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Constant term functors with $\mathbb{F}_p$-coefficients

Robert Cass and Cédric Pépin

Abstract. We study the constant term functor for $\mathbb{F}_p$-sheaves on the affine Grassmannian in characteristic p with respect to a Levi subgroup. Our main result is that the constant term functor induces a tensor functor between categories of equivariant perverse $\mathbb{F}_p$-sheaves. We apply this fact to get information about the Tannakian monoids of the corresponding categories of perverse sheaves. As a byproduct we also obtain geometric proofs of several results due to Herzig on the mod p Satake transform and the structure of the space of mod p Satake parameters.

Contents

1. Introduction

1.1. Constant term functors with $\overline{\mathbb{Q}}_\ell$-coefficients

In the Langlands program over a global field F, the *constant term* and *Eisenstein series* operators relate automorphic functions with respect to a reductive group G/F and its Levi subgroups. When F is the function field of a smooth curve C over a finite field $\mathbb{F}_q$ of characteristic p, it is possible to upgrade these operators to functors on sheaves, cf. [6,13].

For simplicity suppose G arises from a split connected reductive group over $\mathbb{F}_q$. For each $x \in C(\mathbb{F}_q)$ a local Hecke algebra acts on automorphic functions. After choosing an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ and a uniformizing element at x, this local Hecke algebra can be identified with the unramified Hecke algebra $\mathcal{H}_{G,\ell}$ of $G(\mathbb{F}_q((t)))$ with $\overline{\mathbb{Q}}_\ell$-coefficients.

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In order to geometrize $\mathcal{H}_{G,\ell}$, one considers the following functors on $\mathbb{F}_q$-algebras:

$$LG\colon R \longrightarrow G\big(R(\!(t)\!)\big), \qquad L^+G\colon R \longmapsto G\big(R[\![t]\!]\big).$$

The *affine Grassmannian* is the fpqc-quotient $\mathrm{Gr}_G := LG/L^+G$, which is representable by an ind-scheme. Then in the context of the geometric Langlands program, the algebra $\mathcal{H}_{G,\ell}$ is replaced by the tensor category $(P_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell), *)$ of L^+G-equivariant perverse $\overline{\mathbb{Q}}_\ell$-sheaves on Gr_G for $\ell \neq p$.

If P is a parabolic subgroup of $G/\mathbb{F}_q$ with Levi factor L there is a diagram

$$\begin{tikzcd} & \mathrm{Gr}_P \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Gr}_L & & \mathrm{Gr}_G . \end{tikzcd}$$

The local analogue of the constant term functor is

$$\mathrm{CT}^G_{L,\ell}\colon P_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell) \xrightarrow{q_! \circ p^*[\deg_P]} D^b_c(\mathrm{Gr}_L, \overline{\mathbb{Q}}_\ell)$$

for a certain locally constant function $\deg_P\colon \mathrm{Gr}_P \to \mathbb{Z}$, cf. (6.2). The function-sheaf dictionary sends $\mathrm{CT}^G_{L,\ell}$ to the Satake transform $\mathcal{H}_{G,\ell} \to \mathcal{H}_{L,\ell}$ up to a normalization factor. Remarkably, the functor $\mathrm{CT}^G_{L,\ell}$ takes values in $P_{L^+G}(\mathrm{Gr}_L, \overline{\mathbb{Q}}_\ell)$, and is compatible with the tensor structures.

1.2. Constant term functors with $\mathbb{F}_p$-coefficients

Let k be an algebraically closed field of characteristic $p > 0$ and let G be a connected reductive group defined over k. Let Gr_G be the affine Grassmannian of G over k, and let $(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *)$ be the abelian symmetric monoidal category of L^+G-equivariant perverse $\mathbb{F}_p$-sheaves on Gr_G as defined in [9].

Fix a maximal torus and a Borel subgroup $T \subset B \subset G$. Let $B \subset P \subset G$ be a standard parabolic subgroup and L be its Levi factor containing T.

Definition 1.2.1. The *L-constant term functor* is

$$\mathrm{CT}^G_L := Rq_! \circ Rp^*[\deg_P]\colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow D^b_c(\mathrm{Gr}_L, \mathbb{F}_p).$$

Our main result is the following, cf. Section 6.

Theorem 1.2.2. *The functor CT^G_L induces an exact faithful tensor functor*

$$\mathrm{CT}^G_L\colon \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \longrightarrow \big(P_{L^+L}(\mathrm{Gr}_L, \mathbb{F}_p), *\big).$$

Let us start by explaining why CT^G_L preserves perversity. Let $X_*(T)$ be the group of cocharacters of T and $X_*(T)^+$ (resp. $X_*(T)_-$) be the monoid of dominant (resp. antidominant) cocharacters. For $\lambda \in X_*(T)^+$ let $\mathrm{Gr}^{\leq\lambda}_G$ be the reduced closure of the L^+G-orbit of $\lambda(t)$ in Gr_G. By [9, Th. 1.5], the simple objects in $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ are the shifted constant sheaves:

$$\mathrm{IC}_\lambda = \mathbb{F}_p\big[\dim \mathrm{Gr}_G^{\leq\lambda}\big] \in D_c^b\big(\mathrm{Gr}_G^{\leq\lambda}, \mathbb{F}_p\big).$$

Let w_0 be the longest element of the Weyl group of (G, T). In what follows we will use a letter L as a subscript or superscript to denote the corresponding objects for L.

The connected components of Gr_P and Gr_L are in bijection via the map q. If $c \in \pi_0(\mathrm{Gr}_L)$ corresponds to Gr_L^c then we denote the corresponding reduced connected component of Gr_P by S_c. By restricting CT_L^G to S_c we get a decomposition by *weight functors*:

$$\mathrm{CT}_L^G \cong \bigoplus_{c \in \pi_0(\mathrm{Gr}_L)} F_c.$$

Then the fact that CT_L^G preserves perversity is a consequence of the following theorem, which is unique to $\mathbb{F}_p$-sheaves, cf. Theorem 6.2.1.

Theorem 1.2.3. *For $\lambda \in X_*(T)^+$, we have*

$$F_c(\mathrm{IC}_\lambda) = \begin{cases} \mathrm{IC}^L_{w_0^L w_0(\lambda)} & \text{if } w_0(\lambda)(t) \in \mathrm{Gr}_L^c, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, Theorem 1.2.3 computes the relative $\mathbb{F}_p$-cohomology with compact support of the so-called *Mirković–Vilonen cycles* for the Levi L. The proof relies on the dynamics of $\mathbb{G}_m$-schemes of Białynicki-Birula and Drinfeld, together with the existence of $\mathbb{F}_p$-acyclic $\mathbb{G}_m$-equivariant resolutions of singularities of $\mathrm{Gr}_G^{\leq\lambda}$; see Section 1.7 below for more details.

Let us now comment on the tensor property of the functor CT_L^G. The general strategy of proof is similar to the one of Baumann–Riche for $\overline{\mathbb{Q}}_\ell$-coefficients [1, §15]. It involves the Beilinson–Drinfeld global convolution Grassmannian, cf. Section 5.3, and the key step is to show that a certain complex of sheaves is a perverse intermediate extension, cf. Theorem 6.5.1. We achieve it by appealing to the main results regarding perverse $\mathbb{F}_p$-sheaves on F-rational varieties [9, Th. 1.6, Th. 1.7]. In contrast, the analogue of the ingredient used for $\overline{\mathbb{Q}}_\ell$-coefficients fails; see Section 1.6 below.

1.3. Tannakian interpretation

By [9], the functor of tensor endomorphisms of the fiber functor

$$\oplus_i R^i\Gamma : \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \longrightarrow (\mathrm{Vect}_{\mathbb{F}_p}, \otimes)$$

is represented by an affine monoid scheme M_G over $\mathbb{F}_p$. Via the Tannakian formalism this results in an equivalence

$$\big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \cong \big(\mathrm{Rep}_{\mathbb{F}_p}(M_G), \otimes\big).$$

This construction is analogous to the *geometric Satake equivalence* [19] (it is assumed in loc. cit. that G is defined over $\mathbb{C}$, but see [23, 27] for the case where G is defined in positive characteristic). The monoid M_G is pro-solvable, but beyond this little is known. We will apply the functor CT_L^G to deduce more information about M_G.

By Theorem 1.2.3, the functor CT_L^G takes values in the symmetric monoidal subcategory

$$P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-}, \mathbb{F}_p) \subset P_{L^+L}(\mathrm{Gr}_L, \mathbb{F}_p)$$

associated to the submonoid $w_0^L X_*(T)_- \subset X_*(T)_{+/L}$ in the sense of Notation 6.2.2, and by Corollary 6.3.2, it intertwines the fiber functors. Thus denoting by $M_{L,w_0^L X_*(T)_-}$ the Tannakian monoid of $P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-}, \mathbb{F}_p)$, the Tannaka dual to CT_L^G is a morphism of $\mathbb{F}_p$-monoid schemes $M_L \to M_G$ which factors as

$$M_L \longrightarrow M_{L,w_0^L X_*(T)_-} \longrightarrow M_G. \tag{1.1}$$

We currently have a limited understanding of the morphisms in (1.1). This is related to our lack of information on the structure of the Ext groups in the corresponding categories of representations. However, if $L = T$ then we can say more. In this case, the category $P_{L^+T}(\mathrm{Gr}_T, \mathbb{F}_p)$ is semi-simple,

$$M_T = \mathrm{Spec}\left(\mathbb{F}_p\left[X_*(T)\right]\right), \quad M_{T,X_*(T)_-} = \mathrm{Spec}\left(\mathbb{F}_p\left[X_*(T)_-\right]\right),$$

and the following holds, cf. Theorem 7.4.5.

Theorem 1.3.1. *The Tannaka dual of* CT_T^G *induces a morphism of monoids* $M_T \to M_G$ *which factors as an open immersion followed by a closed immersion:*

$$M_T \longrightarrow M_{T,X_*(T)_-} \longrightarrow M_G.$$

Note that M_T is the torus over $\mathbb{F}_p$ with root datum dual to that of T. Thus, the morphism $M_T \to M_G$ in Theorem 1.3.1 is analogous to the reconstruction of the dual maximal torus in the dual group of G in [19].

There is another perspective on the morphism $M_{T,X_*(T)_-} \to M_G$ in Theorem 1.3.1 as follows. By [9, Th. 1.2], the subcategory of semi-simple objects

$$P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)^{\mathrm{ss}} \subset P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$$

is a symmetric monoidal subcategory. Then the Tannakian monoid M_G^{ss} of the category $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)^{\mathrm{ss}}$ identifies canonically with $M_{T,X_*(T)_-}$ by Notation 7.4.1.

Definition 1.3.2. The Tannaka dual of the above inclusion of semi-simple objects is called the *eigenvalues homomorphism*

$$\pi_G \colon M_G \longrightarrow M_G^{\mathrm{ss}}.$$

The morphism

$$\mathfrak{w} \colon M_G^{\mathrm{ss}} \longrightarrow M_G$$

equal to

$$M_{T,X_*(T)_-} \longrightarrow M_G$$

in Theorem 1.3.1 under the canonical identification $M_G^{\mathrm{ss}} = M_{T,X_*(T)_-}$ is called the *weight section*.

By construction these morphisms satisfy

$$\pi_G \circ \mathfrak{w} = \mathrm{id}_{M_G^{\mathrm{ss}}}\,.$$

The Tannaka dual of the weight section can be viewed as a *semi-simplification functor* $(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *) \to (P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *)^{\mathrm{ss}}$. We refer to Section 7.4 for more discussion on this perspective.

1.4. Relation to mod p Hecke algebras

In this subsection alone we view G as a split connected reductive group over $\mathbb{F}_q$. We assume that all relevant subgroups are also defined over $\mathbb{F}_q$. Let $E = \mathbb{F}_q(\!(t)\!)$ and $\mathcal{O} = \mathbb{F}_q[\![t]\!]$, and consider the unramified mod p Hecke algebra

$$\mathcal{H}_G := \big\{f\colon G(E) \to \mathbb{F}_p \mid f \text{ has compact support and is } G(\mathcal{O}) \text{ bi-invariant}\big\}.$$

A basis for $\mathcal{H}_G$ is $\{\mathbb{1}_\lambda\}_{\lambda\in X_*(T)^+}$ where $\mathbb{1}_\lambda$ is the characteristic function of the double coset $G(\mathcal{O})\lambda(t)G(\mathcal{O})$.

Let U_P be the unipotent radical of the parabolic subgroup P. Herzig [16, §2.3] defined the mod p Satake transform

$$\mathcal{S}_L^G\colon \mathcal{H}_G \longrightarrow \mathcal{H}_L, \qquad f \longmapsto \Big(g \longmapsto \sum_{U_P(E)/U_P(\mathcal{O})} f(gu)\Big).$$

As ind-schemes over $\mathbb{F}_q$, for $c \in \pi_0(\mathrm{Gr}_L)$ we have

$$S_c = (LU_P \cdot \mathrm{Gr}_L^c)_{\mathrm{red}} \subset \mathrm{Gr}_G\,.$$

Since $\mathrm{Gr}_G(\mathbb{F}_q) = G(E)/G(\mathcal{O})$, $LU_P(\mathbb{F}_q) = U_P(E)$ and $U_P(E)$ is normal in $P(E)$, then the function-sheaf dictionary sends CT_L^G to $\mathcal{S}_L^G$, cf. [8, §4]. In contrast, for $\overline{\mathbb{Q}}_\ell$-coefficients the two transforms differ by the modulus character of P. The isomorphisms in Theorem 1.2.3 hold over $\mathbb{F}_q$, so by using that the IC-sheaves are constant we obtain a geometric proof of the following result due to Herzig.

Corollary 1.4.1 ([16, Prop. 5.1]). *We have*

$$\mathcal{S}_L^G\Big(\sum_{\mu\le_G\lambda}\mathbb{1}_\mu\Big) = \sum_{\mu\le_L w_0^L w_0(\lambda)}\mathbb{1}_\mu.$$

Note that $\mathcal{H}_T = \mathbb{F}_p[X_*(T)]$ where the characteristic function of $\nu(t)T(\mathcal{O})$ corresponds to $e^\nu \in \mathbb{F}_p[X_*(T)]$ for $\nu \in X_*(T)$. By taking $L = T$, we obtain the following result.

Corollary 1.4.2. *The mod p Satake transform induces an isomorphism*

$$\begin{aligned} \mathcal{S}_T^G \colon \mathcal{H}_G &\xrightarrow{\sim} \mathbb{F}_p[X_*(T)_-] \\ \sum_{\mu \leq \lambda} \mathbb{1}_\mu &\longmapsto e^{w_0(\lambda)}. \end{aligned}$$

Note that Corollaries 1.4.1 and 1.4.2 are ultimately statements about counting $\mathbb{F}_q$-points mod p on the Mirković–Vilonen cycles. From this point of view, the resolutions of singularities which go into the proof of Theorem 1.2.3 allow us to reduce this point counting to one on affine spaces.

Remark 1.4.3. In [8, Prop. 4.5] a particular isomorphism $\varphi \colon \mathcal{H}_G \cong \mathbb{F}_p[X_*(T)_-]$ is constructed using the function-sheaf dictionary and the formula [9, Th. 1.2] for the convolution product in $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$. Herzig's explicit formula [16, Prop. 5.1] is then used to check that $\varphi = \mathcal{S}_T^G$. Here Theorem 1.2.3 gives a purely geometric proof of the fact that $\varphi = \mathcal{S}_T^G$.

1.5. Relation to mod p Satake parameters

As a consequence of Corollary 1.4.2, the $\mathbb{F}_p$-algebra $\mathcal{H}_G$ is commutative and the corresponding affine $\mathbb{F}_p$-scheme is identified with the *space of Satake parameters*

$$\mathcal{P} := \operatorname{Spec}\left(\mathbb{F}_p\left[X_*(T)_-\right]\right).$$

From the geometric theory (Section 1.3), this is the underlying scheme of the semi-simple monoid M_G^{ss}. Now for each standard Levi L as above, the functor CT_L^G preserves the subcategories of semi-simple objects by Theorem 1.2.3, hence by duality the morphism (1.1) admits a semi-simplification $M_L^{\mathrm{ss}} \to M_G^{\mathrm{ss}}$. Then we have the following, cf. Lemma 8.3.1 and Corollary 8.4.1.

Theorem 1.5.1. *The morphism*

$$M_L^{\mathrm{ss}} = \mathcal{P}_L \longrightarrow M_G^{\mathrm{ss}} = \mathcal{P}$$

defined by the constant term functor CT_L^G *is an open immersion.*

Moreover, denoting by $\mathcal{L}$ *the finite set of standard Levi subgroups* $T \subset L \subset G$ *and setting*

$$\forall L \in \mathcal{L}, \quad S_L := \mathcal{P}_L \setminus \bigcup_{\substack{L' \in \mathcal{L} \\ L' \subsetneq L}} \mathcal{P}_{L'} \text{ equipped with its reduced structure},$$

the space of Langlands parameters $\mathcal{P}$ *is stratified as:*

$$\mathcal{P} = \bigcup_{L \in \mathcal{L}} S_L.$$

The stratum S_L *is isomorphic to* $(\mathbb{A}^1 \setminus \{0\})^{\operatorname{rank} \pi_0(\mathrm{Gr}_L)}$ *and the closure relation among the strata is given by* $\overline{S_L} = \bigcup_{L' \supset L} S_{L'}$.

The underlying decomposition of the set $\mathcal{P}(\overline{\mathbb{F}}_p)$ was originally defined by Herzig in [17, §1.5, §2.4]. The construction above makes the link with the category

$$P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p).$$

1.6. Obstructions to adapting proofs for $\overline{\mathbb{Q}}_\ell$-coefficients

Let us now explain why the known proofs that $\mathrm{CT}^G_{L,\ell}$ preserves perversity and is a tensor functor fail for $\mathbb{F}_p$-sheaves. So that we can deal with $\mathbb{F}_p$ and $\overline{\mathbb{Q}}_\ell$-coefficients simultaneously let us set $\mathrm{IC}_{\lambda,\ell}$ to be the ℓ-adic intersection cohomology sheaf of $\mathrm{Gr}_G^{\leq\lambda}$. Then $\mathrm{IC}_{\lambda,\blacktriangle}$ is either an $\mathbb{F}_p$-sheaf or a $\overline{\mathbb{Q}}_\ell$-sheaf depending on the value of $\blacktriangle \in \{\emptyset, \ell\}$.

For both $\overline{\mathbb{Q}}_\ell$-sheaves and $\mathbb{F}_p$-sheaves, there is a homological argument which reduces us to the case $L = T$. Then $\pi_0(\mathrm{Gr}_B) = X_*(T)$ and $(\mathrm{Gr}_T)_{\mathrm{red}}$ is a disjoint union of points indexed by $X_*(T)$, so that the weight functors are

$$F_\nu = R\Gamma_c(S_\nu, \cdot)\big[2\rho(\nu)\big], \quad \nu \in X_*(T),$$

where ρ is half the sum of the positive roots. The fact that F_ν preserves perversity is equivalent to the statement that

$$H^i_c(S_\nu, \mathrm{IC}_{\lambda,\blacktriangle}) \neq 0 \implies i = 2\rho(\nu). \tag{1.2}$$

By dimension estimates, we have $H^i_c(S_\nu, \mathrm{IC}_{\lambda,\blacktriangle}) = 0$ if $i > 2\rho(\nu)$. For the other inequality, one observes that there is a $\mathbb{G}_m$-action on Gr_G such that $S_\nu(k)$ is the set of k-points of the ν-component of the *attractor* in the sense of Definition 2.3.1. Then Braden's hyperbolic localization theorem [5] provides a comparison with the cohomology supported in the ν-component of the *repeller* (i.e. the attractor for the opposite $\mathbb{G}_m$-action), which leads to the other half of the desired vanishing (1.2) for $\overline{\mathbb{Q}}_\ell$-coefficients, cf. [19, Th. 3.5]. However, we show in the appendix that Braden's hyperbolic localization theorem *fails* for $\mathbb{F}_p$-sheaves. Braden's theorem is also the key tool from the proof of the compatibility of $\mathrm{CT}^G_{L,\ell}$ with convolution [1, Prop. 1.15.2] that we lack in the case of $\mathbb{F}_p$-coefficients.

There is another approach to proving (1.2) due to Ngô–Polo [20]. Let $\mathcal{M} \subset X_*(T)^+$ be the subset of cocharacters that are either minuscule or quasi-minuscule. If λ is quasi-minuscule then Ngô–Polo construct a resolution of $\mathrm{Gr}_G^{\leq\lambda}$ and explicitly stratify the fiber over $S_\nu \cap \mathrm{Gr}_G^{\leq\lambda}$ by affine spaces. These stratifications allow one to estimate the dimension of $H^i_c(S_\nu, \mathrm{IC}_{\lambda,\blacktriangle})$ for $(\nu, \lambda) \in X_*(T) \times \mathcal{M}$.

If $\lambda \in X_*(T)^+$ can be decomposed as a sum of elements of $\mathcal{M}$, then by considering the corresponding convolution Grassmannian $m\colon \mathrm{Gr}_G^{\leq\lambda_\bullet} \to \mathrm{Gr}_G^{\leq\lambda}$ the previous estimates allow one to prove (1.2) for any direct summand of $Rm_!(\mathrm{IC}_{\lambda_\bullet,\blacktriangle})$, where $\mathrm{IC}_{\lambda_\bullet,\blacktriangle}$ is the IC-sheaf of $\mathrm{Gr}_G^{\leq\lambda_\bullet}$. This is sufficient to complete the argument for $\overline{\mathbb{Q}}_\ell$-sheaves. However, for $\mathbb{F}_p$-sheaves we have $Rm_!(\mathrm{IC}_{\lambda_\bullet}) = \mathrm{IC}_\lambda$ by [9, Prop. 6.5]. Thus in our situation Ngô–Polo's approach allows us to conclude for groups of type A_n only, since this is the only case where the fundamental coweights freely generating $X_*(T_{\mathrm{ad}})^+$ belong to the subset $\mathcal{M}_{\mathrm{ad}} \subset X_*(T_{\mathrm{ad}})^+$.

1.7. Proof strategy for preservation of perversity

Our approach to proving Theorem 1.2.3 combines ideas from both [19, 20], and works directly for L not necessarily equal to T. We start with the observation that there is a $\mathbb{G}_m$-action on Gr_G such that $\mathrm{Gr}_L(k) = \mathrm{Gr}_G(k)^{\mathbb{G}_m(k)}$ and such that the $S_c(k)$ for $c \in \pi_0(\mathrm{Gr}_L)$ are the sets of k-points of the components of the attractor:

$$\forall c \in \pi_0(\mathrm{Gr}_L), \quad S_c(k) = \big\{x \in \mathrm{Gr}_G(k) \mid \lim_{k^\times \ni z \to 0} z \cdot x \in \mathrm{Gr}_L^c(k)\big\}.$$

Then the (unshifted) weight functor F_c identifies with the hyperbolic localization functor of relative cohomology with compact support flowing in the direction of the fixed points Gr_L^c.

Let $\mathcal{B}$ be the Iwahori group scheme equal to the dilation of $G_{k[\![t]\!]}$ along B_k. The affine flag variety $\mathcal{F}\ell := LG/L^+\mathcal{B}$ is a $\mathbb{G}_m$-equivariant G/B-fibration over Gr_G. Unlike the case of $\overline{\mathbb{Q}}_\ell$-coefficients, the flag variety G/B is acyclic for $\mathbb{F}_p$-coefficients in the sense that $R\Gamma(G/B, \mathbb{F}_p) = \mathbb{F}_p[0]$, thanks to the Bruhat decomposition, cf. Corollary 2.2.3. This allows us to compare $F_c(\mathrm{IC}_\lambda)$ with hyperbolic localizations on the preimage of $S_c \cap \mathrm{Gr}_G^{\leq\lambda}$ in $\mathcal{F}\ell$.

