Ann. Inst. H. Poincaré
Anal. Non Linéaire 41 (2024), 357–403
DOI 10.4171/AIHPC/85

Singularities in L^1-supercritical Fokker–Planck equations: A qualitative analysis

Katharina Hopf

Abstract. A class of nonlinear Fokker–Planck equations with superlinear drift is investigated in the L^1-supercritical regime, which exhibits a finite critical mass. The equations have a formal Wasserstein-like gradient-flow structure with a convex mobility and a free energy functional whose minimising measure has a singular component if above the critical mass. Singularities and concentrations also arise in the evolutionary problem and their finite-time appearance constitutes a primary technical difficulty. This paper aims at a global-in-time qualitative analysis with main focus on the isotropic case, where solutions will be shown to converge to the unique minimiser of the free energy as time tends to infinity. A key step in the analysis consists in properly controlling the singularity profiles during the evolution. Our study covers the three-dimensional Kaniadakis–Quarati model for Bose–Einstein particles, and thus provides a first rigorous result on the continuation beyond blow-up and long-time asymptotic behaviour for this model.

1. Introduction

This manuscript is concerned with a class of Fokker–Planck equations with superlinear drift taking the form

$$\begin{aligned} \partial_t f &= \nabla\cdot(\nabla f + vh(f)), \quad && t>0,\ v\in\mathbb{R}^d, \\ f(0,v) &= f_{\mathrm{in}}(v)\geq 0, && v\in\mathbb{R}^d, \end{aligned} \tag{FP$_\gamma$}$$

where $h(f) = f(1+\sigma|f|^\gamma)$ for some $\gamma\geq 1$ and $\sigma = 1$. For $\gamma = 1$ and $\sigma\in\{\pm 1\}$ this equation was introduced in the 1990s by Kaniadakis and Quarati [24, 25] as a model for the relaxation to equilibrium of quantum particles of Fermi–Dirac ($\sigma = -1$) and Bose–Einstein ($\sigma = 1$) types. We refer to [9, 20] and references therein for more background on the physical model. The interest of the mathematics community in problems of the form (FP$_\gamma$) mainly stems from their variational structure: for densities $f\geq 0$ the first line in (FP$_\gamma$) can formally be written as a continuity equation

$$\partial_t f = \nabla\cdot(h(f)\nabla\delta\mathcal{H}(f)), \tag{1.1}$$

2020 Mathematics Subject Classification. Primary 35Q84; Secondary 35K55, 35A21, 35R06, 35B40.
Keywords. Fokker–Planck equations for bosons, nonlinear mobility, continuation beyond singularities, singular limit, universal blow-up profile, relaxation to minimising measure.

with $\delta\mathcal{H}$ denoting the variational derivative of the convex integral functional

$$\mathcal{H}(f) := \int_{\mathbb{R}^d} \Big(\frac{|v|^2}{2} f + \Phi(f)\Big)\,\mathrm{d}v,$$

where $\Phi(f) := \frac{1}{\gamma}\int_0^f \log(\frac{s^\gamma}{1+\sigma s^\gamma})\,\mathrm{d}s$ and thus $\Phi''(f) = 1/h(f)$. (If $\sigma = -1$ one should restrict to $0 \le f \le 1$.) Thus, the *free energy* $\mathcal{H}(f)$ is formally dissipated along solutions $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f) = -\int_{\mathbb{R}^d} h(f)|\nabla\delta\mathcal{H}(f)|^2\,\mathrm{d}v \le 0$. Let us note that for $\sigma = 1$ the function Φ is sublinear at infinity, and the natural extension of $\mathcal{H}$ to finite, non-negative measures (cf. [17]) vanishes on Dirac deltas centred at the origin. We further observe that for $\sigma = 1$ the nonlinear mobility $h(f) = f(1+\sigma f^\gamma)$ in (1.1) is convex, while it is concave if $\sigma = -1$.

The equation for fermions, where $\sigma = -1$ and $\gamma = 1$, is mathematically well understood. Here, in any dimension, solutions emanating from suitably regular initial data $0 \le f_{\mathrm{in}} \le 1$ remain bounded between 0 and 1, i.e. satisfy $0 \le f \le 1$, consistent with the well-known Pauli exclusion principle. In the long-time limit they converge to the unique minimiser of $\mathcal{H}$ of the given mass [10, 13], namely to the corresponding (smooth) Fermi–Dirac distribution. The concavity of the mobility even allows a rigorous meaning to be given to the above gradient-flow structure with respect to generalised Wasserstein distances [11, 18], which fails for the convex/non-concave mobilities associated with $\sigma = 1$.

The bosonic case, where $\sigma = 1$ (and $\gamma = 1$), is more challenging. Then, equation (FP$_\gamma$) becomes L^1-supercritical in dimension $d > 2$, in which case the large-data long-time analysis has remained open for quite a while. In fact, a first global-in-time rigorous study of the L^1-supercritical regime has only recently been obtained in [8] for a one-dimensional analogue, i.e. for (FP$_\gamma$) with $\sigma = 1$, $d = 1$, and $\gamma > 2$, and is based on a Lagrangian approach and viscosity solution techniques. In the physically most interesting case $d = 3$ and $\gamma = 1$, which will be the main focus of this manuscript, no rigorous long-time analysis exists when $\sigma = 1$, except for [33] which shows finite-time blow-up for large data by a virial-type contradiction argument. In the L^1-critical case, in contrast, solutions are globally regular [6]. For numerical studies on the singularity formation in the supercritical case, we refer to [9, 32]. The qualitative properties obtained in the present manuscript are in agreement with the simulations in [9], although our approximation scheme is different and not restricted to the isotropic case. Let us mention that the uniqueness and stability properties of the present scheme in the isotropic setting may also be of interest numerically.

In this paper we perform a rigorous global-in-time existence and qualitative analysis of (FP$_\gamma$) with $\sigma = 1$ in the L^1-supercritical regime in higher dimensions $d \ge 1$, our main interest being the bosonic three-dimensional Kaniadakis–Quarati model ($\sigma = 1$ and $d = 3$, $\gamma = 1$); thus, hereafter $\sigma = 1$. Preservation of the variational structure beyond finite-time blow-up being a primary concern, we build our analysis on a suitably chosen approximation scheme that respects the basic mass conservation and structural properties of the continuity equation (1.1). To begin with, we note that the stationary mass-constrained minimisation problem for $\mathscr{H}$ is well understood. The minimisers of $\mathscr{H}$ for a given mass have been characterised in [3] and are in fact explicit:

Theorem ([3, Theorem 3.1]). *For every $m \in (0, \infty)$ the functional $\mathscr{H}$ has a unique minimiser $\mu_{\min} = \mu_{\min}^{(m)}$ on the manifold $\{\mu \in \mathcal{M}_+(\mathbb{R}^d) \colon \int \mathrm{d}\mu = m\}$.*[1]

This minimiser takes the form

$$\mu_{\min} = \begin{cases} f_{\infty,\theta}\mathscr{L}^d & \text{if } m \le m_c, \text{ where } \theta \in \mathbb{R}_{\ge 0} \text{ obeys} \\ & \int_{\mathbb{R}^d} f_{\infty,\theta}(v)\,\mathrm{d}v = m, \\ f_c\mathscr{L}^d + (m - m_c)\delta_0 & \text{if } m > m_c. \end{cases} \tag{1.2}$$

Here

$$f_{\infty,\theta}(v) = (\Phi')^{-1}(-\tfrac{1}{2}|v|^2 - \theta) = (\mathrm{e}^{\gamma(\frac{|v|^2}{2}+\theta)} - 1)^{-\frac{1}{\gamma}}, \quad \theta \in \mathbb{R}_{\ge 0}, \tag{1.3}$$

and we abbreviated $f_c := f_{\infty,0}$ as well as $m_c := \int_{\mathbb{R}^d} f_c(v)\,\mathrm{d}v \in (0, \infty]$.

For general $\gamma \ge 1$, the L^1-supercritical regime as determined by a dimensional analysis is given by $d - \frac{2}{\gamma} > 0$. Observe that this is exactly the regime where the *critical mass* m_c appearing in the above theorem is finite and where minimisers with singular parts concentrated at velocity zero emerge. Such singular components are termed Bose–Einstein condensates in the physics literature (at least when $\gamma = 1$).

Let us now put the analysis of the present work into context with existing literature and discuss the main new difficulties. Naturally, several aspects of our approach have their roots in the work [8]. This is particularly true for the fact that our fundamental a priori bound consists in a space-uniform *temporal* Lipschitz estimate (of an integral quantity) that is propagated in time. Both in [8] and in the present paper, such estimates are derived by means of suitable comparison principles. However, the approach in [8] relies on a Lagrangian reformulation of the problem in terms of the pseudo-inverse cumulative distribution function giving access to the powerful instrument of viscosity solution theory [15]. While in higher dimensions such a reformulation is, in principle, still possible [9, 12, 19], the structural properties of the resulting problem greatly deteriorate. Indeed, in higher space dimensions, Lagrangian coordinates are vectorial and the corresponding reformulation leads to a quasilinear degenerate second-order *system* of equations (cf. [9, Section 2.1.3]). In such situations, classical comparison techniques are rarely available. Even in the isotropic case, where a one-dimensional scalar problem can be obtained for the inverse of a rescaled radial cumulative distribution function [9, Section 2.1.2], the comparison technique in [8] does not directly generalise to higher dimensions, since for $d > 1$ the second-order differential operator in the new variables has an explicit dependence on the unknown (see the comments following (7.3) in [23]).

The new challenges we encounter in higher dimensions are thus mainly of a technical nature. In particular, the derivation of the universal space profile at $\{v = 0\}$ for unbounded

[1] We define $\mathscr{H}(\mu) := \infty$ if $\int_{\mathbb{R}^d} |v|^2\,\mathrm{d}\mu = \infty$.

densities in Section 3 (applying to isotropic flows) is significantly more delicate than in the one-dimensional case and requires several intermediate steps. Determining the profile at the *first* blow-up time is still quite feasible and, as in the one-dimensional case, amounts to solving an ordinary differential equation – to be combined, in higher dimensions, with a bootstrap argument. However, in (FP_γ) solutions may regularise after a first blow-up, and such successions of "blow-ups" and "blow-downs" could in principle be highly oscillatory. Thus, for a global-in-time analysis a particular challenge lies in gaining information at general points in time. We should emphasise that the space profile, while of interest in its own right, encodes a certain time-uniform continuity-at-infinity property that appears to be vital for proving relaxation to the minimiser $\mu_{\min}$ in the long-time limit. (Observe that when only looking at equation (FP_γ) from a PDE point of view, other stationary "solutions" consisting of a smooth steady state $f_{\infty,\theta}$ for some $\theta > 0$, plus a suitably weighted non-trivial Dirac measure at zero, are conceivable, though unphysical.) Let us finally point out that, in contrast to [8] where the mass of the condensate component (i.e. of the singular part of the measure solution, which turns out to be supported in $\{v = 0\}$) has only been shown to be a continuous function of time, the present approach allows us to infer Lipschitz continuity in the isotropic case and thus refines [8] (cf. [23]). Some of the basic ideas of this manuscript have been sketched for the one-dimensional model in the author's PhD thesis [23, Chapter 5]. As indicated in [23, Section 5.3], when $d = 1$, the solutions to be constructed below coincide with those obtained from the viscosity solution approach in [8].

1.1. Main results

In the subsequent analysis, unless specified otherwise, we assume the following general hypotheses:

(H1) L^1-supercriticality: $\frac{\gamma d}{2} > 1$, where $\gamma \in [1, \infty)$, $d \in \mathbb{N}_+$ are fixed parameters.

(H2) Initial data:

- $f_{\mathrm{in}} \geq 0$ a.e. in $\mathbb{R}^d$.
- Either $f_{\mathrm{in}} \in (L^\infty_d \cap L^1_{2d+1})(\mathbb{R}^d)$ and f_{in} is isotropic, or $f_{\mathrm{in}} \in (L^\infty_\ell \cap L^1_{\ell+d+1})(\mathbb{R}^d)$ for some $\ell > 3d + 1$ is (possibly) anisotropic.

The spaces $L^p_\ell(\mathbb{R}^d)$ in (H2) are weighted L^p spaces with norm $\|f\|_{L^p_\ell} := \|(1 + |\cdot|^\ell) f\|_{L^p(\mathbb{R}^d)}$; see Section 1.3.

For the asymptotic analysis leading to Theorem 1.2 and all subsequent main results, we further impose the hypothesis $\frac{2}{\gamma} + 2 - d > 0$ (cf. (1.5)), which covers the most interesting case $\gamma = 1$, $d = 3$. For more details on this restriction, we refer to Remark 1.3.

Our results for the nonlinear Fokker–Planck equations (FP_γ) rely on a careful analysis of the proposed approximation scheme, which is devised in such a way as to preserve the Fokker–Planck-type gradient-flow structure (1.1). Approximation schemes for continuation beyond blow-up have been employed in the literature for various other PDE problems. Closest to the present situation are perhaps the constructions in [30, 34] for the two-dimensional Patlak–Keller–Segel model.

Approximation scheme. Pick an even function $\eta \in C^{0,1}(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\{0\})$ that satisfies $\eta(s) = \eta(-s)$ for all $s \in \mathbb{R}$, $\eta(s) = s^\gamma$ for $s \in [0,1]$, $\eta'(s) = 0$ for $s \geq 2$, and which is further such that $(0,\infty) \ni s \mapsto \frac{\eta(s)}{s^\gamma}$ is non-increasing. For $\varepsilon \in (0,1]$ we let $\eta_\varepsilon(s) := \varepsilon^{-\gamma}\eta(\varepsilon s)$ and

$$\begin{aligned} h_\varepsilon(s) &:= s(1+\eta_\varepsilon(s)) \\ &=: s + \vartheta_\varepsilon(s), \quad \text{where } \vartheta_\varepsilon(s) := s\eta_\varepsilon(s). \end{aligned} \tag{1.4}$$

Note that the choice of η implies that $h_\varepsilon(s) \leq h_{\varepsilon'}(s) \leq h(s)$ for all $s \geq 0$ and $0 < \varepsilon' \leq \varepsilon \leq 1$. We then consider the associated Cauchy problem

$$\begin{aligned} \partial_t f_\varepsilon &= \nabla\cdot(\nabla f_\varepsilon + v h_\varepsilon(f_\varepsilon)), \quad && t > 0,\ v \in \mathbb{R}^d, \\ f_\varepsilon(0,v) &= f_{\mathrm{in}}(v) \geq 0, && v \in \mathbb{R}^d. \end{aligned} \tag{FP$_{\gamma.\mathrm{reg}}$}$$

For details on the variational structure of (FP$_{\gamma.\mathrm{reg}}$) we refer to Section 4. The global existence of non-negative mild solutions of (FP$_{\gamma.\mathrm{reg}}$) for suitably regular data can be deduced using the linear growth of h_ε at infinity in conjunction with the fact that Fokker–Planck equations like (FP$_{\gamma.\mathrm{reg}}$) (and (FP$_\gamma$)) propagate moments of order higher than 2 (cf. Proposition 2.4 below). The relatively strong decay hypotheses in (H2) are primarily needed to establish estimates that are independent of ε (cf. Proposition 2.6). The notation $\mathcal{M}_+(\mathbb{R}^d)$, $\mathcal{L}^d$ and further conventions used in the following proposition are specified in Section 1.3.

Proposition 1.1 (Limiting measure for (FP$_\gamma$)). *Suppose (H1), (H2). Then there exists a non-negative Radon measure μ on $[0,\infty)\times\mathbb{R}^d$ with the following properties:*

(i) *Mass-conserving curve: μ can be represented as $\mathrm{d}\mu = \mathrm{d}\mu_t\,\mathrm{d}t$ for a family of measures $\{\mu_t\}_{t\geq 0} \subset \mathcal{M}_+(\mathbb{R}^d)$ with the property that $t \mapsto \mu_t$ is a weakly-$*$ continuous curve in $\mathcal{M}_+(\mathbb{R}^d)$ that satisfies $\mu_t(\mathbb{R}^d) = \|f_{\mathrm{in}}\|_{L^1} =: m$ for all $t \geq 0$ and admits a decomposition according to (ii).*

(ii) *Decomposition: There exists a measurable function $a\colon [0,\infty) \to [0,m]$ and a non-negative function $f \in L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d) \cap C^{1,2}((0,\infty)\times U)$, $U := \mathbb{R}^d\setminus\{0\}$, such that for all $t \geq 0$*

$$\mu_t = a(t)\delta_0 + f(t,\cdot)\mathcal{L}^d,$$

where δ_0 denotes the Dirac measure concentrated at the origin.

The function f is a classical solution of (FP$_\gamma$) in $(0,\infty)\times U$. Moreover, f is strictly positive in $(0,\infty)\times\mathbb{R}^d$ if $\|f_{\mathrm{in}}\|_{L^1} > 0$ in the sense that for all $K \subset\subset (0,\infty)\times\mathbb{R}^d$ there exists $c(K) > 0$ such that $f_{|K} \geq c(K)$ a.e. in K.

(iii) *Approximation property: Denote by $f_\varepsilon \in C([0,\infty);(L^\infty_1 \cap L^1_3)(\mathbb{R}^d))$ the unique mild solution[2] of (FP$_{\gamma.\mathrm{reg}}$) (cf. Section 2.1) and let $\mu^{(\varepsilon)} = f_\varepsilon\mathcal{L}^{1+d}_+$, where $\mathcal{L}^{1+d}_+$ denotes the $(1+d)$-dimensional Lebesgue measure on $[0,\infty)\times\mathbb{R}^d$.*

[2]The approximate solutions f_ε enjoy further regularity properties, which will be needed in the analysis; see Section 2 for details.

Then, along a subsequence $\varepsilon \downarrow 0$,

$$\mu^{(\varepsilon)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}_+([0,T]\times\mathbb{R}^d) \quad \text{for all } T<\infty,$$
$$f_\varepsilon \to f \text{ in } C^{1,2}_{\mathrm{loc}}((0,\infty)\times U),$$

where $U := \mathbb{R}^d \setminus \{0\}$.

(iv) *Unique limit: If* f_{in} *is isotropic, the convergence in (iii) is true along any sequence* $\varepsilon \downarrow 0$.

(v) *Lipschitz continuity of point mass: If* f_{in} *is isotropic,*[3] *the map* $t \mapsto \mu_t(\{0\})$ *is Lipschitz continuous.*

See Section 2.3 for the proof of Proposition 1.1. Later on we show for the isotropic case that the limiting measure μ satisfies (FP_γ) in the sense of renormalised solutions. One of the technical difficulties of problem (FP_γ) is related to the fact that the function $t \mapsto \mu_t(\{0\})$ in general fails to be monotonic (cf. Section 5.2).

Proposition 1.1 (ii) implies that $\operatorname{supp} \mu_t^{\mathrm{sing}} \subset \{v=0\}$. Hence, recalling the sublinearity of $\Phi(s)$ as $s \to \infty$, we infer that for every $t \geq 0$,

$$\mathcal{H}(\mu_t) = \mathcal{H}(f(t)).$$

Since all relevant measures in this work will have singular parts supported at the origin, we (continue to) denote by the symbol $\mathcal{H}$ both the functional acting on densities and the extended functional acting on non-negative finite measures.

The following result provides a sharp characterisation of the space profile near the origin of isotropic solutions and, moreover, it is a key ingredient for uniquely identifying the long-time asymptotic limit. It will be established in Section 3.

Theorem 1.2 (Universal space profile). *In addition to (H1), (H2) suppose that*

$$\frac{2}{\gamma} + 2 - d > 0. \tag{1.5}$$

Further assume that the initial value f_{in} *is isotropic and let* $g(t,|v|) := f(t,v)$, *where* f *denotes the density of the regular part of the limiting measure obtained in Proposition 1.1. There exists* $r_* \in (0,1]$ *and a bounded function* $A \in C_b((0,\infty)\times(0,1))$ *such that for each* $\hat{t} > 0$ either $g(\hat{t},\cdot) \in L^\infty(0,1)$ *and there exists a neighbourhood* $J_{\hat{t}} \subset (0,\infty)$ *of* $\hat{t}$ *such that* $f_{|J_{\hat{t}}\times B_1}$ *is smooth* or

$$g(\hat{t},r) = g_c(r) + A(\hat{t},r)r^{2-d} \quad \text{for } r \in (0,r_*), \tag{1.6}$$

where $g_c(|v|) := f_c(v) = f_{\infty,0}(v)$ *(cf. (1.3)), i.e.* $g_c(r) = (\Phi')^{-1}(-\frac{1}{2}r^2)$.

[3] In the anisotropic case, we will see in Section 3 that $t \mapsto \mu_t(\{0\})$ is at least continuous; see Corollary 3.4.

The radius $r_* > 0$ and the function $A \in C_b((0,\infty)\times(0,1))$ can be chosen in such a way that

$$g(\hat{t}, r) \le g_c(r) + A(\hat{t}, r) r^{2-d} \quad \text{for all } r \in (0, r_*) \text{ and all } \hat{t} \in (0,\infty). \tag{1.7}$$

For all $\hat{t} > 0$ satisfying $\mu_{\hat{t}}(\{0\}) > 0$, the second option, i.e. (1.6), must hold true.

See Section 3.3 for the proof of Theorem 1.2. The main challenge is to show the lower bound $g(\hat{t}, r) \ge g_c(r) + A(\hat{t}, r) r^{2-d}$, $r \in (0, r_*)$, encoded in (1.6). For its proof we combine different techniques: based on the global temporal Lipschitz continuity of the partial mass function, we first establish a partial result on the "stability from below" of the unbounded steady state f_c by employing a bootstrap argument that bears some elements of classical intersection comparison [21, 22]. This step strongly relies on the radial symmetry assumption. It allows us to infer (1.6) at times $\hat{t}$ where $\mu_{\hat{t}}(\{0\}) > 0$, for instance. The full characterisation in Theorem 1.2 is only achieved upon a combination with specially tailored semi-group estimates for mild solutions along with a contradiction-type argument. We refer to Remark 3.2 for more details. The upper bound (1.7) also holds in the anisotropic case (see Remark 1.8 and Corollary 3.4).

We further note that $g_c(r) = (\frac{2}{\gamma})^{\frac{1}{\gamma}} r^{-\frac{2}{\gamma}} + O(r^{-\frac{2}{\gamma}+2})$ for $0 < r \ll 1$, so that the remainder $O(r^{2-d})$ in (1.6) is indeed of lower order under condition (1.5). Moreover, in the expansion for g one can replace the limiting steady state $g_c(r)$ by the power law $(\frac{2}{\gamma})^{\frac{1}{\gamma}} r^{-\frac{2}{\gamma}}$ since $d > \frac{2}{\gamma}$.

