\text{proj}}^\Gamma$, where weak equivalences and fibrations are defined objectwise.*

We will always consider $B(\Gamma, \mathcal{C})$ as a model category with the structure $(\mathcal{C}_{\text{inj}}^{\Gamma^{\text{op}}})_{\text{proj}}^\Gamma$ whenever this structure exists. The model structure on $B(\Gamma, \text{Sp})$ exists for any Γ by Lemma C.2 and Proposition 6.21.

Theorem 6.22 ([2, Theorem 4.1]). *Let \mathcal{C} be a model category and Γ be a small category such that the model structure on $B(\Gamma, \mathcal{C})$ exists. Then the functor*

$$\int^\Gamma : B(\Gamma, \mathcal{C}) \rightarrow \mathcal{C}$$

is a left Quillen functor.

Proof. This is [2, Theorem 4.1]; we sketch the proof for completeness. For $c \in \mathcal{C}$, define $R(c) : \Gamma^{\text{op}} \times \Gamma \rightarrow \mathcal{C}$ by

$$R(c)(s, r) := \prod_{\alpha \in \Gamma(r,s)} c.$$

For morphisms $f : r \rightarrow r'$ and $g : s' \rightarrow s$ in Γ , let f_* and g^* be the unique morphisms making the following diagrams commute for all $\alpha \in \Gamma(r', s)$ and all $\beta \in \Gamma(r, s')$:

$$\begin{array}{ccc}
 R(c)(s, r) & \xrightarrow{f_*} & R(c)(s, r') \\
 \searrow \text{can}_{\alpha \circ f} & & \downarrow \text{can}_{\alpha} \\
 & & c
 \end{array}
 \qquad
 \begin{array}{ccc}
 R(c)(s, r) & \xrightarrow{g^*} & R(c)(s', r) \\
 \searrow \text{can}_{g \circ \beta} & & \downarrow \text{can}_{\beta} \\
 & & c
 \end{array}$$

It is easily verified that $R : \mathcal{C} \rightarrow \mathcal{C}^{\Gamma^{op} \times \Gamma}$ is right adjoint to $\int^{\Gamma} : \mathcal{C}^{\Gamma^{op} \times \Gamma} \rightarrow \mathcal{C}$. Thus, proving the theorem is equivalent to showing that

$$R : \mathcal{C} \rightarrow B(\Gamma, \mathcal{C})$$

preserves fibrations and trivial fibrations. Let $c \rightarrow c'$ be a (trivial) fibration in \mathcal{C} . To prove that $R(c) \rightarrow R(c')$ is a (trivial) fibration in

$$B(\Gamma, \mathcal{C}) = (\mathcal{C}_{\text{inj}}^{\Gamma^{op}})_{\text{proj}}^{\Gamma},$$

it suffices to show that $R(c)(-, r) \rightarrow R(c')(-, r)$ is a (trivial) fibration in $\mathcal{C}_{\text{inj}}^{\Gamma^{op}}$ for every $r \in \Gamma$. But the latter holds since, for every r , there is a Quillen adjunction

$$\text{ev}_r : \mathcal{C}_{\text{inj}}^{\Gamma^{op}} \rightleftarrows \mathcal{C} : R(-)(-, r)$$

where ev_r is the evaluation at r [2, Corollary 2.3 (iii)]. Indeed, this adjunction is Quillen since ev_r clearly preserves cofibrations and trivial cofibrations. ■

Lemma 6.23 ([26, Proposition A.2.8.7]). *Let \mathcal{C} be a model category and let $f : \Gamma \rightarrow \Lambda$ be a functor. Then the restriction functor $f^* : \mathcal{C}^{\Lambda} \rightarrow \mathcal{C}^{\Gamma}$ fits into the following Quillen adjunctions, whenever the model structures in question exist:*

- (1) $f^* : \mathcal{C}_{\text{inj}}^{\Lambda} \rightleftarrows \mathcal{C}_{\text{inj}}^{\Gamma} : f_*$,
- (2) $f_! : \mathcal{C}_{\text{proj}}^{\Gamma} \rightleftarrows \mathcal{C}_{\text{proj}}^{\Lambda} : f^*$.

Remark 6.24. Let \mathcal{C} be a category with small coproducts and let Γ be a small category. Fix $t \in \Gamma$ and let $u_t : \Gamma/t \rightarrow \Gamma$ be the forgetful functor. Then there is an adjunction

$$(u_t)_! : \mathcal{C}^{(\Gamma/t)^{op}} \rightleftarrows \mathcal{C}^{\Gamma^{op}} : (u_t)^*.$$

Moreover, the pushforward functor $(u_t)_!$ can be explicitly described as follows. For $s \in \Gamma$, we have

$$[(u_t)_! F](s) = \coprod_{\alpha \in \text{Hom}(s, t)} F(\alpha).$$

For a morphism $g : s' \rightarrow s$ in Γ , $[(u_t)_! F](g)$ is the unique morphism making the following squares commute, where the vertical arrows are the structural morphisms into the coproducts:

$$\begin{array}{ccc}
 \coprod_{\alpha \in \text{Hom}(s, t)} F(\alpha) & \xrightarrow{[(u_t)_! F](g)} & \coprod_{\alpha' \in \text{Hom}(s', t)} F(\alpha') \\
 \text{can}_{\beta} \uparrow & & \uparrow \text{can}_{\beta \circ g} \\
 F(\beta) & \xrightarrow{F(g)} & F(\beta \circ g).
 \end{array}$$

Lemma 6.25 (cf. [26, Lemma A.2.8.10]). *Let \mathcal{C} be a model category, Γ be a small category, fix $t \in \Gamma$, and let $u_t : \Gamma_{/t} \rightarrow \Gamma$ be the forgetful functor from the slice category. Then the following adjunctions are Quillen adjunctions, whenever the model structures in question exist:*

- (1) $(u_t)^* : \mathcal{C}_{\text{proj}}^\Gamma \rightleftarrows \mathcal{C}_{\text{proj}}^{\Gamma/t} : (u_t)_*$,
- (2) $(u_t)! : \mathcal{C}_{\text{inj}}^{(\Gamma/t)^{\text{op}}} \rightleftarrows \mathcal{C}_{\text{inj}}^{\Gamma^{\text{op}}} : (u_t)^*$.

Proof. To prove (2) is a Quillen adjunction, let us show that $(u_t)!$ preserves cofibrations and trivial cofibrations. Recall from Remark 6.24 that, for $F \in \mathcal{C}^{(\Gamma/t)^{\text{op}}}$ and $s \in \Gamma$, we have

$$[(u_t)!F](s) = \coprod_{\alpha \in \Gamma(s,t)} F(\alpha).$$

Let $\eta : F \rightarrow F'$ be a morphism in $\mathcal{C}^{(\Gamma/t)^{\text{op}}}$. For $s \in \Gamma$, $(u_t)!(\eta)(s)$ is the coproduct of the morphisms

$$\{\eta(\alpha) : F(\alpha) \rightarrow F'(\alpha)\}_{\alpha \in \Gamma(s,t)}.$$

If η is a cofibration (resp. a trivial cofibration) in $\mathcal{C}_{\text{inj}}^{(\Gamma/t)^{\text{op}}}$, the latter are cofibrations (resp. trivial cofibrations) in \mathcal{C} and, thus, $(u_t)!(\eta)(s)$ is again a cofibration (resp. trivial cofibration) in \mathcal{C} . Since this holds for every $s \in \Gamma$, it follows that $(u_t)!(\eta)$ is a cofibration (resp. a trivial cofibration) in $\mathcal{C}_{\text{inj}}^{\Gamma^{\text{op}}}$. The proof of (1) is dual to that of (2), using the fact that

$$[(u_t)_*F](s) = \prod_{\alpha \in \Gamma(t,s)} F(\alpha)$$

for $F \in \mathcal{C}^{\Gamma/t}$ and $s \in \Gamma$. ■

Lemma 6.26 ([26, Remark A.2.8.6]). *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction of combinatorial model categories and let Γ be a small category. Then composition with F and U determines a Quillen adjunction*

$$F^\Gamma : \mathcal{C}^\Gamma \rightleftarrows \mathcal{D}^\Gamma : U^\Gamma$$

with respect to either the injective or the projective model structures.

Lemma 6.27. *Let $\delta : \text{Sp}_{\text{proj}}^\mathcal{O} \rightarrow B(\mathcal{O}, \text{Sp})$ be the functor defined in (6.13). Then δ is a left Quillen functor.*

Proof. Let $\text{const} : \text{Sp} \rightarrow \text{Sp}_{\text{inj}}^{\mathcal{O}^{\text{op}}}$ be the functor that sends a spectrum E to the constant functor on E . Then const is left Quillen since it clearly sends cofibrations (resp. trivial cofibrations) to cofibrations (resp. trivial cofibrations) and it has a right adjoint (taking limit). Note that we have

$$\delta = (\text{const})^\mathcal{O} : \text{Sp}_{\text{proj}}^\mathcal{O} \rightarrow [\text{Sp}_{\text{inj}}^{\mathcal{O}^{\text{op}}}]_{\text{proj}}^\mathcal{O} = B(\mathcal{O}, \text{Sp}).$$

Then δ is left Quillen by Lemma 6.26. ■

Lemma 6.28. *Let \mathcal{C} be a combinatorial model category, Γ be a small category, fix $t \in \Gamma$, and let $u_t : \Gamma_{/t} \rightarrow \Gamma$ be the forgetful functor from the slice category. Then the restriction*

on both variables

$$(u_t)^* : B(\Gamma, \mathcal{C}) \rightarrow B(\Gamma/t, \mathcal{C})$$

is a left Quillen functor.

Proof. Consider the following commutative square:

$$\begin{array}{ccc} B(\Gamma, \mathcal{C}) = (\mathcal{C}_{\text{inj}}^{\Gamma \text{op}})_{\text{proj}}^{\Gamma} & \xrightarrow{(u_t)^*} & (\mathcal{C}_{\text{inj}}^{\Gamma \text{op}})_{\text{proj}}^{\Gamma/t} \\ (u_t)^* \downarrow & \searrow (u_t)^* & \downarrow (u_t)^* \\ [\mathcal{C}_{\text{inj}}^{(\Gamma/t) \text{op}}]_{\text{proj}}^{\Gamma} & \xrightarrow{(u_t)^*} & [\mathcal{C}_{\text{inj}}^{(\Gamma/t) \text{op}}]_{\text{proj}}^{\Gamma/t} = B(\Gamma/t, \mathcal{C}). \end{array}$$

The horizontal morphisms are left Quillen by Lemma 6.25 (1) and the vertical morphisms are left Quillen by Lemma 6.23 and Lemma 6.26. Then $(u_t)^*$ is left Quillen as well, for being the composite of left Quillen functors. ■

6.7. The natural zig-zag

In this section we finally construct a zig-zag of \mathcal{O} -spectra inducing (6.1) upon taking homotopy groups. We begin with the following lemma, that shows that every \mathcal{O} -spectrum can be canonically described as an objectwise coend.

