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A mathematical analysis of the Kakinuma model for interfacial gravity waves. Part I: Structures and well-posedness

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Abstract. We consider a model, which we named the Kakinuma model, for interfacial gravity waves. As is well known, the full model for interfacial gravity waves has a variational structure whose Lagrangian is an extension of Luke's Lagrangian for surface gravity waves, that is, water waves. The Kakinuma model is a system of Euler–Lagrange equations for approximate Lagrangians, which are obtained by approximating the velocity potentials in the Lagrangian for the full model. In this paper we first analyze the linear dispersion relation for the Kakinuma model and show that the dispersion curves highly fit that of the full model in the shallow water regime. We then analyze the linearized equations around constant states and derive a stability condition, which is satisfied for small initial data when the denser water is below the lighter water. We show that the initial value problem is in fact well posed locally in time in Sobolev spaces under the stability condition, the noncavitation assumption, and intrinsic compatibility conditions, in spite of the fact that the initial value problem for the full model does not have any stability domain so that its initial value problem is ill posed in Sobolev spaces. Moreover, it is shown that the Kakinuma model enjoys a Hamiltonian structure and has conservative quantities: mass, total energy, and in the case of a flat bottom, momentum.

1. Introduction

We are concerned with the motion of interfacial gravity waves at the interface between two layers of immiscible waters in a domain of the $(n+1)$-dimensional Euclidean space in the rigid-lid case. Let t be the time, $\boldsymbol{x} = (x_1, \dots, x_n)$ the horizontal spatial coordinates, and z the vertical spatial coordinate. We assume that the interface, the rigid lid of the upper layer, and the bottom of the lower layer are represented as $z = \zeta(\boldsymbol{x}, t)$, $z = h_1$, and $z = -h_2 + b(\boldsymbol{x})$, respectively, where $\zeta(\boldsymbol{x}, t)$ is the elevation of the interface, h_1 and h_2 are mean thicknesses of the upper and lower layers, and $b(\boldsymbol{x})$ represents the bottom topography. The only external force applied to the system is the constant and vertical gravity, and interfacial tension is neglected. Moreover, we assume that the waters in the upper and the lower layers are both incompressible and inviscid fluids with constant densities ρ_1 and ρ_2, respectively, and that the flows are both irrotational. See Figure 1. Then the motion of the

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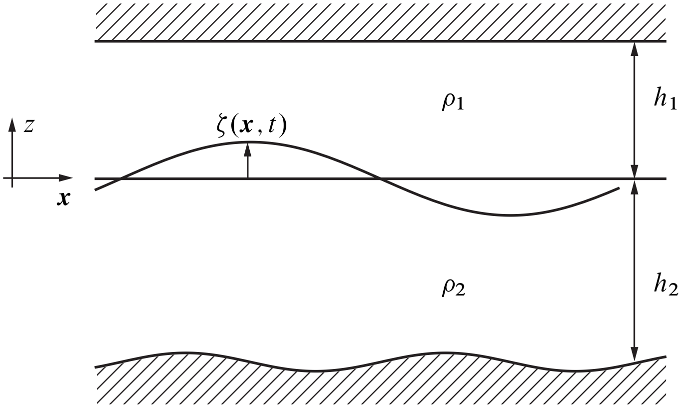

Figure 1. Interfacial gravity waves.

waters is described by the velocity potentials Φ_1 and Φ_2 and the pressures P_1 and P_2 in the upper and the lower layers, respectively, satisfying the basic equations in the theory of fluid dynamics, which will be referred to as the full model for interfacial gravity waves throughout this paper. As shown by Luke [23], the basic equations for the surface gravity waves, that is, the water wave problem, have a variational structure, whose Lagrangian is written in terms of the surface elevation of the water and the velocity potential, and the Lagrangian density is given by the vertical integral of the pressure in the water region. The full model for interfacial gravity waves also has a variational structure and the Lagrangian density $\mathcal{L}(\Phi_1, \Phi_2, \zeta)$ is again given by the vertical integral of the pressure in both water regions. Kakinuma [17–19] proposed a model for interfacial gravity waves and applied his model to numerically simulate the waves. To derive the model, he approximated the velocity potentials Φ_1 and Φ_2 by

$$\Phi_k^{\mathrm{app}}(\boldsymbol{x}, z, t) = \sum_{i=0}^{N} Z_{k,i}(z; \tilde{h}_k(\boldsymbol{x}))\phi_{k,i}(\boldsymbol{x}, t) \tag{1.1}$$

for $k = 1, 2$, where $\{Z_{1,i}\}$ and $\{Z_{2,i}\}$ are appropriate function systems in the vertical coordinate z and may depend on $\tilde{h}_1(\boldsymbol{x})$ and $\tilde{h}_2(\boldsymbol{x})$, respectively, which are the thicknesses of the upper and the lower layers in the rest state, whereas $\boldsymbol{\phi}_k = (\phi_{k,0}, \phi_{k,1}, \dots, \phi_{k,N})^{\mathrm{T}}$, $k = 1, 2$, are unknown variables. Then he derived an approximate Lagrangian density $\mathcal{L}^{\mathrm{app}}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta) = \mathcal{L}(\Phi_1^{\mathrm{app}}, \Phi_2^{\mathrm{app}}, \zeta)$ for unknowns $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta)$. The Kakinuma model is a corresponding system of Euler–Lagrange equations for the approximated Lagrangian density $\mathcal{L}^{\mathrm{app}}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta)$. Different choices of the function systems $\{Z_{1,i}\}$ and $\{Z_{2,i}\}$ give different Kakinuma models and we have to carefully choose the function systems for the Kakinuma model to provide good approximations for interfacial gravity waves.

The Kakinuma model is an extension to interfacial gravity waves of the so-called Isobe–Kakinuma model for surface gravity waves, that is, water waves. In the case of surface gravity waves, the basic equations are known to have a variational structure with Luke's Lagrangian density $\mathscr{L}_{\rm Luke}(\Phi, \zeta)$, where ζ is the surface elevation and Φ is the velocity potential of the water. The Isobe–Kakinuma model is a system of Euler–Lagrange equations for the approximated Lagrangian density $\mathscr{L}^{\rm app}(\boldsymbol{\phi}, \zeta) = \mathscr{L}_{\rm Luke}(\Phi^{\rm app}, \zeta)$, where $\Phi^{\rm app}$ is an approximate velocity potential

$$\Phi^{\rm app}(\boldsymbol{x}, z, t) = \sum_{i=0}^{N} Z_i(z; b(\boldsymbol{x}))\phi_i(\boldsymbol{x}, t) \tag{1.2}$$

and $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_N)^{\rm T}$ are unknown variables. The model was first proposed by Isobe [15, 16] and then applied by Kakinuma to numerically simulate water waves. We note that a similar model was derived by Klopman, van Groesen, and Dingemans [21], and used to simulate water waves. See also Papoutsellis and Athanassoulis [29]. Recently, this model was analyzed from a mathematical point of view. One possible choice of the function system $\{Z_i\}$ is a set of polynomials in z, for example, $Z_i(z; b(\boldsymbol{x})) = (z + h - b(\boldsymbol{x}))^{p_i}$, with integers p_i satisfying $0 = p_0 < p_1 < \cdots < p_N$. Under this choice of the function system $\{Z_i\}$, the initial value problem to the Isobe–Kakinuma model was analyzed by Murakami and Iguchi [27] in a special case and by Nemoto and Iguchi [28] in the general case. The hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Isobe–Kakinuma model, so that one needs to impose some compatibility conditions on the initial data for the existence of the solution. Under these compatibility conditions and a sign condition $-\partial_z P^{\rm app} \geq c_0 > 0$ on the water surface, they showed the well-posedness of the initial value problem locally in time, where $P^{\rm app}$ is an approximate pressure in the Isobe–Kakinuma model calculated from Bernoulli's equation. Moreover, Iguchi [12, 13] showed that under the choice of the function system

$$Z_i(z; b(\boldsymbol{x})) = \begin{cases} (z + h)^{2i} & \text{in the case of a flat bottom,} \\ (z + h - b(\boldsymbol{x}))^i & \text{in the case of a variable bottom,} \end{cases} \tag{1.3}$$

the Isobe–Kakinuma model is a higher-order shallow water approximation for the water wave problem in a strongly nonlinear regime. Furthermore, Duchêne and Iguchi [8] showed that the Isobe–Kakinuma model also enjoys a Hamiltonian structure analogous to the one exhibited by Zakharov [32] on the full water wave problem. Our aim in the present paper is to extend these results on surface gravity waves to interfacial gravity waves.

In view of these results on the Isobe–Kakinuma model, in the present paper we consider the Kakinuma model under the choice of the approximate velocity potentials in (1.1) as

$$\begin{cases} \Phi_1^{\rm app}(\boldsymbol{x}, z, t) = \displaystyle\sum_{i=0}^{N} (-z + h_1)^{2i} \phi_{1,i}(\boldsymbol{x}, t), \\ \Phi_2^{\rm app}(\boldsymbol{x}, z, t) = \displaystyle\sum_{i=0}^{N^*} (z + h_2 - b(\boldsymbol{x}))^{p_i} \phi_{2,i}(\boldsymbol{x}, t), \end{cases} \tag{1.4}$$

where N, N^* and $p_0, p_1, \dots, p_{N^*}$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_{N^*}$. In applications of the Kakinuma model, it would be better to choose $N^* = N$ and $p_i = 2i$ in the case of a flat bottom, and $N^* = 2N$ and $p_i = i$ in the case of a variable bottom. In the case $N = N^* = 0$, that is, if we choose the approximation $\Phi_k^{\mathrm{app}}(\boldsymbol{x}, z, t) = \phi_k(\boldsymbol{x}, t)$ for $k = 1, 2$, functions independent of the vertical coordinate z, then the corresponding Kakinuma model is reduced to the shallow water equations. In the case $N + N^* > 0$, the Kakinuma model is classified into a system of nonlinear dispersive equations.

It is well known that in the case of a flat bottom $b = 0$, the dispersion relation of the linearized equations to the full model around the flow $(\zeta, \Phi_1, \Phi_2) = (0, \boldsymbol{u}_1 \cdot \boldsymbol{x}, \boldsymbol{u}_2 \cdot \boldsymbol{x})$ with constant horizontal velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is given by

$$
\begin{aligned}
&(\rho_1 \coth(h_1|\boldsymbol{\xi}|) + \rho_2 \coth(h_2|\boldsymbol{\xi}|))\omega^2 \\
&\qquad + 2(\rho_1 \boldsymbol{\xi} \cdot \boldsymbol{u}_1 \coth(h_1|\boldsymbol{\xi}|) + \rho_2 \boldsymbol{\xi} \cdot \boldsymbol{u}_2 \coth(h_2|\boldsymbol{\xi}|))\omega \\
&\qquad + \rho_1(\boldsymbol{\xi} \cdot \boldsymbol{u}_1)^2 \coth(h_1|\boldsymbol{\xi}|) + \rho_2(\boldsymbol{\xi} \cdot \boldsymbol{u}_2)^2 \coth(h_2|\boldsymbol{\xi}|) - (\rho_2 - \rho_1) g|\boldsymbol{\xi}| = 0,
\end{aligned}
$$

where $\boldsymbol{\xi} \in \mathbf{R}^n$ is the wave vector, $\omega \in \mathbf{C}$ the angular frequency, and g the gravitational constant. It is easy to see that the roots ω of the above equation are always real for any wave vector $\boldsymbol{\xi} \in \mathbf{R}^n$ if and only if $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\rho_2 \geq \rho_1$. Otherwise, the roots of the above equation have the form $\omega = \omega_r(|\boldsymbol{\xi}|) \pm \mathrm{i}\omega_i(|\boldsymbol{\xi}|)$ satisfying $\omega_i(|\boldsymbol{\xi}|) \to +\infty$ as $|\boldsymbol{\xi}| \to +\infty$, which leads to an instability of the interface. The instabilities in the case $\rho_2 > \rho_1$ and $\boldsymbol{u}_1 \neq \boldsymbol{u}_2$ and in the case $\rho_2 < \rho_1$ and $\boldsymbol{u}_1 = \boldsymbol{u}_2$ are known as the Kelvin–Helmholtz and the Rayleigh–Taylor instabilities, respectively. For more details, see for example Drazin and Reid [7]. In the rest of this paper, we are interested in the situation where

$$
(\rho_2 - \rho_1) g > 0,
$$

that is, the denser water is below the lighter water. In the case $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \mathbf{0}$, the linear dispersion relation is written simply as

$$
\omega^2 = \frac{(\rho_2 - \rho_1) g |\boldsymbol{\xi}|}{\rho_1 \coth(h_1|\boldsymbol{\xi}|) + \rho_2 \coth(h_2|\boldsymbol{\xi}|)}.
$$

We denote the right-hand side by $\omega_{\mathrm{IW}}(\boldsymbol{\xi})^2$. Then the phase speed $c_{\mathrm{IW}}(\boldsymbol{\xi})$ of the plane wave solution related to the wave vector $\boldsymbol{\xi}$ is given by

$$
c_{\mathrm{IW}}(\boldsymbol{\xi}) = \frac{\omega_{\mathrm{IW}}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \pm \sqrt{\frac{(\rho_2 - \rho_1) g}{\rho_1 |\boldsymbol{\xi}| \coth(h_1|\boldsymbol{\xi}|) + \rho_2 |\boldsymbol{\xi}| \coth(h_2|\boldsymbol{\xi}|)}}. \tag{1.5}
$$

As a shallow water limit $h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \to 0$, we have

$$
c_{\mathrm{IW}}(\boldsymbol{\xi}) \simeq c_{\mathrm{SW}} = \pm \sqrt{\frac{(\rho_2 - \rho_1) g h_1 h_2}{\rho_1 h_2 + \rho_2 h_1}}, \tag{1.6}
$$

where c_{SW} is the phase speed of infinitely long and small interfacial gravity waves. In Section 3 we will analyze the linear dispersion relation of the Kakinuma model and calculate the phase speed $c_{\mathrm{K}}(\boldsymbol{\xi})$ of the plane wave solution related to the wave vector $\boldsymbol{\xi}$. Under the choice $N^* = N$ and $p_i = 2i$, or $N^* = 2N$ and $p_i = i$ in the approximation (1.4) of the velocity potentials, it turns out that

$$|c_{\mathrm{IW}}(\boldsymbol{\xi})^2 - c_{\mathrm{K}}(\boldsymbol{\xi})^2| \lesssim (h_1|\boldsymbol{\xi}| + h_2|\boldsymbol{\xi}|)^{4N+2}, \tag{1.7}$$

which indicates that the Kakinuma model may be a good approximation of the full model for interfacial gravity waves in the shallow water regime $h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \ll 1$. We note that the Miyata–Choi–Camassa model derived by Miyata [26] and Choi and Camassa [4] is a model for interfacial gravity waves in the strongly nonlinear regime and can be regarded as a generalization of the Green–Naghdi equations for water waves into a two-layer system. Let $c_{\mathrm{MCC}}(\boldsymbol{\xi})$ be the phase speed of the plane wave solution related to the wave vector $\boldsymbol{\xi}$ for the linearized equations of the Miyata–Choi–Camassa model around the rest state. Then we have

$$|c_{\mathrm{IW}}(\boldsymbol{\xi})^2 - c_{\mathrm{MCC}}(\boldsymbol{\xi})^2| \lesssim (h_1|\boldsymbol{\xi}| + h_2|\boldsymbol{\xi}|)^4,$$

so that the Kakinuma model gives a better approximation of the full model than the Miyata–Choi–Camassa model in the shallow water regime, at least, at the linear level. A rigorous analysis for the consistency of the Kakinuma model in the shallow water regime will be analyzed in the subsequent paper Duchêne and Iguchi [9]. On the other hand, in the deep water limit we have

$$\lim_{h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \to \infty} c_{\mathrm{K}}(\boldsymbol{\xi})^2 > 0,$$

which is not consistent with the limit of the full model

$$\lim_{h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \to \infty} c_{\mathrm{IW}}(\boldsymbol{\xi})^2 = 0.$$

We notice that the Miyata–Choi–Camassa model is only apparently consistent with the full model in this deep water limit since

$$\lim_{h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \to \infty} c_{\mathrm{MCC}}(\boldsymbol{\xi})^2 = 0,$$

but we note also that

$$\lim_{h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \to \infty} \frac{c_{\mathrm{IW}}(\boldsymbol{\xi})^2}{c_{\mathrm{MCC}}(\boldsymbol{\xi})^2} = \infty.$$

We refer to Duchêne, Israwi, and Talhouk [10] for further discussion and the derivation of modified Miyata–Choi–Camassa models having either the same dispersion relation as the full model, or the same behavior as the Kakinuma model in the deep water limit. As we discuss below, thanks to the high-frequency behavior of the linearized equations, and contrarily to both the full model and the Miyata–Choi–Camassa model, the Kakinuma model has a nontrivial stability domain and, as a result, the initial value problem to the Kakinuma model is well posed locally in time in Sobolev spaces under appropriate assumptions on the initial data.

As we have already seen, the roots $\omega \in \mathbf{C}$ of the dispersion relation of the linearized equations of the full model around the rest state are always real, so that the corresponding initial value problem is well posed. However, as for the nonlinear problem, even if the initial velocity is continuous on the interface, a discontinuity of the velocity in the tangential direction on the interface would be created instantaneously in general, so that the Kelvin–Helmholtz instability appears locally in space. As a result, the initial value problem for the full model turns out to be ill posed. For more details, we refer to Iguchi, Tanaka, and Tani [14]. See also Kamotski and Lebeau [20] and Lannes [22]. In Section 4 we consider the linearized equations of the Kakinuma model around an arbitrary flow. After freezing the coefficients and neglecting lower-order terms of the linearized equations, we calculate the linear dispersion relation and derive a stability condition, which is equivalent to

$$-\partial_z(P_2^{\mathrm{app}} - P_1^{\mathrm{app}}) - \frac{\rho_1\rho_2}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}|\nabla\Phi_2^{\mathrm{app}} - \nabla\Phi_1^{\mathrm{app}}|^2 \geq c_0 > 0 \tag{1.8}$$

on the interface, where P_1^{app} and P_2^{app} are approximate pressures of the waters in the upper and the lower layers in the Kakinuma model calculated from Bernoulli's equations, H_1 and H_2 are the thicknesses of the upper and the lower layers, respectively, α_1 is a constant depending only on N, α_2 is a constant determined from $\{p_0, p_1, \dots, p_{N^*}\}$, and $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})^{\mathrm{T}}$ is the nabla with respect to the horizontal spatial coordinates $\boldsymbol{x} = (x_1, \dots, x_n)$. If $\rho_1 = 0$, then (1.8) coincides with the stability condition for the Isobe–Kakinuma model for water waves derived by Nemoto and Iguchi [28].

As in the case of the Isobe–Kakinuma model, the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Kakinuma model, so that one needs to impose some compatibility conditions on the initial data for the existence of the solution. Under these compatibility conditions, the noncavitation assumption $H_1 \geq c_0 > 0$ and $H_2 \geq c_0 > 0$, and the stability condition (1.8), we will show in this paper that the initial value problem to the Kakinuma model is well posed locally in time in Sobolev spaces. Here, we note that the coefficients α_1 and α_2 in the stability condition (1.8) converge to 0 as $N, N^* \to \infty$, so that the domain of stability diminishes as N and N^* grow. This fact is consistent with the aforementioned properties of the full model.

Let us further comment on the significance of approximating an ill-posed system with well-posed systems. Firstly, while the initial value problem for the full model is ill posed in Sobolev spaces, analytic solutions do exist, as shown by Sulem, Sulem, Bardos, and Frisch [31] and Sulem and Sulem [30] in the case where upper and lower boundaries are absent, and we expect that the corresponding solutions to the Kakinuma model provide valid approximations. Secondly, it should be recalled that the full model itself is a simplified model that discards effects that would stabilize the flow, especially vertical mixing across the pycnocline. In [22], Lannes considered another stabilizing effect, namely interfacial tension, and showed the existence and uniqueness of solutions with finite regularity to the corresponding initial value problem over a long time in the shallow water regime. The key physical mechanism at stake is that the Kelvin–Helmholtz instability, which is responsible for ill-posedness issues, occurs at sufficiently small spatial scale, so that it is possible to regularize the equations while being almost transparent to the behavior of the flow at large spatial scale, which is of practical interest for applications. Our results demonstrate that the Kakinuma model inherently incorporates such a stabilizing effect whose strength diminishes as N and N^* grow, consistently with the expectation that the accuracy with respect to the full model increases.

As is well known, the full model for interfacial gravity waves has a conserved energy

$$\begin{aligned}\mathcal{E} = &\int\int_{\Omega_1(t)} \frac{1}{2}\rho_1\big(|\nabla\Phi_1(\boldsymbol{x},z,t)|^2 + (\partial_z\Phi_1(\boldsymbol{x},z,t))^2\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z \\ &+ \int\int_{\Omega_2(t)} \frac{1}{2}\rho_2\big(|\nabla\Phi_2(\boldsymbol{x},z,t)|^2 + (\partial_z\Phi_2(\boldsymbol{x},z,t))^2\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z \\ &+ \int_{\mathbf{R}^n} \frac{1}{2}(\rho_2-\rho_1)g\zeta(\boldsymbol{x},t)^2\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{1.9}$$

where $\Omega_1(t)$ and $\Omega_2(t)$ are the upper and the lower layers, respectively. This is the total energy, that is, the sum of the kinetic energies of the waters in the upper and the lower layers and the potential energy due to gravity. Moreover, Benjamin and Bridges [1] found that the full model can be written in Hamilton's canonical form

$$\partial_t\zeta = \frac{\delta\mathcal{H}}{\delta\phi}, \quad \partial_t\phi = -\frac{\delta\mathcal{H}}{\delta\zeta},$$

where the canonical variable ϕ is defined by

$$\phi(\boldsymbol{x},t) = \rho_2\Phi_2(\boldsymbol{x},\zeta(\boldsymbol{x},t),t) - \rho_1\Phi_1(\boldsymbol{x},\zeta(\boldsymbol{x},t),t) \tag{1.10}$$

and the Hamiltonian $\mathcal{H}$ is the total energy $\mathcal{E}$ written in terms of the canonical variables (ζ,ϕ). Their result can be viewed as a generalization into interfacial gravity waves of Zakharov's Hamiltonian [32] for water waves. For mathematical treatments of the Hamiltonian for interfacial gravity waves, we refer to Craig and Groves [5] and Craig, Guyenne, and Kalisch [6]. The Kakinuma model also has a conserved energy $\mathcal{E}^{\mathrm{K}}$, which is the total energy given by (1.9) with Φ_1 and Φ_2 replaced by Φ_1^{app} and Φ_2^{app}. Moreover, we will show that the Kakinuma model enjoys a Hamiltonian structure with a Hamiltonian $\mathcal{H}^{\mathrm{K}}$ the total energy in terms of canonical variables ζ and ϕ, where ϕ is defined by (1.10) with Φ_1 and Φ_2 replaced by Φ_1^{app} and Φ_2^{app}. This fact can be viewed as a generalization to the Kakinuma model for interfacial gravity waves of a Hamiltonian structure of the Isobe–Kakinuma model for water waves given by Duchêne and Iguchi [8].

The contents of this paper are as follows. In Section 2 we begin with reviewing the full model for interfacial gravity waves and derive the Kakinuma model. Then we state one of the main results of this paper, that is, Theorem 2.1 about the well-posedness of the initial value problem to the Kakinuma model locally in time. In Section 3 we analyze the linear dispersion relation of the linearized equations of the Kakinuma model around the rest state in the case of a flat bottom and show (1.7). In Section 4 we derive the stability condition (1.8) by analyzing the linearized equations of the Kakinuma model around an arbitrary flow. In Section 5 we derive an energy estimate for the linearized equations with frozen coefficients and then transform the equations into a standard positive symmetric system by introducing an appropriate symmetrizer. In Section 6 we introduce several differential operators related to the Kakinuma model and derive elliptic estimates for these operators. In Section 7 we prove one of our main result, Theorem 2.1, by using the method of parabolic regularization of the equations. In Section 8 we prove another main result, Theorem 8.4, which ensures that the Kakinuma model enjoys a Hamiltonian structure. Finally, in Section 9 we derive conservation laws of mass, momentum, and energy for the Kakinuma model together with the corresponding flux functions.

Notation. We denote by $W^{m,p}(\mathbf{R}^n)$ the L^p Sobolev space of order m on $\mathbf{R}^n$ and $H^m = W^{m,2}(\mathbf{R}^n)$. The norm of a Banach space B is denoted by $\|\cdot\|_B$. The L^2-inner product is denoted by $(\cdot,\cdot)_{L^2}$. We put $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$, and $\partial_z = \frac{\partial}{\partial z}$. $[P, Q] = PQ - QP$ denotes the commutator and $[P; \boldsymbol{u}, \boldsymbol{v}] = P(\boldsymbol{u}\cdot\boldsymbol{v}) - (P\boldsymbol{u})\cdot\boldsymbol{v} - \boldsymbol{u}\cdot(P\boldsymbol{v})$ denotes the symmetric commutator. For a matrix A we denote by A^{T} the transpose of A. For a vector $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_N)^{\mathrm{T}}$ we denote the last N components by $\boldsymbol{\phi}' = (\phi_1, \dots, \phi_N)^{\mathrm{T}}$. We use the notational convention $\frac{0}{0} = 0$. We denote by $C(a_1, a_2, \dots)$ a positive constant depending on $a_1, a_2, \dots$. The expression $f \lesssim g$ means that there exists a nonessential positive constant C such that $f \leq Cg$ holds, and $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$ hold.

2. Kakinuma model and well-posedness

We begin with formulating mathematically the full model for interfacial gravity waves. In what follows, the upper layer, the lower layer, the interface, the rigid lid of the upper layer, and the bottom of the lower layer, at time t, are denoted by $\Omega_1(t)$, $\Omega_2(t)$, $\Gamma(t)$, Σ_t, and Σ_b, respectively. Then the motion of the waters is described by the velocity potentials Φ_1 and Φ_2 and the pressures P_1 and P_2 in the upper and the lower layers satisfying the equations of continuity

$$\Delta\Phi_1 + \partial_z^2\Phi_1 = 0 \quad \text{in } \Omega_1(t), \tag{2.1}$$

$$\Delta\Phi_2 + \partial_z^2\Phi_2 = 0 \quad \text{in } \Omega_2(t), \tag{2.2}$$

where $\Delta = \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian with respect to the horizontal spatial coordinates $\boldsymbol{x} = (x_1, \dots, x_n)$, and Bernoulli's equations

$$\rho_1\Big(\partial_t\Phi_1 + \frac{1}{2}(|\nabla\Phi_1|^2 + (\partial_z\Phi_1)^2) + gz\Big) + P_1 = 0 \quad \text{in } \Omega_1(t), \tag{2.3}$$

$$\rho_2\Big(\partial_t\Phi_2 + \frac{1}{2}(|\nabla\Phi_2|^2 + (\partial_z\Phi_2)^2) + gz\Big) + P_2 = 0 \quad \text{in } \Omega_2(t). \tag{2.4}$$

The dynamical boundary condition on the interface is given by

$$P_1 = P_2 \quad \text{on } \Gamma(t). \tag{2.5}$$

The kinematic boundary conditions on the interface, the rigid lid, and the bottom are given by

$$\partial_t \zeta + \nabla\Phi_1 \cdot \nabla\zeta - \partial_z \Phi_1 = 0 \quad \text{on } \Gamma(t), \tag{2.6}$$

$$\partial_t \zeta + \nabla\Phi_2 \cdot \nabla\zeta - \partial_z \Phi_2 = 0 \quad \text{on } \Gamma(t), \tag{2.7}$$

$$\partial_z \Phi_1 = 0 \quad \text{on } \Sigma_t, \tag{2.8}$$

$$\nabla\Phi_2 \cdot \nabla b - \partial_z \Phi_2 = 0 \quad \text{on } \Sigma_b. \tag{2.9}$$

These are the basic equations for interfacial gravity waves. We can remove the pressures P_1 and P_2 from these basic equations. In fact, it follows from Bernoulli's equations (2.3)–(2.4) and the dynamical boundary condition (2.5) that

$$\begin{aligned}
&\rho_1\Big(\partial_t \Phi_1 + \frac{1}{2}(|\nabla\Phi_1|^2 + (\partial_z\Phi_1)^2) + gz\Big) \\
&\quad - \rho_2\Big(\partial_t \Phi_2 + \frac{1}{2}(|\nabla\Phi_2|^2 + (\partial_z\Phi_2)^2) + gz\Big) = 0 \quad \text{on } \Gamma(t).
\end{aligned} \tag{2.10}$$

Then the basic equations consist of (2.1)–(2.2) and (2.6)–(2.10), and we can regard Bernoulli's equations (2.3)–(2.4) as the definition of the pressures P_1 and P_2.

In the case of surface gravity waves, as shown by Luke [23], the basic equations have a variational structure and Luke's Lagrangian density is given by the vertical integral of the pressure $P - P_{\mathrm{atm}}$ in the water region, where P_{atm} is a constant atmospheric pressure. Therefore, it is natural to expect that even in the case of interfacial gravity waves, the vertical integral of the pressure in the water regions would give a Lagrangian density $\mathscr{L}$, so that we first define $\mathscr{L}^{\mathrm{pre}}$ by

$$\mathscr{L}^{\mathrm{pre}} = \int_{\zeta(\boldsymbol{x},t)}^{h_1} P_1(\boldsymbol{x}, z, t)\,\mathrm{d}z + \int_{-h_2+b(\boldsymbol{x})}^{\zeta(\boldsymbol{x},t)} P_2(\boldsymbol{x}, z, t)\,\mathrm{d}z. \tag{2.11}$$

By Bernoulli's equations (2.3)–(2.4), this can be written in terms of the velocity potentials Φ_1, Φ_2, and the elevation of the interface ζ as

$$\begin{aligned}
\mathscr{L}^{\mathrm{pre}} = &-\rho_1 \int_{\zeta}^{h_1} \Big(\partial_t \Phi_1 + \frac{1}{2}(|\nabla\Phi_1|^2 + (\partial_z\Phi_1)^2)\Big)\,\mathrm{d}z \\
&- \rho_2 \int_{-h_2+b}^{\zeta} \Big(\partial_t \Phi_2 + \frac{1}{2}(|\nabla\Phi_2|^2 + (\partial_z\Phi_2)^2)\Big)\,\mathrm{d}z \\
&- \frac{1}{2}(\rho_2 - \rho_1) g\zeta^2 - \frac{1}{2}\rho_1 g h_1^2 + \frac{1}{2}\rho_2 g(-h_2 + b)^2.
\end{aligned}$$

The last two terms do not contribute to the calculus of variations of this Lagrangian, so that we define the Lagrangian density $\mathcal{L}(\Phi_1, \Phi_2, \zeta)$ by

$$\begin{aligned}\mathcal{L}(\Phi_1, \Phi_2, \zeta) = -\rho_1 \int_{\zeta}^{h_1} \Big(\partial_t \Phi_1 + \frac{1}{2}(|\nabla \Phi_1|^2 + (\partial_z \Phi_1)^2)\Big)\, \mathrm{d}z \\ - \rho_2 \int_{-h_2+b}^{\zeta} \Big(\partial_t \Phi_2 + \frac{1}{2}(|\nabla \Phi_2|^2 + (\partial_z \Phi_2)^2)\Big)\, \mathrm{d}z \\ - \frac{1}{2}(\rho_2 - \rho_1) g \zeta^2 \end{aligned} \tag{2.12}$$

and the action function $\mathcal{J}(\Phi_1, \Phi_2, \zeta)$ by

$$\mathcal{J}(\Phi_1, \Phi_2, \zeta) = \int_{t_0}^{t_1} \int_{\mathbf{R}^n} \mathcal{L}(\Phi_1, \Phi_2, \zeta)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t.$$

It is not difficult to check that the corresponding system of Euler–Lagrange equations is exactly the same as the basic equations (2.1)–(2.2) and (2.6)–(2.10) for interfacial gravity waves.