Remark 1.3 (The regime (1.5)). In the present work, we focus on the range $\frac{2}{\gamma} + 2 - d > 0$ as it covers the most interesting case of the three-dimensional Kaniadakis–Quarati model for bosons ($\gamma = 1$, $d = 3$). If $\frac{2}{\gamma} + 2 - d < 0$, the flux into the origin associated with the nonlinear drift term $\operatorname{div}(vh(f_c))$ induced by the unbounded steady state f_c vanishes in the sense that $\lim_{r\downarrow 0} \int_{\partial B_r(0)} h(f_c) v\cdot\nu \, \mathrm{d}\mathcal{H}^{d-1} = 0$. Here, $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure and ν the outer unit normal to $\partial B_r(0)$, so that $v\cdot\nu = r$ for $v \in \partial B_r(0)$. Heuristically, this suggests that an upper bound of order f_c on the space profile of the density $f(t,\cdot)$ near zero (as asserted in Theorem 1.2 for regime (1.5)) might not be compatible with the formation of a point mass at the origin in the case $\frac{2}{\gamma} + 2 - d < 0$. In view of Theorem 1.7, asserting in particular the formation of a Dirac mass in finite time for mass-supercritical data, we conjecture that some new phenomena may be encountered when (1.5) is violated.

Owing to the strong nonlinearity in the drift, one cannot expect the limiting density f to be a distributional solution of (FP$_\gamma$) in $(0,\infty) \times \mathbb{R}^d$. Our analysis leading to Theorem 1.2 allows us to show that the limiting measure satisfies (FP$_\gamma$) in the sense of renormalised solutions.

Definition 1.4 (Renormalised solution of (FP$_\gamma$)). Let μ be a non-negative Radon measure on $[0,\infty) \times \mathbb{R}^d$ and denote by $\mu = \mu^{\text{reg}} + \mu^{\text{sing}} = f(t,v)\mathcal{L}_+^{1+d} + \mu^{\text{sing}}$ its Lebesgue decomposition into regular part μ^{reg} with density $f \in L^1_{\text{loc}}([0,\infty) \times \mathbb{R}^d)$ and singular part μ^{sing}. We call μ a renormalised solution of (FP$_\gamma$) in $(0,\infty)\times\mathbb{R}^d$ with initial data f_{in} if $\mathrm{d}\mu = \mathrm{d}\mu_t\,\mathrm{d}t$ for some weakly-$*$ continuous curve $[0,\infty)\ni t\mapsto\mu_t$ in $\mathcal{M}_+(\mathbb{R}^d)$ with preserved mass $\int\mathrm{d}\mu_t\equiv\|f_{\text{in}}\|_{L^1(\mathbb{R}^d)}$, if $\mathcal{T}_k(f):=\min\{f,k\}\in L^2_{\text{loc}}([0,\infty);H^1_{\text{loc}}(\mathbb{R}^d))$ for every $k>0$, and if for all $\xi\in C^\infty([0,\infty))$ with compactly supported derivative ξ', for a.a. $T\in(0,\infty)$ and all $\psi\in C_c^\infty([0,T]\times\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\xi(f(T,\cdot))\psi(T,\cdot)\,\mathrm{d}v-\int_{\mathbb{R}^d}\xi(f_{\text{in}})\psi(0,\cdot)\,\mathrm{d}v-\int_0^T\int_{\mathbb{R}^d}\xi(f)\partial_t\psi\,\mathrm{d}v\,\mathrm{d}t$$
$$=-\int_0^T\int_{\mathbb{R}^d}(\nabla f+h(f)v)\cdot\nabla(\xi'(f)\psi)\,\mathrm{d}v\,\mathrm{d}t. \tag{1.8}$$

As usual, the gradients of f on the RHS of (1.8) are to be understood as $\nabla\mathcal{T}_k(f)$ for $k=k(\xi)$ large enough such that $\xi'(s)=0$ for $s\geq k$ (cf. [4,16]).

Let us emphasise that the above definition of renormalised solutions should be seen as preliminary. For a "better" and more complete paradigm, the solution concept may have to be complemented by suitable *energy* or *entropy conditions* as it is classical for conservation laws and nonlinear elliptic/parabolic equations, see [4,5,7,27], where they are crucial for uniqueness. A general analysis of the question of uniqueness for (FP$_\gamma$) is, however, beyond the scope of the present manuscript and will be left for future research.

Theorem 1.5 (The limit μ is a renormalised solution). *Assume the hypotheses of Theorem 1.2. Then the limiting measure μ constructed in Proposition 1.1 satisfies (FP$_\gamma$) in the renormalised sense as specified in Definition 1.4.*

The proof of this theorem is given in Section 4.2 and makes use, among other results, of a local and truncated version of the energy dissipation estimate. The following energy dissipation identity is crucial for deducing the long-time asymptotic behaviour.

Proposition 1.6 (Energy dissipation (in)equality). *Assume (H1), (H2) and use the notation of Proposition 1.1. Then for all $t>0$,*

$$\mathcal{H}(f(t))-\mathcal{H}(f_{\text{in}})\leq-\int_0^t\int_{\mathbb{R}^d}\frac{1}{h(f)}|\nabla f+h(f)v|^2\,\mathrm{d}v\,\mathrm{d}\tau. \tag{1.9}$$

When supposing in addition the hypotheses of Theorem 1.2, the stronger balance law holds true: for all $t\geq s\geq 0$,

$$\mathcal{H}(f(t))-\mathcal{H}(f(s))=-\int_s^t\int_{\mathbb{R}^d}\frac{1}{h(f)}|\nabla f+h(f)v|^2\,\mathrm{d}v\,\mathrm{d}\tau. \tag{1.10}$$

See Section 4.3 for the proof of Proposition 1.6.

The long-time behaviour and further transient dynamical properties can be seen as corollaries of the above results (cf. Section 5 for details). Let us here only highlight the long-time asymptotics.

Theorem 1.7 (Convergence to minimiser). *Assume the hypotheses of Theorem 1.2 and denote by $m=\int f_{\text{in}}>0$ the total mass of the initial data. Further let $\mu_{\min}:=\mu^{(m)}_{\min}$ denote* *the unique minimising measure of $\mathcal{H}$ for the given mass m (cf. equation (1.2)). Then, as $t \to \infty$, $\mathcal{H}(\mu_t) \to \mathcal{H}(\mu_{\min})$ and, moreover,*

$$\begin{aligned} &\mu_t \overset{*}{\rightharpoonup} \mu_{\min} \text{ in } \mathcal{M}_+(\mathbb{R}^d) \quad \text{and} \quad \mu_t(\{0\}) \to \mu_{\min}(\{0\}),\\ &f(t) \to f_{\min} \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}),\\ &f(t) \to f_{\min} \text{ in } L^p(\mathbb{R}^d) \quad \text{for any } p \in \big[1, \tfrac{\gamma d}{2}\big), \end{aligned}$$

where $f_{\min}$ denotes the density of the regular part of $\mu_{\min}$ with respect to the Lebesgue measure.

The proof of this result will be completed in Section 5.1.

Remark 1.8 (Anisotropic case). While the main conclusions in Theorems 1.2, 1.5, and 1.7 are restricted to the isotropic setting, several of the intermediate results derived in this paper are proved for anisotropic data. Below, we summarise the most relevant results obtained for anisotropic data satisfying (H1), (H2), where for simplicity we restrict to regime (1.5).

(a) Limiting measure & upper bound for density: Convergence of a sequence of approximate solutions to a mass-conserving curve $t \mapsto \mu_t = a(t)\delta_0 + f(t,\cdot)\mathcal{L}^d$ as detailed in Proposition 1.1 (i)–(iii). The density $f(t,\cdot)$ satisfies the pointwise bound

$$0 \le f(t,v) \le \big(\tfrac{2}{\gamma}\big)^{\frac{1}{\gamma}} |v|^{-\frac{2}{\gamma}} \min\big\{1 + C_1|v|^{\frac{2}{\gamma}+2-d}, C_2|v|^{-(d-\frac{2}{\gamma})}\big\}$$

for all $t > 0$, $v \in \mathbb{R}^d$, and the point mass at the origin $t \mapsto \mu_t(\{0\})$ is continuous; see Corollaries 3.4 and 2.9.

(b) Energy dissipation inequality: The density f obeys inequality (1.9) for all $t > 0$.

(c) Finite-time condensation and relaxation to free energy minimiser for certain data: If f_{in} is bounded below by an admissible isotropic density of supercritical mass, the limiting measure obtained in (a) exhibits a Dirac mass at the origin after a finite time and converges, in the long-time limit, to the (singular) minimiser of the free energy with the same mass.

The crucial compactness property leading to the convergence result in (a) is obtained by pointwise comparison at the level of the approximate densities with an isotropic envelope. It is at this point that the stronger decay hypothesis on the data imposed in the anisotropic case (cf. (H2)) enters, since for this argument the initial data f_{in} need to lie below an admissible isotropic envelope $\hat{f}_{\mathrm{in}}$, i.e. $f_{\mathrm{in}} \le \hat{f}_{\mathrm{in}}$. The assertion in (c) follows from similar comparison arguments combined with the time-asymptotic results obtained in the isotropic case. Finally, key to the energy dissipation inequality (b) are the Fokker–Planck-type variational structure of the regularised problem, lower semi-continuity properties of the convex free energy, and the strong convergence properties away from the origin in Proposition 1.1 (iii). This paper leaves open the question of whether the limiting measure obtained in (a) relaxes to the minimiser of the free energy with the same mass for all admissible anisotropic data. While the energy dissipation inequality allows us to conclude convergence of the density f along a sequence of times $t_k \to \infty$ to $f_{\infty,\theta}$ for some $\theta \geq 0$, our estimates do not allow us to rule out the case of the parameter θ being larger than that of $\mu_{\min}$ for a given mass, i.e. the case of a simultaneous presence of a smooth density and a Dirac mass at zero. The main problem is the lack of sufficiently strong lower bounds for anisotropic densities in the presence of singularities and concentrations. Such bounds might possibly be obtained by suitably controlling the change of mass in small neighbourhoods of the origin. In the isotropic case, the crucial estimate is (2.18). It follows from the comparison principle structure of the equation for the partial mass function (2.13).

1.2. Outline

The remaining part of this paper is structured as follows. In Section 2 we establish global existence for the approximate problem ($\mathrm{FP}_{\gamma,\mathrm{reg}}$) as well as uniform estimates, which allow us to pass to the limit $\varepsilon \to 0$ in Section 2.3. An important ingredient is the uniform bound in Proposition 2.6, which is obtained using a comparison technique. Section 3 lies at the heart of our analysis. Its main purpose is to establish the universal profile asserted in Theorem 1.2 (see Section 3.3). In Section 4 we introduce entropy tools and use the results from Section 3 to show, for the isotropic case, the renormalised solution property of (FP_γ) as well as the energy dissipation identity. Section 5 concludes with a characterisation of the long-time asymptotics and some additional remarks.

1.3. Notation

Unless specified otherwise, we adopt the following notation:

- $L^p_\ell(\mathbb{R}^d)$ for $p \in [1,\infty]$, $\ell \geq 0$: Weighted L^p-space with norm $\|f\|_{L^p_\ell} := \|(1 + |\cdot|^\ell) f\|_{L^p(\mathbb{R}^d)}$, where $|\cdot|$ denotes the function $v \mapsto |v|$. The spaces $L^p_\ell(\mathbb{R}^d)$ are Banach spaces.
- $C^{1,2}((0,\infty) \times \mathbb{R}^d)$: Space of continuously differentiable functions $f = f(t,v)$ that are twice continuously differentiable with respect to $v \in \mathbb{R}^d$.
- $\mathcal{M}_+(G)$: Set of non-negative finite (Radon) measures on a given Borel set $G \subset \mathbb{R}^N$, $N \in \mathbb{N}$. Usually, $G = \mathbb{R}^d$ or $G = I \times \mathbb{R}^d$ for an interval $I \subset [0,\infty)$.
- $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_+(G)$ for $\mu_n, \mu \in \mathcal{M}_+(G)$ stands for the convergence $\int_G \varphi \,\mathrm{d}\mu_n \to \int_G \varphi \,\mathrm{d}\mu$ for all $\varphi \in C_b(G)$. This mode of convergence will be referred to as weak-$*$ convergence of measures. It is induced by a distance on $\mathcal{M}_+(G)$ [2, Section 5.1], [26].
- We write $\mathrm{d}\mu = \mathrm{d}\mu_t\,\mathrm{d}t$ for non-negative Radon measures μ on $[0,\infty) \times \mathbb{R}^d$ and μ_t on $\mathbb{R}^d$, $t \geq 0$, with $\mu_t(\mathbb{R}^d) \equiv \text{const.} \in \mathbb{R}_+$ if for every $\varphi \in C_c([0,\infty) \times \mathbb{R}^d)$ the function $t \mapsto \int_{\mathbb{R}^d} \varphi(t,v)\,\mathrm{d}\mu_t(v)$ is Lebesgue measurable and $\int_{[0,\infty)\times\mathbb{R}^d} \varphi \,\mathrm{d}\mu = \int_{[0,\infty)} \int_{\mathbb{R}^d} \varphi(t,v)\,\mathrm{d}\mu_t(v)\,\mathrm{d}t$.
- μ^{reg}, μ^{sing}: Regular and singular part of $\mu \in \mathcal{M}_+(G)$ with respect to the Lebesgue measure on $G \subset \mathbb{R}^N$.

- $\mathcal{L}^d$: d-dimensional Lebesgue measure.
- $\mathcal{L}^{1+d}_+$: $(1+d)$-dimensional Lebesgue measure on $[0,\infty)\times\mathbb{R}^d$.
- $A \lesssim B$ for non-negative quantities A, B stands for $A \le CB$ for a fixed constant $C \in (0,\infty)$. The relation $A \gtrsim B$ is defined as $B \lesssim A$.
- $s_+ := \max\{s, 0\}$ for $s \in \mathbb{R}$.
- $B_r := B_r(0) := \{v \in \mathbb{R}^d : |v| < r\}$.
- $g_c(r) = f_c(v)$ for $r = |v|$, where $f_c = f_{\infty,0}$ as defined in (1.3). Equivalently, $g_c(r) = (\Phi')^{-1}(-\frac{1}{2}r^2)$.

2. Approximation scheme

As pointed out in the introduction, local-in-time classical solutions of (FP_γ) emanating from initial data that are large in a suitable sense, may cease to exist in $L^\infty(\mathbb{R}^d)$ after a finite time. The main purpose of this section is to establish global existence for the approximation scheme ($\mathrm{FP}_{\gamma,\mathrm{reg}}$) in spaces of suitable regularity, as well as certain compactness and convergence properties for the corresponding approximate solutions. In the isotropic case, our scheme obeys a monotonicity property and, as a consequence, gives rise to a unique limiting measure. Note that this feature may also be of interest from a numerics point of view. A key ingredient in the analysis is a uniform temporal Lipschitz bound for the partial mass function of isotropic solutions (see Proposition 2.6).

2.1. Mild solutions

The local-in-time well-posedness of equations (FP_γ) and ($\mathrm{FP}_{\gamma,\mathrm{reg}}$) in suitably weighted spaces can conveniently be obtained in the framework of mild solutions using the Duhamel integral formulation of (FP_γ), resp. of ($\mathrm{FP}_{\gamma,\mathrm{reg}}$), given by

$$\begin{aligned} f(t,v) &= \int_{\mathbb{R}^d} \mathcal{F}(t,v,w) f_{\mathrm{in}}(w)\,\mathrm{d}w \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \mathcal{F}(t-s,v,w)\big(\mathrm{div}_w(w|f|^\gamma f)\big)|_{(s,w)}\,\mathrm{d}w\,\mathrm{d}s, \end{aligned} \tag{2.1}$$

$$\begin{aligned} f_\varepsilon(t,v) &= \int_{\mathbb{R}^d} \mathcal{F}(t,v,w) f_{\mathrm{in}}(w)\,\mathrm{d}w \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \mathcal{F}(t-s,v,w)\big(\mathrm{div}_w(w\vartheta_\varepsilon(f_\varepsilon))\big)|_{(s,w)}\,\mathrm{d}w\,\mathrm{d}s, \end{aligned} \tag{2.2}$$

where $\mathcal{F} = \mathcal{F}(t,v,w)$ denotes the fundamental solution of the linear Fokker–Planck equation $\partial_t \mathrm{f} = \nabla\cdot(\nabla \mathrm{f} + v\mathrm{f})$, i.e. (cf. [14])

$$\mathcal{F}(t,v,w) = \mathrm{e}^{dt} G_{\nu(t)}(\mathrm{e}^t v - w)$$

with

$$\nu(t) = \mathrm{e}^{2t} - 1, \quad G_\lambda(\xi) = (2\pi\lambda)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|\xi|^2}{2\lambda}}.$$

In this subsection we collect several auxiliary results for mild solutions, many of which can be obtained as in [10]. The reasoning is therefore kept brief.

Using integration by parts, equation (2.1) can formally be rewritten as

$$f(t,v) = \int_{\mathbb{R}^d} \mathcal{F}(t,v,w) f_{\mathrm{in}}(w)\,\mathrm{d}w \\ + \int_0^t \mathrm{e}^{-(t-s)} \int_{\mathbb{R}^d} \nabla_v \mathcal{F}(t-s,v,w)\cdot w|f|^\gamma f|_{(s,w)}\,\mathrm{d}w\,\mathrm{d}s. \tag{2.3}$$

Analogously, we may rewrite equation (2.2). To estimate the integrals appearing on the RHS of (2.3) we use the semi-group estimates in [10, Appendix A]. By [10, Proposition A.1] the linear operator

$$\mathfrak{F}[f](t,v) := \int_{\mathbb{R}^d} \mathcal{F}(t,v,w) f(w)\,\mathrm{d}w$$

enjoys the following smoothing estimates for all $t \in (0,T]$ and $T < \infty$:

$$\|\nabla_v^k \mathfrak{F}[f](t)\|_{L^q_\ell} \le C_T\, \nu(t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{k}{2}} \|f\|_{L^p_\ell} \tag{2.4}$$

for any $1 \le p \le q \le \infty$, $\ell \ge 0$, and $k \in \mathbb{N}_0$, where the constant $C_T = C_T(d,q,k)$ is given by $C_T = C\exp((\frac{d}{q'}+k)T)$ with $\frac{1}{q'}+\frac{1}{q} = 1$ and $C < \infty$ a universal constant. For the definition of $\|\cdot\|_{L^p_\ell}$ we refer to Section 1.3.

In the rest of this section, C denotes a constant that may depend on fixed parameters, but not on time. Constants that additionally depend on the (final) time $T < \infty$ are denoted by C_T. Any such constants may change from line to line.

We begin with a uniqueness result.

Lemma 2.1 (Uniqueness of mild solutions for (FP$_\gamma$) and (FP$_{\gamma,\mathrm{reg}}$)). *Let $p > d$. There exists at most one mild solution $f \in C([0,T];(L^\infty \cap L^p_1)(\mathbb{R}^d))$ of equation (FP$_\gamma$). An analogous result holds for equation (FP$_{\gamma,\mathrm{reg}}$).*

Proof. Let $f, \tilde{f} \in C([0,T];(L^\infty \cap L^p_1)(\mathbb{R}^d))$ both satisfy equation (2.3) for $t \in [0,T]$. Note that since $p > d$, we have $\alpha := \frac{1}{2} + \frac{d}{2}\frac{1}{p} \in [0,1)$. We may thus estimate for $t \in [0,T]$, using bound (2.4) and recalling that $\nu(t) = \mathrm{e}^{2t} - 1$,

$$\begin{aligned} &\|f(t) - \tilde{f}(t)\|_{L^\infty} \\ &\quad \le C_T \int_0^t \nu(t-s)^{-\alpha} \|w(|f|^\gamma f(s,w) - |\tilde{f}|^\gamma \tilde{f}(s,w))\|_{L^p}\,\mathrm{d}s \\ &\quad \le C_T \|\,|f| + |\tilde{f}|\,\|^\gamma_{C([0,T];L^p_1\cap L^\infty)} \int_0^t (t-s)^{-\alpha} \|f(s) - \tilde{f}(s)\|_{L^\infty}\,\mathrm{d}s, \end{aligned}$$

where we used the fact that $\nu(t) \ge 2t$. Invoking the singular Grönwall inequality (see e.g. [1, Theorem 3.3.1]), we infer that $f(t) = \tilde{f}(t)$ for all $t \in (0,T]$, which shows the asserted uniqueness. ∎

We now seek to construct solutions taking values in the Banach space

$$X_{\ell,n} = (L^\infty_\ell \cap L^1_n)(\mathbb{R}^d)$$

for sufficiently large $\ell, n \in [1,\infty)$. Global-in-time existence of non-negative mild solutions to ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) will be obtained under the additional hypothesis that $n = n(\ell, d)$ be sufficiently large; cf. (2.8). The reason for this condition is that our approach to control the L^∞_ℓ-norm of a local solution relies on an a priori control in L^1_n for $n = n(\ell, d)$ large enough; see the proof of Proposition 2.4. The uniform L^1_n-control of non-negative solutions, see Lemma 2.3, is a consequence of the Fokker–Planck structure, which naturally ensures the propagation of higher moment bounds.

The canonical norm on $X := X_{\ell,n}$ will often be abbreviated by $\|\cdot\|_X$, i.e. we let $\|f\|_X := \|f\|_{X_{\ell,n}} := \max\{\|f\|_{L^\infty_\ell}, \|f\|_{L^1_n}\}$.

Lemma 2.2 (Local existence for (FP_γ), ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) in X and basic properties). *Let $\ell, n \geq 1$. For any $L \in (0,\infty)$ there exists $T = T(L) > 0$ such that for every $f_{\mathrm{in}} \in X_{\ell,n}$ with $\|f_{\mathrm{in}}\|_{X_{\ell,n}} \leq L$ there exists a unique mild solution $f \in C([0,T]; X_{\ell,n})$ of equation (FP_γ).*

On any time interval $[0,T^*)$, where the local-in-time mild solution exists, one has the extra regularity $t \mapsto \nu(t)^{\frac{1}{2}}|\nabla f(t)| \in C_b((0,T); X_{\ell,1})$ for every $T \in (0,T^*)$.

Furthermore, if $f_{\mathrm{in}} \geq 0$, the following additional properties hold true:

(i) *Positivity: $f \geq 0$ in $(0,T^*) \times \mathbb{R}^d$.*

(ii) *Smoothness: $f \in C^{1,2}((0,T^*) \times \mathbb{R}^d)$ and (FP_γ) holds in the classical sense.*

(iii) *Mass conservation: $\|f(t)\|_{L^1} = \|f_{\mathrm{in}}\|_{L^1}$ for all $t \in (0,T^*)$.*

(iv) *Preservation of radial symmetry: If f_{in} is isotropic, so is $f(t)$ for all $t \in (0,T^*)$.*

Given non-negative initial data $f^{(i)}_{\mathrm{in}} \in X_{\ell,n}$, $i = 1,2$, denote by $f^{(i)}$, $i = 1,2$, the mild solution emanating from $f^{(i)}_{\mathrm{in}}$ and let $[0,T^*)$ be a common time interval of existence. Then the following properties also hold true:

(v) *L^1-contractivity: $\|f^{(1)}(t) - f^{(2)}(t)\|_{L^1} \leq \|f^{(1)}_{\mathrm{in}} - f^{(2)}_{\mathrm{in}}\|_{L^1}$ for all $t \in (0,T^*)$.*

(vi) *Comparison: If $f^{(1)}_{\mathrm{in}} \leq f^{(2)}_{\mathrm{in}}$, then $f^{(1)} \leq f^{(2)}$ in $(0,T^*) \times \mathbb{R}^d$.*

Completely analogous statements hold for the regularised problem ($\mathrm{FP}_{\gamma.\mathrm{reg}}$).