Next we note that any Schubert variety in $\mathcal{F}\ell$ admits a so-called Demazure resolution, which is both $\mathbb{G}_m$-equivariant and $\mathbb{F}_p$-acyclic.

Then we can appeal to a general result of Białynicki-Birula on the structure of smooth proper $\mathbb{G}_m$-varieties: on the resolution, there is a unique closed attractor component, while the other components are positive-dimensional affine bundles over their fixed points. Such bundles have no relative $\mathbb{F}_p$-cohomology with compact support, so only the closed component contributes.

The final complete determination of $F_c(\mathrm{IC}_\lambda)$ relies on the affineness of Drinfeld's attractor of a not necessarily smooth $\mathbb{G}_m$-scheme [12, Th. 1.4.2 (ii)], cf. also [24, Th. A].

1.8. Outline

In Section 2, we recall results of Białynicki-Birula and Drinfeld on the structure of schemes with a $\mathbb{G}_m$-action. The main result is Corollary 2.3.5 on $\mathbb{F}_p$-cohomology with compact support in the attractors on a general class of $\mathbb{G}_m$-schemes. In Section 3, we apply this result on the affine Grassmannian to prove Theorem 3.7.1, which is the main input in the proof of Theorem 1.2.3. In Sections 4 and 5, we prove Theorems 1.2.2 and 1.2.3 in the case $L = T$. We treat the case of general L in Section 6. In Section 7, we investigate the Tannakian consequences of Theorems 1.2.2 and 1.2.3 for the monoid M_G. In Section 8, we study the stratification of $\mathcal{P}$ induced by the morphisms $M_L^{\mathrm{ss}} \to M_G^{\mathrm{ss}}$. Finally, in the appendix, we show that Braden's hyperbolic localization theorem is false for $\mathbb{F}_p$-coefficients.

Notation. Let k be an algebraically closed field of characteristic $p > 0$ and let G be a connected reductive group over k. Fix a maximal torus and a Borel subgroup $T \subset B \subset G$, and let $U \subset B$ be the unipotent radical of B. Let W be the Weyl group of G and let $w_0 \in W$ be the longest element.

Let $X^*(T)$ and $X_*(T)$ be the lattices of characters and cocharacters of T, and $X_*(T)^+$ (resp. $X_*(T)_-$) the monoid of dominant (resp. antidominant) cocharacters determined by B. Let Φ and $\Phi^\vee$ be the sets of roots and coroots, Φ^+ and $(\Phi^+)^\vee$ the subsets of positive roots and positive coroots, and Δ and $\Delta^\vee$ the subsets of simple roots and simple coroots. For $\nu, \nu' \in X_*(T)$ we write $\nu \leq \nu'$ if $\nu' - \nu$ is a sum of positive coroots with non-negative integer coefficients. Let ρ and $\hat{\rho}$ be respectively half the sum of the positive roots and coroots. For $\nu \in X_*(T)$ let $\rho(\nu) \in \mathbb{Z}$ be the pairing of ρ and ν.

2. Some general computations of $\mathbb{F}_p$-cohomology with compact support

2.1. The affine space

Lemma 2.1.1. *Let $\mathbb{A}^d$ be the affine space over k of dimension d. Then*

$$R\Gamma_c(\mathbb{A}^d, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p[0] & \text{if } d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We can assume $d > 0$. Consider the open immersion $j\colon \mathbb{A}^d \to \mathbb{P}^d$ and the complementary closed immersion $i\colon \mathbb{P}^{d-1} \to \mathbb{P}^d$. This gives rise to an exact triangle

$$Rj_!\mathbb{F}_p[0] \longrightarrow \mathbb{F}_p[0] \longrightarrow Ri_*\mathbb{F}_p[0] \xrightarrow{+1}.$$

From [25], we know that

$$\forall i > 0, \quad H^i(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}) = H^i(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}) = 0.$$

Thus since $H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}) = H^0(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}) = k$, then by the Artin–Schreier sequence the map $R\Gamma(\mathbb{F}_p[0]) \to R\Gamma(Ri_*\mathbb{F}_p[0])$ is a quasi-isomorphism. Hence $R\Gamma(Rj_!\mathbb{F}_p[0]) = 0$, i.e., $R\Gamma_c(\mathbb{A}^d, \mathbb{F}_p) = 0$. ■

2.2. Schemes admitting a decomposition by affine spaces

Notation 2.2.1. Given a scheme X, we denote by $|X|$ its underlying topological space.

Definition 2.2.2. Let X be a scheme.

- A *decomposition* of X is a family of subschemes $X_i \subset X$, $i \in I$, such that

$$|X| = \bigcup_{i \in I} |X_i| \quad \text{and} \quad |X_i| \cap |X_j| = \emptyset \quad \text{for all } i \neq j.$$

- A *filtration* of X is a finite decreasing sequence of closed subschemes

$$X = Z_0 \supset Z_1 \supset \cdots \supset Z_{N-1} \supset Z_N = \emptyset.$$

The subschemes $Z_n \setminus Z_{n+1}$, $n = 0, \dots, N-1$, are the *cells* of the filtration.

Corollary 2.2.3. *Let X be a k-scheme. Assume that X admits a filtration whose cells are positive dimensional affine spaces. Then*

$$R\Gamma_c(X, \mathbb{F}_p) = 0.$$

Proof. This follows from Lemma 2.1.1 and the long exact sequence of $\mathbb{F}_p$-cohomology with compact support associated to the decomposition of a scheme into an open and a complementary closed subscheme. ∎

2.3. Some $\mathbb{G}_m$-schemes

Let X be a scheme of finite type over k, equipped with a $\mathbb{G}_m$-action. Recall from [12] the following definitions and results.

Definition 2.3.1. We define the following fppf sheaves.

- The *space of fixed points* is the fppf sheaf

$$X^0 := \underline{\mathrm{Hom}}_k^{\mathbb{G}_m}\big(\mathrm{Spec}(k), X\big),$$

where $\mathrm{Spec}(k)$ is equipped with the trivial $\mathbb{G}_m$-action.

- The *attractor* is the fppf sheaf

$$X^+ := \underline{\mathrm{Hom}}_k^{\mathbb{G}_m}\big((\mathbb{A}^1)^+, X\big),$$

where $(\mathbb{A}^1)^+$ is the affine line over k equipped with the $\mathbb{G}_m$-action by dilations.

Evaluating at 1 and 0 defines maps p and q:

$$\begin{tikzcd} & X^+ \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X^0 & & X. \end{tikzcd}$$

The space of fixed points is representable by a closed subscheme $X^0 \subset X$. The attractor is representable by a k-scheme. The morphism q is affine, and the section $X^0 \subset X^+$ obtained by precomposing with the structural morphism $(\mathbb{A}^1)^+ \to \mathrm{Spec}(k)$ induces an identification $(X^+)^0 = X^0$; the morphism p restricts to the identity between $X^0 \subset X^+$ and $X^0 \subset X$. Moreover, the morphism q has geometrically connected fibers, cf. [24, Cor. 1.12], so that the decomposition of X^+ as a disjoint union of its connected components is the preimage by q of the corresponding decomposition of X^0:

$$X^+ = \coprod_{i \in \pi_0(X^0)} X_i.$$

For $i \in \pi_0(X^0)$ we will denote by $q_i : X_i \to X_i^0$ the induced retraction.

Remark 2.3.2. Suppose that X is separated over k. Then,

$$p : X^+ \longrightarrow X$$

is a monomorphism, which induces the following identifications of sets:

$$X^+(k) \simeq \big\{x \in X(k) \mid \lim_{k^\times \ni z \to 0} z \cdot x \text{ exists}\big\},$$

$$\begin{aligned} q(k) : X^+(k) &\longrightarrow X^0(k) \\ x &\longmapsto \lim_{k^\times \ni z \to 0} z \cdot x, \end{aligned}$$

and for each $i \in \pi_0(X^0)$,

$$X_i(k) \simeq \big\{x \in X(k) \mid \lim_{k^\times \ni z \to 0} z \cdot x \in X_i^0(k)\big\}.$$

Now consider the following hypothesis:

(H) *for each $i \in \pi_0(X^0)$, the restriction $p|_{X_i} : X_i \to X$ is an* immersion.

Lemma 2.3.3. *The following statements hold true.*

1. *Suppose that* (H) *is satisfied, and that X is proper over k. Then the family of subschemes $(X_i)_{i \in \pi_0(X^0)}$ is a decomposition of X.*
2. *Suppose that there exists a $\mathbb{G}_m$-equivariant immersion of X into some projective space $\mathbb{P}(V)$ where $\mathbb{G}_m$ acts linearly on V. Then* (H) *is satisfied, and if moreover X is proper, there exists a filtration $(Z_n)_{0 \leq n \leq |\pi_0(X^0)|}$ of X having $(X_i)_{i \in \pi_0(X^0)}$ as its family of cells, in the sense of Definition 2.2.2.*

Proof. (1) When X is proper over k, then p is universally bijective by [12, Prop. 1.4.11 (iii)]. In particular

$$|X| = \bigcup_{i \in I} p\big(|X_i|\big) \quad \text{and} \quad p\big(|X_i|\big) \cap p\big(|X_j|\big) = \emptyset \quad \text{for all } i \neq j.$$

When (H) is satisfied, then for each i there exists a unique subscheme $p(X_i) \subset X$ such that $p|_{X_i}$ decomposes as an isomorphism $X_i \xrightarrow{\sim} p(X_i)$ followed by the canonical immersion $p(X_i) \subset X$. Thus, identifying X_i with $p(X_i)$, we get that the family $(X_i)_{i \in \pi_0(X^0)}$ is a decomposition of X.

(2) When X admits a $\mathbb{G}_m$-equivariant immersion into some projective space $\mathbb{P}(V)$ where $\mathbb{G}_m$-acts linearly on V, then, as noted in [12, Th. B.0.3 (iii)], the fact that (H) is satisfied follows from the case $X = \mathbb{P}(V)$. If the immersion is closed, the fact that the decomposition $(X_i)_{i \in \pi_0(X^0)}$ of X can be realized as the cells of a filtration follows again from the case $X = \mathbb{P}(V)$, as proved in [4, Th. 3].[1] ∎

---

[1] As noted in the remark following the proof of the theorem in loc. cit., the smoothness assumption on the closed $\mathbb{G}_m$-subscheme $X \subset \mathbb{P}(V)$ is not used in that proof. The existence of such a filtration is also recorded in [7, Lem. 4.12].

Theorem 2.3.4. *The following statements hold true.*

(1) *Suppose that X is smooth and separated over k. Then* (H) *is satisfied, X^0 and X^+ are smooth over k, and for each $i \in \pi_0(X^0)$, there exists an integer $d_i \geq 0$ such that*

$$\begin{tikzcd} X_i \arrow[rr, "\sim"] \arrow[dr, "q_i"'] & & \mathbb{A}^{d_i} \times X_i^0 \arrow[dl, "\mathrm{pr}_2"] \\ & X_i^0 & \end{tikzcd}$$

Zariski-locally on X_i^0. If moreover X is proper over k, then $X_i \subset X$ is closed if and only if $X_i = X_i^0$, and there exists exactly one such X_i lying in each connected component of X.

(2) *Suppose that X is normal and projective over k. Then there exists a $\mathbb{G}_m$-equivariant closed immersion of X into some projective space $\mathbb{P}(V)$ where $\mathbb{G}_m$-acts linearly on V.*

Proof. (1) The scheme X^0 is smooth over k by [15, Lem. 2.2]. The other results are contained in [3].

(2) This is a result of [26]. ■

Corollary 2.3.5. *Let X be a proper k-scheme equipped with a $\mathbb{G}_m$-action satisfying* (H). *Suppose that there exists a connected smooth projective k-scheme $\widetilde{X}$ equipped with a $\mathbb{G}_m$-action, and a surjective $\mathbb{G}_m$-equivariant morphism of k-schemes*

$$f : \widetilde{X} \longrightarrow X.$$

Then there exists at most one $i =: i_0 \in \pi_0(X^0)$ such that $X_i \subset X$ is closed.

Suppose moreover that $Rf_*\mathbb{F}_p = \mathbb{F}_p[0]$. Then for $i \in \pi_0(X^0)$, we have:

$$R(q_i)_!\mathbb{F}_p = \begin{cases} \mathbb{F}_p|_{X_{i_0}^0}[0] & \text{if } i = i_0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.3.6. If X can be embedded equivariantly into some $\mathbb{P}(V)$ where $\mathbb{G}_m$ acts linearly on V, then by Lemma 2.3.3 (2) there exists at least one $i \in \pi_0(X^0)$ such that $X_i \subset X$ is closed, hence then there is exactly one such i.

Proof of Corollary 2.3.5. Let $i \in \pi_0(X^0)$. Define Y_i and f_i by the fiber product diagram

$$\begin{tikzcd} Y_i \arrow[r, "f_i"] \arrow[d] & X_i \arrow[d, "p|_{X_i}"] \\ \widetilde{X} \arrow[r, "f"] & X. \end{tikzcd}$$

Since $p|_{X_i}$ is an immersion by hypothesis, so is the canonical map $Y_i \to \widetilde{X}$, and we write $X_i \subset X$ and $Y_i \subset \widetilde{X}$ for the corresponding subschemes. Also by Theorem 2.3.4 (1) the schemes $\widetilde{X}_j$, $j \in \pi_0(\widetilde{X}^0)$, are realized as subschemes of $\widetilde{X}$, and they form a decomposition of the latter, cf. Lemma 2.3.3 (1). Then we have the following identity of subspaces of $|\widetilde{X}|$:

$$|Y_i| = \bigcup_{\substack{j \in \pi_0(\widetilde{X}^0) \\ f(j)=i}} |\widetilde{X}_j|;$$

indeed this can be checked on k-points, where it follows from the definitions, cf. Remark 2.3.2. Thus the immersions $\widetilde{X}_j \to \widetilde{X}$, for $f(j) = i$, factor through $Y_i \subset \widetilde{X}$ (note that the schemes $\widetilde{X}_j$ are reduced, cf. Theorem 2.3.4 (1)), and the family $(\widetilde{X}_j)_{f(j)=i}$ is a decomposition of the scheme Y_i. Further, by Theorem 2.3.4 (2) and Lemma 2.3.3 (2), one may form a filtration of $\widetilde{X}$,

$$\widetilde{X} = Z_0 \supset Z_1 \supset \cdots \supset Z_{N-1} \supset Z_N = \emptyset, \quad N := \left|\pi_0(\widetilde{X}^0)\right|,$$

whose family of cells is $(\widetilde{X}_j)_{j \in \pi_0(\widetilde{X}^0)}$. Intersecting with Y_i we get a filtration of Y_i

$$Y_i = Z_{i,0} \supset Z_{i,1} \supset \cdots \supset Z_{i,N-1} \supset Z_{i,N} = \emptyset$$

whose family of nonempty cells is $(\widetilde{X}_j)_{f(j)=i}$.

Now suppose that $X_i \subset X$ is closed. Then so is $Y_i \subset \widetilde{X}$. Moreover the assumption that f is surjective ensures that Y_i is nonempty. Hence, if N_i is the greatest integer $n \leq N$ such that $Z_{i,n}$ is nonempty, then Z_{i,N_i} is equal to some $\widetilde{X}_j$ with $f(j) = i$ which is closed in (Y_i hence in) $\widetilde{X}$. But since $\widetilde{X}$ is connected, there is exactly one $\widetilde{X}_j \subset \widetilde{X}$ which is closed, say $\widetilde{X}_{j_0}$, by Theorem 2.3.4 (1). Thus $i = f(j_0) =: i_0$ is uniquely determined.

Finally, suppose moreover that $Rf_! \mathbb{F}_p = \mathbb{F}_p[0]$. If $i = i_0$, then $R(q_{i_0})_! \mathbb{F}_p = \mathbb{F}_p|_{X^0_{i_0}}[0]$ by Lemma 2.3.7 below. If $i \neq i_0$, consider the commutative diagram

$$\begin{tikzcd} Y_i \arrow[r, "f_i"] \arrow[dr, "q_{Y_i} :="'] & X_i \arrow[d, "q_i"] \\ & X_i^0 . \end{tikzcd}$$

By proper base change $R(f_i)_! \mathbb{F}_p = \mathbb{F}_p[0]$ and

$$R(q_i)_! \mathbb{F}_p = R(q_{Y_i})_! \mathbb{F}_p.$$

Then recall the filtration of Y_i constructed above. For every $0 \leq n \leq N-1$ such that $Z_{i,n} \setminus Z_{i,n+1}$ is nonempty, let $i_n : Z_{i,n+1} \to Z_{i,n}$ be the corresponding closed immersion, $j_n : \widetilde{X}_n \to Z_{i,n}$ be the complementary open immersion, and in $D^b_c(Z_{i,n}, \mathbb{F}_p)$ form the exact triangle

$$Rj_{n!}\mathbb{F}_p[0] \longrightarrow \mathbb{F}_p[0] \longrightarrow Ri_{n*}\mathbb{F}_p[0] \xrightarrow{+1} .$$

Setting $q_{Z_{i,n}} := q_{Y_i}|_{Z_{i,n}} : Z_{i,n} \to X_i^0$ and applying $R(q_{Z_{i,n}})_!$ we get the exact triangle

$$R(q_{Z_{i,n}} \circ j_n)_!\mathbb{F}_p[0] \longrightarrow R(q_{Z_{i,n}})_!\mathbb{F}_p \longrightarrow R(q_{Z_{i,n+1}})_!\mathbb{F}_p \xrightarrow{+1}$$

in $D_c^b(X_i^0, \mathbb{F}_p)$. By construction, the morphism $q_{Z_{i,n}} \circ j_n : \widetilde{X}_n \to X^0$ is equal to $q_i \circ (f_i|_{\widetilde{X}_n})$, and we have the commutative diagram

$$\begin{tikzcd}
\widetilde{X}_n \arrow[r, "f_i|_{\widetilde{X}_n}"] \arrow[d, "q_n"'] & X_i \arrow[d, "q_i"] \\
\widetilde{X}_n^0 \arrow[r] & X_i^0
\end{tikzcd}$$

functorially induced by f. Here $\widetilde{X}_n \neq \widetilde{X}_{j_0}$ since $i \neq i_0$. Consequently $R(q_n)_!\mathbb{F}_p = 0$ by proper base change, Theorem 2.3.4 (1) and Lemma 2.1.1. Thus

$$R(q_{Z_{i,n}})_!\mathbb{F}_p \xrightarrow{\sim} R(q_{Z_{i,n+1}})_!\mathbb{F}_p.$$

Descending in this way along the filtration of Y_i, we obtain

$$R(q_{Y_i})_!\mathbb{F}_p \xrightarrow{\sim} R(q_\emptyset)_!\mathbb{F}_p = 0,$$

which concludes the proof. ■

Lemma 2.3.7. *Let X be a proper k-scheme equipped with a $\mathbb{G}_m$-action satisfying (H). Then for each $i \in \pi_0(X^0)$ such that $X_i \subset X$ is closed, the retraction $q_i : X_i \to X_i^0$ is a universal homeomorphism and the section $X_i^0 \subset X_i$ induces the identity of reduced schemes $(X_i^0)_{\mathrm{red}} = (X_i)_{\mathrm{red}}$.*

Proof. As we have recalled, the retraction

$$q : X^+ \longrightarrow X^0$$

is always affine, [12, Th. 1.4.2 (ii)], with geometrically connected fibers, cf. [24, Cor. 1.12]. In particular its restrictions $q_i : X_i \to X_i^0$ above each X_i^0 have the same properties.

Now let $i \in \pi_0(X^0)$ such that $X_i \subset X$ is closed. Then X_i is proper over k, so that the morphism q_i is proper. Consequently, in this case q_i is a universal homeomorphism. So its canonical section $X_i^0 \subset X_i$ identifies $(X_i^0)_{\mathrm{red}}$ and $(X_i)_{\mathrm{red}}$. ■

3. $\mathbb{F}_p$-cohomology with compact support of the MV-cycles

3.1. The affine Grassmannian

For an affine group scheme H over k (or more generally, over $k[\![t]\!]$) we have the loop group functor

$$LH : k\text{-Algebras} \longrightarrow \text{Sets}, \qquad R \longmapsto H\big(R(\!(t)\!)\big),$$

and the non-negative loop group functor

$$L^+H\colon k\text{-Algebras} \longrightarrow \text{Sets}, \qquad R \longmapsto H\big(R[\![t]\!]\big).$$

The affine Grassmannian of G is the fpqc-quotient $\mathrm{Gr}_G := LG/L^+G$. It is represented by an ind-scheme over k.

3.2. The Cartan decomposition

The set $X_*(T)^+$ embeds in $\mathrm{Gr}_G(k)$ via the identification $\lambda \mapsto \lambda(t)$. For $\lambda \in X^*(T)^+$, denote by Gr_G^λ the reduced L^+G-orbit of $\lambda(t)$ in Gr_G. Then we have the decomposition of the reduced ind-closed subscheme $(\mathrm{Gr}_G)_{\mathrm{red}} \subset \mathrm{Gr}_G$:

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \bigcup_{\lambda \in X_*(T)^+} \mathrm{Gr}_G^\lambda,$$

which on k-points is the quotient of the *Cartan decomposition* of $G(k(\!(t)\!))$:

$$G\big(k(\!(t)\!)\big) = \bigcup_{\lambda \in X_*(T)^+} G\big(k[\![t]\!]\big)\lambda(t)G\big(k[\![t]\!]\big).$$

Let $\mathrm{Gr}_G^{\leq\lambda}$ be the closure of Gr_G^λ in Gr_G with reduced structure. Then $\mathrm{Gr}_G^{\leq\lambda}$ is an integral projective k-scheme, of dimension $2\rho(\lambda)$, which is the union of the Gr_G^μ with $\mu \leq \lambda$. Moreover $(\mathrm{Gr}_G)_{\mathrm{red}}$ is the limit of the $\mathrm{Gr}_G^{\leq\lambda}$:

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \varinjlim_{\lambda \in X_*(T)^+} \mathrm{Gr}_G^{\leq\lambda}.$$

3.3. The Iwasawa decomposition

From our fixed choice $B = U \rtimes T \subset G$, we have the quotient map $B \to T$ and the closed immersion $B \to G$:

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ T & & G. \end{tikzcd}$$

Then by functoriality we get a diagram

$$\begin{tikzcd} & \mathrm{Gr}_B \arrow[dl] \arrow[dr] & \\ \mathrm{Gr}_T & & \mathrm{Gr}_G . \end{tikzcd}$$

Passing to the reductions, we get the decomposition of $(\mathrm{Gr}_B)_{\mathrm{red}}$ into its connected components

$$(\mathrm{Gr}_B)_{\mathrm{red}} = \coprod_{\nu \in X_*(T)} S_\nu$$

and a decomposition of $(\mathrm{Gr}_G)_{\mathrm{red}}$ by ind-subschemes

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \bigcup_{\nu \in X_*(T)} S_\nu,$$

where $X_*(T)$ is embedded in $\mathrm{Gr}_G(k)$ via the identification $\nu \mapsto \nu(t)$. On k-points, it is the quotient of the *Iwasawa decomposition* of $G(k(\!(t)\!))$:

$$G\big(k(\!(t)\!)\big) = \bigcup_{\nu \in X_*(T)} U\big(k(\!(t)\!)\big)\nu(t)G\big(k[\![t]\!]\big).$$

3.4. The Mirković–Vilonen cycles

Definition 3.4.1. Let $(\nu, \lambda) \in X_*(T) \times X_*(T)^+$. The MV-cycle of index (ν, λ) is the reduced k-scheme

$$S_\nu \cap \mathrm{Gr}_G^{\leq \lambda}.$$

The MV-cycles can be reconstructed from the theory of $\mathbb{G}_m$-schemes, as follows.