Lemma 6.29. *Let $\delta : \text{Sp}_{\text{proj}}^{\mathcal{O}} \rightarrow B(\mathcal{O}, \text{Sp})$ be the functor defined in (6.13). For $F \in \text{Sp}^{\mathcal{O}}$ and $t \in \mathcal{O}$, the structural morphisms into the coends*

$$F(t) = [(u_t)^* \delta_F](\text{id}_t, \text{id}_t) \longrightarrow \int^{\mathcal{O}_t} (u_t)^* \delta_F \tag{6.30}$$

are isomorphisms. Moreover, these are natural in $t \in \mathcal{O}$ and in $F \in \text{Sp}^{\mathcal{O}}$.

Proof. Fix $t \in \mathcal{O}$. Since $(u_t)^* \delta_F$ is constant in the contravariant variable, we have

$$\int^{\mathcal{O}_t} (u_t)^* \delta_F \cong \text{colim}_{\alpha \in \mathcal{O}_t} F(u_t(\alpha)).$$

Since id_t is a final object of \mathcal{O}_t , the structural morphism

$$F(t) = F(u_t(\text{id}_t)) \longrightarrow \text{colim}_{\alpha \in \mathcal{O}_t} F(u_t(\alpha))$$

is an isomorphism. Combining the above we get the desired isomorphism,

$$F(t) \xrightarrow{\cong} \int^{\mathcal{O}_t} (u_t)^* \delta_F.$$

It is easily verified that this is natural in t and in F . ■

Lemma 6.31. *Let $F \in \text{Sp}^{\mathcal{O}}$ and fix $t \in \mathcal{O}$. Let $(u_t)_!$ and $(u_t)^*$ be the functors that form the adjunction (6.18). Then there is a morphism of bifunctors $(u_t)^* \delta_F \rightarrow (u_t)^*(u_t)_!(u_t)^* \delta_F$ described as follows. For objects $\alpha : r \rightarrow t$ and $\beta : s \rightarrow t$ of \mathcal{O}_t , the component correspond-*

ing to the pair (β, α) is the structural morphism into the coproduct corresponding to β ,

$$[(u_t)^* \delta_F](\beta, \alpha) = F(r) \xrightarrow{\text{can}_\beta} \coprod_{\mathcal{O}(s,t)} F(r) = [(u_t)^*(u_t)!(u_t)^* \delta_F](\beta, \alpha). \quad (6.32)$$

Moreover, upon taking coend we get a morphism of spectra

$$\int^{\mathcal{O}/t} (u_t)^* \delta_F \longrightarrow \int^{\mathcal{O}/t} (u_t)^*(u_t)!(u_t)^* \delta_F \quad (6.33)$$

that is natural in t .

Proof. The fact that the morphisms (6.32) are natural in α and β is easily verified using the explicit description of the bifunctor $(u_t)!(u_t)^* \delta_F$ given in Remark 6.19.

Let us now prove that (6.33) is natural in t . Let $h : t \rightarrow t'$ be a morphism in \mathcal{O} . By the universal property of the coend, it suffices to show that the outer square in the following diagram commutes for every $\alpha : r \rightarrow t$ in \mathcal{O}/t :

$$\begin{array}{ccc}
 F(r) = [(u_t)^* \delta_F](\alpha, \alpha) & \xrightarrow{\quad} & \int^{\mathcal{O}/t} (u_t)^*(u_t)!(u_t)^* \delta_F \\
 \downarrow \text{can}_\alpha & \searrow & \downarrow h_* \\
 \int^{\mathcal{O}/t} (u_t)^* \delta_F & \xrightarrow{(6.33)} & \int^{\mathcal{O}/t} (u_t)^*(u_t)!(u_t)^* \delta_F \\
 \downarrow h_* & & \downarrow h_* \\
 \int^{\mathcal{O}/t'} (u_{t'})^* \delta_F & \xrightarrow{(6.33)} & \int^{\mathcal{O}/t'} (u_{t'})^*(u_{t'})!(u_{t'})^* \delta_F
 \end{array}$$

Unraveling the definitions of h_* (see Lemma 6.11 and Remark 6.19), it is straightforward to verify that both ways from $F(r)$ to $\int^{\mathcal{O}/t'} (u_{t'})^*(u_{t'})!(u_{t'})^* \delta_F$ in the diagram above equal the composite

$$F(r) \xrightarrow{\text{can}_{h\alpha}} \coprod_{\mathcal{O}(r,t')} F(r) \xrightarrow{\text{can}_{h\alpha}} \int^{\mathcal{O}/t'} (u_{t'})^*(u_{t'})!(u_{t'})^* \delta_F.$$

This finishes the proof. ■

Theorem 6.34. Let $q : Q \xrightarrow{\sim} \text{id}$ and $\bar{q} : \bar{Q} \xrightarrow{\sim} \text{id}$ be, respectively, cofibrant replacements in $B(\mathcal{O}, \text{Sp})$ and $\text{Sp}_{\text{proj}}^{\mathcal{O}}$. Then we have a zig-zag of \mathcal{O} -spectra as follows:

$$\begin{array}{ccccccc}
 \int^{\mathcal{O}/t} (u_t)^* \delta_J & \xleftarrow{\bar{q}} & \int^{\mathcal{O}/t} (u_t)^* \delta_{\bar{Q}J} & \xrightarrow{(6.33)} & \int^{\mathcal{O}/t} (u_t)^*(u_t)!(u_t)^* \delta_{\bar{Q}J} & & \\
 & & & & \uparrow q & & \\
 \int^{\mathcal{O}/t} (u_t)^* Q(M_t) & \xleftarrow{\varphi} & \int^{\mathcal{O}/t} (u_t)^* Q(u_t)!(u_t)^* \delta_J & \xleftarrow{\bar{q}} & \int^{\mathcal{O}/t} (u_t)^* Q(u_t)!(u_t)^* \delta_{\bar{Q}J} & & \\
 \uparrow \psi & & & & & & \\
 \int^{\mathcal{O}/t} (u_t)^* Q(L_t) & \xrightarrow{q} & \int^{\mathcal{O}/t} (u_t)^* L_t & \xrightarrow{\zeta} & \mathbb{K}G(A^{(t)}, M_G B). & &
 \end{array}$$

Moreover, upon identifying

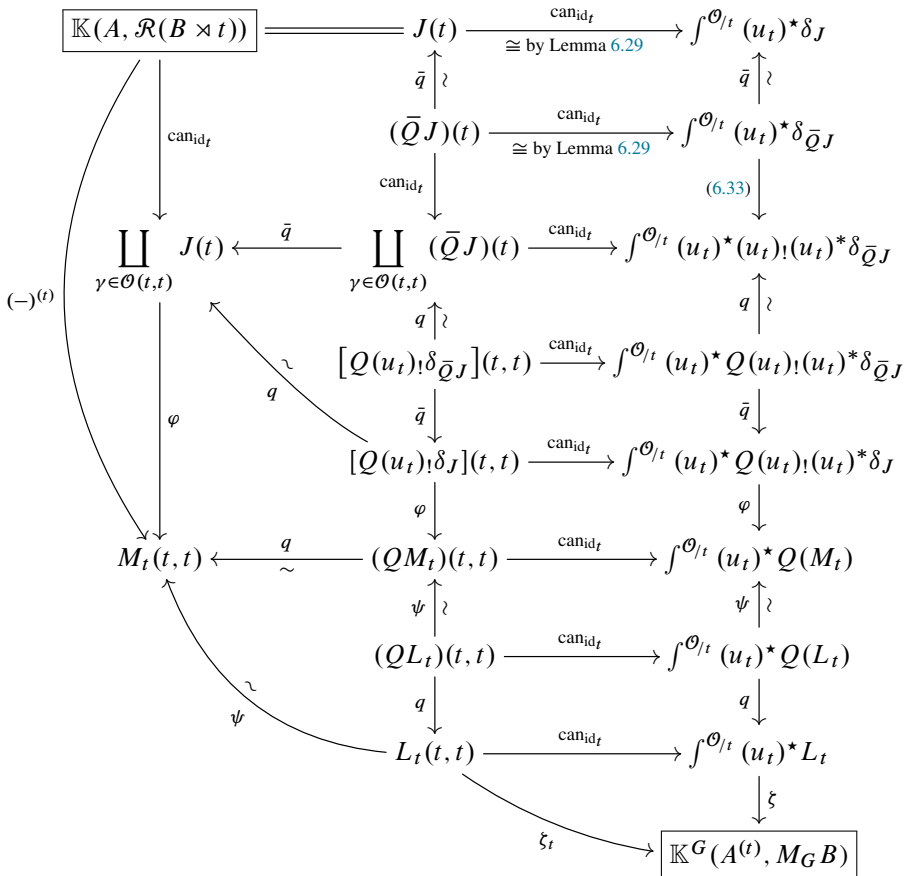
$$\mathbb{K}(A, \mathcal{R}(B \rtimes t)) = J(t) \stackrel{(6.30)}{\cong} \int^{\mathcal{O}/t} (u_t)^* \delta_J$$

and then taking homotopy groups, this zig-zag induces the isomorphism (6.1).

Proof. Let us first show that the morphisms appearing in the zig-zag are indeed natural in t . For (6.33) this is part of Lemma 6.31 and for ζ it is Lemma 6.12. The rest of the morphisms are natural in t because they result from applying the functor \mathcal{C} of Lemma 6.11 to appropriate morphisms in $B(\mathcal{O}, \text{Sp})^{\mathcal{O}}$.

The morphisms in the zig-zag labeled with \sim are indeed weak equivalences by Ken Brown’s lemma [19, Lemma 1.1.12]: they result from applying a left Quillen functor to an appropriate weak equivalence between cofibrant objects. Here we use that the functors δ , $(u_t)^*$ and $\int^{\mathcal{O}/t}$ are left Quillen by Lemmas 6.27, 6.28 and Theorem 6.22 respectively.

The fact that the zig-zag induces (6.1) upon taking homotopy groups follows from the commutativity of the following diagram of spectra.



Upon taking homotopy groups, the morphisms labeled with \sim in the diagram above become isomorphisms and can be inverted. The zig-zag equals the morphism in the top row followed by the composite of the morphisms in the rightmost column. By Lemma 5.5 (3), the composite of the bent morphisms on the left equals, upon taking homotopy groups, the isomorphism (6.1). ■

7. Main theorem

In this section, we prove Theorem 7.11. It will turn out to be an easy consequence of the technical Theorem 6.34 and the following two lemmas.

Lemma 7.1. *Let A be an algebra, B be a G -algebra and X be a $(G, \mathcal{F}in)$ -finite complex. Then there exists a weak equivalence of spectra*

$$\alpha_X : H^G(X; \mathbb{K}^G(A^{(-)}, B)) \rightarrow \mathbb{K}^G(A^{(X)}, B) \tag{7.2}$$

that is natural in X .

