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The compressible Euler equations in a physical vacuum: A comprehensive Eulerian approach

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Abstract. This article is concerned with the local well-posedness problem for the compressible Euler equations in gas dynamics. For this system we consider the free boundary problem which corresponds to a physical vacuum. Despite the clear physical interest in this system, the prior work on this problem is limited to Lagrangian coordinates, in high-regularity spaces. Instead, the objective of the present work is to provide a new, fully Eulerian approach to this problem, which provides a complete, Hadamard-style well-posedness theory for this problem in low-regularity Sobolev spaces. In particular, we give new proofs for existence, uniqueness, and continuous dependence on the data with sharp, scale-invariant energy estimates, and a continuation criterion.

1. Introduction

In this article we study the dynamics of the free boundary problem for a compressible gas. In the simplest form, the gas is contained in a moving domain Ω_t with boundary Γ_t, and is described via its *density* $\rho \geq 0$ and *velocity* v. The evolution of the Eulerian variables (ρ, v) is given by the compressible Euler equations

$$\begin{cases} \rho_t + \nabla(\rho v) = 0, \\ \rho(v_t + (v \cdot \nabla)v) + \nabla p = 0, \end{cases} \tag{1.1}$$

with the constitutive law

$$p = p(\rho).$$

In the present paper we will consider constitutive laws of the form

$$p(\rho) = \rho^{\kappa+1}, \quad \kappa > 0. \tag{1.2}$$

(Here, for expository reasons, we use $\kappa + 1$ rather than κ as the exponent, as it is more common in the literature.)

Heuristically one can view this system as a coupled system consisting of a wave equation for the pair $(\rho, \nabla \cdot v)$ and a transport equation for $\omega = \operatorname{curl} v$. In this interpretation, a

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key physical quantity is the propagation speed c_s for the wave component. This is called the *speed of sound*, and is given by

$$c_s^2 = p'(\rho).$$

We consider this system in the presence of vacuum states, i.e. the density ρ is allowed to vanish. The gas is located in the domain $\Omega_t := \{(t,x) \mid \rho(t,x) > 0\}$, whose boundary Γ_t is moving. The defining characteristic in the case of a gas, versus the fluid case, is that the density vanishes on the free boundary Γ_t, which is thus described by

$$\Gamma_t = \partial\Omega_t := \big\{(t,x) \mid \rho(t,x) = 0\big\}.$$

In this context, the decay rate of the sound speed near the free boundary plays a fundamental role both in the gas dynamics and in the analysis. In essence, one expects that there is a single stable, nontrivial physical regime, which is called *physical vacuum*, and corresponds to the sound speed decay rate

$$c_s^2(t,x) \approx d(x, \Gamma_t). \tag{1.3}$$

Property (1.3) will propagate in time for as long as $\nabla v \in L^\infty$, which will be the case for all solutions considered in this article. We remark that in particular such a bound guarantees a bi-Lipschitz fluid flow.

To provide some intuition for this we note that the acceleration of particles on the free boundary is exactly given by $-\kappa^{-1}\nabla c_s^2$, which is normal to the boundary. Heuristically, because of this, property (1.3) yields the correct balance which allows the free boundary to move with a bounded velocity and acceleration while interacting with the interior, as follows:

- A faster fallout rate for the sound speed would cause the boundary particles to simply move independently and linearly with the outer particle speed. This can only last for a short time, until the faster waves inside overtake the boundary and likely lead to a more stable regime where (1.3) holds. See for instance the results in this direction in [24], but also the dispersive scenario discussed in [11].
- A slower fallout rate would cause an infinite initial acceleration of the boundary, likely leading again to the same pattern.

A fundamental observation concerning physical vacuum is that relation (1.3) guarantees that linear waves with speed c_s can reach the free boundary Γ_t in finite time. Because of this, in the above flow the motion of the boundary is strongly coupled to the wave evolution and is not just a self-contained evolution at leading order.

There are two classical approaches in fluid dynamics, using either Eulerian coordinates, where the reference frame is fixed and the fluid particles are moving, or using Lagrangian coordinates, where the particles are stationary but the frame is moving. Both of these approaches have been extensively developed in the context of the compressible Euler equations, where the local well-posedness problem is very well understood.