Proof. The proof is similar to that of [10, Theorem 2.5]. Abbreviate $X := X_{\ell,n}$. To prove the existence of a mild solution $f \in C([0,T]; X)$ of (FP_γ) we show that the operator

$$\mathcal{T}[f](t) := \mathcal{F}[f_{\mathrm{in}}](t) + \int_0^t \mathrm{e}^{-(t-s)} \int_{\mathbb{R}^d} \nabla_v \mathcal{F}(t-s, v, w) \cdot w|f|^\gamma f|_{(s,w)}\, \mathrm{d}w\, \mathrm{d}s$$

defines a contraction mapping on a closed ball in $C([0,T]; X)$ provided $T = T(\|f_{\mathrm{in}}\|_X) > 0$ is small enough. For this purpose, we rely on (2.4) and estimate

$$\|\mathcal{T}[f](t)\|_{L^1_n} \leq C_T \|f_{\mathrm{in}}\|_{L^1_n} + C_T \int_0^t \nu(t-s)^{-\frac{1}{2}} \|\,|\cdot|\,|f(s)|^{\gamma+1}\|_{L^1_n}\, \mathrm{d}s,$$

where $|\cdot||f(s)|^{\gamma+1}$ denotes the function $w \mapsto |w||f(s,w)|^{\gamma+1}$. We next observe that

$$\begin{aligned} \||\cdot||f|^{\gamma+1}\|_{L^1_n(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d} (1+|w|^n)(1+|w|)|f(w)|^{\gamma+1}\,\mathrm{d}w \\ &\le \int_{\mathbb{R}^d} (1+|w|^n)|f(w)|\,\mathrm{d}w\,\|(1+|\cdot|)|f|^{\gamma}\|_{L^\infty} \\ &\le \|f\|_{L^1_n}\|f\|^{\gamma}_{L^\infty_1}, \end{aligned}$$

where the last step uses the fact that $\gamma \ge 1$.

Next, we estimate, for $p := d+1$,

$$\|\mathcal{T}[f](t)\|_{L^\infty_\ell} \le C_T\|f_{\mathrm{in}}\|_{L^\infty_\ell} + C_T\int_0^t \nu(t-s)^{-\frac{1}{2}-\frac{d}{2p}}\||\cdot||f(s)|^{\gamma+1}\|_{L^p_\ell}\,\mathrm{d}s$$

and

$$\begin{aligned} \||\cdot||f|^{\gamma+1}\|^p_{L^p_\ell(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d}(1+|w|^\ell)^p(1+|w|^p)|f(w)|^{(\gamma+1)p}\,\mathrm{d}w \\ &\le C\int_{\mathbb{R}^d}(1+|w|^{(\ell+1)p})(1+|w|^{\ell[(\gamma+1)p-1]})^{-1}|f|\,\mathrm{d}w \\ &\qquad\times\|(1+|\cdot|^\ell)f\|^{(\gamma+1)p-1}_{L^\infty} \\ &\le C\|f\|_{L^1_n}\|f\|^{(\gamma+1)p-1}_{L^\infty_\ell}, \end{aligned}$$

where the last step uses the fact that

$$(\ell+1)p-\ell[(\gamma+1)p-1] = \ell p + p - \ell p - \ell\gamma p + \ell = \ell + d + 1 - \ell(d+1)\gamma \le 1 \le n,$$

which follows from the choice $p = d+1$, $\ell \ge 1$, and $\gamma \ge 1$.

In combination, this shows that the mapping $\mathcal{T}$ obeys an estimate of the form

$$\|\mathcal{T}[f](t)\|_X \le C_T\|f_{\mathrm{in}}\|_X + C_T\kappa(T)\|f\|^{\gamma+1}_{C([0,T];X)}, \quad t \in [0,T],$$

for some function $\kappa \in C([0,\infty))$ that satisfies $\kappa(0) = 0$.

Using the above estimates and analogous bounds for the difference $\mathcal{T}[f] - \mathcal{T}[\tilde{f}]$, one may now follow [10] to show the contraction mapping property of $\mathcal{T}$ and deduce the existence of a fixed point $f \in C([0,T];X)$ for small enough T as asserted in Lemma 2.2. By construction, this fixed point is a mild solution of (FP_γ). The extra regularity $t \mapsto \nu(t)^{\frac{1}{2}}|\nabla f(t)| \in C_b((0,T);(L^\infty_\ell \cap L^1_1)(\mathbb{R}^d))$ follows from similar arguments (see [10, Section 2.2]) combined with the uniqueness of mild solutions in $C([0,T];(L^\infty \cap L^p_1)(\mathbb{R}^d))$ for $p > d$ shown in Lemma 2.1. (Of course, the contraction mapping property, whose proof we have not presented in full detail, also provides uniqueness.)

Properties (i)–(vi) can be deduced from classical arguments as in [10, Sections 2.3 and 2.4] (see also [28, 31]).

The analogous results for the regularised problem ($\mathrm{FP}_{\gamma,\mathrm{reg}}$) are obtained along the same lines using in particular the bound $0 \le \eta_\varepsilon(f) \le |f|^\gamma$, where η_ε is defined in the line above (1.4). ∎

Lemma 2.3 (Uniform moment bound). *Assume that* $\ell \geq 1$, $n \geq 2$, *and let* $f_{\text{in}} \in X_{\ell,n}$ *be non-negative. Denote by* $f_\varepsilon \in C([0,T^*); X_{\ell,n})$ *the non-negative (local-in-time) mild solution of* ($\text{FP}_{\gamma,\text{reg}}$) *as obtained in Lemma* 2.2. *Then, for all* $t \in [0,T^*)$,

$$\int_{\mathbb{R}^d} f_\varepsilon(t,v)(1+|v|^n)\,\mathrm{d}v \leq C(n,d)\|f_{\text{in}}\|_{L^1_n}. \tag{2.5}$$

We emphasise that the constant $C(n,d) < \infty$ in the above lemma is independent of ε and T^*. Moreover, the bound (2.5) equally holds for the local mild solution of (FP_γ).

Proof of Lemma 2.3. Let us first provide the formal argument leading to the above estimate. We abbreviate $f := f_\varepsilon$ and define for $k \in [0,\infty)$,

$$E_k(t) = \int_{\mathbb{R}^d} |v|^k f(t,v)\,\mathrm{d}v.$$

Then, by Lemma 2.2, $E_0(t) \equiv \|f_{\text{in}}\|_{L^1(\mathbb{R}^d)} =: B_0$. Clearly, $B_0 \leq \|f_{\text{in}}\|_{L^1_n}$.

We now argue inductively and assume that $\sup_{t\in[0,T^*)} E_{k-2}(t) \leq B_{k-2}$ for some $k \in [2,n]$ and a positive constant B_{k-2} obeying the bound $B_{k-2} \leq C\|f_{\text{in}}\|_{L^1_n}$ with $C = C(n,d)$. Formally, we may then compute

$$\begin{aligned}
\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}E_k(t) &= -\int_{\mathbb{R}^d} |v|^{k-2}v\cdot(\nabla f + vh_\varepsilon(f))\,\mathrm{d}v \\
&= \int_{\mathbb{R}^d} \operatorname{div}(|v|^{k-2}v) f\,\mathrm{d}v - \int_{\mathbb{R}^d} |v|^k h_\varepsilon(f)\,\mathrm{d}v \\
&\leq (k-2+d)\int_{\mathbb{R}^d} |v|^{k-2} f\,\mathrm{d}v - \int_{\mathbb{R}^d} |v|^k f\,\mathrm{d}v \\
&= (k-2+d)E_{k-2}(t) - E_k(t),
\end{aligned} \tag{2.6}$$

which implies that

$$E_k(t) \leq \max\{E_k(0), (k-2+d)B_{k-2}\} =: B_k, \quad t \in [0,T^*). \tag{2.7}$$

Since $k \leq n$, the new upper bound B_k again satisfies the estimate $B_k \leq C\|f_{\text{in}}\|_{L^1_n}$ for some possibly larger constant $C = C(n,d)$.

We now let $k = 2$ in the above step to find that $\sup_t E_2(t) \leq \max\{E_2(0), dB_0\} = B_2$. By interpolation we infer that $E_k(t) \leq B_2^{\frac{k}{2}} B_0^{\frac{2-k}{2}} =: B_k$ for all $k \in (0,2)$ and all $t \in [0,T^*)$. Observe that $B_k \leq C\|f_{\text{in}}\|_{L^1_n}$ for all $k \in (0,2)$. We may now complete the induction argument: starting with $k-2 = n - 2\lfloor n/2\rfloor$ (which lies in $[0,2)$) and iterating the above induction step $\lfloor n/2\rfloor$ times, we arrive at the bound $\sup_t E_n(t) \leq C(n,d)\|f_{\text{in}}\|_{L^1_n}$.

Finally, let us note that the computation (2.6) can be made rigorous by introducing a smooth, compactly supported cut-off function φ_R, $R \geq 1$, with $\varphi_R(v) = \varphi(R^{-1}v)$ for some $\varphi \in C_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $\{|v| \leq 1\}$. The time derivative of $t \mapsto \frac{1}{k}\int |v|^k f(t,v)\varphi_R(v)\,\mathrm{d}v$ then satisfies an inequality which leads to (2.6)–(2.7) in the limit $R \to \infty$. ∎

Global existence for (FP$_{\gamma.\text{reg}}$) in $X_{\ell,n}$ will be obtained under the decay conditions

$$\begin{aligned} &\ell \geq 1, \\ &n \geq \ell + d + 1. \end{aligned} \tag{2.8}$$

Proposition 2.4 (Global existence for (FP$_{\gamma.\text{reg}}$)). *Let $\varepsilon \in (0,1]$. Let ℓ, n satisfy (2.8), and suppose that $f_{\text{in}} \in X_{\ell,n}$ is non-negative. There exists a unique global-in-time mild solution $f_\varepsilon \in C([0,\infty); X_{\ell,n})$ of the Cauchy problem (FP$_{\gamma.\text{reg}}$).*

Note that, as a consequence of Lemma 2.2, the function f_ε in the above proposition enjoys additional properties (i)–(iv). In particular, it is a classical solution of (FP$_{\gamma.\text{reg}}$) in $(0,\infty)\times\mathbb{R}^d$.

Proof of Proposition 2.4. Local-in-time well-posedness of (FP$_{\gamma.\text{reg}}$) in $X := X_{\ell,n}$ follows from Lemma 2.2. Thus, for proving global existence it suffices to show that, for $\varepsilon > 0$ fixed, $\|f_\varepsilon(t)\|_X$ cannot blow up in finite time. For this purpose, let $T < \infty$ and suppose that $f_\varepsilon \in C([0,T); X)$ is a mild solution of (FP$_{\gamma.\text{reg}}$) on the interval $[0,T)$. Since $n \geq 2$, we may invoke Lemma 2.3 to infer that $\|f_\varepsilon(t)\|_{L^1_n}$ remains bounded uniformly in time:

$$\sup_{t\in[0,T)} \|f_\varepsilon(t)\|_{L^1_n} \leq C\|f_{\text{in}}\|_{L^1_n} < \infty. \tag{2.9}$$

Next, we let $p := d+1$ and estimate for $t \in [0,T)$,

$$\|f_\varepsilon(t)\|_{L^\infty_\ell} \leq C_T\|f_{\text{in}}\|_{L^\infty_\ell} + C_T\int_0^t \nu(t-s)^{-\frac{1}{2}-\frac{d}{2p}}\|f_\varepsilon\eta_\varepsilon(f_\varepsilon)\|_{L^p_{\ell+1}}\,\mathrm{d}s, \tag{2.10}$$

where we used the fact that $\||\cdot|\tilde f(\cdot)\|_{L^p_\ell} \leq 2\|\tilde f\|_{L^p_{\ell+1}}$ for $\tilde f \in L^p_{\ell+1}(\mathbb{R}^d)$. Since

$$(\ell+1)p - \ell(p-1) = \ell + d + 1 \leq n$$

and hence $(1+|w|^{\ell+1})^p \lesssim (1+|w|^n)(1+|w|^\ell)^{p-1}$, we further have

$$\begin{aligned} \|f_\varepsilon\eta_\varepsilon(f_\varepsilon)\|^p_{L^p_{\ell+1}} &\leq C(\varepsilon)^p\int_{\mathbb{R}^d}(1+|w|^{\ell+1})^p|f_\varepsilon|^p\,\mathrm{d}w \\ &\leq C(\varepsilon)^p\|f_\varepsilon\|_{L^1_n}\|f_\varepsilon\|^{p-1}_{L^\infty_\ell}. \end{aligned}$$

Hence, using the Young inequality $ab \leq \frac{1}{p}a^p + \frac{p-1}{p}b^{\frac{p}{p-1}}$, we deduce

$$\|f_\varepsilon\eta_\varepsilon(f_\varepsilon)\|_{L^p_{\ell+1}} \leq C(\varepsilon)\|f_\varepsilon\|_{L^1_n} + C(\varepsilon)\|f_\varepsilon\|_{L^\infty_\ell}.$$

Inserting this bound into (2.10), using (2.9), and applying the generalised Grönwall inequality [1, Theorem 3.3.1] yields

$$\sup_{t\in[0,T)}\|f_\varepsilon(t)\|_{L^\infty_\ell} \leq C_{T,\varepsilon}(\|f_{\text{in}}\|_{L^\infty_\ell} + \|f_{\text{in}}\|_{L^1_n})\exp(C_{T,\varepsilon}),$$

where the constants $C_{T,\varepsilon}$ depend on ε, T, and fixed parameters. This shows that the unique local-in-time mild solution can be extended beyond the time T, and since $T \in (0,\infty)$ was arbitrary, the function f_ε extends to a unique global-in-time mild solution $f_\varepsilon \in C([0,\infty); X)$. ∎

For later reference, let us note the following consequence of the above theory.

Corollary 2.5 (Short-time consistency). Assume the hypotheses of Proposition 2.4. Now let $f \in C([0,T]; X_{\ell,n})$ be a local-in-time mild solution of (FP_γ), let $0 < \epsilon_* < (\|f\|_{C([0,T];L^\infty)})^{-1}$, and $\varepsilon \in (0,\epsilon_*]$. Then, since $h_\varepsilon(s) = h(s)$ for $s \le \varepsilon^{-1}$, the function f is also the unique mild solution $f = f_\varepsilon \in C([0,T]; X_{\ell,n})$ of ($\mathrm{FP}_{\gamma,\mathrm{reg}}$) in $[0,T]$. In particular, as long as the mild solution of (FP_γ) obtained in Lemma 2.2 exists, the scheme $\{f_\varepsilon\}_\varepsilon$ trivially converges to this solution as $\varepsilon \downarrow 0$.

2.2. Uniform bounds

2.2.1. Preliminaries. From now on we assume hypothesis (H2), which imposes a somewhat stronger decay condition on the initial data as compared to Proposition 2.4. In particular, $f_{\mathrm{in}} \in X_{\ell,\ell+d+1}$ with $\ell = d$, resp. $\ell > 3d+1$, if f_{in} is isotropic, resp. anisotropic. Let us note that the specific regularity conditions in (H2) have been made for convenience, and we have not attempted to optimise them.

Let $f \in C([0,T]; X_{\ell,\ell+d+1})$ denote the local mild solution of (FP_γ) obtained in Lemma 2.2. Then, replacing f by the time-shifted solution $f(t_0+\cdot)$ emanating from $f(t_0)$ for some small $t_0 \in (0,T/2)$, we may henceforth assume, without loss of generality, the additional regularity $f \in C^{1,2}([0,T/2]\times\mathbb{R}^d)$ with $\nabla f \in C([0,T/2]; L^\infty_d(\mathbb{R}^d))$ and, moreover, that f is strictly positive in $[0,T/2]\times\mathbb{R}^d$ (if $m>0$, the strict positivity of $f(t_0)$ follows from [31, Proposition 52.7]).

Thus, from now on we may assume the following stronger version of hypothesis (H2):

$$\left.\begin{array}{l}\text{(H2) and the local regular solution } f \text{ of } (\mathrm{FP}_\gamma) \text{ with } f(0) = f_{\mathrm{in}}\\ \text{satisfies } f \in C^{1,2}([0,\tau_*]\times\mathbb{R}^d),\ \nabla f \in C([0,\tau_*]; L^\infty_d(\mathbb{R}^d)),\\ f > 0 \text{ in } [0,\tau_*]\times\mathbb{R}^d \text{ for some fixed } \tau_* > 0.\end{array}\right\} \tag{H2$'$}$$

Furthermore, we henceforth denote by f_ε, $\varepsilon \in (0,\epsilon_*]$, the global mild solution of the regularised equation ($\mathrm{FP}_{\gamma,\mathrm{reg}}$) as obtained in Proposition 2.4, where $\epsilon_* \in (0,1]$ is chosen small enough such that

$$f_\varepsilon \equiv f \text{ in } [0,\tau_*] \quad \text{for all } \varepsilon \in (0,\epsilon_*]. \tag{2.11}$$

Such an ϵ_* exists by virtue of Corollary 2.5.

2.2.2. Isotropic solutions. In this subsection we assume f_{in} to be isotropic and write $g_{\mathrm{in}}(r) = f_{\mathrm{in}}(v)$, $|v| = r$. By Proposition 2.4 (iv), the global mild solution f_ε of ($\mathrm{FP}_{\gamma,\mathrm{reg}}$) is isotropic, allowing us to write $g_\varepsilon(t,r) := f_\varepsilon(t,v)$ for $r = |v| \ge 0$. Observe that g_ε satisfies the equation

$$\begin{aligned} \partial_t g_\varepsilon &= r^{-(d-1)}\partial_r(r^{d-1}\partial_r g_\varepsilon + r^d h_\varepsilon(g_\varepsilon)) \text{ in } \mathbb{R}_+ \times \mathbb{R}_+, \\ 0 &= \lim_{r\to 0}(r^{d-1}\partial_r g_\varepsilon + r^d h_\varepsilon(g_\varepsilon)), \end{aligned} \tag{2.12}$$

where the limit in the last line holds locally uniformly in $t \in [0,\infty)$.

Our fundamental a priori bound for (FP_γ) relies on the fact that, in the isotropic case, equation ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) can be expressed as an evolution equation for the partial mass function

$$M_\varepsilon(t,r) := \int_0^r g_\varepsilon(t,\rho)\rho^{d-1}\,\mathrm{d}\rho = c_d^{-1}\int_{B_r} f_\varepsilon(t,v)\,\mathrm{d}v \le c_d^{-1}\|f_{\mathrm{in}}\|_{L^1(\mathbb{R}^d)}, \tag{2.13}$$

where c_d denotes the area of the unit sphere ∂B_1 in $\mathbb{R}^d$. The equation for M_ε is obtained by multiplying (2.12) by r^{d-1} and integrating in r:

$$\partial_t M_\varepsilon = r^{d-1}\partial_r g_\varepsilon + r^d h_\varepsilon(g_\varepsilon). \tag{2.14}$$

Using the relations

$$\begin{aligned} \partial_r M_\varepsilon &= r^{d-1} g_\varepsilon, \\ \partial_r^2 M_\varepsilon &= r^{d-1}\partial_r g_\varepsilon + \frac{(d-1)}{r}\partial_r M_\varepsilon, \end{aligned}$$

one arrives at

$$\begin{cases} \partial_t M_\varepsilon = \partial_r^2 M_\varepsilon - \dfrac{(d-1)}{r}\partial_r M_\varepsilon + r^d h_\varepsilon(r^{1-d}\partial_r M_\varepsilon), & t > 0,\ r \in \mathbb{R}_+, \\ M_\varepsilon(t,0) = 0, & t > 0, \\ M_\varepsilon(0,r) = M_{\mathrm{in}}(r), & r \in \mathbb{R}_+. \end{cases} \tag{2.15}$$

We note that, as a consequence of (2.11),

$$M_\varepsilon \equiv M \text{ in } [0,\tau_*]\times[0,\infty) \quad \text{for all } \varepsilon \in (0,\epsilon_*], \tag{2.16}$$

where $M(t,r) = c_d^{-1}\int_{B_r} f(t,v)\,\mathrm{d}v$ with $f \in C([0,\tau_*]; X_{d,2d+1})$ denoting the local-in-time mild solution of (FP_γ). Hence, thanks to the regularity established in Lemma 2.2 and hypothesis (H2′) we can ensure that

$$M \in C^{1,2}([0,\tau_*]\times[0,\infty)) \quad \text{with} \quad \sup_{\tau\in[0,\tau_*]}\|\partial_t M(\tau,\cdot)\|_{L^\infty([0,\infty))} \le K < \infty, \tag{2.17}$$

where the last estimate follows from (2.14) and the regularity $f \in C([0,\tau_*]; L^\infty_d(\mathbb{R}^d))$, $\nabla f \in C([0,\tau_*]; L^\infty_d(\mathbb{R}^d))$.

Proposition 2.6 (Global Lipschitz regularity in time). *Suppose that f_{in} is isotropic and satisfies the hypotheses in (H2′). Denote by M_ε the partial mass function (2.13) of the* *global solution f_ε of* (FP$_{\gamma,\mathrm{reg}}$) *obtained in Proposition* 2.4*. In particular, M_ε is a classical solution of equation* (2.15) *satisfying* (2.16)*,* (2.17) *and is such that f_ε enjoys the uniform moment bound* (2.5) *for $n = 2d + 1$. Then*

$$\sup_{\varepsilon\in(0,\epsilon_*]}\sup_{t,r>0}|\partial_t M_\varepsilon(t,r)| \le K_*, \tag{2.18}$$

where

$$K_* := \max\left\{K, \frac{\widetilde{m}}{\tau_*}\right\} < \infty \tag{2.19}$$

with K as in (2.17) *and $\widetilde{m} := c_d^{-1} m = c_d^{-1}\|f_{\mathrm{in}}\|_{L^1(\mathbb{R}^d)}$.*

Proof. Let K_* be as in (2.19). We will show by contradiction that

$$\sup_{\varepsilon\in(0,\epsilon_*]}(M_\varepsilon(t,r) - M_\varepsilon(s,r)) \le K_*|t-s|$$

for all $t, s, r > 0$.

Suppose the last inequality is false for some $\varepsilon > 0$. Then there exist $t_1, s_1, r_1 > 0$ such that

$$M_\varepsilon(t_1,r_1) - M_\varepsilon(s_1,r_1) - K_*|t_1 - s_1| > 0.$$

Pick some $T \ge \max\{t_1, s_1\} + 1$. Without loss of generality we further assume that $T > \tau_*$ with τ_* being as in (2.16), (2.17). Then, for $\delta > 0$ small enough, we have

$$M_\varepsilon(t_1,r_1) - \frac{\delta}{T-t_1} - \frac{\delta}{T-s_1} - M_\varepsilon(s_1,r_1) - K_*|t_1-s_1| > 0$$

and hence

$$\sup_{(t,s,r)\in Q}\left(M_\varepsilon(t,r) - M_\varepsilon(s,r) - K_*|t-s| - \frac{\delta}{T-t} - \frac{\delta}{T-s}\right) > 0,$$

where $Q = (0,T)\times(0,T)\times(0,\infty)$.