The adjoint action of the torus T on LG normalizes L^+G and hence induces an action on Gr_G. Fixing a regular dominant cocharacter $\mathbb{G}_m \to T$, we equip Gr_G with the resulting $\mathbb{G}_m$-action.

Let $\lambda \in X_*(T)^+$. Then Gr_G^{λ} and $\mathrm{Gr}_G^{\leq\lambda}$ are stable under the $\mathbb{G}_m$-action. Thus

$$X := \mathrm{Gr}_G^{\leq\lambda}$$

is a projective $\mathbb{G}_m$-scheme over k. Moreover, it can be embedded equivariantly in some $\mathbb{P}(V)$ where $\mathbb{G}_m$ acts linearly on V: indeed, one can construct on the affine Grassmannian Gr_G some G-*equivariant* very ample line bundle, cf. [27, §1.5]. Consequently, by Lemma 2.3.3 (2), the connected components of the attractor X^+ are realized as subschemes of X. Then, it follows from Remark 2.3.2 and the Iwasawa decomposition of $G(k(\!(t)\!))$ that

$$X^0(k) = X_*(T) \cap X \quad \text{and} \quad \forall \nu \in X^0(k),\ X_\nu(k) = (S_\nu \cap \mathrm{Gr}_G^{\leq\lambda})(k).$$

Thus the MV-cycles indexed by (ν, λ) for varying ν are precisely the $(X_\nu)_{\mathrm{red}} \subset X$, which decompose X as

$$X = \bigcup_{\nu \in X_*(T) \cap X} (X_\nu)_{\mathrm{red}}.$$

3.5. Generalization to the standard Levi subgroups

Let $P = U_P \rtimes L \subset G$ be a parabolic subgroup of G containing B with unipotent radical U_P and Levi factor L. Then

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ L & & G \end{tikzcd}$$

induces

$$\begin{tikzcd} & \mathrm{Gr}_P \arrow[dl] \arrow[dr] & \\ \mathrm{Gr}_L & & \mathrm{Gr}_G, \end{tikzcd}$$

the decomposition of $(\mathrm{Gr}_P)_{\mathrm{red}}$ into its connected components

$$(\mathrm{Gr}_P)_{\mathrm{red}} = \coprod_{c \in \pi_0(\mathrm{Gr}_L)} S_c$$

and a decomposition of $(\mathrm{Gr}_G)_{\mathrm{red}}$ by ind-subschemes

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \bigcup_{c \in \pi_0(\mathrm{Gr}_L)} S_c.$$

Definition 3.5.1. Let $(c, \lambda) \in \pi_0(\mathrm{Gr}_L) \times X_*(T)^+$. The MV-cycle of index (c, λ) is the reduced k-scheme

$$S_c \cap \mathrm{Gr}_G^{\leq \lambda}.$$

Fix a dominant cocharacter $\mathbb{G}_m \to T$ whose centralizer in G is equal to L, and equip Gr_G with the restriction to $\mathbb{G}_m$ of the adjoint action of T along this cocharacter.

Let $\lambda \in X_*(T)^+$ and $X := \mathrm{Gr}_G^{\leq \lambda}$. The connected components of the attractor X^+ are realized as subschemes of X, and $X^0(k) = (\mathrm{Gr}_L \cap X)(k)$.

Lemma 3.5.2. *Let $c \in \pi_0(\mathrm{Gr}_L)$. Then $\mathrm{Gr}_L^c \cap X$ is connected.*

Proof. Indeed $\mathrm{Gr}_L^c \cap X = \mathrm{Gr}_L^c \cap \mathrm{Gr}_G^{\leq \lambda}$ is a closed L^+L-stable subscheme of Gr_L^c, hence a union of Cartan closures for the affine Grassmannian Gr_L which are contained in the connected component Gr_L^c. Such Cartan closures are irreducible, and all contain the unique minimal L^+L-orbit of Gr_L^c, so any union of them is connected. ■

It follows that

$$\pi_0(X^0) = \big\{ |\mathrm{Gr}_L^c \cap X| \mid c \in \pi_0(\mathrm{Gr}_L) \text{ and } \mathrm{Gr}_L^c \cap X \neq \emptyset \big\}.$$

Next, the bijection $\mathrm{Gr}_P(k) \xrightarrow{\sim} \mathrm{Gr}_G(k)$ corresponds to the decomposition

$$G\big(k(\!(t)\!)\big)/G\big(k[\![t]\!]\big) = \bigcup_{c \in \pi_0(\mathrm{Gr}_L)} S_c(k) = \bigcup_{c \in \pi_0(\mathrm{Gr}_L)} U_P\big(k(\!(t)\!)\big)\,\mathrm{Gr}_L^c(k),$$

and so we compute using Remark 2.3.2 that

$$\forall c \in \pi_0(X^0), \quad (X_c)_{\mathrm{red}} = S_c \cap \mathrm{Gr}_G^{\leq \lambda}.$$

Thus the MV-cycles indexed by (c, λ) for varying c are precisely the $(X_c)_{\mathrm{red}} \subset X$, and they decompose X as

$$X = \bigcup_{\substack{c \in \pi_0(\mathrm{Gr}_L) \\ \mathrm{Gr}_L^c \cap X \neq \emptyset}} (X_c)_{\mathrm{red}}.$$

3.6. Equivariant resolutions of Schubert varieties

Let

$$W_a := \mathbb{Z}\Phi^\vee \rtimes W \subset \widetilde{W} := X_*(T) \rtimes W$$

be the affine Weyl group and the Iwahori-Weyl group. Consider the *length function*

$$\begin{aligned} \ell : \widetilde{W} &\longrightarrow \mathbb{N} \\ \nu w &\longmapsto \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)>0}} \big|\langle \nu, \alpha\rangle\big| + \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)<0}} \big|\langle \nu, \alpha\rangle + 1\big|. \end{aligned}$$

Let S_a be the set of elements of length 1 which are contained in W_a. Then (W_a, S_a) is a Coxeter system. Let $\Omega \subset \widetilde{W}$ be the set of elements of length 0. This is a subgroup and $\widetilde{W} = W_a \rtimes \Omega$. Finally, denote by $\mathcal{B}$ the Iwahori group scheme equal to the dilation of $G_{k[\![t]\!]}$ along B_k, and for each $s \in S_a$, by $\mathcal{P}_s$ the parahoric group scheme increasing $\mathcal{B}$ determined by s.

Now let $\lambda \in X_*(T)^+$. Choose a reduced expression of $\lambda w_0 \in \widetilde{W}$, i.e., an $(n+1)$-tuple $(s_1, \ldots, s_n, \omega) \in S_a^n \times \Omega$ such that $s_1 \cdots s_n \omega = \lambda w_0$ and $n = \ell(\lambda w_0)$. In the next proposition, we denote by $\mathcal{F}\ell_G^{\leq \lambda w_0}$ the *Schubert variety* of λw_0 in the *affine flag variety* $\mathcal{F}\ell_G := LG/L^+\mathcal{B}$, i.e., the closure of $\mathcal{F}\ell_G^{\lambda w_0} := L^+\mathcal{B} \cdot \lambda w_0 \subset \mathcal{F}\ell_G$ with reduced structure.

Proposition 3.6.1. *The fpqc quotient $\widetilde{X} := L^+\mathcal{P}_{s_1} \times^{L^+\mathcal{B}} \cdots \times^{L^+\mathcal{B}} L^+\mathcal{P}_{s_n}/L^+\mathcal{B}$ is representable by a connected smooth projective scheme over k, and it is equipped with a T-action by multiplication on the left on the factor $L^+\mathcal{P}_{s_1}$. The morphism*

$$\begin{aligned} L^+\mathcal{P}_{s_1} \times^{L^+\mathcal{B}} \cdots \times^{L^+\mathcal{B}} L^+\mathcal{P}_{s_n}/L^+\mathcal{B} &\longrightarrow LG/L^+\mathcal{B} =: \mathcal{F}\ell_G \\ [p_1, \ldots, p_n] &\longmapsto p_1 \cdots p_n \omega \end{aligned}$$

factors through $\mathcal{F}\ell_G^{\leq \lambda w_0}$. The canonical projection

$$\mathcal{F}\ell_G := LG/L^+\mathcal{B} \longrightarrow LG/L^+G =: \mathrm{Gr}_G$$

induces a morphism $\mathcal{F}\ell_G^{\leq \lambda w_0} \to \mathrm{Gr}_G^{\leq \lambda} =: X$. The composition

$$f : \widetilde{X} \longrightarrow X$$

is surjective, T-equivariant, and satisfies $Rf_*\mathbb{F}_p = \mathbb{F}_p[0]$.

Proof. The morphism $f_1 : \widetilde{X} \to \mathcal{F}\ell_G^{\leq \lambda w_0}$ spelled out in the proposition is nothing but the well-known affine Demazure resolution of the Schubert variety $\mathcal{F}\ell_G^{\leq \lambda w_0}$ [21, Prop. 8.8]. It satisfies $R(f_1)_*\mathbb{F}_p = \mathbb{F}_p[0]$. Indeed, decompose it as

$$\widetilde{X} \xrightarrow{f_1'} (\mathcal{F}\ell_G^{\leq \lambda w_0})^{\mathrm{nor}} \xrightarrow{f_1''} \mathcal{F}\ell_G^{\leq \lambda w_0},$$

where f_1'' is the normalization. Then f_1'' is a universal homeomorphism by [21, Prop. 9.7(a)]. Moreover, $R(f_1')_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{(\mathcal{F}\ell_G^{\leq\lambda w_0})^{\mathrm{nor}}}[0]$ by [21, Prop. 9.7 (d)], whence $R(f_1')_*\mathbb{F}_p = \mathbb{F}_p[0]$ by considering the Artin–Schreier short exact sequences on $\widetilde{X}$ and on $(\mathcal{F}\ell_G^{\leq\lambda w_0})^{\mathrm{nor}}$.

On the other hand, the morphism $f_2 : \mathcal{F}\ell_G^{\leq\lambda w_0} \to \mathrm{Gr}_G^{\leq\lambda} =: X$ is the restriction over $\mathrm{Gr}_G^{\leq\lambda}$ of the canonical projection $\mathcal{F}\ell_G \to \mathrm{Gr}_G$. In particular it is a G/B-bundle, whence $R(f_2)_*\mathbb{F}_p = \mathbb{F}_p[0]$ by proper base change and the Bruhat decomposition of the flag variety G/B (which can be filtered), cf. Corollary 2.2.3.

Thus $Rf_*\mathbb{F}_p = R(f_2)_*R(f_1)_*\mathbb{F}_p = \mathbb{F}_p[0]$. ■

Remark 3.6.2. The morphism $\widetilde{X} \to \mathcal{F}\ell_G^{\leq\lambda w_0}$ in Proposition 3.6.1 is moreover birational, so that it is a *resolution of singularities* of the Schubert variety $\mathcal{F}\ell_G^{\leq\lambda w_0}$, and $\widetilde{X} \to X$ in Proposition 3.6.1 is the composition of the latter with the G/B-fibration

$$\mathcal{F}\ell_G^{\leq\lambda w_0} \longrightarrow \mathrm{Gr}_G^{\leq\lambda}.$$

Instead, we could also have used a T-equivariant resolution of singularities of the variety $\mathrm{Gr}_G^{\leq\lambda}$ itself, e.g. the affine Demazure resolution of $\mathcal{F}\ell_G^{\leq\lambda}$ followed by the birational projection $\mathcal{F}\ell_G^{\leq\lambda} \to \mathrm{Gr}_G^{\leq\lambda}$.

In fact, this resolution of $\mathrm{Gr}_G^{\leq\lambda}$ is a very particular case of the equivariant resolutions of singularities of Schubert varieties in the twisted affine flag variety associated to any connected reductive group over $k((t))$ constructed in [22]; precisely it is a particular case of [22, Ex. 3.2 (i)].[2] If the reductive group over $k((t))$ splits over a tamely ramified extension and the order of the fundamental group of its derived subgroup is prime-to-p, then any Schubert variety has rational singularities by [21, Th. 8.4]; since "having rational singularities" is an intrinsic notion by [10, Th. 1] (see also [18]), then in this case all the resolutions f from [22] satisfy $Rf_*\mathbb{F}_p = \mathbb{F}_p[0]$ (using Artin–Schreier).

3.7. $\mathbb{F}_p$-direct images with compact support of the MV-cycles

Theorem 3.7.1. *Let* $(\nu,\lambda) \in X_*(T) \times X_*(T)^+$. *Then*

$$R\Gamma_c(S_\nu \cap \mathrm{Gr}_G^{\leq\lambda}, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p|_{\{w_0(\lambda)\}}[0] & \textit{if } \nu = w_0(\lambda), \\ 0 & \textit{otherwise}. \end{cases}$$

More generally, let $(c,\lambda) \in \pi_0(\mathrm{Gr}_L) \times X_*(T)^+$. *Let*

$$q_{c,\lambda} : S_c \cap \mathrm{Gr}_G^{\leq\lambda} \longrightarrow \mathrm{Gr}_L^c \cap \mathrm{Gr}_G^{\leq\lambda}$$

be the morphism of k-schemes defined by the diagram

$$\begin{tikzcd} & \mathrm{Gr}_P \arrow[dl] \arrow[dr] & \\ \mathrm{Gr}_L & & \mathrm{Gr}_G . \end{tikzcd}$$

---

[2] For the normalization of the Kottwitz map as in [21], which is opposite to the one in [22].

Then

$$R(q_{c,\lambda})_!\mathbb{F}_p = \begin{cases} \mathbb{F}_p|_{\mathrm{Gr}_L^{\leq w_0^L w_0(\lambda)}}[0] & \textit{if } c = c\big(w_0(\lambda)\big), \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Let $\eta_T : \mathbb{G}_m \to T$ be a regular dominant cocharacter. We start by applying Corollary 2.3.5 to $X := \mathrm{Gr}_G^{\leq\lambda}$ equipped the $\mathbb{G}_m$-action $\eta_T(\mathbb{G}_m)$ obtained by restriction of the adjoint T-action along η_T; it does apply thanks to Lemma 2.3.3 (2) combined with [27, §1.5], and Proposition 3.6.1.

Recall from [19, Th. 3.2 (a)] (see also [27, Th. 5.3.9]) that the MV-cycle $S_{w_0(\lambda)} \cap \mathrm{Gr}_G^{\leq\lambda}$ is 0-dimensional. Hence

$$(X_{w_0(\lambda)})_{\mathrm{red}} = S_{w_0(\lambda)} \cap \mathrm{Gr}_G^{\leq\lambda} = \big\{w_0(\lambda)\big\} \subset X_{\mathrm{red}}$$

is closed, and the theorem in the case of the torus T follows.

Next let L be a standard Levi. We have the canonical commutative diagram

$$\begin{tikzcd} \mathrm{Gr}_B \arrow[r] \arrow[d] & \mathrm{Gr}_P \arrow[r] \arrow[d] & \mathrm{Gr}_G \\ \mathrm{Gr}_T \arrow[r] & \mathrm{Gr}_L\,. & \end{tikzcd}$$

It shows that for each $c \in \pi_0(\mathrm{Gr}_L)$,

$$S_c(k) = \bigcup_{\nu \in X_*(T) \cap \mathrm{Gr}_L^c} S_\nu(k) \subset \mathrm{Gr}_G(k).$$

Intersecting with $X = \mathrm{Gr}_G^{\leq\lambda} \subset \mathrm{Gr}_G$ we get

$$X_c(k) = \bigcup_{\nu \in X_*(T) \cap \mathrm{Gr}_L^c \cap X} X_\nu(k) \quad \subset X(k).$$

Consequently, the subscheme $(X_c)_{\mathrm{red}} \subset X$ is $\eta_T(\mathbb{G}_m)$-stable, and the reduced connected components of its attractor are realized by the subschemes $(X_\nu)_{\mathrm{red}}$, $\nu \in X_*(T) \cap (\mathrm{Gr}_L^c \cap X)$. In particular, by Lemma 2.3.3 (2), there exists at least one nonempty closed $(X_\nu)_{\mathrm{red}} \subset (X_c)_{\mathrm{red}}$.

Now let $\eta_L : \mathbb{G}_m \to T$ be a dominant cocharacter whose centralizer in G is L, and equip $X := \mathrm{Gr}_G^{\leq\lambda}$ with the $\mathbb{G}_m$-action $\eta_L(\mathbb{G}_m)$ obtained by restriction of the adjoint T-action along η_L. Thanks to Lemma 2.3.3 (2) combined with [27, §1.5], there exists at least one nonempty $(X_{c_0})_{\mathrm{red}} := (X_c)_{\mathrm{red}} \subset X$ which is closed. Choosing $(X_{\nu_0})_{\mathrm{red}} \subset (X_{c_0})_{\mathrm{red}}$ nonempty and closed, then we get $(X_{\nu_0})_{\mathrm{red}} \subset X_{\mathrm{red}}$ nonempty and closed, so that $\nu_0 = w_0(\lambda)$ by the torus case. Hence $c_0 = c(w_0(\lambda))$. And by Lemma 3.7.2 (2) below,

$$|X^0_{c(w_0(\lambda))}| = |\mathrm{Gr}_L^{c(w_0(\lambda))} \cap X| = |\mathrm{Gr}_L^{\leq w_0^L w_0(\lambda)}|.$$

The theorem in the case of the standard Levi L follows by Corollary 2.3.5, which applies thanks to Proposition 3.6.1. ■

Lemma 3.7.2. *Let $c \in \pi_0(\mathrm{Gr}_L)$ and $\lambda \in X_*(T)^+$.*

(1) *If $\lambda \in c$ then $\mathrm{Gr}_L^c \cap \mathrm{Gr}_G^{\leq\lambda} = \mathrm{Gr}_L^{\leq\lambda}$.*

(2) *If $w_0(\lambda) \in c$ then $\mathrm{Gr}_L^c \cap \mathrm{Gr}_G^{\leq\lambda} = \mathrm{Gr}_L^{\leq w_0^L w_0(\lambda)}$.*

Proof. Let $\Delta^\vee \subset \Phi^\vee$ be the set of simple coroots of G with respect to the pair (B, T), and let $\Delta_L^\vee \subset \Delta^\vee$ be the subset of simple coroots of the Levi L with respect to $(B \cap L, T)$. By the Cartan decomposition

$$\mathrm{Gr}_L \cap \mathrm{Gr}_G^{\leq\lambda} = \bigcup_{\substack{\lambda' \in X_*(T)^+ \\ \lambda' \leq \lambda}} \quad \bigcup_{\mu \in X_*(T)_{+/L} \cap W\lambda'} \quad \mathrm{Gr}_L^\mu .$$

As $\mathrm{Gr}_L^c \cap \mathrm{Gr}_G^{\leq\lambda} \subset \mathrm{Gr}_L$ is closed and L^+L-stable, to prove (1) it suffices to show that, for $\lambda \in c$ and μ as above, $\mathrm{Gr}_L^c \cap \mathrm{Gr}_L^\mu = \emptyset$ unless $\mu \leq_L \lambda$. To prove this, suppose $\mathrm{Gr}_L^c \cap \mathrm{Gr}_L^\mu \neq \emptyset$. Then $\lambda - \mu \in \mathbb{Z}\Delta_L^\vee$, and moreover since $\mu \in W\lambda'$ we have $\lambda - \mu \in \mathbb{N}\Delta^\vee$. Because $\Delta^\vee$ is linearly independent then $\lambda - \mu \in \mathbb{Z}\Delta_L^\vee \cap \mathbb{N}\Delta^\vee = \mathbb{N}\Delta_L^\vee$. Thus $\mu \leq_L \lambda$ and hence the claim follows. Finally, (2) can be proved similarly, since then $\mathrm{Gr}_L^c \cap \mathrm{Gr}_L^\mu \neq \emptyset$ implies $w_0(\lambda) - w_0^L(\mu) \in \mathbb{Z}\Delta_L^\vee$ and $\mu \in W\lambda'$ implies $w_0^L(\mu) - w_0(\lambda) \in \mathbb{N}\Delta^\vee$, and hence $w_0^L(\mu) - w_0(\lambda) \in \mathbb{Z}\Delta_L^\vee \cap \mathbb{N}\Delta^\vee = \mathbb{N}\Delta_L^\vee$. ∎

Finally, we record from the proof of Theorem 3.7.1 (and Lemma 2.3.7) the following result.

Corollary 3.7.3. *For all $\lambda \in X_*(T)^+$,*

$$S_{c(w_0(\lambda))} \cap \mathrm{Gr}_G^{\leq\lambda} = \mathrm{Gr}_L^{\leq w_0^L w_0(\lambda)} .$$

4. Hyperbolic localization on the affine Grassmannian

4.1. Perverse $\mathbb{F}_p$-sheaves on the affine Grassmannian

For a separated scheme X of finite type over k let $P_c^b(X, \mathbb{F}_p)$ be the abelian category of perverse $\mathbb{F}_p$-sheaves on X as defined in [9, §2]. This is an abelian subcategory of $D_c^b(X, \mathbb{F}_p)$ in which all objects have finite length. The definition of perverse sheaves extends to ind-schemes of ind-finite type as in [9, Rem. 3.13].

Let $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \subset P_c^b(\mathrm{Gr}_G, \mathbb{F}_p)$ be the full abelian subcategory of L^+G-equivariant perverse $\mathbb{F}_p$-sheaves on Gr_G as defined in [9, §6.1]. This category consists of objects $\mathcal{F}^\bullet \in P_c^b(\mathrm{Gr}_G, \mathbb{F}_p)$ that are equivariant in the naive sense. In other words, $\mathcal{F}^\bullet$ is equivariant if there exists some $\lambda \in X_*(T)^+$ and some finite-type jet quotient $L^+G \to L^nG$, $n \in \mathbb{Z}_{\geq 0}$, acting on $\mathrm{Gr}_G^{\leq\lambda}$ such that $\mathcal{F}^\bullet$ is supported on $\mathrm{Gr}_G^{\leq\lambda}$ and there exists an isomorphism $Ra^*\mathcal{F}^\bullet \cong Rp^*\mathcal{F}^\bullet$, where a and p are the action and projection maps. Similarly to ℓ-adic sheaves, this naive notion of equivariance coincides with the correct notion. Indeed, there is a unique such isomorphism which satisfies the associated cocycle condition [9, Lem. 3.7], and maps between equivariant objects automatically respect the equivariance data [9, Prop. 3.9]. Additionally, $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ is stable under subqoutients in $P_c^b(\mathrm{Gr}_G, \mathbb{F}_p)$ [9, Prop. 3.10].

By [9, Th. 1.1], the category $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ is symmetric monoidal and the functor

$$H = \bigoplus_{i\in\mathbb{Z}} R^i\Gamma : \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \longrightarrow (\mathrm{Vect}_{\mathbb{F}_p}, \otimes)$$

is an exact faithful tensor functor. The definition of the convolution product $*$ will be reviewed in Section 5.3.