By contrast, the free boundary problem corresponding to the physical vacuum has been far less studied and understood. Because of the difficulties related to the need to track the evolution of the free boundary, all the prior work is in the Lagrangian setting and in high-regularity spaces which are only indirectly defined.

Our goal in this paper is to provide a *new, complete, low-regularity approach* for this free boundary problem which is *fully within the Eulerian framework*. In particular, our work contains the following steps, each of which represents original, essential advances in the study of this problem:

(a) We prove the *uniqueness* of solutions with very limited regularity[1] $v \in \text{Lip}$, $\rho \in \text{Lip}$. More generally, at the same regularity level we prove *stability*, by showing that bounds for a certain distance between different solutions can be propagated in time.

(b) We develop the Eulerian Sobolev *function space structure* where this problem should be considered, providing the correct, natural scale of spaces for this evolution.

(c) We prove sharp, *scale-invariant energy estimates* within the above-mentioned scale of spaces, which show that the appropriate Sobolev regularity of solutions can be continued for as long as we have uniform bounds at the same scale $v \in \text{Lip}$.

(d) We give a simpler, more elegant proof of *existence* for regular solutions, fully within the Eulerian setting, based on the above energy estimates.

(e) We devise a nonlinear Littlewood–Paley-type method to obtain *rough solutions* as unique limits of smooth solutions, also proving the *continuous dependence* of the solutions on the initial data.

At a conceptual level, we also remark that in our approach the study of the linearized problem plays the main role, whereas the energy bounds for the full system are seen as secondary, derived estimates. This is unlike in prior works, where the linearized equation is relegated to a secondary role if it appears at all.

1.1. The material derivative and the Hamiltonian

The derivative along the particle trajectories D_t is called the material derivative and is defined as

$$D_t = \partial_t + v \cdot \nabla.$$

With this notation system (1.1) is rewritten as

$$\begin{cases} D_t \rho + \rho \nabla v = 0, \\ \rho D_t v + \nabla p = 0. \end{cases}$$

[1] In an appropriately weighted sense in the case of ρ; see Theorem 1.

Differentiating once more in the first equation we obtain

$$D_t^2\rho - \rho\nabla(\rho^{-1}p'(\rho)\nabla\rho) = \rho[(\nabla\cdot v)^2 - \mathrm{Tr}(\nabla v)^2],$$

which at leading order is a wave equation for ρ with propagation speed c_s, and where $\nabla\cdot v$ can be viewed as a dependent variable.

On the other hand, for the vorticity $\omega = \operatorname{curl} v$ one can use the second equation to obtain the transport equation

$$D_t\omega = -\omega\cdot\nabla v - (\nabla v)^{\mathsf{T}}\omega.$$

The last two equations show that indeed one can interpret the Euler equations as a coupled system consisting of a wave equation for the pair $(\rho, \nabla v)$ and a transport equation for $\omega = \operatorname{curl} v$.

This problem admits a conserved energy, which in a suitable setting can be interpreted as a Hamiltonian (see [3, 10, 22])

$$E = \int_{\Omega_t} e\,dx,$$

where the energy density e is given by

$$e = \frac{1}{2}\rho v^2 + \rho h(\rho),$$

with the specific enthalpy h defined by

$$h(\rho) = \int_0^\rho \frac{p(\lambda)}{\lambda^2}\,d\lambda.$$

1.2. The good variables

The pair of variables (ρ, v) is convenient to use if $\kappa = 1$. However, for other values of κ in (1.2) we can make a better choice. To understand that, we compute the sound speed

$$c_s^2 = (\kappa+1)\rho^\kappa.$$

This should have linear behavior near the boundary. Because of this, it is more convenient to use $r = r(\rho)$ defined by

$$r' = \rho^{-1}p'(\rho),$$

which gives

$$r = \frac{\kappa+1}{\kappa}\rho^\kappa$$

as a good variable instead of ρ.

Written in terms of (r, v) the equations become

$$\begin{cases} r_t + v\nabla r + \rho r'\nabla v = 0, \\ v_t + (v\cdot\nabla)v + \nabla r = 0. \end{cases}$$

In our case we have $\rho r' = \kappa r$ so we rewrite the above system as

$$\begin{cases} r_t + v\nabla r + \kappa r \nabla v = 0, \\ v_t + (v \cdot \nabla) v + \nabla r = 0, \end{cases} \tag{1.4}$$

or, using material derivatives,

$$\begin{cases} D_t r + \kappa r \nabla v = 0, \\ D_t v + \nabla r = 0. \end{cases} \tag{1.5}$$

We will work with this system for the rest of the paper.