We assert that the function

$$P(t,s,r) := M_\varepsilon(t,r) - M_\varepsilon(s,r) - K_*|t-s| - \frac{\delta}{T-t} - \frac{\delta}{T-s}$$

attains its (positive) supremum in the interior of Q. This can be seen as follows: By the uniform continuity of M_ε on $[0,T]\times[0,1]$ and the fact that $M_\varepsilon(\cdot,0) \equiv 0$, there exists $r' > 0$ such that $P < 0$ in $[0,T]\times[0,T]\times[0,r']$. Moreover, by (2.16) and (2.17) one has $P < 0$ in $[0,\tau_*]\times[0,\tau_*]\times[0,\infty)$. The bound $M_\varepsilon \le \widetilde{m}$ further shows that $P < 0$ in $[0,T]\times[T-\epsilon,T]\times[0,\infty)$ and in $[T-\epsilon,T]\times[0,T]\times[0,\infty)$ for some $\epsilon = \epsilon(\delta,\widetilde{m}) > 0$. Next, for all $\bar{s}\in[\tau_*,T]$ and $r\in[0,\infty)$, we have $P(0,\bar{s},r) \le \widetilde{m} - K_*\tau_* - \frac{2\delta}{T} < 0$ thanks to the choice of K_*. Likewise, $P(\bar{t},0,r) \le -\frac{2\delta}{T}$ for all $\bar{t}\in[t^*,T]$ and $r\in[0,\infty)$. Hence, it remains to rule out the existence of a maximising sequence (t_n, s_n, r_n) with $r_n \to \infty$. To this end, we take advantage of the bound (2.5) (for $n = 2$) to estimate

$$\begin{aligned} P(t,s,r) &\le c_d^{-1} \int_{\mathbb{R}^d \setminus B_r} f_\varepsilon(s,v)\,\mathrm{d}v - \frac{2\delta}{T} \\ &\le \frac{1}{c_d(1+r)} \int_{\mathbb{R}^d \setminus B_r} f_\varepsilon(s,v)(1+|v|)\,\mathrm{d}v - \frac{2\delta}{T} \\ &\le \frac{1}{c_d(1+r)} \|f_{\mathrm{in}}\|_{L^1_2(\mathbb{R}^d)} - \frac{2\delta}{T}. \end{aligned}$$

Observe that the RHS is negative whenever $r \ge R_*$ for a finite radius $R_* = R_*(\|f_{\mathrm{in}}\|_{L^1_2}, T, \delta)$ large enough. Hence, the same is true for $P(t,s,r)$.

Thus, the supremum of P must be attained at some interior point $p^* = (t,s,r) \in Q$. At the point p^* we have the optimality conditions

$$\begin{aligned} \partial_t M_\varepsilon(t,r) - K_* \frac{t-s}{|t-s|} &= \frac{\delta}{(T-t)^2}, \\ -\partial_s M_\varepsilon(s,r) + K_* \frac{t-s}{|t-s|} &= \frac{\delta}{(T-s)^2}, \end{aligned}$$

and hence

$$\partial_t M_\varepsilon(t,r) - \partial_s M_\varepsilon(s,r) = \frac{\delta}{(T-t)^2} + \frac{\delta}{(T-s)^2}.$$

Moreover,

$$\partial_r M_\varepsilon(t,r) = \partial_r M_\varepsilon(s,r)$$

and thus

$$h_\varepsilon(r^{1-d}\partial_r M_\varepsilon(t,r)) - h_\varepsilon(r^{1-d}\partial_r M_\varepsilon(s,r)) = 0. \tag{2.20}$$

Further note that $0 \ge \partial_r^2 P(t,s,r) = \partial_r^2 M_\varepsilon(t,r) - \partial_r^2 M_\varepsilon(s,r)$.

In combination with equation (2.15) we deduce at the point $(t,s,r) = p^*$,

$$\begin{aligned} 0 &= \partial_t M_\varepsilon(t,r) - \partial_s M_\varepsilon(s,r) - (\partial_r^2 M_\varepsilon(t,r) - \partial_r^2 M_\varepsilon(s,r)) \\ &\ge \frac{\delta}{(T-t)^2} + \frac{\delta}{(T-s)^2} > 0, \end{aligned}$$

which is a contradiction. This completes the proof of Proposition 2.6.

Let us remark that, thanks to the smoothness of M_ε, estimate (2.18) may alternatively be proved by directly considering the equation satisfied by $N_\varepsilon := \partial_t M_\varepsilon$, at least if one assumes a slightly stronger decay hypothesis on f_{in}. Indeed, notice that positive constants above $\sup N_\varepsilon(0,\cdot)$ of the problem for N_ε are supersolutions, while negative constants below $\inf N_\varepsilon(0,\cdot)$ are subsolutions. And if $f_\varepsilon \in C([0,T]; L^\infty_\ell(\mathbb{R}^d))$ and $\nabla f_\varepsilon \in C([0,T]; L^\infty_{\ell-1}(\mathbb{R}^d))$ for some $\ell > d$, we may use (2.14) to find that $N_\varepsilon(t,r) \to 0$ as $r \to \infty$, uniformly in $t \in [0,T]$. ∎

The comparison principle underlying the proof of Proposition 2.6 can further be used to deduce monotonicity in ε of $M_\varepsilon(t,r)$.

Proposition 2.7 (Monotonicity of the scheme). *Let the hypotheses of Proposition 2.6 hold. For any* $0<\varepsilon'\le\varepsilon\le\epsilon_*$,

$$M_{\varepsilon'}(t,r)\ge M_\varepsilon(t,r),\quad t,r>0.$$

Proof. To begin with, we recall that $h_\varepsilon\le h_{\varepsilon'}$ whenever $0<\varepsilon'\le\varepsilon$ because of the non-increase of the function $(0,\infty)\ni s\mapsto s^{-\gamma}\eta(s)$.

The remaining reasoning is similar to the proof of Proposition 2.6. By contradiction, one assumes that there exist $t_1,r_1>0$ such that $M_\varepsilon(t_1,r_1)-M_{\varepsilon'}(t_1,r_1)$ is positive. Next, one fixes a finite-time horizon $T\ge t_1+1$ and picks $\delta>0$ small enough such that the function

$$\widetilde{P}(t,r):=M_\varepsilon(t,r)-M_{\varepsilon'}(t,r)-\frac{\delta}{T-t}$$

has a positive supremum on $(0,T)\times(0,\infty)$. At an interior maximum point, one uses elementary calculus as before, where the main difference is that instead of line (2.20), we have now an inequality

$$h_\varepsilon(r^{1-d}\partial_r M_\varepsilon(t,r))-h_{\varepsilon'}(r^{1-d}\partial_r M_{\varepsilon'}(t,r))\le 0.$$

The conclusion is then obtained by conceptually following the proof of Proposition 2.6. ■

The bound in Proposition 2.6 combined with the conservation of mass allows us to infer a uniform pointwise bound of the family $\{f_\varepsilon\}_\varepsilon$ away from the origin. Let us emphasise that, at this stage, we do not aim for optimal blow-up rates as $r\downarrow 0$. Such sharp rates will be derived in Section 3.

Lemma 2.8 (Bound away from origin: isotropic case). *Assume the hypotheses of Proposition 2.6 and let K_* be as in (2.19). Then for all $\varepsilon\in(0,\epsilon_*]$, all $t>0$, and all $r>0$,*

$$g_\varepsilon(t,r)\le 2\max\{K_*,d\widetilde{m}\}r^{-d},$$

where, as before, we let $g_\varepsilon(t,|v|):=f_\varepsilon(t,v)$ for $f_\varepsilon(t,\cdot)$ isotropic.

Proof. The inequality $s\le h_\varepsilon(s)$ and (2.14) imply the bound $g_\varepsilon(t,r)\le r^{-d}K_*-r^{-1}\partial_r g_\varepsilon$. Hence,

$$g_\varepsilon(t,r)\le K_* r^{-d}\quad\text{whenever } \partial_r g_\varepsilon(t,r)\ge 0.$$

Suppose now that $\partial_r g_\varepsilon(t,r)<0$ for some $t,r>0$. If $\partial_r g_\varepsilon(t,\cdot)\le 0$ on $[2^{-\frac{1}{d}}r,r]$, then

$$g_\varepsilon(t,r)\frac{r^d}{2d}=g_\varepsilon(t,r)\int_{2^{-\frac{1}{d}}r}^{r}\rho^{d-1}\,\mathrm{d}\rho\le\int_{2^{-\frac{1}{d}}r}^{r}g_\varepsilon(t,\rho)\rho^{d-1}\,\mathrm{d}\rho\le\widetilde{m},$$

where the second step uses the monotonicity of $g_\varepsilon(t,\cdot)$ on $[2^{-\frac{1}{d}}r, r]$ and the third step follows from mass conservation. Otherwise, there exists $r_0 \in [2^{-\frac{1}{d}}r, r]$ such that $\partial_r g_\varepsilon(t,\rho) < 0$ for all $\rho \in (r_0, r]$ and $\partial_r g_\varepsilon(t, r_0) = 0$. In this case, we estimate

$$g_\varepsilon(t,r) \le g_\varepsilon(t,r_0) \le K_* r_0^{-d} \le 2K_* r^{-d}.$$

In combination, this shows the bound $g_\varepsilon(t,r) \le 2\max\{K_*, d\tilde{m}\} r^{-d}$ for all $r > 0$ and every $t > 0$. ∎

2.2.3. Anisotropic case. For non-isotropic initial data f_{in} satisfying (H2′) and thus in particular $f_{\text{in}} \in L^\infty_\ell(\mathbb{R}^d)$ for some $\ell > 3d + 1$, we consider as in [6] an isotropic envelope $\hat{f}_{\text{in}}(v) \ge f_{\text{in}}(v)$ given by

$$\hat{f}_{\text{in}}(v) = \frac{\|f_{\text{in}}\|_{L^\infty_\ell}}{(1 + |v|^\ell)}.$$

Since $\ell - (2d+1) > d$, the isotropic function $\hat{f}_{\text{in}}$ satisfies (H2′) and thus in particular the hypotheses of Proposition 2.4. Invoking this proposition, we obtain non-negative global-in-time (mild) solutions f_ε and $\hat{f}_\varepsilon$ of ($\text{FP}_{\gamma,\text{reg}}$) emanating from f_{in}, resp. $\hat{f}_{\text{in}}$, where by the comparison property, Proposition 2.4 (vi), $f_\varepsilon \le \hat{f}_\varepsilon$ in $[0,\infty) \times \mathbb{R}^d$. Thus, the uniform bound away from zero in the isotropic case (cf. Lemma 2.8) implies a similar result for anisotropic solutions:

Corollary 2.9 (Bound away from origin: anisotropic case). Assume (H2′); thus in particular $f_{\text{in}} \in L^\infty_\ell(\mathbb{R}^d)$ for some $\ell > 3d + 1$ if f_{in} is non-isotropic. There exists a finite (non-explicit) constant $\hat{K}_*$ only depending on $\|f_{\text{in}}\|_{L^\infty_\ell}$, ℓ and fixed parameters such that for all $t > 0$ and all $v \in \mathbb{R}^d \setminus \{0\}$,

$$f_\varepsilon(t,v) \le 2\max\left\{\hat{K}_*, d\hat{m}\right\}|v|^{-d},$$

where $\hat{m} = c_d^{-1}\|\hat{f}_{\text{in}}\|_{L^1}$.

2.3. Passage to the limit

Proof of Proposition 1.1. For $\varepsilon \in (0, \epsilon_*]$ let f_ε be the global-in-time mild solution of ($\text{FP}_{\gamma,\text{reg}}$) emanating from f_{in} as constructed in Proposition 2.4. In the rest of this proof we abbreviate $U := \mathbb{R}^d \setminus \{0\}$.

We first show assertions (i)–(iii).

Approximation property (iii) and regularity of f. We assert that for every compact set $G \subset\subset (0,\infty) \times U$, we have an ε-uniform bound of the form

$$\|f_\varepsilon\|_{H^{1+\frac{\alpha}{2},2+\alpha}(G)} \le C_G \tag{2.21}$$

for some $\alpha \in (0,1)$, where $H^{1+\frac{\alpha}{2},2+\alpha}(G)$ denotes the parabolic Hölder space with $\frac{\alpha}{2}$-Hölder continuous first-order temporal and α-Hölder continuous second-order spatial derivatives. Inequality (2.21) can be shown using standard results on parabolic regularity [28, 29]. To sketch the main points, we first observe that each f_ε is strictly positive unless $f_{\text{in}} \equiv 0$ (cf. [31, Proposition 52.7]) and smooth in $(0,\infty) \times \mathbb{R}^d$. Moreover, as a consequence of Lemma 2.8, resp. Corollary 2.9, the family $\{f_\varepsilon\}_\varepsilon$ is ε-uniformly bounded in $L^\infty(G)$. Hence, rewriting (FP$_{\gamma,\text{reg}}$) as $\partial_t f_\varepsilon = \Delta f_\varepsilon + h'_\varepsilon(f_\varepsilon) v \cdot \nabla f_\varepsilon + d h_\varepsilon(f_\varepsilon)$, Theorem 11.1 in [28, Chapter III] on linear parabolic equations provides us with an ε-uniform gradient bound $\|\nabla f_\varepsilon\|_{C^0(G)} \le C_G$. For higher-order spatial derivatives, ε-uniform bounds on G are obtained by applying a similar reasoning to the equation satisfied by $\partial_{v_i} f_\varepsilon$ etc., and time regularity follows from the equation itself.

Hence, by the Arzelà–Ascoli theorem, there exists a function $f \in C^{1,2}((0,\infty) \times U)$, $f \ge 0$, such that, upon passing to a subsequence $\varepsilon \downarrow 0$ (not relabelled),

$$f_\varepsilon \to f \text{ in } C^{1,2}(G) \text{ for every } G \subset\subset (0,\infty) \times U, \tag{2.22}$$

and f is a classical solution of (FP$_\gamma$) in $(0,\infty) \times U$.

Combining (2.22) with the moment bound in Lemma 2.3 yields, for all $\rho > 0$,

$$\lim_{\varepsilon \to 0} \|f_\varepsilon(t) - f(t)\|_{L^1(\mathbb{R}^d \setminus B_\rho(0))} = 0 \tag{2.23}$$

locally uniformly in $t \in [0,\infty)$. Moreover, Fatou's lemma implies $\int_{\mathbb{R}^d} f(t) \le m$ for all t.

Let us now show that f is strictly positive for non-trivial initial data f_{in}, i.e. whenever $m > 0$. For this purpose, we pick some $\theta > 0$, define $f_{\text{in}}^\# := \min\{f_{\infty,\theta}, f_{\text{in}}\}$, and let $\{f_\varepsilon^\#\}_{\varepsilon \in (0,\epsilon_\theta]}$ with $\epsilon_\theta := (\|f_{\infty,\theta}\|_{L^\infty(\mathbb{R}^d)})^{-1} > 0$ denote the family of global-in-time mild solutions of (FP$_{\gamma,\text{reg}}$) starting from $f_{\text{in}}^\#$. For $\varepsilon \in (0, \epsilon_\theta]$ the steady state $f_{\infty,\theta}$ is a classical solution of (FP$_{\gamma,\text{reg}}$) with rapid decay as $|v| \to \infty$, and thus in particular a mild solution. Hence, the comparison principle in Lemma 2.2 (vi) implies that

$$f_\varepsilon^\# \le \min\{f_{\infty,\theta}, f_\varepsilon\}, \tag{2.24}$$

showing in particular that $f^\# := f_\varepsilon^\#$ is independent of ε for $\varepsilon \in (0,\epsilon_\theta]$ (cf. the argument in Corollary 2.5). By Lemma 2.2 (ii), $f^\#$ is a non-negative classical solution of (FP$_{\gamma,\text{reg}}$) (and (FP$_\gamma$)) with initial datum $f_{\text{in}}^\# \not\equiv 0$. From a classical strong comparison principle (see e.g. [31, Proposition 52.7]), comparing $f^\#$ with the zero solution, we deduce that $f^\#$ is strictly positive in $(0,\infty) \times \mathbb{R}^d$. Taking the limit $\varepsilon \to 0$ in (2.24) along the subsequence obtained in (2.22) yields $f^\# \le f$, and thus provides us with a locally uniform positive lower bound for f away from zero.

The family of measures $\{\mu^{(\varepsilon)}\}_\varepsilon$, $\mu^{(\varepsilon)} := f_\varepsilon \mathcal{L}_+^{1+d}$, is tight on any finite-time horizon as ensured by Lemma 2.3. Hence, by Prokhorov's theorem, there exists a non-negative Radon measure μ on $[0,\infty) \times \mathbb{R}^d$ such that after possibly passing to another subsequence $\varepsilon \downarrow 0$,

$$\mu^{(\varepsilon)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}_+([0,T] \times \mathbb{R}^d) \tag{2.25}$$

for any $T < \infty$. In fact, due to (2.22), (2.23) and $\int_{\mathbb{R}^d} f_\varepsilon(t) \equiv m$, the passage to a subsequence $\varepsilon \downarrow 0$ would not have been necessary at this point (see also the next paragraph).

Mass-conserving curve and decomposition. By (2.23) the family $\mu_t^{(\varepsilon)} := f_\varepsilon(t)\mathcal{L}^d$ satisfies

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^d}\psi\,\mathrm{d}\mu_t^{(\varepsilon)} = \int_{\mathbb{R}^d}\psi(v)f(t,v)\,\mathrm{d}v \tag{2.26}$$

for all $\psi\in C_b(\mathbb{R}^d)$ with $\operatorname{supp}\psi\subset\mathbb{R}^d\setminus\{0\}$, where the limit is taken along the same sequence $\varepsilon\downarrow0$ as in (2.23). At the same time, the tightness of the family $\{\mu_t^{(\varepsilon)}\}_\varepsilon$ ensures, upon passing to a subsequence $\varepsilon_j\downarrow0$ which may (initially) depend on t, that $\mu_t^{(\varepsilon_j)}\overset{*}{\rightharpoonup}\mu_t$ in $\mathcal{M}_+(\mathbb{R}^d)$ for some $\mu_t\in\mathcal{M}_+(\mathbb{R}^d)$ with $\mu_t(\mathbb{R}^d)=m$. As a consequence of (2.26), we have $\operatorname{supp}(\mu_t-f(t)\mathcal{L}^d)\subset\{0\}$, independent of the chosen subsequence $\varepsilon_j\downarrow0$. Since $\mu_t(\mathbb{R}^d)=m$, this entails that $\mu_t(\{0\})=m-\|f(t)\|_{L^1(\mathbb{R}^d)}:=a(t)$. Hence,

$$\mu_t = a(t)\delta_0 + f(t)\mathcal{L}^d,\quad t\ge0, \tag{2.27}$$

and the convergence

$$\mu_t^{(\varepsilon)}\overset{*}{\rightharpoonup}\mu_t \text{ in } \mathcal{M}_+(\mathbb{R}^d) \tag{2.28}$$

holds for the entire sequence $\varepsilon\downarrow0$ as obtained in (2.22)–(2.25). Now let $\varphi\in C_c([0,\infty)\times\mathbb{R}^d)$. On the one hand, identity (2.25) implies that

$$\lim_{\varepsilon\downarrow0}\int_{[0,\infty)\times\mathbb{R}^d}\varphi\,\mathrm{d}\mu^{(\varepsilon)} = \int_{[0,\infty)\times\mathbb{R}^d}\varphi\,\mathrm{d}\mu.$$

On the other hand, the function $\iota_\varepsilon(t):=\int_{\mathbb{R}^d}\varphi(t,v)\,\mathrm{d}\mu_t^{(\varepsilon)}(v)$ admits the uniform bound $|\iota_\varepsilon(t)|\le m\|\varphi\|_{L^\infty}$ for all $t\ge0$ and converges pointwise to $\int_{\mathbb{R}^d}\varphi(t,v)\,\mathrm{d}\mu_t(v)$ as $\varepsilon\downarrow0$. Hence, using dominated convergence for the RHS (in conjunction with the compact support of φ in time), we may pass to the limit $\varepsilon\downarrow0$ in the identity

$$\int_{[0,\infty)\times\mathbb{R}^d}\varphi\,\mathrm{d}\mu^{(\varepsilon)} = \int_{[0,\infty)}\int_{\mathbb{R}^d}\varphi(t,v)\,\mathrm{d}\mu_t^{(\varepsilon)}(v)\,\mathrm{d}t$$

to deduce the representation $\mathrm{d}\mu=\mathrm{d}\mu_t\,\mathrm{d}t$.

To prove the asserted weak-$*$ continuity of the mapping $[0,\infty)\ni t\mapsto\mu_t\in\mathcal{M}_+(\mathbb{R}^d)$, let us first recall that $\mathcal{M}_+(\mathbb{R}^d)$ endowed with the weak-$*$ topology is metrisable (see e.g. [2, 26]), so that it suffices to show sequential continuity: given $\hat t\ge0$ and a sequence (t_j) satisfying $\lim_{j\to\infty}t_j=\hat t$, we need to prove that $\mu_{t_j}\overset{*}{\rightharpoonup}\mu_{\hat t}$ in $\mathcal{M}_+(\mathbb{R}^d)$. By the portmanteau theorem (see e.g. [26, Theorem 13.16]), it suffices to show for every open subset $O\subset\mathbb{R}^d$ the estimate

$$\int_O\mathrm{d}\mu_{\hat t}\le\liminf_{j\to\infty}\int_O\mathrm{d}\mu_{t_j}. \tag{2.29}$$

If $0\in O$, then $B_\rho(0)\subset O$ for $\rho>0$ small enough, and (2.29) holds with an equality. Indeed, the smoothness of f away from $v=0$ and the moment bound in Lemma 2.3 (which implies an analogous bound for the pointwise limit f of f_ε) ensure that $f(t_j)\to f(\hat t)$ in $L^1(\mathbb{R}^d\setminus O)$, and hence

$$\int_O\mathrm{d}\mu_{\hat t} = m-\int_{\mathbb{R}^d\setminus O}f(\hat t) = \lim_{j\to\infty}\left(m-\int_{\mathbb{R}^d\setminus O}f(t_j)\right) = \lim_{j\to\infty}\int_O\mathrm{d}\mu_{t_j}.$$

If $0 \notin O$, inequality (2.29) is equivalent to $\int_O f(\hat{t}, v)\,dv \le \liminf_{j\to\infty} \int_O f(t_j, v)\,dv$ (by virtue of (2.27)), and this bound is a consequence of Fatou's lemma since $f(t_j) \to f(\hat{t})$ a.e. in $\mathbb{R}^d$. This establishes (2.29).

It remains to prove assertions (iv) and (v).

Unique limit. We now show (iv). In the isotropic case, Proposition 2.7 ensures that the limit $M(t,r) := \lim_{\varepsilon\to 0} M_\varepsilon(t,r) = c_d^{-1} \lim_{\varepsilon\to 0} \mu_t^{(\varepsilon)}(B_r)$ is well defined for all $t, r > 0$. Thus, in this case, the limiting density f in (2.22) and hence μ can be uniquely recovered from M, which is independent of the choice of the sequence $\varepsilon \downarrow 0$. In view of the above compactness properties, this implies assertion (iv).