By [9, Th. 1.5], the simple objects in $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ are the shifted constant sheaves:

$$\mathrm{IC}_\lambda = \mathbb{F}_p\big[2\rho(\lambda)\big] \in P_c^b(\mathrm{Gr}_G^{\leq\lambda}, \mathbb{F}_p), \quad \lambda \in X_*(T)^+.$$

Furthermore, if $\lambda_i \in X_*(T)^+$ then by [9, Th. 1.2] there is a natural isomorphism

$$\mathrm{IC}_{\lambda_1} * \mathrm{IC}_{\lambda_2} \cong \mathrm{IC}_{\lambda_1+\lambda_2}.$$

4.2. The hyperbolic localization functor

Definition 4.2.1. Let $\nu \in X_*(T)$ and $\mathcal{F}^\bullet \in D_c^b(\mathrm{Gr}_G, \mathbb{F}_p)$. Denote by $s_\nu \colon S_\nu \to \mathrm{Gr}_G$ the ind-immersion of the corresponding connected component of $(\mathrm{Gr}_B)_{\mathrm{red}}$ and define

$$R\Gamma_c(S_\nu, \mathcal{F}^\bullet) := R\Gamma_c(S_\nu, Rs_\nu^*\mathcal{F}^\bullet) \in D_c^b(\mathrm{Vect}_{\mathbb{F}_p}),$$

and

$$\forall i \in \mathbb{Z}, \quad H_c^i(S_\nu, \mathcal{F}^\bullet) := H^i\big(R\Gamma_c(S_\nu, \mathcal{F}^\bullet)\big) = H_c^i(S_\nu, Rs_\nu^*\mathcal{F}^\bullet) \in \mathrm{Vect}_{\mathbb{F}_p}.$$

Theorem 4.2.2. *Let $\nu \in X_*(T)$ and $\lambda \in X_*(T)^+$.*

(1) *We have*

$$H_c^{2\rho(\nu)}(S_\nu, \mathrm{IC}_\lambda) = \begin{cases} H^0\big(\{w_0(\lambda)\}, \mathbb{F}_p\big) = \mathbb{F}_p & \text{if } \nu = w_0(\lambda), \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If $i \neq 2\rho(\nu)$ then*

$$H_c^i(S_\nu, \mathrm{IC}_\lambda) = 0.$$

(3) *If $\mathcal{F}^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$, then*

$$R\Gamma_c(S_\nu, \mathcal{F}^\bullet) \in D_c^{\leq 2\rho(\nu)}(\mathrm{Vect}_{\mathbb{F}_p}) \cap D_c^{\geq 2\rho(\nu)}(\mathrm{Vect}_{\mathbb{F}_p}) = \mathrm{Vect}_{\mathbb{F}_p}\big[-2\rho(\nu)\big].$$

Proof. Since IC_λ is the shifted constant sheaf $\mathbb{F}_p[2\rho(\lambda)]$ supported on $\mathrm{Gr}_G^{\leq\lambda}$ then parts (1) and (2) follow immediately from Theorem 3.7.1. To prove part (3), by dévissage we can assume that $\mathcal{F}^\bullet = \mathrm{IC}_\lambda$ for some $\lambda \in X_*(T)^+$. Then part (3) follows from (1) and (2). ■

Remark 4.2.3. We claim that

$$H_c^{2\rho(\nu)}(S_\nu, \mathrm{IC}_\lambda) \cong H_c^{2\rho(\nu+\lambda)}(S_\nu \cap \mathrm{Gr}_G^\lambda, \mathbb{F}_p),$$

which is also true for characteristic 0 coefficients, see e.g. [1, proof of Prop. 1.5.13]. To prove the claim, note that it suffices to show that the canonical map

$$H_c^{2\rho(\nu+\lambda)}(S_\nu \cap \mathrm{Gr}_G^\lambda, \mathbb{F}_p) \longrightarrow H_c^{2\rho(\nu+\lambda)}(S_\nu \cap \mathrm{Gr}_G^{\leq\lambda}, \mathbb{F}_p)$$

is an isomorphism. If $\nu = w_0(\lambda)$ then $S_\nu \cap \mathrm{Gr}_G^\lambda = S_\nu \cap \mathrm{Gr}_G^{\leq\lambda} = \{\nu\}$, so the claim follows in this case. If $\nu \neq w_0(\lambda)$ then by Theorem 3.7.1, we must show that

$$H_c^{2\rho(\nu+\lambda)}(S_\nu \cap \mathrm{Gr}_G^\lambda, \mathbb{F}_p) = 0.$$

Note that $\dim S_\nu \cap \mathrm{Gr}_G^\lambda = \rho(\nu + \lambda) > 0$ by [19, Th. 3.2]. The desired vanishing then follows from the following general fact (cf. [14, Th. 7.2.11]): if X is a separated scheme of finite type over k, then

$$\forall i > \dim X, \quad H_c^i(X, \mathbb{F}_p) = 0.$$

4.3. An alternative description of the hyperbolic localization functor

Definition 4.3.1. Let $\nu \in X_*(T)$ and $\mathcal{F}^\bullet \in D_c^b(\mathrm{Gr}_G, \mathbb{F}_p)$. Denote by $i_\nu \colon \{\nu\} \to \mathrm{Gr}_G$ the inclusion of the k-point $\nu(t)$ and define

$$R\Gamma\big(\{\nu\}, \mathcal{F}^\bullet\big) := Ri_\nu^* \mathcal{F}^\bullet \in D_c^b(\mathrm{Vect}_{\mathbb{F}_p}),$$

and

$$\forall i \in \mathbb{Z}, \quad H^i\big(\{\nu\}, \mathcal{F}^\bullet\big) := H^i\big(R\Gamma\big(\{\nu\}, \mathcal{F}^\bullet\big)\big) = H^i(Ri_\nu^* \mathcal{F}^\bullet) \in \mathrm{Vect}_{\mathbb{F}_p}.$$

Lemma 4.3.2. *Let $\nu \in X_*(T)$ and $\mathcal{F}^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$.*

(1) *If $\lambda \in X_*(T)^+$ then*

$$H^{2\rho(\nu)}\big(\{\nu\}, \mathrm{IC}_\lambda\big) = \begin{cases} H^0\big(\{\nu\}, \mathbb{F}_p\big) = \mathbb{F}_p & \text{if } \nu = w_0(\lambda), \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If $H^i(\{\nu\}, \mathcal{F}^\bullet) \neq 0$ then*

$$i \equiv 2\rho(\nu) \mod 2.$$

Proof. For part (1), we have

$$H^{2\rho(\nu)}\big(\{\nu\}, \mathrm{IC}_\lambda\big) = H^{2\rho(\nu+\lambda)}\big(\{\nu\} \cap \mathrm{Gr}_G^{\leq\lambda}, \mathbb{F}_p\big).$$

This is zero unless $\{\nu\} \in \mathrm{Gr}_G^{\leq\lambda}$ and $2\rho(\nu + \lambda) = 0$, in which case $w_0(\lambda) \leq \nu \leq \lambda$ and $2\rho(\nu - w_0(\lambda)) = 0$, i.e., $\nu = w_0(\lambda)$.

By dévissage, to prove part (2) we can assume that $\mathcal{F}^\bullet = \mathrm{IC}_\lambda$ for some $\lambda \in X_*(T)^+$. Then for all $i \in \mathbb{Z}$ we have

$$H^i\big(\{\nu\}, \mathrm{IC}_\lambda\big) = H^{i+2\rho(\lambda)}\big(\{\nu\} \cap \mathrm{Gr}_G^{\leq\lambda}, \mathbb{F}_p\big).$$

If this is nonzero then $\{\nu\} \in \mathrm{Gr}_G^{\leq\lambda}$ and $i + 2\rho(\lambda) = 0$, so $\rho(\lambda - \nu)$ is an integer and

$$i + 2\rho(\nu) = i + 2\rho(\lambda) - 2\rho(\lambda - \nu) \equiv 0 \mod 2. \qquad \blacksquare$$

Theorem 4.3.3. *For $\nu \in X_*(T)$ there is an isomorphism of functors*

$$H_c^{2\rho(\nu)}(S_\nu, \cdot) \xrightarrow{\sim} H^{2\rho(\nu)}\big(\{\nu\}, \cdot\big)\colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow \mathrm{Vect}_{\mathbb{F}_p}.$$

Proof. By the adjunction between Ri_ν^* and $Ri_{\nu *}$ there is a natural map

$$H_c^{2\rho(\nu)}(S_\nu, \mathcal{F}^\bullet) \longrightarrow H^{2\rho(\nu)}\big(\{\nu\}, \mathcal{F}^\bullet\big).$$

If $\mathcal{F}^\bullet = \mathrm{IC}_\lambda$ for $\lambda \in X_*(T)^+$ then it is an isomorphism by Theorem 4.2.2 (1) and Lemma 4.3.2 (1). For the general case, note that $H^{2\rho(\nu)-1}(\{\nu\}, \mathcal{F}^\bullet) = H^{2\rho(\nu)+1}(\{\nu\}, \mathcal{F}^\bullet) = 0$ for all $\mathcal{F}^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ by Lemma 4.3.2 (2). Since $H_c^{2\rho(\nu)+1}(S_\nu, \mathcal{F}^\bullet) = 0$ for all $\mathcal{F}^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ by Theorem 4.2.2 (3), then by induction on the length of $\mathcal{F}^\bullet$ and the five lemma we see that the map $H_c^{2\rho(\nu)}(S_\nu, \mathcal{F}^\bullet) \to H^{2\rho(\nu)}(\{\nu\}, \mathcal{F}^\bullet)$ is an isomorphism in general. ■

5. The total weight functor

5.1. The definition of the total weight functor

Definition 5.1.1. For $\nu \in X_*(T)$, the *weight functor associated to* ν is

$$F_\nu := H_c^{2\rho(\nu)}(S_\nu, \cdot) \xrightarrow{\sim} H^{2\rho(\nu)}\big(\{\nu\}, \cdot\big)\colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow \mathrm{Vect}_{\mathbb{F}_p}.$$

Proposition 5.1.2. *The functor F_ν is exact. Furthermore, if $\nu \notin X_*(T)_-$ then $F_\nu = 0$.*

Proof. Exactness follows from Theorem 4.2.2 (3). Since for $\nu \notin X_*(T)_-$ we have $F_\nu(\mathcal{F}^\bullet) = 0$ for all simple $\mathcal{F}^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ by Theorem 4.2.2 (1), we may conclude by induction on the length that $F_\nu = 0$ in this case. ■

Notation 5.1.3. Given an abstract abelian monoid A, we will denote by $(\mathrm{Vect}_{\mathbb{F}_p}(A), \otimes)$ the symmetric monoidal category of finite dimensional A-graded $\mathbb{F}_p$-vector spaces equipped with the tensor product

$$\mathbb{F}_p(a) \otimes \mathbb{F}_p(b) := \mathbb{F}_p(a+b),$$

where $\mathbb{F}_p(a)$ denotes the vector space $\mathbb{F}_p$ placed in "degree" $a \in A$.

Definition 5.1.4. The *total weight functor* is

$$F_- := \bigoplus\nolimits_{\nu \in X_*(T)_-} F_\nu\colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow \mathrm{Vect}_{\mathbb{F}_p}\big(X_*(T)_-\big).$$

Remark 5.1.5. Recall from Section 3.3 the diagram

$$\begin{tikzcd} & \mathrm{Gr}_B \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Gr}_T & & \mathrm{Gr}_G . \end{tikzcd}$$

Since

$$(\mathrm{Gr}_T)_{\mathrm{red}} = \coprod_{\nu \in X_*(T)} \{\nu\}$$

then F_- can be obtained from the functor

$$Rq_! \circ Rp^* \colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow D_c^b(\mathrm{Gr}_T, \mathbb{F}_p)$$

by taking the direct sum of the stalks over the $\{\nu\}$ in degree $2\rho(\nu)$. This identifies F_- with the T-constant term functor CT_T^G defined in Definition 6.1.1.

5.2. Relation to the Satake equivalence

Recall the exact faithful symmetric monoidal functor

$$H = \bigoplus_{i \in \mathbb{Z}} R^i\Gamma : \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \longrightarrow (\mathrm{Vect}_{\mathbb{F}_p}, \otimes)$$

from [9, Th. 6.11, Th. 7.11]. Our goal in this subsection is to construct a natural isomorphism between H and F_- composed with the forgetful functor $\mathrm{Vect}_{\mathbb{F}_p}(X_*(T)_-) \to \mathrm{Vect}_{\mathbb{F}_p}$.

Remark 5.2.1. In the case of characteristic 0 coefficients, Baumann and Riche construct an isomorphism between H and $\bigoplus_{\nu \in X_*(T)} F_\nu$ in the proof of [1, Th. 1.5.9]. In our proof of Theorem 5.2.2 below we use Theorem 4.3.3, which is unique to $\mathbb{F}_p$-sheaves, to compare the functors H and F_-.

By [9, Th. 6.9], $R^i\Gamma(\mathcal{F}^\bullet) = 0$ for all $\mathcal{F}^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ and $i > 0$. Set $\mathbb{Z}_- := \mathbb{Z}_{\leq 0}$. For all $i \in \mathbb{Z}_-$, the adjunction between Ri_ν^* and $Ri_{\nu *}$ induces a natural transformation of functors

$$R^i\Gamma \longrightarrow \bigoplus_{\substack{\nu \in X_*(T)_- \\ 2\rho(\nu) = i}} H^{2\rho(\nu)}\big(\{\nu\}, \cdot\big)$$

from $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ to $\mathrm{Vect}_{\mathbb{F}_p}$. Hence there is a natural transformation of functors

$$H = \bigoplus_{i \in \mathbb{Z}_-} R^i\Gamma \longrightarrow \bigoplus_{i \in \mathbb{Z}_-} \bigoplus_{\substack{\nu \in X_*(T)_- \\ 2\rho(\nu) = i}} F_\nu = \bigoplus_{\nu \in X_*(T)_-} F_\nu$$

from $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ to $\mathrm{Vect}_{\mathbb{F}_p}$.

Theorem 5.2.2. *The natural transformation of functors*

$$H \to \bigoplus_{\nu \in X_*(T)_-} F_\nu \colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow \mathrm{Vect}_{\mathbb{F}_p}$$

is an isomorphism. In particular, for all $i \in \mathbb{Z}$ it restricts to an isomorphism

$$R^i\Gamma \cong \bigoplus_{\substack{\nu \in X_*(T)_- \\ 2\rho(\nu) = i}} F_\nu \colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow \mathrm{Vect}_{\mathbb{F}_p}\,.$$

Proof. Let $\lambda \in X_*(T)^+$. Combining [9, Th. 6.9] and Theorem 4.2.2 (1), taking the stalk at $\{w_0(\lambda)\}$ defines an isomorphism in $\mathrm{Vect}_{\mathbb{F}_p}$

$$\begin{aligned} H(\mathrm{IC}_\lambda) = R^{-2\rho(\lambda)}\Gamma(\mathrm{IC}_\lambda) &= H^{-2\rho(\lambda)}\big(\mathrm{Gr}_G^{\leq\lambda}, \mathbb{F}_p[2\rho(\lambda)]\big) \\ &\xrightarrow{\sim} H^{2\rho(w_0(\lambda))}\big(Ri^*_{w_0(\lambda)}\mathbb{F}_p[2\rho(\lambda)]\big) = F_{w_0(\lambda)}(\mathrm{IC}_\lambda). \end{aligned}$$

Thus since $F_\nu(\mathrm{IC}_\lambda) = 0$ if $\nu \neq w_0(\lambda)$ then the natural map

$$H(\mathcal{F}^\bullet) \longrightarrow \bigoplus_{\nu \in X_*(T)_-} F_\nu(\mathcal{F}^\bullet)$$

is an isomorphism if $\mathcal{F}^\bullet$ is simple. Now H is exact by [9, Th. 6.11] and each F_ν is exact by Proposition 5.1.2. Hence it follows by induction on the length of $\mathcal{F}^\bullet$ that the above map is an isomorphism in general. ■

By Theorem 5.2.2, composing F_- with the forgetful functor $\mathrm{Vect}_{\mathbb{F}_p}(X_*(T)_-) \to \mathrm{Vect}_{\mathbb{F}_p}$ gives H.

Remark 5.2.3. Using the method in [19, Th. 3.6] one can show that the decomposition $H \cong \oplus_{\nu \in X_*(T)_-} F_\nu$ is independent of the choice of the pair (T, B).

5.3. Recollections on convolution

We first recall the definition of the convolution product in $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ following [9, §6.2]. There is a diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times \mathrm{Gr}_G \xrightarrow{q} LG \overset{L^+G}{\times} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G\,.$$

Here p is the quotient map on the first factor, q is the quotient by the diagonal action of L^+G, and m is induced by multiplication in LG. We set

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G := LG \overset{L^+G}{\times} \mathrm{Gr}_G\,.$$

For $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$, by [9, Lem. 6.2] there exists a unique perverse sheaf

$$\mathcal{F}_1^\bullet \mathbin{\tilde{\boxtimes}} \mathcal{F}_2^\bullet \in P_c^b(\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G, \mathbb{F}_p)$$

such that

$$Rp^*(\mathcal{F}_1^\bullet \overset{L}{\boxtimes} \mathcal{F}_2^\bullet) \cong Rq^*(\mathcal{F}_1^\bullet \mathbin{\tilde{\boxtimes}} \mathcal{F}_2^\bullet).$$

The convolution product is

$$\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet := Rm_!(\mathcal{F}_1^\bullet \mathbin{\tilde{\boxtimes}} \mathcal{F}_2^\bullet).$$

Note that because $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ is ind-projective we have $m_! = m_*$.

We now recall the construction of the monoidal structure on H following [9, §7]. Let $X = \mathbb{A}^1$. The construction uses the Beilinson–Drinfeld Grassmannians Gr_{G,X^I} and the global convolution Grassmannians $\widetilde{\mathrm{Gr}}_{G,X^I}$ for $I = \{*\}$ and $I = \{1,2\}$ (see also [27, §3.1]). There is a convolution morphism

$$m_I\colon \widetilde{\mathrm{Gr}}_{G,X^I} \longrightarrow \mathrm{Gr}_{G,X^I}$$

and a projection $f_I\colon \mathrm{Gr}_{G,X^I} \to X^I$. Since $X = \mathbb{A}^1$, for $I = \{*\}$ there are canonical isomorphisms

$$\mathrm{Gr}_{G,X} \cong \mathrm{Gr}_G \times X, \quad \widetilde{\mathrm{Gr}}_{G,X} \cong (\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G) \times X, \quad m_{\{*\}} = m \times \mathrm{id}, \quad f_{\{*\}} = \mathrm{pr}_2 .$$

So in the sequel we keep the notation I for the set $\{1,2\}$ only. Let $U \subset X^2$ be the complement of the image of the diagonal embedding $\Delta\colon X \to X^2$. Then we have the following commutative diagram with Cartesian squares:

$$\begin{tikzcd}
\mathrm{Gr}_G \times \mathrm{Gr}_G \times U \arrow[r] \arrow[d, "\mathrm{id}"] & \widetilde{\mathrm{Gr}}_{G,X^2} \arrow[d, "m_I"] & (\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G) \times X \arrow[l] \arrow[d, "m\times\mathrm{id}"] \\
\mathrm{Gr}_G \times \mathrm{Gr}_G \times U \arrow[r, "j_I"] \arrow[d] & \mathrm{Gr}_{G,X^2} \arrow[d, "f_I"] & \mathrm{Gr}_G \times X \arrow[l, "i_I"'] \arrow[d] \\
U \arrow[r] & X^2 & X. \arrow[l, "\Delta"']
\end{tikzcd}$$

Let $\tau\colon \mathrm{Gr}_{G,X} = \mathrm{Gr}_G \times X \to \mathrm{Gr}_G$ be the projection and let

$$\tau^\circ := R\tau^*[1]\colon D_c^b(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow D_c^b(\mathrm{Gr}_{G,X}, \mathbb{F}_p).$$

Fix $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$. By [9, Lem. 7.6, Prop. 7.10], there is a perverse sheaf

$$\mathcal{F}_{1,2}^\bullet := \tau^\circ \mathcal{F}_1^\bullet \,\tilde{\boxtimes}\, \tau^\circ \mathcal{F}_2^\bullet \in P_c^b(\widetilde{\mathrm{Gr}}_{G,X^2}, \mathbb{F}_p)$$

such that for $x_1, x_2 \in X(k)$,

$$H^{n-2}\big(Rf_{I,!}(Rm_{I,!}\mathcal{F}_{1,2}^\bullet)\big)\big|_{(x_1,x_2)} \cong \begin{cases} \bigoplus\limits_{i+j=n} R^i\Gamma(\mathcal{F}_1^\bullet) \otimes R^j\Gamma(\mathcal{F}_2^\bullet) & \text{if } x_1 \neq x_2 \\ R^n\Gamma(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) & \text{if } x_1 = x_2. \end{cases} \tag{5.1}$$

The sheaf $H^{n-2}(Rf_{I,!}(Rm_{I,!}\mathcal{F}_{1,2}^\bullet))$ is constant by [9, Prop. 7.9]. Therefore, by summing (5.1) over n we get an isomorphism

$$H(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \cong H(\mathcal{F}_1^\bullet) \otimes H(\mathcal{F}_2^\bullet).$$

This gives H the structure of a monoidal functor.

We finally recall that the associativity constraint in $(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *)$ is constructed using the one of the bifunctor $\overset{L}{\boxtimes}$ and proper base change [9, Th. 6.8], and the commutativity constraint as follows. There is a morphism $\mathrm{Gr}_{G,X^2} \to \mathrm{Gr}_{G,X^2}$ which swaps the factors in X^2. Using that this morphism restricts to the identity map over $\Delta(X)$, it is shown in the proof of [9, Th. 7.11] that there is a canonical isomorphism

$$j_{I,!*}\big(\tau^\circ \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \tau^\circ \mathcal{F}_2^\bullet|_U\big)\big|_{\Delta(X)} \cong j_{I,!*}\big(\tau^\circ \mathcal{F}_2^\bullet \overset{L}{\boxtimes} \tau^\circ \mathcal{F}_1^\bullet|_U\big)\big|_{\Delta(X)}.$$

On the other hand, we have the following proposition.

Proposition 5.3.1. *There is a canonical isomorphism*

$$\tau^\circ(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \cong Ri_I^* \circ j_{I,!*}\big(\tau^\circ \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \tau^\circ \mathcal{F}_2^\bullet|_U\big)[-1].$$

Proof. By the arguments in the proof of [9, Prop. 7.10 (ii)], there is a canonical isomorphism

$$Ri_I^*\big(Rm_{I,!}(\mathcal{F}_{1,2}^\bullet)\big)[-1] \cong \tau^\circ(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet).$$

On the other hand, by [9, Lem. 7.8] we have

$$Rm_{I,!}(\mathcal{F}_{1,2}^\bullet) \cong j_{I,!*}\big(\tau^\circ \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \tau^\circ \mathcal{F}_2^\bullet|_U\big). \tag{5.2}$$

■

Consequently, we get a commutativity isomorphism

$$\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet \cong \mathcal{F}_2^\bullet * \mathcal{F}_1^\bullet.$$

In order to make this commutativity isomorphism compatible with that of ⊗ it must be modified by certain sign changes which depend on the parities of the dimensions of the strata occurring in the support of the $\mathcal{F}_i^\bullet$; see the proof of [9, Th. 7.11] for more details.