1.3. Energies and function spaces

Given the constitutive law (1.2), the conserved energy is

$$E = \int \frac{1}{\kappa} \rho^{\kappa+1} + \frac{1}{2} \rho v^2 \, dx.$$

Switching to the (r, v) variables and adjusting constants, we obtain

$$E = \int r^{\frac{1-\kappa}{\kappa}} \Big(r^2 + \frac{\kappa + 1}{2} r v^2 \Big) \, dx. \tag{1.6}$$

This will not be directly useful in solving the equation, but will give us a good idea for the higher-order function spaces we will have to employ. Based on this, we introduce the energy space $\mathcal{H}$ with norm

$$\|(s, w)\|_{\mathcal{H}}^2 = \int r^{\frac{1-\kappa}{\kappa}} (|s|^2 + \kappa r |w|^2) \, dx \tag{1.7}$$

for functions (s, v) defined a.e. within the fluid domain Ω_t. Importantly, we note that the constants above do not match (1.6), and instead have been adjusted to match the energy functional for the linearized equation, which is discussed in Section 3. The two components of the $\mathcal{H}$ space as weighted L^2 spaces are given by

$$\mathcal{H} = L^2(r^{\frac{1-\kappa}{\kappa}}) \times L^2(r^{\frac{1}{\kappa}}).$$

For higher regularity, we take our cue from the second-order wave equation, which has the leading operator $c_s^2 \Delta = r\Delta$, which is naturally associated to the acoustic metric[2]

$$g = r^{-1} \, dx^2 \quad \text{in } \Omega_t. \tag{1.8}$$

[2]Technically one should add a k^{-1} factor here.

Correspondingly, we define the higher-order Sobolev spaces $\mathcal{H}^{2k}$ for distributions within the fluid domain Ω_t to have norms

$$\|(s,w)\|^2_{\mathcal{H}^{2k}} = \sum_{|\beta|\leq 2k}^{|\beta|-\alpha\leq k} \|r^\alpha \partial^\beta (s,w)\|^2_{\mathcal{H}},$$

where α is implicitly restricted to $0 \leq \alpha \leq k$. More generally, for all real $k \geq 0$ one can define by interpolation the spaces $\mathcal{H}^{2k}$. These spaces and their properties are further discussed in the next section.

1.4. Scaling and control parameters

Equation (1.4) admits the scaling law

$$(r(t,x), v(t,x)) \to (\lambda^{-2} r(\lambda t, \lambda^2 x), \lambda^{-1} v(\lambda t, \lambda^2 x)). \tag{1.9}$$

We use this scaling to track the *order* of factors in multilinear expressions, introducing a counting device based on scaling:

(i) r and v have degree -1, respectively $-\frac{1}{2}$.

(ii) ∇ has order 1 and D_t has order $\frac{1}{2}$.

The order of a multilinear expression is defined as the sum of the orders of each factor. In this way, all terms in each of the equations have the same order. This property remains valid if we either differentiate the equations in x, t or apply the material derivative D_t.

Corresponding to the above spaces and scaling we identify the *critical space* $\mathcal{H}^{2k_0}$ where k_0 is given by[3]

$$2k_0 = d + 1 + \frac{1}{\kappa}.$$

This has the property that its (homogeneous) norm is invariant with respect to the above scaling.

Associated to this Sobolev exponent we introduce the scale-invariant time-dependent pointwise control norm

$$A = \|\nabla r - N\|_{L^\infty} + \|v\|_{\dot{C}^{\frac{1}{2}}}, \tag{1.10}$$

where N is a given nonzero vector. Here, N can be chosen as $N = \nabla r(x_0)$ for some fixed point x_0, where $r(x_0) = 0$. The motivation for using such an N, rather than just $\|\nabla r\|_{L^\infty}$, is that the latter is a scale-invariant quantity of fixed, unit size. On the other hand, the A defined above can be harmlessly assumed to be small simply by working in a small neighborhood of the reference point x_0. Such a localization is allowed in the study of compressible Euler systems because of the finite speed of propagation. The control parameter A will play a leading role in elliptic estimates at fixed time, and, in order to avoid cumbersome notation, will be implicitly assumed to be small in all of our analysis.