Lipschitz continuity of point mass. Restricting to isotropic data, we have for $r > 0$,

$$\begin{aligned} c_d M_\varepsilon(t,r) = \mu_t^{(\varepsilon)}(B_r) \to \mu_t(B_r) &\quad \text{as } \varepsilon \to 0, \\ \mu_t(B_r) \to a(t) &\quad \text{as } r \to 0, \end{aligned}$$

where the first line follows from (2.28) and the fact that $\operatorname{supp} \mu_t^{\text{sing}} \subseteq \{0\}$. Thus, the Lipschitz bound (2.18) implies that $|a(t) - a(s)| \le c_d K_* |t - s|$, hence part (v). ∎

3. Universal space profile

Equipped with the uniform control (2.18), we will now combine ODE and bootstrap arguments with localised semi-group estimates to study the regularity and the space profile of the density f near the origin. A rigorous analysis is achieved by working with the family of approximate solutions f_ε constructed in Section 2.1. We will show that for isotropic data the solution at any fixed positive time is either regular and smooth, or the density of the regular part follows, up to a lower-order term with explicit rate, a universal profile at the origin that is uniquely determined by the limiting steady state f_c. This even slightly improves the profile obtained in [8] for $d = 1$.

Throughout this section we assume (H2′) and let K_* denote the least upper bound such that inequality (2.18) holds true, i.e.

$$K_* := \sup_{\varepsilon\in(0,\epsilon_*]} \sup_{t>0, r>0} |\partial_t M_\varepsilon(t,r)|. \tag{3.1}$$

By virtue of Section 2.2.1, it is clear that the main conclusion of the present section, Theorem 1.2, only requires hypothesis (H2) and not its stronger version (H2′).

3.1. Lower and upper bounds

The analysis in this subsection mostly concerns isotropic solutions, for which the uniform bound (2.18) is available. As introduced in Section 2.2.2, in the isotropic case we write $g_\varepsilon(t,r) := f_\varepsilon(t,v)$ whenever $r = |v| > 0$, and likewise $g(t,r) := f(t,v)$ for the pointwise limit obtained upon sending $\varepsilon \downarrow 0$.

Proposition 3.1 (Lower bound). *Abbreviate* $\alpha_c = \frac{2}{\gamma}$. *In addition to* (H1), (H2′) *suppose that*

$$\alpha_c + 2 - d > 0.$$

Further assume that the initial value f_{in} *is isotropic. Pick any* $\underline{\alpha} \in ((d-2)_+, \alpha_c)$. *If* $d = 1$, *assume in addition that*[4] $\underline{\alpha} > \frac{1}{\gamma-1}$. *For* $\alpha \in [\underline{\alpha}, \alpha_c]$ *let* $\tilde{g}(r) = c_\gamma r^{-\alpha}$, *where* $c_\gamma = (2/\gamma)^{1/\gamma}$. *For* $t > 0$ *and* $\varepsilon \in (0, \epsilon_*]$ *define*

$$\tilde{r}_\varepsilon = \tilde{r}_\varepsilon(t) = \sup\{r > 0 \colon g_\varepsilon(t, \rho) < \tilde{g}(\rho) \text{ for all } \rho \in (0, r)\}.$$

There exists a constant $B < \infty$ *and a radius* $r_* \in (0, 1]$ *only depending on* K_* *(cf.* (3.1)*) and on* $\gamma, d, \underline{\alpha}$ *(but not on* α*) such that for all* $t > 0$ *and all* $\varepsilon \in (0, \epsilon_*]$ *the following holds: whenever* $\tilde{r}_\varepsilon(t) \in (0, r_*)$, *then*

$$g_\varepsilon(t, r) \geq \tilde{g}(r) - B r^{2-d} \quad \text{for } r \in (\tilde{r}_\varepsilon(t), r_*).$$

Remark 3.2. Let us note that for $\alpha = \alpha_c$ it is (a priori) not clear whether the unboundedness of the limiting function f at some time t, i.e. $\|f(t)\|_{L^\infty(\mathbb{R}^d)} = \infty$, implies that $\liminf_{\varepsilon \downarrow 0} \tilde{r}_\varepsilon(t) = 0$. This is the main reason why the derivation of the universal lower bound on the spatial singularity profile in Theorem 1.2 requires several further steps (cf. Sections 3.2 and 3.3). To rule out fine spike-like singularities in $f(t)$ near the origin that are dominated by a subcritical power law, i.e. by $C r^{-\alpha}$ for some $\alpha < \alpha_c$, we exploit the fact that such subcritical singularities are smoothed out instantaneously (cf. Proposition 3.5) and so cannot form at a positive time. To deal with potential intermediate situations (e.g. oscillatory power laws), it is crucial that the stability result in Proposition 3.1 is valid not only for $\alpha = \alpha_c$ but also for a small range of subcritical exponents α near α_c; see Section 3.3 for details.

Proof of Proposition 3.1. To begin with, we note that for any $\alpha \in [\underline{\alpha}, \alpha_c]$,

$$-\alpha \leq -\underline{\alpha} < 2 - d \leq 4 - d - \alpha\gamma.$$

Now let $t > 0$ and $\varepsilon \in (0, \epsilon_*]$. Observe that the radius $\tilde{r}_\varepsilon(t)$ may be infinite and that the assertion of Proposition 3.1 only concerns the case where $\tilde{r}_\varepsilon(t) > 0$ is small. (If $\tilde{r}_\varepsilon(t) \geq 1$ for all $\varepsilon \in (0, \epsilon_*]$ and $t > 0$, the assertion is trivially satisfied for $r_* = 1$.) Hence, in the following we may assume that $\tilde{r}_\varepsilon(t) \in (0, 1)$. Then, by continuity, $g_\varepsilon(t, \tilde{r}_\varepsilon(t)) = \tilde{g}(\tilde{r}_\varepsilon(t))$, and we may define a radius $\tilde{r}_{1,\varepsilon} > \tilde{r}_\varepsilon$ via

$$\tilde{r}_{1,\varepsilon}(t) := \sup\{r \in (\tilde{r}_\varepsilon(t), 1) \colon g_\varepsilon(t, \rho) \geq \tfrac{1}{2}\tilde{g}(\rho) \text{ for all } \rho \in (\tilde{r}_\varepsilon(t), r)\}.$$

To proceed, we abbreviate $b_\varepsilon(t, r) := \partial_t M_\varepsilon(t, r)$ and note that (cf. (2.14))

$$r^{d-1}\partial_r g_\varepsilon + r^d h_\varepsilon(g_\varepsilon) = b_\varepsilon.$$

[4] In dimension $d = 1$ condition (H1) reduces to $\gamma > 2$, which implies that $\alpha_c > \frac{1}{\gamma-1}$.

In the rest of the proof we are concerned with suitably estimating an integrated version of this differential equation. The following calculations being of a purely spatial type, we henceforth omit the fixed time argument t. Recall that Φ' is a primitive of $\frac{1}{h}$, i.e. $\Phi'' = \frac{1}{h}$, whence

$$\frac{\mathrm{d}}{\mathrm{d}r}\Phi'(g_\varepsilon) = \frac{\partial_r g_\varepsilon}{h(g_\varepsilon)} = -r\frac{h_\varepsilon(g_\varepsilon)}{h(g_\varepsilon)} + b_\varepsilon r^{1-d}\frac{1}{h(g_\varepsilon)} \geq -r - K_* r^{1-d} g_\varepsilon^{-(\gamma+1)},$$

where we used the fact that $\frac{h_\varepsilon(s)}{h(s)} \leq 1$ and $h(s) \geq s^{\gamma+1}$ for all $s > 0$. Renaming r by ρ and integrating the above inequality in space over $\rho \in (\tilde{r}_\varepsilon, r)$ for $r \in (\tilde{r}_\varepsilon, 1]$ yields

$$\Phi'(g_\varepsilon(r)) - \Phi'(g_\varepsilon(\tilde{r}_\varepsilon)) \geq -\frac{1}{2}r^2 + \frac{1}{2}\tilde{r}_\varepsilon^2 - K_* \int_{\tilde{r}_\varepsilon}^{r} \rho^{1-d} g_\varepsilon(\rho)^{-(\gamma+1)}\,\mathrm{d}\rho. \tag{3.2}$$

We next expand for $s \gg 1$,

$$\Phi'(s) = -\frac{1}{\gamma}\log(s^{-\gamma}+1) = -\frac{1}{\gamma}s^{-\gamma} + O(s^{-2\gamma}) = -\frac{1}{\gamma}s^{-\gamma}(1 + O(s^{-\gamma})). \tag{3.3}$$

Note that the increasing function $\Phi'\colon (0,\infty) \to (-\infty, 0)$ is bijective and for $-1 \ll \hat{s} < 0$,

$$(\Phi')^{-1}(\hat{s}) = (\exp(-\gamma\hat{s}) - 1)^{-\frac{1}{\gamma}} = (-\gamma\hat{s})^{-\frac{1}{\gamma}}(1 + O(\hat{s})). \tag{3.4}$$

Furthermore, we assert that there exists $r_\circ = r_\circ(\underline{\alpha}, \gamma) \in (0, \mathrm{e}^{-1}]$ such that

$$r^2 - \rho^2 \leq r^{\alpha\gamma} - \rho^{\alpha\gamma} \quad \text{for all } 0 < \rho < r \leq r_\circ. \tag{3.5}$$

Inequality (3.5) can be shown as follows: since $\beta := \frac{\alpha\gamma}{2} \leq 1$, concavity yields

$$\beta r^{\beta-1}(r-\rho) \leq r^\beta - \rho^\beta \quad \text{for all } 0 < \rho < r < 1.$$

Upon multiplication by $r + \rho \leq r^\beta + \rho^\beta$, we deduce $\beta r^{\beta-1}(r^2 - \rho^2) \leq r^{2\beta} - \rho^{2\beta}$. This implies (3.5), since $\beta r^{\beta-1} \geq 1$ for all $\beta \in [\frac{\alpha\gamma}{2}, 1]$ and $r \in (0, r_\circ]$ if $r_\circ > 0$ is small enough as above. (To see the latter, note that $\beta r^{\beta-1} \geq 1$ is equivalent to $r \leq \beta^{\frac{1}{1-\beta}}$, where $\beta^{\frac{1}{1-\beta}} \uparrow \mathrm{e}^{-1}$ as $\beta \uparrow 1$.)

Letting $\rho = \tilde{r}_\varepsilon$ in (3.5) we infer from (3.2), also using (3.3) and the identity $g_\varepsilon(\tilde{r}_\varepsilon) = c_\gamma \tilde{r}_\varepsilon^{-\alpha}$,

$$\Phi'(g_\varepsilon(r)) \geq -\frac{1}{2}r^{\alpha\gamma} + O(r^{2\alpha\gamma}) - K_* \int_{\tilde{r}_\varepsilon}^{r} \rho^{1-d} g_\varepsilon(\rho)^{-(\gamma+1)}\,\mathrm{d}\rho$$

whenever $\tilde{r}_\varepsilon < r < r_\circ$. For $\rho \in (\tilde{r}_\varepsilon, \tilde{r}_{1,\varepsilon})$ we have $g_\varepsilon(\rho)^{-(\gamma+1)} \leq 2^{\gamma+1}\tilde{g}(\rho)^{-(\gamma+1)} =: C_1(\gamma)\rho^{\alpha\gamma+\alpha}$. We will now show that there exists $r_* \in (0, r_\circ]$ only depending on K_*, $\underline{\alpha}$ and fixed parameters such that $\tilde{r}_{1,\varepsilon}(t) \geq r_*$ for all $t > 0$ and $\varepsilon \in (0, \epsilon_*]$ for which $\tilde{r}_\varepsilon(t) \in (0,1)$. To this end, we let $\tilde{r}_{2,\varepsilon} := \min\{\tilde{r}_{1,\varepsilon}, r_*\}$ for some $r_* \in (0, r_\circ]$ to be fixed later. For $r \in (\tilde{r}_\varepsilon, \tilde{r}_{2,\varepsilon}]$ we have

$$\Phi'(g_\varepsilon(r)) \geq -\frac{1}{2}r^{\alpha\gamma}(1 + K_* C_1(\gamma) r^{2-d+\alpha} + O(r^{\alpha\gamma})).$$

The last two terms in the brackets on the RHS behave like $O(r^{2-d+\alpha})$ for $0 < r \ll 1$, because $2-d+\alpha < \alpha\gamma$ for all $\alpha \in [\underline{\alpha}, \alpha_c]$ (if $d = 1$ it follows from the choice $\underline{\alpha} > \frac{1}{\gamma-1}$; if $d \geq 2$ this follows from the condition $\gamma > \frac{2}{d}$ in (H1), which implies that $\alpha(\gamma-1) > \alpha(\frac{2}{d}-1) = \frac{\alpha}{d}(2-d) \geq (2-d)$ since $2-d \leq 0$ and $\alpha \leq \frac{2}{\gamma} < d$), where the hidden constants in $O(\cdot)$ only depend on K_* and fixed parameters. Hence, using the fact that Φ' is increasing and recalling the expansion (3.4), we infer for $r \in (\tilde{r}_\varepsilon, \tilde{r}_{2,\varepsilon}]$,

$$\begin{aligned} g_\varepsilon(r) &\geq (\Phi')^{-1}\big(-\tfrac{1}{2}r^{\alpha\gamma}(1+O(r^{2-d+\alpha}))\big) \\ &= c_\gamma r^{-\alpha}(1+O(r^{2-d+\alpha})) = c_\gamma r^{-\alpha} + O(r^{2-d}). \end{aligned} \tag{3.6}$$

Since $2-d+\underline{\alpha} > 0$, this shows that after possibly decreasing $r_* \in (0, r_\circ]$ (only depending on K_*, $\underline{\alpha}$, and fixed parameters) we can ensure that $g_\varepsilon(t,r) \geq \frac{3}{4}\tilde{g}(r)$ for all $r \in (\tilde{r}_\varepsilon, \tilde{r}_{2,\varepsilon}]$, $t > 0$, and $\varepsilon \in (0, \epsilon_*]$. As a consequence, $\tilde{r}_{1,\varepsilon}(t) > \tilde{r}_{2,\varepsilon}(t)$ and $\tilde{r}_{2,\varepsilon}(t) = r_*$. This, in turn, means that inequality (3.6) is valid for all $r \in (\tilde{r}_\varepsilon, r_*]$ whenever $\tilde{r}_\varepsilon(t) < 1$, completing the proof of Proposition 3.1. ∎

Proposition 3.3 (Upper bound). *Use the notation and assume the hypotheses of Proposition 3.1. There exists a finite constant B and a radius r_* only depending on K_*, γ, d such that for all $t > 0$ and all $r \in (0, r_*)$,*

$$g(t,r) \leq g_c(r) + Br^{2-d}, \tag{3.7}$$

where $g_c(r) = (\Phi')^{-1}(-\frac{1}{2}r^2)$.

Note that, in contrast to the lower bound in Proposition 3.1, the upper bound (3.7) is formulated only for the limiting function g obtained after sending $\varepsilon \downarrow 0$.

Proof of Proposition 3.3. We adopt the notation of Proposition 3.1 and its proof, where here it will suffice to consider the choice $\alpha = \frac{2}{\gamma}$. Thus, we let $\tilde{g}(r) = c_\gamma r^{-\frac{2}{\gamma}}$ and set

$$r_\varepsilon = r_\varepsilon(t) = \sup\big\{r > 0\colon g_\varepsilon(t,\rho) < \tilde{g}(\rho) \text{ for all } \rho \in (0,r)\big\}.$$

Let r_* be the radius obtained in Proposition 3.1. For $r \in (0, r_\varepsilon(t))$ we trivially have $g_\varepsilon(t,r) \leq \tilde{g}(r) = (\frac{\gamma}{2}r^2)^{-\frac{1}{\gamma}}$, or equivalently (cf. (3.3))

$$\Phi'(g_\varepsilon(r)) \leq \Phi'(\tilde{g}(r)) = -\frac{1}{2}r^2(1+O(r^2)), \quad r \in (0, r_\varepsilon(t)). \tag{3.8}$$

The main step in the proof of the upper bound (3.7) is to establish a bound similar to (3.8) on the interval $r \in [r_\varepsilon(t), r_*)$ in the case where $r_\varepsilon(t) < r_*$. Of course, due to the possible formation of a point mass at the origin, such a bound can in general only be expected to hold true up to some error term that tends to zero as $\varepsilon \downarrow 0$.

If $r_\varepsilon(t) < r_*$, we note that as in the proof of Proposition 3.1 we have the formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\Phi'(g_\varepsilon) = \frac{\partial_r g_\varepsilon}{h(g_\varepsilon)} = -r\frac{h_\varepsilon(g_\varepsilon)}{h(g_\varepsilon)} + b_\varepsilon r^{1-d}\frac{1}{h(g_\varepsilon)}.$$

Hence, for all $r \in [r_\varepsilon, r_*)$,

$$\Phi'(g_\varepsilon(r)) = \Phi'(\tilde{g}(r_\varepsilon)) - \int_{(r_\varepsilon,r)} \rho \frac{h_\varepsilon(g_\varepsilon)}{h(g_\varepsilon)} \,\mathrm{d}\rho + \int_{(r_\varepsilon,r)} b_\varepsilon \rho^{1-d} \frac{1}{h(g_\varepsilon)} \,\mathrm{d}\rho,$$

where we omitted the (fixed) time argument t. To proceed, we define the set

$$J_\varepsilon = J_\varepsilon(t) = \{\rho \in [r_\varepsilon(t), r_*) : g_\varepsilon(t,\rho) \geq \varepsilon^{-1}\}.$$

On $[r_\varepsilon, r_*) \setminus J_\varepsilon$ we have $\frac{h_\varepsilon(g_\varepsilon)}{h(g_\varepsilon)} \equiv 1$, while on J_ε we only know that $0 \leq \frac{h_\varepsilon(g_\varepsilon)}{h(g_\varepsilon)} \leq 1$. Hence, we may estimate for $r \in [r_\varepsilon, r_*)$, also using the bound $g_\varepsilon(\rho) \gtrsim \rho^{-\frac{2}{\gamma}}$ for $\rho \in (r_\varepsilon, r_*)$ from Proposition 3.1,

$$\begin{aligned}
\Phi'(g_\varepsilon(r)) &\leq \Phi'(c_\gamma r_\varepsilon^{-\frac{2}{\gamma}}) - \int_{(r_\varepsilon,r)\setminus J_\varepsilon} \rho \,\mathrm{d}\rho + CK_* r^{2-d+(\gamma+1)\frac{2}{\gamma}} \\
&\leq -\frac{1}{2} r_\varepsilon^2 + O(r^4) - \frac{1}{2}(r^2 - r_\varepsilon^2) + r_* \mathcal{L}^1(J_\varepsilon) + CK_* r^{4-d+\frac{2}{\gamma}} \\
&\leq -\frac{1}{2} r^2 + C(K_*) r^{4-d+\frac{2}{\gamma}} + r_* \mathcal{L}^1(J_\varepsilon).
\end{aligned} \tag{3.9}$$

Here, we further used (3.3) in the second step and $d > \frac{2}{\gamma}$ in the third step.

Combining (3.9) with (3.8), we deduce (independently of whether $r_\varepsilon < r_*$ or $r_\varepsilon \geq r_*$)

$$\Phi'(g_\varepsilon(r)) \leq -\frac{1}{2} r^2 (1 + O(r^{2-d+\frac{2}{\gamma}})) + r_* \mathcal{L}^1(J_\varepsilon) \quad \text{for all } r \in (0, r_*),$$

where $O = O(\cdot)$ only depends on K_* and γ. Mass conservation, i.e. $\int_{\mathbb{R}^d} f_\varepsilon(t) \equiv \int_{\mathbb{R}^d} f_{\text{in}}$, implies that $\lim_{\varepsilon \to 0} \mathcal{L}^1(J_\varepsilon(t)) = 0$. Thus, sending $\varepsilon \to 0$, we infer the bound $\Phi'(g(r)) \leq -\frac{1}{2} r^2 (1 + O(r^{2-d+\frac{2}{\gamma}}))$ for all $r \in (0, r_*)$. Finally, we invoke (3.4) and arrive at

$$g(r) \leq c_\gamma r^{-\frac{2}{\gamma}} (1 + O(r^{2-d+\frac{2}{\gamma}}))(1 + O(r^2)) = g_c(r) + O(r^{2-d}), \quad r \in (0, r_*). \qquad \blacksquare$$

For anisotropic data, the approximate solutions $\{f_\varepsilon\}$ are dominated by an isotropic scheme $\{\hat{f}_\varepsilon\}$ (cf. Section 2.2.3). Hence, the density $f(t,v)$ of the regular part of the limiting measure in Proposition 1.1 inherits the upper bound obtained above for the isotropic case.

Corollary 3.4 (Upper bound on space profile: anisotropic case). In addition to (H1), (H2) suppose that $\frac{2}{\gamma} + 2 - d > 0$. There exists a finite constant $\widehat{B}$ and a radius $\hat{r}_*$ only depending on f_{in} (non-explicitly) and on γ, d such that for all $t > 0$ and all v with $|v| \in (0, \hat{r}_*)$,

$$f(t,v) \leq f_c(v) + \widehat{B} |v|^{2-d}.$$

In particular, the point mass at the origin $t \mapsto \mu_t(\{0\}) = m - \int f(t,\cdot)$ is continuous (as a consequence of Lebesgue's dominated convergence theorem).

3.2. Instantaneous regularisation

For the nonlinear problem (FP$_\gamma$) the Lebesgue space $L^{p_c}(\mathbb{R}^d)$, $p_c := \frac{\gamma d}{2}$ is critical (as regards high values of the density). Thus, for $p > p_c$ one would expect equation (FP$_\gamma$) to enjoy a smoothing property in L^p. The following result formalises these heuristics.

Proposition 3.5 (Smoothing out subcritical singularities). *Let $\{f_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a family of (suitably regular) non-negative mild solutions of the ε-regularised problems* (FP$_{\gamma,\mathrm{reg}}$)[5] *with uniformly controlled mass $\|f_\varepsilon(t)\|_{L^1} \le m$. Let $p > p_c := \frac{\gamma d}{2}$, let $t_0 \ge 0$, and suppose the following conditions:*

- (C1) *There exists $L < \infty$ such that $\|f_\varepsilon(t_0,\cdot)\|_{L^p(\mathbb{R}^d)} \le L$ for all $\varepsilon \in (0,\varepsilon_0]$.*
- (C2) *There exists $t_1 \in (t_0, t_0+1]$, a constant $L' < \infty$, and a radius $r_0 \in (0,1]$ such that $f_\varepsilon(t,v) \le L'|v|^{-\frac{2}{\gamma}}$ for all $t \in [t_0,t_1]$, all $v \in \mathbb{R}^d$ with $|v| \le r_0$, and all $\varepsilon \in (0,\varepsilon_0]$.*
- (C3) *For all $\tilde r_0 > 0$ there exists $L'' = L''(\tilde r_0) < \infty$ such that $f_\varepsilon \le L''$ in $[t_0,\infty) \times \{v\colon |v| \ge \tilde r_0\}$ for all $\varepsilon \in (0,\varepsilon_0]$.*

Then there exists $T = T(L, L', L''(r_0), p, d, \gamma, m) \in (0,1]$ such that for $\widehat T := \min\{T, t_1 - t_0\}$ and for all $\tau \in (0,\widehat T]$,

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \sup_{t\in[t_0+\tau,t_0+\widehat T]} \|f_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} < \infty. \tag{3.10}$$

Proof. We proceed in two steps. In a first step, we derive smoothing estimates based on the mild formulation (2.2) satisfied by f_ε, where, as in Section 2, the nonlinear term is to be rewritten analogously to (2.3).