We proceed to derive the Kakinuma model for interfacial gravity waves. Let Φ_1^{app} and Φ_2^{app} be approximate velocity potentials defined by (1.4) and define an approximate Lagrangian density $\mathcal{L}^{\mathrm{app}}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta)$ for $\boldsymbol{\phi}_1 = (\phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,N})^{\mathrm{T}}$, $\boldsymbol{\phi}_2 = (\phi_{2,0}, \phi_{2,1}, \ldots, \phi_{2,N^*})^{\mathrm{T}}$, and ζ by

$$\mathcal{L}^{\mathrm{app}}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta) = \mathcal{L}(\Phi_1^{\mathrm{app}}, \Phi_2^{\mathrm{app}}, \zeta), \tag{2.13}$$

which can be written explicitly as

$$\begin{aligned}\mathcal{L}^{\mathrm{app}} = \rho_1 \Bigg\{ \sum_{i=0}^{N} \frac{1}{2i+1} H_1^{2i+1} \partial_t \phi_{1,i} \\ + \frac{1}{2} \sum_{i,j=0}^{N} \Big(\frac{1}{2(i+j)+1} H_1^{2(i+j)+1} \nabla \phi_{1,i} \cdot \nabla \phi_{1,j} \\ + \frac{4ij}{2(i+j)-1} H_1^{2(i+j)-1} \phi_{1,i} \phi_{1,j} \Big) \Bigg\} \\ - \rho_2 \Bigg\{ \sum_{i=0}^{N^*} \frac{1}{p_i+1} H_2^{p_i+1} \partial_t \phi_{2,i} \\ + \frac{1}{2} \sum_{i,j=0}^{N^*} \Big(\frac{1}{p_i+p_j+1} H_2^{p_i+p_j+1} \nabla \phi_{2,i} \cdot \nabla \phi_{2,j} \\ - \frac{2p_i}{p_i+p_j} H_2^{p_i+p_j} \phi_{2,i} \nabla b \cdot \nabla \phi_{2,j} \\ + \frac{p_i p_j}{p_i+p_j-1} H_2^{p_i+p_j-1} (1 + |\nabla b|^2) \phi_{2,i} \phi_{2,j} \Big) \Bigg\} \\ - \frac{1}{2}(\rho_2 - \rho_1) g \zeta^2, \end{aligned}$$

where H_1 and H_2 are thicknesses of the upper and the lower layers, that is,

$$H_1(\boldsymbol{x},t) = h_1 - \zeta(\boldsymbol{x},t), \quad H_2(\boldsymbol{x},t) = h_2 + \zeta(\boldsymbol{x},t) - b(\boldsymbol{x}).$$

The corresponding system of Euler–Lagrange equations is the Kakinuma model, which consists of the equations

$$\begin{aligned} H_1^{2i}\partial_t\zeta - \sum_{j=0}^{N}\Big\{\nabla\cdot\Big(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,j}\Big) \\ - \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\phi_{1,j}\Big\} = 0 \end{aligned} \tag{2.14}$$

for $i = 0, 1, \dots, N$,

$$\begin{aligned} H_2^{p_i}\partial_t\zeta + \sum_{j=0}^{N^*}\Big\{\nabla\cdot\Big(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,j} - \frac{p_j}{p_i+p_j}H_2^{p_i+p_j}\phi_{2,j}\nabla b\Big) \\ + \frac{p_i}{p_i+p_j}H_2^{p_i+p_j}\nabla b\cdot\nabla\phi_{2,j} \\ - \frac{p_i p_j}{p_i+p_j-1}H_2^{p_i+p_j-1}(1+|\nabla b|^2)\phi_{2,j}\Big\} = 0 \end{aligned} \tag{2.15}$$

for $i = 0, 1, \dots, N^*$, and

$$\begin{aligned} \rho_1\Big\{\sum_{j=0}^{N}H_1^{2j}\partial_t\phi_{1,j} + g\zeta + \frac{1}{2}\Big(\Big|\sum_{j=0}^{N}H_1^{2j}\nabla\phi_{1,j}\Big|^2 + \Big(\sum_{j=0}^{N}2jH_1^{2j-1}\phi_{1,j}\Big)^2\Big)\Big\} \\ - \rho_2\Big\{\sum_{j=0}^{N^*}H_2^{p_j}\partial_t\phi_{2,j} + g\zeta \\ + \frac{1}{2}\Big(\Big|\sum_{j=0}^{N^*}(H_2^{p_j}\nabla\phi_{2,j} - p_jH_2^{p_j-1}\phi_{2,j}\nabla b)\Big|^2 \\ + \Big(\sum_{j=0}^{N^*}p_jH_2^{p_j-1}\phi_{2,j}\Big)^2\Big)\Big\} = 0. \end{aligned} \tag{2.16}$$

Here and in what follows we use the notational convention $\frac{0}{0} = 0$. This system of equations is the Kakinuma model that we are going to consider in this paper. We consider the initial value problem to the Kakinuma model (2.14)–(2.16) under the initial condition

$$(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = (\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)}) \quad \text{at } t = 0. \tag{2.17}$$

For notational convenience, we decompose $\boldsymbol{\phi}_k$ as $\boldsymbol{\phi}_k = (\phi_{k,0}, \boldsymbol{\phi}_k')^{\mathrm{T}}$ for $k = 1, 2$ with $\boldsymbol{\phi}_1' = (\phi_{1,1}, \dots, \phi_{1,N})$ and $\boldsymbol{\phi}_2' = (\phi_{2,1}, \dots, \phi_{2,N^*})$. Accordingly, we decompose the initial data $\boldsymbol{\phi}_{k(0)}$ as $\boldsymbol{\phi}_{k(0)} = (\phi_{k,0(0)}, \boldsymbol{\phi}_{k(0)}')^{\mathrm{T}}$ for $k = 1, 2$.

The hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Kakinuma model (2.14)–(2.16), so that the initial value problem (2.14)–(2.17) is not solvable in general. In fact, by eliminating the time derivative $\partial_t \zeta$ from the equations, we see that if the problem has a solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$, then the solution has to satisfy the $N + N^* + 1$ relations

$$\begin{aligned} &H_1^{2i} \sum_{j=0}^{N} \nabla \cdot \Big(\frac{1}{2j+1} H_1^{2j+1} \nabla \phi_{1,j} \Big) \\ &\quad - \sum_{j=0}^{N} \Big\{ \nabla \cdot \Big(\frac{1}{2(i+j)+1} H_1^{2(i+j)+1} \nabla \phi_{1,j} \Big) \\ &\qquad - \frac{4ij}{2(i+j)-1} H_1^{2(i+j)-1} \phi_{1,j} \Big\} = 0 \end{aligned} \tag{2.18}$$

for $i = 1, 2, \ldots, N$,

$$\begin{aligned} &H_2^{p_i} \sum_{j=0}^{N^*} \nabla \cdot \Big(\frac{1}{p_j+1} H_2^{p_j+1} \nabla \phi_{2,j} - \frac{p_j}{p_j} H_2^{p_j} \phi_{2,j} \nabla b \Big) \\ &\quad - \sum_{j=0}^{N^*} \Big\{ \nabla \cdot \Big(\frac{1}{p_i+p_j+1} H_2^{p_i+p_j+1} \nabla \phi_{2,j} - \frac{p_j}{p_i+p_j} H_2^{p_i+p_j} \phi_{2,j} \nabla b \Big) \\ &\qquad + \frac{p_i}{p_i+p_j} H_2^{p_i+p_j} \nabla b \cdot \nabla \phi_{2,j} \\ &\qquad - \frac{p_i p_j}{p_i+p_j-1} H_2^{p_i+p_j-1} (1 + |\nabla b|^2) \phi_{2,j} \Big\} = 0 \end{aligned} \tag{2.19}$$

for $i = 1, 2, \ldots, N^*$, and

$$\begin{aligned} &\sum_{j=0}^{N} \nabla \cdot \Big(\frac{1}{2j+1} H_1^{2j+1} \nabla \phi_{1,j} \Big) \\ &\quad + \sum_{j=0}^{N^*} \nabla \cdot \Big(\frac{1}{p_j+1} H_2^{p_j+1} \nabla \phi_{2,j} - \frac{p_j}{p_j} H_2^{p_j} \phi_{2,j} \nabla b \Big) = 0. \end{aligned} \tag{2.20}$$

Therefore, as a necessary condition, the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b have to satisfy relations (2.18)–(2.20) for the existence of the solution. These necessary conditions will be referred to as the compatibility conditions.

The following theorem is one of our main results in this paper, which guarantees the well-posedness of the initial value problem to the Kakinuma model (2.14)–(2.17) locally in time.

Theorem 2.1. *Let g, ρ_1, ρ_2, h_1, h_2, c_0, M_0 be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a time $T > 0$ such that for any initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$*

and bottom topography b satisfying the compatibility conditions (2.18)–(2.20)*, the stability condition* (1.8)*, and*

$$\begin{cases} \|(\zeta_{(0)}, \nabla\phi_{1,0(0)}, \nabla\phi_{2,0(0)})\|_{H^m} + \|(\boldsymbol{\phi}'_{1(0)}, \boldsymbol{\phi}'_{2(0)})\|_{H^{m+1}} + \|b\|_{W^{m+2,\infty}} \le M_0, \\ h_1 - \zeta_{(0)}(\boldsymbol{x}) \ge c_0, \quad h_2 + \zeta_{(0)}(\boldsymbol{x}) - b(\boldsymbol{x}) \ge c_0 \quad \textit{for } \boldsymbol{x} \in \mathbf{R}^n, \end{cases} \tag{2.21}$$

the initial value problem (2.14)–(2.17) *has a unique solution* $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ *satisfying*

$$\begin{cases} \zeta, \nabla\phi_{1,0}, \nabla\phi_{2,0} \in C([0,T]; H^m) \cap C^1([0,T]; H^{m-1}), \\ \boldsymbol{\phi}'_1, \boldsymbol{\phi}'_2 \in C([0,T]; H^{m+1}) \cap C^1([0,T]; H^m). \end{cases}$$

Remark 2.2. The term $(\partial_z(P_2^{\mathrm{app}} - P_1^{\mathrm{app}}))|_{z=\zeta}$ in the stability condition (1.8) is explicitly given in (4.4). It includes the terms $\partial_t \boldsymbol{\phi}_k(\boldsymbol{x}, 0)$ for $k = 1, 2$. Although the hypersurface $t = 0$ is characteristic for the Kakinuma model, we can uniquely determine them in terms of the initial data and b. For details, we refer to Remark 7.1. Under the condition $(\rho_2 - \rho_1)g > 0$ and if the initial data and the bottom topography are suitably small, the stability condition (1.8) is automatically satisfied at $t = 0$.

Remark 2.3. In the case $N = N^* = 0$, that is, if we approximate the velocity potentials in the Lagrangian by functions independent of the vertical spatial variable z as $\Phi_k^{\mathrm{app}}(\boldsymbol{x}, z, t) = \phi_k(\boldsymbol{x}, t)$ for $k = 1, 2$, then the Kakinuma model (2.14)–(2.16) is reduced to the nonlinear shallow water equations

$$\begin{cases} \partial_t \zeta - \nabla \cdot ((h_1 - \zeta)\nabla\phi_1) = 0, \\ \partial_t \zeta + \nabla \cdot ((h_2 + \zeta - b)\nabla\phi_2) = 0, \\ \rho_1\Big(\partial_t\phi_1 + g\zeta + \dfrac{1}{2}|\nabla\phi_1|^2\Big) - \rho_2\Big(\partial_t\phi_2 + g\zeta + \dfrac{1}{2}|\nabla\phi_2|^2\Big) = 0. \end{cases} \tag{2.22}$$

The compatibility conditions (2.18)–(2.20) are reduced to

$$\nabla \cdot ((h_1 - \zeta)\nabla\phi_1) + \nabla \cdot ((h_2 + \zeta - b)\nabla\phi_2) = 0$$

and the stability condition (1.8) is reduced to

$$g(\rho_2 - \rho_1) - \frac{\rho_1\rho_2}{\rho_1 H_2 + \rho_2 H_1}|\nabla\phi_2 - \nabla\phi_1|^2 \ge c_0 > 0.$$

Therefore, we recover the conditions for well-posedness in Sobolev spaces of the initial value problem to the nonlinear shallow water equations (2.22) proved by Bresch and Renardy [3].

Remark 2.4. By analogy with the canonical variable (1.10) for interfacial gravity waves introduced by Benjamin and Bridges [1], we introduce a canonical variable for the Kakinuma model:

$$\phi = \rho_2 \sum_{j=0}^{N^*} H_2^{p_j}\phi_{2,j} - \rho_1 \sum_{j=0}^{N} H_1^{2j}\phi_{1,j}. \tag{2.23}$$

Given the initial data $(\zeta_{(0)}, \phi_{(0)})$ for the canonical variables (ζ, ϕ) and the bottom topography b, the compatibility conditions (2.18)–(2.20) and relation (2.23) determine the initial data $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ for the Kakinuma model, which is unique up to an additive constant of the form $(\mathcal{C}\rho_2, \mathcal{C}\rho_1)$ to $(\phi_{1,0(0)}, \phi_{2,0(0)})$. In fact, we have the following proposition, which is a simple corollary of Lemma 6.4 given in Section 6.

Proposition 2.5. *Let ρ_1, ρ_2, h_1, h_2, c_0, M_0 be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a positive constant C such that for any initial data $(\zeta_{(0)}, \phi_{(0)})$ and bottom topography b satisfying*

$$\begin{cases} \|\zeta_{(0)}\|_{H^m} + \|b\|_{W^{m,\infty}} \leq M_0, \quad \|\nabla\phi_{(0)}\|_{H^{m-1}} < \infty, \\ h_1 - \zeta_{(0)}(\boldsymbol{x}) \geq c_0, \ h_2 + \zeta_{(0)}(\boldsymbol{x}) - b(\boldsymbol{x}) \geq c_0 \quad \text{for } \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

the compatibility conditions (2.18)–(2.20) and relation (2.23) determine the initial data $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ for the Kakinuma model, uniquely up to an additive constant of the form $(\mathcal{C}\rho_2, \mathcal{C}\rho_1)$ to $(\phi_{1,0(0)}, \phi_{2,0(0)})$. Moreover, we have

$$\|(\nabla\phi_{1,0(0)}, \nabla\phi_{2,0(0)})\|_{H^{m-1}} + \|(\boldsymbol{\phi}'_{1(0)}, \boldsymbol{\phi}'_{2(0)})\|_{H^m} \leq C\|\nabla\phi_{(0)}\|_{H^{m-1}}.$$

Therefore, given the initial data $(\zeta_{(0)}, \phi_{(0)})$, we infer initial data for the Kakinuma model, which satisfy the compatibility conditions (2.18)–(2.20).

3. Linear dispersion relation

In this section we consider the linearized equations of the Kakinuma model (2.14)–(2.16) around the flow $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = (0, \mathbf{0}, \mathbf{0})$ in the case of a flat bottom. The linearized equations have the form

$$\begin{cases} \partial_t\zeta - \displaystyle\sum_{j=0}^{N}\Big(\frac{h_1^{2j+1}}{2(i+j)+1}\Delta\phi_{1,j} \\ \qquad\qquad - \dfrac{4ij}{2(i+j)-1}h_1^{2j-1}\phi_{1,j}\Big) = 0 \quad \text{for } i = 0, 1, \ldots, N, \\ \partial_t\zeta + \displaystyle\sum_{j=0}^{N^*}\Big(\frac{h_2^{p_j+1}}{p_i+p_j+1}\Delta\phi_{2,j} \\ \qquad\qquad - \dfrac{p_i p_j}{p_i+p_j-1}h_2^{p_j-1}\phi_{2,j}\Big) = 0 \quad \text{for } i = 0, 1, \ldots, N^*, \\ \rho_1\Big(\displaystyle\sum_{j=0}^{N} h_1^{2j}\partial_t\phi_{1,j} + g\zeta\Big) \\ \quad -\rho_2\Big(\displaystyle\sum_{j=0}^{N^*} h_2^{p_j}\partial_t\phi_{2,j} + g\zeta\Big) = 0. \end{cases} \tag{3.1}$$

Putting $\boldsymbol{\psi}_1 = (\phi_{1,0}, h_1^2\phi_{1,1}, \dots, h_1^{2N}\phi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\psi}_2 = (\phi_{2,0}, h_1^{p_1}\phi_{2,1}, \dots, h_1^{p_{N^*}}\phi_{2,N^*})^{\mathrm{T}}$, we can rewrite the above equations in the simple matrix form

$$
\begin{pmatrix} 0 & -\rho_1\mathbf{1}^{\mathrm{T}} & \rho_2\mathbf{1}^{\mathrm{T}} \\ h_1\mathbf{1} & O & O \\ -h_2\mathbf{1} & O & O \end{pmatrix} \partial_t \begin{pmatrix} \zeta \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{pmatrix}
+ \begin{pmatrix} (\rho_2-\rho_1)g & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & -h_1^2 A_{1,0}\Delta + A_{1,1} & O \\ \mathbf{0} & O & -h_2^2 A_{2,0}\Delta + A_{2,1} \end{pmatrix} \begin{pmatrix} \zeta \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{pmatrix} = \mathbf{0},
$$

where $\mathbf{1} = (1, \dots, 1)^{\mathrm{T}}$ and matrices $A_{k,0}$ and $A_{k,1}$ for $k = 1, 2$ are given by

$$
A_{1,0} = \Big(\frac{1}{2(i+j)+1}\Big)_{0\le i,j\le N}, \quad A_{1,1} = \Big(\frac{4ij}{2(i+j)-1}\Big)_{0\le i,j\le N},
$$
$$
A_{2,0} = \Big(\frac{1}{p_i+p_j+1}\Big)_{0\le i,j\le N^*}, \quad A_{2,1} = \Big(\frac{p_ip_j}{p_i+p_j-1}\Big)_{0\le i,j\le N^*}.
$$

Therefore, the linear dispersion relation is given by

$$
\det \begin{pmatrix} (\rho_2-\rho_1)g & \mathrm{i}\rho_1\omega\mathbf{1}^{\mathrm{T}} & -\mathrm{i}\rho_2\omega\mathbf{1}^{\mathrm{T}} \\ -\mathrm{i}h_1\omega\mathbf{1} & \mathscr{A}_1(h_1\boldsymbol{\xi}) & O \\ \mathrm{i}h_2\omega\mathbf{1} & O & \mathscr{A}_2(h_2\boldsymbol{\xi}) \end{pmatrix} = 0,
$$

where $\boldsymbol{\xi} \in \mathbf{R}^n$ is the wave vector, $\omega \in \mathbf{C}$ is the angular frequency, and $\mathscr{A}_k(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 A_{k,0} + A_{k,1}$ for $k = 1, 2$. We can expand this dispersion relation as

$$
\begin{aligned}
&\big(\rho_1h_1 \det\widetilde{\mathscr{A}}_1(h_1\boldsymbol{\xi}) \det\mathscr{A}_2(h_2\boldsymbol{\xi}) + \rho_2h_2 \det\widetilde{\mathscr{A}}_2(h_2\boldsymbol{\xi}) \det\mathscr{A}_1(h_1\boldsymbol{\xi})\big)\omega^2 \\
&\qquad - (\rho_2-\rho_1)g \det\mathscr{A}_1(h_1\boldsymbol{\xi}) \det\mathscr{A}_2(h_2\boldsymbol{\xi}) = 0.
\end{aligned} \tag{3.2}
$$

Here and in what follows, we use the notation

$$
\widetilde{\mathscr{A}} = \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & \mathscr{A} \end{pmatrix}
$$

for a matrix $\mathscr{A}$. Concerning the determinants appearing in the above dispersion relation, we have the following proposition, which was proved by Nemoto and Iguchi [28].

Proposition 3.1. (1) *For any $\boldsymbol{\xi} \in \mathbf{R}^n \setminus \{\mathbf{0}\}$, the symmetric matrices $\mathscr{A}_1(\boldsymbol{\xi})$ and $\mathscr{A}_2(\boldsymbol{\xi})$ are positive.*

(2) *There exists $c_0 > 0$ such that for any $\boldsymbol{\xi} \in \mathbf{R}^n$ we have $\det\widetilde{\mathscr{A}}_k(\boldsymbol{\xi}) \ge c_0$ for $k = 1, 2$.*

(3) *$|\boldsymbol{\xi}|^{-2}\det\mathscr{A}_1(\boldsymbol{\xi})$ and $|\boldsymbol{\xi}|^{-2}\det\mathscr{A}_2(\boldsymbol{\xi})$ are polynomials in $|\boldsymbol{\xi}|^2$ of degree N and N^* and their leading coefficients are $\det A_{1,0}$ and $\det A_{2,0}$, respectively.*

(4) *$\det\widetilde{\mathscr{A}}_1(\boldsymbol{\xi})$ and $\det\widetilde{\mathscr{A}}_2(\boldsymbol{\xi})$ are polynomials in $|\boldsymbol{\xi}|^2$ of degree N and N^* and their leading coefficients are $\det\widetilde{A}_{1,0}$ and $\det\widetilde{A}_{2,0}$, respectively.*

Thanks to this proposition and the dispersion relation (3.2), the linearized system (3.1) is classified into the dispersive system in the case $N + N^* > 0$, so that the Kakinuma model (2.14)–(2.16) is a nonlinear dispersive system of equations.

Therefore, we can define the phase speed $c_{\mathrm{K}}(\boldsymbol{\xi})$ of the plane wave solution to (3.1) related to the wave vector $\boldsymbol{\xi} \in \mathbf{R}^n$ by

$$c_{\mathrm{K}}(\boldsymbol{\xi})^2 = \frac{(\rho_2 - \rho_1) g |\boldsymbol{\xi}|^{-2} \det \mathscr{A}_1(h_1\boldsymbol{\xi}) \det \mathscr{A}_2(h_2\boldsymbol{\xi})}{\rho_1 h_1 \det \widetilde{\mathscr{A}}_1(h_1\boldsymbol{\xi}) \det \mathscr{A}_2(h_2\boldsymbol{\xi}) + \rho_2 h_2 \det \widetilde{\mathscr{A}}_2(h_2\boldsymbol{\xi}) \det \mathscr{A}_1(h_1\boldsymbol{\xi})}. \tag{3.3}$$

It follows from Proposition 3.1 that

$$\lim_{h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \to \infty} c_{\mathrm{K}}(\boldsymbol{\xi})^2 = \frac{(\rho_2 - \rho_1) g h_1 h_2 \det A_{1,0} \det A_{2,0}}{\rho_1 h_2 \det \widetilde{A}_{1,0} \det A_{2,0} + \rho_2 h_1 \det \widetilde{A}_{2,0} \det A_{1,0}} > 0,$$

which is not consistent with the linear interfacial gravity waves

$$\lim_{h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \to \infty} c_{\mathrm{IW}}(\boldsymbol{\xi})^2 = 0.$$

However, as shown by the following theorems, the Kakinuma model gives a very precise approximation in the shallow water regime $h_1|\boldsymbol{\xi}|, h_2|\boldsymbol{\xi}| \ll 1$ under an appropriate choice of the indices p_i for $i = 0, 1, \ldots, N^*$.

Theorem 3.2. *If we choose $N^* = N$ and $p_i = 2i$ for $i = 0, 1, \ldots, N^*$ or $N^* = 2N$ and $p_i = i$ for $i = 0, 1, \ldots, N^*$, then for any $\boldsymbol{\xi} \in \mathbf{R}^n$ and any $h_1, h_2, g > 0$ we have*

$$\left| \left(\frac{c_{\mathrm{IW}}(\boldsymbol{\xi})}{c_{\mathrm{SW}}} \right)^2 - \left(\frac{c_{\mathrm{K}}(\boldsymbol{\xi})}{c_{\mathrm{SW}}} \right)^2 \right| \le C (h_1|\boldsymbol{\xi}| + h_2|\boldsymbol{\xi}|)^{4N+2},$$

where C is a positive constant depending only on N.

Proof. The phase speeds $c_{\mathrm{IW}}(\boldsymbol{\xi})$ and $c_{\mathrm{K}}(\boldsymbol{\xi})$ can be written in the form

$$\left(\frac{c_{\mathrm{IW}}(\boldsymbol{\xi})}{c_{\mathrm{SW}}} \right)^2 = \frac{\dfrac{\tanh(h_1|\boldsymbol{\xi}|)}{h_1|\boldsymbol{\xi}|} \dfrac{\tanh(h_2|\boldsymbol{\xi}|)}{h_2|\boldsymbol{\xi}|}}{\theta \dfrac{\tanh(h_1|\boldsymbol{\xi}|)}{h_1|\boldsymbol{\xi}|} + (1-\theta) \dfrac{\tanh(h_2|\boldsymbol{\xi}|)}{h_2|\boldsymbol{\xi}|}}$$

and

$$\left(\frac{c_{\mathrm{K}}(\boldsymbol{\xi})}{c_{\mathrm{SW}}} \right)^2 = \frac{\dfrac{\det \mathscr{A}_1(h_1\boldsymbol{\xi})}{(h_1|\boldsymbol{\xi}|)^2 \det \widetilde{\mathscr{A}}_1(h_1\boldsymbol{\xi})} \dfrac{\det \mathscr{A}_2(h_2\boldsymbol{\xi})}{(h_2|\boldsymbol{\xi}|)^2 \det \widetilde{\mathscr{A}}_2(h_2\boldsymbol{\xi})}}{\theta \dfrac{\det \mathscr{A}_1(h_1\boldsymbol{\xi})}{(h_1|\boldsymbol{\xi}|)^2 \det \widetilde{\mathscr{A}}_1(h_1\boldsymbol{\xi})} + (1-\theta) \dfrac{\det \mathscr{A}_2(h_2\boldsymbol{\xi})}{(h_2|\boldsymbol{\xi}|)^2 \det \widetilde{\mathscr{A}}_2(h_2\boldsymbol{\xi})}},$$

respectively, where $\theta = \frac{\rho_2 h_1}{\rho_2 h_1 + \rho_1 h_2} \in (0, 1)$. It has been shown by Nemoto and Iguchi [28] that

$$\left| \frac{\tanh|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|} - \frac{\det \mathscr{A}_k(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2 \det \widetilde{\mathscr{A}}_k(\boldsymbol{\xi})} \right| \le C |\boldsymbol{\xi}|^{4N+2}$$

for $k = 1, 2$, so that we obtain the desired inequality. ∎

4. Stability condition

In this section we will derive the stability condition (1.8) by analyzing a system of linearized equations to the Kakinuma model (2.14)–(2.16). We linearize the Kakinuma model around an arbitrary flow $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ and denote the variation by $(\dot{\zeta}, \dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2)$. After neglecting lower-order terms, the linearized equations have the form

$$\begin{cases} \partial_t \dot{\zeta} + \boldsymbol{u}_1 \cdot \nabla \dot{\zeta} - \displaystyle\sum_{j=0}^{N} \frac{1}{2(i+j)+1} H_1^{2j+1} \Delta \dot{\phi}_{1,j} = 0 \quad \text{for } i = 0, 1, \ldots, N, \\ \partial_t \dot{\zeta} + \boldsymbol{u}_2 \cdot \nabla \dot{\zeta} + \displaystyle\sum_{j=0}^{N^*} \frac{1}{p_i + p_j + 1} H_2^{p_j+1} \Delta \dot{\phi}_{2,j} = 0 \quad \text{for } i = 0, 1, \ldots, N^*, \\ \rho_1 \displaystyle\sum_{j=0}^{N} H_1^{2j} (\partial_t \dot{\phi}_{1,j} + \boldsymbol{u}_1 \cdot \nabla \dot{\phi}_{1,j}) \\ \qquad - \rho_2 \displaystyle\sum_{j=0}^{N^*} H_2^{p_j} (\partial_t \dot{\phi}_{2,j} + \boldsymbol{u}_2 \cdot \nabla \dot{\phi}_{2,j}) - a \dot{\zeta} = 0, \end{cases} \tag{4.1}$$

where $H_1 = h_1 - \zeta$ and $H_2 = h_2 + \zeta - b$ are the thicknesses of the layers,

$$\begin{cases} \boldsymbol{u}_1 = (\nabla \Phi_1^{\mathrm{app}})|_{z=\zeta} = \displaystyle\sum_{j=0}^{N} H_1^{2j} \nabla \phi_{1,j}, \\ \boldsymbol{u}_2 = (\nabla \Phi_2^{\mathrm{app}})|_{z=\zeta} = \displaystyle\sum_{j=0}^{N^*} (H_2^{p_j} \nabla \phi_{2,j} - p_j H_2^{p_j-1} \phi_{2,j} \nabla b) \end{cases} \tag{4.2}$$

are approximate horizontal velocities in the upper and the lower layers at the interface,

$$\begin{cases} w_1 = (\partial_z \Phi_1^{\mathrm{app}})|_{z=\zeta} = -\displaystyle\sum_{j=0}^{N} 2j H_1^{2j-1} \phi_{1,j}, \\ w_2 = (\partial_z \Phi_2^{\mathrm{app}})|_{z=\zeta} = \displaystyle\sum_{j=0}^{N^*} p_j H_2^{p_j-1} \phi_{2,j} \end{cases} \tag{4.3}$$

are approximate vertical velocities in the upper and the lower layers at the interface, and

$$\begin{aligned} a &= \rho_2 \Biggl(\sum_{j=0}^{N^*} p_j H_2^{p_j-1} (\partial_t \phi_{2,j} + \boldsymbol{u}_2 \cdot \nabla \phi_{2,j}) \\ &\qquad + (w_2 - \boldsymbol{u}_2 \cdot \nabla b) \sum_{j=0}^{N^*} p_j (p_j - 1) H_2^{p_j-2} \phi_{2,j} + g \Biggr) \\ &\quad + \rho_1 \Biggl(\sum_{j=0}^{N} 2j H_1^{2j-1} (\partial_t \phi_{1,j} + \boldsymbol{u}_1 \cdot \nabla \phi_{1,j}) - w_1 \sum_{j=0}^{N} 2j(2j-1) H^{2(j-1)} \phi_{1,j} - g \Biggr) \\ &= -(\partial_z (P_2^{\mathrm{app}} - P_1^{\mathrm{app}}))|_{z=\zeta}. \end{aligned} \tag{4.4}$$

Here, $P_1^{\rm app}$ and $P_2^{\rm app}$ are approximate pressures in the upper and the lower layers calculated from Bernoulli's equations (2.3)–(2.4), that is,

$$P_k^{\rm app} = -\rho_k\Big(\partial_t\Phi_k^{\rm app} + \frac{1}{2}(|\nabla\Phi_k^{\rm app}|^2 + (\partial_z\Phi_k^{\rm app})^2) + gz\Big)$$

for $k = 1, 2$. Now we freeze the coefficients in the linearized equations (4.1) and put

$$\begin{cases} \dot{\boldsymbol{\psi}}_1 = (\dot\phi_{1,0}, H_1^2\dot\phi_{1,1}, \dots, H_1^{2N}\dot\phi_{1,N})^{\rm T}, \\ \dot{\boldsymbol{\psi}}_2 = (\dot\phi_{2,0}, H_2^{p_1}\dot\phi_{2,1}, \dots, H_2^{p_{N^*}}\dot\phi_{2,N^*})^{\rm T}. \end{cases} \tag{4.5}$$

Then (4.1) can be written in the form

$$\begin{pmatrix} 0 & -\rho_1\mathbf{1}^{\rm T} & \rho_2\mathbf{1}^{\rm T} \\ H_1\mathbf{1} & O & O \\ -H_2\mathbf{1} & O & O \end{pmatrix} \partial_t \begin{pmatrix} \dot\zeta \\ \dot{\boldsymbol{\psi}}_1 \\ \dot{\boldsymbol{\psi}}_2 \end{pmatrix}$$
$$+ \begin{pmatrix} a & -\rho_1\mathbf{1}^{\rm T}(\boldsymbol{u}_1\cdot\nabla) & \rho_2\mathbf{1}^{\rm T}(\boldsymbol{u}_2\cdot\nabla) \\ H_1\mathbf{1}(\boldsymbol{u}_1\cdot\nabla) & -H_1^2A_{1,0}\Delta & O \\ -H_2\mathbf{1}(\boldsymbol{u}_2\cdot\nabla) & O & -H_2^2A_{2,0}\Delta \end{pmatrix} \begin{pmatrix} \dot\zeta \\ \dot{\boldsymbol{\psi}}_1 \\ \dot{\boldsymbol{\psi}}_2 \end{pmatrix} = \mathbf{0}.$$