5.4. Compatibility with convolution

Remark 5.4.1. In this subsection we use Theorem 4.3.3 in order to take $H^{2\rho(\nu)}(\{\nu\}, \cdot)$ as our definition of F_ν. This allows us to give a proof that F_- is a tensor functor which is unique to $\mathbb{F}_p$-sheaves and simpler than that in [19, Prop. 6.4]. In particular, we need only globalize the points $\{\nu\}$ relative to a curve instead of the S_ν. In Section 6.6, we globalize the S_ν to give a proof of the compatibility between convolution and the constant term functor CT_L^G with respect to a general Levi subgroup $L \subset G$. By taking $L = T$ this provides an alternative proof of Theorem 5.4.2 below which is analogous to that in [19, Prop. 6.4].

For $\nu \in X_*(T)_-$ let $\{\nu\}(X^2) \subset \mathrm{Gr}_{G,X^2}$ be the reduced closure of

$$\bigcup_{\substack{\nu_1,\nu_2 \in X_*(T)_- \\ \nu_1+\nu_2=\nu}} \{\nu_1\} \times \{\nu_2\} \times U.$$

The reduced fiber of $\{\nu\}(X^2)$ over $\Delta(X)$ is isomorphic to $\{\nu\} \times X \subset \mathrm{Gr}_G \times X$. Denote by

$$i_{\nu,X^2}\colon \{\nu\}(X^2) \longrightarrow \mathrm{Gr}_{G,X^2}$$

the inclusion. For $\nu \in X_*(T)_-$ and $\mathcal{F}^\bullet \in D_c^b(\mathrm{Gr}_{G,X^2}, \mathbb{F}_p)$ set

$$\widetilde{F}_\nu(\mathcal{F}^\bullet) := Rf_{I,!}\big(Ri_{\nu,X^2,*}(Ri^*_{\nu,X^2}\mathcal{F}^\bullet)\big) \in D_c^b(X^2, \mathbb{F}_p).$$

Theorem 5.4.2. *The total weight functor is a tensor functor*

$$F_-\colon \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \longrightarrow \big(\mathrm{Vect}_{\mathbb{F}_p}\big(X_*(T)_-\big), \otimes\big).$$

Proof. By the same considerations as in the proof of (5.1) in [9, Prop. 7.10], we have

$$H^{2\rho(\nu)-2}\big(\widetilde{F}_\nu(Rm_{I,!}\mathcal{F}^\bullet_{1,2})\big)\big|_{(x_1,x_2)} \cong \begin{cases} \bigoplus\limits_{\nu_1+\nu_2=\nu} F_{\nu_1}(\mathcal{F}_1^\bullet)\otimes F_{\nu_2}(\mathcal{F}_2^\bullet) & \text{if } x_1 \neq x_2 \\ F_\nu(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) & \text{if } x_1 = x_2. \end{cases} \tag{5.3}$$

From the adjunction between Ri^*_{ν,X^2} and $Ri_{\nu,X^2,*}$ we get a natural map

$$H^{n-2}\big(Rf_{I,!}(Rm_{I,!}\mathcal{F}^\bullet_{1,2})\big) \longrightarrow \bigoplus_{2\rho(\nu)=n} H^{n-2}\big(\widetilde{F}_\nu(Rm_{I,!}\mathcal{F}^\bullet_{1,2})\big). \tag{5.4}$$

By Theorem 5.2.2 and the description of the stalks in (5.1), (5.3), the above map (5.4) is an isomorphism over closed points in X^2. Since each of the sheaves in (5.4) is constructible then this is an isomorphism of sheaves on X^2. As $H^{n-2}(Rf_{I,!}(Rm_{I,!}\mathcal{F}^\bullet_{1,2}))$ is constant by [9, Prop. 7.9], then each of the sheaves $H^{n-2}(\widetilde{F}_\nu(Rm_{I,!}\mathcal{F}^\bullet_{1,2}))$ is also constant. Hence by (5.3), we get a natural isomorphism

$$F_\nu(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \cong \bigoplus_{\nu_1+\nu_2=\nu} F_{\nu_1}(\mathcal{F}_1^\bullet) \otimes F_{\nu_2}(\mathcal{F}_2^\bullet).$$

By summing over $\nu \in X_*(T)_-$ we get an isomorphism

$$F_-(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \cong F_-(\mathcal{F}_1^\bullet) \otimes F_-(\mathcal{F}_2^\bullet).$$

The associativity isomorphism in $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ is constructed from the associativity of the operation $\boxtimes$ (see the proof of [9, Th. 7.11]), so the above isomorphism is compatible with the usual associativity isomorphism in $\mathrm{Vect}_{\mathbb{F}_p}(X_*(T)_-)$. Moreover, using (5.2) and (5.3), one can verify directly from the construction in [9, Th. 7.11] that the commutativity isomorphism in $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ is compatible with the commutativity isomorphism in $\mathrm{Vect}_{\mathbb{F}_p}(X_*(T)_-)$. Thus F_- is a tensor functor. ■

We denote by $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)^{\mathrm{ss}}$ the full subcategory of $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ consisting of semi-simple objects. By [9, Th. 1.2], it is a Tannakian subcategory with fiber functor given by the restriction of H.

Corollary 5.4.3. *The functor*

$$F_-|_{(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*)} : \big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*\big) \longrightarrow \big(\mathrm{Vect}_{\mathbb{F}_p}\big(X_*(T)_-\big),\otimes\big)$$

is an equivalence of symmetric monoidal categories. We have

$$\forall\lambda\in X_*(T)^+,\quad F_-(\mathrm{IC}_\lambda)=\mathbb{F}_p\big(w_0(\lambda)\big).$$

Remark 5.4.4. We can summarize this section as follows. Let $2\rho_- : X_*(T)_- \to \mathbb{Z}_-$ be the additive map induced by the group homomorphism $2\rho : X_*(T)\to\mathbb{Z}$, and let $2\rho_- : \mathrm{Vect}_{\mathbb{F}_p}(X_*(T)_-)\to\mathrm{Vect}_{\mathbb{F}_p}(\mathbb{Z}_-)$ be the induced functor. Then the exact faithful symmetric monoidal functor

$$H : \big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p),*\big)\longrightarrow(\mathrm{Vect}_{\mathbb{F}_p},\otimes)$$

factors as a composition of exact faithful symmetric monoidal functors

$$\begin{aligned}\big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p),*\big) &\xrightarrow{F_-}\big(\mathrm{Vect}_{\mathbb{F}_p}\big(X_*(T)_-\big),\otimes\big)\\ &\xrightarrow{2\rho_-}\big(\mathrm{Vect}_{\mathbb{F}_p}(\mathbb{Z}_-),\otimes\big)\xrightarrow{\mathrm{Forget}}(\mathrm{Vect}_{\mathbb{F}_p},\otimes).\end{aligned}$$

6. The constant term functor

6.1. The definition of CT_L^G

We return to the setup in Section 3.5 following the geometric setting explained in [2, §5.3.27]; see also [1, §1.15.1], [15, §5.1]. In particular, $P\subset G$ is a parabolic subgroup containing B, and $L\subset P$ is the Levi factor containing T. We may consider for L all the objects that we consider for G; we will denote them using a letter L as a subscript or a superscript. There is a diagram

$$\begin{tikzcd} & \mathrm{Gr}_P \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Gr}_L & & \mathrm{Gr}_G . \end{tikzcd} \tag{6.1}$$

The connected components of Gr_L are parametrized by

$$\pi_0(\mathrm{Gr}_L)=\pi_1(L)=X_*(T)/\mathbb{Z}\Phi_L^\vee,$$

where $\Phi_L^\vee$ is the set of coroots of L with respect to T. For $c\in\pi_0(\mathrm{Gr}_L)$ let Gr_L^c and Gr_P^c be the corresponding connected components of Gr_L and Gr_P.

Let ρ_L be half the sum of the positive roots of L. Then $2(\rho-\rho_L)(c)$ is a well-defined integer for $c\in\pi_0(\mathrm{Gr}_L)$ since $\rho=\rho_L$ on $\Phi_L^\vee$. Define the locally constant function

$$\deg_P : \mathrm{Gr}_P\longrightarrow\pi_0(\mathrm{Gr}_P)\xrightarrow{2(\rho-\rho_L)}\mathbb{Z}, \tag{6.2}$$

where $\mathrm{Gr}_P\to\pi_0(\mathrm{Gr}_P)$ sends Gr_P^c to c.

Definition 6.1.1. The *L-constant term functor* is

$$\mathrm{CT}_L^G := Rq_! \circ Rp^*[\deg_P]\colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow D_c^b(\mathrm{Gr}_L, \mathbb{F}_p).$$

Let $c \in \pi_0(\mathrm{Gr}_L)$. Since

$$(\mathrm{Gr}_P^c)_{\mathrm{red}} = S_c,$$

then by restricting (6.1) to S_c, we get a diagram

$$\begin{tikzcd} & S_c \arrow[dl, "\sigma_c"'] \arrow[dr, "s_c"] & \\ \mathrm{Gr}_L^c & & \mathrm{Gr}_G . \end{tikzcd}$$

Definition 6.1.2. The *weight functor associated to c* is

$$F_c := R\sigma_{c!} \circ Rs_c^*\big[2(\rho - \rho_L)(c)\big]\colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow D_c^b(\mathrm{Gr}_L^c, \mathbb{F}_p).$$

Lemma 6.1.3. *There is a natural isomorphism of functors*

$$\mathrm{CT}_L^G \cong \bigoplus_{c \in \pi_0(\mathrm{Gr}_L)} F_c.$$

Proof. This follows from the definitions and the topological invariance of the étale site. ■

6.2. Preservation of perversity

Theorem 6.2.1. *Let $c \in \pi_0(\mathrm{Gr}_L)$ and $\mathcal{F}^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$. Then*

$$F_c(\mathcal{F}^\bullet) \in P_{L^+L}(\mathrm{Gr}_L, \mathbb{F}_p).$$

Furthermore, for $\lambda \in X_*(T)^+$ we have

$$F_c(\mathrm{IC}_\lambda) = \begin{cases} \mathrm{IC}^L_{w_0^L w_0(\lambda)} & \text{if } c = c\big(w_0(\lambda)\big), \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The description of $F_c(\mathrm{IC}_\lambda)$ follows from Theorem 3.7.1 since $\mathrm{IC}_\lambda = \mathbb{F}_p[2\rho(\lambda)]$ supported on $\mathrm{Gr}_G^{\leq\lambda}$ and

$$\mathrm{IC}^L_{w_0^L w_0(\lambda)} = \mathbb{F}_p\big[2\rho_L\big(w_0^L w_0(\lambda)\big)\big]$$

supported on $\mathrm{Gr}_L^{\leq w_0^L w_0(\lambda)}$. Then the perversity of $F_c(\mathcal{F}^\bullet)$ for general $\mathcal{F}^\bullet$ follows by induction on the length of $\mathcal{F}^\bullet$. For equivariance, we observe that $\mathcal{F}^\bullet$ is L^+L-equivariant, and that S_c is L^+L-stable and $\sigma_c\colon S_c \to \mathrm{Gr}_L^c$ is L^+L-equivariant. As pullback along a smooth morphism is t-exact (up to a shift) for the perverse t-structure by [9, Lem. 2.15], then it follows that $F_c(\mathcal{F}^\bullet) \in P_{L^+L}(\mathrm{Gr}_L^c, \mathbb{F}_p)$ by the proper base change theorem (cf. [9, Lem. 3.2]). ■

Notation 6.2.2. Given a subset $A \subset X_*(T)^+$, equipped with the induced partial order, we set

$$\mathrm{Gr}_{G,A} := \varinjlim_{\lambda \in A} \mathrm{Gr}_G^{\leq \lambda} .$$

This is an ind-closed subscheme of Gr_G, which is stable under the L^+G-action. There is a natural embedding

$$P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p) \subset P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$$

which identifies $P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p)$ with the full subcategory of $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ whose objects are supported on $\mathrm{Gr}_{G,A}$. Let

$$\bar{A} = \left\{\lambda \in X_*(T)^+ \mid \lambda \leq \mu \text{ for some } \mu \in A\right\}.$$

Then the simple objects in $P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p)$ are the IC_λ for $\lambda \in \bar{A}$. Moreover, if $A \subset X_*(T)^+$ is a *submonoid*, then so is $\bar{A}$ and it follows from [9, Th. 1.2, Cor. 6.7] that the full subcategory $P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p)$ inherits from $P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ the structure of a symmetric monoidal category.

Corollary 6.2.3. *If $\mathcal{F}^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ and $c \cap X_*(T)_- = \emptyset$, then $F_c(\mathcal{F}^\bullet) = 0$. In general,*

$$F_c(\mathcal{F}^\bullet) \in P_{L^+L}(\mathrm{Gr}_{L, w_0^L X_*(T)_-}, \mathbb{F}_p).$$

Proof. If $\mathcal{F}^\bullet$ is simple this follows from Theorem 6.2.1. The general case follows by induction on the length of $\mathcal{F}^\bullet$. ∎

Corollary 6.2.4. *The L-constant term functor is an exact functor*

$$\mathrm{CT}_L^G \colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow P_{L^+L}(\mathrm{Gr}_{L, w_0^L X_*(T)_-}, \mathbb{F}_p).$$

Proof. This follows from Corollary 6.2.3 and Lemma 6.1.3. ∎

Note that for $L = T$, we recover the functor F_-, i.e.,

$$\mathrm{CT}_T^G = F_- := \bigoplus_{\nu \in X_*(T)_-} F_\nu \colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow \mathrm{Vect}_{\mathbb{F}_p}\left(X_*(T)_-\right).$$

In particular, $\mathrm{CT}_T^L = F_-^L$.

Remark 6.2.5. Let us set

$$\begin{aligned}\pi_0(\mathrm{Gr}_L)_- &:= \left\{c \in \pi_0(\mathrm{Gr}_L) \mid c \cap X_*(T)_- \neq \emptyset\right\} \\ &= \mathrm{Im}\left(X_*(T)_- \longrightarrow X_*(T)/\mathbb{Z}\Phi_L^\vee\right),\end{aligned}$$

which is a submonoid of the abelian group $\pi_0(\mathrm{Gr}_L)$, and

$$\mathrm{Gr}_L^- := \coprod_{c \in \pi_0(\mathrm{Gr}_L)_-} \mathrm{Gr}_L^c .$$

Then $\pi_0(\mathrm{Gr}_L^-) = \pi_0(\mathrm{Gr}_L)_-$, $\mathrm{CT}_L^G \cong \bigoplus_{c\in\pi_0(\mathrm{Gr}_L)_-} F_c$ and we have the inclusion

$$\mathrm{Gr}_{L,w_0^L X_*(T)_-} \subset \mathrm{Gr}_L^- .$$

The latter is an equality for $L = T$, but it is *strict* in general. Indeed, for any $\alpha^\vee \in \Phi_L^\vee$, we have $\{\alpha^\vee\} \in \mathrm{Gr}_L^0 \subset \mathrm{Gr}_L^-$, while $\{\alpha^\vee\} \notin \mathrm{Gr}_{L,w_0^L X_*(T)_-}$ in general, e.g. for $L = \mathrm{GL}_2 \times \mathrm{GL}_1 \subset G = \mathrm{GL}_3$,

$$\alpha^\vee = (1,-1,0) = w_0^L(-1,1,0) \in X_*(T)_{+/L} \setminus w_0^L X_*(T)_-.$$

Remark 6.2.6. There is a more general version of Theorem 4.3.3 as follows. Let $c \in \pi_0(\mathrm{Gr}_L)$ and denote by $i_c\colon \mathrm{Gr}_L^c \to \mathrm{Gr}_G$ the inclusion. Then one can show that there is a natural isomorphism of functors

$$F_c \cong {}^pH^{2(\rho-\rho_L)(c)} \circ Ri_c^*\colon P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p) \longrightarrow P_{L^+L}(\mathrm{Gr}_L^c, \mathbb{F}_p).$$

We will only use the functor F_c because it does not require a perverse truncation.

6.3. Relation to the Satake equivalence

Proposition 6.3.1. *Let $c \in \pi_0(\mathrm{Gr}_L)$ and $\nu \in X_*(T)$. If $\nu \notin c$, then*

$$F_\nu^L \circ F_c = 0,$$

and if $\nu \in c$ then

$$F_\nu^L \circ F_c \cong F_\nu.$$

Proof. If $\nu \notin c$, then $S_\nu \cap \mathrm{Gr}_L^c = \emptyset$ so that $F_\nu^L \circ F_c = 0$. If $\nu \in c$, then up to possible non-reducedness of the fiber product we have a Cartesian diagram

$$\begin{tikzcd} S_\nu \arrow[r] \arrow[d, "\sigma_{\nu,c}:="'] & S_c \arrow[d, "\sigma_c"] \\ S_\nu^L \arrow[r] & \mathrm{Gr}_L^c . \end{tikzcd}$$

Hence by the proper base change theorem $(R\sigma_{c!}(Rs_c^*\mathcal{F}^\bullet))|_{S_\nu^L} \cong R\sigma_{\nu,c!}(\mathcal{F}^\bullet|_{S_\nu})$, so that

$$R\Gamma_c\big(S_\nu^L, F_c(\mathcal{F}^\bullet)\big) \cong R\Gamma_c(S_\nu, \mathcal{F}^\bullet)\big[2(\rho-\rho_L)(\nu)\big].$$

Now take the cohomology of both sides in degree $2\rho_L(\nu)$. ■

Corollary 6.3.2. *For all $\nu \in X_*(T)$,*

$$F_\nu^L \circ \mathrm{CT}_L^G \cong F_\nu.$$

In particular, there is a canonical transitivity isomorphism

$$H^L \circ \mathrm{CT}_L^G \cong H,$$

and the functor CT_L^G is faithful.

Proof. The first part follows from Lemma 6.1.3 and Proposition 6.3.1. Then the transitivity isomorphism is obtained by summing over ν (in $X_*(T)_-$). Finally the faithfulness of CT_L^G follows from the transitivity isomorphism and the faithfulness of H. ■

6.4. The ind-schemes $S_c(X)$ and $S_c(X^2)$

For $c \in \pi_0(\mathrm{Gr}_L)$ let $S_c(X) \subset \mathrm{Gr}_{G,X}$ and $S_c(X^2) \subset \mathrm{Gr}_{G,X^2}$ be the reduced ind-subschemes realizing relative versions of S_c as in [1, §1.15.1] (see also [15, Th. 5.6] for a base field of arbitrary characteristic). They can be identified with the corresponding connected components of $(\mathrm{Gr}_{P,X})_{\mathrm{red}}$ and $(\mathrm{Gr}_{P,X^2})_{\mathrm{red}}$. Let $\mathrm{Gr}^c_{L,X}$ and Gr^c_{L,X^2} denote the connected components of $\mathrm{Gr}_{L,X}$ and Gr_{L,X^2} determined by c. We denote the relative versions of the ind-immersion $s_c\colon S_c \to \mathrm{Gr}_G$ and the projection $\sigma_c\colon S_c \to \mathrm{Gr}^c_L$ as follows:

$$\tilde{s}_c\colon S_c(X) \longrightarrow \mathrm{Gr}_{G,X}, \qquad \tilde{\sigma}_c\colon S_c(X) \longrightarrow \mathrm{Gr}^c_{L,X},$$

$$\tilde{s}^2_c\colon S_c(X^2) \longrightarrow \mathrm{Gr}_{G,X^2}, \qquad \tilde{\sigma}^2_c\colon S_c(X^2) \longrightarrow \mathrm{Gr}^c_{L,X^2}.$$

Since $X = \mathbb{A}^1$ there are canonical isomorphisms

$$\mathrm{Gr}_{G,X} \cong \mathrm{Gr}_G \times X, \quad \mathrm{Gr}_{L,X} \cong \mathrm{Gr}_L \times X, \quad S_c(X) \cong S_c \times X,$$

in particular we have the projection $\tau\colon \mathrm{Gr}_{G,X} \to \mathrm{Gr}_G$ and the associated shifted pull-back $\tau^\circ := R\tau^*[1]\colon D^b_c(\mathrm{Gr}_G, \mathbb{F}_p) \to D^b_c(\mathrm{Gr}_{G,X}, \mathbb{F}_p)$.

The important facts about the geometry of these ind-schemes are summarized in the following commutative diagram from [1, §1.15.1] whose squares are Cartesian (up to possible non-reducedness of fiber products) and are obtained by restriction to $U \subset X^2$ or its complementary diagonal $\Delta(X) \subset X^2$:

$$\begin{tikzcd}
(\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X})\big|_U \arrow[r, "j_I"] & \mathrm{Gr}_{G,X^2} & \mathrm{Gr}_{G,X} \arrow[l, "i_I"'] \\
\coprod_{c_1+c_2=c} (S_{c_1}(X) \times S_{c_2}(X))\big|_U \arrow[u, "\tilde{s}^2_c|_U"] \arrow[d, "\tilde{\sigma}^2_c|_U"'] \arrow[r, "j_c"] & S_c(X^2) \arrow[u, "\tilde{s}^2_c"'] \arrow[d, "\tilde{\sigma}^2_c"] & S_c(X) \arrow[l, "i_c"'] \arrow[u, "\tilde{s}_c"'] \arrow[d, "\tilde{\sigma}_c"'] \\
\coprod_{c_1+c_2=c} (\mathrm{Gr}^{c_1}_{L,X} \times \mathrm{Gr}^{c_2}_{L,X})\big|_U \arrow[r, "j^c_L"] & \mathrm{Gr}^c_{L,X^2} & \mathrm{Gr}^c_{L,X} \arrow[l, "i^c_L"']
\end{tikzcd}$$

We have canonical identifications

$$\tilde{s}_c = s_c \times \mathrm{id}_X\colon S_c \times X \longrightarrow \mathrm{Gr}_G \times X, \qquad \tilde{\sigma}_c = \sigma_c \times \mathrm{id}_X\colon S_c \times X \longrightarrow \mathrm{Gr}^c_L \times X$$

and

$$\tilde{s}^2_c\big|_U = \coprod_{c_1+c_2=c} (\tilde{s}_{c_1} \times \tilde{s}_{c_2})\big|_U, \quad \tilde{\sigma}^2_c\big|_U = \coprod_{c_1+c_2=c} (\tilde{\sigma}_{c_1} \times \tilde{\sigma}_{c_2})\big|_U.$$

Definition 6.4.1. Let $c \in \pi_0(\mathrm{Gr}_L)$. Set

$$\widetilde{F}_c := R\tilde{\sigma}_{c!} \circ R\tilde{s}_c^*\big[2(\rho - \rho_L)(c)\big]\colon D_c^b(\mathrm{Gr}_{G,X}, \mathbb{F}_p) \longrightarrow D_c^b(\mathrm{Gr}_{L,X}^c, \mathbb{F}_p),$$

and

$$\widetilde{F}_c^2 := R\tilde{\sigma}_{c!}^2 \circ R\tilde{s}_c^{2*}\big[2(\rho - \rho_L)(c)\big]\colon D_c^b(\mathrm{Gr}_{G,X^2}, \mathbb{F}_p) \longrightarrow D_c^b(\mathrm{Gr}_{L,X^2}^c, \mathbb{F}_p).$$

6.5. The key isomorphism for the compatibility with convolution

Theorem 6.5.1. *There is a canonical isomorphism*

$$\widetilde{F}_c^2 \circ j_{I,!*}\big(\tau^\circ \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \tau^\circ \mathcal{F}_2^\bullet\big|_U\big) \cong j_{L,!*}^c\Big(\bigoplus_{c_1+c_2=c} \tau_L^\circ F_{c_1}(\mathcal{F}_1^\bullet) \overset{L}{\boxtimes} \tau_L^\circ F_{c_2}(\mathcal{F}_1^\bullet)\big|_U\Big).$$

Contrary to the case of characteristic 0 coefficients, we cannot appeal to Braden's theorem to compute the co-restriction of the left side of Theorem 6.5.1 over $\Delta(X)$ as in [1, Prop. 1.15.2]. This complication is the primary obstacle we must overcome in order to prove Theorem 6.5.1. We begin by reducing to the case where the $\mathcal{F}_i^\bullet$ are simple.