Step 1: Localised smoothing estimate. Fix some sufficiently small $\epsilon_1 > 0$ such that

$$\frac{p_c}{p} \le 1 - 2\epsilon_1.$$

Let $\tilde p, \tilde q \in [p,\infty]$ with $\tilde p \le \tilde q$. Then

$$a := \frac{d\gamma}{4}\frac{1}{\tilde q} + \frac{1}{2} = \frac{1}{2}\Big(\frac{p_c}{\tilde q} + 1\Big) \le 1 - \epsilon_1. \tag{3.11}$$

Defining $b := \frac{d}{2}(\frac{1}{\tilde p} - \frac{1}{\tilde q})(\frac{\gamma}{2} + 1)$ we further have

$$a + b - \frac{d}{2}\Big(\frac{1}{\tilde p} - \frac{1}{\tilde q}\Big) = \frac{d}{2}\Big(\frac{1}{\tilde p} - \frac{1}{\tilde q}\Big)\frac{\gamma}{2} + \frac{d\gamma}{4}\frac{1}{\tilde q} + \frac{1}{2} = \frac{1}{2}\Big(\frac{p_c}{\tilde p} + 1\Big) \le 1 - \epsilon_1. \tag{3.12}$$

We assert that if $\tilde p, \tilde q$ are sufficiently close in the sense that

$$b = \frac{d}{2}\Big(\frac{1}{\tilde p} - \frac{1}{\tilde q}\Big)\Big(\frac{\gamma}{2} + 1\Big) \le 1 - \hat\epsilon_1 \tag{3.13}$$

[5] The family $\{f_\varepsilon\}$ does not have to take the same initial data.

for some $\hat{\epsilon}_1 > 0$, there exists an (explicit) strictly increasing function $\kappa \in C([0,1])$ only depending on $\epsilon_1, \hat{\epsilon}_1, d$ and on L' and $L'' := L''(r_0)$ with $\kappa(0) = 0$, and a finite constant $C_1 = C_1(d)$ such that for all $t \in [0, t_1 - t_0]$,

$$\|\chi_{\{|v|\le r_0\}} f_\varepsilon^{(t_0)}\|_{Z_t} \le C_1 \|f_\varepsilon(t_0,\cdot)\|_{L^{\tilde{p}}(\mathbb{R}^d)} + \kappa(t)\|f_\varepsilon^{(t_0)}\|_{Z_t}(\|f_\varepsilon^{(t_0)}\|_{Z_t}^{\frac{\gamma}{2}} + 1), \tag{3.14}$$

where $f_\varepsilon^{(t_0)}(\tau,\cdot) := f_\varepsilon(t_0+\tau,\cdot)$ and

$$\|\tilde{f}\|_{Z_t} := \|\tilde{f}\|_{Z_t^{(\tilde{p},\tilde{q})}} := \sup_{s\in[0,t]} \nu(s)^{\frac{d}{2}(\frac{1}{\tilde{p}}-\frac{1}{\tilde{q}})}\|\tilde{f}(s,\cdot)\|_{L^{\tilde{q}}(\mathbb{R}^d)}.$$

Proof of Step 1. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ with $0 \le \zeta \le 1$, $\zeta = 1$ on $\{|v| \le r_0\}$, $\operatorname{supp}\zeta \subset B_{2r_0}(0)$.

By the mild solution property of f_ε, we have (cf. Section 2.1)

$$\begin{aligned} f_\varepsilon^{(t_0)}(\tau, v) = &\int_{\mathbb{R}^d} \mathcal{F}(\tau, v, w) f_\varepsilon(t_0, w)\,\mathrm{d}w \\ &+ \int_0^\tau \mathrm{e}^{-(\tau-s)} \int_{\mathbb{R}^d} \nabla_v \mathcal{F}(\tau-s, v, w)\cdot w\vartheta_\varepsilon(f_\varepsilon^{(t_0)}(s,w))\,\mathrm{d}w\,\mathrm{d}s. \end{aligned} \tag{3.15}$$

Using the bound $|\vartheta_\varepsilon(g)| \le |g|^{\gamma+1}$ (cf. definition (1.4)), the fact that[6] $|w| f_\varepsilon^{\frac{\gamma}{2}}(s,w) \le C(L',\gamma)$ for $|w| \le r_0$ and $|f_\varepsilon(s,w)| \le L''$ for $|w| \ge r_0$ for all $s \in [t_0, t_1]$ (cf. (C2) and (C3)), we now estimate for $0 \le s < \tau \le t \le t_1 - t_0$,

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} \nabla_v \mathcal{F}(\tau-s, v, w)\cdot(w\vartheta_\varepsilon(f_\varepsilon^{(t_0)}(s,w)))\,\mathrm{d}w\right| \\ &\quad\le C(L')\int_{\{|w|\le r_0\}} |\nabla_v \mathcal{F}(\tau-s,v,w)|(f_\varepsilon^{(t_0)})^{\frac{\gamma}{2}+1}(s,w)\,\mathrm{d}w \\ &\qquad + C(L'')\mathrm{e}^{\tau-s}\int_{\{|w|>r_0\}} |\nabla_v \mathcal{F}(\tau-s,v,w)|(|v| + |\mathrm{e}^{-(\tau-s)}w - v|) f_\varepsilon^{(t_0)}(s,w)\,\mathrm{d}w. \end{aligned}$$

The integrals on the RHS will be handled similarly to the proof of [10, Proposition A.1]. To estimate the $L^{\tilde{q}}(\mathbb{R}^d)$-norm, Young's convolution inequality is employed. For the first term on the RHS we invoke inequality (2.4) and estimate

$$\begin{aligned} &\left\|\int_{\mathbb{R}^d} |\nabla_v \mathcal{F}(\tau-s,v,w)|(f_\varepsilon^{(t_0)})^{\frac{\gamma}{2}+1}(s,w)\,\mathrm{d}w\right\|_{L^{\tilde{q}}(\mathbb{R}^d)} \\ &\quad\le C\nu(\tau-s)^{-\frac{1}{2}-\frac{d}{2}\left(\frac{\frac{\gamma}{2}+1}{\tilde{q}}-\frac{1}{\tilde{q}}\right)}\|(f_\varepsilon^{(t_0)})^{\frac{\gamma}{2}+1}(s)\|_{L^{\tilde{q}/(\frac{\gamma}{2}+1)}(\mathbb{R}^d)} \\ &\quad\le C\nu(\tau-s)^{-\frac{1}{2}-\frac{d\gamma}{4\tilde{q}}}\nu(s)^{-\frac{d}{2}(\frac{1}{\tilde{p}}-\frac{1}{\tilde{q}})(\frac{\gamma}{2}+1)}\|f_\varepsilon^{(t_0)}\|_{Z_t}^{\frac{\gamma}{2}+1} \\ &\quad= C\nu(\tau-s)^{-a}\nu(s)^{-b}\|f_\varepsilon^{(t_0)}\|_{Z_t}^{\frac{\gamma}{2}+1}. \end{aligned}$$

[6] Any dependence on the fixed parameter γ will henceforth not be explicitly indicated.

Here and below, C denotes a positive constant that only depends on fixed parameters, but which may change from line to line.

We next estimate

$$\begin{aligned}&\left\|\zeta(v)\int_{\{|w|>r_0\}}|\nabla_v\mathcal{F}(\tau-s,v,w)|\,|v|\,f_\varepsilon^{(t_0)}(s,w)\,\mathrm{d}w\right\|_{L^{\tilde q}(\mathbb{R}^d)}\\&\quad\le C\left\|\int_{\mathbb{R}^d}|\nabla_v\mathcal{F}(\tau-s,v,w)|\,f_\varepsilon^{(t_0)}(s,w)\,\mathrm{d}w\right\|_{L^{\tilde q}(\mathbb{R}^d)}\\&\quad\le C\nu(\tau-s)^{-\frac12}\nu(s)^{-\frac d2(\frac1{\tilde p}-\frac1{\tilde q})}\|f_\varepsilon^{(t_0)}\|_{Z_t},\end{aligned}$$

where the second step follows from (2.4).

Finally, the rapid decay of the Fokker–Planck kernel allows us to further estimate

$$\begin{aligned}&\left\|\int_{\mathbb{R}^d}|\nabla_v\mathcal{F}(\tau-s,v,w)|\,|\mathrm{e}^{-(\tau-s)}w-v|\,f_\varepsilon^{(t_0)}(s,w)\,\mathrm{d}w\right\|_{L^{\tilde q}(\mathbb{R}^d)}\\&\quad\le C\nu(\tau-s)^{-\frac12}\nu(s)^{-\frac d2(\frac1{\tilde p}-\frac1{\tilde q})}\|f_\varepsilon^{(t_0)}\|_{Z_t};\end{aligned}$$

see Lemma A.1 for details.

Inserting the above estimates into (3.15), we infer for $\tau\le t\in[0,t_1-t_0]\subseteq[0,1]$,

$$\begin{aligned}&\nu(\tau)^{\frac d2(\frac1{\tilde p}-\frac1{\tilde q})}\|f_\varepsilon^{(t_0)}(\tau)\zeta\|_{L^{\tilde q}(\mathbb{R}^d)}\\&\quad\le C_1\|f_\varepsilon(t_0)\|_{L^{\tilde p}(\mathbb{R}^d)}+C(L',L'')\nu(\tau)^{\frac d2(\frac1{\tilde p}-\frac1{\tilde q})}\int_0^\tau\nu(\tau-s)^{-a}\nu(s)^{-b}\,\mathrm{d}s\\&\qquad\qquad\times\|f_\varepsilon^{(t_0)}\|_{Z_t}(\|f_\varepsilon^{(t_0)}\|_{Z_t}^{\frac\gamma2}+1),\end{aligned}$$

where once more we used inequality (2.4) as well as the fact that $a\ge\frac12$ and $b\ge\frac d2(\frac1{\tilde p}-\frac1{\tilde q})$. To proceed we estimate for $t\in[0,1]$, using the bound $s\le2s\le\nu(s)\le2\mathrm{e}^2s$ for all $s\in[0,1]$ and a change of variables,

$$\begin{aligned}&\sup_{\tau\in[0,t]}C\nu(\tau)^{\frac d2(\frac1{\tilde p}-\frac1{\tilde q})}\int_0^\tau\nu(\tau-s)^{-a}\nu(s)^{-b}\,\mathrm{d}s\\&\quad\le C\sup_{\tau\in[0,t]}\tau^{\frac d2(\frac1{\tilde p}-\frac1{\tilde q})+1-a-b}\int_0^1(1-\tilde s)^{-(1-\epsilon_1)}\tilde s^{-(1-\hat\epsilon_1)}\,\mathrm{d}\tilde s\\&\quad\le Ct^{\epsilon_1}\int_0^1(1-\tilde s)^{-(1-\epsilon_1)}\tilde s^{-(1-\hat\epsilon_1)}\,\mathrm{d}\tilde s=:\kappa(t),\end{aligned}$$

where we abbreviated $C=C(L',L'')\in(0,\infty)$. In the first line we used inequality (3.11) and hypothesis (3.13); in the second line we used (3.12).

Hence, for all $t\in[0,t_1-t_0]$,

$$\|\zeta f_\varepsilon^{(t_0)}\|_{Z_t}\le C_1\|f_\varepsilon(t_0)\|_{L^{\tilde p}(\mathbb{R}^d)}+\kappa(t)\|f_\varepsilon^{(t_0)}\|_{Z_t}(\|f_\varepsilon^{(t_0)}\|_{Z_t}^{\frac\gamma2}+1),$$

which proves the assertion of Step 1. ∎

Step 2. We are now ready to *complete the proof of Proposition* 3.5 using estimate (3.14) and property (C3) with $L'' := L''(r_0)$. The idea is to perform a finite number of iterations in the integrability exponents to eventually upgrade the ε-uniform L^p bound on $f_\varepsilon(t_0)$ to an ε-uniform L^∞ bound on $f_\varepsilon(t_0+\tau)$ for given $\tau > 0$ small.

It is elementary to verify that for any $\tilde p \geq p > p_c$ and for $\tilde q := 2\tilde p$ the tuple $(\tilde p, \tilde q)$ satisfies the hypotheses of Step 1 with parameter ϵ_1 only depending on p, γ, d and with $\hat\epsilon_1 = \frac14$. Indeed, for this choice we have $\frac{d}{2}(\frac{1}{\tilde p} - \frac{1}{\tilde q})(\frac{\gamma}{2}+1) = \frac{p_c}{\tilde p}\frac12(\frac12 + \frac1\gamma) \leq \frac34 = 1-\hat\epsilon_1$, showing (3.13). Hence, by Step 1, there exists a strictly increasing function $\kappa \in C([0,1])$ with $\kappa(0)=0$ only depending on p, γ, d and on L', L'' such that for any $\tilde p \geq p$, for $\tilde q = 2\tilde p$, and all $t \in [0, t_1 - t_0]$,

$$\|\chi_{\{|v|\leq r_0\}} f_\varepsilon^{(t_0)}\|_{Z_t} \leq C_1 \|f_\varepsilon(t_0,\cdot)\|_{L^{\tilde p}(\mathbb{R}^d)} + \kappa(t)\|f_\varepsilon^{(t_0)}\|_{Z_t}(\|f_\varepsilon^{(t_0)}\|_{Z_t}^{\frac{\gamma}{2}} + 1), \tag{3.16}$$

where $f_\varepsilon^{(t_0)}(\tau,\cdot) := f_\varepsilon(t_0+\tau,\cdot)$ and $\|\cdot\|_{Z_t} := \|\cdot\|_{Z_t^{(\tilde p,\tilde q)}}$.

In the following, we abbreviate $F(t) := \|\chi_{\{|v|\leq r_0\}} f_\varepsilon^{(t_0)}\|_{Z_t^{(\tilde p,\tilde q)}}$. Thanks to mass control and (C3), we have for any $t \in [0,1]$ and any $\tilde q \geq \tilde p \geq 1$ the estimate

$$\begin{aligned}\|f_\varepsilon^{(t_0)}\|_{Z_t} &\leq F(t) + \sup_{s\in[0,t]} \nu(s)^{\frac{d}{2}(\frac{1}{\tilde p}-\frac{1}{\tilde q})}\|\chi_{\{|v|>r_0\}} f_\varepsilon^{(t_0)}(s)\|_{L^{\tilde q}(\mathbb{R}^d)} \\ &\leq F(t) + \sup_{s\in[0,t]} \nu(s)^{\frac{d}{2}(\frac{1}{\tilde p}-\frac{1}{\tilde q})}(m + L'') \\ &\leq F(t) + C_\#, \end{aligned} \tag{3.17}$$

where $C_\# = C(d)(m+L'')$.

Inequalities (3.16) and (3.17) show that for $(\tilde p, \tilde q) = (p, 2p)$ the function $F(t)$ obeys a bound of the form

$$F(t) \leq B + \kappa(t)(F(t)+C_\#)\big((F(t)+C_\#)^{\frac{\gamma}{2}}+1\big), \quad t \in [0, t_1 - t_0], \tag{3.18}$$

where B only depends on fixed parameters (here one may choose $B = C_1 L$). Since $\kappa(0) = 0$, there exists, for every $B > 0$, a unique maximal time $T_B \in (0,1]$ such that

$$\sup_{t\in[0,T_B]} \kappa(t) \leq \frac{B}{(2B+C_\#)((2B+C_\#)^{\frac{\gamma}{2}}+1)},$$

i.e.

$$T_B := \kappa^{-1}\Big(\min\Big\{\kappa(1), \frac{B}{(2B+C_\#)((2B+C_\#)^{\gamma/2}+1)}\Big\}\Big),$$

where κ^{-1} denotes the inverse of κ. With this choice, we deduce from (3.18) that $F(t) \leq 2B$ for all $t \in [0, \widehat{T}_B]$, where $\widehat{T}_B := \min\{T_B, t_1 - t_0\}$. In particular, for $B = B_1 := C_1 L$ we infer

$$\|\chi_{\{|v|\leq r_0\}} f_\varepsilon(t_0+t)\|_{L^{2p}} \leq \nu(t)^{-\frac{d}{4p}} 2B_1, \quad t \in [0, \widehat{T}],$$

where $\widehat{T} := \widehat{T}_{B_1}$. Combined with (C3) and mass control this shows that

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \|f_\varepsilon^{(t_0)}\|_{L^\infty([\tau,\widehat{T}];L^{2p}(\mathbb{R}^d))} \le C(\tau) \quad \text{for all } \tau \in (0,\widehat{T}], \tag{3.19}$$

for some non-increasing function $C(\cdot)$, which depends on further fixed parameters. This argument can be iterated to give the asserted bound (3.10) for the same time $\widehat{T}$ $(= \widehat{T}_{B_1})$. Let us provide some details. Fix some $N \in \mathbb{N}_+$ large enough such that $2^N p > d$. For $\tau > 0$ small and $(\tilde{p},\tilde{q}) = (2^i p, 2^{i+1} p)$, $i = 1$, the (time-shifted) function $F(t) := \|\chi_{\{|v|\le r_0\}} f_\varepsilon^{(t_0+\tau)}\|_{Z_t}$ obeys a bound of the form

$$F(t) \le B + \kappa(t)(F(t) + C_\#)\big((F(t)+C_\#)^{\frac{\gamma}{2}} + 1\big), \quad t \in [0, t_1-(t_0+\tau)], \tag{3.20}$$

where $B = B(\tau) < \infty$ is non-increasing in $\tau > 0$. This allows us to infer (for $i = 1$) that

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \|f_\varepsilon^{(t_0)}\|_{L^\infty([\tau,\widehat{T}];L^{2^{i+1}p}(\mathbb{R}^d))} \le C(\tau) \quad \text{for all } \tau \in (0,\widehat{T}], \tag{3.21}$$

for a non-increasing function $C(\cdot)$. Observe that, thanks to the non-increase with respect to τ of the constants $C(\cdot)$ and B appearing in (3.19) and (3.20), the locally uniform bound (3.21) can indeed be achieved on the entire time interval $(0,\widehat{T}]$ (by iteration), so that the final time $\widehat{T}$ does not need to be decreased. Repeating the argument for $i = 2,\dots,N-1$, we deduce a bound of the form $\sup_{\varepsilon\in(0,\varepsilon_0]} \|f_\varepsilon^{(t_0)}\|_{L^\infty([\tau,\widehat{T}];L^{2^N p}(\mathbb{R}^d))} \le C(\tau)$ for all $\tau \in (0,\widehat{T}]$. For $\tau \in (0,\widehat{T})$ and $f_\varepsilon^{(t_0)}$ replaced by $f_\varepsilon^{(t_0+\tau)}$ we may now take $\tilde{p} := 2^N p > \max\{d, p_c\}$ in Step 1, in which case the choice $\tilde{q} = \infty$ is admissible. (Indeed, with this choice we have $b = \frac{d}{2}(\frac{\gamma}{2}\frac{1}{\tilde{p}} + \frac{1}{\tilde{p}}) < \frac{d}{2}(\frac{1}{d} + \frac{1}{d}) = 1$, so that (3.13) is fulfilled.) Arguing similarly to before we infer (3.10). ∎

3.3. Space profile

Finally, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Thanks to the short-time regularity for (FP$_\gamma$), we may assume without loss of generality the strengthened version (H2′) of (H2). We fix some $\underline{\alpha} < \alpha_c := \frac{2}{\gamma}$ as in Proposition 3.1 and let $r_* > 0$ denote the associated radius obtained in Proposition 3.1. Now let $\hat{t} > 0$. We assert that the behaviour of $g(\hat{t},\cdot)$ near zero is determined by whether or not the hypotheses of Case 1 are fulfilled, where Case 1 is determined as follows:

Case 1. There exists $\alpha \in [\underline{\alpha},\alpha_c)$, a time $\hat{t}_0 < \hat{t}$, a radius $r_0 \in (0,r_*)$, and $\varepsilon_0 \in (0,\epsilon_*]$ such that for all $\varepsilon \in (0,\varepsilon_0]$, all $r \in (0,r_0]$, and all $t \in [\hat{t}_0,\hat{t}]$,

$$g_\varepsilon(t,r) \le \tilde{g}^{(\alpha)}(r) := c_\gamma r^{-\alpha}, \quad \text{where } c_\gamma := \left(\tfrac{2}{\gamma}\right)^{\frac{1}{\gamma}}. \tag{3.22}$$

Here, $\{g_\varepsilon\}$ denotes the family of isotropic approximate solutions in radial coordinates.

If Case 1 is fulfilled, Proposition 3.5 implies the existence of a constant $\delta > 0$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \sup_{t\in(\hat{t}-\delta,\hat{t}]} \|f_\varepsilon(t,\cdot)\|_{L^\infty} < \infty. \tag{3.23}$$

Indeed, since $\alpha < \alpha_c$, we can choose $p > p_c$ such that $f^{(\alpha)}(v) := c_\gamma |v|^{-\alpha} \in L^p(B_1)$, where $B_1 := \{v : |v| \le 1\}$. Hence, combining (3.22) with mass conservation and the uniform bound away from the origin (cf. Lemma 2.8), we find that

$$\sup_{\varepsilon \in (0,\varepsilon_0]} \|f_\varepsilon(t,\cdot)\|_{L^p} \le L$$

for all $t \in [\hat{t}_0, \hat{t}]$ and some finite constant L. Property (3.22) further guarantees the bound $\sup_{\varepsilon\in(0,\varepsilon_0]} f_\varepsilon(t,v) \le L'|v|^{-\frac{2}{\gamma}}$ for all $v \in B_{r_0}$, all $t \in [\hat{t}_0, \hat{t}]$, and suitable $L' < \infty$. Finally, Lemma 2.8 ensures that for all $\tilde{r}_0 > 0$ there exists $L''(\tilde{r}_0) < \infty$ such that $\sup_{t>0} \sup_{\varepsilon\in(0,\varepsilon_0]} f_\varepsilon(t,v) \le L''(\tilde{r}_0)$ whenever $|v| \ge \tilde{r}_0$. Hence, for every $t_0 \in [\hat{t}_0, \hat{t}) \cap [\hat{t}-1, \hat{t})$ and for p as above, conditions (C1)–(C3) of Proposition 3.5 are satisfied with $t_1 = \hat{t}$, which implies (3.23) for suitable $\delta > 0$.