Therefore, the linear dispersion relation for (4.1) is given by

$$\det\begin{pmatrix} a & {\rm i}\rho_1(\omega-\boldsymbol{u}_1\cdot\boldsymbol{\xi})\mathbf{1}^{\rm T} & -{\rm i}\rho_2(\omega-\boldsymbol{u}_2\cdot\boldsymbol{\xi})\mathbf{1}^{\rm T} \\ -{\rm i}H_1(\omega-\boldsymbol{u}_1\cdot\boldsymbol{\xi})\mathbf{1} & (H_1|\boldsymbol{\xi}|)^2A_{1,0} & O \\ {\rm i}H_2(\omega-\boldsymbol{u}_2\cdot\boldsymbol{\xi})\mathbf{1} & O & (H_2|\boldsymbol{\xi}|)^2A_{2,0} \end{pmatrix} = 0,$$

where $\boldsymbol{\xi}\in\mathbf{R}^n$ is the wave vector and $\omega\in\mathbf{C}$ the angular frequency. The left-hand side can be expanded as

$$\begin{aligned}
{\rm LHS} &= \det\begin{pmatrix} a & {\rm i}\rho_1(\omega-\boldsymbol{u}_1\cdot\boldsymbol{\xi})\mathbf{1}^{\rm T} & -{\rm i}\rho_2(\omega-\boldsymbol{u}_2\cdot\boldsymbol{\xi})\mathbf{1}^{\rm T} \\ \mathbf{0} & (H_1|\boldsymbol{\xi}|)^2A_{1,0} & O \\ \mathbf{0} & O & (H_2|\boldsymbol{\xi}|)^2A_{2,0} \end{pmatrix} \\
&\quad + \det\begin{pmatrix} 0 & {\rm i}\rho_1(\omega-\boldsymbol{u}_1\cdot\boldsymbol{\xi})\mathbf{1}^{\rm T} & -{\rm i}\rho_2(\omega-\boldsymbol{u}_2\cdot\boldsymbol{\xi})\mathbf{1}^{\rm T} \\ -{\rm i}H_1(\omega-\boldsymbol{u}_1\cdot\boldsymbol{\xi})\mathbf{1} & (H_1|\boldsymbol{\xi}|)^2A_{1,0} & O \\ {\rm i}H_2(\omega-\boldsymbol{u}_2\cdot\boldsymbol{\xi})\mathbf{1} & O & (H_2|\boldsymbol{\xi}|)^2A_{2,0} \end{pmatrix} \\
&= a\det((H_1|\boldsymbol{\xi}|)^2A_{1,0})\det((H_2|\boldsymbol{\xi}|)^2A_{2,0}) \\
&\quad + \det\begin{pmatrix} 0 & {\rm i}\rho_1(\omega-\boldsymbol{u}_1\cdot\boldsymbol{\xi})\mathbf{1}^{\rm T} \\ -{\rm i}H_1(\omega-\boldsymbol{u}_1\cdot\boldsymbol{\xi})\mathbf{1} & (H_1|\boldsymbol{\xi}|)^2A_{1,0} \end{pmatrix}\det((H_2|\boldsymbol{\xi}|)^2A_{2,0}) \\
&\quad + \det\begin{pmatrix} 0 & -{\rm i}\rho_2(\omega-\boldsymbol{u}_2\cdot\xi)\mathbf{1}^{\rm T} \\ {\rm i}H_2(\omega-\boldsymbol{u}_2\cdot\boldsymbol{\xi})\mathbf{1} & (H_2|\boldsymbol{\xi}|)^2A_{2,0} \end{pmatrix}\det\Big((H_1|\boldsymbol{\xi}|)^2A_{1,0}\Big) \\
&= H_1^{2N+1}H_2^{2N^*+1}|\boldsymbol{\xi}|^{2(N+N^*+1)}\{aH_1H_2|\boldsymbol{\xi}|^2\det A_{1,0}\det A_{2,0} \\
&\qquad - \rho_1H_2(\omega-\boldsymbol{u}_1\cdot\boldsymbol{\xi})^2\det\widetilde{A}_{1,0}\det A_{2,0} \\
&\qquad - \rho_2H_1(\omega-\boldsymbol{u}_2\cdot\boldsymbol{\xi})^2\det\widetilde{A}_{2,0}\det A_{1,0}\},
\end{aligned}$$

so that the linear dispersion relation is given simply as

$$\frac{\rho_1}{H_1\alpha_1}(\omega - \boldsymbol{u}_1\cdot\boldsymbol{\xi})^2 + \frac{\rho_2}{H_2\alpha_2}(\omega - \boldsymbol{u}_2\cdot\boldsymbol{\xi})^2 - a|\boldsymbol{\xi}|^2 = 0, \tag{4.6}$$

where

$$\alpha_k = \frac{\det A_{k,0}}{\det \widetilde{A}_{k,0}}, \quad \widetilde{A}_{k,0} = \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A_{k,0} \end{pmatrix} \tag{4.7}$$

for $k = 1, 2$. The discriminant of this quadratic equation in ω is

$$\begin{aligned}
&\Big(\frac{\rho_1}{H_1\alpha_1}\boldsymbol{u}_1\cdot\boldsymbol{\xi} + \frac{\rho_2}{H_2\alpha_2}\boldsymbol{u}_2\cdot\boldsymbol{\xi}\Big)^2 \\
&\qquad - \Big(\frac{\rho_1}{H_1\alpha_1} + \frac{\rho_2}{H_2\alpha_2}\Big)\Big(\frac{\rho_1}{H_1\alpha_1}(\boldsymbol{u}_1\cdot\boldsymbol{\xi})^2 + \frac{\rho_2}{H_2\alpha_2}(\boldsymbol{u}_2\cdot\boldsymbol{\xi})^2 - a|\boldsymbol{\xi}|^2\Big) \\
&\quad = \Big(\frac{\rho_1}{H_1\alpha_1} + \frac{\rho_2}{H_2\alpha_2}\Big)\Big(a|\boldsymbol{\xi}|^2 - \frac{\rho_1\rho_2}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}\big((\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot\boldsymbol{\xi}\big)^2\Big).
\end{aligned}$$

Therefore, the solutions ω to the dispersion relation (4.6) are real for any wave vector $\boldsymbol{\xi} \in \mathbf{R}^n$ if and only if

$$a - \frac{\rho_1\rho_2}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}|\boldsymbol{u}_2 - \boldsymbol{u}_1|^2 \geq 0.$$

Otherwise, the roots of the linear dispersion relation (4.6) have the form $\omega = \omega_r(\boldsymbol{\xi}) \pm \mathrm{i}\omega_i(\boldsymbol{\xi})$ satisfying $\omega_i(\boldsymbol{\xi}) \to +\infty$ as $\boldsymbol{\xi} = (\boldsymbol{u}_2 - \boldsymbol{u}_1)\xi$ and $\xi \to +\infty$, which leads to an instability of the problem. These considerations leads us to the stability condition

$$a - \frac{\rho_1\rho_2}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}|\boldsymbol{u}_2 - \boldsymbol{u}_1|^2 \geq c_0 > 0, \tag{4.8}$$

which is equivalent to

$$-(\partial_z(P_2^{\mathrm{app}} - P_1^{\mathrm{app}}))|_{z=\zeta} - \frac{\rho_1\rho_2}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}(|\nabla\Phi_2^{\mathrm{app}} - \nabla\Phi_1^{\mathrm{app}}|^2)|_{z=\zeta} \geq c_0.$$

Here, we note that α_1 and α_2 are positive constants depending only on N and $\{p_0, p_1, \dots, p_{N^*}\}$ and converge to 0 as $N, N^* \to \infty$. Therefore, as N and N^* go to infinity the domain of stability diminishes.

5. Analysis of the linearized system

In this section we still analyze the system of linearized equations (4.1) with frozen coefficients. We first derive an energy estimate for solutions to the linearized system by defining a suitable energy function, and then transform the linearized system into a standard symmetric form, for which the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is noncharacteristic. These results motivate the subsequent analysis on the nonlinear equations.

5.1. Energy estimate

With the notation (4.5), the linearized system (4.1) with frozen coefficients can be written in a symmetric form as

$$\mathcal{A}_1 \partial_t \dot{\boldsymbol{U}} + \mathcal{A}_0 \dot{\boldsymbol{U}} = \boldsymbol{0}, \tag{5.1}$$

where $\dot{\boldsymbol{U}} = (\dot{\zeta}, \dot{\boldsymbol{\psi}}_1, \dot{\boldsymbol{\psi}}_2)^{\mathrm{T}}$ and

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -\rho_1 \mathbf{1}^{\mathrm{T}} & \rho_2 \mathbf{1}^{\mathrm{T}} \\ \rho_1 \mathbf{1} & O & O \\ -\rho_2 \mathbf{1} & O & O \end{pmatrix},$$

$$\mathcal{A}_0 = \begin{pmatrix} a & -\rho_1 \mathbf{1}^{\mathrm{T}}(\boldsymbol{u}_1 \cdot \nabla) & \rho_2 \mathbf{1}^{\mathrm{T}}(\boldsymbol{u}_2 \cdot \nabla) \\ \rho_1 \mathbf{1}(\boldsymbol{u}_1 \cdot \nabla) & -\rho_1 H_1 A_{1,0} \Delta & O \\ -\rho_2 \mathbf{1}(\boldsymbol{u}_2 \cdot \nabla) & O & -\rho_2 H_2 A_{2,0} \Delta \end{pmatrix}.$$

We note that $\mathcal{A}_0$ is symmetric in $L^2(\mathbf{R}^n)$ whereas $\mathcal{A}_1$ is skew-symmetric. Therefore, by taking L^2-inner product of (5.1) with $\partial_t \dot{\boldsymbol{U}}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{U}}, \mathcal{A}_0 \dot{\boldsymbol{U}})_{L^2} = 0$$

for any regular solution $\dot{\boldsymbol{U}}$ to (5.1), so that $(\dot{\boldsymbol{U}}, \mathcal{A}_0 \dot{\boldsymbol{U}})_{L^2}$ would give a mathematical energy function to the linearized system (5.1) if we show the positivity of the symmetric operator $\mathcal{A}_0$ in $L^2(\mathbf{R}^n)$. We proceed to check the positivity. For simplicity, we consider first the case $N = N^* = 0$ so that $A_{1,0} = A_{2,0} = 1$. Then we see that

$$(\dot{\boldsymbol{U}}, \mathcal{A}_0 \dot{\boldsymbol{U}})_{L^2} = \int_{\mathbf{R}^n} \begin{pmatrix} \dot{\zeta} \\ \nabla \dot{\phi}_{1,0} \\ \nabla \dot{\phi}_{2,0} \end{pmatrix} \cdot \begin{pmatrix} a & -\rho_1 \boldsymbol{u}_1^{\mathrm{T}} & \rho_2 \boldsymbol{u}_2^{\mathrm{T}} \\ -\rho_1 \boldsymbol{u}_1 & \rho_1 H_1 \mathrm{Id} & O \\ \rho_2 \boldsymbol{u}_2 & O & \rho_2 H_2 \mathrm{Id} \end{pmatrix} \begin{pmatrix} \dot{\zeta} \\ \nabla \dot{\phi}_{1,0} \\ \nabla \dot{\phi}_{2,0} \end{pmatrix} \mathrm{d}\boldsymbol{x}.$$

Therefore, it is sufficient to analyze the positivity of this $(2n+1) \times (2n+1)$ matrix. The characteristic polynomial of this matrix is given by

$$\begin{aligned} 0 &= \det \begin{pmatrix} \lambda - a & \rho_1 \boldsymbol{u}_1^{\mathrm{T}} & -\rho_2 \boldsymbol{u}_2^{\mathrm{T}} \\ \rho_1 \boldsymbol{u}_1 & (\lambda - \rho_1 H_1)\mathrm{Id} & O \\ -\rho_2 \boldsymbol{u}_2 & O & (\lambda - \rho_2 H_2)\mathrm{Id} \end{pmatrix} \\ &= (\lambda - a)(\lambda - \rho_1 H_1)^n (\lambda - \rho_2 H_2)^n \\ &\quad - \rho_1^2 |\boldsymbol{u}_1|^2 (\lambda - \rho_1 H_1)^{n-1} (\lambda - \rho_2 H_2)^n - \rho_2^2 |\boldsymbol{u}_2|^2 (\lambda - \rho_1 H_1)^n (\lambda - \rho_2 H_2)^{n-1} \\ &= (\lambda - \rho_1 H_1)^{n-1} (\lambda - \rho_2 H_2)^{n-1} \bigl\{ (\lambda - a)(\lambda - \rho_1 H_1)(\lambda - \rho_2 H_2) \\ &\qquad - \rho_1^2 |\boldsymbol{u}_1|^2 (\lambda - \rho_2 H_2) - \rho_2^2 |\boldsymbol{u}_2|^2 (\lambda - \rho_1 H_1) \bigr\}. \end{aligned}$$

Therefore, the eigenvalues of the matrix are $\rho_1 H_1$ and $\rho_2 H_2$ of multiplicity $n-1$ and λ_1, λ_2, λ_3, which are the roots of the polynomial

$$(\lambda - a)(\lambda - \rho_1 H_1)(\lambda - \rho_2 H_2) - \rho_1^2 |\boldsymbol{u}_1|^2 (\lambda - \rho_2 H_2) - \rho_2^2 |\boldsymbol{u}_2|^2 (\lambda - \rho_1 H_1) = 0.$$

Here, we see that

$$\lambda_1\lambda_2\lambda_3 = \rho_1\rho_2(aH_1H_2 - \rho_1H_2|\boldsymbol{u}_1|^2 - \rho_2H_1|\boldsymbol{u}_2|^2),$$

which is not necessarily positive even if $\boldsymbol{u}_1 = \boldsymbol{u}_2$. Therefore, for the positivity of the symmetric operator $\mathcal{A}_0$ we need a smallness of the horizontal velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Such a condition is, of course, a stronger restriction than the stability condition (4.8). This means that $(\dot{\boldsymbol{U}}, \mathcal{A}_0\dot{\boldsymbol{U}})_{L^2}$ is not an optimal energy function and we proceed to find another one.

We are now considering the linearized system (5.1) with frozen coefficients, that is,

$$\begin{cases} H_1\mathbf{1}(\partial_t\dot{\zeta} + \boldsymbol{u}_1\cdot\nabla\dot{\zeta}) - H_1^2A_{1,0}\Delta\dot{\boldsymbol{\psi}}_1 = \mathbf{0}, \\ H_2\mathbf{1}(\partial_t\dot{\zeta} + \boldsymbol{u}_2\cdot\nabla\dot{\zeta}) + H_2^2A_{2,0}\Delta\dot{\boldsymbol{\psi}}_2 = \mathbf{0}, \\ \rho_1\mathbf{1}\cdot(\partial_t\dot{\boldsymbol{\psi}}_1 + (\boldsymbol{u}_1\cdot\nabla)\dot{\boldsymbol{\psi}}_1) - \rho_2\mathbf{1}\cdot(\partial_t\dot{\boldsymbol{\psi}}_2 + (\boldsymbol{u}_2\cdot\nabla)\dot{\boldsymbol{\psi}}_2) - a\dot{\zeta} = 0. \end{cases} \tag{5.2}$$

Applying Δ to the last equation in (5.2) we have

$$\begin{aligned} &\rho_1(A_{1,0})^{-1}\mathbf{1}\cdot(\partial_t + \boldsymbol{u}_1\cdot\nabla)A_{1,0}\Delta\dot{\boldsymbol{\psi}}_1 \\ &\qquad - \rho_2(A_{2,0})^{-1}\mathbf{1}\cdot(\partial_t + \boldsymbol{u}_2\cdot\nabla)A_{2,0}\Delta\dot{\boldsymbol{\psi}}_2 - a\Delta\dot{\zeta} = 0. \end{aligned} \tag{5.3}$$

Plugging the first and second equations in (5.2) into (5.3) to remove $\dot{\boldsymbol{\psi}}_1$ and $\dot{\boldsymbol{\psi}}_2$, we obtain

$$\Big(\frac{\rho_1(A_{1,0})^{-1}\mathbf{1}\cdot\mathbf{1}}{H_1}(\partial_t + \boldsymbol{u}_1\cdot\nabla)^2 + \frac{\rho_2(A_{2,0})^{-1}\mathbf{1}\cdot\mathbf{1}}{H_2}(\partial_t + \boldsymbol{u}_2\cdot\nabla)^2\Big)\dot{\zeta} - a\Delta\dot{\zeta} = 0.$$

In view of the relation following from Cramer's rule

$$(A_{k,0})^{-1}\mathbf{1}\cdot\mathbf{1} = \frac{\det\widetilde{A}_{k,0}}{\det A_{k,0}} = \frac{1}{\alpha_k}$$

for $k = 1, 2$, the above equation for $\dot{\zeta}$ can be written as

$$\begin{aligned} &\Big(\frac{\rho_1}{H_1\alpha_1} + \frac{\rho_2}{H_2\alpha_2}\Big)(\partial_t + \boldsymbol{u}\cdot\nabla)^2\dot{\zeta} \\ &\qquad - \Big(a\Delta - \frac{\rho_1\rho_2}{\rho_1H_2\alpha_2 + \rho_2H_1\alpha_1}((\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot\nabla)^2\Big)\dot{\zeta} = 0, \end{aligned} \tag{5.4}$$

where $\boldsymbol{u}$ is an averaged horizontal velocity on the interface defined by

$$\boldsymbol{u} = \frac{\rho_1H_2\alpha_2}{\rho_1H_2\alpha_2 + \rho_2H_1\alpha_1}\boldsymbol{u}_1 + \frac{\rho_2H_1\alpha_1}{\rho_1H_2\alpha_2 + \rho_2H_1\alpha_1}\boldsymbol{u}_2. \tag{5.5}$$

Taking (5.4) into account, we consider the constant coefficient second-order partial differential equation

$$c_1(\partial_t + \boldsymbol{u}\cdot\nabla)^2\dot{\zeta} - (c_2\Delta - (\boldsymbol{v}\cdot\nabla)^2)\dot{\zeta} = 0, \tag{5.6}$$

where c_1 and c_2 are positive constants. By taking the L^2-inner product of (5.6) with $(\partial_t + \boldsymbol{u}\cdot\nabla)\dot{\zeta}$ and using integration by parts, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_1\|\partial_t\dot{\zeta} + \boldsymbol{u}\cdot\nabla\dot{\zeta}\|_{L^2}^2 + c_2\|\nabla\dot{\zeta}\|_{L^2}^2 - \|\boldsymbol{v}\cdot\nabla\dot{\zeta}\|_{L^2}^2) = 0$$

for any regular solution $\dot{\zeta}$ to (5.6). Here, we have

$$c_2\|\nabla\dot{\zeta}\|_{L^2}^2 - \|\boldsymbol{v}\cdot\nabla\dot{\zeta}\|_{L^2}^2 = (\nabla\dot{\zeta}, (c_2\mathrm{Id} - \boldsymbol{v}\otimes\boldsymbol{v})\nabla\dot{\zeta})_{L^2}.$$

The matrix $c_2\mathrm{Id} - \boldsymbol{v}\otimes\boldsymbol{v}$ is positive if and only if $c_2 - |\boldsymbol{v}|^2 > 0$. Under this assumption, we obtain an energy estimate for the solutions to (5.6). Applying this consideration to (5.4), we see that the positivity condition is exactly the same as the stability condition (4.8), under which we can obtain an energy estimate for (5.4).

In [3] (see also [2]), Bresch and Renardy rewrote the nonlinear shallow water equations (2.22), corresponding to the case $N = N^* = 0$, as a scalar second-order partial differential equation analogous to (5.4), and then used the abstract theory of Hughes, Kato, and Marsden [11] to obtain the local well-posedness of the initial value problem under sharp hyperbolicity conditions, as mentioned in Remark 2.3. Our strategy is different as we rely on the symmetrization of the system and parabolic regularization to prove Theorem 2.1.

In view of (5.4) and the subsequent observation we rewrite the linearized system (5.1) with frozen coefficients in the form

$$\mathcal{A}_1(\partial_t + \boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{U}} + \mathcal{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}} = \boldsymbol{0},$$

where

$$\begin{aligned}\mathcal{A}_0^{\mathrm{mod}} &= \mathcal{A}_0 - \mathcal{A}_1(\boldsymbol{u}\cdot\nabla)\\ &= \begin{pmatrix} a & \frac{\rho_1\rho_2H_1\alpha_1}{\rho_1H_2\alpha_2+\rho_2H_1\alpha_1}\boldsymbol{1}^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla) & \frac{\rho_1\rho_2H_2\alpha_2}{\rho_1H_2\alpha_2+\rho_2H_1\alpha_1}\boldsymbol{1}^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla) \\ -\frac{\rho_1\rho_2H_1\alpha_1}{\rho_1H_2\alpha_2+\rho_2H_1\alpha_1}\boldsymbol{1}(\boldsymbol{v}\cdot\nabla) & -\rho_1H_1A_{1,0}\Delta & O \\ -\frac{\rho_1\rho_2H_2\alpha_2}{\rho_1H_2\alpha_2+\rho_2H_1\alpha_1}\boldsymbol{1}(\boldsymbol{v}\cdot\nabla) & O & -\rho_2H_2A_{2,0}\Delta \end{pmatrix}\end{aligned}$$

and $\boldsymbol{v} = \boldsymbol{u}_2 - \boldsymbol{u}_1$. By taking the L^2-inner product of this equation with $(\partial_t + \boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{U}}$ and using integration by parts, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} = 0$$

for any regular solution to (5.1). We proceed to check the positivity of the symmetric operator $\mathcal{A}_0^{\mathrm{mod}}$ in $L^2(\mathbf{R}^n)$ under the stability condition (4.8). We see that

$$\begin{aligned}(\mathcal{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} &= (a\dot{\zeta}, \dot{\zeta})_{L^2} + \sum_{l=1}^{n}\{(\rho_1H_1A_{1,0}\partial_l\dot{\boldsymbol{\psi}}_1, \partial_l\dot{\boldsymbol{\psi}}_1)_{L^2} + (\rho_2H_2A_{2,0}\partial_l\dot{\boldsymbol{\psi}}_2, \partial_l\dot{\boldsymbol{\psi}}_2)_{L^2}\}\\ &\quad + 2\Big(\frac{\rho_1\rho_2H_1\alpha_1}{\rho_1H_2\alpha_2 + \rho_2H_1\alpha_1}(\boldsymbol{v}\cdot\nabla)(\boldsymbol{1}\cdot\dot{\boldsymbol{\psi}}_1), \dot{\zeta}\Big)_{L^2}\\ &\quad + 2\Big(\frac{\rho_1\rho_2H_2\alpha_2}{\rho_1H_2\alpha_2 + \rho_2H_1\alpha_1}(\boldsymbol{v}\cdot\nabla)(\boldsymbol{1}\cdot\dot{\boldsymbol{\psi}}_2), \dot{\zeta}\Big)_{L^2}.\end{aligned}$$

On the other hand, the matrix $\widetilde{A}_{k,0}$ is nonsingular and its inverse matrix can be written as

$$(\widetilde{A}_{k,0})^{-1} = \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A_{k,0} \end{pmatrix}^{-1} = \begin{pmatrix} q_{k,0} & (\boldsymbol{q}_{k,0})^{\mathrm{T}} \\ -\boldsymbol{q}_{k,0} & Q_{k,0} \end{pmatrix},$$

with a symmetric matrix $Q_{k,0}$ for $k = 1, 2$. Moreover, $q_{k,0} = \frac{\det A_{k,0}}{\det \widetilde{A}_{k,0}} = \alpha_k$ is positive and $Q_{k,0}$ is nonnegative. In fact, for any $\boldsymbol{\psi}$, putting $\binom{\zeta}{\boldsymbol{\phi}} = (\widetilde{A}_{k,0})^{-1}\binom{0}{\boldsymbol{\psi}}$, we have

$$Q_{k,0}\boldsymbol{\psi}\cdot\boldsymbol{\psi} = \begin{pmatrix} q_{k,0} & (\boldsymbol{q}_{k,0})^{\mathrm{T}} \\ -\boldsymbol{q}_{k,0} & Q_{k,0} \end{pmatrix}\begin{pmatrix} 0 \\ \boldsymbol{\psi} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \zeta \\ \boldsymbol{\phi} \end{pmatrix}\cdot\widetilde{A}_{k,0}\begin{pmatrix} \zeta \\ \boldsymbol{\phi} \end{pmatrix} = \boldsymbol{\phi}\cdot A_{k,0}\boldsymbol{\phi} \geq 0.$$

We note that $Q_{k,0}$ is not positive because it has a zero eigenvalue with an eigenvector $\mathbf{1}$. Now, for any $\boldsymbol{\phi}$, putting $\eta = \mathbf{1}\cdot\boldsymbol{\phi}$ and $\boldsymbol{\psi} = A_{k,0}\boldsymbol{\phi}$, we have $\widetilde{A}_{k,0}\binom{0}{\boldsymbol{\phi}} = \binom{\eta}{\boldsymbol{\psi}}$ so that

$$A_{k,0}\boldsymbol{\phi}\cdot\boldsymbol{\phi} = \widetilde{A}_{k,0}\begin{pmatrix} 0 \\ \boldsymbol{\phi} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} \eta \\ \boldsymbol{\psi} \end{pmatrix}\cdot(\widetilde{A}_{k,0})^{-1}\begin{pmatrix} \eta \\ \boldsymbol{\psi} \end{pmatrix} = q_{k,0}\eta^2 + Q_{k,0}\boldsymbol{\psi}\cdot\boldsymbol{\psi},$$

from which we deduce the identity

$$A_{k,0}\boldsymbol{\phi}\cdot\boldsymbol{\phi} = \alpha_k(\mathbf{1}\cdot\boldsymbol{\phi})^2 + Q_{k,0}A_{k,0}\boldsymbol{\phi}\cdot A_{k,0}\boldsymbol{\phi}. \tag{5.7}$$

By using decomposition (5.7) we see that

$$\begin{aligned}
(\mathcal{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} &= \sum_{l=1}^{n}\{(\rho_1 H_1 Q_{1,0}A_{1,0}\partial_l\dot{\boldsymbol{\psi}}_1, A_{1,0}\partial_l\dot{\boldsymbol{\psi}}_1)_{L^2} \\
&\qquad + (\rho_2 H_2 Q_{2,0}A_{2,0}\partial_l\dot{\boldsymbol{\psi}}_2, A_{2,0}\partial_l\dot{\boldsymbol{\psi}}_2)_{L^2}\} \\
&\quad + \Big\{(a\dot{\zeta}, \dot{\zeta})_{L^2} + (\rho_1 H_1\alpha_1\nabla(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_1), \nabla(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_1))_{L^2} \\
&\qquad + (\rho_2 H_2\alpha_2\nabla(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_2), \nabla(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_2))_{L^2} \\
&\qquad + \Big(\frac{2\rho_1\rho_2 H_1\alpha_1}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}(\boldsymbol{v}\cdot\nabla)(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_1), \dot{\zeta}\Big)_{L^2} \\
&\qquad + \Big(\frac{2\rho_1\rho_2 H_2\alpha_2}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}(\boldsymbol{v}\cdot\nabla)(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_2), \dot{\zeta}\Big)_{L^2}\Big\} \\
&=: I_1 + I_2.
\end{aligned}$$

Here, $I_1 \geq 0$ since $Q_{1,0}$ and $Q_{2,0}$ are nonnegative, and

$$\begin{aligned}
I_2 \geq \int_{\mathbf{R}^n}\Big\{&a\dot{\zeta}^2 + \rho_1 H_1\alpha_1|\nabla(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_1)|^2 + \rho_2 H_2\alpha_2|\nabla(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_2)|^2 \\
&- \frac{2\rho_1\rho_2|\boldsymbol{v}|}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}(H_1\alpha_1|\nabla(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_1)| + H_2\alpha_2|\nabla(\mathbf{1}\cdot\dot{\boldsymbol{\psi}}_2)|)|\dot{\zeta}|\Big\}\,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

so that it is sufficient to show the positivity of the matrix

$$\mathfrak{A}_0 := \begin{pmatrix} a & -\frac{\rho_1\rho_2 H_1\alpha_1}{\rho_1 H_2\alpha_2+\rho_2 H_1\alpha_1}|\boldsymbol{v}| & -\frac{\rho_1\rho_2 H_2\alpha_2}{\rho_1 H_2\alpha_2+\rho_2 H_1\alpha_1}|\boldsymbol{v}| \\ -\frac{\rho_1\rho_2 H_1\alpha_1}{\rho_1 H_2\alpha_2+\rho_2 H_1\alpha_1}|\boldsymbol{v}| & \rho_1 H_1\alpha_1 & 0 \\ -\frac{\rho_1\rho_2 H_2\alpha_2}{\rho_1 H_2\alpha_2+\rho_2 H_1\alpha_1}|\boldsymbol{v}| & 0 & \rho_2 H_2\alpha_2 \end{pmatrix}.$$

From Sylvester's criterion and since $\rho_k H_k \alpha_k$ is positive for $k = 1, 2$, the positivity of the matrix $\mathfrak{A}_0$ is equivalent to

$$
\begin{aligned}
\det \mathfrak{A}_0 &= a(\rho_1 H_1 \alpha_1)(\rho_2 H_2 \alpha_2) \\
&\quad - \rho_1 H_1 \alpha_1 \Big(\frac{\rho_1 \rho_2 H_2 \alpha_2}{\rho_1 H_2 \alpha_2 + \rho_2 H_1 \alpha_1} |\boldsymbol{v}| \Big)^2 - \rho_2 H_2 \alpha_2 \Big(\frac{\rho_1 \rho_2 H_1 \alpha_1}{\rho_1 H_2 \alpha_2 + \rho_2 H_1 \alpha_1} |\boldsymbol{v}| \Big)^2 \\
&= (\rho_1 H_1 \alpha_1)(\rho_2 H_2 \alpha_2) \Big(a - \frac{\rho_1 \rho_2}{\rho_1 H_2 \alpha_2 + \rho_2 H_1 \alpha_1} |\boldsymbol{v}|^2 \Big) > 0.
\end{aligned}
$$

Since $\boldsymbol{v} = \boldsymbol{u}_2 - \boldsymbol{u}_1$, under the stability condition (4.8) we have the positivity of $\mathfrak{A}_0$, so that in view of (5.7) and the positivity of the matrix $A_{k,0}$ for $k = 1, 2$ we finally obtain the equivalence

$$
(\mathcal{A}_0^{\mathrm{mod}} \dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} \simeq \|\dot{\zeta}\|_{L^2}^2 + \|\nabla \dot{\boldsymbol{\phi}}_1\|_{L^2}^2 + \|\nabla \dot{\boldsymbol{\phi}}_2\|_{L^2}^2 .
$$

Therefore, $(\mathcal{A}_0^{\mathrm{mod}} \dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2}$ would provide a useful mathematical energy function.