Reduction of Theorem 6.5.1 to the case of simple $\mathcal{F}_i^\bullet$. By a diagram chase involving the proper base change theorem and the Künneth formula, the two complexes in Theorem 6.5.1 are canonically identified over U. Once we show that the complex on the left is isomorphic to the one on the right, by [9, Lem. 2.11] there will be a unique isomorphism which restricts to our canonical identification over U.

We claim that it suffices to show the left side is the intermediate extension of its restriction to U in the case where the $\mathcal{F}_i^\bullet$ are simple. By the properties characterizing $j_{L!*}^c$ in [9, Lem. 2.7], it follows that if the outer two terms in an exact triangle are intermediate extensions, then so is the middle term (cf. the proof of [9, Lem. 7.8]). While $j_{I,!*}$ may not be exact in general, (5.2) allows us to replace $j_{I,!*}$ by the triangulated functor $Rm_{I,!}$. Thus, by induction on the lengths of the $\mathcal{F}_i^\bullet$ we can assume that $\mathcal{F}_i^\bullet = \mathrm{IC}_{\lambda_i}$ for $\lambda_i \in X_*(T)^+$. ■

The remainder of the proof will be an explicit computation of both sides of Theorem 6.5.1 in the special case $\mathcal{F}_i^\bullet = \mathrm{IC}_{\lambda_i}$ for $\lambda_i \in X_*(T)^+$. For convenience we denote

$$\lambda_\bullet := (\lambda_1, \lambda_2), \quad |\lambda_\bullet| := \lambda_1 + \lambda_2.$$

Let $\mathrm{Gr}_{G,X^2}^{\leq\lambda_\bullet}$ be the closure of

$$\mathrm{Gr}_G^{\leq\lambda_1} \times \mathrm{Gr}_G^{\leq\lambda_2} \times U \subset \mathrm{Gr}_{G,X^2}$$

with its reduced scheme structure. If $p \nmid |\pi_1(G_{\mathrm{der}})|$ then by [27, Prop. 3.1.14] we have

$$\mathrm{Gr}_{G,X^2}^{\leq\lambda_\bullet}\big|_{\Delta(X)} \cong \mathrm{Gr}_G^{\leq|\lambda_\bullet|} \times X.$$

If $p \mid |\pi_1(G_{\mathrm{der}})|$ this isomorphism should be modified by passing to the reduced subscheme on the left side.

Lemma 6.5.2. *There is a canonical isomorphism*

$$j_{I,!*}\big(\tau^\circ \mathrm{IC}_{\lambda_1} \overset{L}{\boxtimes} \tau^\circ \mathrm{IC}_{\lambda_2} \big|_U\big) \cong \mathbb{F}_p\big[2\rho\big(|\lambda_\bullet|\big) + 2\big] \in P_c^b(\mathrm{Gr}_{G,X^2}^{\leq \lambda_\bullet}, \mathbb{F}_p).$$

Proof. We first observe that $\tau^\circ \mathrm{IC}_{\lambda_1} \overset{L}{\boxtimes} \tau^\circ \mathrm{IC}_{\lambda_2} \big|_U$ is canonically identified with a shifted constant sheaf supported on $\mathrm{Gr}_G^{\leq \lambda_1} \times \mathrm{Gr}_G^{\leq \lambda_2} \times U \subset \mathrm{Gr}_{G,X^2}$. If $p \nmid |\pi_1(G_{\mathrm{der}})|$ then $\mathrm{Gr}_{G,X^2}^{\leq \lambda_\bullet}$ is integral and F-rational by [9, Th. 7.4], so

$$j_{I,!*}\big(\tau^\circ \mathrm{IC}_{\lambda_1} \overset{L}{\boxtimes} \tau^\circ \mathrm{IC}_{\lambda_2} \big|_U\big)$$

is a shifted constant sheaf supported on $\mathrm{Gr}_{G,X^2}^{\leq \lambda_\bullet}$ by [9, Th. 1.7]. If $p \mid |\pi_1(G_{\mathrm{der}})|$, choose a z-extension $G' \to G$ and choose lifts λ_1', λ_2' of λ_1, λ_2 to dominant cocharacters of G'. The induced morphism

$$\mathrm{Gr}_{G',X^2}^{\leq \lambda_\bullet'} \to \mathrm{Gr}_{G,X^2}^{\leq \lambda_\bullet}$$

is a universal homeomorphism (see [9, Rem. 7.12] for more details), so by topological invariance of the étale site it follows that

$$j_{I,!*}\big(\tau^\circ \mathrm{IC}_{\lambda_1} \overset{L}{\boxtimes} \tau^\circ \mathrm{IC}_{\lambda_2} \big|_U\big)$$

is still a shifted constant sheaf supported on $\mathrm{Gr}_{G,X^2}^{\leq \lambda_\bullet}$. Hence in any case there is a canonical isomorphism as stated. ∎

Lemma 6.5.3. *If $\mathcal{F}_i^\bullet = \mathrm{IC}_{\lambda_i}$ for $\lambda_i \in X_*(T)^+$ and $w_0(|\lambda_\bullet|) \notin c$ then both sides of Theorem 6.5.1 are zero.*

Proof. By the assumption of the lemma, if $c_1 + c_2 = c$ then $w_0(\lambda_i) \notin c_i$ for $i = 1$ or 2. For such i we have $F_{c_i}(\mathrm{IC}_{\lambda_i}) = 0$ by Theorem 6.2.1, so both sides of Theorem 6.5.1 vanish over U. Therefore the right side of Theorem 6.5.1 vanishes. On the other hand, by Lemma 6.5.2 and the proper base change theorem,

$$\widetilde{F}_c^2\big(j_{I,!*}\big(\tau^\circ \mathrm{IC}_{\lambda_1} \overset{L}{\boxtimes} \tau^\circ \mathrm{IC}_{\lambda_2} \big|_U\big)\big)\big|_{\Delta(X)} \cong \tau^\circ F_c(\mathrm{IC}_{|\lambda_\bullet|}).$$

This complex is also zero by Theorem 6.2.1, so the left side of Theorem 6.5.1 is zero. ∎

Lemma 6.5.4. *If $\mathcal{F}_i^\bullet = \mathrm{IC}_{\lambda_i}$ for $\lambda_i \in X_*(T)^+$ and $w_0(|\lambda_\bullet|) \in c$, then the right side of Theorem 6.5.1 is canonically isomorphic to the shifted constant sheaf*

$$\mathbb{F}_p\big[2\rho_L\big(w_0^L w_0\big(|\lambda_\bullet|\big)\big) + 2\big] \in P_c^b\big(\mathrm{Gr}_{L,X^2}^{\leq w_0^L w_0(\lambda_\bullet)}, \mathbb{F}_p\big).$$

Proof. By Theorem 6.2.1, the right side of Theorem 6.5.1 is canonically isomorphic to

$$j_{L,!*}^c\big(\tau_L^\circ \mathrm{IC}^L_{w_0^L w_0(\lambda_1)} \overset{L}{\boxtimes} \tau_L^\circ \mathrm{IC}^L_{w_0^L w_0(\lambda_2)} \big|_U\big).$$

Now apply Lemma 6.5.2 to L instead of G. ∎

From now on we assume $w_0(|\lambda_\bullet|) \in c$. Let

$$V^c_{\lambda_\bullet} := \coprod_{\substack{c_1+c_2=c \\ w_0(\lambda_i)\notin c_i \text{ for some } i}} (S_{c_1} \cap \mathrm{Gr}_G^{\leq\lambda_1}) \times (S_{c_2} \cap \mathrm{Gr}_G^{\leq\lambda_2}) \times U.$$

Then $V^c_{\lambda_\bullet}$ is an open subscheme of $(S_c(X^2) \cap \mathrm{Gr}^{\leq\lambda_\bullet}_{G,X^2})_{\mathrm{red}}$. Let $Z^c_{\lambda_\bullet} \subset S_c(X^2) \cap \mathrm{Gr}^{\leq\lambda_\bullet}_{G,X^2}$ be its complement with the reduced scheme structure. Then $Z^c_{\lambda_\bullet}$ is a locally closed subscheme of $\mathrm{Gr}^{\leq\lambda_\bullet}_{G,X^2}$ such that

$$\left(Z^c_{\lambda_\bullet}|_{\Delta(X)}\right)_{\mathrm{red}} \cong \left(S_c \cap \mathrm{Gr}_G^{\leq|\lambda_\bullet|}\right) \times X$$

and

$$Z^c_{\lambda_\bullet}|_U \cong \left(S_{c(w_0(\lambda_1))} \cap \mathrm{Gr}_G^{\leq\lambda_1}\right) \times \left(S_{c(w_0(\lambda_2))} \cap \mathrm{Gr}_G^{\leq\lambda_2}\right) \times U.$$

By Lemma 3.7.2 (2), $\tilde{\sigma}^2_c$ restricts to a morphism

$$\tilde{\sigma}^2_{c,\lambda_\bullet} := \tilde{\sigma}^2_c|_{Z^c_{\lambda_\bullet}} : Z^c_{\lambda_\bullet} \longrightarrow \mathrm{Gr}^{\leq w_0^L w_0(\lambda_\bullet)}_{L,X^2} .$$

Lemma 6.5.5. *The morphism* $\tilde{\sigma}^2_{c,\lambda_\bullet} : Z^c_{\lambda_\bullet} \to \mathrm{Gr}^{\leq w_0^L w_0(\lambda_\bullet)}_{L,X^2}$ *is a universal homeomorphism.*

Proof. By Corollary 3.7.3, $\tilde{\sigma}^2_{c,\lambda_\bullet}$ restricts to a universal homeomorphism over U and $\Delta(X)$, so it is universally bijective. The natural morphism $(\mathrm{Gr}^c_{L,X^2})_{\mathrm{red}} \to S_c(X^2)$ coming from the morphism $L \to P$ induces a section to $\tilde{\sigma}^2_{c,\lambda_\bullet}$, so it is a universal homeomorphism. ■

Lemma 6.5.6. *If* $\mathcal{F}_i^\bullet = \mathrm{IC}_{\lambda_i}$ *for* $\lambda_i \in X_*(T)^+$ *and* $w_0(|\lambda_\bullet|) \in c$, *then the left side of Theorem 6.5.1 is canonically isomorphic to the shifted constant sheaf*

$$\mathbb{F}_p\left[2\rho_L\left(w_0^L w_0\left(|\lambda_\bullet|\right)\right) + 2\right] \in P^b_c\left(\mathrm{Gr}^{\leq w_0^L w_0(\lambda_\bullet)}_{L,X^2}, \mathbb{F}_p\right).$$

Proof. By abuse of notation, let us view $\tilde{\sigma}^2_c$ as a morphism

$$S_c(X^2) \cap \mathrm{Gr}^{\leq\lambda_\bullet}_{G,X^2} \xrightarrow{\tilde{\sigma}^2_c} \mathrm{Gr}^c_{L,X^2} .$$

Then by the definition of $\widetilde{F}^2_c$ and Lemma 6.5.2, the left side of Theorem 6.5.1 is

$$R\tilde{\sigma}^2_{c!}(\mathbb{F}_p)\left[2\rho_L\left(w_0^L w_0\left(|\lambda_\bullet|\right)\right) + 2\right].$$

Let $j_{V^c_{\lambda_\bullet}} : V^c_{\lambda_\bullet} \to S_c(X^2) \cap \mathrm{Gr}^{\leq\lambda_\bullet}_{G,X^2}$ be the inclusion. By Lemma 3.7.2 (2), we have

$$\tilde{\sigma}^2_c(V^c_{\lambda_\bullet}) \cap \mathrm{Gr}^{\leq w_0^L w_0(\lambda_\bullet)}_{L,X^2} = \emptyset.$$

The scheme $V^c_{\lambda_\bullet}$ is open and closed in $S_c(X^2) \cap \mathrm{Gr}^{\leq\lambda_\bullet}_{G,X^2}|_U$ by [1, (1.15.2)], so that

$$R\tilde{\sigma}^2_{c!} \circ R(j_{V^c_{\lambda_\bullet}})_!\left(\mathbb{F}_p\left[2\rho_L\left(w_0^L w_0(|\lambda_\bullet|)\right) + 2\right]\right)$$

is a direct summand of the restriction to U of the complex

$$R\tilde{\sigma}^2_{c!}(\mathbb{F}_p)\big[2\rho_L\big(w_0^L w_0\big(|\lambda_\bullet|\big)\big)+2\big].$$

Hence the former complex is supported in $\mathrm{Gr}_{L,X^2}^{\leq w_0^L w_0(\lambda_\bullet)}$ by Lemma 6.5.4 since the left and right sides of Theorem 6.5.1 agree over U. It follows that

$$\begin{aligned} &R(\tilde{\sigma}^2_c \circ j_{V^c_{\lambda_\bullet}})_!\big(\mathbb{F}_p\big[2\rho_L\big(w_0^L w_0\big(|\lambda_\bullet|\big)\big)+2\big]\big) \\ &\quad = R\tilde{\sigma}^2_{c!} \circ R(j_{V^c_{\lambda_\bullet}})_!\big(\mathbb{F}_p\big[2\rho_L\big(w_0^L w_0\big(|\lambda_\bullet|\big)\big)+2\big]\big) = 0. \end{aligned}$$

Consequently, by applying $R\tilde{\sigma}^2_{c!}$ to the exact triangle associated to the decomposition of $S_c(X^2) \cap \mathrm{Gr}_{G,X^2}^{\leq \lambda_\bullet}$ into $V^c_{\lambda_\bullet}$ and $Z^c_{\lambda_\bullet}$, the left side of Theorem 6.5.1 is

$$R(\tilde{\sigma}^2_{c,\lambda_\bullet})_!(\mathbb{F}_p)\big[2\rho_L\big(w_0^L w_0\big(|\lambda_\bullet|\big)\big)+2\big].$$

Now we conclude by using Lemma 6.5.5. ■

Proof of Theorem 6.5.1. We have reduced to the case where $\mathcal{F}_i^\bullet = \mathrm{IC}_{\lambda_i}$ for $\lambda_i \in X_*(T)^+$. Then if $w_0(|\lambda_\bullet|) \notin c$ both sides of Theorem 6.5.1 vanish by Lemma 6.5.3, and if $w_0(|\lambda_\bullet|) \in c$ both sides are canonically identified with the same complex

$$\mathbb{F}_p\big[2\rho_L\big(w_0^L w_0\big(|\lambda_\bullet|\big)\big)+2\big] \in P_c^b\big(\mathrm{Gr}_{L,X^2}^{\leq w_0^L w_0(\lambda_\bullet)}, \mathbb{F}_p\big)$$

by Lemmas 6.5.4 and 6.5.6. ■

6.6. Compatibility with convolution

Theorem 6.6.1. *The L-constant term functor is a tensor functor*

$$\mathrm{CT}_L^G \colon \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \longrightarrow \big(P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-}, \mathbb{F}_p), *\big).$$

Proof. Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$. Recall from Proposition 5.3.1 the canonical isomorphism

$$\tau^\circ(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \cong Ri_I^* \circ j_{I,!*}\big(\tau^\circ \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \tau^\circ \mathcal{F}_2^\bullet|_U\big)[-1].$$

Let $c \in \pi_0(\mathrm{Gr}_L)$. First, apply $\widetilde{F}_c$. After unwinding the definitions and using the proper base change theorem, there is a canonical isomorphism

$$\widetilde{F}_c\big(\tau^\circ(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet)\big) \cong \tau_L^\circ\big(F_c(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet)\big).$$

A similar diagram chase yields a canonical isomorphism

$$\widetilde{F}_c\big(Ri_I^* \circ j_{I,!*}\big(\tau^\circ \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \tau^\circ \mathcal{F}_2^\bullet|_U\big)[-1]\big) \cong Ri_L^{c*}\big(\widetilde{F}_c^2 \circ j_{I,!*}\big(\tau^\circ \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \tau^\circ \mathcal{F}_2^\bullet|_U\big)\big)[-1].$$

Whence

$$\tau_L^\circ\big(F_c(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet)\big) \cong Ri_L^{c*}\big(\widetilde{F}_c^2 \circ j_{I,!*}\big(\tau^\circ\mathcal{F}_1^\bullet \overset{L}{\boxtimes} \tau^\circ\mathcal{F}_2^\bullet\big|_U\big)\big)[-1].$$

Second, use the key isomorphism Theorem 6.5.1 to get

$$\tau_L^\circ\big(F_c(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet)\big) \cong Ri_L^{c*} \circ j^c_{L,!*}\Big(\bigoplus_{c_1+c_2=c} \tau_L^\circ F_{c_1}(\mathcal{F}_1^\bullet) \overset{L}{\boxtimes} \tau_L^\circ F_{c_2}(\mathcal{F}_1^\bullet)\big|_U\Big)[-1].$$

Third, use Proposition 5.3.1 for L instead of G to get

$$\tau_L^\circ\big(F_c(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet)\big) \cong \bigoplus_{c_1+c_2=c} \tau_L^\circ\big(F_{c_1}(\mathcal{F}_1^\bullet) * F_{c_2}(\mathcal{F}_2^\bullet)\big).$$

By taking the sum over the $c \in \pi_0(\mathrm{Gr}_L)$ we obtain finally (cf. Lemma 6.1.3)

$$\mathrm{CT}_L^G(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \cong \mathrm{CT}_L^G(\mathcal{F}_1^\bullet) * \mathrm{CT}_L^G(\mathcal{F}_2^\bullet).$$

By appealing to the constructions in Section 5.3 one can verify that this isomorphism is compatible with the associativity and commutativity constraints. The arguments are analogous to the case of characteristic 0 coefficients as in [1, Prop. 1.15.2]; we leave the details to the reader. ∎

Corollary 6.6.2. *The functor* CT_L^G *induces an equivalence of symmetric monoidal categories*

$$\mathrm{CT}_L^G\,|_{(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*)} : \big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}}, *\big) \xrightarrow{\sim} \big(P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-},\mathbb{F}_p)^{\mathrm{ss}}, *\big).$$

We have

$$\forall \lambda \in X_*(T)^+, \quad \mathrm{CT}_L^G(\mathrm{IC}_\lambda) = \mathrm{IC}^L_{w_0^L w_0(\lambda)}.$$

Proof. The last assertion follows from Lemma 6.1.3 and Theorem 6.2.1. In particular, it implies that the restriction $\mathrm{CT}_L^G\,|_{P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}}}$ factors through

$$P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-},\mathbb{F}_p)^{\mathrm{ss}} \subset P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-},\mathbb{F}_p).$$

Combined with [9, Th. 1.2], it also implies that $\mathrm{CT}_L^G\,|_{(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*)}$ is a tensor functor, which is also a consequence of Theorem 6.6.1.

To conclude the proof, it remains to see that CT_L^G induces a bijection between the sets of (isomorphism classes of) simple objects, in other words, that the inclusion

$$w_0^L X_*(T)_- \subset \overline{w_0^L X_*(T)_-} = \big\{\lambda \in X_*(T)_{+/L} \mid \lambda \leq_L \mu \text{ for some } \mu \in w_0^L X_*(T)_-\big\}$$

is an equality. So let $\lambda \in \overline{w_0^L X_*(T)_-}$, and pick $\mu \in w_0^L X_*(T)_-$ such that $\lambda \leq_L \mu$. Set

$$\lambda' := w_0^L(\lambda) \in X_*(T)_{-/L} \quad \text{and} \quad \mu' := w_0^L(\mu) \in X_*(T)_-.$$

We need to check that $\lambda' \in X_*(T)_-$, which means that $\langle \alpha, \lambda' \rangle \leq 0$ for all the simple roots $\alpha \in \Delta \subset \Phi$. The inequality holds if $\alpha \in \Delta_L \subset \Delta$ since $\lambda' \in X_*(T)_{-/L}$. Now assume that $\alpha \in \Delta \setminus \Delta_L$. As $\lambda \leq_L \mu$, we have $\mu' \leq_L \lambda'$ i.e.

$$\lambda' \in \mu' + \mathbb{N}\Delta_L^\vee.$$

Moreover, as $\mu' \in X_*(T)_-$, we have $\langle \alpha, \mu' \rangle \leq 0$. Lastly, if $\beta \in \Delta_L$, then $\beta \in \Delta$ and $\beta \neq \alpha$, so that α and β are two distinct elements of a root basis, which implies $\langle \alpha, \beta^\vee \rangle \leq 0$. ■

Remark 6.6.3. We can summarize this section as follows. The exact faithful symmetric monoidal functor

$$F_- : \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \longrightarrow \big(\mathrm{Vect}_{\mathbb{F}_p}\,\big(X_*(T)_-\big), \otimes\big)$$

can be rewritten as

$$\mathrm{CT}_T^G : \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \longrightarrow \big(P_{L^+T}(\mathrm{Gr}_{T, X_*(T)_-}, \mathbb{F}_p), *\big)$$

and factors as a composition of exact faithful symmetric monoidal functors

$$\begin{tikzcd}
\big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \arrow[d, "\mathrm{CT}_L^G"] & \\
\big(P_{L^+L}(\mathrm{Gr}_{L, w_0^L X_*(T)_-}, \mathbb{F}_p), *\big) \arrow[d, phantom, "\cap"] & \\
\big(P_{L^+L}(\mathrm{Gr}_L, \mathbb{F}_p), *\big) \arrow[r, "\mathrm{CT}_T^L"] & \big(P_{L^+T}(\mathrm{Gr}_{T, X_*(T)_{-/L}}, \mathbb{F}_p), *\big) \arrow[d, phantom, "\cap"] \\
& \big(P_{L^+T}(\mathrm{Gr}_T, \mathbb{F}_p), *\big)
\end{tikzcd}$$

(with values in $P_{L^+T}(\mathrm{Gr}_{T, X_*(T)_-}, \mathbb{F}_p) \subset P_{L^+T}(\mathrm{Gr}_T, \mathbb{F}_p)$).

7. Tannakian interpretation

7.1. The Satake equivalence

Recall from [9, Th. 1.1] the Tannaka equivalence given by the geometric Satake equivalence with $\mathbb{F}_p$-coefficients:

$$\begin{tikzcd}
\big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \arrow[rr, "\sim", "S_G"'] \arrow[dr, "H"'] & & \big(\mathrm{Rep}_{\mathbb{F}_p}(M_G), \otimes\big) \arrow[dl, "\text{forget}"] \\
& (\mathrm{Vect}_{\mathbb{F}_p}, \otimes). &
\end{tikzcd}$$

In particular M_G is an affine monoid scheme over $\mathbb{F}_p$ which represents the functor of tensor endomorphisms of the fiber functor H.

Remark 7.1.1. The Tannaka equivalence in [9, Th. 1.1] is constructed using the following observation just before [11, Rem. II.2.17]: If one omits the assumption of rigidity (i.e., the existence of tensor duals) in the definition of a neutral Tannakian category, then one gets an equivalence with the category of representations of an affine monoid scheme instead of a group scheme. Below we will use the fact that a morphism of fiber functors induces a morphism of affine monoid schemes (the analogue for affine group schemes is [11, Cor. II.2.9]). This morphism of monoid schemes can already be constructed at the level of coalgebras (e.g., [1, Prop. 1.2.6]), and then the fact that it is also an algebra homomorphism follows from the compatibility with the tensor structures.

Notation 7.1.2. We will use the following notation.