Proof. Let $\mathbf{E} : \text{Or}(G, \mathcal{F}in) \rightarrow \text{Sp}$ be defined by

$$\mathbf{E}(G/H) = \mathbb{K}^G(A^{(G/H)}, B)$$

and let X be a $(G, \mathcal{F}in)$ -complex. Since coends and smash products commute with colimits, we have

$$\begin{aligned} H^G(X; \mathbf{E}) &= \int^{G/H} X_+^H \wedge \mathbf{E}(G/H) \\ &\cong \int^{G/H} \text{colim}_{G/K \times \Delta^n \downarrow X} (G/K \times \Delta^n)_+^H \wedge \mathbf{E}(G/H) \quad (\text{by Lemma A.6}) \\ &\cong \text{colim}_{G/K \times \Delta^n \downarrow X} \Delta_+^n \wedge \int^{G/H} (G/K)_+^H \wedge \mathbf{E}(G/H) \quad (\text{by Lemma A.1}) \\ &= \text{colim}_{G/K \times \Delta^n \downarrow X} \Delta_+^n \wedge H^G(G/K; \mathbf{E}) \\ &\cong \text{colim}_{G/K \times \Delta^n \downarrow X} \Delta_+^n \wedge \mathbf{E}(G/K). \end{aligned}$$

Thus, to construct the morphism (7.2) it suffices to define compatible morphisms

$$\Delta_+^n \wedge \mathbb{K}^G(A^{(G/K)}, B) \rightarrow \mathbb{K}^G(A^{(X)}, B) \tag{7.3}$$

for every $f : G/K \times \Delta^n \rightarrow X$. Define (7.3) as the following composite:

$$\Delta_+^n \wedge \mathbb{K}^G(A^{(G/K)}, B) \xrightarrow{\text{(C.4)}} \mathbb{K}^G(A^{(G/K \times \Delta^n)}, B) \xrightarrow{f_*} \mathbb{K}^G(A^{(X)}, B).$$

The compatibility of these morphisms is immediate from the naturality of (C.4) in G/K . This defines the morphism (7.2).

We claim that α_X is a weak equivalence of spectra. We will prove the claim by considering the following increasingly more general cases:

- (1) $X = G/H$ with $H \in \mathcal{F}in$,
- (2) $X = G/H \times \Delta^n$ with $H \in \mathcal{F}in$ and $n \in \mathbb{N}$,
- (3) $X = \bigsqcup_{i=1}^m G/H_i \times \Delta^n$ with $H_i \in \mathcal{F}in$ and $m, n \in \mathbb{N}$,
- (4) X any $(G, \mathcal{F}in)$ -finite complex.

To prove the case (1), note that the identity of $X \cong G/H \times \Delta^0$ is a final object among those $G/K \times \Delta^n \downarrow X$. Then, taking colimit over $G/K \times \Delta^n \downarrow X$ boils down to evaluating at the final object $\text{id}_{G/H \times \Delta^0}$ and the result follows from Lemma C.3.

For the case (2), let $\pi : G/H \times \Delta^n \rightarrow G/H$ be the projection and consider the following commutative diagram:

$$\begin{CD}
 H^G(G/H \times \Delta^n; \mathbf{E}) @>\alpha_{G/H \times \Delta^n}>> \mathbb{K}^G(A^{(G/H \times \Delta^n)}, B) \\
 @V\pi_* \wr VV @VV\wr \pi_* V \\
 H^G(G/H; \mathbf{E}) @>\alpha_{G/H}>> \mathbb{K}^G(A^{(G/H)}, B).
 \end{CD}$$

The left vertical arrow is an equivalence by homotopy invariance of equivariant homology theories. The right vertical arrow is a weak equivalence by homotopy invariance of $kk_*^G(-, B)$, since we have

$$A^{(G/H \times \Delta^n)} \cong A^{(\bigsqcup_{G/H} \Delta^n)} \cong \bigoplus_{G/H} A^{\Delta^n} \cong \left(\bigoplus_{G/H} A \right) \otimes \ell^{\Delta^n} \cong A^{(G/H)} \otimes \ell^{\Delta^n}.$$

The bottom arrow is a weak equivalence by the case (1).

Now let X be as in case (3). This case follows from the previous one since we have

$$\begin{aligned}
 H_*^G(X; \mathbf{E}) &\cong \bigoplus_{i=1}^n H_*^G(G/H_i \times \Delta^n; \mathbf{E}), \\
 kk_*^G(A^{(X)}, B) &\cong \bigoplus_{i=1}^n kk_*^G(A^{(G/H_i \times \Delta^n)}, B), \\
 \alpha_X &= \bigoplus \alpha_{G/H_i \times \Delta^n}.
 \end{aligned}$$

To prove the case (4) we proceed by induction on $n = \dim X$. The base case $n = 0$ holds by the case (3). Now suppose $n \geq 1$ and assume that α_Z is a weak equivalence for every $(G, \mathcal{F}in)$ -finite complex Z with $\dim Z \leq n - 1$. Since X is $(G, \mathcal{F}in)$ -finite, there is a pushout diagram as follows:

$$\begin{CD}
 \bigsqcup_{i=1}^n G/H_i \times \partial \Delta^n @>>> \text{sk}_{n-1} X \\
 @VVV @VVV \\
 \bigsqcup_{i=1}^n G/H_i \times \Delta^n @>>> X.
 \end{CD}$$

By Lemma B.2, upon applying the functor $A^{(-)}$ we obtain the following Milnor square:

$$\begin{array}{ccc} A(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n) & \longleftarrow & A(\text{sk}_{n-1} X) \\ \uparrow & & \uparrow \\ A(\bigsqcup_{i=1}^n G/H_i \times \Delta^n) & \longleftarrow & A(X). \end{array}$$

This induces a morphism of Mayer–Vietoris sequences as follows; see Lemma B.5:

$$\begin{array}{ccc} H_*^G(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n; \mathbf{E}) & \xrightarrow{\alpha_{\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n}} & kk_*^G(A(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n), B) \\ \downarrow & & \downarrow \\ H_*^G(\bigsqcup_{i=1}^n G/H_i \times \Delta^n; \mathbf{E}) & \xrightarrow{\alpha_{\bigsqcup_{i=1}^n G/H_i \times \Delta^n} \oplus \alpha_{\text{sk}_{n-1} X}} & kk_*^G(A(\bigsqcup_{i=1}^n G/H_i \times \Delta^n), B) \\ \oplus & & \oplus \\ H_*^G(\text{sk}_{n-1} X; \mathbf{E}) & & kk_*^G(A(\text{sk}_{n-1} X), B) \\ \downarrow & & \downarrow \\ H_*^G(X; \mathbf{E}) & \xrightarrow{\alpha_X} & kk_*^G(A(X), B) \\ \downarrow & & \downarrow \\ H_{*-1}(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n; \mathbf{E}) & \longrightarrow & kk_{*-1}^G(A(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n), B). \end{array}$$

The morphisms $\alpha_{\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n}$ and $\alpha_{\text{sk}_{n-1} X}$ are isomorphisms by the inductive hypothesis. The morphism $\alpha_{\bigsqcup_{i=1}^n G/H_i \times \Delta^n}$ is an isomorphism by the case (3). We conclude by the Five Lemma that α_X is an isomorphism. ■

Remark 7.4. The morphism α_X of Lemma 7.1 can be defined for any $(G, \mathcal{F}in)$ -complex X . However, the hypothesis of X being $(G, \mathcal{F}in)$ -finite is needed for stating that α_X is natural in X – the codomain of α_X is a functor on X only when restricted to the full subcategory of S^G whose objects are $(G, \mathcal{F}in)$ -finite complexes.

Lemma 7.5. *Let G be a group satisfying (5.3). Let A be an algebra, B be a G -algebra and Z be a $(G, \mathcal{F}in)$ -complex. Then there is a natural isomorphism*

$$H_*^G(Z; \mathbb{K}(A, \mathcal{R}(B \rtimes -))) \cong H_*^G(Z; \mathbb{K}^G(A^{(-)}, B))$$

induced by a natural zig-zag of spectra.

Proof. Recall the notation and definitions introduced in Section 6. By Theorem 6.34, we have a zig-zag of $\text{Or}(G, \mathcal{F}in)$ -spectra as follows, where the morphisms labeled with \sim are objectwise weak equivalences of spectra:

$$\begin{array}{ccc} \mathbb{K}(A, \mathcal{R}(B \rtimes t)) & \xleftarrow{\sim} & \int^{\mathcal{O}/t} (u_t)^* \delta_{\bar{Q}J} \longrightarrow \int^{\mathcal{O}/t} (u_t)^* (u_t)_! (u_t)^* \delta_{\bar{Q}J} \\ & & \uparrow \wr \\ \int^{\mathcal{O}/t} (u_t)^* Q(M_t) & \xleftarrow{\varphi \circ \bar{q}} & \int^{\mathcal{O}/t} (u_t)^* Q(u_t)_! (u_t)^* \delta_{\bar{Q}J} \quad (7.6) \\ \uparrow \wr & & \\ \int^{\mathcal{O}/t} (u_t)^* Q(L_t) & \xrightarrow{\zeta \circ q} & \mathbb{K}^G(A^{(t)}, M_G B). \end{array}$$

We also have a zig-zag of $\text{Or}(G, \mathcal{F}in)$ -spectra as follows, induced by the G -stability zig-zag (2.13):

$$\mathbb{K}^G(A^{(t)}, M_G B) \xrightarrow[\sim]{\iota'} \mathbb{K}^G(A^{(t)}, M_{G_+} B) \xleftarrow[\sim]{\iota} \mathbb{K}^G(A^{(t)}, B) \tag{7.7}$$

In (7.7), both morphisms are objectwise weak equivalences of spectra since the G -algebra homomorphisms of (2.13) induce isomorphisms in kk^G . To ease notation, we name the $\text{Or}(G, \mathcal{F}in)$ -spectra appearing in the zig-zags above as follows:

$$\begin{aligned} \mathbf{F}(t) &:= \mathbb{K}(A, \mathcal{R}(B \rtimes t)), \\ \mathbf{R}_1(t) &:= \int^{\mathcal{O}/t} (u_t)^* \delta_{\overline{Q}J}, \\ \mathbf{R}_2(t) &:= \int^{\mathcal{O}/t} (u_t)^* (u_t)! (u_t)^* \delta_{\overline{Q}J}, \\ \mathbf{R}_3(t) &:= \int^{\mathcal{O}/t} (u_t)^* Q(u_t)! (u_t)^* \delta_{\overline{Q}J}, \\ \mathbf{R}_4(t) &:= \int^{\mathcal{O}/t} (u_t)^* Q(M_t), \\ \mathbf{R}_5(t) &:= \int^{\mathcal{O}/t} (u_t)^* Q(L_t), \\ \mathbf{R}_6(t) &:= \mathbb{K}^G(A^{(t)}, M_G B), \\ \mathbf{R}_7(t) &:= \mathbb{K}^G(A^{(t)}, M_{G_+} B), \\ \mathbf{E}(t) &:= \mathbb{K}^G(A^{(t)}, B). \end{aligned}$$

Upon concatenating (7.6) and (7.7) we get a zig-zag of $\text{Or}(G, \mathcal{F}in)$ -spectra as follows, where the morphisms labeled with \sim are objectwise weak equivalences of spectra:

$$\mathbf{F} \xleftarrow[\sim]{} \mathbf{R}_1 \longrightarrow \mathbf{R}_2 \xleftarrow[\sim]{} \mathbf{R}_3 \longrightarrow \mathbf{R}_4 \xleftarrow[\sim]{} \mathbf{R}_5 \longrightarrow \mathbf{R}_6 \xrightarrow[\sim]{} \mathbf{R}_7 \xleftarrow[\sim]{} \mathbf{E}. \tag{7.8}$$

Moreover, by Theorem (6.34), this zig-zag induces the isomorphism (6.1) upon taking homotopy groups.