5.2. Symmetrization of the linearized equations

We still consider the linearized equations (4.1) with frozen coefficients. However, for later use we define $\dot{\boldsymbol{\phi}}_1$ and $\dot{\boldsymbol{\phi}}_2$ in place of (4.5) by

$$
\begin{cases}
\dot{\boldsymbol{\phi}}_1 = (\dot{\phi}_{1,0}, \dot{\phi}_{1,1}, \ldots, \dot{\phi}_{1,N})^{\mathrm{T}}, \\
\dot{\boldsymbol{\phi}}_2 = (\dot{\phi}_{2,0}, \dot{\phi}_{2,1}, \ldots, \dot{\phi}_{2,N^*})^{\mathrm{T}}.
\end{cases}
$$

Then the linearized equations have the form

$$
\begin{cases}
\boldsymbol{l}_1(H_1)(\partial_t \dot{\zeta} + \boldsymbol{u}_1 \cdot \nabla \dot{\zeta}) - A_1(H_1) \Delta \dot{\boldsymbol{\phi}}_1 = \boldsymbol{0}, \\
-\boldsymbol{l}_2(H_2)(\partial_t \dot{\zeta} + \boldsymbol{u}_2 \cdot \nabla \dot{\zeta}) - A_2(H_2) \Delta \dot{\boldsymbol{\phi}}_2 = \boldsymbol{0}, \\
-\rho_1 \boldsymbol{l}_1(H_1) \cdot (\partial_t \dot{\boldsymbol{\phi}}_1 + (\boldsymbol{u}_1 \cdot \nabla) \dot{\boldsymbol{\phi}}_1) \\
\qquad + \rho_2 \boldsymbol{l}_2(H_2) \cdot (\partial_t \dot{\boldsymbol{\phi}}_2 + (\boldsymbol{u}_2 \cdot \nabla) \dot{\boldsymbol{\phi}}_2) + a \dot{\zeta} = 0,
\end{cases}
\tag{5.8}
$$

where

$$
\boldsymbol{l}_1(H_1) = (1, H_1^2, H_1^4, \ldots, H_1^{2N})^{\mathrm{T}}, \quad \boldsymbol{l}_2(H_2) = (1, H_2^{p_1}, H_2^{p_2}, \ldots, H_2^{p_{N^*}})^{\mathrm{T}}, \tag{5.9}
$$

and

$$
\begin{cases}
A_1(H_1) = \Big(\dfrac{1}{2(i+j)+1} H_1^{2(i+j)+1} \Big)_{0 \le i,j \le N}, \\
A_2(H_2) = \Big(\dfrac{1}{p_i + p_j + 1} H_2^{p_i + p_j + 1} \Big)_{0 \le i,j \le N^*}.
\end{cases}
\tag{5.10}
$$

In the following, for simplicity we abbreviate $\boldsymbol{l}_k(H_k)$ and $A_k(H_k)$ as $\boldsymbol{l}_k$ and A_k for $k = 1, 2$. We are going to show that the system can be transformed into a positive symmetric system of the form

$$
\mathcal{A}_0^{\mathrm{mod}} \partial_t \dot{\boldsymbol{U}} + \mathcal{A} \dot{\boldsymbol{U}} = \boldsymbol{0}, \tag{5.11}
$$

where $\dot{\boldsymbol{U}} = (\dot{\zeta}, \dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2)^{\mathrm{T}}$, $\mathcal{A}_0^{\mathrm{mod}}$ is the positive operator defined in the previous section with slight modification, and $\mathcal{A}$ is a skew-symmetric operator in $L^2(\mathbf{R}^n)$. As before, we put $\boldsymbol{v} = \boldsymbol{u}_2 - \boldsymbol{u}_1$ and define $\boldsymbol{u}$ by (5.5). Furthermore, we introduce the notation

$$\theta_1 = \frac{\rho_2 H_1 \alpha_1}{\rho_1 H_2 \alpha_2 + \rho_2 H_1 \alpha_1}, \quad \theta_2 = \frac{\rho_1 H_2 \alpha_2}{\rho_1 H_2 \alpha_2 + \rho_2 H_1 \alpha_1}, \tag{5.12}$$

where α_1 and α_2 are positive constants defined by (4.7). Then we have $\boldsymbol{u} = \theta_2 \boldsymbol{u}_1 + \theta_1 \boldsymbol{u}_2$ and $\theta_1 + \theta_2 = 1$. We can also express $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$ as

$$\boldsymbol{u}_1 = \boldsymbol{u} - \theta_1 \boldsymbol{v}, \quad \boldsymbol{u}_2 = \boldsymbol{u} + \theta_2 \boldsymbol{v}.$$

Applying Δ to the third equation in (5.8) and differentiating the first and the second equations with respect to t, we obtain

$$\begin{pmatrix} 0 & -\rho_1 \boldsymbol{l}_1^{\mathrm{T}} & \rho_2 \boldsymbol{l}_2^{\mathrm{T}} \\ -\rho_1 \boldsymbol{l}_1 & \rho_1 A_1 & O \\ \rho_2 \boldsymbol{l}_2 & O & \rho_2 A_2 \end{pmatrix} \begin{pmatrix} \partial_t^2 \dot{\zeta} \\ \Delta \partial_t \dot{\boldsymbol{\phi}}_1 \\ \Delta \partial_t \dot{\boldsymbol{\phi}}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\rho_1 \boldsymbol{l}_1 (\boldsymbol{u}_1 \cdot \nabla) \\ \rho_2 \boldsymbol{l}_2 (\boldsymbol{u}_2 \cdot \nabla) \end{pmatrix} \partial_t \dot{\zeta}$$
$$+ \begin{pmatrix} a & -\rho_1 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & \rho_2 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \\ \mathbf{0} & O & O \\ \mathbf{0} & O & O \end{pmatrix} \Delta \dot{\boldsymbol{U}} = \mathbf{0}.$$

In view of this, we introduce a symmetric matrix

$$\begin{pmatrix} q_0 & \boldsymbol{q}_1^{\mathrm{T}} & \boldsymbol{q}_2^{\mathrm{T}} \\ \boldsymbol{q}_1 & Q_{11} & Q_{12} \\ \boldsymbol{q}_2 & Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} 0 & -\rho_1 \boldsymbol{l}_1^{\mathrm{T}} & \rho_2 \boldsymbol{l}_2^{\mathrm{T}} \\ -\rho_1 \boldsymbol{l}_1 & \rho_1 A_1 & O \\ \rho_2 \boldsymbol{l}_2 & O & \rho_2 A_2 \end{pmatrix}^{-1}, \tag{5.13}$$

where $Q_{11}^{\mathrm{T}} = Q_{11}$, $Q_{22}^{\mathrm{T}} = Q_{22}$, and $Q_{12}^{\mathrm{T}} = Q_{21}$. Moreover, we have

$$\begin{cases} -\rho_1 \boldsymbol{l}_1 \cdot \boldsymbol{q}_1 + \rho_2 \boldsymbol{l}_2 \cdot \boldsymbol{q}_2 = 1, \quad A_1 \boldsymbol{q}_1 = q_0 \boldsymbol{l}_1, \quad A_2 \boldsymbol{q}_2 = -q_0 \boldsymbol{l}_2, \\ \rho_1 A_1 Q_{11} = \mathrm{Id} + \rho_1 \boldsymbol{l}_1 \boldsymbol{q}_1^{\mathrm{T}}, \quad \rho_2 A_2 Q_{22} = \mathrm{Id} - \rho_2 \boldsymbol{l}_2 \boldsymbol{q}_2^{\mathrm{T}}, \\ A_1 Q_{12} = \boldsymbol{l}_1 \boldsymbol{q}_2^{\mathrm{T}}, \quad A_2 Q_{21} = -\boldsymbol{l}_2 \boldsymbol{q}_1^{\mathrm{T}} \end{cases}$$

and by Cramer's rule,

$$q_0 = -\frac{H_1 H_2 \alpha_1 \alpha_2}{\rho_1 H_2 \alpha_2 + \rho_2 H_1 \alpha_1}, \quad \boldsymbol{l}_1 \cdot \boldsymbol{q}_1 = \frac{-q_0}{H_1 \alpha_1} = -\frac{\theta_2}{\rho_1}, \quad \boldsymbol{l}_2 \cdot \boldsymbol{q}_2 = \frac{q_0}{H_2 \alpha_2} = \frac{\theta_1}{\rho_2}.$$

Using this notation we have

$$\begin{pmatrix} -\rho_1 A_1 \Delta \partial_t \dot{\boldsymbol{\phi}}_1 \\ -\rho_2 A_2 \Delta \partial_t \dot{\boldsymbol{\phi}}_2 \end{pmatrix} + \begin{pmatrix} -\rho_1 A_1 & O \\ O & -\rho_2 A_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_1 & Q_{11} & Q_{12} \\ \boldsymbol{q}_2 & Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} 0 \\ -\rho_1 \boldsymbol{l}_1 (\boldsymbol{u}_1 \cdot \nabla) \\ \rho_2 \boldsymbol{l}_2 (\boldsymbol{u}_2 \cdot \nabla) \end{pmatrix} \partial_t \dot{\zeta}$$
$$+ \begin{pmatrix} -\rho_1 A_1 & O \\ O & -\rho_2 A_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_1 & Q_{11} & Q_{12} \\ \boldsymbol{q}_2 & Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} a & -\rho_1 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & \rho_2 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \\ \mathbf{0} & O & O \\ \mathbf{0} & O & O \end{pmatrix} \Delta \dot{\boldsymbol{U}}$$
$$= \mathbf{0}.$$

Here, we see that

$$\begin{pmatrix} -\rho_1 A_1 & O \\ O & -\rho_2 A_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_1 & Q_{11} & Q_{12} \\ \boldsymbol{q}_2 & Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} 0 \\ -\rho_1 \boldsymbol{l}_1 (\boldsymbol{u}_1 \cdot \nabla) \\ \rho_2 \boldsymbol{l}_2 (\boldsymbol{u}_2 \cdot \nabla) \end{pmatrix} = - \begin{pmatrix} \theta_1 \rho_1 \boldsymbol{l}_1 \\ \theta_2 \rho_2 \boldsymbol{l}_2 \end{pmatrix} (\boldsymbol{v} \cdot \nabla),$$

$$\begin{aligned} &\begin{pmatrix} -\rho_1 A_1 & O \\ O & -\rho_2 A_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_1 & Q_{11} & Q_{12} \\ \boldsymbol{q}_2 & Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} a & -\rho_1 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & \rho_2 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \\ \boldsymbol{0} & O & O \\ \boldsymbol{0} & O & O \end{pmatrix} \\ &\quad = q_0 \begin{pmatrix} -a\rho_1 \boldsymbol{l}_1 & \rho_1^2 \boldsymbol{l}_1 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & -\rho_1 \rho_2 \boldsymbol{l}_1 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \\ a\rho_2 \boldsymbol{l}_2 & -\rho_1 \rho_2 \boldsymbol{l}_2 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & \rho_2^2 \boldsymbol{l}_2 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \end{pmatrix}, \end{aligned}$$

so that

$$\begin{aligned} &\begin{pmatrix} -\rho_1 A_1 \Delta \partial_t \dot{\boldsymbol{\phi}}_1 - \theta_1 \rho_1 \boldsymbol{l}_1 (\boldsymbol{v} \cdot \nabla) \partial_t \dot{\zeta} \\ -\rho_2 A_2 \Delta \partial_t \dot{\boldsymbol{\phi}}_2 - \theta_2 \rho_2 \boldsymbol{l}_2 (\boldsymbol{v} \cdot \nabla) \partial_t \dot{\zeta} \end{pmatrix} \\ &\quad = q_0 a \begin{pmatrix} \rho_1 \boldsymbol{l}_1 \\ -\rho_2 \boldsymbol{l}_2 \end{pmatrix} \Delta \dot{\zeta} + q_0 \begin{pmatrix} -\rho_1^2 \boldsymbol{l}_1 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & \rho_1 \rho_2 \boldsymbol{l}_1 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \\ \rho_1 \rho_2 \boldsymbol{l}_2 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & -\rho_2^2 \boldsymbol{l}_2 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \end{pmatrix} \Delta \begin{pmatrix} \dot{\boldsymbol{\phi}}_1 \\ \dot{\boldsymbol{\phi}}_2 \end{pmatrix}. \end{aligned} \tag{5.14}$$

On the other hand, taking the Euclidean inner product of the first and the second equations in (5.8) with $-\rho_1 \boldsymbol{q}_1$ and $\rho_2 \boldsymbol{q}_2$, respectively, we obtain

$$\begin{cases} \theta_2 (\partial_t \dot{\zeta} + \boldsymbol{u}_1 \cdot \nabla \dot{\zeta}) + q_0 \rho_1 \boldsymbol{l}_1 \cdot \Delta \dot{\boldsymbol{\phi}}_1 = 0, \\ \theta_1 (\partial_t \dot{\zeta} + \boldsymbol{u}_2 \cdot \nabla \dot{\zeta}) - q_0 \rho_2 \boldsymbol{l}_2 \cdot \Delta \dot{\boldsymbol{\phi}}_2 = 0, \end{cases}$$

which are equivalent to

$$\begin{cases} \partial_t \dot{\zeta} + \boldsymbol{u} \cdot \nabla \dot{\zeta} + q_0 \Delta (\rho_1 \boldsymbol{l}_1 \cdot \dot{\boldsymbol{\phi}}_1 - \rho_2 \boldsymbol{l}_2 \cdot \dot{\boldsymbol{\phi}}_2) = 0, \\ \theta_1 \theta_2 \boldsymbol{v} \cdot \nabla \dot{\zeta} - q_0 \Delta (\theta_1 \rho_1 \boldsymbol{l}_1 \cdot \dot{\boldsymbol{\phi}}_1 + \theta_2 \rho_2 \boldsymbol{l}_2 \cdot \dot{\boldsymbol{\phi}}_2) = 0. \end{cases} \tag{5.15}$$

It follows from the second equation in (5.15) that

$$\theta_1 \rho_1 \boldsymbol{l}_1 \cdot \partial_t \dot{\boldsymbol{\phi}}_1 + \theta_2 \rho_2 \boldsymbol{l}_2 \cdot \partial_t \dot{\boldsymbol{\phi}}_2 = q_0^{-1} \theta_1 \theta_2 (\boldsymbol{v} \cdot \nabla) \Delta^{-1} \partial_t \dot{\zeta}.$$

Therefore, we obtain

$$\begin{aligned} &a \partial_t \dot{\zeta} + (\boldsymbol{v} \cdot \nabla)(\theta_1 \rho_1 \boldsymbol{l}_1 \cdot \partial_t \dot{\boldsymbol{\phi}}_1 + \theta_2 \rho_2 \boldsymbol{l}_2 \cdot \partial_t \dot{\boldsymbol{\phi}}_2) \\ &\quad = -a \big((\boldsymbol{u} \cdot \nabla) \dot{\zeta} + q_0 \Delta (\rho_1 \boldsymbol{l}_1 \cdot \dot{\boldsymbol{\phi}}_1 - \rho_2 \boldsymbol{l}_2 \cdot \dot{\boldsymbol{\phi}}_2) \big) \\ &\qquad - \theta_1 \theta_2 (\boldsymbol{v} \cdot \nabla)^2 \big(q_0^{-1} (\boldsymbol{u} \cdot \nabla) \Delta^{-1} \dot{\zeta} + (\rho_1 \boldsymbol{l}_1 \cdot \dot{\boldsymbol{\phi}}_1 - \rho_2 \boldsymbol{l}_2 \cdot \dot{\boldsymbol{\phi}}_2) \big). \end{aligned} \tag{5.16}$$

We proceed to symmetrize the second term in the right-hand side of (5.14):

$$\begin{aligned} &q_0 \begin{pmatrix} -\rho_1^2 \boldsymbol{l}_1 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & \rho_1 \rho_2 \boldsymbol{l}_1 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \\ \rho_1 \rho_2 \boldsymbol{l}_2 \boldsymbol{l}_1^{\mathrm{T}} (\boldsymbol{u}_1 \cdot \nabla) & -\rho_2^2 \boldsymbol{l}_2 \boldsymbol{l}_2^{\mathrm{T}} (\boldsymbol{u}_2 \cdot \nabla) \end{pmatrix} \Delta \begin{pmatrix} \dot{\boldsymbol{\phi}}_1 \\ \dot{\boldsymbol{\phi}}_2 \end{pmatrix} \\ &\quad = q_0 \begin{pmatrix} -\rho_1^2 \boldsymbol{l}_1 \boldsymbol{l}_1^{\mathrm{T}} & \rho_1 \rho_2 \boldsymbol{l}_1 \boldsymbol{l}_2^{\mathrm{T}} \\ \rho_1 \rho_2 \boldsymbol{l}_2 \boldsymbol{l}_1^{\mathrm{T}} & -\rho_2^2 \boldsymbol{l}_2 \boldsymbol{l}_2^{\mathrm{T}} \end{pmatrix} (\boldsymbol{u} \cdot \nabla) \Delta \begin{pmatrix} \dot{\boldsymbol{\phi}}_1 \\ \dot{\boldsymbol{\phi}}_2 \end{pmatrix} \\ &\qquad + q_0 \begin{pmatrix} \theta_1 \rho_1^2 \boldsymbol{l}_1 \boldsymbol{l}_1^{\mathrm{T}} & \theta_2 \rho_1 \rho_2 \boldsymbol{l}_1 \boldsymbol{l}_2^{\mathrm{T}} \\ -\theta_1 \rho_1 \rho_2 \boldsymbol{l}_2 \boldsymbol{l}_1^{\mathrm{T}} & -\theta_2 \rho_2^2 \boldsymbol{l}_2 \boldsymbol{l}_2^{\mathrm{T}} \end{pmatrix} (\boldsymbol{v} \cdot \nabla) \Delta \begin{pmatrix} \dot{\boldsymbol{\phi}}_1 \\ \dot{\boldsymbol{\phi}}_2 \end{pmatrix}, \end{aligned}$$

where

$$q_0 \begin{pmatrix} \theta_1\rho_1^2 \boldsymbol{l}_1\boldsymbol{l}_1^{\mathrm{T}} & \theta_2\rho_1\rho_2 \boldsymbol{l}_1\boldsymbol{l}_2^{\mathrm{T}} \\ -\theta_1\rho_1\rho_2 \boldsymbol{l}_2\boldsymbol{l}_1^{\mathrm{T}} & -\theta_2\rho_2^2 \boldsymbol{l}_2\boldsymbol{l}_2^{\mathrm{T}} \end{pmatrix} \Delta \begin{pmatrix} \dot{\boldsymbol{\phi}}_1 \\ \dot{\boldsymbol{\phi}}_2 \end{pmatrix} = \begin{pmatrix} \rho_1\boldsymbol{l}_1 \\ -\rho_2\boldsymbol{l}_2 \end{pmatrix} q_0\Delta(\theta_1\rho_1\boldsymbol{l}_1\cdot\dot{\boldsymbol{\phi}}_1 + \theta_2\rho_2\boldsymbol{l}_2\cdot\dot{\boldsymbol{\phi}}_2)$$
$$= \theta_1\theta_2 \begin{pmatrix} \rho_1\boldsymbol{l}_1 \\ -\rho_2\boldsymbol{l}_2 \end{pmatrix} (\boldsymbol{v}\cdot\nabla)\dot{\zeta}.$$

In the above calculation, we used the second equation in (5.15). Therefore,

$$\begin{pmatrix} -\rho_1 A_1\Delta\partial_t\dot{\boldsymbol{\phi}}_1 - \theta_1\rho_1\boldsymbol{l}_1(\boldsymbol{v}\cdot\nabla)\partial_t\dot{\zeta} \\ -\rho_2 A_2\Delta\partial_t\dot{\boldsymbol{\phi}}_2 - \theta_2\rho_2\boldsymbol{l}_2(\boldsymbol{v}\cdot\nabla)\partial_t\dot{\zeta} \end{pmatrix} = q_0 a \begin{pmatrix} \rho_1\boldsymbol{l}_1 \\ -\rho_2\boldsymbol{l}_2 \end{pmatrix}\Delta\dot{\zeta} + \theta_1\theta_2 \begin{pmatrix} \rho_1\boldsymbol{l}_1 \\ -\rho_2\boldsymbol{l}_2 \end{pmatrix}(\boldsymbol{v}\cdot\nabla)^2\dot{\zeta}$$
$$+ q_0 \begin{pmatrix} -\rho_1^2\boldsymbol{l}_1\boldsymbol{l}_1^{\mathrm{T}} & \rho_1\rho_2\boldsymbol{l}_1\boldsymbol{l}_2^{\mathrm{T}} \\ \rho_1\rho_2\boldsymbol{l}_2\boldsymbol{l}_1^{\mathrm{T}} & -\rho_2^2\boldsymbol{l}_2\boldsymbol{l}_2^{\mathrm{T}} \end{pmatrix} (\boldsymbol{u}\cdot\nabla)\Delta \begin{pmatrix} \dot{\boldsymbol{\phi}}_1 \\ \dot{\boldsymbol{\phi}}_2 \end{pmatrix}.$$

Summarizing the above calculations, if we define the symmetrizer $\mathcal{A}_0^{\mathrm{mod}}$ by

$$\mathcal{A}_0^{\mathrm{mod}} = \begin{pmatrix} a & \theta_1\rho_1\boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla) & \theta_2\rho_2\boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla) \\ -\theta_1\rho_1\boldsymbol{l}_1(\boldsymbol{v}\cdot\nabla) & -\rho_1 A_1\Delta & O \\ -\theta_2\rho_2\boldsymbol{l}_2(\boldsymbol{v}\cdot\nabla) & O & -\rho_2 A_2\Delta \end{pmatrix}, \tag{5.17}$$

then we obtain

$$\begin{aligned} &\mathcal{A}_0^{\mathrm{mod}}\partial_t\dot{\boldsymbol{U}} \\ &= \begin{pmatrix} a\partial_t\dot{\zeta} + (\boldsymbol{v}\cdot\nabla)(\theta_1\rho_1\boldsymbol{l}_1\cdot\partial_t\dot{\boldsymbol{\phi}}_1 + \theta_2\rho_2\boldsymbol{l}_2\cdot\partial_t\dot{\boldsymbol{\phi}}_2) \\ -\theta_1\rho_1\boldsymbol{l}_1(\boldsymbol{v}\cdot\nabla)\partial_t\dot{\zeta} - \rho_1 A_1\Delta\partial_t\dot{\boldsymbol{\phi}}_1 \\ -\theta_2\rho_2\boldsymbol{l}_2(\boldsymbol{v}\cdot\nabla)\partial_t\dot{\zeta} - \rho_2 A_2\Delta\partial_t\dot{\boldsymbol{\phi}}_2 \end{pmatrix} \\ &= a\begin{pmatrix} -\boldsymbol{u}\cdot\nabla & -q_0\rho_1\boldsymbol{l}_1^{\mathrm{T}}\Delta & q_0\rho_2\boldsymbol{l}_2^{\mathrm{T}}\Delta \\ q_0\rho_1\boldsymbol{l}_1\Delta & O & O \\ -q_0\rho_2\boldsymbol{l}_2\Delta & O & O \end{pmatrix}\dot{\boldsymbol{U}} \\ &\quad + \begin{pmatrix} -q_0^{-1}\theta_1\theta_2(\boldsymbol{v}\cdot\nabla)^2(\boldsymbol{u}\cdot\nabla)\Delta^{-1} & -\theta_1\theta_2\rho_1\boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla)^2 & \theta_1\theta_2\rho_2\boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla)^2 \\ \theta_1\theta_2\rho_1\boldsymbol{l}_1(\boldsymbol{v}\cdot\nabla)^2 & -q_0\rho_1^2\boldsymbol{l}_1\boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{u}\cdot\nabla)\Delta & q_0\rho_1\rho_2\boldsymbol{l}_1\boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{u}\cdot\nabla)\Delta \\ -\theta_1\theta_2\rho_2\boldsymbol{l}_2(\boldsymbol{v}\cdot\nabla)^2 & q_0\rho_1\rho_2\boldsymbol{l}_2\boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{u}\cdot\nabla)\Delta & -q_0\rho_2^2\boldsymbol{l}_2\boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{u}\cdot\nabla)\Delta \end{pmatrix}\dot{\boldsymbol{U}}. \end{aligned}$$

Therefore, $\dot{\boldsymbol{U}}$ satisfies the symmetric system (5.11) with a skew-symmetric operator $\mathcal{A}$ defined by

$$\begin{aligned} \mathcal{A} &= a\begin{pmatrix} \boldsymbol{u}\cdot\nabla & q_0\rho_1\boldsymbol{l}_1^{\mathrm{T}}\Delta & -q_0\rho_2\boldsymbol{l}_2^{\mathrm{T}}\Delta \\ -q_0\rho_1\boldsymbol{l}_1\Delta & O & O \\ q_0\rho_2\boldsymbol{l}_2\Delta & O & O \end{pmatrix} \\ &\quad + \begin{pmatrix} q_0^{-1}\theta_1\theta_2(\boldsymbol{v}\cdot\nabla)^2(\boldsymbol{u}\cdot\nabla)\Delta^{-1} & \theta_1\theta_2\rho_1\boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla)^2 & -\theta_1\theta_2\rho_2\boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla)^2 \\ -\theta_1\theta_2\rho_1\boldsymbol{l}_1(\boldsymbol{v}\cdot\nabla)^2 & q_0\rho_1^2\boldsymbol{l}_1\boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{u}\cdot\nabla)\Delta & -q_0\rho_1\rho_2\boldsymbol{l}_1\boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{u}\cdot\nabla)\Delta \\ \theta_1\theta_2\rho_2\boldsymbol{l}_2(\boldsymbol{v}\cdot\nabla)^2 & -q_0\rho_1\rho_2\boldsymbol{l}_2\boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{u}\cdot\nabla)\Delta & q_0\rho_2^2\boldsymbol{l}_2\boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{u}\cdot\nabla)\Delta \end{pmatrix}. \end{aligned}$$

For the positive symmetric system (5.11), we can apply the standard theory for partial differential equations to show its well-posedness of the initial value problem. Moreover, these considerations help us to analyze the nonlinear problem (2.14)–(2.16).

6. Analysis of related operators

We go back to consider the nonlinear problem, that is, the Kakinuma model (2.14)–(2.16). We introduce the following second-order differential operators $L_{1,ij} = L_{1,ij}(H_1)$ $(i,j = 0,1,\dots,N)$ and $L_{2,ij} = L_{2,ij}(H_2,b)$ $(i,j = 0,1,\dots,N^*)$:

$$L_{1,ij}\varphi_{1,j} = -\nabla\cdot\Big(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\varphi_{1,j}\Big) + \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\varphi_{1,j}, \tag{6.1}$$

$$\begin{aligned} L_{2,ij}\varphi_{2,j} &= -\nabla\cdot\Big(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\varphi_{2,j} - \frac{p_j}{p_i+p_j}H_2^{p_i+p_j}\varphi_{2,j}\nabla b\Big) \\ &\quad - \frac{p_i}{p_i+p_j}H_2^{p_i+p_j}\nabla b\cdot\nabla\varphi_{2,j} \\ &\quad + \frac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}(1+|\nabla b|^2)\varphi_{2,j}. \end{aligned} \tag{6.2}$$

Then we have $(L_{k,ij})^* = L_{k,ji}$ for $k=1,2$, where $(L_{k,ij})^*$ is the adjoint operator of $L_{k,ij}$ in $L^2(\mathbf{R}^n)$. We also use $\boldsymbol{u}_k$ and w_k for $k=1,2$ defined by (4.2) and (4.3), which represent approximately the horizontal and the vertical components of the velocity field on the interface from the water region $\Omega_k(t)$, respectively. Then the Kakinuma model (2.14)–(2.16) can be written simply as

$$\begin{cases} H_1^{2i}\partial_t\zeta + \displaystyle\sum_{j=0}^{N} L_{1,ij}(H_1)\phi_{1,j} = 0 & \text{for } i = 0,1,\dots,N, \\ -H_2^{p_i}\partial_t\zeta + \displaystyle\sum_{j=0}^{N^*} L_{2,ij}(H_2,b)\phi_{2,j} = 0 & \text{for } i = 0,1,\dots,N^*, \\ -\rho_1\Big\{\displaystyle\sum_{j=0}^{N} H_1^{2j}\partial_t\phi_{1,j} + g\zeta + \frac12(|\boldsymbol{u}_1|^2 + w_1^2)\Big\} & \\ \qquad +\rho_2\Big\{\displaystyle\sum_{j=0}^{N^*} H_2^{p_j}\partial_t\phi_{2,j} + g\zeta + \frac12(|\boldsymbol{u}_2|^2 + w_2^2)\Big\} = 0. & \end{cases}$$

Moreover, introducing $\boldsymbol{\phi}_1 = (\phi_{1,0},\phi_{1,1},\dots,\phi_{1,N})^{\mathrm{T}}$, $\boldsymbol{\phi}_2 = (\phi_{2,0},\phi_{2,1},\dots,\phi_{2,N^*})^{\mathrm{T}}$, and

$$\begin{cases} \boldsymbol{l}_1(H_1) = (1, H_1^2, H_1^4, \dots, H_1^{2N})^{\mathrm{T}}, \ L_1(H_1) = (L_{1,ij}(H_1))_{0\le i,j\le N}, \\ \boldsymbol{l}_2(H_2) = (1, H_2^{p_1}, H_2^{p_2}, \dots, H_2^{p_{N^*}})^{\mathrm{T}}, \ L_2(H_2,b) = (L_{2,ij}(H_2,b))_{0\le i,j\le N^*}, \end{cases} \tag{6.3}$$

we can write the Kakinuma model (2.14)–(2.16) more simply as

$$\begin{cases} \boldsymbol{l}_1(H_1)\partial_t\zeta + L_1(H_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\ -\boldsymbol{l}_2(H_2)\partial_t\zeta + L_2(H_2,b)\boldsymbol{\phi}_2 = \boldsymbol{0}, \\ -\rho_1\Big\{\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + g\zeta + \frac12(|\boldsymbol{u}_1|^2 + w_1^2)\Big\} \\ \qquad +\rho_2\Big\{\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 + g\zeta + \frac12(|\boldsymbol{u}_2|^2 + w_2^2)\Big\} = 0. \end{cases} \tag{6.4}$$

By eliminating $\partial_t \zeta$ from the Kakinuma model, we obtain $N + N^* + 1$ scalar relations

$$\begin{cases} \displaystyle\sum_{j=0}^{N}(L_{1,ij}(H_1)\phi_{1,j} - H_1^{2i} L_{1,0j}(H_1)\phi_{1,j}) = 0 & \text{for } i = 1,2,\dots,N, \\ \displaystyle\sum_{j=0}^{N^*}(L_{2,ij}(H_2,b)\phi_{2,j} - H_2^{p_i} L_{2,0j}(H_2,b)\phi_{2,j}) = 0 & \text{for } i = 1,2,\dots,N^*, \\ \displaystyle\sum_{j=0}^{N} L_{1,0j}(H_1)\phi_{1,j} + \sum_{j=0}^{N^*} L_{2,0j}(H_2,b)\phi_{2,j} = 0. \end{cases}$$

These are compatibility conditions for the existence of the solution to the Kakinuma model, and exactly the same as the compatibility conditions (2.18)–(2.20). Introducing further linear operators $\mathcal{L}_{1,i} = \mathcal{L}_{1,i}(H_1)$ $(i = 0, 1, \dots, N)$ acting on $\boldsymbol{\varphi}_1 = (\varphi_{1,0}, \dots, \varphi_{1,N})^{\mathrm{T}}$ and $\mathcal{L}_{2,i} = \mathcal{L}_{2,i}(H_2, b)$ $(i = 0, 1, \dots, N^*)$ acting on $\boldsymbol{\varphi}_2 = (\varphi_{2,0}, \dots, \varphi_{2,N^*})^{\mathrm{T}}$ as

$$\begin{cases} \mathcal{L}_{1,0}(H_1)\boldsymbol{\varphi}_1 = \displaystyle\sum_{j=0}^{N} L_{1,0j}(H_1)\varphi_{1,j}, \\ \mathcal{L}_{1,i}(H_1)\boldsymbol{\varphi}_1 \\ \quad = \displaystyle\sum_{j=0}^{N}(L_{1,ij}(H_1)\varphi_{1,j} - H_1^{2i} L_{1,0j}(H_1)\varphi_{1,j}) & \text{for } i = 1,2,\dots,N, \\ \mathcal{L}_{2,0}(H_2,b)\boldsymbol{\varphi}_2 = \displaystyle\sum_{j=0}^{N^*} L_{2,0j}(H_2,b)\varphi_{2,j}, \\ \mathcal{L}_{2,i}(H_2,b)\boldsymbol{\varphi}_2 \\ \quad = \displaystyle\sum_{j=0}^{N^*}(L_{2,ij}(H_2,b)\varphi_{2,j} - H_2^{p_i} L_{2,0j}(H_2,b)\varphi_{2,j}) & \text{for } i = 1,2,\dots,N^*, \end{cases} \tag{6.5}$$

the compatibility conditions can be written simply as

$$\begin{cases} \mathcal{L}_{1,i}(H_1)\boldsymbol{\phi}_1 = 0 & \text{for } i = 1,2,\dots,N, \\ \mathcal{L}_{2,i}(H_2,b)\boldsymbol{\phi}_2 = 0 & \text{for } i = 1,2,\dots,N^*, \\ \mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1 + \mathcal{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2 = 0. \end{cases} \tag{6.6}$$

We proceed to derive evolution equations for $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$. To this end, we differentiate the above compatibility conditions with respect to t and use equations of the Kakinuma model to eliminate $\partial_t \zeta$. Then we obtain

$$\begin{cases} \mathcal{L}_{1,i}(H_1)\partial_t\boldsymbol{\phi}_1 = F_{1,i} & \text{for } i = 1,2,\dots,N, \\ \mathcal{L}_{2,i}(H_2,b)\partial_t\boldsymbol{\phi}_2 = F_{2,i} & \text{for } i = 1,2,\dots,N^*, \\ \mathcal{L}_{1,0}(H_1)\partial_t\boldsymbol{\phi}_1 + \mathcal{L}_{2,0}(H_2,b)\partial_t\boldsymbol{\phi}_2 = F_3, \end{cases} \tag{6.7}$$

where

$$\begin{cases} F_{1,i} = -\dfrac{\partial \mathcal{L}_{1,i}}{\partial H_1}(H_1)[\mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1]\boldsymbol{\phi}_1 & \text{for } i = 1,2,\dots,N, \\ F_{2,i} = -\dfrac{\partial \mathcal{L}_{2,i}}{\partial H_2}(H_2,b)[\mathcal{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2]\boldsymbol{\phi}_2 & \text{for } i = 1,2,\dots,N^*, \\ F_3 = -\dfrac{\partial \mathcal{L}_{1,0}}{\partial H_1}(H_1)[\mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1]\boldsymbol{\phi}_1 \\ \qquad -\dfrac{\partial \mathcal{L}_{2,0}}{\partial H_2}(H_2,b)[\mathcal{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2]\boldsymbol{\phi}_2. \end{cases}$$

Here, we note that F_3 can be written in divergence form as

$$F_3 = \nabla \cdot \left\{ (\mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1) \sum_{j=0}^{N} H_1^{2j} \nabla \phi_{1,j} + (\mathcal{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2) \sum_{j=0}^{N^*} H_2^{p_j} \nabla \phi_{2,j} \right\}.$$

On the other hand, the last equation in the Kakinuma model can be written as

$$-\rho_1 \boldsymbol{l}_1(H_1) \cdot \partial_t \boldsymbol{\phi}_1 + \rho_2 \boldsymbol{l}_2(H_2) \cdot \partial_t \boldsymbol{\phi}_2 = F_4, \tag{6.8}$$

where

$$F_4 = \rho_1 \left\{ g\zeta + \frac{1}{2}(|\boldsymbol{u}_1|^2 + w_1^2) \right\} - \rho_2 \left\{ g\zeta + \frac{1}{2}(|\boldsymbol{u}_2|^2 + w_2^2) \right\}.$$

In view of these evolution equations (6.7)–(6.8) for $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$, we will consider the following equations for $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$:

$$\begin{cases} \mathcal{L}_{1,i}(H_1)\boldsymbol{\varphi}_1 = f_{1,i} & \text{for } i = 1,2,\dots,N, \\ \mathcal{L}_{2,i}(H_2,b)\boldsymbol{\varphi}_2 = f_{2,i} & \text{for } i = 1,2,\dots,N^*, \\ \mathcal{L}_{1,0}(H_1)\boldsymbol{\varphi}_1 + \mathcal{L}_{2,0}(H_2,b)\boldsymbol{\varphi}_2 = \nabla \cdot \boldsymbol{f}_3, \\ -\rho_1 \boldsymbol{l}_1(H_1) \cdot \boldsymbol{\varphi}_1 + \rho_2 \boldsymbol{l}_2(H_2) \cdot \boldsymbol{\varphi}_2 = f_4. \end{cases} \tag{6.9}$$

In the following we will use the notation $\boldsymbol{\varphi}_1' = (\varphi_{1,1}, \dots, \varphi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\varphi}_2' = (\varphi_{2,1}, \dots, \varphi_{2,N^*})^{\mathrm{T}}$, and we put $\boldsymbol{f}_1' = (f_{1,1}, \dots, f_{1,N})^{\mathrm{T}}$ and $\boldsymbol{f}_2' = (f_{2,1}, \dots, f_{2,N^*})^{\mathrm{T}}$.