It is easy to see that, after possibly decreasing $\delta > 0$, the bound (3.23) even holds with $\sup_t$ being taken over $t \in J_{\hat{t}} := (\hat{t} - \delta, \hat{t} + \delta)$. (To this end, one may adapt the estimates in the proof of Proposition 3.5 and choose $p = \infty$ in (C1). In this case, the proof greatly simplifies, condition (C2) is not needed, and one may deduce an estimate of the form (3.10) even with $\tau = 0$.) Given this uniform bound, we can argue classically as in the proof of Proposition 1.1 (iii) to infer the smoothness of the limiting density f on $J_{\hat{t}} \times \mathbb{R}^d \supset J_{\hat{t}} \times B_1$.

Case 2. It remains to consider the situation where the hypotheses of Case 1 are not satisfied. In this case, we can find sequences

$$\alpha_j \uparrow \alpha_c, \quad t_j \uparrow \hat{t}, \quad r_j \downarrow 0, \quad \varepsilon_j \downarrow 0,$$

with $\underline{\alpha} \le \alpha_j$ and $r_j < r_*$ for all j, in such a way that

$$g_{\varepsilon_j}(t_j, r_j) \ge \tilde{g}^{(\alpha_j)}(r_j) \quad \text{for all } j \in \mathbb{N}.$$

Thus, invoking Proposition 3.1, we infer

$$g_{\varepsilon_j}(t_j, r) \ge \tilde{g}^{(\alpha_j)}(r) - C r^{2-d} \quad \text{for all } r \in (r_j, r_*). \tag{3.24}$$

By construction, $\lim_{j\to\infty} g_{\varepsilon_j}(t_j, r) = g(\hat{t}, r)$ for every $r > 0$. Hence, sending $j \to \infty$ in inequality (3.24) yields

$$g(\hat{t}, r) \ge \tilde{g}^{(\alpha_c)}(r) - C r^{2-d} \quad \text{for all } r \in (0, r_*),$$

which implies that

$$g(\hat{t}, r) \ge g_c(r) - C r^{2-d} \quad \text{for all } r \in (0, r_*).$$

In view of the upper bound in Proposition 3.3 this completes the proof of the main assertion in Theorem 1.2.

Now let $\alpha = \alpha_c$ in Proposition 3.1 and define $\tilde{r}_\varepsilon$ correspondingly. If $\hat{t}$ is such that $\mu_{\hat{t}}(\{0\}) > 0$, we must have $\lim_{\varepsilon\to 0} \tilde{r}_\varepsilon(\hat{t}) = 0$. Proposition 3.1 (combined with Proposition 3.3) thus implies the assertion concerning this case. ∎

4. Renormalised form

4.1. Variational structure

Our subsequent analysis relies on the following gradient-flow structure of the regularised Fokker–Planck equation ($\mathrm{FP}_{\gamma.\mathrm{reg}}$). Such a structure was previously used in [8, Section 3.3] for the proof of an energy dissipation identity. To proceed, let us recall that $\Phi'(s) = -\int_s^\infty \frac{1}{h(\sigma)}\,\mathrm{d}\sigma$ and $\Phi(0) = 0$.

We define the approximate free energy functional by

$$\mathcal{H}_\varepsilon(f) = \int_{\mathbb{R}^d} \Big(\frac{|v|^2}{2} f + \Phi_\varepsilon(f)\Big)\,\mathrm{d}v,$$

where $\Phi_\varepsilon \in C([0,\infty)) \cap C^\infty((0,\infty))$ satisfies

$$\Phi_\varepsilon(s) = \Phi(s) \quad \text{for } s \in [0, \varepsilon^{-1}] \tag{4.1}$$

and

$$\Phi_\varepsilon'' = \frac{1}{h_\varepsilon}, \qquad \Phi_\varepsilon \geq \Phi. \tag{4.2}$$

The function Φ_ε with the above properties can now be obtained by setting $\Phi_\varepsilon(s) = \int_0^s \Phi_\varepsilon'(\sigma)\,\mathrm{d}\sigma$, where $\Phi_\varepsilon'(s)$ is given by

$$\Phi_\varepsilon'(s) = -\int_s^{B_\varepsilon} \frac{1}{h_\varepsilon(\sigma)}\,\mathrm{d}\sigma,$$

with the constant $B_\varepsilon > \frac{1}{\varepsilon}$ being such that

$$\int_{\frac{1}{\varepsilon}}^{B_\varepsilon} \frac{1}{h_\varepsilon(\sigma)}\,\mathrm{d}\sigma = \int_{\frac{1}{\varepsilon}}^{\infty} \frac{1}{h(\sigma)}\,\mathrm{d}\sigma.$$

Identity (4.1) is a consequence of the fact that $h_\varepsilon(s) = h(s)$ in $[0, \varepsilon^{-1}]$, while the second property in (4.2) follows from the inequality $h_\varepsilon \leq h$.

Notice that the functional derivative of $\mathcal{H}_\varepsilon$ is given by $\delta\mathcal{H}_\varepsilon(f) = \frac{1}{2}|v|^2 + \Phi_\varepsilon'(f)$, which allows us to rewrite ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) as

$$\partial_t f_\varepsilon = \operatorname{div}(h_\varepsilon(f_\varepsilon)\nabla\delta\mathcal{H}_\varepsilon(f_\varepsilon)).$$

Lemma 4.1 (Energy dissipation balance for ($\mathrm{FP}_{\gamma.\mathrm{reg}}$)). *Under the hypotheses of Proposition 2.4, the solutions f_ε of ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) obtained therein satisfy for all $0 \leq s \leq t < \infty$,*

$$\mathcal{H}_\varepsilon(f_\varepsilon(t)) + \int_s^t \int_{\mathbb{R}^d} \frac{1}{h_\varepsilon(f_\varepsilon)} |\nabla f_\varepsilon + v h_\varepsilon(f_\varepsilon)|^2\,\mathrm{d}v\,\mathrm{d}\tau = \mathcal{H}_\varepsilon(f_\varepsilon(s)). \tag{4.3}$$

Proof. Recall that $f_\varepsilon \in C^{1,2}((0,\infty) \times \mathbb{R}^d)$ is a classical solution of ($\mathrm{FP}_{\gamma.\mathrm{reg}}$). Hence, the only task in deriving equation (4.3) lies in appropriately controlling the tails as $|v| \to \infty$. This is a consequence of the moment control of the bounded function f_ε and follows from classical arguments; see e.g. [10]. ∎

Lemma 4.2. *Suppose (H1), (H2) and use the notation in Proposition 1.1. For any $t \geq 0$,*

$$\liminf_{\varepsilon \to 0} \mathcal{H}_\varepsilon(f_\varepsilon(t)) \geq \mathcal{H}(f(t)),$$

where the $\liminf$ *is taken along the sequence $\varepsilon \downarrow 0$ selected in Proposition 1.1 (iii).*

Proof. Since $\Phi_\varepsilon \geq \Phi$ (cf. (4.2)), we have $\int \Phi_\varepsilon(f_\varepsilon(t)) \geq \int \Phi(f_\varepsilon(t))$. Next, given $\delta > 0$, we let $L = L(\delta) > 0$ be large enough such that $|\Phi(s)| \leq \delta s$ for $s \geq L$. Then

$$\begin{aligned}\int \Phi(f_\varepsilon(t,v))\,\mathrm{d}v &\geq \int \Phi(f_\varepsilon(t,v))\chi_{\{f_\varepsilon \leq L\}}\,\mathrm{d}v + \int \Phi(f_\varepsilon(t,v))\chi_{\{f_\varepsilon > L\}}\,\mathrm{d}v \\ &\geq \int \Phi(f_\varepsilon(t,v))\chi_{\{f_\varepsilon \leq L\}}\,\mathrm{d}v - \delta m,\end{aligned}$$

where $m = \int f_{\text{in}}$. First sending $\varepsilon \to 0$ (using dominated convergence) and then $L \to \infty$, we infer $\liminf_{\varepsilon\to 0} \int \Phi_\varepsilon(f_\varepsilon(t)) \geq \int \Phi(f(t)) - \delta m$ and hence

$$\liminf_{\varepsilon \to 0} \int \Phi_\varepsilon(f_\varepsilon(t)) \geq \int \Phi(f(t)).$$

For the kinetic part, we let $\mathcal{A}_\rho := \{\rho \leq |v| \leq \rho^{-1}\}$ for $0 < \rho \ll 1$ and estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \frac{1}{2}|v|^2 f_\varepsilon(t) - \int_{\mathbb{R}^d} \frac{1}{2}|v|^2 f(t)\right| &\leq \left|\int_{\mathcal{A}_\rho} \frac{1}{2}|v|^2 (f_\varepsilon(t) - f(t))\,\mathrm{d}v\right| \\ &\quad + \rho^2 m + C\rho \|f_{\text{in}}\|_{L^1_3},\end{aligned}$$

where we used mass conservation and the bound $\|f(t)\|_{L^1_3}, \|f_\varepsilon(t)\|_{L^1_3} \leq C\|f_{\text{in}}\|_{L^1_3} < \infty$ (cf. Lemma 2.3). We deduce $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \frac{1}{2}|v|^2 f_\varepsilon(t,v)\,\mathrm{d}v = \int_{\mathbb{R}^d} \frac{1}{2}|v|^2 f(t,v)\,\mathrm{d}v$, where we used the locally uniform convergence in Proposition 1.1 (iii) and the fact that $0 < \rho \ll 1$ can be taken arbitrarily small. ∎

4.2. The limiting measure is a renormalised solution

Proof of Theorem 1.5. The weak-$*$ continuity of the mass-conserving curve $t \mapsto \mu_t$ in $\mathcal{M}_+(\mathbb{R}^d)$ has already been established in Proposition 1.1.

We next show that $\mathcal{T}_k(f) = \min\{f, k\}$ has a weak derivative $\nabla \mathcal{T}_k(f) \in L^2_{\text{loc}}([0,\infty) \times \mathbb{R}^d)$. For this purpose, we choose $s = 0$ and $t = T$ in estimate (4.3) and, letting $\epsilon_* > 0$ be small enough so that, by Lemma 4.2, $-\mathcal{H}_\varepsilon(f_\varepsilon(T)) \leq -\mathcal{H}(f(T)) + 1$ for all $\varepsilon \in (0, \epsilon_*]$, we infer the ε-uniform bound

$$\int_0^T \int_{\mathbb{R}^d} \frac{1}{h_\varepsilon(f_\varepsilon)} |\nabla f_\varepsilon + v h_\varepsilon(f_\varepsilon)|^2 \,\mathrm{d}v\,\mathrm{d}\tau \leq \mathcal{H}(f_{\text{in}}) - \mathcal{H}(f(T)) + 1. \tag{4.4}$$

To deduce a bound on $\nabla \mathcal{T}_k(f_\varepsilon)$, we note that

$$|\nabla \mathcal{T}_k(f_\varepsilon)|^2 \leq 2|\mathcal{T}_k'(f_\varepsilon)[\nabla f_\varepsilon + v h_\varepsilon(f_\varepsilon)]|^2 + 2|\mathcal{T}_k'(f_\varepsilon) v h_\varepsilon(f_\varepsilon)|^2.$$

Hence, using the fact that $|\mathcal{T}_k'| \le 1$ and $\mathcal{T}_k'(s) = 0$ for $s > k$, we deduce from (4.4) for any $R \in (0, \infty)$,

$$\int_0^T \int_{\{|v|\le R\}} |\nabla \mathcal{T}_k(f_\varepsilon)|^2 \,\mathrm{d}v\,\mathrm{d}t \le C(k)(\mathcal{H}(f_{\mathrm{in}}) - \mathcal{H}(f(T)) + 1) + C(k,R)T. \tag{4.5}$$

Thanks to the convergence in Proposition 1.1 (iii),

$$\begin{aligned} &\mathcal{T}_k(f_\varepsilon) \to \mathcal{T}_k(f) \quad \text{a.e. in } [0,\infty)\times\mathbb{R}^d, \\ &\mathcal{T}_k(f_\varepsilon) \to \mathcal{T}_k(f) \quad \text{in } L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d) \quad \text{for all } p \in [1,\infty), \\ \text{and thus, by (4.5),}\quad &\nabla\mathcal{T}_k(f_\varepsilon) \rightharpoonup \nabla\mathcal{T}_k(f) \text{ in } L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d). \end{aligned}$$

As a consequence,

$$\int_0^T \int_{\{|v|\le R\}} |\nabla \mathcal{T}_k(f)|^2 \,\mathrm{d}v\,\mathrm{d}t \le C(k)\big(\mathcal{H}(f_{\mathrm{in}}) - \mathcal{H}(f(T)) + 1\big) + C(k,R)T. \tag{4.6}$$

If the gradients $\nabla\mathcal{T}_k(f_\varepsilon)$ were known to converge strongly in L^2_{loc}, the renormalised formulation (1.8) could easily be derived from that for f_ε in the limit $\varepsilon \to 0$. For general anisotropic solutions such a result is, however, not available at the moment. The proof of (1.8) presented below in the isotropic case uses a somewhat different argument that will be taken up when deriving the entropy balance law (1.10).

Now let $\xi \in C^\infty([0,\infty))$ have a compactly supported derivative ξ', let $T < \infty$, and let $\psi \in C_c^\infty([0,T]\times\mathbb{R}^d)$. Further let $\varphi \in C^\infty([0,\infty);[0,1])$ satisfy $\varphi(r) = 0$ for $r \in [0,1]$ and $\varphi(r) = 1$ for $r \ge 2$, and abbreviate $\varphi_\rho(r) = \varphi(r/\rho)$ for $\rho \in (0,1]$. Then, since f is a classical solution of (FP_γ) in $(0,\infty)\times(\mathbb{R}^d\setminus\{0\})$, a direct calculation gives

$$\begin{aligned} &\int_{\mathbb{R}^d} \xi(f(T,\cdot))\psi(T,\cdot)\varphi_\rho(|v|)\,\mathrm{d}v - \int_{\mathbb{R}^d} \xi(f_{\mathrm{in}})\psi(0,\cdot)\varphi_\rho(|v|)\,\mathrm{d}v \\ &\qquad - \int_0^T\int_{\mathbb{R}^d} \xi(f)\partial_t\psi\varphi_\rho(|v|)\,\mathrm{d}v\,\mathrm{d}t \\ &\quad = -\int_0^T\int_{\mathbb{R}^d} (\nabla f + h(f)v)\cdot\big[\xi''(f)\nabla f\psi\varphi_\rho(|v|) + \xi'(f)\nabla\psi\varphi_\rho(|v|)\big]\,\mathrm{d}v\,\mathrm{d}t \\ &\qquad - \int_0^T\int_{\mathbb{R}^d} (\nabla f + h(f)v)\cdot\Big[\xi'(f)\psi\varphi_\rho'(|v|)\cdot\frac{v}{|v|}\Big]\,\mathrm{d}v\,\mathrm{d}t. \end{aligned} \tag{4.7}$$

By the dominated convergence theorem and since $\varphi_\rho(r) \xrightarrow{\rho\downarrow 0} 1$ for all $r > 0$, the LHS of (4.7) converges, as $\rho \to 0$, to

$$\int_{\mathbb{R}^d} \xi(f(T,\cdot))\psi(T,\cdot)\,\mathrm{d}v - \int_{\mathbb{R}^d} \xi(f_{\mathrm{in}})\psi(0,\cdot)\,\mathrm{d}v - \int_0^T\int_{\mathbb{R}^d} \xi(f)\partial_t\psi\,\mathrm{d}v\,\mathrm{d}t.$$

Likewise, thanks to the bound (4.6) and the compact support of ξ'', ξ' and of ψ, the dominated convergence theorem allows us to pass to the limit in the first integral on the

RHS of (4.7), giving the term

$$-\int_0^T \int_{\mathbb{R}^d} (\nabla f + h(f)v) \cdot \big[\xi''(f)\nabla f \psi + \xi'(f)\nabla\psi\big] \, \mathrm{d}v \, \mathrm{d}t.$$

We are left to show that the last integral in (4.7) vanishes in the limit $\rho \downarrow 0$. First, since $|h(f)\xi'(f)| \leq C(\operatorname{supp}\xi') < \infty$ and $\varphi_\rho'(|v|) = 0$ for $|v| \geq 2\rho$ as well as $|v\varphi_\rho'(|v|)| = |\rho^{-1}v\varphi'(|\rho^{-1}v|)| \lesssim 1$, the dominated convergence theorem yields

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^d} h(f)v \cdot \left[\xi'(f)\psi\varphi_\rho'(|v|) \cdot \frac{v}{|v|}\right] \mathrm{d}v \, \mathrm{d}t\right| \\
&\qquad \leq \int_0^T \int_{\mathbb{R}^d} |h(f)\xi'(f)| \, |\psi| \, |v\varphi_\rho'(|v|)| \, \mathrm{d}v \, \mathrm{d}t \to 0 \quad \text{as } \rho \to 0.
\end{aligned}$$

The remaining part of the integral is more delicate. We estimate using the radial symmetry of $f(t, v)$ $(=: g(t, |v|))$,

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^d} \nabla f \cdot \left[\xi'(f)\psi\varphi_\rho'(|v|) \cdot \frac{v}{|v|}\right] \mathrm{d}v \, \mathrm{d}t\right| \\
&\qquad \leq C \int_0^T \int_0^{2\rho} |\xi'(g)\partial_r g| \rho^{-1} |\varphi'(\rho^{-1}r)| r^{d-1} \, \mathrm{d}r \, \mathrm{d}t \\
&\qquad = C \int_0^T A(t, \rho) \, \mathrm{d}t,
\end{aligned}$$

where we abbreviated

$$A(t, \rho) = \int_0^{2\rho} |\xi'(g)\partial_r g| \rho^{-1} |\varphi'(\rho^{-1}r)| r^{d-1} \, \mathrm{d}r.$$

As a consequence of the bound (2.18), we have $|\xi'(g)r^{d-1}\partial_r g| \leq CK_* + C(\operatorname{supp}\xi')r^d$. We hence infer the following (t, ρ)-uniform bound on $|A(t, \rho)|$:

$$|A(t, \rho)| \leq C \int_0^{2\rho} \rho^{-1} |\varphi'(\rho^{-1}r)| \, \mathrm{d}r = C \int_0^2 |\varphi'(\hat{r})| \, \mathrm{d}\hat{r}.$$

Thus, to show that $\lim_{\rho\to 0} \int_0^T A(t, \rho) \, \mathrm{d}t = 0$ it suffices to prove the pointwise convergence $\lim_{\rho\to 0} A(t, \rho) = 0$ for (almost) all $t \in (0, T]$. Thanks to Theorem 1.2, only the following two cases may occur.

Case 1: $g(t, 0+) = +\infty$. In this case, there exists $r_* > 0$ such that $\xi'(g(t, r)) = 0$ for all $r \in (0, r_*)$. Hence, we trivially have $\lim_{\rho\to 0} A(t, \rho) = 0$.

Case 2: $g(t, \cdot) \in L^\infty$. In this case, by Theorem 1.2, there exists a neighbourhood J of t such that $f_{|J\times\mathbb{R}^d}$ is smooth. In particular, $\partial_r g(t, \cdot) \in L^\infty(0, 1)$. If $d > 1$, the conclusion $\lim_{\rho\to 0} A(t, \rho) = 0$ then directly follows from the definition of $A(t, \rho)$, while for $d = 1$ we resort to the fact that $\sup_{r\in(0,\rho)} |\partial_r g(t, r)| \to 0$ as $\rho \to 0$. ■

4.3. Energy dissipation identity

An argument similar to that in the proof of Theorem 1.5 shows that isotropic solutions satisfy the energy dissipation balance. In the anisotropic case, we obtain an inequality.

Proof of Proposition 1.6. Combining Lemmas 4.1 and 4.2 with the convergence properties of f_ε to f in Proposition 1.1 (iii), we readily infer for all $t > 0$ the inequality

$$\mathcal{H}(f(t)) + \int_0^t \mathcal{D}(f(\tau))\,\mathrm{d}\tau \le \mathcal{H}(f_{\mathrm{in}}),$$

where

$$\mathcal{D}(f) := \int_{\mathbb{R}^d} \frac{1}{h(f)} |\nabla f + h(f)v|^2\,\mathrm{d}v.$$

It remains to prove that in the isotropic case the above inequality holds with an equality. Then the asserted identity (1.10) follows by subtracting on both sides the quantity $\mathcal{H}(f(s))$, which is then known to equal $\mathcal{H}(f_{\mathrm{in}}) - \int_0^s \mathcal{D}(f(\tau))\,\mathrm{d}\tau$. Thus, in the remainder, we assume that f_{in} is isotropic. Moreover, without loss of generality, we may assume hypothesis (H2′). Otherwise we replace f_{in} by $f(t_0)$ for small $t_0 > 0$. From the arguments below we will then obtain the identity $\mathcal{H}(f(t)) + \int_{t_0}^t \mathcal{D}(f(\tau))\,\mathrm{d}\tau = \mathcal{H}(f(t_0))$, and taking the limit $t_0 \downarrow 0$, using monotone convergence for the term $\int_{t_0}^t \mathcal{D}(f(\tau))\,\mathrm{d}\tau$ and dominated convergence for $\mathcal{H}(f(t_0))$ we will arrive at the assertion.

As in the proof of Theorem 1.5 (cf. Section 4.2), we pick a non-decreasing function $\varphi \in C^\infty([0,\infty);[0,1])$ satisfying $\varphi(r) = 0$ for $r \in [0,1]$ and $\varphi(r) = 1$ for $r \ge 2$, and abbreviate $\varphi_\rho(r) = \varphi(r/\rho)$ for $\rho \in (0,1]$. Then, defining

$$\mathcal{H}^{(\rho)}(f) := \int_{\mathbb{R}^d} \Big[\frac{1}{2}|v|^2 f + \Phi(f)\Big]\varphi_\rho(|v|)\,\mathrm{d}v,$$

one has

$$\begin{aligned}\mathcal{H}^{(\rho)}(f(t)) - \mathcal{H}^{(\rho)}(f_{\mathrm{in}}) &= \int_0^t \int_{\mathbb{R}^d} \Big[\frac{1}{2}|v|^2 + \Phi'(f)\Big] \operatorname{div}(\nabla f + h(f)v)\varphi_\rho(|v|)\,\mathrm{d}v\,\mathrm{d}\tau \\ &= -\int_0^t \int_{\mathbb{R}^d} \frac{1}{h(f)} |\nabla f + h(f)v|^2 \varphi_\rho(|v|)\,\mathrm{d}v\,\mathrm{d}\tau \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \Big[\frac{1}{2}|v|^2 + \Phi'(f)\Big](\nabla f + h(f)v)\cdot \frac{v}{|v|}\varphi_\rho'(|v|)\,\mathrm{d}v\,\mathrm{d}\tau.\end{aligned}$$

We note that

$$\lim_{\rho\to 0} \mathcal{H}^{(\rho)}(f(t)) = \mathcal{H}(f(t)) \quad \text{for all } t \ge 0.$$

Furthermore, monotone convergence gives

$$\lim_{\rho\to 0} \int_0^t \int_{\mathbb{R}^d} \frac{1}{h(f)} |\nabla f + h(f)v|^2 \varphi_\rho(|v|)\,\mathrm{d}v\,\mathrm{d}\tau = \int_0^t \int_{\mathbb{R}^d} \frac{1}{h(f)} |\nabla f + h(f)v|^2\,\mathrm{d}v\,\mathrm{d}\tau.$$

Hence, it remains to prove that the quantity

$$B(\tau,\rho) := \int_{\mathbb{R}^d} \Big[\frac{1}{2}|v|^2 + \Phi'(f)\Big](\nabla f + h(f)v)\cdot \frac{v}{|v|}\varphi_\rho'(|v|)\,\mathrm{d}v$$

satisfies

$$\lim_{\rho\to 0}\int_0^t B(\tau,\rho)\,\mathrm{d}\tau = 0. \tag{4.8}$$

Using the isotropy of $f(\tau,\cdot)$ we write

$$B(\tau,\rho) = c_d \int_0^{2\rho}\Big[\frac{1}{2}r^2 + \Phi'(g)\Big](\partial_r g + h(g)r)\varphi_\rho'(r)r^{d-1}\,\mathrm{d}r,$$

where c_d denotes the area of the unit sphere. We can now argue similarly to the proof of Theorem 1.5. The function $|B(\tau,\rho)|$ is uniformly bounded for $(\tau,\rho)\in[0,t]\times(0,1]$ thanks to the estimate $|(\partial_r g + rh(g))r^{d-1}| \le K_*$ and the fact that, by (H2′) and Proposition 1.1 (ii), $\inf_{[0,t]\times(0,1]} g =: \iota > 0$. Indeed, note that for $(\tau,\rho)\in[0,t]\times(0,1]$,

$$|B(\tau,\rho)| \le C(\Phi'(\iota))K_* \int_0^{2\rho} |\varphi'(\rho^{-1}r)|\rho^{-1}\,\mathrm{d}r = C(\Phi'(\iota))K_*\int_0^2 |\varphi'(\hat r)|\,\mathrm{d}\hat r.$$

Identity (4.8) therefore follows from the dominated convergence theorem provided we can prove the pointwise convergence $\lim_{\rho\to 0} B(\tau,\rho) = 0$ for a.e. $\tau > 0$.