Let Z be a $(G, \mathcal{F}in)$ -complex. After applying $H_*^G(Z; -)$ to (7.8) we get a zig-zag of graded abelian groups

$$H_*^G(Z; \mathbf{F}) \xleftarrow[\cong]{} H_*^G(Z; \mathbf{R}_1) \longrightarrow H_*^G(Z; \mathbf{R}_2) \xleftarrow[\cong]{} \dots \xleftarrow[\cong]{} H_*^G(Z; \mathbf{E}). \tag{7.9}$$

Each arrow in (7.9) is natural in Z since it is induced by a morphism of $\text{Or}(G, \mathcal{F}in)$ -spectra; see Remark 2.9. Moreover, by Remark 2.9, those arrows labeled with \sim in (7.8) induce natural isomorphisms in (7.9) and thus can be uniquely inverted. Upon inverting the natural isomorphisms in (7.9) we get a chain of natural transformations as follows:

$$H_*^G(Z; \mathbf{F}) \longrightarrow H_*^G(Z; \mathbf{R}_1) \longrightarrow H_*^G(Z; \mathbf{R}_2) \longrightarrow \dots \longrightarrow H_*^G(Z; \mathbf{E}). \tag{7.10}$$

Write $\beta_Z : H_*^G(Z; \mathbf{F}) \rightarrow H_*^G(Z; \mathbf{E})$ for the composite of the morphisms above. It is clear that β_Z is natural in Z since each one of the morphisms appearing in (7.10) is. We claim, moreover, that β_Z is an isomorphism. Since homology commutes with filtered unions, it suffices to prove this for $(G, \mathcal{F}in)$ -finite Z . The claim holds for $Z = G/N$ since, in this case, β_Z is the isomorphism (6.1) by Theorem 6.34. Now we can continue as in the proof of Lemma 7.1: the case $Z = G/H \times \Delta^n$ follows from homotopy invariance and the general case follows from excision upon considering the skeletal filtration of Z . ■

Combining Lemma 7.1 and Lemma 7.5 with the fact that homology commutes with filtered unions we get the main result of this paper.

Theorem 7.11. *Let G be a group satisfying (5.3). Let A be an algebra, B be a G -algebra and Z be a $(G, \mathcal{F}in)$ -complex. Then there is a natural isomorphism*

$$H_*^G(Z; \mathbb{K}(A, \mathcal{R}(B \rtimes -))) \cong \operatorname{colim}_{\substack{X \subseteq Z \\ G\text{-finite}}} kk_*^G(A^{(X)}, B)$$

induced by a natural zig-zag of spectra.

Remark 7.12. Let G be a group satisfying (5.3) and let B be a G -algebra. Define an $\operatorname{Or}(G)$ -spectrum \mathbf{KH}_B by $\mathbf{KH}_B(G/H) := \mathbb{K}(\ell, \mathcal{R}(B \rtimes G/H))$ and note that it satisfies (2.11). By Theorem 7.11, for every $(G, \mathcal{F}in)$ -complex Z we have

$$H_*^G(Z; \mathbf{KH}_B) \cong \operatorname{colim}_{\substack{X \subseteq Z \\ G\text{-finite}}} kk_*^G(\ell^{(X)}, B).$$

When taking $Z = E_{\mathcal{F}in}(G)$, this shows that the domain of the KH-assembly map can be expressed in terms of kk^G -groups in a way that is completely analogous to that of the Baum–Connes assembly map.

8. Towards a kk -theoretic assembly map

8.1. The Baum–Connes assembly map

Let us briefly recall the definition of the Baum–Connes assembly map as formulated in [4, Section 9]. Fix a G - C^* -algebra B and write $B \rtimes_r G$ for the reduced crossed product. Let $E_{\mathcal{F}in}(G)$ be a topological model of the classifying space of G with respect to $\mathcal{F}in$. To every proper G -compact G -space X one associates a canonical class $e_X \in KK(\mathbb{C}, C_0(X) \rtimes G)$, where $C_0(X)$ denotes the algebra of continuous functions $X \rightarrow \mathbb{C}$ that vanish at infinity. Now consider the following composites:

$$KK_*^G(C_0(X), B) \xrightarrow{-\rtimes_r G} KK_*(C_0(X) \rtimes_r G, B \rtimes_r G) \xrightarrow{e_X^*} KK_*(\mathbb{C}, B \rtimes_r G).$$

As X varies over the closed G -compact subsets of $E_{\mathcal{F}in}(G)$, these maps are compatible and define the *Baum–Connes assembly map*,

$$\operatorname{colim}_{\substack{X \subseteq E_{\mathcal{F}in}(G) \\ G\text{-compact}}} KK_*^G(C_0(X), B) \rightarrow KK_*(\mathbb{C}, B \rtimes_r G) \cong K_*^{\operatorname{top}}(B \rtimes_r G). \quad (8.1)$$

The canonical class e_X can be obtained as we proceed to explain. Let $C_c(X)$ denote the algebra of continuous functions $X \rightarrow \mathbb{C}$ with compact support. A *cut-off function* for X is a nonnegative $f \in C_c(X)$ such that $\sum_{g \in G} f(g \cdot x) = 1$ for all $x \in X$; such functions always exist for proper and G -compact X . Let f be a cut-off function for X and define $p_f \in C_c(G \times X)$ by

$$p_f(g, x) := \sqrt{f(x)f(g^{-1} \cdot x)}.$$

If we consider $p_f \in C_0(X) \rtimes G$, then p_f is a projection and its class

$$e_X = [p_f] \in KK_0(\mathbb{C}, C_0(X) \rtimes G)$$

is independent of the choice of f [24, Lemma 2.2].

Example 8.2. Let G be a (discrete) group and let $H \subseteq G$ be a finite subgroup. For every $uH \in G/H$ we have a cut-off function $f_{uH} := \frac{1}{|H|}\chi_{uH} \in C_c(G/H)$. The corresponding projection is

$$p_{uH} := \frac{1}{|H|}\chi_{uH} \rtimes \sum_{h \in H} uhu^{-1} \in C_0(G/H) \rtimes G. \tag{8.3}$$

8.2. The KH-assembly map

Let G be a group satisfying (5.3) and let B be a G -algebra. Taking Remark 7.12 into account, it is a natural question whether the KH-assembly map admits a kk -theoretic description analogous to (8.1). We could expect to find, for each $(G, \mathcal{F}in)$ -finite complex X , a natural class $e_X \in kk_0(\ell, \ell^{(X)} \rtimes G)$ such that the composites

$$kk_*^G(\ell^{(X)}, B) \xrightarrow{-\rtimes G} kk_*(\ell^{(X)} \rtimes G, B \rtimes G) \xrightarrow{e_X^*} kk_*(\ell, B \rtimes G)$$

are natural in X and define a morphism \mathcal{A} making the following diagram commute:

$$\begin{array}{ccc} H_*^G(E_{\mathcal{F}in}(G); \mathbb{K}(\ell, \mathcal{R}(B \rtimes -))) & \xrightarrow{\text{pr}} & kk_*(\ell, B \rtimes G) \cong \text{KH}_*(B \rtimes G). \\ \downarrow \cong & & \uparrow \mathcal{A} \\ \text{Theorem 7.11} & & \\ \downarrow & & \\ \text{colim}_{\substack{X \subseteq E_{\mathcal{F}in}(G) \\ G\text{-compact}}} & & kk_*^G(\ell^{(X)}, B) \end{array}$$

The authors have no results in this generality although it is clear how to define e_X for $X = G/H$, as we explain below.

Let $H \subseteq G$ be a finite subgroup. The formula (8.3) defines an idempotent

$$p_{uH} = \frac{1}{|H|}\chi_{uH} \rtimes \sum_{h \in H} uhu^{-1} \in \ell^{(G/H)} \rtimes G \tag{8.4}$$

and thus an element $[p_{uH}] \in kk_0(\ell, \ell^{(G/H)} \rtimes G)$. It is immediate that the latter does not depend upon uH since we have

$$(1 \rtimes g)p_H(1 \rtimes g^{-1}) = p_{uH} \in \ell^{G/H} \rtimes G.$$

Lemma 8.5. *Let G be a group satisfying (5.3) and let B be a G -algebra. Let $H \subseteq G$ be a finite subgroup and let $p_H \in \ell^{(G/H)} \rtimes G$ be the idempotent defined in (8.4). Then the following diagram commutes:*

$$\begin{array}{ccc}
 H_*^G(G/H; \mathbb{K}(\ell, \mathcal{R}(B \rtimes -))) & \xrightarrow{\text{pr}} & kk_*(\ell, B \rtimes G) \cong \text{KH}_*(B \rtimes G). \\
 \downarrow \text{Theorem 7.11} \cong & & \nearrow \\
 kk_*^G(\ell^{(G/H)}, B) & \xrightarrow{(- \circ [p_H]) \circ (- \rtimes G)} &
 \end{array} \tag{8.6}$$

Proof. For every algebra A we have a morphism $p_H \otimes A : A \rightarrow A^{(G/H)} \rtimes G$. We will prove that the following diagram commutes for every algebra A ; note that the commutativity of the outer square for $A = \ell$ is equivalent to that of (8.6).

$$\begin{array}{ccc}
 H_*^G(G/H; \mathbb{K}(A, \mathcal{R}(B \rtimes -))) & \xrightarrow{\text{pr}} & H_*^G(G/G; \mathbb{K}(A, \mathcal{R}(B \rtimes -))) \\
 \downarrow \text{Theorem 7.11} \cong & \searrow \cong & \downarrow \cong \\
 & kk_*(A, \mathcal{R}(B \rtimes G/H)) & \xrightarrow{\text{pr}} kk_*(A, \mathcal{R}(B \rtimes G/G)) \\
 & \swarrow \cong (6.1) & \parallel \\
 kk_*^G(A^{(G/H)}, B) & \xrightarrow{(p_H \otimes A)^* \circ (- \rtimes G)} & kk_*(A, B \rtimes G).
 \end{array} \tag{8.7}$$