Lemma 6.1. *Let c_0 and c_1 be positive constants. There exists a positive constant $C = C(c_0, c_1)$ depending only on c_0 and c_1 such that for any $H_1, H_2, \nabla b \in L^\infty(\mathbf{R}^n)$ satisfying $H_1(x), H_2(x) \geq c_0$ and $|\nabla b(x)| \leq c_1$, any regular solution $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ to (6.9) satisfies*

$$\begin{aligned} &\rho_1(\|\nabla \varphi_{1,0}\|_{L^2}^2 + \|\boldsymbol{\varphi}_1'\|_{H^1}^2) + \rho_2(\|\nabla \varphi_{2,0}\|_{L^2}^2 + \|\boldsymbol{\varphi}_2'\|_{H^1}^2) \\ &\quad \leq C \biggl(-\sum_{j=0}^{N} \Bigl(\nabla f_4, \frac{1}{2j+1} H_1^{2j+1} \nabla \varphi_{1,j} \Bigr)_{L^2} \\ &\qquad\qquad + \rho_1 (\boldsymbol{f}_1', \boldsymbol{\varphi}_1')_{L^2} + \rho_2 (\boldsymbol{f}_2', \boldsymbol{\varphi}_2')_{L^2} + \rho_2 (\nabla \cdot \boldsymbol{f}_3, \boldsymbol{l}_2(H_2) \cdot \boldsymbol{\varphi}_2)_{L^2} \biggr). \end{aligned}$$

Proof. We introduce a dummy variable η as

$$\eta = -\mathscr{L}_{1,0}(H_1)\boldsymbol{\varphi}_1.$$

Then we can rewrite the equations in (6.9) as

$$\begin{cases} \eta \boldsymbol{l}_1(H_1) + L_1(H_1)\boldsymbol{\varphi}_1 = \boldsymbol{f}_1 = (0, f_{1,1}, \dots, f_{1,N})^{\mathrm{T}}, \\ -\eta \boldsymbol{l}_2(H_2) + L_2(H_2, b)\boldsymbol{\varphi}_2 = \boldsymbol{f}_2 = (0, f_{2,1}, \dots, f_{2,N^*})^{\mathrm{T}} + (\nabla \cdot \boldsymbol{f}_3)\boldsymbol{l}_2(H_2), \\ -\rho_1 \boldsymbol{l}_1(H_1) \cdot \boldsymbol{\varphi}_1 + \rho_2 \boldsymbol{l}_2(H_2) \cdot \boldsymbol{\varphi}_2 = f_4, \end{cases}$$

that is,

$$\begin{pmatrix} 0 & -\rho_1 \boldsymbol{l}_1(H_1)^{\mathrm{T}} & \rho_2 \boldsymbol{l}_2(H_2)^{\mathrm{T}} \\ \rho_1 \boldsymbol{l}_1(H_1) & \rho_1 L_1(H_1) & O \\ -\rho_2 \boldsymbol{l}_2(H_2) & O & \rho_2 L_2(H_2, b) \end{pmatrix} \begin{pmatrix} \eta \\ \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \end{pmatrix} = \begin{pmatrix} f_4 \\ \rho_1 \boldsymbol{f}_1 \\ \rho_2 \boldsymbol{f}_2 \end{pmatrix}.$$

By taking the L^2-inner product of this equation with $(\eta, \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)^{\mathrm{T}}$, we see that

$$\begin{aligned} &\rho_1 (L_1(H_1)\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1)_{L^2} + \rho_2 (L_2(H_2, b)\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_2)_{L^2} \\ &\quad = (f_4, \eta)_{L^2} + \rho_1 (\boldsymbol{f}_1, \boldsymbol{\varphi}_1)_{L^2} + \rho_2 (\boldsymbol{f}_2, \boldsymbol{\varphi}_2)_{L^2} \\ &\quad = -\sum_{j=0}^{N} \Big(\nabla f_4, \frac{1}{2j+1} H_1^{2j+1} \nabla \varphi_{1,j} \Big)_{L^2} \\ &\qquad + \rho_1 (\boldsymbol{f}_1', \boldsymbol{\varphi}_1')_{L^2} + \rho_2 (\boldsymbol{f}_2', \boldsymbol{\varphi}_2')_{L^2} + \rho_2 (\nabla \cdot \boldsymbol{f}_3, \boldsymbol{l}_2(H_2) \cdot \boldsymbol{\varphi}_2)_{L^2}. \end{aligned}$$

Here, by direct calculation we have

$$\begin{aligned} (L_1(H_1)\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1)_{L^2} &= \sum_{i,j=0}^{N} (L_{1,ij}(H_1)\varphi_{1,j}, \varphi_{1,i})_{L^2} \\ &= \int_{\mathbf{R}^n} \mathrm{d}\boldsymbol{x} \int_0^{H_1} \left\{ \left| \sum_{i=0}^{N} (z^{2i} \nabla \varphi_{1,i}) \right|^2 + \left(\sum_{i=0}^{N} 2i z^{2i-1} \varphi_{1,i} \right)^2 \right\} \mathrm{d}z \\ &\simeq \int_{\mathbf{R}^n} \mathrm{d}\boldsymbol{x} \int_0^{H_1} \sum_{i=0}^{N} (z^{4i} |\nabla \varphi_{1,i}|^2 + i^2 z^{4i-2} \varphi_{1,i}^2) \, \mathrm{d}z \\ &\simeq \int_{\mathbf{R}^n} \sum_{i=0}^{N} (H_1^{4i+1} |\nabla \varphi_{1,i}|^2 + i^2 H_1^{4i-1} \varphi_{1,i}^2) \, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{6.10}$$

where we used the fact that $\{z^{2i}\}_{i=0,\dots,N}$ and $\{z^{2i-1}\}_{i=1,\dots,N}$ are both linearly independent. We also have

$$\begin{aligned} (L_2(H_2, b)\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_2)_{L^2} &= \sum_{i,j=0}^{N^*} (L_{2,ij}(H_2, b)\varphi_{2,j}, \varphi_{2,i})_{L^2} \\ = \int_{\mathbf{R}^n} \mathrm{d}\boldsymbol{x} \int_0^{H_2} &\left\{ \left| \sum_{i=0}^{N^*} (z^{p_i} \nabla \varphi_{2,i} - p_i z^{p_i - 1} \varphi_{2,i} \nabla b) \right|^2 + \left(\sum_{i=0}^{N^*} p_i z^{p_i - 1} \varphi_{2,i} \right)^2 \right\} \mathrm{d}z. \end{aligned}$$

If $\{z^{p_i}, z^{p_i-1}\}_{i=0,\dots,N}$ are linearly independent, then we have

$$
\begin{aligned}
&(L_2(H_2,b)\boldsymbol{\varphi}_2,\boldsymbol{\varphi}_2)_{L^2}\\
&\quad\simeq \int_{\mathbf{R}^n}\mathrm{d}\boldsymbol{x}\int_0^{H_2}\sum_{i=0}^{N^*}\{(z^{2p_i}|\nabla\varphi_{2,i}|^2 + p_i^2 z^{2p_i-2}|\nabla b|^2\varphi_{2,i}^2) + p_i^2 z^{2p_i-2}\varphi_{2,i}^2\}\,\mathrm{d}z\\
&\quad\simeq \int_{\mathbf{R}^n}\sum_{i=0}^{N^*}\{H_2^{2p_i+1}|\nabla\varphi_{2,i}|^2 + p_i^2 H_2^{2p_i-1}(1+|\nabla b|^2)\varphi_{2,i}^2\}\,\mathrm{d}\boldsymbol{x}. \qquad (6.11)
\end{aligned}
$$

Otherwise, for example, in the case $p_i = i$ $(i = 0,\dots,N)$ we obtain

$$
\begin{aligned}
&(L_2(H_2,b)\boldsymbol{\varphi}_2,\boldsymbol{\varphi}_2)_{L^2}\\
&\quad= \int_{\mathbf{R}^n}\mathrm{d}\boldsymbol{x}\int_0^{H_2}\Bigg\{\Bigg|\sum_{i=0}^{N^*-1} z^i(\nabla\varphi_{2,i} - (i+1)\varphi_{2,i+1}\nabla b) + z^{N^*}\nabla\varphi_{2,N^*}\Bigg|^2\\
&\qquad\qquad + \Bigg(\sum_{i=0}^{N^*} p_i z^{p_i-1}\varphi_{2,i}\Bigg)^2\Bigg\}\,\mathrm{d}z\\
&\quad\simeq \int_{\mathbf{R}^n}\mathrm{d}\boldsymbol{x}\int_0^{H_2}\Bigg\{\sum_{i=0}^{N^*-1} z^{2i}|\nabla\varphi_{2,i} - (i+1)\varphi_{2,i+1}\nabla b|^2 + z^{2N^*}|\nabla\varphi_{2,N^*}|^2\\
&\qquad\qquad + \sum_{i=0}^{N^*} i^2 z^{2(i-1)}\varphi_{2,i}^2\Bigg\}\,\mathrm{d}z\\
&\quad\simeq \int_{\mathbf{R}^n}\Bigg\{\sum_{i=0}^{N^*-1} H_2^{2i+1}|\nabla\varphi_{2,i} - (i+1)\varphi_{2,i+1}\nabla b|^2 + H_2^{2N^*+1}|\nabla\varphi_{2,N^*}|^2\\
&\qquad\qquad + \sum_{i=0}^{N^*} i^2 H_2^{2i-1}\varphi_{2,i}^2\Bigg\}\,\mathrm{d}\boldsymbol{x}. \qquad (6.12)
\end{aligned}
$$

A similar estimate holds in other cases. These estimates give the desired one. ∎

Although this lemma gives an a priori bound of the solution to (6.9), the equations in (6.9) do not have good symmetry. In order to give an existence theorem to (6.9) with robust elliptic estimates, it is better to rewrite them in a symmetric form by introducing a good unknown variable. We introduce scalar functions φ_1 and φ_2 as

$$
\varphi_1 = \boldsymbol{l}_1(H_1)\cdot\boldsymbol{\varphi}_1, \quad \varphi_2 = \boldsymbol{l}_2(H_2)\cdot\boldsymbol{\varphi}_2. \qquad (6.13)
$$

We also introduce the second-order differential operators $P_{1,i}(H_1)$ $(i = 1,\dots,N)$ and $Q_1(H_1)$ acting on $\mathbf{R}^N$-valued functions $\boldsymbol{\varphi}_1' = (\varphi_{1,1},\dots,\varphi_{1,N})^{\mathrm{T}}$ and $P_{2,i}(H_2,b)$ $(i = 1,\dots,N^*)$ and $Q_2(H_2)$ acting on $\mathbf{R}^{N^*}$-valued functions $\boldsymbol{\varphi}_2' = (\varphi_{2,1},\dots,\varphi_{2,N^*})^{\mathrm{T}}$

as

$$
\begin{cases}
P_{1,i}(H_1)\boldsymbol{\varphi}_1' = \displaystyle\sum_{j=1}^{N}\bigl\{(L_{1,ij}(H_1) - H_1^{2i}L_{1,0j}(H_1))\varphi_{1,j} \\
\qquad\qquad\qquad\qquad -(L_{1,i0}(H_1) - H_1^{2i}L_{1,00}(H_1))(H_1^{2j}\varphi_{1,j})\bigr\}, \\
Q_1(H_1)\boldsymbol{\varphi}_1' = \displaystyle\sum_{j=1}^{N}\bigl\{L_{1,0j}(H_1)\varphi_{1,j} - L_{1,00}(H_1)(H_1^{2j}\varphi_{1,j})\bigr\},
\end{cases}
\tag{6.14}
$$

and

$$
\begin{cases}
P_{2,i}(H_2,b)\boldsymbol{\varphi}_2' = \displaystyle\sum_{j=1}^{N^*}\bigl\{(L_{2,ij}(H_2,b) - H_2^{p_i}L_{2,0j}(H_2,b))\varphi_{2,j} \\
\qquad\qquad\qquad\qquad -(L_{2,i0}(H_2,b) - H_2^{p_i}L_{2,00}(H_2,b))(H_2^{p_j}\varphi_{2,j})\bigr\}, \\
Q_2(H_2,b)\boldsymbol{\varphi}_2' = \displaystyle\sum_{j=1}^{N^*}\bigl\{L_{2,0j}(H_2,b)\varphi_{2,j} - L_{2,00}(H_2,b)(H_2^{p_j}\varphi_{2,j})\bigr\},
\end{cases}
\tag{6.15}
$$

respectively, and put

$$
\begin{cases}
P_1(H_1)\boldsymbol{\varphi}_1' = (P_{1,1}(H_1)\boldsymbol{\varphi}_1', \dots, P_{1,N}(H_1)\boldsymbol{\varphi}_1')^{\mathrm{T}}, \\
P_2(H_2,b)\boldsymbol{\varphi}_2' = (P_{2,1}(H_2,b)\boldsymbol{\varphi}_2', \dots, P_{2,N^*}(H_2,b)\boldsymbol{\varphi}_2')^{\mathrm{T}}.
\end{cases}
$$

Then we see easily that $P_1(H_1)$ and $P_2(H_2,b)$ are symmetric in $L^2(\mathbf{R}^n)$ and that

$$
\mathscr{L}_{1,i}(H_1)\boldsymbol{\varphi}_1 = \begin{cases}
Q_1(H_1)\boldsymbol{\varphi}_1' + L_{1,00}(H_1)(\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\varphi}_1) & \text{for } i=0, \\
P_{1,i}(H_1)\boldsymbol{\varphi}_1' + ((Q_1(H_1))^*(\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\varphi}_1))_i & \text{for } i=1,\dots,N,
\end{cases}
$$

$$
\mathscr{L}_{2,i}(H_2,b)\boldsymbol{\varphi}_2 = \begin{cases}
Q_2(H_2,b)\boldsymbol{\varphi}_2' + L_{2,00}(H_2,b)(\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\varphi}_2) & \text{for } i=0, \\
P_{2,i}(H_2,b)\boldsymbol{\varphi}_2' + ((Q_2(H_2,b))^*(\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\varphi}_2))_i & \text{for } i=1,\dots,N^*,
\end{cases}
$$

where Q^* denotes an adjoint operator of Q in $L^2(\mathbf{R}^n)$. Therefore, we can rewrite (6.9) as

$$
\begin{cases}
P_1(H_1)\boldsymbol{\varphi}_1' + (Q_1(H_1))^*\varphi_1 = \boldsymbol{f}_1, \\
P_2(H_2,b)\boldsymbol{\varphi}_2' + (Q_2(H_2,b))^*\varphi_2 = \boldsymbol{f}_2, \\
Q_1(H_1)\boldsymbol{\varphi}_1' + L_{1,00}(H_1)\varphi_1 + Q_2(H_2,b)\boldsymbol{\varphi}_2' + L_{2,00}(H_2,b)\varphi_2 = \nabla\cdot\boldsymbol{f}_3, \\
-\rho_1\varphi_1 + \rho_2\varphi_2 = f_4.
\end{cases}
$$

These equations for $(\boldsymbol{\varphi}_1', \varphi_1, \boldsymbol{\varphi}_2', \varphi_2)$ do not yet have good symmetry. But, it follows from the last equation that

$$
\rho_2\varphi_2 = \rho_1\varphi_1 + f_4.
$$

Using this we can remove φ_2 from the equations and obtain

$$\begin{cases} \rho_1 P_1(H_1)\boldsymbol{\varphi}_1' + \rho_1(Q_1(H_1))^*\varphi_1 = \rho_1\boldsymbol{F}_1, \\ \rho_2 P_2(H_2,b)\boldsymbol{\varphi}_2' + \rho_1(Q_2(H_2,b))^*\varphi_1 = \rho_2\boldsymbol{F}_2, \\ \rho_1 Q_1(H_1)\boldsymbol{\varphi}_1' + \rho_1 Q_2(H_2,b)\boldsymbol{\varphi}_2' + \rho_1(L_{1,00}(H_1) + \frac{\rho_1}{\rho_2}L_{2,00}(H_2,b))\varphi_1 \\ \qquad = \rho_1\nabla\cdot\boldsymbol{F}_3, \end{cases}$$

where

$$\boldsymbol{F}_1 = \boldsymbol{f}_1, \quad \boldsymbol{F}_2 = \boldsymbol{f}_2 - \frac{1}{\rho_2}(Q_2(H_2,b))^* f_4, \quad \boldsymbol{F}_3 = \boldsymbol{f}_3 + \frac{1}{\rho_2}H_2\nabla f_4. \tag{6.16}$$

These equations for $(\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2', \varphi_1)$ have good symmetry and can be written in the matrix form

$$\mathcal{P}(\zeta,b)\begin{pmatrix}\boldsymbol{\varphi}_1' \\ \boldsymbol{\varphi}_2' \\ \varphi_1\end{pmatrix} = \begin{pmatrix}\rho_1\boldsymbol{F}_1 \\ \rho_2\boldsymbol{F}_2 \\ \rho_1\nabla\cdot\boldsymbol{F}_3\end{pmatrix}, \tag{6.17}$$

where

$$\mathcal{P}(\zeta,b) = \begin{pmatrix} \rho_1 P_1(H_1) & O & \rho_1(Q_1(H_1))^* \\ O & \rho_2 P_2(H_2,b) & \rho_1(Q_2(H_2,b))^* \\ \rho_1 Q_1(H_1) & \rho_1 Q_2(H_2,b) & \rho_1(L_{1,00}(H_1) + \frac{\rho_1}{\rho_2}L_{2,00}(H_2,b)) \end{pmatrix}, \tag{6.18}$$

which is symmetric in $L^2(\mathbf{R}^n)$. Moreover, $\mathcal{P}(\zeta,b)$ is positive in $L^2(\mathbf{R}^n)$ as shown in the following lemma.

Lemma 6.2. *Let c_0, c_1 be positive constants. There exists a positive constant $C = C(c_0, c_1)$ depending only on c_0 and c_1 such that if $\zeta, b \in W^{1,\infty}(\mathbf{R}^n)$ satisfy $H_1(x), H_2(x) \geq c_0$ and $H_1(x) + |\nabla H_1(x)| + |\nabla b(x)| \leq c_1$, then for any $\tilde{\boldsymbol{\varphi}} = (\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2', \varphi_1)^{\mathrm{T}}$ we have*

$$(\mathcal{P}(\zeta,b)\tilde{\boldsymbol{\varphi}}, \tilde{\boldsymbol{\varphi}})_{L^2} \geq C^{-1}(\rho_1\|\boldsymbol{\varphi}_1'\|_{H^1}^2 + \rho_2\|\boldsymbol{\varphi}_2'\|_{H^1}^2 + \rho_1\|\nabla\varphi_1\|_{L^2}^2).$$

Proof. Given $\tilde{\boldsymbol{\varphi}} = (\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2', \varphi_1)^{\mathrm{T}}$, we define $\varphi_{1,0}$ and $\varphi_{2,0}$ by

$$\varphi_{1,0} = \varphi_1 - \sum_{j=1}^{N} H_1^{2j}\varphi_{1,j}, \quad \varphi_{2,0} = \frac{\rho_1}{\rho_2}\varphi_1 - \sum_{j=1}^{N^*} H_2^{p_j}\varphi_{2,j}$$

and put $\boldsymbol{\varphi}_1 = (\varphi_{1,0}, \varphi_{1,1}, \dots, \varphi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\varphi}_2 = (\varphi_{2,0}, \varphi_{2,1}, \dots, \varphi_{2,N^*})^{\mathrm{T}}$. Then we have $\varphi_1 = \boldsymbol{l}_1(H_1)\cdot\boldsymbol{\varphi}_1 = \frac{\rho_2}{\rho_1}\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\varphi}_2$, so that

$$\rho_1\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\varphi}_1 - \rho_2\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\varphi}_2 = 0.$$

We also define $\boldsymbol{F}_1 = (F_{1,1}, \dots, F_{1,N})^{\mathrm{T}}$, $\boldsymbol{F}_2 = (F_{2,1}, \dots, F_{2,N^*})^{\mathrm{T}}$, and F_3 by

$$\begin{pmatrix}\boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ F_3\end{pmatrix} = \mathcal{P}(\zeta,b)\begin{pmatrix}\boldsymbol{\varphi}_1' \\ \boldsymbol{\varphi}_2' \\ \varphi_1\end{pmatrix}.$$

Then we have

$$\begin{cases} \rho_1 \mathscr{L}_{1,i}(H_1)\boldsymbol{\varphi}_1 = F_{1,i} & \text{for } i = 1,2,\ldots,N, \\ \rho_2 \mathscr{L}_{2,i}(H_2,b)\boldsymbol{\varphi}_2 = F_{2,i} & \text{for } i = 1,2,\ldots,N^*, \\ \rho_1 \mathscr{L}_{1,0}(H_1)\boldsymbol{\varphi}_1 + \rho_2 \mathscr{L}_{2,0}(H_2,b)\boldsymbol{\varphi}_2 = F_3. \end{cases}$$

Now we introduce a dummy variable η as

$$\eta = -\mathscr{L}_{1,0}(H_1)\boldsymbol{\varphi}_1.$$

Then it follows from the above equations that

$$\begin{cases} -\rho_1 \boldsymbol{l}_1(H_1)\cdot\boldsymbol{\varphi}_1 + \rho_2 \boldsymbol{l}_2(H_2)\cdot\boldsymbol{\varphi}_2 = 0, \\ \rho_1(\eta \boldsymbol{l}_1(H_1) + L_1(H_1)\boldsymbol{\varphi}_1) = \boldsymbol{f}_1, \\ \rho_2(-\eta \boldsymbol{l}_2(H_2) + L_2(H_2,b)\boldsymbol{\varphi}_2) = \boldsymbol{f}_2 + \frac{\rho_2}{\rho_1}\boldsymbol{l}_2(H_2)F_3, \end{cases}$$

where $\boldsymbol{f}_1 = (0, F_{1,1},\ldots,F_{1,N})^{\mathrm{T}}$ and $\boldsymbol{f}_2 = (0, F_{2,1},\ldots,F_{2,N^*})^{\mathrm{T}}$. These equations can be written in the matrix form

$$\begin{pmatrix} 0 & -\rho_1 \boldsymbol{l}_1(H_1)^{\mathrm{T}} & \rho_2 \boldsymbol{l}_2(H_2)^{\mathrm{T}} \\ \rho_1 \boldsymbol{l}_1(H_1) & \rho_1 L_1(H_1) & O \\ -\rho_2 \boldsymbol{l}_2(H_2) & O & \rho_2 L_2(H_2,b) \end{pmatrix} \begin{pmatrix} \eta \\ \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{f}_1 \\ \boldsymbol{f}_2 + \frac{\rho_2}{\rho_1}\boldsymbol{l}_2(H_2)F_3 \end{pmatrix}.$$

By taking the L^2-inner product of this equation with $(\eta, \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)^{\mathrm{T}}$ we see that

$$\begin{aligned} &\rho_1(L_1(H_1)\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1)_{L^2} + \rho_2(L_2(H_2,b)\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_2)_{L^2} \\ &\quad = (\boldsymbol{f}_1, \boldsymbol{\varphi}_1)_{L^2} + (\boldsymbol{f}_2, \boldsymbol{\varphi}_2)_{L^2} + \frac{\rho_2}{\rho_1}(\boldsymbol{l}_2(H_2)F_3, \boldsymbol{\varphi}_2)_{L^2} \\ &\quad = (\boldsymbol{F}_1, \boldsymbol{\varphi}_1')_{L^2} + (\boldsymbol{F}_2, \boldsymbol{\varphi}_2')_{L^2} + (F_3, \varphi_1)_{L^2} \\ &\quad = (\mathcal{P}(\zeta,b)\tilde{\boldsymbol{\varphi}}, \tilde{\boldsymbol{\varphi}})_{L^2}, \end{aligned}$$

which gives, by (6.10) and (6.11) or (6.12),

$$(\mathcal{P}(\zeta,b)\tilde{\boldsymbol{\varphi}}, \tilde{\boldsymbol{\varphi}})_{L^2} \simeq \rho_1(\|\boldsymbol{\varphi}_1'\|_{H^1}^2 + \|\nabla\varphi_{1,0}\|_{L^2}^2) + \rho_2(\|\boldsymbol{\varphi}_2'\|_{H^1}^2 + \|\nabla\varphi_{2,0}\|_{L^2}^2).$$

Since $\|\nabla\varphi_1\|_{L^2}^2 \lesssim \|\boldsymbol{\varphi}_1'\|_{H^1}^2 + \|\nabla\varphi_{1,0}\|_{L^2}^2$, we obtain the desired estimate. ∎

By this lemma, the explicit expression (6.18) of the operator $\mathcal{P}(\zeta,b)$, and the standard theory of elliptic partial differential equations, we can obtain the following lemma.

Lemma 6.3. *Let ρ_1, ρ_2, h_1, h_2, c_0, M be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a positive constant $C = C(\rho_1,\rho_2,h_1,h_2,c_0,m)$ such that if ζ and b satisfy*

$$\begin{cases} \|\zeta\|_{H^m} + \|b\|_{W^{m,\infty}} \le M, \\ H_1(\boldsymbol{x}) = h_1 - \zeta(\boldsymbol{x}) \ge c_0, \quad H_2(\boldsymbol{x}) = h_2 + \zeta(\boldsymbol{x}) - b(\boldsymbol{x}) \ge c_0 \quad \textit{for } \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

then for any $\boldsymbol{F}_1, \boldsymbol{F}_2 \in H^{k-1}$ *and* $\boldsymbol{F}_3 \in H^k$ *with* $k \in \{0, 1, \ldots, m-1\}$*, there exists a solution* $(\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2', \varphi_1)$ *of* (6.17) *satisfying*

$$\|(\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2')\|_{H^{k+1}} + \|\nabla\varphi_1\|_{H^k} \leq C(\|(\boldsymbol{F}_1, \boldsymbol{F}_2)\|_{H^{k-1}} + \|\boldsymbol{F}_3\|_{H^k}).$$

Moreover, the solution is unique up to an additive constant to φ_1*.*

We proceed to consider the solvability of (6.9). Given $\boldsymbol{f}_1'$, $\boldsymbol{f}_2'$, $\boldsymbol{f}_3$, f_4, we define $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ by (6.16), for which there exists a solution $(\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2', \varphi_1)$ to (6.17), define $\varphi_{1,0}$ and $\varphi_{2,0}$ by

$$\varphi_{1,0} = \varphi_1 - \sum_{j=1}^{N} H_1^{2j}\varphi_{1,j}, \quad \varphi_{2,0} = \frac{\rho_1}{\rho_2}\varphi_1 - \sum_{j=1}^{N^*} H_2^{p_j}\varphi_{2,j} + \frac{1}{\rho_2} f_4,$$

and put $\boldsymbol{\varphi}_1 = (\varphi_{1,0}, \varphi_{1,1}, \ldots, \varphi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\varphi}_2 = (\varphi_{2,0}, \varphi_{2,1}, \ldots, \varphi_{2,N^*})^{\mathrm{T}}$. Then we see that $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ is a solution to (6.9). More precisely, we obtain the following lemma.

Lemma 6.4. *Under the hypothesis of Lemma* 6.3*, for any* $\boldsymbol{f}_1' = (f_{1,1}, \ldots, f_{1,N})^{\mathrm{T}}$*,* $\boldsymbol{f}_2' = (f_{2,1}, \ldots, f_{2,N^*})^{\mathrm{T}}$*,* $\boldsymbol{f}_3$*, and* f_4 *satisfying* $\boldsymbol{f}_1', \boldsymbol{f}_2' \in H^{k-1}$ *and* $\boldsymbol{f}_3, \nabla f_4 \in H^k$ *with* $k \in \{0, 1, \ldots, m-1\}$*, there exists a solution* $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ *to* (6.9) *satisfying*

$$\|(\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2')\|_{H^{k+1}} + \|(\nabla\varphi_{1,0}, \nabla\varphi_{2,0})\|_{H^k} \leq C(\|(\boldsymbol{f}_1', \boldsymbol{f}_2')\|_{H^{k-1}} + \|(\boldsymbol{f}_3, \nabla f_4)\|_{H^k}),$$

where $C = C(\rho_1, \rho_2, h_1, h_2, c_0, m)$*. Moreover, the solution is unique up to an additive constant of the form* $(\mathscr{C}\rho_2, \mathscr{C}\rho_1)$ *to* $(\varphi_{1,0}, \varphi_{2,0})$*.*

7. Construction of the solution

In this section we will prove Theorem 2.1, one of the main theorems in this paper. One possible strategy to construct the solution of the initial value problem to the Kakinuma model (2.14)–(2.16) would consist in firstly transforming the equations into a quasilinear positive symmetric system, that is, a quasilinear version of the positive symmetric system (5.11), secondly applying the method of parabolic regularization to construct the solution of the transformed system, and finally to show that the solution to the transformed system is in fact the solution of the Kakinuma model if we further impose the compatibility conditions (2.18)–(2.18) on the initial data. Here, in order to avoid the heavy computations that would be involved when following this strategy, we find it more convenient to instead apply the method of parabolic regularization to the Kakinuma model directly.