Case 1: $g(\tau,0+) = +\infty$. In this case, we estimate

$$|B(\tau,\rho)| \le CK_*\int_0^2 |\varphi'(\hat r)|\,\mathrm{d}\hat r \cdot \sup_{r\in(0,\rho)} |\tfrac{1}{2}r^2 + \Phi'(g(\tau,r))|$$

and note that the sublinearity of Φ at infinity implies

$$\sup_{r\in(0,\rho)} |\tfrac{1}{2}r^2 + \Phi'(g(\tau,r))| \to 0 \quad \text{as } \rho\to 0.$$

Hence, $\lim_{\rho\to 0}|B(\tau,\rho)| = 0$.

Case 2: $g(\tau,\cdot)\in L^\infty$. Here, the assertion $\lim_{\rho\to 0}|B(\tau,\rho)| = 0$ is obtained similarly to Case 2 of the proof of Theorem 1.5 using the regularity of $g(\tau,\cdot)$ shown in Theorem 1.2. ∎

5. Long-time behaviour

5.1. Relaxation to equilibrium

Proof of Theorem 1.7. Let

$$\mathcal{D}(f) := \int_{\mathbb{R}^d} \frac{1}{h(f)}|\nabla f + h(f)v|^2\,\mathrm{d}v = \int_{\mathbb{R}^d} h(f)|\nabla\delta\mathcal{H}(f)|^2\,\mathrm{d}v.$$

Proposition 1.6 implies that $\int_0^\infty \mathcal{D}(f(t))\,\mathrm{d}t \le \mathcal{H}(f_{\mathrm{in}}) - \inf_{\mathcal{M}_+}\mathcal{H} < \infty$, and hence there exists an increasing sequence $t_k \to \infty$ such that

$$\lim_{k\to\infty} \mathcal{D}(f(t_k)) = 0. \tag{5.1}$$

The sequence $\{\mu_{t_k}\}_k \subset \mathcal{M}_+(\mathbb{R}^d)$ of measures of mass m is tight since $\sup_k \int |v|^3\,\mathrm{d}\mu_{t_k} \le C\|f_{\mathrm{in}}\|_{L^1_3} < \infty$. Prokhorov's theorem thus ensures the existence of a measure $\mu_\infty \in \mathcal{M}_+(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \mathrm{d}\mu_\infty = m$ such that, along a subsequence (not relabelled),

$$\mu_{t_k} \overset{*}{\rightharpoonup} \mu_\infty \text{ in } \mathcal{M}_+(\mathbb{R}^d) \tag{5.2}$$

$$\text{and, moreover,} \quad \int |v|^2\,\mathrm{d}\mu_{t_k} \to \int |v|^2\,\mathrm{d}\mu_\infty.$$

At the same time, by Proposition 1.1 (ii), $\mu_{t_k} = a(t_k)\delta_0 + f(t_k)\mathcal{L}^d$, where f satisfies a time-uniform bound of the form $|f(t,r)| \le C(\rho)$ for all $r \ge \rho > 0$ (cf. Lemma 2.8, resp. Corollary 2.9). Thus, the sequence $f_k(t) := f(t_k + t)$ obeys an estimate analogous to (2.21) for $G \subset\subset (-1,\infty)\times(\mathbb{R}^d \setminus \{0\})$, where we assume without loss of generality that $t_1 \ge 1$. We may therefore argue similarly to the proof of Proposition 1.1 and invoke the Arzelà–Ascoli theorem to infer the existence of $f_\infty \in C^2(\mathbb{R}^d \setminus \{0\}) \cap L^1_3(\mathbb{R}^d)$ such that, after possibly passing to another subsequence,

$$f(t_k) \to f_\infty \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}). \tag{5.3}$$

Notice that $\mu_\infty^{\mathrm{reg}} = f_\infty \mathcal{L}^d$ and $\operatorname{supp}\mu_\infty^{\mathrm{sing}} \subseteq \{0\}$, as a consequence of (5.2) and (5.3).

We will now show that μ_∞ agrees with the minimiser of $\mathcal{H}$ of mass m. To this end, let $\lambda(s) = \exp(\Phi'(s)) = \frac{s}{(1+s^\gamma)^{1/\gamma}}$ and note that

$$\frac{h(s)}{\lambda^2(s)} = \frac{s(1+s^\gamma)^{1+2/\gamma}}{s^2} \ge (1+s^\gamma)^{1+\frac{1}{\gamma}} \ge 1$$

for all $s \in (0,\infty)$. Hence,

$$\begin{aligned}
\mathcal{D}(f) &= \int_{\mathbb{R}^d} h(f)|\nabla\Phi'(f) + v|^2\,\mathrm{d}v \\
&= \int_{\mathbb{R}^d} \frac{h(f)}{\lambda(f)^2}|\lambda(f)\nabla\Phi'(f) + \lambda(f)v|^2\,\mathrm{d}v \\
&\ge \int_{\mathbb{R}^d} |\nabla\lambda(f) + v\lambda(f)|^2\,\mathrm{d}v.
\end{aligned}$$

Thanks to the convergence (5.1), the last estimate applied to $f := f(t_k)$ implies that

$$\nabla\lambda(f(t_k)) + v\lambda(f(t_k)) \to 0 \text{ in } L^2(\mathbb{R}^d).$$

At the same time, using (5.3), we may pass to the limit in the sense of distributions

$$\nabla\lambda(f(t_k)) + v\lambda(f(t_k)) \to \nabla\lambda(f_\infty) + v\lambda(f_\infty) \text{ in } \mathcal{D}'(\mathbb{R}^d\setminus\{0\}).$$

Thus, $\nabla\lambda(f_\infty) + v\lambda(f_\infty) = 0$ in $\mathcal{D}'(\mathbb{R}^d\setminus\{0\})$, and hence $\nabla(e^{\frac{1}{2}|v|^2}\lambda(f_\infty)) = 0$ in $\mathbb{R}^d\setminus\{0\}$. This implies that $\Phi'(f_\infty) + \frac{1}{2}|v|^2 \equiv -\theta$ for a constant $\theta \in \mathbb{R}_{\geq 0}$, where the sign of θ follows from the fact that $\Phi' \leq 0$. Hence, $f_\infty(v) = (\Phi')^{-1}(-\frac{1}{2}|v|^2 - \theta) = f_{\infty,\theta}(v)$. To determine θ, recall that $\mu_\infty(\mathbb{R}^d) = m$. Thus, if $\|f_{\infty,\theta}\|_{L^1(\mathbb{R}^d)} < m$, then $\mu_\infty(\{0\}) > 0$. In this case the convergence (5.2), combined with the time-uniform upper bound $\sup_{t>0} f(t,v) \leq C|v|^{-\frac{2}{\gamma}} \in L^1(B_{r_*})$, which follows from Theorem 1.2, resp. Corollary 3.4, implies the existence of $\underline{k} \in \mathbb{N}$ such that $\mu_{t_k}(\{0\}) > 0$ for all $k \geq \underline{k}$. Invoking Theorem 1.2 once more (now using the radial symmetry assumption), we find that (1.6) holds true for all such t_k and, owing to the convergence (5.3), we conclude that $\theta = 0$. If on the other hand $\|f_{\infty,\theta}\|_{L^1(\mathbb{R}^d)} = m$, there is no excess mass and we must have $\mu_\infty = f_{\infty,\theta}\mathcal{L}^d$. In conclusion, we have shown that the measure μ_∞ coincides with the unique minimiser $\mu_{\min} = \mu_{\min}^{(m)}$ of mass m.

From the convergence properties established so far we infer $\lim_{k\to\infty} \mathcal{H}(\mu_{t_k}) = \mathcal{H}(\mu_{\min})$. Since $t \mapsto \mathcal{H}(\mu_t)$ is non-increasing, this immediately yields

$$\lim_{t\to\infty} \mathcal{H}(\mu_t) = \mathcal{H}(\mu_{\min}).$$

Combining this result with the above compactness properties, mass conservation, and the uniqueness of the minimiser $\mu_{\min}^{(m)}$, one can easily deduce the remaining convergence properties along any sequence $t \to \infty$ as asserted in Theorem 1.7. Here, also recall the bounds $\sup_{t>0} f(t,v) \lesssim |v|^{-\frac{2}{\gamma}}$ for $|v| \leq r_*$, $f(t,v) \lesssim 1$ for $|v| \geq r_*$ and $\sup_{t>0} \|f(t)\|_{L^1_3} \lesssim \|f_{\text{in}}\|_{L^1_3}$, which guarantee $\lim_{t\to\infty} \mu_t(\{0\}) = \mu_\infty(\{0\})$ and $\lim_{t\to\infty} \|f(t) - f_{\infty,\theta}\|_{L^p(\mathbb{R}^d)} = 0$ for $p \in [1, \frac{\gamma d}{2})$. ∎

5.2. Long-time and transient properties

Let us briefly point out some implications of the above analysis on further qualitative dynamical properties, restricting for consistency to the isotropic case. If $m < m_c$, Theorem 1.7 along with Theorem 1.2 implies the eventual regularity of μ_t after some sufficiently large time $T \gg 1$. However, using a contradiction argument, finite-time blow-up and the formation of a condensate (i.e. $\mu_t(\{0\}) > 0$ for some $t > 0$) can be shown to occur for any size of the mass $m > 0$ by choosing the smooth initial data sufficiently concentrated near the origin (cf. [8,33]). Hence, there exist flows exhibiting *transient condensates* with singular parts compactly supported in time. On the other hand, whenever $m > m_c$, the above theory implies the eventual formation of a condensate: there exists $T \gg 1$ such that $\mu_t(\{0\}) > 0$ for all $t \geq T$. This is a consequence of the convergence $\lim_{t\to\infty} \mu_t(\{0\}) = \mu_{\min}(\{0\})$. It is also possible to infer information on the spatiotemporal features of singularity formation and regularisation using rescaling methods. We refer to [23, Chapter 5.2], where such dynamics have been shown to be of “type II” for the one-dimensional case.

Finally, we note that finite-time condensation in the mass-supercritical case and convergence to the entropy minimiser can also be deduced in the anisotropic setting if f_{in} admits a mass-supercritical isotropic lower barrier, i.e. $f_{\text{in}} \geq f_{\text{in}}^{\#}$ for some non-negative radially symmetric function $f^\#_{\mathrm{in}}$ with $\int f^\#_{\mathrm{in}} > m_c$. In this case, the density $f(t,\cdot)$ is squeezed between two isotropic barriers which, by virtue of Theorem 1.7, both converge to f_c as $t \to \infty$.

5.3. Concluding remark

The comparison principle structure provides us with a priori bounds that allow for a detailed characterisation of the singularities which isotropic flows starting from regular data may exhibit (and even gives uniqueness in the one-dimensional case [8, 23], resp. convergence of the scheme to a unique limit in higher dimensions). However, one may not expect such a structure to persist in more complex situations. Particularly with regard to the study of uniqueness and stability properties in the presence of singularities, it would be interesting to see whether variational problems like (FP$_\gamma$) allow for more robust approaches.

A. Auxiliary estimate

Recall that $\mathcal{F}(t,v,w) = \mathrm{e}^{dt} G_{\nu(t)}(\mathrm{e}^t v - w)$, $\nu(t) = \mathrm{e}^{2t} - 1$, $G_\lambda(\xi) = (2\pi\lambda)^{-\frac{d}{2}} \mathrm{e}^{-\frac{|\xi|^2}{2\lambda}}$.

Lemma A.1. *Let $T \le 1$ and let $\tilde q \in [1,\infty]$. There exists $C = C(\tilde q, d) < \infty$ such that for all $t \in (0,T]$,*

$$\left\| \int_{\mathbb{R}^d} |\nabla_v \mathcal{F}(t,v,w)|\, |\mathrm{e}^{-t} w - v|\, |f(w)|\, \mathrm{d}w \right\|_{L^{\tilde q}(\mathbb{R}^d)} \le C \|f\|_{L^{\tilde q}(\mathbb{R}^d)} \le C\nu(t)^{-\frac{1}{2}} \|f\|_{L^{\tilde q}(\mathbb{R}^d)}. \tag{A.1}$$

Proof. The second bound in (A.1) is trivial.

To verify the first inequality, we compute for $t \in (0,T]$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla_v \mathcal{F}(t,v,w)|\, |\mathrm{e}^{-t} w - v|\, |f(w)|\, \mathrm{d}w \\
&\quad = (2\pi\nu(t))^{-\frac{d}{2}} \mathrm{e}^{dt} \mathrm{e}^t (2\nu(t))^{-\frac{1}{2}} \int_{\mathbb{R}^d} 2\frac{|\mathrm{e}^t v - w|}{\sqrt{2\nu(t)}} \mathrm{e}^{-\frac{|\mathrm{e}^t v - w|^2}{2\nu(t)}} |\mathrm{e}^{-t} w - v|\, |f(w)|\, \mathrm{d}w \\
&\quad = (2\pi\nu(t))^{-\frac{d}{2}} \mathrm{e}^{2dt} \mathrm{e}^t (2\nu(t))^{-\frac{1}{2}} \int_{\mathbb{R}^d} 2\frac{|\mathrm{e}^t (v - \tilde w)|}{\sqrt{2\nu(t)}} \mathrm{e}^{-\frac{|\mathrm{e}^t (v - \tilde w)|^2}{2\nu(t)}} |\tilde w - v|\, |f(\mathrm{e}^t \tilde w)|\, \mathrm{d}\tilde w \\
&\quad = (2\pi\nu(t))^{-\frac{d}{2}} \mathrm{e}^{2dt} \int_{\mathbb{R}^d} 2\frac{|\mathrm{e}^t (v - \tilde w)|^2}{2\nu(t)} \mathrm{e}^{-\frac{|\mathrm{e}^t (v - \tilde w)|^2}{2\nu(t)}} |f(\mathrm{e}^t \tilde w)|\, \mathrm{d}\tilde w.
\end{aligned}$$

Now the asserted inequality follows upon an application of Young's convolution inequality, $\|a * b\|_{L^{\tilde q}} \le \|a\|_{L^1} \|b\|_{L^{\tilde q}}$. ∎

Acknowledgements

The author would like to thank Prof. John King for valuable discussions on this topic.

References

[1] H. Amann, *Linear and quasilinear parabolic problems. Vol. I*. Monographs in Mathematics 89, Birkhäuser, Boston, MA, 1995 Zbl 0819.35001 MR 1345385

[2] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*. 2nd edn., Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2008 Zbl 1145.35001 MR 2401600

[3] N. Ben Abdallah, I. M. Gamba, and G. Toscani, On the minimization problem of sub-linear convex functionals. *Kinet. Relat. Models* **4** (2011), no. 4, 857–871 Zbl 1251.35168 MR 2861577

[4] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), no. 2, 241–273 Zbl 0866.35037 MR 1354907

[5] D. Blanchard and F. Murat, Renormalised solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness. *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), no. 6, 1137–1152 Zbl 0895.35050 MR 1489429

[6] J. A. Cañizo, J. A. Carrillo, P. Laurençot, and J. Rosado, The Fokker–Planck equation for bosons in 2D: well-posedness and asymptotic behavior. *Nonlinear Anal.* **137** (2016), 291–305 Zbl 1339.35321 MR 3485127

[7] J. A. Carrillo, M. Di Francesco, and G. Toscani, Condensation phenomena in nonlinear drift equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **15** (2016), 145–171 Zbl 1355.35134 MR 3495424

[8] J. A. Carrillo, K. Hopf, and J. L. Rodrigo, On the singularity formation and relaxation to equilibrium in 1D Fokker–Planck model with superlinear drift. *Adv. Math.* **360** (2020), article no. 106883 Zbl 1433.35408 MR 4031115

[9] J. A. Carrillo, K. Hopf, and M.-T. Wolfram, Numerical study of Bose–Einstein condensation in the Kaniadakis–Quarati model for bosons. *Kinet. Relat. Models* **13** (2020), no. 3, 507–529 Zbl 1441.35235 MR 4097723

[10] J. A. Carrillo, P. Laurençot, and J. Rosado, Fermi–Dirac–Fokker–Planck equation: Well-posedness & long-time asymptotics. *J. Differential Equations* **247** (2009), no. 8, 2209–2234 Zbl 1181.35292 MR 2561276

[11] J. A. Carrillo, S. Lisini, G. Savaré, and D. Slepčev, Nonlinear mobility continuity equations and generalized displacement convexity. *J. Funct. Anal.* **258** (2010), no. 4, 1273–1309 Zbl 1225.49043 MR 2565840

[12] J. A. Carrillo, H. Ranetbauer, and M.-T. Wolfram, Numerical simulation of nonlinear continuity equations by evolving diffeomorphisms. *J. Comput. Phys.* **327** (2016), 186–202 Zbl 1373.82070 MR 3564334

[13] J. A. Carrillo, J. Rosado, and F. Salvarani, 1D nonlinear Fokker–Planck equations for fermions and bosons. *Appl. Math. Lett.* **21** (2008), no. 2, 148–154 Zbl 1151.35044 MR 2426970

[14] J. A. Carrillo and G. Toscani, Exponential convergence toward equilibrium for homogeneous Fokker–Planck-type equations. *Math. Methods Appl. Sci.* **21** (1998), no. 13, 1269–1286 Zbl 0922.35131 MR 1639292

[15] M. G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67 Zbl 0755.35015 MR 1118699

[16] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet, Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **28** (1999), no. 4, 741–808 Zbl 0958.35045 MR 1760541

[17] F. Demengel and R. Temam, Convex functions of a measure and applications. *Indiana Univ. Math. J.* **33** (1984), no. 5, 673–709 Zbl 0581.46036 MR 756154

[18] J. Dolbeault, B. Nazaret, and G. Savaré, A new class of transport distances between measures. *Calc. Var. Partial Differential Equations* **34** (2009), no. 2, 193–231 Zbl 1157.49042 MR 2448650

[19] L. C. Evans, O. Savin, and W. Gangbo, Diffeomorphisms and nonlinear heat flows. *SIAM J. Math. Anal.* **37** (2005), no. 3, 737–751 Zbl 1096.35061 MR 2191774

[20] T. D. Frank, *Nonlinear Fokker-Planck equations*. Springer Ser. Synergetics, Springer, Berlin, 2005 Zbl 1071.82001 MR 2118870

[21] V. A. Galaktionov, *Geometric Sturmian theory of nonlinear parabolic equations and applications*. Chapman & Hall/CRC Appl. Math. Nonlinear Sci. Ser. 3, Chapman & Hall/CRC, Boca Raton, FL, 2004 Zbl 1075.35017 MR 2059317

[22] V. A. Galaktionov and J. L. Vázquez, *A stability technique for evolution partial differential equations*. Prog. Nonlinear Differ. Equ. Appl. 56, Birkhäuser, Boston, MA, 2004 Zbl 1065.35002 MR 2020328

[23] K. Hopf, *On the singularity formation and long-time asymptotics in a class of nonlinear Fokker–Planck equations*, Ph.D. thesis, University of Warwick, 2019

[24] G. Kaniadakis, Classical model of bosons and fermions. *Phys. Rev. E* **49** (1994) 5103–5110

[25] G. Kaniadakis and P. Quarati, Kinetic equation for classical particles obeying an exclusion principle. *Phys. Rev. E* **48** (1993), no. 6, article no. 4263

[26] A. Klenke, *Probability theory*, 2nd edn., Universitext, Springer, London, 2014 Zbl 1295.60001 MR 3112259

[27] S. N. Kružkov, First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)* **81 (123)** (1970), 228–255 Zbl 0202.11203 MR 0267257

[28] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs 23, American Mathematical Society, Providence, RI 1968 Zbl 0174.15403 MR 0241822

[29] G. M. Lieberman, *Second order parabolic differential equations*. World Scientific, River Edge, NJ, 1996 Zbl 0884.35001 MR 1465184

[30] S. Luckhaus, Y. Sugiyama, and J. J. L. Velázquez, Measure valued solutions of the 2D Keller–Segel system. *Arch. Ration. Mech. Anal.* **206** (2012), no. 1, 31–80 Zbl 1256.35180 MR 2968590

[31] P. Quittner and P. Souplet, *Superlinear parabolic problems*. Birkhäuser Adv. Texts, Basler Lehrbüch., Birkhäuser/Springer, Cham, 2019 MR 3967048 Zbl 1423.35004

[32] J. Sopik, C. Sire, and P.-H. Chavanis, Dynamics of the Bose-Einstein condensation: analogy with the collapse dynamics of a classical self-gravitating Brownian gas. *Phys. Rev. E (3)* **74** (2006), no. 1, article no. 011112 MR 2276587

[33] G. Toscani, Finite time blow up in Kaniadakis–Quarati model of Bose–Einstein particles. *Comm. Partial Differential Equations* **37** (2012), no. 1, 77–87 Zbl 1252.35264 MR 2864807

[34] J. J. L. Velázquez, Point dynamics in a singular limit of the Keller–Segel model. I. Motion of the concentration regions. *SIAM J. Appl. Math.* **64** (2004), no. 4, 1198–1223 Zbl 1058.35021 MR 2068667

Received 10 February 2022; revised 23 November 2022; accepted 5 January 2023.

Katharina Hopf
Weierstrass Institute for Applied Analysis and Stochastics (WIAS), Mohrenstrasse 39, 10117 Berlin, Germany; hopf@wias-berlin.de