To see that the triangle in (8.7) commutes, recall that the isomorphism of Theorem 7.11 is induced by the zig-zag of $\text{Or}(G, \mathcal{F}in)$ -spectra from Theorem 6.34 and that the latter recovers the isomorphism (6.1) upon taking homotopy groups. As the upper right square in (8.7) clearly commutes, we have to prove the commutativity of the lower right one. After identifying $kk_*(A, \mathcal{R}(B \rtimes G/H)) \cong kk_*(A, B \rtimes H)$ using (4.14) and unraveling the details of the isomorphism (6.1), we are left to show that the following diagram commutes for all n :

$$\begin{array}{ccc}
 kk_n(A, B \rtimes H) & \xrightarrow{\text{incl}_*} & kk_n(A, B \rtimes G) \\
 \downarrow (-)^{(G/H)} & & \downarrow \iota_* \cong (2.13) \\
 kk_n^G(A^{(G/H)}, (R \rtimes H)^{(G/H)}) & & kk_n(A, (M_{G^+} B) \rtimes G) \\
 \downarrow (\psi_B)_* (3.2) & & \uparrow (\iota')_* \cong (2.13) \\
 kk_n^G(A^{(G/H)}, M_G B) & \xrightarrow{(p_H \otimes A)^* \circ (- \rtimes G)} & kk_n(A, (M_G B) \rtimes G).
 \end{array} \tag{8.8}$$

It suffices to consider the case $n = 0$ since the general case follows upon replacing A by $\Sigma^n A$ if $n > 0$ and by $\Omega^{-n} A$ if $n < 0$; see [9, Corollary 6.4.2]. Let $\alpha \in kk(A, B \rtimes H)$ and put

$$\begin{aligned}
 \alpha_1 &:= (\iota_* \circ \text{incl}_*)(\alpha), \\
 \alpha_2 &:= ((\iota')_* \circ (p_H \otimes A)^* \circ (- \rtimes G) \circ (\psi_B)_* \circ (-)^{(G/H)})(\alpha).
 \end{aligned}$$

We will show that $\alpha_1 = \alpha_2$. Recall from (2.15) that we have an isomorphism

$$(R_{G_+,B})_* : kk(A, (M_{G_+} B) \rtimes G) \rightarrow kk(A, M_{|G_+|}(B \rtimes G)).$$

We claim that $(R_{G_+,B})_*(\alpha_1) = (R_{G_+,B})_*(\alpha_2)$. First suppose that α is represented by an algebra homomorphism $f : A \rightarrow B \rtimes H$. Fix $a \in A$ and write

$$f(a) = \sum_{k \in H} b_k \rtimes k.$$

It is straightforward to verify that $(R_{G_+,B})_*(\alpha_i)$ is represented by the algebra homomorphism $f_i : A \rightarrow M_{|G_+|}(B \rtimes G)$, where f_1 and f_2 are given by

$$\begin{aligned} f_1(a) &= \sum_{k \in H} e_{+,+} \otimes (b_k \rtimes k), \\ f_2(a) &= \frac{1}{|H|} \sum_{p,t,k \in H} e_{p,t} \otimes (p \cdot b_k \rtimes pkt^{-1}). \end{aligned}$$

An easy computation shows that

$$(V_H \otimes 1)V^{-1}f_2V(V_H^{-1} \otimes 1) = f_1$$

where V is defined in (3.5) and V_H in (3.9). Thus, f_1 and f_2 induce the same morphism in kk . Suppose now that α is represented by an algebra homomorphism

$$f : J^r A \rightarrow (B \rtimes H)^{S^r}.$$

Let $\Omega : kk \rightarrow kk$ denote the translation functor in kk and recall that we have natural isomorphisms as follows; see [9, Lemma 6.3.11] and [29, Lemma 7.10]:

$$kk(A, B) \cong kk(\Omega^r A, \Omega^r B) \cong kk(J^r A, B^{S^r})$$

Since these are compatible with all the morphisms appearing in (8.8), we may replace A by $J^r A$, B by B^{S^r} and reduce to the case $r = 0$ that we have already addressed. ■

A. G -Simplicial sets

Let G be a group. We recall some definitions and properties concerning the G -simplicial sets. A G -simplicial set is a simplicial set with a left action of G . We write \mathbb{S}^G for the category of G -simplicial sets with equivariant morphisms. Every G -simplicial set X has a skeletal filtration such that the n -skeleton $sk_n X$ is obtained from $sk_{n-1} X$ upon attaching cells of the form $G/H \times \Delta^n$ with H a subgroup of G . We say that X is G -finite if X can be built from a finite number of these cells; it is easily verified that X is G -finite if and only if $G \backslash X$ is a finite simplicial set. Let \mathcal{F} be a nonempty family of subgroups of G closed under conjugation and subgroups; we are interested in the family $\mathcal{F}in$ of finite subgroups. A G -simplicial set X is called a (G, \mathcal{F}) -complex if X can be built from cells

of the form $G/H \times \Delta^n$ with $H \in \mathcal{F}$. The (G, \mathcal{F}) -complexes are the cofibrant objects for a certain model structure on \mathbb{S}^G ; see [8, Proposition 2.3]. A G -simplicial set X is called (G, \mathcal{F}) -finite if it can be built from a finite number of cells of the form $G/H \times \Delta^n$ with $H \in \mathcal{F}$. It is easily verified that X is (G, \mathcal{F}) -finite if and only if X is a G -finite (G, \mathcal{F}) -complex. In the rest of this section we gather some technical results that are used in Section 7.

Lemma A.1. *Let G be a group and let $\mathbb{S}_c \subset \mathbb{S}$ denote the full subcategory of connected simplicial sets. Let $G/H, G/K \in \text{Or}(G)$ and $X, Y \in \mathbb{S}_c$. Then there is a natural isomorphism*

$$\text{Hom}_{\mathbb{S}^G}(G/H \times X, G/K \times Y) \cong \text{Hom}_{\text{Or}(G)}(G/H, G/K) \times \text{Hom}_{\mathbb{S}}(X, Y).$$

In other words, the full subcategory of \mathbb{S}^G whose objects are $G/H \times X$ with $G/H \in \text{Or}(G)$ and $X \in \mathbb{S}_c$ is equivalent to the product category $\text{Or}(G) \times \mathbb{S}_c$.

Proof. Let $f : G/H \times X \rightarrow G/K \times Y$ be a morphism in \mathbb{S}^G . We claim that there exist a unique coset $uK \in G/K$ and a unique morphism $h : X \rightarrow Y$ that fit into a commutative square as follows, as we proceed to explain.

$$\begin{array}{ccc} \bigsqcup_{G/H} X & \xrightarrow{f} & \bigsqcup_{G/K} Y \\ \text{can}_H \uparrow & & \uparrow \text{can}_{uK} \\ X & \xrightarrow{h} & Y. \end{array} \tag{A.2}$$

Since Y is a connected simplicial set, the set of connected components of $\bigsqcup_{G/K} Y$ is $\{\text{can}_{uK}(Y)\}_{uK \in G/K}$. Since X is a connected simplicial set, there is a unique connected component $\text{can}_{uK}(Y)$ of $\bigsqcup_{G/K} Y$ such that $(f \circ \text{can}_H)(X) \subseteq \text{can}_{uK}(Y)$. Now define h as the following composite:

$$X \xrightarrow{f \circ \text{can}_H} \text{can}_{uK}(Y) \xrightarrow[\cong]{(\text{can}_{uK})^{-1}} Y.$$

It is clear that h makes the square (A.2) commute. Moreover, it follows from the equivariance of f that $g(tH) = tuK$ defines a morphism $g : G/H \rightarrow G/K$. Conversely, every pair $(g, h) \in \text{Hom}_{\text{Or}(G)}(G/H, G/K) \times \text{Hom}_{\mathbb{S}}(X, Y)$ defines a unique G -equivariant morphism f making the following squares commute for all t :

$$\begin{array}{ccc} \bigsqcup_{G/H} X & \xrightarrow{f} & \bigsqcup_{G/K} Y \\ \text{can}_t \uparrow & & \uparrow \text{can}_{g(tH)} \\ X & \xrightarrow{h} & Y. \end{array}$$

It is easily verified that both constructions are mutually inverse. ■

Lemma A.3. *Let \mathcal{F} be a family of subgroups of G and let $\psi : Y \rightarrow X$ be a morphism of G -simplicial sets. If X is a (G, \mathcal{F}) -complex then Y is a (G, \mathcal{F}) -complex too.*

Proof. Let $\sigma \in Y_n$; it is easily verified that $\text{Stab}(\sigma) \subseteq \text{Stab}(\psi(\sigma)) \in \mathcal{F}$. ■

Lemma A.4. *Let G be an infinite group, let $H \subseteq G$ be a subgroup, let X be a $(G, \mathcal{F}in)$ -complex and let K be a finite simplicial set. Then every G -equivariant morphism $\psi : G/H \times K \rightarrow X$ is proper, i.e. $\psi^{-1}(L)$ is a finite simplicial set for every finite simplicial subset $L \subseteq X$.*

Proof. First of all notice that $H \in \mathcal{F}in$ by Lemma A.3. Let $L \subseteq X$ be a finite simplicial set and suppose that $\psi^{-1}(L)$ is not finite. Then there is an infinite number of non-degenerate simplices in $\psi^{-1}(L) \subseteq G/H \times K$. Since every non-degenerate simplex of $G/H \times K$ has dimension $\leq d := \dim K$, there exists $0 \leq p \leq d$ such that there is an infinite number of non-degenerate p -simplices in $\psi^{-1}(L)$. Let $\{g_i, i \in I\} \subseteq G$ be a system of representatives for the cosets in G/H ; notice that I is infinite because H is finite. Every non-degenerate p -simplex of $G/H \times K$ is of the form $(g_i H, \sigma)$ for some $i \in I$ and some non-degenerate p -simplex σ of K . Since K has finitely many non-degenerate p -simplices, there exist a non-degenerate p -simplex σ of K and an infinite subset $J \subseteq I$ such that

$$\{(g_i H, \sigma), i \in J\} \subseteq \psi^{-1}(L).$$

Then

$$\{\psi(g_i H, \sigma), i \in J\} \subseteq L_p.$$

Since L_p is a finite set, replacing J by a smaller but still infinite subset, we can assume without loss of generality that there is $\tau \in L_p$ such that $\psi(g_i H, \sigma) = \tau$ for every $i \in J$. Fix $i_0 \in J$. Then

$$g_{i_0} \cdot \psi(H, \sigma) = \psi(g_{i_0} H, \sigma) = \tau = \psi(g_i H, \sigma) = g_i \cdot \psi(H, \sigma)$$

for every $i \in J$ and it follows that $\{g_i^{-1} g_{i_0}, i \in J\} \subseteq \text{Stab}(\psi(H, \sigma)) \in \mathcal{F}in$; this is a contradiction since J is infinite. ■

Lemma A.5. *Let X be a $(G, \mathcal{F}in)$ -complex and Y be a $(G, \mathcal{F}in)$ -finite simplicial set. Then every morphism $\phi : Y \rightarrow X$ is proper.*