7.1. Parabolic regularization of the equations

We recall that the Kakinuma model (2.14)–(2.16) can be written compactly as (6.4), that is,

$$\begin{cases} \boldsymbol{l}_1(H_1)\partial_t\zeta + L_1(H_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\ -\boldsymbol{l}_2(H_2)\partial_t\zeta + L_2(H_2, b)\boldsymbol{\phi}_2 = \boldsymbol{0}, \\ -\rho_1\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + \rho_2\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 = F, \end{cases} \tag{7.1}$$

where $\boldsymbol{\phi}_1 = (\phi_{1,0}, \phi_{1,1}, \dots, \phi_{1,N})^{\mathrm{T}}$, $\boldsymbol{\phi}_2 = (\phi_{2,0}, \phi_{2,1}, \dots, \phi_{2,N^*})^{\mathrm{T}}$, $\boldsymbol{l}_k$ and L_k for $k = 1, 2$ are defined in (6.3), and

$$F = \rho_1 \Big\{ g\zeta + \frac{1}{2}(|\boldsymbol{u}_1|^2 + w_1^2) \Big\} - \rho_2 \Big\{ g\zeta + \frac{1}{2}(|\boldsymbol{u}_2|^2 + w_2^2) \Big\}. \tag{7.2}$$

Here $\boldsymbol{u}_k$ and w_k for $k = 1, 2$ are defined by (4.2) and (4.3) respectively. We regularize the Kakinuma model by adding artificial viscosity terms as

$$\begin{cases} \boldsymbol{l}_1(H_1)(\partial_t \zeta - \varepsilon \Delta \zeta) + L_1(H_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\ -\boldsymbol{l}_2(H_2)(\partial_t \zeta - \varepsilon \Delta \zeta) + L_2(H_2, b)\boldsymbol{\phi}_2 = \boldsymbol{0}, \\ -\rho_1 \boldsymbol{l}_1(H_1) \cdot (\partial_t \boldsymbol{\phi}_1 - \varepsilon \Delta \boldsymbol{\phi}_1) + \rho_2 \boldsymbol{l}_2(H_2) \cdot (\partial_t \boldsymbol{\phi}_2 - \varepsilon \Delta \boldsymbol{\phi}_2) = F. \end{cases} \tag{7.3}$$

We are going to show the existence of the solution to the initial value problem for this regularized Kakinuma model under the initial conditions

$$(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)|_{t=0} = (\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)}). \tag{7.4}$$

For this regularized Kakinuma model, the compatibility conditions for the existence of the solution have the same form as the original Kakinuma model, that is,

$$\begin{cases} \mathscr{L}_{1,i}(H_1)\boldsymbol{\phi}_1 = 0 & \text{for } i = 1, 2, \dots, N, \\ \mathscr{L}_{2,i}(H_2, b)\boldsymbol{\phi}_2 = 0 & \text{for } i = 1, 2, \dots, N^*, \\ \mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1 + \mathscr{L}_{2,0}(H_2, b)\boldsymbol{\phi}_2 = 0, \end{cases} \tag{7.5}$$

where $\mathscr{L}_{1,i}(H_1)$ for $i = 0, 1, \dots, N$ and $\mathscr{L}_{2,i}(H_2, b)$ for $i = 0, 1, \dots, N^*$ are defined in (6.5). Here, we note the identities

$$\begin{cases} [\partial_t, \mathscr{L}_{1,i}(H_1)]\boldsymbol{\phi}_1 = f_{1,i}(\zeta, \boldsymbol{\phi}_1)\partial_t \zeta & \text{for } i = 1, 2, \dots, N, \\ [\partial_t, \mathscr{L}_{2,i}(H_2, b)]\boldsymbol{\phi}_2 = f_{2,i}(\zeta, \boldsymbol{\phi}_2, b)\partial_t \zeta & \text{for } i = 1, 2, \dots, N^*, \\ [\partial_t, \mathscr{L}_{1,0}(H_1)]\boldsymbol{\phi}_1 + [\partial_t, \mathscr{L}_{2,0}(H_2, b)]\boldsymbol{\phi}_2 = -\nabla \cdot (\boldsymbol{v}\partial_t \zeta), \end{cases}$$

where $\boldsymbol{v} = \boldsymbol{u}_2 - \boldsymbol{u}_1$ and

$$\begin{cases} f_{1,i}(\zeta, \boldsymbol{\phi}_1) = -\displaystyle\sum_{j=0}^{N} \Big\{ \frac{2i}{2j+1} H_1^{2(i+j)} \Delta \phi_{1,j} + 4ij H_1^{2(i+j-1)} \phi_{1,j} \Big\}, \\ f_{2,i}(\zeta, \boldsymbol{\phi}_2, b) = \displaystyle\sum_{j=0}^{N^*} \Big\{ \frac{p_i}{p_j + 1} H_2^{p_i + p_j} \Delta \phi_{2,j} - \frac{p_i p_j}{p_j} H_2^{p_i + p_j - 1} \nabla \cdot (\phi_{2,j} \nabla b) \\ \qquad\qquad - p_i H_2^{p_i + p_j - 1} \nabla b \cdot \nabla \phi_{2,j} + p_i p_j H_2^{p_i + p_j - 2} (1 + |\nabla b|^2) \phi_{2,j} \Big\}, \end{cases}$$

and

$$\begin{cases} [\Delta, \mathscr{L}_{1,i}(H_1)]\boldsymbol{\phi}_1 = f_{1,i}(\zeta, \boldsymbol{\phi}_1)\Delta \zeta + \tilde{f}_{1,i}(\zeta, \boldsymbol{\phi}_1) & \text{for } i = 1, 2, \dots, N, \\ [\Delta, \mathscr{L}_{2,i}(H_2, b)]\boldsymbol{\phi}_2 = f_{2,i}(\zeta, \boldsymbol{\phi}_2, b)\Delta \zeta + \tilde{f}_{2,i}(\zeta, \boldsymbol{\phi}_2, b) & \text{for } i = 1, 2, \dots, N^*, \\ [\Delta, \mathscr{L}_{1,0}(H_1)]\boldsymbol{\phi}_1 + [\Delta, \mathscr{L}_{2,0}(H_2, b)]\boldsymbol{\phi}_2 = -\nabla \cdot (\boldsymbol{v}\Delta \zeta) + f_3(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2, b), \end{cases}$$

where

$$\begin{cases} \tilde{f}_{1,i}(\zeta,\boldsymbol{\phi}_1) = \displaystyle\sum_{l=1}^{n}\{[\partial_l,\mathscr{L}_{1,i}(H_1)]\partial_l\boldsymbol{\phi}_1 + (\partial_l\zeta)\partial_l f_{1,i}(\zeta,\boldsymbol{\phi}_1)\}, \\ \tilde{f}_{2,i}(\zeta,\boldsymbol{\phi}_2,b) = \displaystyle\sum_{l=1}^{n}\Big\{[\partial_l,\mathscr{L}_{2,i}(H_2,b)]\partial_l\boldsymbol{\phi}_2 + (\partial_l\zeta)f_{2,i}(\zeta,\boldsymbol{\phi}_2,b) \\ \qquad -\partial_l((\partial_l b)f_{2,i}(\zeta,\boldsymbol{\phi}_2,b)) \\ \qquad + \displaystyle\sum_{j=0}^{N^*}\partial_l\Big(-\frac{p_i p_j}{(p_i+p_j)p_j}H_2^{p_i+p_j}\nabla\cdot(\phi_{2,j}\nabla\partial_l b) \\ \qquad -\dfrac{p_i}{p_i+p_j}H_2^{p_i+p_j}\nabla\partial_l b\cdot\nabla\phi_{2,j} \\ \qquad +\dfrac{p_i p_j}{p_i+p_j-1}H_2^{p_i+p_j-1}2(\nabla b\cdot\nabla\partial_l b)\Big)\Big\}, \\ f_3(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,b) = \displaystyle\sum_{l=1}^{n}\Big\{[\partial_l,\mathscr{L}_{1,0}(H_1)]\partial_l\boldsymbol{\phi}_1 + [\partial_l,\mathscr{L}_{2,0}(H_2,b)]\partial_l\boldsymbol{\phi}_2 \\ \qquad +\nabla\cdot\Big(-(\partial_l\zeta)(\partial_l\boldsymbol{v}) + \partial_l\Big((\partial_l b)\boldsymbol{u}_2 + \displaystyle\sum_{j=1}^{N^*}H_2^{p_j}\phi_{2,j}\nabla\partial_l b\Big)\Big)\Big\}. \end{cases}$$

We also note that $f_3(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,b)$ can be written in a divergence form as

$$f_3(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,b) = \nabla\cdot\boldsymbol{f}_3(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,b),$$

where

$$\begin{aligned} \boldsymbol{f}_3(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,b) = \sum_{l=1}^{n}\Big\{&(\partial_l\zeta)\sum_{j=0}^{N}H_1^{2j}\nabla\partial_l\phi_{1,j} + \sum_{j=1}^{N^*}H_2^{p_j}(\partial_l\phi_{2,j})\nabla\partial_l b \\ &+ (\partial_l b - \partial_l\zeta)\sum_{j=0}^{N^*}(H_2^{p_j}\nabla\partial_l\phi_{2,j} - p_j H_2^{p_j-1}(\partial_l\phi_{2,j})\nabla b) \\ &- (\partial_l\zeta)(\partial_l\boldsymbol{v}) + \partial_l\Big((\partial_l b)\boldsymbol{u}_2 + \sum_{j=1}^{N^*}H_2^{p_j}\phi_{2,j}\nabla\partial_l b\Big)\Big\}. \end{aligned}$$

Therefore, applying the operator $\partial_t - \varepsilon\Delta$ to (7.5) we obtain

$$\begin{cases} \mathscr{L}_{1,i}(H_1)(\partial_t\boldsymbol{\phi}_1 - \varepsilon\Delta\boldsymbol{\phi}_1) \\ \qquad = -f_{1,i}(\zeta,\boldsymbol{\phi}_1)(\partial_t\zeta - \varepsilon\Delta\zeta) + \varepsilon\tilde{f}_{1,i}(\zeta,\boldsymbol{\phi}_1) \qquad \text{for } i = 1,2,\dots,N, \\ \mathscr{L}_{2,i}(H_2,b)(\partial_t\boldsymbol{\phi}_2 - \varepsilon\Delta\boldsymbol{\phi}_2) \\ \qquad = -f_{2,i}(\zeta,\boldsymbol{\phi}_2,b)(\partial_t\zeta - \varepsilon\Delta\zeta) + \varepsilon\tilde{f}_{2,i}(\zeta,\boldsymbol{\phi}_2,b) \quad \text{for } i = 1,2,\dots,N^*, \\ \mathscr{L}_{1,0}(H_1)(\partial_t\boldsymbol{\phi}_1 - \varepsilon\Delta\boldsymbol{\phi}_1) + \mathscr{L}_{2,0}(H_2,b)(\partial_t\boldsymbol{\phi}_2 - \varepsilon\Delta\boldsymbol{\phi}_2) \\ \qquad = \nabla\cdot(\boldsymbol{v}(\partial_t\zeta - \varepsilon\Delta\zeta) + \varepsilon\boldsymbol{f}_3(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,b)). \end{cases} \tag{7.6}$$

On the other hand, we have $N + N^* + 2$ evolution equations for one scalar function ζ. To select an appropriate evolution equation for ζ, we will use the notation defined by (5.13). We note that they depend on the unknown functions H_1 and H_2. Taking Euclidean inner products of the first and the second equations in (7.3) with $\rho_1 \boldsymbol{q}_1$ and $\rho_2 \boldsymbol{q}_2$, respectively, adding the resulting equations, and using the relation $-\rho_1 \boldsymbol{l}_1 \cdot \boldsymbol{q}_1 + \rho_2 \boldsymbol{l}_2 \cdot \boldsymbol{q}_2 = 1$, we obtain

$$\partial_t \zeta - \varepsilon \Delta \zeta = G_0, \tag{7.7}$$

where

$$G_0 = \rho_1 \boldsymbol{q}_1 \cdot L_1(H_1)\boldsymbol{\phi}_1 + \rho_2 \boldsymbol{q}_2 \cdot L_2(H_2, b)\boldsymbol{\phi}_2.$$

Plugging this into (7.6) and noting the last equation in (7.3), we have

$$\begin{cases} \mathcal{L}_{1,i}(H_1)(\partial_t \boldsymbol{\phi}_1 - \varepsilon \Delta \boldsymbol{\phi}_1) \\ \qquad = -f_{1,i}(\zeta, \boldsymbol{\phi}_1) G_0 + \varepsilon \tilde{f}_{1,i}(\zeta, \boldsymbol{\phi}_1) \qquad \text{for } i = 1, 2, \ldots, N, \\ \mathcal{L}_{2,i}(H_2, b)(\partial_t \boldsymbol{\phi}_2 - \varepsilon \Delta \boldsymbol{\phi}_2) \\ \qquad = -f_{2,i}(\zeta, \boldsymbol{\phi}_2, b) G_0 + \varepsilon \tilde{f}_{2,i}(\zeta, \boldsymbol{\phi}_2, b) \quad \text{for } i = 1, 2, \ldots, N^*, \\ \mathcal{L}_{1,0}(H_1)(\partial_t \boldsymbol{\phi}_1 - \varepsilon \Delta \boldsymbol{\phi}_1) + \mathcal{L}_{2,0}(H_2, b)(\partial_t \boldsymbol{\phi}_2 - \varepsilon \Delta \boldsymbol{\phi}_2) \\ \qquad = \nabla \cdot (\boldsymbol{v} G_0 + \varepsilon \boldsymbol{f}_3(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2, b)), \\ -\rho_1 \boldsymbol{l}_1(H_1) \cdot (\partial_t \boldsymbol{\phi}_1 - \varepsilon \Delta \boldsymbol{\phi}_1) \\ \qquad + \rho_2 \boldsymbol{l}_2(H_2) \cdot (\partial_t \boldsymbol{\phi}_2 - \varepsilon \Delta \boldsymbol{\phi}_2) = F. \end{cases} \tag{7.8}$$

Therefore, thanks to Lemma 6.4 we obtain

$$\begin{cases} \partial_t \boldsymbol{\phi}_1 - \varepsilon \Delta \boldsymbol{\phi}_1 = \boldsymbol{G}_1, \\ \partial_t \boldsymbol{\phi}_2 - \varepsilon \Delta \boldsymbol{\phi}_2 = \boldsymbol{G}_2, \end{cases} \tag{7.9}$$

where $\boldsymbol{G}_1 = (G_{1,0}, G_{1,1}, \ldots, G_{1,N})^{\mathrm{T}}$ and $\boldsymbol{G}_2 = (G_{2,0}, G_{2,1}, \ldots, G_{2,N^*})^{\mathrm{T}}$ are defined as a solution to the following equations:

$$\begin{cases} \mathcal{L}_{1,i}(H_1)\boldsymbol{G}_1 = -f_{1,i}(\zeta, \boldsymbol{\phi}_1) G_0 + \varepsilon \tilde{f}_{1,i}(\zeta, \boldsymbol{\phi}_1) \quad \text{for } i = 1, 2, \ldots, N, \\ \mathcal{L}_{2,i}(H_2, b)\boldsymbol{G}_2 \\ \qquad = -f_{2,i}(\zeta, \boldsymbol{\phi}_2, b) G_0 + \varepsilon \tilde{f}_{2,i}(\zeta, \boldsymbol{\phi}_2, b) \qquad \text{for } i = 1, 2, \ldots, N^*, \\ \mathcal{L}_{1,0}(H_1)\boldsymbol{G}_1 + \mathcal{L}_{2,0}(H_2, b)\boldsymbol{G}_2 = \nabla \cdot (\boldsymbol{v} G_0 + \varepsilon \boldsymbol{f}_3(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2, b)), \\ -\rho_1 \boldsymbol{l}_1(H_1) \cdot \boldsymbol{G}_1 + \rho_2 \boldsymbol{l}_2(H_2) \cdot \boldsymbol{G}_2 = F. \end{cases} \tag{7.10}$$

Precisely speaking, $(\boldsymbol{G}_1, \boldsymbol{G}_2)$ are defined uniquely up to an additive constant of the form $(\mathscr{C}\rho_2, \mathscr{C}\rho_1)$ to $(G_{1,0}, G_{2,0})$. However, this indeterminacy does not cause any difficulties in the following arguments.

Remark 7.1. The equations in (7.9) are valid even in the case $\varepsilon = 0$, that is, any regular solutions to the Kakinuma model (2.14)–(2.15) satisfy (7.9) with $\varepsilon = 0$. Particularly, $\partial_t \boldsymbol{\phi}_k(\boldsymbol{x}, 0)$ for $k = 1, 2$ can be expressed in terms of the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b.

7.2. Existence of the solution to the regularized problem

Lemma 7.2. *Let g, ρ_1, ρ_2, h_1, h_2, c_0 be positive constants and m an integer such that $m > \frac{n}{2} + 1$. For any initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and bottom topography b satisfying*

$$\begin{cases} \zeta_{(0)}, \nabla\phi_{1,0(0)}, \nabla\phi_{2,0(0)} \in H^m, \quad \boldsymbol{\phi}'_{1(0)}, \boldsymbol{\phi}'_{2(0)} \in H^{m+1}, \quad b \in W^{m+2,\infty}, \\ h_1 - \zeta_{(0)}(\boldsymbol{x}) \geq c_0, \quad h_2 + \zeta_{(0)}(\boldsymbol{x}) - b(\boldsymbol{x}) \geq c_0 \quad \text{for } \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

and for any $\varepsilon > 0$ there exists a maximal existence time $T_\varepsilon \in (0, +\infty]$ such that the initial value problem (7.7), (7.9), and (7.4) has a unique solution $(\zeta^\varepsilon, \boldsymbol{\phi}_1^\varepsilon, \boldsymbol{\phi}_2^\varepsilon)$ satisfying

$$\zeta^\varepsilon, \nabla\phi_{1,0}^\varepsilon, \nabla\phi_{2,0}^\varepsilon \in C([0, T_\varepsilon); H^m), \quad \boldsymbol{\phi}_1^{\varepsilon\prime}, \boldsymbol{\phi}_2^{\varepsilon\prime} \in C([0, T_\varepsilon); H^{m+1}).$$

Proof. We evaluate the right-hand sides of the equations, that is, the terms G_0, $\boldsymbol{G}_1$, and $\boldsymbol{G}_2$. To this end, suppose that $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ and b satisfy

$$\begin{cases} \|(\zeta, \nabla\phi_{1,0}, \nabla\phi_{2,0})\|_{H^m} + \|(\boldsymbol{\phi}'_1, \boldsymbol{\phi}'_2)\|_{H^{m+1}} + \|b\|_{W^{m+2,\infty}} \leq M, \\ h_1 - \zeta(\boldsymbol{x}) \geq c_1, \quad h_2 + \zeta(\boldsymbol{x}) - b(\boldsymbol{x}) \geq c_1 \quad \text{for } \boldsymbol{x} \in \mathbf{R}^n. \end{cases} \tag{7.11}$$

Then we see that

$$\|G_0\|_{H^{m-1}} + \|(\boldsymbol{f}'_1, \boldsymbol{f}'_2, f_3)\|_{H^{m-1}} + \|(\tilde{\boldsymbol{f}}'_1, \tilde{\boldsymbol{f}}'_2)\|_{H^{m-2}} + \|F\|_{H^m} \leq C(M, c_1),$$

where $\boldsymbol{f}'_1 = (f_{1,1}(\zeta, \boldsymbol{\phi}_1), \ldots, f_{1,N}(\zeta, \boldsymbol{\phi}_1))$ and so on. Therefore, by Lemma 6.4 we have

$$\|(\nabla G_{1,0}, \nabla G_{2,0})\|_{H^{m-1}} + \|(\boldsymbol{G}'_1, \boldsymbol{G}'_2)\|_{H^m} \leq C(M, c_1, \varepsilon),$$

where we notice for further use that $C(M, c_1, \varepsilon)$ is bounded uniformly with respect to $\varepsilon \in (0, 1]$. We obtain the desired result by the standard theory of the heat equation. ∎

Lemma 7.3. *Suppose that the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b satisfy the hypotheses in Lemma 7.2 and the compatibility conditions (7.5). Then the solution $(\zeta^\varepsilon, \boldsymbol{\phi}_1^\varepsilon, \boldsymbol{\phi}_2^\varepsilon)$ constructed in Lemma 7.2 satisfies the regularized Kakinuma model (7.3).*

Proof. By the construction of the solution, we easily see that it satisfies (7.8) and in particular the last equation in (7.3). Therefore, it is sufficient to show that it also satisfies the first two equations in (7.3). By (7.7) and (7.8), we have

$$\begin{cases} (\partial_t - \varepsilon\Delta)(\mathscr{L}_{1,i}(H_1)\boldsymbol{\phi}_1) = 0 & \text{for } i = 1, 2, \ldots, N, \\ (\partial_t - \varepsilon\Delta)(\mathscr{L}_{2,i}(H_2, b)\boldsymbol{\phi}_2) = 0 & \text{for } i = 1, 2, \ldots, N^*, \\ (\partial_t - \varepsilon\Delta)(\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1 + \mathscr{L}_{2,0}(H_2, b)\boldsymbol{\phi}_2) = 0, \end{cases}$$

so that by the uniqueness of the solution to the initial value problem of the heat equation, if the initial data satisfy the compatibility conditions (7.5), then the solution also satisfies (7.5) for all $t \in [0, T_\varepsilon)$. Particularly, we obtain

$$\begin{cases} -\boldsymbol{l}_1(H_1)(\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1) + L_1(H_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\ -\boldsymbol{l}_2(H_2)(\mathscr{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2) + L_2(H_2,b)\boldsymbol{\phi}_2 = \boldsymbol{0}, \end{cases}$$

so that by the last equation in the compatibility conditions (7.5) we have

$$\begin{cases} -\boldsymbol{l}_1(H_1)(\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1) + L_1(H_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\ \boldsymbol{l}_2(H_2)(\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1) + L_2(H_2,b)\boldsymbol{\phi}_2 = \boldsymbol{0}. \end{cases} \tag{7.12}$$

Taking Euclidean inner products of the first and the second equations with $\rho_1\boldsymbol{q}_1$ and $\rho_2\boldsymbol{q}_2$, respectively, adding the resulting equations, and using the relation $-\rho_1\boldsymbol{l}_1\cdot\boldsymbol{q}_1 + \rho_2\boldsymbol{l}_2\cdot\boldsymbol{q}_2 = 1$, we obtain

$$\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1 + \rho_1\boldsymbol{q}_1\cdot L_1(H_1)\boldsymbol{\phi}_1 + \rho_2\boldsymbol{q}_2\cdot L_2(H_2,b)\boldsymbol{\phi}_2 = 0,$$

which together with (7.7) implies

$$\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1 = -(\partial_t\zeta - \varepsilon\Delta\zeta).$$

Plugging this into (7.12), we see that the solution satisfies the first two equations in (7.3). ■

7.3. Uniform bound of the solution to the regularized problem

We proceed to derive estimates concerning solutions $(\zeta^\varepsilon, \boldsymbol{\phi}_1^\varepsilon, \boldsymbol{\phi}_2^\varepsilon)$ to the regularized Kakinuma model (7.3), uniform with respect to the regularized parameter $\varepsilon \in (0,1]$ and for a time interval independent of ε. To this end, we make use of the good symmetric structure of the Kakinuma model based on the analysis of Section 5.1. In order to simplify the notation we write $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ in place of $(\zeta^\varepsilon, \boldsymbol{\phi}_1^\varepsilon, \boldsymbol{\phi}_2^\varepsilon)$.

In view of (6.1) and (6.2) we decompose $L_1(H_1)\boldsymbol{\phi}_1$ and $L_2(H_2,b)\boldsymbol{\phi}_2$ into their principal parts and remainder parts as

$$L_1(H_1)\boldsymbol{\phi}_1 = -A_1(H_1)\Delta\boldsymbol{\phi}_1 + \boldsymbol{l}_1(H_1)(\boldsymbol{u}_1\cdot\nabla\zeta) + L_1^{\mathrm{low}}(H_1)\boldsymbol{\phi}_1, \tag{7.13}$$
$$L_2(H_2,b)\boldsymbol{\phi}_2 = -A_2(H_2)\Delta\boldsymbol{\phi}_2 - \boldsymbol{l}_2(H_2)(\boldsymbol{u}_2\cdot\nabla\zeta) + L_2^{\mathrm{low}}(H_2,b)\boldsymbol{\phi}_2, \tag{7.14}$$

where the matrices $A_1(H_1)$, $A_2(H_2)$ are given by (5.10), $L_1^{\mathrm{low}}(H_1) = (L_{1,ij}^{\mathrm{low}}(H_1))_{0\le i,j\le N}$ and $L_2^{\mathrm{low}}(H_2,b) = (L_{2,ij}^{\mathrm{low}}(H_2,b))_{0\le i,j\le N^*}$ are given by

$$\begin{aligned} L_{1,ij}^{\mathrm{low}}(H_1)\varphi_{1,j} &= \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\varphi_{1,j}, \\ L_{2,ij}^{\mathrm{low}}(H_2,b)\varphi_{2,j} &= \nabla b\cdot(H_2^{p_i+p_j}\nabla\varphi_{2,j} - p_jH_2^{p_i+p_j-1}\varphi_{2,j}\nabla b) \\ &\quad + \frac{p_j}{p_i+p_j}H_2^{p_i+p_j}\nabla\cdot(\varphi_{2,j}\nabla b) - \frac{p_i}{p_i+p_j}H_2^{p_i+p_j}\nabla b\cdot\nabla\varphi_{2,j} \\ &\quad + \frac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}(1+|\nabla b|^2)\varphi_{2,j}. \end{aligned}$$

Let us recall the definitions of $\boldsymbol{u}$ in (5.5), and θ_1 and θ_2 in (5.12), so that

$$\boldsymbol{u}_1 = \boldsymbol{u} - \theta_1 \boldsymbol{v}, \quad \boldsymbol{u}_2 = \boldsymbol{u} + \theta_2 \boldsymbol{v}.$$

Therefore, we can rewrite the first two equations in (7.3) as

$$\begin{cases} \boldsymbol{l}_1(H_1)(\partial_t \zeta - \varepsilon\Delta\zeta + (\boldsymbol{u} - \theta_1\boldsymbol{v})\cdot\nabla\zeta) - A_1(H_1)\Delta\boldsymbol{\phi}_1 + L_1^{\mathrm{low}}(H_1)\boldsymbol{\phi}_1 = \mathbf{0}, \\ -\boldsymbol{l}_2(H_2)(\partial_t \zeta - \varepsilon\Delta\zeta + (\boldsymbol{u} + \theta_2\boldsymbol{v})\cdot\nabla\zeta) - A_2(H_2)\Delta\boldsymbol{\phi}_2 + L_2^{\mathrm{low}}(H_2, b)\boldsymbol{\phi}_2 = \mathbf{0}. \end{cases}$$

Let $\beta = (\beta_1, \dots, \beta_n)$ be a multi-index satisfying $|\beta| \le m$. Applying the differential operator ∂^β to these equations and noting the relation $(\boldsymbol{v}\cdot\nabla) = -(\boldsymbol{v}\cdot\nabla)^* - (\nabla\cdot\boldsymbol{v})$, we have

$$\begin{cases} \rho_1\boldsymbol{l}_1(\partial_t\zeta^\beta - \varepsilon\Delta\zeta^\beta + \boldsymbol{u}\cdot\nabla\zeta^\beta) + (\boldsymbol{v}\cdot\nabla)^*(\rho_1\theta_1\boldsymbol{l}_1\zeta^\beta) \\ \qquad -\displaystyle\sum_{l=1}^n \partial_l(\rho_1 A_1 \partial_l \boldsymbol{\phi}_1^\beta) = \boldsymbol{F}_{1,\beta}, \\ -\rho_2\boldsymbol{l}_2(\partial_t\zeta^\beta - \varepsilon\Delta\zeta^\beta + \boldsymbol{u}\cdot\nabla\zeta^\beta) + (\boldsymbol{v}\cdot\nabla)^*(\rho_2\theta_2\boldsymbol{l}_2\zeta^\beta) \\ \qquad -\displaystyle\sum_{l=1}^n \partial_l(\rho_2 A_2 \partial_l \boldsymbol{\phi}_2^\beta) = \boldsymbol{F}_{2,\beta}, \end{cases} \tag{7.15}$$

where $\zeta^\beta = \partial^\beta\zeta$, $\boldsymbol{\phi}_k^\beta = \partial^\beta\boldsymbol{\phi}_k$ for $k = 1, 2$, and

$$\begin{cases} \boldsymbol{F}_{1,\beta} = \rho_1\Big\{ -[\partial^\beta, \boldsymbol{l}_1]G_0 - [\partial^\beta, \boldsymbol{l}_1\boldsymbol{u}_1^{\mathrm{T}}]\nabla\zeta \\ \qquad\qquad -(\nabla\cdot\boldsymbol{v})\theta_1\boldsymbol{l}_1\zeta^\beta + [\boldsymbol{v}\cdot\nabla, \theta_1\boldsymbol{l}_1]\zeta^\beta \\ \qquad\qquad -\displaystyle\sum_{l=1}^n (\partial_l A_1)\partial_l\boldsymbol{\phi}_1^\beta + [\partial^\beta, A_1]\Delta\boldsymbol{\phi}_1 - \partial^\beta L_1^{\mathrm{low}}(H_1)\boldsymbol{\phi}_1 \Big\}, \\ \boldsymbol{F}_{2,\beta} = \rho_2\Big\{ [\partial^\beta, \boldsymbol{l}_2]G_0 + [\partial^\beta, \boldsymbol{l}_2\boldsymbol{u}_2^{\mathrm{T}}]\nabla\zeta \\ \qquad\qquad -(\nabla\cdot\boldsymbol{v})\theta_2\boldsymbol{l}_2\zeta^\beta + [\boldsymbol{v}\cdot\nabla, \theta_2\boldsymbol{l}_2]\zeta^\beta \\ \qquad\qquad -\displaystyle\sum_{l=1}^n (\partial_l A_2)\partial_l\boldsymbol{\phi}_2^\beta + [\partial^\beta, A_2]\Delta\boldsymbol{\phi}_2 - \partial^\beta L_2^{\mathrm{low}}(H_2, b)\boldsymbol{\phi}_2 \Big\}. \end{cases}$$

In the above calculation, we used (7.7). Similarly, applying the differential operator ∂^β to the last equation in (7.3), we have

$$\begin{aligned} &-\rho_1\boldsymbol{l}_1\cdot(\partial_t\boldsymbol{\phi}_1^\beta - \varepsilon\Delta\boldsymbol{\phi}_1^\beta + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\phi}_1^\beta) + \rho_1\theta_1\boldsymbol{l}_1\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{\phi}_1^\beta \\ &\qquad + \rho_2\boldsymbol{l}_2\cdot(\partial_t\boldsymbol{\phi}_2^\beta - \varepsilon\Delta\boldsymbol{\phi}_2^\beta + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\phi}_2^\beta) + \rho_2\theta_2\boldsymbol{l}_2\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{\phi}_2^\beta + a\zeta^\beta \\ &\quad = F_{0,\beta}, \end{aligned} \tag{7.16}$$

where

$$
\begin{aligned}
a = \rho_2\Big(& \sum_{i=0}^{N^*} p_i H_2^{p_i-1}(G_{2,i} + \boldsymbol{u}_2 \cdot \nabla\phi_{2,i}) \\
& + \sum_{i=0}^{N^*} p_i(p_i-1)H_2^{p_i-2}(w_2 - \boldsymbol{u}_2 \cdot \nabla b)\phi_{2,i} + g\Big) \\
& + \rho_1\Big(\sum_{i=0}^{N} 2iH_1^{2i-1}(G_{1,i} + \boldsymbol{u}_1 \cdot \nabla\phi_{1,i}) - w_1 \sum_{i=0}^{N} 2i(2i-1)H^{2(i-1)}\phi_{1,i} - g\Big),
\end{aligned}
$$

$$
\begin{aligned}
F_{0,\beta} = \rho_1\Big\{ & \big(\partial^\beta \boldsymbol{l}_1(H_1) - (\partial_{H_1}\boldsymbol{l}_1(H_1))\partial^\beta H_1\big) \cdot \boldsymbol{G}_1 + [\partial^\beta; \boldsymbol{l}_1(H_1), \boldsymbol{G}_1] \\
& + \boldsymbol{u}_1 \cdot \sum_{j=0}^{N}\big([\partial^\beta; l_{1,j}(H_1), \nabla\phi_{1,j}] \\
& \qquad + \big(\partial^\beta l_{1,j}(H_1) - (\partial_{H_1} l_{1,j}(H_1))\partial^\beta H_1\big)\nabla\phi_{1,j}\big) \\
& - w_1 \sum_{j=0}^{N}\big([\partial^\beta, \phi_{1,j}]\partial_{H_1} l_{1,j}(H_1) \\
& \qquad + \big(\partial^\beta \partial_{H_1} l_{1,j}(H_1) - (\partial_{H_1}^2 l_{1,j}(H_1))\partial^\beta H_1\big)\phi_{1,j}\big) \\
& + \frac{1}{2}([\partial^\beta; \boldsymbol{u}_1, \boldsymbol{u}_1] + [\partial^\beta; w_1, w_1])\Big\} \\
- \rho_2\Big\{ & \big(\partial^\beta \boldsymbol{l}_2(H_2) - (\partial_{H_2}\boldsymbol{l}_2(H_2))\partial^\beta \zeta\big) \cdot \boldsymbol{G}_2 + [\partial^\beta; \boldsymbol{l}_2(H_2), \boldsymbol{G}_2] \\
& + \boldsymbol{u}_2 \cdot \sum_{j=0}^{N^*}\Big([\partial^\beta; l_{2,j}(H_2), \nabla\phi_{2,j}] \\
& \qquad + \big(\partial^\beta l_{2,j}(H_2) - (\partial_{H_2} l_{2,j}(H_2))\partial^\beta H_2\big)\nabla\phi_{2,j} \\
& \qquad - [\partial^\beta, \partial_{H_2} l_{2,j}(H_2)]\phi_{2,j}\nabla b \\
& \qquad - \big(\partial^\beta \partial_{H_2} l_{2,j}(H_2) - (\partial_{H_2}^2 l_{2,j}(H_2))\partial^\beta H_2\big)\phi_{2,j}\nabla b\Big) \\
& + w_2 \sum_{j=0}^{N^*}\big([\partial^\beta, \phi_{2,j}]\partial_{H_2} l_{2,j}(H_2) \\
& \qquad + \big(\partial^\beta \partial_{H_2} l_{2,j}(H_2) - (\partial_{H_2}^2 l_{2,j}(H_2))\partial^\beta H_2\big)\phi_{2,j}\big) \\
& + \frac{1}{2}([\partial^\beta; \boldsymbol{u}_2, \boldsymbol{u}_2] + [\partial^\beta; w_2, w_2])\Big\}.
\end{aligned}
$$

In the above calculation, we used (7.9) and the notation

$$
\begin{cases}
\boldsymbol{l}_1(H_1) = (l_{1,0}(H_1), l_{1,1}(H_1), \dots, l_{1,N}(H_1))^{\mathrm{T}}, \\
\boldsymbol{l}_2(H_2) = (l_{2,0}(H_2), l_{2,1}(H_2), \dots, l_{2,N^*}(H_2))^{\mathrm{T}},
\end{cases}
$$

and the notation for the symmetric commutator $[\partial^\beta; \boldsymbol{u}, \boldsymbol{v}] = \partial^\beta(\boldsymbol{u}\cdot\boldsymbol{v}) - (\partial^\beta \boldsymbol{u})\cdot\boldsymbol{v} - \boldsymbol{u}\cdot(\partial^\beta \boldsymbol{v})$. We can rewrite (7.15) and (7.16) in matrix form as

$$\mathcal{A}_1(\partial_t \boldsymbol{U}^\beta - \varepsilon\Delta \boldsymbol{U}^\beta + (\boldsymbol{u}\cdot\nabla)\boldsymbol{U}^\beta) + \mathcal{A}_0^{\mathrm{mod}}\boldsymbol{U}^\beta = \boldsymbol{F}_\beta, \tag{7.17}$$

where

$$\boldsymbol{U}^\beta = \begin{pmatrix} \zeta^\beta \\ \boldsymbol{\phi}_1^\beta \\ \boldsymbol{\phi}_2^\beta \end{pmatrix}, \quad \boldsymbol{F}_\beta = \begin{pmatrix} F_{0,\beta} \\ \boldsymbol{F}_{1,\beta} \\ \boldsymbol{F}_{2,\beta} \end{pmatrix},$$

and

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -\rho_1 \boldsymbol{l}_1^{\mathrm{T}} & \rho_2 \boldsymbol{l}_2^{\mathrm{T}} \\ \rho_1 \boldsymbol{l}_1 & O & O \\ -\rho_2 \boldsymbol{l}_2 & O & O \end{pmatrix},$$

$$\mathcal{A}_0^{\mathrm{mod}} = \begin{pmatrix} a & \rho_1\theta_1 \boldsymbol{l}_1^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla) & \rho_2\theta_2 \boldsymbol{l}_2^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla) \\ (\boldsymbol{v}\cdot\nabla)^*(\rho_1\theta_1\boldsymbol{l}_1\,\cdot) & -\sum_{l=1}^n \partial_l(\rho_1 A_1 \partial_l\,\cdot) & O \\ (\boldsymbol{v}\cdot\nabla)^*(\rho_2\theta_2\boldsymbol{l}_2\,\cdot) & O & -\sum_{l=1}^n \partial_l(\rho_2 A_2 \partial_l\,\cdot) \end{pmatrix}.$$

Here, we note that $\mathcal{A}_1$ is a skew-symmetric matrix and $\mathcal{A}_0^{\mathrm{mod}}$ is symmetric in $L^2(\mathbf{R}^n)$. Concerning the positivity of $\mathcal{A}_0^{\mathrm{mod}}$, we have the following lemma.