- Let $A \subset X_*(T)^+$ be a submonoid. The full subcategory

  $$P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p) \subset P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$$

  introduced in Notation 6.2.2 is a Tannakian subcategory with fiber functor given by the restriction of H. We denote by $M_{G,A}$ the corresponding $\mathbb{F}_p$-monoid scheme and by $\mathcal{S}_{G,A}$ the resulting Tannaka equivalence. It fits into a commutative diagram

  $$\begin{tikzcd}
  \big(P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p), *\big) \arrow[r, "\sim", "\mathcal{S}_{G,A}"'] \arrow[d, phantom, "\cap"] & \big(\mathrm{Rep}_{\mathbb{F}_p}(M_{G,A}), \otimes\big) \arrow[d, phantom, "\cap"] \\
  \big(P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p), *\big) \arrow[r, "\sim", "\mathcal{S}_G"'] & \big(\mathrm{Rep}_{\mathbb{F}_p}(M_G), \otimes\big).
  \end{tikzcd}$$

  We have a canonical homomorphism

  $$M_G \to M_{G,A},$$

  which for $A = X_*(T)^+$ is the identity.

- Given an arbitrary abstract abelian monoid A, the category $(\mathrm{Vect}_{\mathbb{F}_p}(A), \otimes)$ introduced in Notation 5.1.3 is Tannakian with fiber functor given by forgetting the grading. Its Tannaka monoid is the diagonalizable $\mathbb{F}_p$-monoid scheme

  $$D(A) := \mathrm{Spec}\,\big(\mathbb{F}_p[A]\big).$$

Remark 7.1.3. In the case $G = T$, we have

$$M_{T,A} = D(A)$$

for all submonoids $A \subset X_*(T)$. In particular,

$$M_T = M_{T,X_*(T)} = D\big(X_*(T)\big) = T^\vee,$$

the torus dual to T.

7.2. The dual of the torus embedding

As noticed in Remark 5.4.4, we have obtained a factorization of H as

$$\begin{aligned}\big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p),*\big) &\xrightarrow{F_-} \big(\mathrm{Vect}_{\mathbb{F}_p}\big(X_*(T)_-\big),\otimes\big)\\ &\xrightarrow{2\rho_-} \big(\mathrm{Vect}_{\mathbb{F}_p}(\mathbb{Z}_-),\otimes\big)\\ &\xrightarrow{\mathrm{Forget}} (\mathrm{Vect}_{\mathbb{F}_p},\otimes).\end{aligned}$$

Under the equivalences $\mathcal{S}_G$ and $\mathcal{S}_T$ it corresponds to a sequence of tensor functors

$$\begin{aligned}\big(\mathrm{Rep}_{\mathbb{F}_p}(M_G),\otimes\big) &\longrightarrow \big(\mathrm{Rep}_{\mathbb{F}_p}\big(D\big(X_*(T)_-\big)\big),\otimes\big)\\ &\longrightarrow \big(\mathrm{Rep}_{\mathbb{F}_p}(\mathbb{A}^1_-),\otimes\big)\\ &\longrightarrow \big(\mathrm{Rep}_{\mathbb{F}_p}(1_{\mathbb{F}_p}),\otimes\big),\end{aligned}$$

i.e., by Tannaka duality to a sequence of morphisms of $\mathbb{F}_p$-monoid schemes

$$\begin{tikzcd}
1_{\mathbb{F}_p} \arrow[r] & \mathbb{A}^1_- \arrow[r, "2\rho_-"] & D(X_*(T)_-) \arrow[r, "D(F_-)"] & M_G. \\
& \mathbb{G}_m \arrow[u, hook] \arrow[r, "2\rho"] & T^\vee \arrow[u, hook] &
\end{tikzcd}$$

Remark 7.2.1. We show in Theorem 7.4.5 that $D(F_-)$, denoted there by $\mathfrak{w}$, is a closed immersion, and that $T^\vee \to D(X_*(T)_-)$ is an open immersion.

7.3. The dual of the Levi embedding

As noticed in Remark 6.6.3, we have obtained a factorization of F_- as

$$\begin{aligned}&\big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p),*\big) \xrightarrow{\mathrm{CT}_L^G} \big(P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-},\mathbb{F}_p),*\big)\\ &\quad\subset \big(P_{L^+L}(\mathrm{Gr}_L,\mathbb{F}_p),*\big) \xrightarrow{\mathrm{CT}_T^L=F_-^L} \big(P_{L^+T}\big(\mathrm{Gr}_{T,X_*(T)_{-/L}},\mathbb{F}_p\big),*\big) \subset \big(P_{L^+T}(\mathrm{Gr}_T,\mathbb{F}_p),*\big).\end{aligned}$$

Under the equivalences $\mathcal{S}_G$, $\mathcal{S}_L$, and $\mathcal{S}_T$ it corresponds to a diagram

$$\begin{aligned}&\big(\mathrm{Rep}_{\mathbb{F}_p}(M_G),\otimes\big) \longrightarrow \big(\mathrm{Rep}_{\mathbb{F}_p}\big(M_{L,w_0^L X_*(T)_-}\big),\otimes\big)\\ &\quad\subset \big(\mathrm{Rep}_{\mathbb{F}_p}(M_L),\otimes\big) \longrightarrow \big(\mathrm{Rep}_{\mathbb{F}_p}\big(M_{T,X_*(T)_{-/L}}\big),\otimes\big) \subset \big(\mathrm{Rep}_{\mathbb{F}_p}(T^\vee),\otimes\big),\end{aligned}$$

i.e., by Tannaka duality to a sequence of morphisms of $\mathbb{F}_p$-monoid schemes

$$\begin{tikzcd}
T^\vee \arrow[r] \arrow[d, hook] & M_{T,X_*(T)_{-/L}} \arrow[r, "D(F_-^L)"] & M_L \arrow[r] & M_{L,w_0^L X_*(T)_-} \arrow[r, "D(\mathrm{CT}_L^G)"] & M_G. \\
M_{T,X_*(T)_-} \arrow[urrrr, bend right, "D(F_-)"'] & & & &
\end{tikzcd}$$

Remark 7.3.1. The morphisms $T^\vee \to M_{T,X_*(T)_{-/L}}$ and $T^\vee \to M_{T,X_*(T)_-}$ are open immersions, and the morphisms $D(F_-^L)$ and $D(F_-)$ are closed immersions (cf. Remark 7.2.1). Deciding whether $M_L \to M_{L,w_0^L X_*(T)_-}$ and $D(\mathrm{CT}_L^G)$ are open or closed immersions in general seems to require a greater understanding of the extensions between representations of M_G (and M_L), and how these extensions interact with the constant term functors.

7.4. Semi-simplification

Notation 7.4.1. Let $A \subset X_*(T)^+$ be a submonoid. Similarly as in [9, proof of Prop. 7.14], we denote by $P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p)^{\mathrm{ss}}$ the full subcategory of $P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p)$ consisting of semi-simple objects. As noticed in Notation 6.2.2, the simple objects are the IC_λ for $\lambda \in \bar{A}$, and hence it follows from [9, Th. 1.2] that $P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p)^{\mathrm{ss}}$ is a Tannakian subcategory with fiber functor given by the restriction of H. Then by Theorem 5.2.2 the corresponding $\mathbb{F}_p$-monoid scheme is

$$M_{G,A}^{\mathrm{ss}} := D(w_0 \bar{A}).$$

The resulting Tannaka equivalence $\mathcal{S}_{G,A}^{\mathrm{ss}}$ fits into the commutative diagram

$$\begin{tikzcd}
\big(P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p)^{\mathrm{ss}}, *\big) \arrow[r, "\sim", "\mathcal{S}_{G,A}^{\mathrm{ss}}"'] \arrow[d, phantom, "\cap"] & \big(\mathrm{Rep}_{\mathbb{F}_p}(M_{G,A}^{\mathrm{ss}}), \otimes\big) \arrow[d, phantom, "\cap"] \\
\big(P_{L^+G}(\mathrm{Gr}_{G,A}, \mathbb{F}_p), *\big) \arrow[r, "\sim", "\mathcal{S}_{G,A}"'] & \big(\mathrm{Rep}_{\mathbb{F}_p}(M_{G,A}), \otimes\big).
\end{tikzcd}$$

We have a canonical homomorphism $\pi_{G,M} : M_{G,A} \to M_{G,A}^{\mathrm{ss}}$. As $M_{G,X_*(T)^+} = M_G$, we write simply M_G^{ss} for $M_{G,X_*(T)^+}^{\mathrm{ss}} = D(X_*(T)_-)$ and $\pi_G : M_G \to M_G^{\mathrm{ss}}$ for the corresponding canonical homomorphism.

Remark 7.4.2. Since every simple object of $\mathrm{Rep}_{\mathbb{F}_p}(M_G) \cong P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ is 1-dimensional, the monoid M_G is pro-solvable, cf. [9, Prop. 7.15]. Let $\{V_\lambda, \lambda \in X_*(T)_-\}$ be (a set of representatives of) the set of irreducible finite dimensional representations of M_G. For any $V \in \mathrm{Rep}_{\mathbb{F}_p}(M_G)$, denote by $d_\lambda(V)$ the multiplicity of V_λ as a subquotient in any Jordan–Hölder filtration of V. Then the canonical homomorphism $\pi_G : M_G \to M_G^{\mathrm{ss}}$ admits the following explicit description: it maps $m \in M_G$ to the unique $\pi_G(m) \in M_G^{\mathrm{ss}} = D(X_*(T)_-)$ acting

$$\text{on} \quad \bigoplus_{\lambda \in X_*(T)_-} V_\lambda^{d_\lambda(V)} \quad \text{by} \quad \bigoplus_{\lambda \in X_*(T)_-} \lambda\big(\pi_G(m)\big)$$

for all $V \in \mathrm{Rep}_{\mathbb{F}_p}(M_G)$.

Definition 7.4.3. We call the canonical homomorphism

$$\pi_G : M_G \longrightarrow M_G^{\mathrm{ss}}$$

the *eigenvalues homomorphism.*

From Corollary 5.4.3, we have the equivalence

$$F_-|_{(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*)} : \big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*\big) \xrightarrow{\sim} \big(\mathrm{Vect}_{\mathbb{F}_p}\big(X_*(T)_-\big),\otimes\big),$$

such that $F_-(\mathrm{IC}_\lambda) = \mathbb{F}_p(w_0(\lambda))$. By Tannaka duality, it corresponds to the identity

$$M_{T,X_*(T)_-} = D\big(X_*(T)_-\big) =\!=\!= D\big(X_*(T)_-\big) = M_G^{\mathrm{ss}}.$$

Definition 7.4.4. By the above equivalence we can make the following definitions.

- We call the composition

$$(\cdot)^{\mathrm{ss}} := F_-|^{-1}_{(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*)} \circ F_- \colon \big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p),*\big) \longrightarrow \big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*\big)$$

  the *semi-simplification functor*.
- We call its Tannaka dual

$$\mathfrak{w} := D\big((\cdot)^{\mathrm{ss}}\big) : M_G^{\mathrm{ss}} \longrightarrow M_G$$

  the *weight section*.

Thus the functor $(\cdot)^{\mathrm{ss}}$ is a retraction to $P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}} \subset P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)$ and the morphism $\mathfrak{w}$ is a section to $\pi_G : M_G \to M_G^{\mathrm{ss}}$. Moreover $\mathfrak{w}$ identifies with the morphism $D(F_-)$ from Section 7.2.

Theorem 7.4.5. *The morphisms π_G and $\mathfrak{w}$ satisfy the following properties.*

- *The morphism $\pi_G\colon M_G \to M_G^{\mathrm{ss}}$ is surjective.*
- *The weight section $\mathfrak{w} : M_G^{\mathrm{ss}} \to M_G$ is a closed immersion. The dual torus embedding $T^\vee \to M_G$ factors through $\mathfrak{w}$ by an open immersion.*

Proof. The weight section $\mathfrak{w}$ of π_G is a closed immersion since the morphism π_G is affine. Conversely, the fact that π_G admits a section implies that π_G is surjective.

By construction, we have the commutative diagram

$$\begin{tikzcd}
M_G^{\mathrm{ss}} \arrow[r, equal] \arrow[rr, bend left, "\mathfrak{w}"] & M_{T,X_*(T)_-} \arrow[r, "D(F_-)"'] & M_G \\
& T^\vee. \arrow[u, hook] &
\end{tikzcd}$$

The fact that $T^\vee \to M_{T,X_*(T)_-} = M_G^{\mathrm{ss}}$ is an open immersion will be shown in Lemma 8.3.1. ■

From Corollary 6.6.2, we have the equivalence

$$\mathrm{CT}_L^G|_{(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*)} : \big(P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}},*\big) \xrightarrow{\sim} \big(P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-},\mathbb{F}_p)^{\mathrm{ss}},*\big),$$

such that $\mathrm{CT}_L^G(\mathrm{IC}_\lambda) = \mathrm{IC}^L_{w_0^L w_0(\lambda)}$. By Tannaka duality, it corresponds to the identity

$$M^{\mathrm{ss}}_{L,w_0^L X_*(T)_-} = D\big(X_*(T)_-\big) =\!=\!= D\big(X_*(T)_-\big) = M^{\mathrm{ss}}_G.$$

8. The space of Satake parameters

8.1. The definition of Satake parameters

The *space of Satake parameters* is the $\mathbb{F}_p$-scheme

$$\mathcal{P} := \operatorname{Spec}\big(\mathbb{F}_p\big[X_*(T)_-\big]\big)$$

underlying the $\mathbb{F}_p$-monoid scheme $D(X_*(T)_-) = M^{\mathrm{ss}}_G$.

A *Satake parameter* is an $\mathbb{F}_p$-point of $\mathcal{P}$.

Definition 8.1.1. Let X be a scheme. A *stratification* of X is a decomposition $X = \bigcup_{i\in I} X_i$ as in Definition 2.2.2 such that for all $i \in I$, the closure of X_i in X is a union of some X_j's, i.e., there exists $J_i \subset I$ such that

$$|\overline{X_i}| = \bigcup_{j\in J_i} |X_j|.$$

We are going to define a stratification of the space of Satake parameters by first defining the relevant categories of equivariant perverse sheaves on the affine Grassmannian and then applying Tannaka duality.

8.2. The closed stratum

Let us set

$$\Delta^\perp := \big\{\lambda \in X_*(T) \mid \langle \alpha, \lambda\rangle = 0 \ \forall \alpha \in \Delta\big\}.$$

Then for all $\lambda \in \Delta^\perp$, we have $\dim \mathrm{Gr}_G^{\leq\lambda} = 2\rho(\lambda) = 0$, so that $\mathrm{Gr}_G^{\leq\lambda} = \{\lambda\}$ and hence

$$\mathrm{Gr}_{G,\Delta^\perp} = \coprod_{\lambda\in\Delta^\perp} \{\lambda\}.$$

Consequently, the embedding $P_{L^+G}(\mathrm{Gr}_{G,\Delta^\perp}, \mathbb{F}_p) \subset P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)$ factors as

$$P_{L^+G}(\mathrm{Gr}_{G,\Delta^\perp}, \mathbb{F}_p) \subset P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)^{\mathrm{ss}} \subset P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p),$$

and the equivalence of tensor categories

$$\begin{aligned} \operatorname{Vect}\big(X_*(T)^+\big) &\xrightarrow{\sim} P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)^{\mathrm{ss}} \\ \mathbb{F}_p(\lambda) &\longmapsto \mathrm{IC}_\lambda \end{aligned}$$

restricts to an equivalence of tensor categories

$$\begin{aligned} \mathrm{Vect}(\Delta^{\perp}) &\xrightarrow{\sim} P_{L^+G}(\mathrm{Gr}_{G,\Delta^{\perp}}, \mathbb{F}_p) \\ \mathbb{F}_p(\lambda) &\longmapsto \mathrm{IC}_{\lambda}\,. \end{aligned}$$

We define a retraction

$$\begin{tikzcd} P_{L^+G}(\mathrm{Gr}_{G,\Delta^{\perp}}, \mathbb{F}_p) \arrow[r, hook] & P_{L^+G}(\mathrm{Gr}_{G}, \mathbb{F}_p)^{\mathrm{ss}} \arrow[l, bend right, "r"'] \end{tikzcd}$$

by the rule

$$\begin{aligned} r : P_{L^+G}(\mathrm{Gr}_{G}, \mathbb{F}_p)^{\mathrm{ss}} &\longrightarrow P_{L^+G}(\mathrm{Gr}_{G,\Delta^{\perp}}, \mathbb{F}_p) \\ \mathrm{IC}_{\lambda} &\longmapsto \begin{cases} \mathrm{IC}_{\lambda} & \text{if } \lambda \in \Delta^{\perp} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 8.2.1. *The $\mathbb{F}_p$-linear functor r is a tensor functor.*

Proof. Indeed, for $\lambda, \mu \in X_*(T)^+$, we have $\mathrm{IC}_{\lambda} * \mathrm{IC}_{\mu} = \mathrm{IC}_{\lambda+\mu}$ and

$$(\lambda \in \Delta^{\perp} \text{ and } \mu \in \Delta^{\perp}) \iff \lambda + \mu \in \Delta^{\perp}.$$

Moreover $r(\mathrm{IC}_0) = \mathrm{IC}_0$. ■

Applying the Satake equivalence $\mathcal{S}_{G,\Delta^{\perp}}$ from Notation 7.1.2, we get a tensor retraction

$$\begin{tikzcd} \mathrm{Rep}_{\mathbb{F}_p}(M_{G,\Delta^{\perp}}) \arrow[r, hook] & \mathrm{Rep}_{\mathbb{F}_p}(M_G^{\mathrm{ss}}), \arrow[l, bend right, "r"'] \end{tikzcd}$$

which by Tannaka duality corresponds to a multiplicative section

$$\begin{tikzcd} M_G^{\mathrm{ss}} = D\big(X_*(T)_-\big) \arrow[r, two heads] & M_{G,\Delta^{\perp}} = D(\Delta^{\perp})\,. \arrow[l, bend right, "s"'] \end{tikzcd}$$

In particular

$$S_G := s\big(D(\Delta^{\perp})\big)$$

is a closed subsemigroup of $D(X_*(T)_-)$.

Lemma 8.2.2. *Let A be an abstract, right cancellative monoid. Let $B \subset A$ be a subgroup and let R be a ring. Then $R[A]$ is a free $R[B]$-module. In particular, the inclusion of rings $R[B] \subset R[A]$ is flat.*

Proof. Because B is a group then the right cosets of B in A give a partition of A. Thus, if $\{a_i\}_i$ is a collection of representatives for these cosets then

$$R[A] = \bigoplus_i R[Ba_i].$$

Since A is right cancellative then each morphism $R[B] \to R[Ba_i]$, $b \mapsto ba_i$, is an isomorphism. ■

Proposition 8.2.3. *The morphism $M_G^{\mathrm{ss}} \to M_{G,\Delta^\perp}$ is faithfully flat.*

Proof. It is flat by Lemma 8.2.2 applied to the monoid $X_*(T)_-$ and the subgroup $\Delta^\perp$. It is surjective since it admits a section, namely s. ■

Remark 8.2.4. If G is not a torus then the functor r does *not* intertwine the fiber functors $H|_{P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}}}$ and $H|_{P_{L^+G}(\mathrm{Gr}_{G,\Delta^\perp},\mathbb{F}_p)}$. Correspondingly, the section s does not send the unit of the group scheme $D(\Delta^\perp)$ to the unit of the monoid scheme $D(X_*(T)_-)$.

8.3. The open complement to the closed stratum

Recall that a standard Levi subgroup of G is the Levi factor containing T of a parabolic subgroup of G containing B. We denote by $\mathscr{L}$ the set of standard Levi subgroups of G. It is in 1-1 correspondence with the power set of the set Δ of simple roots corresponding to the pair (B, T):

$$\begin{aligned} \mathscr{L} &\xrightarrow{\sim} \mathscr{P}(\Delta) \\ L &\longmapsto \Delta_L, \end{aligned}$$

where Δ_L is the set of simple roots of L with respect to the pair $(B \cap L, T)$. In particular $\Delta_T = \emptyset$ and $\Delta_G = \Delta$.

For each $L \in \mathscr{L}$, we have constructed the functor

$$P_{L^+G}(\mathrm{Gr}_G, \mathbb{F}_p)^{\mathrm{ss}} \xrightarrow[\mathrm{CT}_L^G|_{P_{L^+G}(\mathrm{Gr}_G,\mathbb{F}_p)^{\mathrm{ss}}}]{\sim} P_{L^+L}(\mathrm{Gr}_{L,w_0^L X_*(T)_-}, \mathbb{F}_p)^{\mathrm{ss}} \subset P_{L^+L}(\mathrm{Gr}_L, \mathbb{F}_p)^{\mathrm{ss}},$$

which corresponds to

$$\begin{aligned} j_L : M_L^{\mathrm{ss}} = D\big(X_*(T)_{-/L}\big) &\longrightarrow M^{\mathrm{ss}}_{L,w_0^L X_*(T)_-} \\ = D\big(X_*(T)_-\big) &=\!=\!=\!= D\big(X_*(T)_-\big) = M_G^{\mathrm{ss}}, \end{aligned}$$

cf. end of Section 7.4.

Lemma 8.3.1. *The following statements hold true.*

- *The morphism of $\mathbb{F}_p$-monoid schemes j_L is an open immersion.*
- *For all $L, L' \in \mathscr{L}$, we have*

$$j_L\big(D(X_*(T)_{-/L})\big) \cap j_{L'}\big(D\big(X_*(T)_{-/L'}\big)\big) = j_{L''}\big(D\big(X_*(T)_{-/L''}\big)\big)$$

  with $\Delta_{L''} := \Delta_L \cap \Delta_{L'}$.

- *We have*

$$\mathcal{P} \setminus S_G = \bigcup_{L \in \mathscr{L}\setminus\{G\}} j_L\big(D\big(X_*(T)_{-/L}\big)\big).$$

Proof. By construction, $j_L^* : \mathbb{F}_p[X_*(T)_-] \to \mathbb{F}_p[X_*(T)_{-/L}]$ is the morphism of $\mathbb{F}_p$-algebras induced by the canonical inclusion $X_*(T)_- \subset X_*(T)_{-/L}$. Let $\lambda_\alpha, \alpha \in \Delta$, be elements of $X_*(T)_-$ such that

$$\forall \alpha, \beta \in \Delta, \quad \langle \alpha, \lambda_\beta \rangle \begin{cases} \in \mathbb{Z}_{\leq -1} & \text{if } \alpha = \beta \\ = 0 & \text{otherwise} \end{cases}$$

(complete Δ into a basis of $X^*(T) \otimes \mathbb{Q}$ and consider the dual basis of $X_*(T) \otimes \mathbb{Q}$ under the perfect pairing $\langle\ ,\ \rangle$). Then, for all $\lambda \in X_*(T)_{-/L}$, we can find some $n_\alpha \in \mathbb{Z}_{\geq 0}$, $\alpha \in \Delta \setminus \Delta_L$, such that

$$\Big(\lambda + \sum_{\alpha \in \Delta \setminus \Delta_L} n_\alpha \lambda_\alpha\Big) \in X_*(T)_-,$$

i.e.,

$$\mathbb{F}_p\big[X_*(T)_{-/L}\big] = \mathbb{F}_p\big[X_*(T)_-\big]\big[(e^{\lambda_\alpha})^{-1}, \alpha \in \Delta \setminus \Delta_L\big].$$

Hence j_L is an open immersion, and the complement of $j_L(D(X_*(T)_{-/L}))$ in $\mathcal{P} = D(X_*(T)_-)$ is the closed subset defined by the equation $\prod_{\alpha \in \Delta \setminus \Delta_L} e^{\lambda_\alpha} = 0$.