Proof. Let $K \subseteq X$ be a finite simplicial subset and suppose that $\phi^{-1}(K)$ is not finite. Note that $\phi^{-1}(K)$ has finite dimension since $\phi^{-1}(K) \subseteq Y$ and Y has finite dimension. It follows that, for some $n \geq 0$, $\phi^{-1}(K)$ has infinitely many non-degenerate n -simplices. Let $\{\sigma_k\}_{k \in \mathbb{N}} \subseteq \phi^{-1}(K)$ be a list of distinct non-degenerate n -simplices. Consider a pushout diagram

$$\begin{array}{ccc} \bigsqcup_{i=1}^r G/H_i \times \partial \Delta^n & \longrightarrow & \text{sk}_{n-1} Y \\ \downarrow & & \downarrow \\ \bigsqcup_{i=1}^r G/H_i \times \Delta^n & \longrightarrow & \text{sk}_n Y \end{array}$$

where the H_i are finite. For every $k \in \mathbb{N}$, there exist $1 \leq i_k \leq r$ and a non-degenerate n -simplex τ_k of $G/H_{i_k} \times \Delta^n$ such that σ_k is the image of τ_k under $G/H_{i_k} \times \Delta^n \rightarrow \text{sk}_n Y$. Then there exists some $1 \leq j \leq r$ such that $i_k = j$ for infinitely many values of k . We may assume without loss of generality that $i_k = j$ for all k . Let ψ be the following composite:

$$G/H_j \times \Delta^n \longrightarrow \text{sk}_n Y \xrightarrow{\text{incl}} Y \xrightarrow{\phi} X.$$

Then $\{\tau_k\}_{k \in \mathbb{N}}$ is a list of distinct non-degenerate n -simplices of $\psi^{-1}(K) \subseteq G/H_j \times \Delta^n$. But the latter is not possible since ψ is proper by Lemma A.4. ■

Lemma A.6. *Let G be a group, $K \subseteq G$ be a subgroup and X be a G -simplicial set. Then*

$$X^K \cong \text{colim}_{G/H \times \Delta^n \downarrow X} (G/H \times \Delta^n)^K.$$

Proof. There is a natural morphism

$$\text{colim}_{G/H \times \Delta^n \downarrow X} (G/H \times \Delta^n)^K \rightarrow X^K. \tag{A.7}$$

Let us prove that it is surjective. Let $\sigma \in (X^K)_p = (X_p)^K$ and put $H := \text{Stab}(\sigma)$; note that we have $K \subseteq H$. Let $f : G/H \times \Delta^p \rightarrow X$ be the G -equivariant morphism determined by $(H, \iota_p) \mapsto \sigma$. Since $(H, \iota_p) \in (G/H \times \Delta^p)^K$, we have that $\sigma = f^K(H, \iota_p)$, showing that σ is in the image of (A.7). We still have to prove that (A.7) is injective. Let us first show that every p -simplex of

$$\text{colim}_{G/H \times \Delta^n \downarrow X} (G/H \times \Delta^n)^K \tag{A.8}$$

is represented by one of the form $(L, \iota_p) \in (G/L \times \Delta^p)^K$. Let $f : G/H \times \Delta^n \rightarrow X$ be a G -equivariant morphism and let (gH, τ) be a p -simplex of $(G/H \times \Delta^n)^K$. Then $K \subseteq gHg^{-1}$ and $\tau = \tau_*(\iota_p)$ for some nondecreasing function $\tau : [p] \rightarrow [n]$. The commutativity of the following triangle implies that (gH, τ) and (gH, ι_p) represent the same simplex of (A.8):

$$\begin{array}{ccc} (gH, \tau) & G/H \times \Delta^n & \xrightarrow{f} X \\ \uparrow & \uparrow \text{id} \times \tau_* & \nearrow f \circ (\text{id} \times \tau_*) \\ (gH, \iota_p) & G/H \times \Delta^p & \end{array}$$

Write $L := gHg^{-1}$ and note that there is a G -equivariant bijection $\beta : G/H \rightarrow G/L$ determined by $\beta(H) = g^{-1}L$. The commutativity of the following triangle implies that (gH, ι_p) and (L, ι_p) represent the same simplex of (A.8):

$$\begin{array}{ccc} (gH, \iota_p) & G/H \times \Delta^p & \xrightarrow{f \circ (\text{id} \times \tau_*)} X \\ \downarrow & \downarrow \beta \times \text{id} \cong & \nearrow \\ (L, \iota_p) & G/L \times \Delta^p & \end{array}$$

Now let (L, ι_p) and (M, ι_p) represent two p -simplices of (A.8) having the same image $\sigma \in (X^K)_p$ under (A.7). Put $S := \text{Stab}(\sigma)$ and note that $L, M \subseteq S$. The commutativity of the following diagram shows that (L, ι_p) and (M, ι_p) represent the same simplex of (A.7):

$$\begin{array}{ccc}
 (L, \iota_p) & G/L \times \Delta^p & \searrow \\
 \downarrow & \downarrow & \searrow \\
 (S, \iota_p) & G/S \times \Delta^p & \longrightarrow X \\
 \uparrow & \uparrow & \nearrow \\
 (M, \iota_p) & G/M \times \Delta^p & \nearrow
 \end{array}$$

This finishes the proof. ■

B. A Mayer–Vietoris sequence in kk -theory

Definition B.1. A *Milnor square of G -algebras* is a pullback square

$$\begin{array}{ccc}
 A & \longrightarrow & B \\
 \downarrow & & \downarrow f \\
 C & \longrightarrow & D
 \end{array}$$

where f is surjective and has a G -linear section.

Lemma B.2. Let A be an algebra and let X be a $(G, \mathcal{F}in)$ -finite G -simplicial set. For each $n \geq 1$ there is a pushout diagram

$$\begin{array}{ccc}
 \bigsqcup_{i=1}^r G/H_i \times \partial\Delta^n & \longrightarrow & \text{sk}_{n-1} X \\
 \downarrow & & \downarrow \\
 \bigsqcup_{i=1}^r G/H_i \times \Delta^n & \longrightarrow & \text{sk}_n X
 \end{array} \tag{B.3}$$

of G -simplicial sets with $H_i \in \mathcal{F}in$ for all i . Then all the morphisms appearing in (B.3) are proper and this diagram induces the following Milnor square of G -algebras:

$$\begin{array}{ccc}
 A\left(\bigsqcup_{i=1}^r G/H_i \times \partial\Delta^n\right) & \longleftarrow & A^{(\text{sk}_{n-1} X)} \\
 \uparrow & & \uparrow \\
 A\left(\bigsqcup_{i=1}^r G/H_i \times \Delta^n\right) & \longleftarrow & A^{(\text{sk}_n X)}.
 \end{array} \tag{B.4}$$

Proof. All the morphisms in (B.3) are proper by Lemma A.5. Then we can apply $A^{(-)}$ to get a commutative diagram of G -algebras like (B.4) that it is easily seen to be a pullback;

see Remark 2.3. Write $i : \partial\Delta^n \rightarrow \Delta^n$ for the inclusion. By Lemma 2.1, the morphism $i^* : A^{\Delta^n} \rightarrow A^{\partial\Delta^n}$ admits a linear section. Then the left vertical morphism in (B.4) admits a G -linear section, since it identifies with

$$\bigoplus_i \ell^{(G/H_i)} \otimes i^* : \bigoplus_i \ell^{(G/H_i)} \otimes A^{\Delta^n} \longrightarrow \bigoplus_i \ell^{(G/H_i)} \otimes A^{\partial\Delta^n}$$

by Remark 2.3. ■

Lemma B.5. *Let E be a G -algebra. Then every Milnor square of G -algebras*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

induces a long exact Mayer–Vietoris sequence as follows:

$$kk_*^G(D, E) \longrightarrow kk_*^G(B, E) \oplus kk_*^G(C, E) \longrightarrow kk_*^G(A, E) \longrightarrow kk_{*-1}^G(D, E).$$

Proof. It follows from excision in kk^G [15, Theorem 4.1.1] and from the argument given in [12, Theorem 2.41]. ■

C. The model category of spectra and spectra representing kk -theory

C.1. The stable model category of spectra

In this section, we recall the definition of the stable model category of spectra and discuss some of its properties.

Definition C.1 ([5, Definition 2.1]). A *spectrum* X is a sequence of pointed simplicial sets X^0, X^1, X^2, \dots together with pointed morphisms

$$S^1 \wedge X^n \rightarrow X^{n+1} \quad \text{for all } n,$$

called *bonding maps*. Here, $S^1 = \Delta^1/\partial\Delta^1$. A *morphism of spectra* $f : X \rightarrow Y$ is a sequence of pointed morphisms $f^n : X^n \rightarrow Y^n$ that commute with the bonding maps as follows:

$$\begin{array}{ccc} S^1 \wedge X^n & \longrightarrow & X^{n+1} \\ S^1 \wedge f^n \downarrow & & \downarrow f^{n+1} \\ S^1 \wedge Y^n & \longrightarrow & Y^{n+1}. \end{array}$$

We write Sp for the category of spectra and morphisms of spectra.

We endow Sp with its stable model structure, that we proceed to describe; see [5, Section 2] and [20, Section 3] for details:

- Let X be a spectrum and let $m \in \mathbb{Z}$. The m -th stable homotopy group of X is defined as

$$\pi_m(X) = \operatorname{colim}_k \pi_{m+k}(X^k).$$

A morphism of spectra $f : X \rightarrow Y$ is a *weak equivalence* if $\pi_m(f)$ is an isomorphism for all $m \in \mathbb{Z}$.

- A morphism of spectra $f : X \rightarrow Y$ is a *fibration* if $f^n : X^n \rightarrow Y^n$ is a fibration of simplicial sets for all n .
- A morphism of spectra is a *cofibration* if it has the left lifting property with respect to trivial fibrations.

Lemma C.2 ([30, Example 3.6 (iii)]). *The stable model structure on Sp is combinatorial.*

Proof. Recall from [14, Definition 2.1] that a model category is *combinatorial* if it is cofibrantly generated and its underlying category is locally presentable. The stable model structure on Sp is cofibrantly generated by [20, Definition 3.3 and Corollary 3.5]. We have to show that the underlying category is locally presentable. We claim that Sp is locally *finitely* presentable. By [1, Theorem 1.11], to prove this claim we have to show that Sp has a strong generator formed by finitely presentable objects. By [1, Example 1.12], the set $\{\Delta^m : m \geq 0\}$ is a strong generator for \mathbb{S} formed by finitely presentable objects. Since the forgetful functor $\mathbb{S}_* \rightarrow \mathbb{S}$ commutes with filtered colimits, the latter is easily seen to imply that $\{\Delta^m_+ : m \geq 0\}$ is a strong generator for \mathbb{S}_* formed by finitely presentable objects. For $n \geq 0$, let $F_n : \mathbb{S}_* \rightarrow \operatorname{Sp}$ be the left adjoint to evaluation at n . Explicitly, for a pointed simplicial set X , let $F_n(X)$ be the spectrum whose level k is $(S^1)^{\wedge(k-n)} \wedge X$ if $k \geq n$ and $*$ otherwise. The bonding maps are the obvious ones. It is easily verified that $\{F_n(\Delta^m_+) : m, n \geq 0\}$ is a strong generator for Sp formed by finitely presentable spectra. ■

C.2. Bivariant K -theory spectra

In this section, we recall the definitions of spectra representing kk -theory [17, Theorem 9.8] and kk^G -theory [27, Theorem 5.3.11].