Lemma 7.4. *Let c_0 and C_0 be positive constants. Then there exists $C = C(c_0, C_0) > 0$ such that if a, H_1, H_2, and $\boldsymbol{v}$ satisfy*

$$\begin{cases} \|a\|_{L^\infty} + \|(H_1, H_2)\|_{L^\infty} + \|\boldsymbol{v}\|_{L^\infty} \le C_0, \\ H_1(\boldsymbol{x}) \ge c_0, \quad H_2(\boldsymbol{x}) \ge c_0 \quad \text{for } \boldsymbol{x}\in\mathbf{R}^n, \end{cases} \tag{7.18}$$

and the stability condition

$$a(\boldsymbol{x}) - \frac{\rho_1\rho_2}{\rho_1 H_2(\boldsymbol{x})\alpha_2 + \rho_2 H_1(\boldsymbol{x})\alpha_1}|\boldsymbol{v}(\boldsymbol{x})|^2 \ge c_0 > 0 \quad \text{for } \boldsymbol{x}\in\mathbf{R}^n, \tag{7.19}$$

then for any $\dot{\boldsymbol{U}} = (\dot\zeta, \dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2)^{\mathrm{T}}$, we have the equivalence

$$C^{-1}\|(\dot\zeta, \nabla\dot{\boldsymbol{\phi}}_1, \nabla\dot{\boldsymbol{\phi}}_2)\|_{L^2}^2 \le (\mathcal{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}}, \dot{\boldsymbol{U}})_{L^2} \le C\|(\dot\zeta, \nabla\dot{\boldsymbol{\phi}}_1, \nabla\dot{\boldsymbol{\phi}}_2)\|_{L^2}^2.$$

Proof. Introducing diagonal matrices $D_1(H_1)$ and $D_2(H_2)$ as

$$\begin{cases} D_1(H_1) = \mathrm{diag}(1, H_1^2, H_1^4, \dots, H_1^{2N}), \\ D_2(H_2) = \mathrm{diag}(1, H_2^{p_1}, H_2^{p_2}, \dots, H_2^{p_{N^*}}), \end{cases}$$

we have

$$A_k(H_k) = H_k D_k(H_k) A_{k,0} D_k(H_k), \quad k = 1, 2,$$

where $A_{1,0}$ and $A_{2,0}$ are constant matrices defined by

$$A_{1,0} = \left(\frac{1}{2(i+j)+1}\right)_{0\le i,j\le N}, \quad A_{2,0} = \left(\frac{1}{p_i + p_j + 1}\right)_{0\le i,j\le N^*}.$$

We also have

$$\mathbf{1}\cdot D_k(H_k)\boldsymbol{\phi}_k = \boldsymbol{l}_k(H_k)\cdot\boldsymbol{\phi}_k, \quad k=1,2.$$

Therefore,

$$\begin{aligned}
(\mathcal{A}_0^{\mathrm{mod}}\dot{\boldsymbol{U}},\dot{\boldsymbol{U}})_{L^2} &= (a\dot{\zeta},\dot{\zeta})_{L^2} + \sum_{l=1}^{n}\sum_{k=1,2}(\rho_k H_k A_{k,0} D_k\partial_l\dot{\boldsymbol{\phi}}_k, D_k\partial_l\dot{\boldsymbol{\phi}}_k)_{L^2} \\
&\quad + 2\sum_{k=1,2}(\rho_k\theta_k\boldsymbol{l}_k\cdot(\boldsymbol{v}\cdot\nabla)\dot{\boldsymbol{\phi}}_k,\dot{\zeta})_{L^2} \\
&= \sum_{l=1}^{n}\sum_{k=1,2}(\rho_k H_k Q_{k,0}A_{k,0}D_k\partial_l\dot{\boldsymbol{\phi}}_k, A_{k,0}D_k\partial_l\dot{\boldsymbol{\phi}}_k)_{L^2} \\
&\quad + (a\dot{\zeta},\dot{\zeta})_{L^2} + \sum_{k=1,2}\{(\rho_k H_k\alpha_k(\boldsymbol{l}_k\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_k,(\boldsymbol{l}_k\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_k)_{L^2} \\
&\qquad\qquad + 2(\rho_k\theta_k\boldsymbol{v}\cdot(\boldsymbol{l}_k\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_k,\dot{\zeta})_{L^2}\} \\
&=: I_1 + I_2,
\end{aligned}$$

where we used identity (5.7). Since $Q_{1,0}$ and $Q_{2,0}$ are nonnegative and in view of

$$I_2 \ge \int_{\mathbf{R}^n}\left\{a\dot{\zeta}^2 + \sum_{k=1,2}\{\rho_k H_k\alpha_k|(\boldsymbol{l}_k\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_k|^2 - 2\rho_k\theta_k|\boldsymbol{v}|\,|(\boldsymbol{l}_k\otimes\nabla)^{\mathrm{T}}\dot{\boldsymbol{\phi}}_k|\,|\dot{\zeta}|\}\right\}\mathrm{d}\boldsymbol{x}$$

and the analysis in Section 5.1, we can show the desired equivalence. ■

Lemma 7.5. *Let g, ρ_1, ρ_2, h_1, h_2, c_0, M_0 be positive constants and m an integer such that $m > \frac{n}{2}+1$. There exist a positive time T and a positive constant C such that if initial data $(\zeta_{(0)},\boldsymbol{\phi}_{1(0)},\boldsymbol{\phi}_{2(0)})$ and bottom topography b satisfy*

$$\begin{cases}
\|(\zeta_{(0)},\nabla\phi_{1,0(0)},\nabla\phi_{2,0(0)})\|_{H^m} + \|(\boldsymbol{\phi}_{1(0)}',\boldsymbol{\phi}_{2(0)}')\|_{H^{m+1}} + \|b\|_{W^{m+2,\infty}} \le M_0, \\
h_1 - \zeta_{(0)}(\boldsymbol{x}) \ge 2c_0, \quad h_2 + \zeta_{(0)}(\boldsymbol{x}) - b(\boldsymbol{x}) \ge 2c_0 \quad \textit{for } \boldsymbol{x}\in\mathbf{R}^n,
\end{cases}$$

the stability condition (7.19) with c_0 replaced by $2c_0$, and the compatibility conditions (7.5), then for any $\varepsilon\in(0,1]$ the solution $(\zeta^\varepsilon,\boldsymbol{\phi}_1^\varepsilon,\boldsymbol{\phi}_2^\varepsilon)$ constructed in Lemmas 7.2 and 7.3 satisfies

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\|(\zeta^\varepsilon(t),\nabla\phi_{1,0}^\varepsilon(t),\nabla\phi_{2,0}^\varepsilon(t))\|_{H^m}^2 + \|(\boldsymbol{\phi}_1^{\varepsilon\prime},\boldsymbol{\phi}_2^{\varepsilon\prime})\|_{H^{m+1}}^2\right) \\
&\quad + \varepsilon\int_0^T \|(\zeta^\varepsilon(t),\nabla\boldsymbol{\phi}_1^\varepsilon(t),\nabla\boldsymbol{\phi}_2^\varepsilon(t))\|_{H^{m+1}}^2\,\mathrm{d}t \le C.
\end{aligned}$$

Proof. Once again we simply write $\boldsymbol{U} = (\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)^{\mathrm{T}}$ in place of $(\zeta^\varepsilon,\boldsymbol{\phi}_1^\varepsilon,\boldsymbol{\phi}_2^\varepsilon)^{\mathrm{T}}$. We define an energy function $\mathcal{E}_m(t)$ by

$$\mathcal{E}_m(t) = \sum_{|\beta|\le m}\{(\mathcal{A}_0^{\mathrm{mod}}\partial^\beta\boldsymbol{U}(t),\partial^\beta\boldsymbol{U}(t))_{L^2} + \|(\partial^\beta\boldsymbol{\phi}_1'(t),\partial^\beta\boldsymbol{\phi}_2'(t))\|_{L^2}^2\}.$$

We assume that the solution $(\zeta(t), \boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t))$ satisfies (7.18) and the stability condition (7.19) for $0 \le t \le T$. Then the energy function $\mathcal{E}_m(t)$ is equivalent to

$$E_m(t) = \|(\zeta(t), \nabla\phi_{1,0}(t), \nabla\phi_{2,0}(t))\|_{H^m}^2 + \|(\boldsymbol{\phi}_1'(t), \boldsymbol{\phi}_2'(t))\|_{H^{m+1}}^2.$$

Furthermore, we assume that

$$E_m(t) + \varepsilon \int_0^t E_{m+1}(\tau)\,\mathrm{d}\tau \le M_1 \tag{7.20}$$

for $0 \le t \le T$, where the constant M_1 and the time T will be determined later. In the following we simply write the constants depending only on $(g, \rho_1, \rho_2, h_1, h_2, c_0, C_0, M_0)$ as C_1 and the constants depending also on M_1 as C_2. They may change from line to line. Then it holds that

$$C_1^{-1} E_j(t) \le \mathcal{E}_j(t) \le C_1 E_j(t)$$

for $j = 0, 1, 2, \ldots$. We are going to evaluate the evolution of the energy function $\mathcal{E}_m(t)$. To this end, we take the L^2-inner product of (7.17) with $\partial_t \boldsymbol{U}^\beta - \varepsilon\Delta\boldsymbol{U}^\beta + (\boldsymbol{u}\cdot\nabla)\boldsymbol{U}^\beta$ and use integration by parts to get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{A}_0^{\mathrm{mod}}\boldsymbol{U}^\beta, \boldsymbol{U}^\beta)_{L^2} + \varepsilon\sum_{l=1}^n (\mathcal{A}_0^{\mathrm{mod}}\partial_l\boldsymbol{U}^\beta, \partial_l\boldsymbol{U}^\beta)_{L^2} \\
&\quad = \frac{1}{2}([\partial_t, \mathcal{A}_0^{\mathrm{mod}}]\boldsymbol{U}^\beta, \boldsymbol{U}^\beta)_{L^2} - \varepsilon\sum_{l=1}^n([\partial_l, \mathcal{A}_0^{\mathrm{mod}}]\boldsymbol{U}^\beta, \partial_l\boldsymbol{U}^\beta)_{L^2} - (\mathcal{A}_0^{\mathrm{mod}}\boldsymbol{U}^\beta, (\boldsymbol{u}\cdot\nabla)\boldsymbol{U}^\beta)_{L^2} \\
&\qquad + (F_{0,\beta}, \partial^\beta G_0 + (\boldsymbol{u}\cdot\nabla)\zeta^\beta)_{L^2} + \sum_{k=1,2}(\boldsymbol{F}_{k,\beta}, \partial^\beta\boldsymbol{G}_k + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\phi}_k^\beta)_{L^2}.
\end{aligned}$$

Here, we see that

$$\begin{aligned}
&([\partial_t, \mathcal{A}_0^{\mathrm{mod}}]\boldsymbol{U}^\beta, \boldsymbol{U}^\beta)_{L^2} \\
&\quad = ((\partial_t a)\zeta^\beta, \zeta^\beta)_{L^2} \\
&\qquad + 2\sum_{k=1,2}\rho_k([\partial_t, \theta_k\boldsymbol{l}_k^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla)]\boldsymbol{\phi}_k^\beta, \zeta^\beta)_{L^2} + \sum_{l=1}^n\sum_{k=1,2}\rho_k((\partial_t A_k)\partial_l\boldsymbol{\phi}_k^\beta, \partial_l\boldsymbol{\phi}_k^\beta)_{L^2}, \\
&([\partial_l, \mathcal{A}_0^{\mathrm{mod}}]\boldsymbol{U}^\beta, \partial_l\boldsymbol{U}^\beta)_{L^2} \\
&\quad = ((\partial_l a)\zeta^\beta, \zeta^\beta)_{L^2} \\
&\qquad + \sum_{k=1,2}\rho_k\{([\partial_l, \theta_k\boldsymbol{l}_k^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla)]\boldsymbol{\phi}_k^\beta, \partial_l\zeta^\beta)_{L^2} + (\zeta^\beta, [\partial_l, \theta_k\boldsymbol{l}_k^{\mathrm{T}}(\boldsymbol{v}\cdot\nabla)]\partial_l\boldsymbol{\phi}_k^\beta)_{L^2}\} \\
&\qquad + \sum_{k=1,2}\sum_{j=1}^n\rho_k((\partial_j A_k)\partial_l\boldsymbol{\phi}_k^\beta, \partial_j\partial_l\boldsymbol{\phi}_k^\beta)_{L^2},
\end{aligned}$$

$$
\begin{aligned}
&(\mathcal{A}_0^{\rm mod}\boldsymbol{U}^\beta, (\boldsymbol{u}\cdot\nabla)\boldsymbol{U}^\beta)_{L^2} \\
&\quad = -\frac{1}{2}\big((\nabla\cdot(a\boldsymbol{u}))\zeta^\beta, \zeta^\beta\big)_{L^2} \\
&\qquad - \sum_{k=1,2}\rho_k\{((\nabla\cdot\boldsymbol{u})\zeta^\beta, \theta_k\boldsymbol{l}_k\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{\phi}_k^\beta)_{L^2} + (\zeta^\beta, [(\boldsymbol{u}\cdot\nabla), \theta_k\boldsymbol{l}_k^{\rm T}(\boldsymbol{v}\cdot\nabla)]\boldsymbol{\phi}_k^\beta)_{L^2}\} \\
&\qquad - \sum_{k=1,2}\sum_{l=1}^n\rho_k\Big\{\big(A_k\partial_l\boldsymbol{\phi}_k^\beta, ((\partial_l\boldsymbol{u})\cdot\nabla)\boldsymbol{\phi}_k^\beta\big)_{L^2} + \frac{1}{2}\big(((\boldsymbol{u}\cdot\nabla)^*A_k)\partial_l\boldsymbol{\phi}_k^\beta, \partial_l\boldsymbol{\phi}_k^\beta\big)_{L^2}\Big\},
\end{aligned}
$$

so that for $1 \le |\beta| \le m$ we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{A}_0^{\rm mod}\boldsymbol{U}^\beta, \boldsymbol{U}^\beta)_{L^2} + \varepsilon\sum_{l=1}^n(\mathcal{A}_0^{\rm mod}\partial_l\boldsymbol{U}^\beta, \partial_l\boldsymbol{U}^\beta)_{L^2} \\
&\quad \le C_2(1+\varepsilon E_{m+1}(t)^{\frac{1}{2}}) + \|F_{0,\beta}\|_{H^1}\|\partial^\beta G_0 + (\boldsymbol{u}\cdot\nabla)\zeta^\beta\|_{H^{-1}} \\
&\qquad + \sum_{k=1,2}\|\boldsymbol{F}_{k,\beta}\|_{L^2}\|\partial^\beta\boldsymbol{G}_k + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\phi}_k^\beta\|_{L^2} \\
&\quad \le C_2(1+\varepsilon E_{m+1}(t)^{\frac{1}{2}}).
\end{aligned}
\tag{7.21}
$$

A similar estimate can be obtained in the case $|\beta| = 0$ more directly. On the other hand, it follows from (7.9) that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\boldsymbol{\phi}_1^{\beta\prime}, \boldsymbol{\phi}_2^{\beta\prime})\|_{L^2}^2 + \varepsilon\|(\nabla\boldsymbol{\phi}_1^{\beta\prime}, \nabla\boldsymbol{\phi}_2^{\beta\prime})\|_{L^2}^2 = \sum_{k=1,2}(\partial^\beta\boldsymbol{G}_k', \boldsymbol{\phi}_k^{\beta\prime})_{L^2}^2 \le C_2.
$$

Therefore, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_m(t) + \varepsilon E_{m+1}(t) \le C_2(1+\varepsilon E_{m+1}(t)^{\frac{1}{2}}),
$$

which yields

$$
E_m(t) + \varepsilon\int_0^t E_{m+1}(\tau)\,\mathrm{d}\tau \le C_1 + C_2 t.
$$

Putting $M_1 = 2C_1$ and taking $T > 0$ so that $C_2T \le C_1$, we obtain by a continuity argument that (7.20) holds for $0 \le t \le T$.

It remains to show that $(\zeta(t), \boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t))$ satisfies (7.18) and the stability condition (7.19) for $0 \le t \le T$. By the Sobolev embedding theorem, (7.7), and (7.9), we see that

$$
\begin{aligned}
&|\zeta(\boldsymbol{x},t) - \zeta_{(0)}(\boldsymbol{x})| + \sum_{k=1,2}(|\nabla\boldsymbol{\phi}_k(\boldsymbol{x},t) - \nabla\boldsymbol{\phi}_{k(0)}(\boldsymbol{x})| + |\boldsymbol{\phi}_k'(\boldsymbol{x},t) - \boldsymbol{\phi}_{k(0)}'(\boldsymbol{x})|) \\
&\quad \le C_1\Big(\|\zeta(t) - \zeta_{(0)}\|_{H^{m-1}} \\
&\qquad + \sum_{k=1,2}(\|\nabla\boldsymbol{\phi}_k(t) - \nabla\boldsymbol{\phi}_{k(0)}\|_{H^{m-1}} + \|\boldsymbol{\phi}_k'(t) - \boldsymbol{\phi}_{k(0)}'\|_{H^{m-1}})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le C_1 \int_0^t \Big(\|\partial_t \zeta(\tau)\|_{H^{m-1}} + \sum_{k=1,2} (\|\nabla \partial_t \boldsymbol{\phi}_k(\tau)\|_{H^{m-1}} + \|\partial_t \boldsymbol{\phi}_k'(\tau)\|_{H^{m-1}}) \Big) \mathrm{d}\tau \\
&\le C_1 \int_0^t (\|(G_0, \nabla G_{1,0}, \nabla G_{2,0})(\tau)\|_{H^{m-1}} + \|(\boldsymbol{G}_1', \boldsymbol{G}_2')(\tau)\|_{H^m} + \varepsilon E_{m+1}(\tau)^{\frac{1}{2}}) \, \mathrm{d}\tau \\
&\le C_2(t + \sqrt{\varepsilon t}),
\end{aligned} \tag{7.22}
$$

which yields (7.18), except for the estimate for a, by taking $T > 0$ sufficiently small. We now turn to the stability condition (7.19). In order to evaluate $\partial_t a$, we need to obtain estimates for $\partial_t \boldsymbol{G}_k'$ for $k = 1, 2$. Differentiating (7.10) with respect to t, we have

$$
\begin{cases}
\mathscr{L}_{1,i}(H_1) \partial_t \boldsymbol{G}_1 = g_{1,i} & \text{for } i = 1, 2, \dots, N, \\
\mathscr{L}_{2,i}(H_2, b) \partial_t \boldsymbol{G}_2 = g_{2,i} & \text{for } i = 1, 2, \dots, N^*, \\
\mathscr{L}_{1,0}(H_1) \partial_t \boldsymbol{G}_1 + \mathscr{L}_{2,0}(H_2, b) \partial_t \boldsymbol{G}_2 = \nabla \cdot \boldsymbol{g}_3, \\
-\rho_1 \boldsymbol{l}_1(H_1) \cdot \partial_t \boldsymbol{G}_1 + \rho_2 \boldsymbol{l}_2(H_2) \cdot \partial_t \boldsymbol{G}_2 = g_4,
\end{cases}
$$

where

$$
\begin{cases}
g_{1,i} = -[\partial_t, \mathscr{L}_{1,i}(H_1)] \boldsymbol{G}_1 \\
\qquad\quad -\partial_t (f_{1,i}(\zeta, \boldsymbol{\phi}_1) G_0 - \varepsilon \tilde{f}_{1,i}(\zeta, \boldsymbol{\phi}_1)) & \text{for } i = 1, 2, \dots, N, \\
g_{2,i} = -[\partial_t, \mathscr{L}_{2,i}(H_2, b)] \boldsymbol{G}_2 \\
\qquad\quad -\partial_t (f_{2,i}(\zeta, \boldsymbol{\phi}_2, b) G_0 - \varepsilon \tilde{f}_{2,i}(\zeta, \boldsymbol{\phi}_2, b)) & \text{for } i = 1, 2, \dots, N^*, \\
\boldsymbol{g}_3 = (\partial_t \zeta) \Big(-\sum_{j=0}^{N} H_1^{2j} \nabla G_{1,j} + \sum_{j=0}^{N^*} (H_2^{p_j} \nabla G_{2,j} - p_j H_2^{p_j - 1} G_{2,j} \nabla b) \Big) \\
\qquad + \partial_t (\boldsymbol{v} G_0 + \varepsilon \boldsymbol{f}_3(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2, b)), \\
g_4 = \rho_1 [\partial_t, \boldsymbol{l}_1(H_1)^{\mathrm{T}}] \boldsymbol{G}_1 - \rho_2 [\partial_t, \boldsymbol{l}_2(H_2)^{\mathrm{T}}] \boldsymbol{G}_2 + \partial_t F.
\end{cases}
$$

Therefore, by Lemma 6.4 with $k = m - 2$ we obtain

$$
\begin{aligned}
&\|(\nabla \partial_t G_{1,0}, \nabla \partial_t G_{2,0})\|_{H^{m-2}} + \|(\partial_t \boldsymbol{G}_1', \partial_t \boldsymbol{G}_2')\|_{H^{m-1}} \\
&\quad \le C_2 (\|(\boldsymbol{g}_1, \boldsymbol{g}_2)\|_{H^{m-3}} + \|(\boldsymbol{g}_3, \nabla g_4)\|_{H^{m-2}}) \\
&\quad \le C_2 (\|(\partial_t \zeta, \nabla \partial_t \phi_{1,0}, \nabla \partial_t \phi_{2,0})\|_{H^{m-1}} + \|(\partial_t \boldsymbol{\phi}_1', \partial_t \boldsymbol{\phi}_2')\|_{H^m}).
\end{aligned}
$$

On the other hand, it follows from (7.7) and (7.9) that

$$
\begin{aligned}
&\|(\partial_t \zeta, \nabla \partial_t \phi_{1,0}, \nabla \partial_t \phi_{2,0})\|_{H^{m-1}} + \|(\partial_t \boldsymbol{\phi}_1', \partial_t \boldsymbol{\phi}_2')\|_{H^m} \\
&\quad \le \|(G_0, \nabla G_{1,0}, \nabla G_{2,0})\|_{H^{m-1}} + \|(\boldsymbol{G}_1', \boldsymbol{G}_2')\|_{H^m} \\
&\qquad + \varepsilon (\|(\zeta, \nabla \phi_{1,0}, \nabla \phi_{2,0})\|_{H^{m+1}} + \|(\boldsymbol{\phi}_1', \boldsymbol{\phi}_2')\|_{H^{m+2}}) \\
&\quad \le C_2 (1 + \varepsilon E_{m+1}(t)^{\frac{1}{2}}).
\end{aligned}
$$

Thus,

$$\begin{aligned}\|\partial_t a\|_{H^{m-1}} &\le C_2(\|(\partial_t\zeta, \nabla\partial_t\phi_{1,0}, \nabla\partial_t\phi_{2,0}, \partial_t \boldsymbol{G}_1', \partial_t \boldsymbol{G}_2')\|_{H^{m-1}} + \|(\partial_t\boldsymbol{\phi}_1', \partial_t\boldsymbol{\phi}_2')\|_{H^m}) \\ &\le C_2(1+\varepsilon E_{m+1}(t)^{\frac{1}{2}}),\end{aligned}$$

so that

$$|a(x,t)-a(x,0)| \le C_1\int_0^t \|\partial_t a(\tau)\|_{H^{m-1}}\,\mathrm{d}\tau \le C_2(t+\sqrt{\varepsilon t}).$$

This together with (7.22) yields (7.18) and the stability condition (7.19) by taking $T>0$ sufficiently small. This completes the proof. ■

Once we obtain this kind of uniform estimate, compactness arguments allow us to pass to the limit $\varepsilon \to +0$ in the regularized problem (7.3) and (7.4). By construction, the limit $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ satisfies (2.14)–(2.17) and

$$\begin{cases}\zeta, \nabla\phi_{1,0}, \nabla\phi_{2,0} \in L^\infty(0,T;H^m)\cap C([0,T];H^{m-1}),\\ \boldsymbol{\phi}_1', \boldsymbol{\phi}_2' \in L^\infty(0,T;H^{m+1})\cap C([0,T];H^m),\\ \partial^\beta\zeta, \partial^\beta\nabla\boldsymbol{\phi}_1, \partial^\beta\nabla\boldsymbol{\phi}_2 \in C_{\mathrm{w}}([0,T];L^2)\end{cases}$$

for any multi-index β satisfying $|\beta|=m$. It remains to show that the above weak continuity in time can be replaced by strong continuity. To this end, we use the technique by Majda [24], that is, we make use of the energy estimate. See also Majda and Bertozzi [25]. For each $t\in[0,T]$ we introduce an inner product

$$\langle(\eta, \nabla\boldsymbol{\psi}_1, \nabla\boldsymbol{\psi}_2), (\tilde\eta, \nabla\tilde{\boldsymbol{\psi}}_1, \nabla\tilde{\boldsymbol{\psi}}_2)\rangle_t := (\mathscr{A}_0^{\mathrm{mod}}(t)\boldsymbol{V}, \widetilde{\boldsymbol{V}})_{L^2}$$

with $\boldsymbol{V}=(\eta, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)^{\mathrm{T}}$ and $\widetilde{\boldsymbol{V}}=(\tilde\eta, \tilde{\boldsymbol{\psi}}_1, \tilde{\boldsymbol{\psi}}_2)^{\mathrm{T}}$, and denote the corresponding norm by $\|\cdot\|_t$, which is equivalent to the standard L^2-norm by Lemma 7.4. By using the energy estimate corresponding to (7.21), for any multi-index β satisfying $|\beta|=m$ we can show the continuity of $\|(\partial^\beta\zeta(t), \partial^\beta\nabla\boldsymbol{\phi}_1(t), \partial^\beta\nabla\boldsymbol{\phi}_2(t))\|_t$ in $t\in[0,T]$. Particularly, for each $t_0\in[0,T]$ we have

$$\lim_{t\to t_0}\|(\partial^\beta\zeta(t), \partial^\beta\nabla\boldsymbol{\phi}_1(t), \partial^\beta\nabla\boldsymbol{\phi}_2(t))\|_{t_0} = \|(\partial^\beta\zeta(t_0), \partial^\beta\nabla\boldsymbol{\phi}_1(t_0), \partial^\beta\nabla\boldsymbol{\phi}_2(t_0))\|_{t_0}.$$

Since we already knew weak continuity, this gives strong continuity, that is, we have $\partial^\beta\zeta, \partial^\beta\nabla\boldsymbol{\phi}_1, \partial^\beta\nabla\boldsymbol{\phi}_2 \in C([0,T];L^2)$. Thus, Theorem 2.1 follows.

8. Hamiltonian structure

In this section we will show that the Kakinuma model (2.14)–(2.16) also enjoys a Hamiltonian structure analogous to the one exhibited by Benjamin and Bridges [1] on the full interfacial gravity waves. We recall that the Kakinuma model can be written simply as

$$\begin{cases} \boldsymbol{l}_1(H_1)\partial_t\zeta + L_1(H_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\ -\boldsymbol{l}_2(H_2)\partial_t\zeta + L_2(H_2,b)\boldsymbol{\phi}_2 = \boldsymbol{0}, \\ -\rho_1\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + \rho_2\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 = F, \end{cases} \tag{8.1}$$

where $\boldsymbol{\phi}_1 = (\phi_{1,0},\phi_{1,1},\dots,\phi_{1,N})^{\mathrm{T}}$, $\boldsymbol{\phi}_2 = (\phi_{2,0},\phi_{2,1},\dots,\phi_{2,N^*})^{\mathrm{T}}$, $\boldsymbol{l}_k$ and L_k for $k=1,2$ are defined by (6.3), and F is defined by

$$\begin{aligned} F &= \rho_1\Big\{g\zeta + \frac{1}{2}\big(|(\nabla\Phi_1^{\mathrm{app}})|_{z=\zeta}|^2 + ((\partial_z\Phi_1^{\mathrm{app}})|_{z=\zeta})^2\big)\Big\} \\ &\quad - \rho_2\Big\{g\zeta + \frac{1}{2}\big(|(\nabla\Phi_2^{\mathrm{app}})|_{z=\zeta}|^2 + ((\partial_z\Phi_2^{\mathrm{app}})|_{z=\zeta})^2\big)\Big\}. \end{aligned} \tag{8.2}$$

Here, Φ_1^{app} and Φ_2^{app} are approximate velocity potentials defined by (1.4).