Consequently,

$$\mathcal{P} \setminus j_L\big(D\big(X_*(T)_{-/L}\big)\big) \cap j_{L'}\big(D\big(X_*(T)_{-/L'}\big)\big)$$

is the closed subset defined by the equation $\prod_{\alpha \in \Delta \setminus (\Delta_L \cap \Delta_{L'})} e^{\lambda_\alpha} = 0$, and hence

$$j_L\big(D\big(X_*(T)_{-/L}\big)\big) \cap j_{L'}\big(D\big(X_*(T)_{-/L'}\big)\big) = j_{L''}\big(D\big(X_*(T)_{-/L''}\big)\big)$$

with $\Delta_{L''} := \Delta_L \cap \Delta_{L'}$.

Finally,

$$\mathcal{P} \setminus \bigcup_{L \in \mathcal{L} \setminus \{G\}} j_L\big(D\big(X_*(T)_{-/L}\big)\big)$$

is the closed subset defined by the equations

$$\forall \alpha \in \Delta, \quad e^{\lambda_\alpha} = 0.$$

On the other hand,

$$s(S_G) = V\big(e^\lambda, \lambda \in X_*(T)_- \setminus \Delta^\perp\big) \subset D\big(X_*(T)_-\big) = \mathcal{P}$$

by construction. We claim that

$$(e^{\lambda_\alpha}, \alpha \in \Delta) \subset \big(e^\lambda, \lambda \in X_*(T)_- \setminus \Delta^\perp\big) \subset \sqrt{(e^{\lambda_\alpha}, \alpha \in \Delta)}.$$

The first inclusion follows from the definition of the elements λ_α. For the second one, note that for $\lambda \in X_*(T)_- \setminus \Delta^\perp$ we can find integers $m > 0$, $m_\alpha \geq 0$, such that $m\lambda - \sum_\alpha m_\alpha \lambda_\alpha \in \Delta^\perp$. Since the elements e^μ for $\mu \in \Delta^\perp$ are units, the second inclusion follows. Hence $\mathcal{P} \setminus \bigcup_{L \in \mathcal{L} \setminus \{G\}} j_L(D(X_*(T)_{-/L}))$ is equal to the subset underlying the closed subscheme $s(S_G)$. ■

From now on we will write simply $D(X_*(T)_{-/L})$ for

$$j_L\big(D\big(X_*(T)_{-/L}\big)\big) \subset D\big(X_*(T)_-\big).$$

Remark 8.3.2. We have seen in the proof of Lemma 8.3.1 that $T^\vee = \mathrm{Spec}(\mathbb{F}_p[X_*(T)])$ is the open complement in $\mathcal{P} = D(X_*(T)_-)$ of the Cartier divisor defined by the regular element

$$\prod_{\alpha\in\Delta} e^{\lambda_\alpha} = e^{\sum_{\alpha\in\Delta}\lambda_\alpha} \in \mathbb{F}_p\big[X_*(T)_-\big].$$

In particular, the scheme $\mathcal{P}$ is integral.

Example 8.3.3. If $G = \mathrm{GL}_n$ then $X_*(T)_- = \oplus_{i=1}^{n-1}\mathbb{Z}_{\geq 0}\omega_{i-} \oplus \mathbb{Z}\omega_{n-}$ where $\omega_{i-} \in \mathbb{Z}^n$ has its first $n-i$ entries equal to 0 and last i entries equal to 1, so

$$\mathcal{P} = \mathrm{Spec}\left(\mathbb{F}_p[T_1, \ldots, T_{n-1}, T_n^{\pm 1}]\right).$$

If $G = \mathrm{SL}_2$ then $X_*(T)_- = \mathbb{Z}_{\geq 0}(-\alpha^\vee)$ where $-\alpha^\vee = (-1, 1)$, so $\mathcal{P} = \mathrm{Spec}(\mathbb{F}_p[T])$. In particular, $\mathcal{P}$ is smooth in both of these examples.

Example 8.3.4. In general $\mathcal{P}$ is not smooth. For example, let $G = \mathrm{SL}_3$. Then $X_*(T) = \{(a,b,c) \in \mathbb{Z}^3 : a+b+c = 0\}$ and the simple roots are $\alpha = (1,-1,0)$, $\beta = (0,1,-1) \in X^*(T) = \mathbb{Z}^3/\mathbb{Z}$. Then $X_*(T) = \mathbb{Z}\alpha^\vee \oplus \mathbb{Z}\beta^\vee$ and

$$X_*(T)^+ = \{a\alpha^\vee + b\beta^\vee \mid 2a \geq b, 2b \geq a\}.$$

The monoid $X_*(T)^+$ is generated by the elements

$$\alpha^\vee + \beta^\vee, \quad \alpha^\vee + 2\beta^\vee, \quad 2\alpha^\vee + \beta^\vee.$$

By sending the indeterminates x, y, z to the corresponding generators in $\mathbb{F}_p[X_*(T)^+]$, we get a surjection

$$\mathbb{F}_p[x,y,z]/I \twoheadrightarrow \mathbb{F}_p\big[X_*(T)^+\big], \quad I = (x^3 - yz).$$

Since I is a prime ideal and $\mathbb{F}_p[X_*(T)^+]$ is an integral domain of dimension 2 then this map is an isomorphism. In particular, the ring $\mathbb{F}_p[X_*(T)^+]$, equivalently the ring $\mathbb{F}_p[X_*(T)_-]$, is not regular.

8.4. The Herzig stratification

For all $L \in \mathcal{L}$, set

$$S_L := s_L\big(D(\Delta_L^\perp)\big).$$

Corollary 8.4.1. *The space of Satake parameters admits the following stratification by subsemigroups:*

$$\mathcal{P} = \bigcup_{L\in\mathcal{L}} S_L.$$

The stratum S_L is isomorphic to a torus of rank equal to

$$\operatorname{rank} T - |\Delta_L| = \operatorname{rank} \pi_1(L) = \operatorname{rank} \pi_0(\mathrm{Gr}_L).$$

The closure of S_L in $\mathcal{P}$ is

$$\overline{S_L} = \bigcup_{L' \supset L} S_{L'}.$$

Proof. The decomposition is a consequence of Lemma 8.3.1.

Let $L \in \mathcal{L}$. Since $\Delta_L^\perp$ is a subgroup of the finitely generated free abelian group $X_*(T)$ then $\Delta_L^\perp$ is also finitely generated and free. Hence $D(\Delta_L^\perp)$ is a torus, of rank equal to

$$\operatorname{rank} \Delta_L^\perp = \dim_{\mathbb{Q}}(\mathbb{Z}\Delta_L \otimes \mathbb{Q})^\perp = \operatorname{rank} T - |\Delta_L| = \operatorname{rank} X_*(T)/\mathbb{Z}\Phi_L^\vee.$$

Finally, with the notation of the proof of Lemma 8.3.1, we have

$$\begin{aligned} S_L &:= \operatorname{Spec}\left(\mathbb{F}_p\left[X_*(T)_{-/L}\right]/\left(e^\lambda, \lambda \in X_*(T)_{-/L} \setminus \Delta_L^\perp\right)\right) \\ &= \operatorname{Spec}\left(\mathbb{F}_p\left[X_*(T)_-\right]\left[(e^{\lambda_\alpha})^{-1}, \alpha \in \Delta \setminus \Delta_L\right]/(e^{\lambda_\beta}, \beta \in \Delta_L)\right)_{\mathrm{red}}. \end{aligned}$$

Thus, setting

$$f_L := \prod_{\alpha \in \Delta \setminus \Delta_L} e^{\lambda_\alpha} = e^{\sum_{\alpha \in \Delta \setminus \Delta_L} \lambda_\alpha} \in \mathbb{F}_p\left[X_*(T)_-\right]$$

and

$$V_L := \operatorname{Spec}\left(\mathbb{F}_p\left[X_*(T)_-\right]/(e^{\lambda_\beta}, \beta \in \Delta_L)\right) \subset \operatorname{Spec}\left(\mathbb{F}_p\left[X_*(T)_-\right]\right) = \mathcal{P},$$

we have

$$|S_L| = |D(f_L)| \cap |V_L| \subset \mathcal{P}$$

and

$$|V_L| = \bigcup_{L' \supset L} |D(f_{L'})| \cap |V_{L'}| = \bigcup_{L' \supset L} |S_{L'}|.$$

Now let us show that $|\overline{S_L}| = |V_L|$. Since $|S_L| = |D(f_L)| \cap |V_L|$, it suffices to show that f_L defines a Cartier divisor *after restriction to* V_L, i.e., that its image in the ring of functions on V_L is a regular element. So let $a = \sum_\lambda a_\lambda e^\lambda \in \mathbb{F}_p[X_*(T)_-]$ such that

$$f_L a = \sum_{\beta \in \Delta_L} g_\beta e^{\lambda_\beta} \in (e^{\lambda_\beta}, \beta \in \Delta_L).$$

If $a_\lambda \neq 0$ then

$$\sum_{\alpha \in \Delta \setminus \Delta_L} \lambda_\alpha + \lambda = \mu + \lambda_\beta$$

for some $\mu \in X_*(T)_-$ and $\beta \in \Delta_L$. The cocharacter

$$\nu := \lambda - \lambda_\beta = \mu - \sum_{\alpha \in \Delta \setminus \Delta_L} \lambda_\alpha$$

satisfies $\langle \gamma, \nu \rangle \leq 0$ for all $\gamma \in \Delta$, i.e., $\nu \in X_*(T)_-$. Hence $a \in (e^{\lambda_\beta}, \beta \in \Delta_L)$, as desired. ∎

We call the stratification Corollary 8.4.1 the *Herzig stratification*, since it corresponds to the stratification of the set $\mathcal{P}(\overline{\mathbb{F}}_p)$ defined in [17, §1.5, §2.4].

Definition 8.4.2. We call the open stratum

$$S_T = T^\vee = D\big(X_*(T)\big) \subset \mathcal{P}$$

the *ordinary locus*, and the closed stratum

$$S_G = s_G\big(D(\Delta^\perp)\big) \subset \mathcal{P}$$

the *supersingular locus*.

Example 8.4.3. For $G = \mathrm{GL}_2$, we have $X_*(T)_- = \mathbb{N}(0,1) \oplus \mathbb{Z}(1,1)$, the space of Satake parameters is

$$M^{\mathrm{ss}}_{GL_2} = D\big(X_*(T)_-\big) = \mathrm{Spec}\,\big(\mathbb{F}_p[e^{(0,1)}, e^{\pm(1,1)}]\big) = \mathbb{A}^1 \times \mathbb{G}_m,$$

and the Herzig stratification consists only in the ordinary and the supersingular loci

$$S_T \cup S_G = (\mathbb{G}_m \times \mathbb{G}_m) \cup \big(\{0\} \times \mathbb{G}_m\big).$$

Example 8.4.4. The supersingular locus S_G is 0-dimensional if and only if G is semi-simple, in which case it is just one $\mathbb{F}_p$-point.

Lemma 8.4.5. *The ordinary locus $T^\vee$ is the group of invertible elements of the monoid M^{ss}_G.*

Proof. Let $s \in M^{\mathrm{ss}}_G(\overline{\mathbb{F}}_p)$. Let L be the element of $\mathcal{L}$ such that $s \in S_L(\overline{\mathbb{F}}_p)$. Then, for all $\lambda \in X_*(T)_{-/L} \setminus \Delta_L^\perp$,

$$\lambda(s) = s^*(e^\lambda) = 0 \in \overline{\mathbb{F}}_p,$$

i.e., the character $\lambda : D(X_*(T)_{-/L}) \to \mathbb{A}^1$ vanishes on s. Hence, if s is invertible in $M^{\mathrm{ss}}_G(\overline{\mathbb{F}}_p) = D(X_*(T)_-)(\overline{\mathbb{F}}_p)$, then $(X_*(T)_{-/L} \setminus \Delta_L^\perp) \cap X_*(T)_- = \emptyset$, i.e., $X_*(T)_- \subset \Delta_L^\perp$, which occurs only if $\Delta_L = \emptyset$, in which case $L = T$. ■

Lemma 8.4.6. *The supersingular locus S_G is absorbing in the monoid M^{ss}_G, i.e., the restriction of the multiplication $M^{\mathrm{ss}}_G \times M^{\mathrm{ss}}_G \to M^{\mathrm{ss}}_G$ to $S_G \times M^{\mathrm{ss}}_G$ factors through S_G.*

Proof. Let $s \in S_G(\overline{\mathbb{F}}_p)$ and $s' \in M^{\mathrm{ss}}_G(\overline{\mathbb{F}}_p)$. Then, for all $\lambda \in X_*(T)_- \setminus \Delta^\perp$,

$$(ss')^*(e^\lambda) = \lambda(ss') = \lambda(s)\lambda(s') = s^*(e^\lambda)\lambda(s') = 0 \in \overline{\mathbb{F}}_p.$$

Thus the $\overline{\mathbb{F}}_p$-algebra morphism

$$(ss')^* : \overline{\mathbb{F}}_p\big[X_*(T)_-\big] \longrightarrow \overline{\mathbb{F}}_p$$

vanishes on the ideal $(e^\lambda, \lambda \in X_*(T)_- \setminus \Delta^\perp)$ of S_G in M^{ss}_G, which means precisely that $ss' \in S_G(\overline{\mathbb{F}}_p)$. ■

Corollary 8.4.7. *Let $\pi_G : M_G \to M_G^{\mathrm{ss}}$ be the canonical eigenvalues homomorphism. Then $\pi_G^{-1}(T^\vee) \subset M_G$ is open and is the group of invertible elements, and $\pi_G^{-1}(S_G) \subset M_G$ is closed and is an absorbing subsemigroup.*

Proof. The only part left to check is that $\pi_G^{-1}(T^\vee)$ consists of units. This follows from the fact that an endomorphism of the forgetful tensor functor $\mathrm{Rep}_{\mathbb{F}_p}(M_G) \to \mathrm{Vect}_{\mathbb{F}_p}$ is an automorphism if and only if it is an automorphism on simple objects. ■

Appendix: Cohomology with support in T_ν

Let U^- be the unipotent radical of the opposite Borel B^-. For $\nu \in X_*(T)$, let

$$T_\nu := \big(LU^- \cdot \nu(t)\big)_{\mathrm{red}} \subset \mathrm{Gr}_G$$

be the reduced ind-subscheme of the corresponding connected component of the *repeller* [12] with respect to the $\mathbb{G}_m$-action on Gr_G from Section 3.4. For $\lambda \in X_*(T)^+$, we denote by $i_{T_{\nu,\lambda}} : T_\nu \cap \mathrm{Gr}_G^{\leq\lambda} \to \mathrm{Gr}_G^{\leq\lambda}$ the canonical immersion (where $T_\nu \cap \mathrm{Gr}_G^{\leq\lambda}$ is equipped with its reduced structure) and define

$$\forall i \in \mathbb{Z}, \quad H^i_{T_\nu}(\mathrm{Gr}_G, \mathrm{IC}_\lambda) := R^i\Gamma(T_\nu \cap \mathrm{Gr}_G^{\leq\lambda}, Ri^!_{T_{\nu,\lambda}}\, \mathrm{IC}_\lambda).$$

Proposition A.1. *Let $\lambda \in X_*(T)^+$. If $\nu = w_0(\lambda)$ then*

$$H^{2\rho(\nu)}_{T_\nu}(\mathrm{Gr}_G, \mathrm{IC}_\lambda) = R^{-2\rho(\lambda)}\Gamma(\mathrm{IC}_\lambda) \cong \mathbb{F}_p.$$

Proof. By [19, Th. 3.2], $T_{w_0(\lambda)} \cap \mathrm{Gr}_G^{\leq\lambda}$ is of pure dimension

$$-\rho\big(w_0(\lambda) + w_0(\lambda)\big) = 2\rho(\lambda) = \dim \mathrm{Gr}_G^{\leq\lambda}.$$

Thus $T_{w_0(\lambda)} \cap \mathrm{Gr}_G^{\leq\lambda}$ is open in $\mathrm{Gr}_G^{\leq\lambda}$, so $Ri^!_{T_{w_0(\lambda),\lambda}} = Ri^*_{T_{w_0(\lambda),\lambda}}$ and the proposition follows. ■

Proposition A.2. *Let $\lambda \in X_*(T)^+$ be such that $\rho(\lambda) \neq 0$. If $\nu = \lambda$ then*

$$H^{2\rho(\nu)}_{T_\nu}(\mathrm{Gr}_G, \mathrm{IC}_\lambda) = 0.$$

Proof. By [19, Th. 3.2], $T_\lambda \cap \mathrm{Gr}_G^{\leq\lambda}$ is a point. Let $U := \mathrm{Gr}_G^{\leq\lambda} \backslash (T_\lambda \cap \mathrm{Gr}_G^{\leq\lambda})$ and $j : U \to \mathrm{Gr}_G^{\leq\lambda}$ be the canonical open immersion. We claim that, as a complex of sheaves, $Rj_*\mathcal{O}_U$ is concentrated in degrees $\leq 2\rho(\lambda) - 1$. To prove the claim, note that we may replace $\mathrm{Gr}_G^{\leq\lambda}$ by the local ring $(A, \mathfrak{m})$ at $T_\lambda \cap \mathrm{Gr}_G^{\leq\lambda}$ in $\mathrm{Gr}_G^{\leq\lambda}$. For $n \geq 1$ we have $R^n j_*\mathcal{O}_U = H^{n+1}_{\mathfrak{m}}(A)$. Since $H^i_{\mathfrak{m}}(A) = 0$ for $i > \dim A$ then $R^n j_*\mathcal{O}_U = 0$ unless $n \leq 2\rho(\lambda) - 1$.

Now by the Artin–Schreier sequence $Rj_*(\mathbb{F}_p[2\rho(\lambda)])$ is concentrated in degrees ≤ 0. Hence by the exact triangle

$$Ri_{T_{\lambda,\lambda},*}Ri^!_{T_{\lambda,\lambda}}\big(\mathbb{F}_p\big[2\rho(\lambda)\big]\big) \longrightarrow \mathbb{F}_p\big[2\rho(\lambda)\big] \longrightarrow Rj_*Rj^*\big(\mathbb{F}_p\big[2\rho(\lambda)\big]\big) \xrightarrow{+1}.$$

it follows that $Ri^!_{T_{\lambda,\lambda}}(\mathbb{F}_p[2\rho(\lambda)])$ is concentrated in degrees ≤ 1. Now we are done because $2\rho(\lambda) > 1$ and $T_\lambda \cap \mathrm{Gr}_G^{\leq\lambda}$ is a point. ■

Proposition A.3. *Suppose $G = \mathrm{SL}_2$, and that T and B are the diagonal maximal torus and the upper triangular Borel subgroup; in particular $X_*(T)^+ \cong \mathbb{Z}_{\geq 0}$. If $\lambda = 1$ and $\nu = 0$, then $H^{2\rho(\nu)}_{T_\nu}(\mathrm{Gr}_G, \mathrm{IC}_\lambda)$ is infinite-dimensional.*

Proof. The scheme $\mathrm{Gr}_G^{\leq\lambda}$ is stratified by $T_{-\lambda} \cap \mathrm{Gr}_G^{\leq\lambda}$, $T_0 \cap \mathrm{Gr}_G^{\leq\lambda}$, and $T_\lambda \cap \mathrm{Gr}_G^{\leq\lambda}$. These strata have dimensions 2, 1, and 0, respectively. Let

$$Z = \overline{T}_0 \cap \mathrm{Gr}_G^{\leq\lambda} = (T_0 \cap \mathrm{Gr}_G^{\leq\lambda}) \cup (T_\lambda \cap \mathrm{Gr}_G^{\leq\lambda})$$

and let $i\colon Z \to \mathrm{Gr}_G^{\leq\lambda}$ be the corresponding closed immersion. Let $j\colon T_{-\lambda} \cap \mathrm{Gr}_G^{\leq\lambda} \to \mathrm{Gr}_G^{\leq\lambda}$ be the complementary open immersion. Then there is an exact triangle

$$Ri_*Ri^!(\mathrm{IC}_\lambda) \longrightarrow \mathrm{IC}_\lambda \longrightarrow Rj_*Rj^*(\mathrm{IC}_\lambda) \xrightarrow{+1} .$$

By [9, Th. 6.9], $R\Gamma(\mathrm{IC}_\lambda) \cong \mathbb{F}_p[2]$, and the map $R^{-2}\Gamma(\mathrm{IC}_\lambda) \to R^{-2}\Gamma(Rj_*Rj^*(\mathrm{IC}_\lambda))$ is an isomorphism. By [20, Lem. 5.2], $T_{-\lambda} \cap \mathrm{Gr}_G^{\leq\lambda}$ is isomorphic to $\mathbb{A}^2$. Thus by a computation with the Artin–Schreier sequence we find that $R\Gamma(Rj_*Rj^*(\mathrm{IC}_\lambda))$ is concentrated in degrees -2 and -1, and it is infinite-dimensional in degree -1. Hence $R\Gamma(Ri_*Ri^!(\mathrm{IC}_\lambda))$ is concentrated in degree 0, and $R^0\Gamma(Ri_*Ri^!(\mathrm{IC}_\lambda))$ is infinite-dimensional.

Now let $p\colon T_0 \cap \mathrm{Gr}_G^{\leq\lambda} \to Z$ be the open immersion. There is an exact triangle

$$R\Gamma\big(Ri^!_{T_\lambda}(\mathrm{IC}_\lambda)\big) \longrightarrow R\Gamma\big(Ri^!(\mathrm{IC}_\lambda)\big) \longrightarrow R\Gamma\big(Rp^*\big(Ri^!(\mathrm{IC}_\lambda)\big)\big) \xrightarrow{+1} .$$

By [9, Lem. 2.10], IC_λ is the intermediate extension of its restriction to $\mathrm{Gr}_G^{\leq\lambda} \backslash (T_\lambda \cap \mathrm{Gr}_G^{\leq\lambda})$, so by [9, Lem. 2.7], $Ri^!_{T_\lambda}(\mathrm{IC}_\lambda)$ is concentrated in degrees ≥ 1. Thus the map

$$R^0\Gamma\big(Ri^!(\mathrm{IC}_\lambda)\big) \longrightarrow R^0\Gamma\big(Rp^*\big(Ri^!(\mathrm{IC}_\lambda)\big)\big)$$

is injective. Now we are done because there is a natural isomorphism

$$R^0\Gamma\big(Rp^*\big(Ri^!(\mathrm{IC}_\lambda)\big)\big) \cong H^{2\rho(0)}_{T_0}(\mathrm{Gr}_G, \mathrm{IC}_\lambda).$$ ■

By comparing Propositions A.1 and A.2 with Theorem 4.2.2, we see that the groups

$$H^{2\rho(\nu)}_c(S_\nu, \mathrm{IC}_\lambda) \quad \text{and} \quad H^{2\rho(\nu)}_{T_\nu}(\mathrm{Gr}_G, \mathrm{IC}_\lambda)$$

agree in some cases. However, by Proposition A.3, these groups are not isomorphic in general. In other words, *Braden's hyperbolic localization theorem fails for $\mathbb{F}_p$-coefficients.*

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Robert Cass
Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA; cassr@umich.edu

Cédric Pépin
Département de Mathématiques, Institut Galilée, Université Sorbonne Paris Nord, 99 avenue Jean-Baptiste Clément, 93 430 Villetaneuse, France; cpepin@math.univ-paris13.fr