Let \mathcal{C} denote either Alg_ℓ or $G\operatorname{Alg}_\ell$. For two objects A and B of \mathcal{C} , the *bivariant K -theory space* of the pair (A, B) [28, Definition 4.10] is defined as the fibrant simplicial set

$$\mathcal{H}(A, B) := \operatorname{colim}_n \Omega^n \operatorname{Ex}^\infty \operatorname{Hom}_{\mathcal{C}}(J^n A, B^\Delta).$$

This definition is equivalent to the original one given in [17, Section 4]. By [17, Theorem 5.1], there is a natural isomorphism of simplicial sets

$$\mathcal{H}(A, B) \cong \Omega \mathcal{H}(JA, B).$$

Thus, we have an Ω -spectrum $\mathbb{K}(A, B)$ defined by the sequence

$$\mathcal{H}(A, B), \mathcal{H}(JA, B), \mathcal{H}(J^2 A, B), \dots$$

The spectrum $\mathbb{K}(A, B)$ (denoted by $\mathbb{K}^{\text{unst}}(A, B)$ in [17]) represents a universal bivariant K -theory introduced by Garkusha that is excisive, homotopy invariant but matrix-unstable [17, Comparison Theorem B]. Different matrix-stabilizations can be performed in order to obtain spectra representing kk - and kk^G -theories:

- (1) *Stabilization by finite matrices.* For two objects A and B of \mathcal{C} put

$$\mathbb{K}_f(A, B) := \operatorname{colim}_n \mathbb{K}(A, M_n B)$$

where the transition maps are induced by the inclusion $M_n B \rightarrow M_{n+1} B$ into the upper left corner. These spectra represent a universal bivariant K -theory that is excisive, homotopy invariant and stable by finite matrices [17, Theorem 9.8]; see [17, Section 9] and [27, Section 5.1] for details.

- (2) *Stabilization by finite matrices indexed on an infinite set.* Let X be an infinite set. For two objects A and B of \mathcal{C} put [27, Definition 5.2.21]

$$\mathbb{K}_X(A, B) := \mathbb{K}_f(A, M_X B).$$

The spectra $\mathbb{K}_X(A, B)$ represent a universal bivariant K -theory that is excisive, homotopy invariant and M_X -stable [27, Theorem 5.2.22]. For any X , Weibel’s homotopy K -theory KH is the functor represented by the base ring ℓ [27, Theorem 5.2.20]. For $X = \mathbb{N}$, this theory coincides with the kk -theory defined in [9].

- (3) *G -stabilization.* Let G be a group and let $X = \mathbb{N} \times |G|$. For two G -algebras A and B put

$$\mathbb{K}^G(A, B) := \mathbb{K}_X(M_G A, M_G B).$$

These spectra represent kk^G -theory [27, Theorem 5.3.11].

Lemma C.3 (cf. [27, Section 4.4]). *Let $\mathbb{S}_f \subset \mathbb{S}$ denote the full subcategory of finite simplicial sets. Let G be a group, X be an infinite set, $G/K \in \text{Or}(G, \mathcal{F}in)$, $A, B \in G\text{Alg}_\ell$ and $S \in \mathbb{S}_f$. Then, for $\mathbb{E} \in \{\mathbb{K}, \mathbb{K}_f, \mathbb{K}_X, \mathbb{K}^G\}$, there is a morphism of spectra*

$$S_+ \wedge \mathbb{E}(A^{(G/K)}, B) \rightarrow \mathbb{E}(A^{(G/K \times S)}, B) \tag{C.4}$$

that is natural in $A, B, G/K$ and S . Moreover, for $S = \Delta^0$ this is an isomorphism.

Proof. It suffices to prove the Lemma for $\mathbb{E} = \mathbb{K}$: the case $\mathbb{E} = \mathbb{K}_f$ follows from this upon taking colimit over the inclusions $M_n B \rightarrow M_{n+1} B$ and the rest of the cases follow from the latter upon replacing A and B with appropriate matrix algebras.

Let us prove the case $\mathbb{E} = \mathbb{K}$. We will define (C.4) levelwise. At level $p \geq 0$, we have to define a morphism of simplicial sets

$$S_+ \wedge \mathcal{K}(J^p(A^{(G/K)}), B) \rightarrow \mathcal{K}(J^p(A^{(G/K \times S)}), B). \tag{C.5}$$

Let us describe it in dimension $q \geq 0$. A q -simplex of

$$S_+ \wedge \mathcal{K}(J^p(A^{(G/K)}), B)$$

is represented by a pair (σ, α) where σ is a q -simplex of S and α is a q -simplex of $\mathcal{K}(J^p(A^{(G/K)}), B)$. Below we use the notation of [28, Section 2.9]. Let α be represented by a G -algebra homomorphism

$$\alpha : J^{p+v}(A^{(G/K)}) \rightarrow B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}$$

for some $v, r \geq 0$. Then the morphism (C.5) sends the pair (σ, α) to the q -simplex of $\mathcal{K}(J^p(A^{(G/K \times S)}), B)$ represented by the following composite in $G\text{Alg}_\ell$:

$$\begin{array}{ccc} J^{p+v}(A^{(G/K \times S)}) & \xlongequal{\quad} & J^{p+v}((A^{(G/K)})^S) \xrightarrow{\text{clas}} [J^{p+v}(A^{(G/K)})]^S \\ & & \downarrow \alpha_* \\ B_r^{(I^v \times \Delta^q \times \Delta^q, \partial I^v \times \Delta^q \times \Delta^q)} & \xleftarrow{\mu} & [B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}]^{\Delta^q} \xleftarrow{\sigma^*} [B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}]^S \\ & \downarrow \text{diag}^* & \\ B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)} & & \end{array}$$

Here, μ is the morphism defined in [28, Remark 3.4]. This clearly defines a morphism (C.5) that is natural in $A, B, G/K$ and S . Let us now show that (C.5) is an isomorphism for $S = \Delta^0$. First note that the classifying map

$$\text{clas} : J^{p+v}((A^{(G/K)})^{\Delta^0}) \rightarrow [J^{p+v}(A^{(G/K)})]^{\Delta^0}$$

is an isomorphism. Moreover, it follows from the naturality of μ that the composite $\text{diag}^* \circ \mu \circ \sigma^*$ equals the obvious isomorphism

$$[B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}]^{\Delta^0} \xrightarrow{\cong} B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}.$$

Together, these observations imply that, for $S = \Delta^0$ and making the obvious identifications, the morphism (C.5) is the identity of $\mathcal{K}(J^p(A^{(G/K)}), B)$. This finishes the proof. ■

Lemma C.6. *Let A and B be two G -algebras and let $f : C \rightarrow D$ be a morphism of G -algebras. Then the following square of spectra commutes:*

$$\begin{array}{ccc} \mathbb{K}^G(A, B) & \xrightarrow{-\otimes C} & \mathbb{K}^G(A \otimes C, B \otimes C) \\ -\otimes D \downarrow & & \downarrow f_* \\ \mathbb{K}^G(A \otimes D, B \otimes D) & \xrightarrow{f^*} & \mathbb{K}^G(A \otimes C, B \otimes D). \end{array}$$

Proof. Unraveling the definitions of the spectra $\mathbb{K}^G, \mathbb{K}_X$ and \mathbb{K}_f , it suffices to show that the following square commutes:

$$\begin{array}{ccc} \mathbb{K}(A, B) & \xrightarrow{-\otimes C} & \mathbb{K}(A \otimes C, B \otimes C) \\ -\otimes D \downarrow & & \downarrow f_* \\ \mathbb{K}(A \otimes D, B \otimes D) & \xrightarrow{f^*} & \mathbb{K}(A \otimes C, B \otimes D). \end{array}$$

At level $p \geq 0$ the latter is the following square of simplicial sets:

$$\begin{array}{ccc}
 \mathcal{K}(J^p A, B) & \xrightarrow{-\otimes C} & \mathcal{K}(J^p(A \otimes C), B \otimes C) \\
 \downarrow -\otimes D & & \downarrow f_* \\
 \mathcal{K}(J^p(A \otimes D), B \otimes D) & \xrightarrow{f^*} & \mathcal{K}(J^p(A \otimes C), B \otimes D).
 \end{array} \tag{C.7}$$

Let $q \geq 0$. Write $B_r^{S^n \times \Delta^q}$ instead of $B_r^{(I^n \times \Delta^q, \partial \Delta^n \times \Delta^q)}$ to ease notation. Let α be a q -simplex of $\mathcal{K}(J^p A, B)$, represented by an algebra homomorphism $\alpha : J^{p+n} A \rightarrow B_r^{S^n \times \Delta^q}$ for some $n, r \geq 0$. Consider the following commutative diagram of algebras:

$$\begin{array}{ccccccc}
 J^{p+n}(A \otimes C) & \xrightarrow{\text{clas}} & J^{p+n}(A) \otimes C & \xrightarrow{\alpha \otimes \text{id}} & B_r^{S^n \times \Delta^q} \otimes C & \xlongequal{\quad} & (B \otimes C)_r^{S^n \times \Delta^q} \\
 \downarrow J^{p+n}(\text{id} \otimes f) & & \downarrow J^{p+n}(\text{id}) \otimes f & & \downarrow \text{id} \otimes f & & \downarrow (\text{id} \otimes f)_r^{S^n \times \Delta^q} \\
 J^{p+n}(A \otimes D) & \xrightarrow{\text{clas}} & J^{p+n}(A) \otimes D & \xrightarrow{\alpha \otimes \text{id}} & B_r^{S^n \times \Delta^q} \otimes D & \xlongequal{\quad} & (B \otimes D)_r^{S^n \times \Delta^q}.
 \end{array}$$

The composite of the top morphisms followed by the rightmost vertical morphism represents the q -simplex $f_*(\alpha \otimes C)$ of $\mathcal{K}(J^p(A \otimes C), B \otimes D)$. The leftmost vertical morphism followed by the composite of the bottom morphisms represents the q -simplex $f^*(\alpha \otimes D)$ of $\mathcal{K}(J^p(A \otimes C), B \otimes D)$. The commutativity of the diagram shows that $f_*(\alpha \otimes C) = f^*(\alpha \otimes D)$ and, thus, that (C.7) commutes. ■

D. Equivariant kk -theory as a universal δ -functor into a graded category

Let G be a group. Equivariant kk^G -theory was introduced in [15, Theorem 4.1.1] as the universal homotopy invariant, G -stable and excisive functor from $G\text{Alg}_\ell$ into a triangulated category. This universal property allows us, for example, to define the crossed product with a subgroup $H \subseteq G$ – or better, with an orbit space G/H – at the level of kk -theory. Indeed, by Proposition 4.17, for every subgroup $H \subseteq G$ there exists a unique triangulated functor $\overline{- \rtimes G/H} : kk^G \rightarrow kk$ making the following diagram commute:

$$\begin{array}{ccc}
 G\text{Alg}_\ell & \xrightarrow{\mathcal{R}(- \rtimes G/H)} & \text{Alg}_\ell \\
 j^G \downarrow & & \downarrow j \\
 kk^G & \xrightarrow{\overline{- \rtimes G/H}} & kk.
 \end{array}$$

A morphism $f : G/H \rightarrow G/K$ of G -sets induces a natural transformation $f_* : \mathcal{R}(- \rtimes G/H) \rightarrow \mathcal{R}(- \rtimes G/K)$ of functors $G\text{Alg}_\ell \rightarrow \text{Alg}_\ell$. We would like f to induce as well a natural transformation $f_* : \overline{- \rtimes G/H} \rightarrow \overline{- \rtimes G/K}$ of triangulated functors $kk^G \rightarrow kk$. Such a natural transformation can be thought of as a functor $kk^G \rightarrow kk^I$, where kk^I

is the category of functors from the interval category I into kk . We cannot expect to define a functor $kk^G \rightarrow kk^I$ by the universal property of kk^G mentioned above [15, Theorem 4.1.1] since kk^I is not a triangulated category in an obvious way. To get around this problem, we work in the setting of graded categories.