8.1. Hamiltonian

As was expected, the Hamiltonian would be the total energy. In terms of our variables $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$, the total energy $\mathcal{E}^{\mathrm{K}}$ is given by

$$\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2) = \int_{\mathbf{R}^n} e^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\,\mathrm{d}\boldsymbol{x}, \tag{8.3}$$

where the density of the energy $e^{\mathrm{K}} = e^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ is given by

$$\begin{aligned} e^{\mathrm{K}} &= \int_\zeta^{h_1} \frac{1}{2}\rho_1(|\nabla\Phi_1^{\mathrm{app}}|^2 + (\partial_z\Phi_1^{\mathrm{app}})^2)\,\mathrm{d}z + \int_{-h_2+b}^{\zeta} \frac{1}{2}\rho_2(|\nabla\Phi_2^{\mathrm{app}}|^2 + (\partial_z\Phi_2^{\mathrm{app}})^2)\,\mathrm{d}z \\ &\quad + \frac{1}{2}(\rho_2-\rho_1)g\zeta^2 \\ &= \frac{1}{2}\rho_1\sum_{i,j=0}^{N}\Big(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,i}\cdot\nabla\phi_{1,j} \\ &\qquad\qquad + \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\phi_{1,i}\phi_{1,j}\Big) \\ &\quad + \frac{1}{2}\rho_2\sum_{i,j=0}^{N^*}\Big(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,i}\cdot\nabla\phi_{2,j} \\ &\qquad\qquad - \frac{2p_i}{p_i+p_j}H_2^{p_i+p_j}\phi_{2,i}\nabla b\cdot\nabla\phi_{2,j} \\ &\qquad\qquad + \frac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}(1+|\nabla b|^2)\phi_{2,i}\phi_{2,j}\Big) \\ &\qquad\qquad + \frac{1}{2}(\rho_2-\rho_1)g\zeta^2. \end{aligned} \tag{8.4}$$

By integration by parts, we also have

$$\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=\int_{\mathbf{R}^n}\Bigl(\frac{1}{2}\rho_1 L_1(H_1)\boldsymbol{\phi}_1\cdot\boldsymbol{\phi}_1+\frac{1}{2}\rho_2 L_2(H_2,b)\boldsymbol{\phi}_2\cdot\boldsymbol{\phi}_2+\frac{1}{2}(\rho_2-\rho_1)g\zeta^2\Bigr)\,\mathrm{d}\boldsymbol{x}.$$

In view of the symmetry of the operators $L_1(H_1)$ and $L_2(H_2,b)$, we can easily calculate the variational derivatives of this energy functional and obtain

$$\begin{cases}\delta_\zeta\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=-F,\\ \delta_{\boldsymbol{\phi}_1}\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=\rho_1 L_1(H_1)\boldsymbol{\phi}_1,\\ \delta_{\boldsymbol{\phi}_2}\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)=\rho_2 L_2(H_2,b)\boldsymbol{\phi}_2.\end{cases}\tag{8.5}$$

Therefore, the Kakinuma model (8.1) can be written as

$$\begin{pmatrix}0 & \rho_1\boldsymbol{l}_1(H_1)^{\mathrm{T}} & -\rho_2\boldsymbol{l}_2(H_2)^{\mathrm{T}}\\ -\rho_1\boldsymbol{l}_1(H_1) & O & O\\ \rho_2\boldsymbol{l}_2(H_2) & O & O\end{pmatrix}\partial_t\begin{pmatrix}\zeta\\ \boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\end{pmatrix}=\begin{pmatrix}\delta_\zeta\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\\ \delta_{\boldsymbol{\phi}_1}\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\\ \delta_{\boldsymbol{\phi}_2}\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)\end{pmatrix}.\tag{8.6}$$

As we will see later, the canonical variables of the Kakinuma model are the surface elevation ζ and ϕ given by

$$\phi=\rho_2\Phi_2^{\mathrm{app}}|_{z=\zeta}-\rho_1\Phi_1^{\mathrm{app}}|_{z=\zeta}=\rho_2\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2-\rho_1\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1,\tag{8.7}$$

which is the canonical variable for the full interfacial gravity waves found by Benjamin and Bridges [1] with (Φ_1,Φ_2) replaced by $(\Phi_1^{\mathrm{app}},\Phi_2^{\mathrm{app}})$. Then the compatibility conditions (2.18)–(2.20) and (8.7) are written in the form

$$\begin{cases}\mathcal{L}_{1,i}(H_1)\boldsymbol{\phi}_1=0 & \text{for } i=1,2,\dots,N,\\ \mathcal{L}_{2,i}(H_2,b)\boldsymbol{\phi}_2=0 & \text{for } i=1,2,\dots,N^*,\\ \mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1+\mathcal{L}_{2,0}(H_2,b)\boldsymbol{\phi}_2=0, \\ -\rho_1\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1+\rho_2\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2=\phi.\end{cases}\tag{8.8}$$

Therefore, it follows from Lemma 6.4 that once the canonical variables (ζ,ϕ) are given in an appropriate class of functions, $\boldsymbol{\phi}_1'=(\phi_{1,1},\dots,\phi_{1,N})^{\mathrm{T}}$, $\boldsymbol{\phi}_2'=(\phi_{2,1},\dots,\phi_{2,N^*})^{\mathrm{T}}$, $\nabla\phi_{1,0}$, $\nabla\phi_{2,0}$ can be determined uniquely. In other words, these variables depend on the canonical variables (ζ,ϕ) and b, and furthermore they depend on ϕ linearly. Although the solution $(\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ to the above equations is not unique, we will denote the solution by

$$\boldsymbol{\phi}_1=\boldsymbol{S}_1(\zeta,b)\phi,\quad \boldsymbol{\phi}_2=\boldsymbol{S}_2(\zeta,b)\phi.$$

This abbreviation causes no confusion in the following calculations. Since we will fix b, we simply write $\boldsymbol{S}_1(\zeta)$ and $\boldsymbol{S}_2(\zeta)$ in place of $\boldsymbol{S}_1(\zeta,b)$ and $\boldsymbol{S}_2(\zeta,b)$ for simplicity. Now, we define the Hamiltonian to the Kakinuma model as

$$\mathcal{H}^{\mathrm{K}}(\zeta,\phi)=\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{S}_1(\zeta)\phi,\boldsymbol{S}_2(\zeta)\phi),\tag{8.9}$$

which is uniquely determined from (ζ,ϕ).

8.2. Hamilton's canonical form

We proceed to show that the Kakinuma model (8.1) is equivalent to Hamilton's canonical form with the Hamiltonian defined by (8.9). In the following, we fix $b \in W^{m,\infty}$ with $m > \frac{n}{2}+1$ and put

$$U_b^m = \bigl\{\zeta \in H^m ; \inf_{\boldsymbol{x}\in\mathbf{R}^n}(h_1-\zeta(\boldsymbol{x}))>0 \text{ and } \inf_{\boldsymbol{x}\in\mathbf{R}^n}(h_2+\zeta(\boldsymbol{x})-b(\boldsymbol{x}))>0\bigr\},$$

which is an open set in H^m. We also use the function space $\mathring{H}^k = \{\phi ; \nabla\phi \in H^{m-1}\}$. For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $B(\mathcal{X};\mathcal{Y})$ the set of all linear and bounded operators from $\mathcal{X}$ into $\mathcal{Y}$. By Lemma 6.4, we easily see the following lemma.

Lemma 8.1. *Let m be an integer such that $m > \frac{n}{2}+1$ and $b \in W^{m,\infty}$. For each $\zeta \in U_b^m$ and for $k = 1,2,\dots,m$, the linear operators*

$$\begin{cases} \mathbf{S}_1(\zeta) : \mathring{H}^k \ni \phi \mapsto \boldsymbol{\phi}_1 \in \mathring{H}^k \times (H^k)^N, \\ \mathbf{S}_2(\zeta) : \mathring{H}^k \ni \phi \mapsto \boldsymbol{\phi}_2 \in \mathring{H}^k \times (H^k)^{N^*}, \end{cases}$$

where $(\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ is the solution to (8.8), are defined. Moreover, we have $\boldsymbol{S}_1(\zeta) \in B(\mathring{H}^k; \mathring{H}^k \times (H^k)^N)$ and $\boldsymbol{S}_2(\zeta) \in B(\mathring{H}^k; \mathring{H}^k \times (H^k)^{N^*})$.

Formally, $\dot{\boldsymbol{\psi}}_k = D_\zeta \boldsymbol{S}_k(\zeta)[\dot{\zeta}]\phi$, the Fréchet derivative of $\boldsymbol{S}_k(\zeta)\phi$ with respect to ζ applied to $\dot{\zeta}$ for $k=1,2$ satisfy

$$\begin{cases} \mathcal{L}_{1,i}(H_1)\dot{\boldsymbol{\psi}}_1 = D_{H_1}\mathcal{L}_{1,i}(H_1)[\dot{\zeta}]\boldsymbol{\phi}_1 & \text{for } i=1,2,\dots,N, \\ \mathcal{L}_{2,i}(H_2,b)\dot{\boldsymbol{\psi}}_2 = -D_{H_2}\mathcal{L}_{2,i}(H_2,b)[\dot{\zeta}]\boldsymbol{\phi}_2 & \text{for } i=1,2,\dots,N^*, \\ \mathcal{L}_{1,0}(H_1)\dot{\boldsymbol{\psi}}_1 + \mathcal{L}_{2,0}(H_2,b)\dot{\boldsymbol{\psi}}_2 & \\ \qquad = D_{H_1}\mathcal{L}_{1,0}(H_1)[\dot{\zeta}]\boldsymbol{\phi}_1 - D_{H_2}\mathcal{L}_{2,0}(H_2,b)[\dot{\zeta}]\boldsymbol{\phi}_2, & \\ -\rho_1\boldsymbol{l}_1(H_1)\cdot\dot{\boldsymbol{\psi}}_1 + \rho_2\boldsymbol{l}_2(H_2)\cdot\dot{\boldsymbol{\psi}}_2 & \\ \qquad = -\bigl(\rho_1(\partial_{H_1}\boldsymbol{l}_1(H_1))\cdot\boldsymbol{\phi}_1 + \rho_2(\partial_{H_2}\boldsymbol{l}_2(H_2))\cdot\boldsymbol{\phi}_2\bigr)\dot{\zeta} & \end{cases} \tag{8.10}$$

with $\boldsymbol{\phi}_j = \boldsymbol{S}_j(\zeta)\phi$ for $j=1,2$, where for $i=1,\dots,N$,

$$\begin{aligned} D_{H_1}\mathcal{L}_{1,i}(H_1)[\dot{\zeta}]\boldsymbol{\phi}_1 &= \sum_{j=0}^{N}\bigl(D_{H_1}L_{1,ij}(H_1)[\dot{\zeta}] - H_1^{2i}D_{H_1}L_{1,0j}(H_1)[\dot{\zeta}] \\ &\qquad - 2iH_1^{2i-1}\dot{\zeta}L_{1,0j}(H_1)\bigr)\phi_{1,j}, \\ D_{H_1}L_{1,ij}(H_1)[\dot{\zeta}]\phi_{1,j} &= -\nabla\cdot(\dot{\zeta}H_1^{2(i+j)}\nabla\phi_{1,j}) + 4ij\dot{\zeta}H_1^{2(i+j-1)}\phi_{1,j}, \end{aligned}$$

and so on. By using these equations together with Lemma 6.4 and standard arguments, we can justify the Fréchet differentiability of $\boldsymbol{S}_k(\zeta)$ with respect to ζ for $k=1,2$. More precisely, we have the following lemma.

Lemma 8.2. *Let m be an integer such that $m > \frac{n}{2}+1$ and $b \in W^{m,\infty}$. Then the maps $U_b^m \ni \zeta \mapsto \boldsymbol{S}_1(\zeta) \in B(\mathring{H}^k; \mathring{H}^k \times (H^k)^N)$ and $U_b^m \ni \zeta \mapsto \boldsymbol{S}_2(\zeta) \in B(\mathring{H}^k; \mathring{H}^k \times (H^k)^{N^*})$ are Fréchet differentiable for $k=1,2,\dots,m$, and (8.10) holds.*

We proceed to calculate the variational derivatives of the Hamiltonian $\mathcal{H}^{\mathrm{K}}(\zeta,\phi)$, which are given by the following lemma.

Lemma 8.3. *Let m be an integer such that $m > \frac{n}{2}+1$ and $b \in W^{m,\infty}$. Then the map $U_b^m \times \mathring{H}^1 \ni (\zeta,\phi) \mapsto \mathcal{H}^{\mathrm{K}}(\zeta,\phi) \in \mathbf{R}$ is Fréchet differentiable and the variational derivatives of the Hamiltonian are*

$$\begin{cases} \delta_\phi \mathcal{H}^{\mathrm{K}}(\zeta,\phi) = -\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1, \\ \delta_\zeta \mathcal{H}^{\mathrm{K}}(\zeta,\phi) = (\delta_\zeta \mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2) \\ \qquad\qquad\qquad + (\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1)\big(\rho_1(\partial_{H_1}\boldsymbol{l}_1)(H_1)\cdot\boldsymbol{\phi}_1 + \rho_2(\partial_{H_2}\boldsymbol{l}_2)(H_2)\cdot\boldsymbol{\phi}_2\big), \end{cases}$$

where $\boldsymbol{\phi}_k = \boldsymbol{S}_k(\zeta)$ for $k=1,2$.

Proof. Let us calculate Fréchet derivatives of the Hamiltonian $\mathcal{H}^{\mathrm{K}}(\zeta,\phi)$. Let us consider first $U_b^m \times H^2 \ni (\zeta,\phi) \mapsto \mathcal{H}^{\mathrm{K}}(\zeta,\phi)$. For any $\dot{\phi} \in H^2$, we see that

$$\begin{aligned} &D_\phi \mathcal{H}^{\mathrm{K}}(\zeta,\phi)[\dot{\phi}] \\ &\quad = (D_{\boldsymbol{\phi}_1}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{S}_1(\zeta)\phi,\boldsymbol{S}_2(\zeta)\phi)[\boldsymbol{S}_1(\zeta)\dot{\phi}] + (D_{\boldsymbol{\phi}_2}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{S}_1(\zeta)\phi,\boldsymbol{S}_2(\zeta)\phi)[\boldsymbol{S}_2(\zeta)\dot{\phi}] \\ &\quad = ((\delta_{\boldsymbol{\phi}_1}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2),\boldsymbol{S}_1(\zeta)\dot{\phi})_{L^2} + ((\delta_{\boldsymbol{\phi}_2}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2),\boldsymbol{S}_2(\zeta)\dot{\phi})_{L^2} \\ &\quad = (\rho_1 L_1(H_1)\boldsymbol{\phi}_1,\boldsymbol{S}_1(\zeta)\dot{\phi})_{L^2} + (\rho_2 L_2(H_2,b)\boldsymbol{\phi}_2,\boldsymbol{S}_2(\zeta)\dot{\phi})_{L^2} \\ &\quad = \big(\rho_1\boldsymbol{l}_1(H_1)(\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1),\boldsymbol{S}_1(\zeta)\dot{\phi}\big)_{L^2} - \big(\rho_2\boldsymbol{l}_2(H_2)(\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1),\boldsymbol{S}_2(\zeta)\dot{\phi}\big)_{L^2} \\ &\quad = (\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1,\rho_1\boldsymbol{l}_1(H_1)\cdot\boldsymbol{S}_1(\zeta)\dot{\phi} - \rho_2\boldsymbol{l}_2(H_2)\cdot\boldsymbol{S}_2(\zeta)\dot{\phi})_{L^2} \\ &\quad = -(\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1,\dot{\phi})_{L^2}, \end{aligned}$$

where we used (8.5) and Lemma 8.1. The above calculations are also valid when $(\phi,\dot{\phi}) \in \mathring{H}^1 \times \mathring{H}^1$, provided we replace the L^2-inner products with the $\mathcal{X}'$–$\mathcal{X}$ duality product, where $\mathcal{X} = \mathring{H}^1 \times (H^1)^N$ or $\mathcal{X} = \mathring{H}^1 \times (H^1)^{N^*}$ for the first lines, and $\mathcal{X} = \mathring{H}^1$ for the last line. This gives the first equation of the lemma.

Similarly, for any $(\zeta,\phi) \in U_b^m \times \mathring{H}^2$ and $\dot{\zeta} \in H^m$ we see that

$$\begin{aligned} D_\zeta \mathcal{H}^{\mathrm{K}}(\zeta,\phi)[\dot{\zeta}] &= (D_\zeta \mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{S}_1(\zeta)\phi,\boldsymbol{S}_2(\zeta)\phi)[\dot{\zeta}] \\ &\quad + (D_{\boldsymbol{\phi}_1}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{S}_1(\zeta)\phi,\boldsymbol{S}_2(\zeta)\phi)[D_\zeta\boldsymbol{S}_1(\zeta)[\dot{\zeta}]\phi] \\ &\quad + (D_{\boldsymbol{\phi}_2}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{S}_1(\zeta)\phi,\boldsymbol{S}_2(\zeta)\phi)[D_\zeta\boldsymbol{S}_2(\zeta)[\dot{\zeta}]\phi] \\ &= ((\delta_\zeta \mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2),\dot{\zeta})_{L^2} + ((\delta_{\boldsymbol{\phi}_1}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2),D_\zeta\boldsymbol{S}_1(\zeta)[\dot{\zeta}]\phi)_{L^2} \\ &\quad + ((\delta_{\boldsymbol{\phi}_2}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2),D_\zeta\boldsymbol{S}_2(\zeta)[\dot{\zeta}]\phi)_{L^2}. \end{aligned}$$

Here, we have

$$\begin{aligned} &\big((\delta_{\boldsymbol{\phi}_1}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2),D_\zeta\boldsymbol{S}_1(\zeta)[\dot{\zeta}]\phi\big)_{L^2} + \big((\delta_{\boldsymbol{\phi}_2}\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2),D_\zeta\boldsymbol{S}_2(\zeta)[\dot{\zeta}]\phi\big)_{L^2} \\ &\quad = \big(\rho_1 L_1(H_1)\boldsymbol{\phi}_1,D_\zeta\boldsymbol{S}_1(\zeta)[\dot{\zeta}]\phi\big)_{L^2} + \big(\rho_2 L_2(H_2,b)\boldsymbol{\phi}_2,D_\zeta\boldsymbol{S}_2(\zeta)[\dot{\zeta}]\phi\big)_{L^2} \\ &\quad = \big(\mathscr{L}_{1,0}(H_1)\boldsymbol{\phi}_1,\rho_1\boldsymbol{l}_1(H_1)\cdot D_\zeta\boldsymbol{S}_1(\zeta)[\dot{\zeta}]\phi - \rho_2\boldsymbol{l}_2(H_2)\cdot D_\zeta\boldsymbol{S}_2(\zeta)[\dot{\zeta}]\phi\big)_{L^2} \end{aligned}$$

$$
\begin{aligned}
&= \big(\mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1, \big(\rho_1(\partial_{H_1}\boldsymbol{l}_1)(H_1)\cdot\boldsymbol{\phi}_1 + \rho_2(\partial_{H_2}\boldsymbol{l}_2)(H_2)\cdot\boldsymbol{\phi}_2\big)\dot{\zeta}\big)_{L^2} \\
&= \big((\mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1)\big(\rho_1(\partial_{H_1}\boldsymbol{l}_1)(H_1)\cdot\boldsymbol{\phi}_1 + \rho_2(\partial_{H_2}\boldsymbol{l}_2)(H_2)\cdot\boldsymbol{\phi}_2\big), \dot{\zeta}\big)_{L^2},
\end{aligned}
$$

where we used the identity

$$
\begin{aligned}
&\rho_1\boldsymbol{l}_1(H_1)\cdot D_\zeta\boldsymbol{S}_1(\zeta)[\dot{\zeta}]\phi - \rho_2\boldsymbol{l}_2(H_2)\cdot D_\zeta\boldsymbol{S}_2(\zeta)[\dot{\zeta}]\phi \\
&\quad = \big(\rho_1(\partial_{H_1}\boldsymbol{l}_1)(H_1)\cdot\boldsymbol{\phi}_1 + \rho_2(\partial_{H_2}\boldsymbol{l}_2)(H_2)\cdot\boldsymbol{\phi}_2\big)\dot{\zeta},
\end{aligned}
$$

stemming from (8.10). Again, the above identities are still valid for $(\zeta,\phi)\in U_b^m\times\mathring{H}^1$ provided we replace the L^2-inner products with suitable duality products. This concludes the proof of the Fréchet differentiability, and the second equation of the lemma. ■

Now we are ready to show another main result in this paper.

Theorem 8.4. *Let m be an integer such that $m>\frac{n}{2}+1$ and $b\in W^{m,\infty}$. Then the Kakinuma model (2.14)–(2.16) is equivalent to Hamilton's canonical equations*

$$
\partial_t\zeta = \frac{\delta\mathcal{H}^{\mathrm{K}}}{\delta\phi}, \quad \partial_t\phi = -\frac{\delta\mathcal{H}^{\mathrm{K}}}{\delta\zeta}, \tag{8.11}
$$

with $\mathcal{H}^{\mathrm{K}}$ defined by (8.9) as long as $\zeta(\cdot,t)\in U_b^m$ and $\phi(\cdot,t)\in\mathring{H}^1$. More precisely, for any regular solution $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ to the Kakinuma model (2.14)–(2.16), if we define ϕ by (8.7), then (ζ,ϕ) satisfies Hamilton's canonical equations (8.11). Conversely, for any regular solution (ζ,ϕ) to Hamilton's canonical equations (8.11), if we define $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ by $\boldsymbol{\phi}_k=\boldsymbol{S}_k(\zeta)\phi$ for $k=1,2$, then $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ satisfies the Kakinuma model (2.14)–(2.16).

Proof. Suppose that $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ is a solution to the Kakinuma model (2.14)–(2.16). Then it satisfies (8.6), and in particular

$$
\partial_t\zeta = -\mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1. \tag{8.12}
$$

Moreover, it follows from (8.7) and (8.6) that

$$
\begin{aligned}
\partial_t\phi &= \rho_2\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 - \rho_1\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 \\
&\qquad + \big(\rho_2(\partial_{H_2}\boldsymbol{l}_2(H_2))\cdot\boldsymbol{\phi}_2 + \rho_1(\partial_{H_1}\boldsymbol{l}_1(H_1))\cdot\boldsymbol{\phi}_1\big)\partial_t\zeta \\
&= -(\delta_\zeta\mathcal{E}^{\mathrm{K}})(\zeta,\boldsymbol{\phi}_1\boldsymbol{\phi}_2) \\
&\qquad - (\mathcal{L}_{1,0}(H_1)\boldsymbol{\phi}_1)\big(\rho_1(\partial_{H_1}\boldsymbol{l}_1(H_1))\cdot\boldsymbol{\phi}_1 + \rho_2(\partial_{H_2}\boldsymbol{l}_2(H_2))\cdot\boldsymbol{\phi}_2\big).
\end{aligned}
$$

These equations together with Lemma 8.3 show that (ζ,ϕ) satisfies (8.11).

Conversely, suppose that (ζ,ϕ) satisfies Hamilton's canonical equations (8.11) and put $\boldsymbol{\phi}_k=\boldsymbol{S}_k(\zeta)\phi$ for $k=1,2$. Then it follows from (8.11) and Lemma 8.3 that we have (8.12). This fact and Lemma 8.1 imply the equations

$$
\begin{cases}
\boldsymbol{l}_1(H_1)\partial_t\zeta + L_1(H_1)\boldsymbol{\phi}_1 = \boldsymbol{0}, \\
-\boldsymbol{l}_2(H_2)\partial_t\zeta + L_2(H_2,b)\boldsymbol{\phi}_2 = \boldsymbol{0}.
\end{cases}
$$

We see also that

$$\begin{aligned}
& -\rho_1 \boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + \rho_2\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 \\
&\quad = \partial_t\phi - \big(\rho_1(\partial_{H_1}\boldsymbol{l}_1)(H_1)\cdot\boldsymbol{\phi}_1 + \rho_2(\partial_{H_2}\boldsymbol{l}_2)(H_2)\cdot\boldsymbol{\phi}_2\big)\partial_t\zeta \\
&\quad = -\delta_\zeta\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1\boldsymbol{\phi}_2) = F,
\end{aligned}$$

where we used (8.11), (8.12), Lemma 8.3, and (8.5). Therefore, $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ satisfies (8.1), that is, the Kakinuma model (2.14)–(2.16). ∎

9. Conservation laws

The Kakinuma model (2.14)–(2.16) has conservative quantities: the excess of mass $\int_{\mathbf{R}^n}\zeta\,\mathrm{d}\boldsymbol{x}$ and the total energy $\mathcal{E}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ given by (8.3). Moreover, in the case of a flat bottom in the lower layer, the momentum given by

$$\begin{aligned}
\mathcal{M}^{\mathrm{K}}(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2) &= \iint_{\Omega_1(t)}\rho_1\nabla\Phi_1^{\mathrm{app}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z + \iint_{\Omega_2(t)}\rho_2\nabla\Phi_2^{\mathrm{app}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z \\
&= \int_{\mathbf{R}^n}\zeta\nabla(-\rho_1\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1 + \rho_2\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2)\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbf{R}^n}\zeta\nabla\phi\,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

is also conserved for the Kakinuma model. Here, we also give the corresponding flux functions to these conservative quantities.

We have two forms of conservation of mass by (2.14) and (2.15) with $i=0$, that is,

$$\partial_t\zeta + \nabla\cdot\sum_{j=0}^{N}\Big(-\frac{1}{2j+1}H_1^{2j+1}\nabla\phi_{1,j}\Big) = 0, \tag{9.1}$$

$$\partial_t\zeta + \nabla\cdot\sum_{j=0}^{N^*}\Big(\frac{1}{p_j+1}H_2^{p_j+1}\nabla\phi_{2,j} - \frac{p_j}{p_j}H_2^{p_j}\phi_{2,j}\nabla b\Big) = 0. \tag{9.2}$$

Proposition 9.1. *Any regular solution $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ to the Kakinuma model (2.14)–(2.16) satisfies the conservation of energy*

$$\partial_t e^{\mathrm{K}} + \nabla\cdot\boldsymbol{f}_e^{\mathrm{K}} = 0,$$

where the energy density e^{K} is defined by (8.4) and the corresponding flux $\boldsymbol{f}_e^{\mathrm{K}}$ is given by

$$\begin{aligned}
\boldsymbol{f}_e^{\mathrm{K}} &= \rho_1\sum_{i,j=0}^{N}\Big(-\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,j}\Big)(\partial_t\phi_{1,i}) \\
&\quad + \rho_2\sum_{i,j=0}^{N^*}\Big(-\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,j} + \frac{p_j}{p_i+p_j}H_2^{p_i+p_j}\phi_{2,j}\nabla b\Big)(\partial_t\phi_{2,i}).
\end{aligned}$$

Proof. By using F defined by (8.2), we see that

$$
\begin{aligned}
\partial_t e^{\mathrm{K}} &= -F\partial_t\zeta \\
&\quad + \rho_1 \sum_{i,j}^{N}\Big(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,j}\cdot\nabla\partial_t\phi_{1,i} \\
&\qquad\qquad + \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\phi_{1,j}\partial_t\phi_{1,i}\Big) \\
&\quad + \rho_2 \sum_{i,j=0}^{N^*}\Big\{\Big(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,j} \\
&\qquad\qquad - \frac{p_j}{p_i+p_j}H_2^{p_i+p_j}\phi_{2,j}\nabla b\Big)\cdot\nabla\partial_t\phi_{2,i} \\
&\qquad\qquad + \Big(-\frac{p_i}{p_i+p_j}H_2^{p_i+p_j}\nabla b\cdot\nabla\phi_{2,j} \\
&\qquad\qquad\quad + \frac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}(1+|\nabla b|^2)\phi_{2,j}\Big)\partial_t\phi_{2,i}\Big\} \\
&= -F\partial_t\zeta - \nabla\cdot\boldsymbol{f}_e^{\mathrm{K}} + \rho_1 L_1(H_1)\boldsymbol{\phi}_1\cdot\partial_t\boldsymbol{\phi}_1 + \rho_2 L_2(H_2,b)\boldsymbol{\phi}_2\cdot\partial_t\boldsymbol{\phi}_2,
\end{aligned}
$$

so that, by (8.1),

$$
\begin{aligned}
\partial_t e^{\mathrm{K}} + \nabla\cdot\boldsymbol{f}_e^{\mathrm{K}} &= -F\partial_t\zeta + \rho_1 L_1(H_1)\boldsymbol{\phi}_1\cdot\partial_t\boldsymbol{\phi}_1 + \rho_2 L_2(H_2,b)\boldsymbol{\phi}_2\cdot\partial_t\boldsymbol{\phi}_2 \\
&= (-F - \rho_1\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1 + \rho_2\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2)\partial_t\zeta \\
&= 0,
\end{aligned}
$$

which is the desired identity. ■

Proposition 9.2. *Suppose that the bottom in the lower layer is flat, that is, $b=0$. Then any regular solution $(\zeta,\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ to the Kakinuma model (2.14)–(2.16) satisfies the conservation of momentum*

$$
\partial_t\boldsymbol{m}^{\mathrm{K}} + \nabla\cdot F_{\boldsymbol{m}}^{\mathrm{K}} = 0,
$$

where the momentum density $\boldsymbol{m}^{\mathrm{K}}$ and the corresponding flux matrix $F_{\boldsymbol{m}}^{\mathrm{K}}$ are given by

$$
\begin{aligned}
\boldsymbol{m}^{\mathrm{K}} &= \zeta\nabla\phi = \zeta\nabla(\rho_2\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2 - \rho_1\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1), \\
F_{\boldsymbol{m}}^{\mathrm{K}} &= -\big(\zeta\partial_t(\rho_2\boldsymbol{l}_2(H_2)\cdot\boldsymbol{\phi}_2 - \rho_1\boldsymbol{l}_1(H_1)\cdot\boldsymbol{\phi}_1) + e^{\mathrm{K}}\big)\mathrm{Id} \\
&\quad + \rho_1\sum_{i,j=0}^{N}\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,i}\otimes\nabla\phi_{1,j} \\
&\quad + \rho_1\sum_{i,j=0}^{N^*}\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,i}\otimes\nabla\phi_{2,j}.
\end{aligned}
$$

Proof. For $l = 1, 2, \dots, n$, we see by (8.1) that

$$
\begin{aligned}
\partial_t(\zeta\partial_l\phi) - \partial_l(\zeta\partial_t\phi) &= (\partial_t\zeta)(\rho_2\boldsymbol{l}_2(H_2)\cdot\partial_l\boldsymbol{\phi}_2 - \rho_1\boldsymbol{l}_1(H_1)\cdot\partial_l\boldsymbol{\phi}_1) \\
&\quad - (\partial_l\zeta)(\rho_2\boldsymbol{l}_2(H_2)\cdot\partial_t\boldsymbol{\phi}_2 - \rho_1\boldsymbol{l}_1(H_1)\cdot\partial_t\boldsymbol{\phi}_1) \\
&= \rho_2 L_2(H_2,0)\boldsymbol{\phi}_2\cdot\partial_l\boldsymbol{\phi}_2 + \rho_1 L_1(H_1)\boldsymbol{\phi}_1\cdot\partial_l\boldsymbol{\phi}_1 - (\partial_l\zeta)F \\
&= -\nabla\cdot\biggl\{\rho_1\sum_{i,j=0}^{N}\Bigl(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,i}\Bigr)\partial_l\phi_{1,j} \\
&\qquad + \rho_2\sum_{i,j=0}^{N^*}\Bigl(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,i}\Bigr)\partial_l\phi_{2,j}\biggr\} + R_1,
\end{aligned}
$$

where F is given by (8.2) and

$$
\begin{aligned}
R_1 &= \rho_1\sum_{i,j=0}^{N}\Bigl(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,i}\cdot\nabla\partial_l\phi_{1,j} \\
&\qquad + \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\phi_{1,i}\partial_l\phi_{1,j}\Bigr) \\
&\quad + \rho_2\sum_{i,j=0}^{N^*}\Bigl(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,i}\cdot\nabla\partial_l\phi_{2,j} \\
&\qquad + \frac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}\phi_{2,i}\partial_l\phi_{2,j}\Bigr) - (\partial_l\zeta)F \\
&= \partial_l\biggl\{\frac{1}{2}\rho_1\sum_{i,j=0}^{N}\Bigl(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,i}\cdot\nabla\phi_{1,j} \\
&\qquad + \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\phi_{1,i}\phi_{1,j}\Bigr) \\
&\quad + \frac{1}{2}\rho_2\sum_{i,j=0}^{N^*}\Bigl(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,i}\cdot\nabla\phi_{2,j} \\
&\qquad + \frac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}\phi_{2,i}\phi_{2,j}\Bigr)\biggr\} + R_2.
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
R_2 &= \frac{1}{2}\rho_1\sum_{i,j=0}^{N}\bigl(H_1^{2(i+j)}\nabla\phi_{1,i}\cdot\nabla\phi_{1,j} + 4ijH_1^{2(i+j-1)}\phi_{1,i}\phi_{1,j}\bigr)\partial_l\zeta \\
&\quad - \frac{1}{2}\rho_2\sum_{i,j=0}^{N^*}\bigl(H_2^{p_i+p_j}\nabla\phi_{2,i}\cdot\nabla\phi_{2,j} + p_ip_jH_2^{p_i+p_j-2}\phi_{2,i}\phi_{2,j}\bigr)\partial_l\zeta - F\partial_l\zeta \\
&= (\rho_2-\rho_1)g\zeta\partial_l\zeta = \partial_l\Bigl(\frac{1}{2}(\rho_2-\rho_1)g\zeta^2\Bigr),
\end{aligned}
$$

so that $R_1 = \partial_l e^{\mathrm{K}}$. These identities yield the desired one. ∎

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