Definition D.1 ([29, Section 10]). A *graded category* is a pair (\mathcal{A}, Ω) where \mathcal{A} is an additive category and Ω is an automorphism of \mathcal{A} . A *graded functor* $F : (\mathcal{A}, \Omega) \rightarrow (\mathcal{A}', \Omega')$ is an additive functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ such that $F \circ \Omega = \Omega' \circ F$. Let $F, G : (\mathcal{A}, \Omega) \rightarrow (\mathcal{A}', \Omega')$ be graded functors. A *graded natural transformation* $\nu : F \rightarrow G$ is a natural transformation ν such that $\Omega'(\nu_X) = \nu_{\Omega(X)} : \Omega'F(X) \rightarrow \Omega'G(X)$ for all $X \in \mathcal{A}$.

Every triangulated category is a graded category. If \mathcal{A} is a graded category, then \mathcal{A}^I is a graded category as well.

Definition D.2 (cf. [29, Definition 10.6]). Let \mathcal{C} denote either Alg_ℓ or $G\text{Alg}_\ell$ and let (\mathcal{A}, Ω) be a graded category. A δ -*functor* with values in \mathcal{A} consists of the following data:

- (1) a functor $X : \mathcal{C} \rightarrow \mathcal{A}$ that preserves finite products;
- (2) a morphism $\delta_\mathcal{E} \in \text{Hom}_{\mathcal{A}}(\Omega X(C), X(A))$ for every extension \mathcal{E} as in (2.16).

These morphisms $\delta_\mathcal{E}$ are subject to the following conditions:

- (1) $\delta_\mathcal{E} : \Omega X(C) \rightarrow X(A)$ is an isomorphism if $X(B) = 0$;
- (2) the morphisms $\delta_\mathcal{E}$ are natural with respect to morphisms of extensions.

Any excisive homology theory in the sense of Cortiñas and Thom [9, Section 6.6] is a δ -functor. By [29, Theorem 10.15 and Remark 10.17], the excisive, homotopy invariant and matrix-unstable bivariant K -theory introduced by Garkusha in [16] is the universal homotopy invariant δ -functor with values in a graded category. By [27, Theorem 5.2.15], for any infinite set X , M_X -stable algebraic kk -theory is the universal homotopy invariant and M_X -stable δ -functor with values in a graded category. A similar result holds in the equivariant context.

Theorem D.3. *Let G be a group, let (\mathcal{A}, Ω) be a graded category and let $X : G\text{Alg}_\ell \rightarrow \mathcal{A}$ be a homotopy invariant and G -stable δ -functor. Then there exists a unique graded functor $\bar{X} : kk^G \rightarrow \mathcal{A}$ such that $\bar{X}(\partial_\mathcal{E}^G) = \delta_\mathcal{E}$ for every extension \mathcal{E} and such that the following diagram commutes:*

$$\begin{array}{ccc}
 G\text{Alg}_\ell & \xrightarrow{j^G} & kk^G \\
 & \searrow X & \downarrow \exists! \bar{X} \\
 & & \mathcal{A}
 \end{array}
 \tag{D.4}$$

Here, the morphisms $\partial_\mathcal{E}^G$ are those that make $j^G : G\text{Alg}_\ell \rightarrow kk^G$ into an excisive homology theory.

Proof. The proof is similar to that of [15, Theorem 4.1.1]. We start by recalling some details of the construction of kk^G . Following [9, Section 6.6], an *excisive homology theory*

of G -algebras consists of a triangulated category (\mathcal{T}, Ω) , a functor $X : G\text{Alg}_\ell \rightarrow \mathcal{T}$ and a morphism $\delta_\mathcal{E} : \Omega X(C) \rightarrow X(A)$ for every extension \mathcal{E} as in (2.16) such that:

- (1) For every extension \mathcal{E} , the following is a distinguished triangle in \mathcal{T} :

$$\Omega X(C) \xrightarrow{\delta_\mathcal{E}} X(A) \longrightarrow X(B) \longrightarrow X(C).$$

- (2) The morphisms $\delta_\mathcal{E}$ are natural with respect to morphisms of extensions.

Fix $j : G\text{Alg}_\ell \rightarrow kk^{G\text{Alg}_\ell}, \{\partial_\mathcal{E}\}_\mathcal{E}$, a universal excisive, homotopy invariant and $M_{\mathbb{N} \times |G|}$ -stable homology theory. The existence of such a homology theory follows from [9, Theorem 6.6.2] in the case $G = 1$, from [15, Theorem 2.6.5] for countable G and from [27, Theorem 5.2.16] in the general case. Since the composite

$$G\text{Alg}_\ell \xrightarrow{M_G \otimes -} G\text{Alg}_\ell \xrightarrow{j} kk^{G\text{Alg}_\ell}$$

is an excisive, homotopy invariant and $M_{\mathbb{N} \times |G|}$ -stable homology theory, there exists a unique triangulated functor $M_G : kk^{G\text{Alg}_\ell} \rightarrow kk^{G\text{Alg}_\ell}$ making the following square commute:

$$\begin{array}{ccc} G\text{Alg}_\ell & \xrightarrow{j} & kk^{G\text{Alg}_\ell} \\ M_G \otimes - \downarrow & & \downarrow M_G \\ G\text{Alg}_\ell & \xrightarrow{j} & kk^{G\text{Alg}_\ell}. \end{array}$$

Now define kk^G (see [15, Section 4.1] and [27, Definition 5.3.4]) as the category whose objects are those of $kk^{G\text{Alg}_\ell}$ and whose morphism sets are given by

$$\text{Hom}_{kk^G}(A, B) := \text{Hom}_{kk^{G\text{Alg}_\ell}}(M_G A, M_G B).$$

Let $f \in \text{Hom}_{kk^{G\text{Alg}_\ell}}(M_G A, M_G B)$. We will often consider f both as a morphism in $kk^{G\text{Alg}_\ell}$ and as a morphism $A \rightarrow B$ in kk^G . To avoid ambiguity, we will write $[f]$ instead of f when considering f as a morphism in kk^G . There is a functor $t^G : kk^{G\text{Alg}_\ell} \rightarrow kk^G$ that is the identity on objects and that sends $f \in \text{Hom}_{kk^{G\text{Alg}_\ell}}(A, B)$ to $[M_G(f)] \in \text{Hom}_{kk^G}(A, B)$. Let j^G be the following composite:

$$G\text{Alg}_\ell \xrightarrow{j} kk^{G\text{Alg}_\ell} \xrightarrow{t^G} kk^G.$$

By [15, Theorem 4.1.1] and [27, Theorem 5.3.8],

$$j^G : G\text{Alg}_\ell \longrightarrow kk^G$$

endowed with $\{\partial_\mathcal{E}^G := t^G(\partial_\mathcal{E})\}_\mathcal{E}$ is the universal excisive, homotopy invariant and G -stable homology theory.

Let us show that $j^G : G\text{Alg}_\ell \rightarrow kk^G, \{\partial_\mathcal{E}^G\}_\mathcal{E}$, is also the universal homotopy invariant and G -stable δ -functor into a graded category. Let $X : G\text{Alg}_\ell \rightarrow (\mathcal{A}, \Omega), \{\delta_\mathcal{E}\}$, be a homotopy invariant and G -stable δ -functor. Since X is G -stable, then it is $M_{\mathbb{N} \times |G|}$ -stable; see [15, Section 3]. By [27, Theorem 5.2.15] there exists a unique graded functor

$$\widehat{X} : kk^{G\text{Alg}_\ell} \longrightarrow \mathcal{A}$$

making the following diagram commute and such that $\widehat{X}(\partial_{\mathcal{E}}) = \delta_{\mathcal{E}}$ for every extension \mathcal{E} :

$$\begin{array}{ccc} G\text{Alg}_{\ell} & \xrightarrow{j} & kk^{G\text{Alg}_{\ell}} \\ & \searrow X & \downarrow \exists! \widehat{X} \\ & & \mathcal{A}. \end{array}$$

Suppose that there exists a graded functor

$$\bar{X} : kk^G \rightarrow \mathcal{A}$$

making the triangle (D.4) commute and such that $\bar{X}(\partial_{\mathcal{E}}^G) = \delta_{\mathcal{E}}$ for every extension \mathcal{E} . Then we have $X = \bar{X} \circ j^G = (\bar{X} \circ t^G) \circ j$ and $\delta_{\mathcal{E}} = \bar{X}(\partial_{\mathcal{E}}^G) = (\bar{X} \circ t^G)(\partial_{\mathcal{E}})$ for every extension \mathcal{E} . By the uniqueness of \widehat{X} we must have $\widehat{X} = \bar{X} \circ t^G$. By [27, Lemma 5.3.6], for any $f \in \text{Hom}_{kk^{G\text{Alg}_{\ell}}}(M_G A, M_G B)$ we have a commutative diagram in kk^G as follows:

$$\begin{array}{ccc} M_G A & \xrightarrow{t^G(f)} & M_G B \\ \downarrow t^G(t'_A) \cong & & \cong \downarrow t^G(t'_B) \\ M_{G_+} A & & M_{G_+} B \\ \uparrow t^G(t_A) \cong & & \cong \uparrow t^G(t_B) \\ A & \xrightarrow{[f]} & B. \end{array}$$

Here t and t' are the morphisms appearing in the G -stability zig-zag (2.13). Upon applying \bar{X} to the diagram above we get

$$\bar{X}([f]) = \widehat{X}(t_B)^{-1} \circ \widehat{X}(t'_B) \circ \widehat{X}(f) \circ \widehat{X}(t'_A)^{-1} \circ \widehat{X}(t_A).$$

This shows that \bar{X} is uniquely determined. Moreover, it is straightforward to verify that the last equation defines a graded functor $\bar{X} : kk^G \rightarrow \mathcal{A}$ with the desired properties. ■

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