[3] In general this will not be an integer.

For the energy estimates we will also introduce a second time-dependent control norm which is associated with the space $\mathscr{H}^{2k_0+1}$, namely

$$B = \|\nabla r\|_{\tilde{C}^{0,\frac{1}{2}}} + \|\nabla v\|_{L^\infty}, \tag{1.11}$$

where the $\tilde{C}^{0,\frac{1}{2}}$ norm is given by

$$\|f\|_{\tilde{C}^{0,\frac{1}{2}}} = \sup_{x,y\in\Omega_t} \frac{|f(x)-f(y)|}{r(x)^{\frac{1}{2}} + r(y)^{\frac{1}{2}} + |x-y|^{\frac{1}{2}}}.$$

This scales like the $\dot{C}^{\frac{1}{2}}$ norm, but it is weaker in that it only uses one derivative of r away from the free boundary.

The role of B will be to control the growth rate for our energies, while also allowing for a secondary dependence of the implicit constants on A.

1.5. The main results

Our main result is a well-posedness result for the compressible Euler evolution (1.4). However, it is more revealing to break the result down into several components. We begin with the uniqueness result, which requires least regularity.

Theorem 1 (Uniqueness). *For every Lipschitz initial data (r_0, v_0) satisfying the nondegeneracy condition $|\nabla r_0| > 0$ on Γ_0, system (1.4) admits at most one solution (r, v) in the class*

$$v \in C^1_x, \quad \nabla r \in \tilde{C}^{0,\frac{1}{2}}_x.$$

In other words, uniqueness holds in the class of solutions (r, v) for which B remains finite. One can further relax this to $B \in L^1_t$. We note that only the spatial regularity is specified in the theorem, as the time regularity can then be obtained from the equations. Also the nondegeneracy condition is only given at the initial time, but it can be easily propagated to later times given our regularity assumptions.

To the best of our knowledge, this is the first uniqueness proof for this problem which applies directly in the Eulerian setting, and also the first uniqueness result at low, scale-invariant[4] regularity.

Remark 1.1. The result in Theorem 1 can be seen as a subset of Theorem 5 in Section 4. There we go one step further, and prove that a suitable nonlinear distance between two solutions is propagated along the flow, under the same assumptions as in Theorem 1.

Next we consider the well-posedness question. Here we define the phase space

$$\mathbf{H}^{2k} = \{(r, v) \mid (r, v) \in \mathscr{H}^{2k}\}.$$

[4]Scale invariance corresponds to the assumption $B \in L^1_t$.

One should think of this in a nonlinear fashion, as an infinite-dimensional manifold, as the $\mathcal{H}^{2k}$ norms depend on Ω_t and thus on r. The topology on this manifold is discussed in the next section. Now we can state our main well-posedness result:

Theorem 2 (Well-posedness). *System (1.1) is locally well posed in the space $\mathbf{H}^{2k}$ for $k \in \mathbb{R}$ with*

$$2k > 2k_0 + 1. \tag{1.12}$$

The well-posedness result should be interpreted in a quasilinear fashion, i.e. including

- existence of solutions $(r, v) \in C[0, T; \mathbf{H}^{2k}]$;
- uniqueness of solutions in a larger class; see Theorem 1 above;
- weak Lipschitz dependence on the initial data, relative to a new, nonlinear distance functional introduced in Section 4;
- continuous dependence of the solutions on the initial data in the $\mathbf{H}^{2k}$ topology.

The last question we consider is that of continuation of the solutions, which is where our control norms are critically used. This is closely related to the energy estimates for our system:

Theorem 3. *For each integer $k \geq 0$ there exists an energy functional E^{2k} with the following properties:*

(a) *Coercivity: As long as[5] $A \ll 1$, we have*

$$E^{2k}(r, v) \approx \|(r, v)\|^2_{\mathcal{H}^{2k}}.$$

(b) *Energy estimates for solutions to (1.1):*

$$\frac{d}{dt} E^{2k}(r, v) \lesssim_A B\|(r, v)\|^2_{\mathcal{H}^{2k}}.$$

By Grönwall's inequality this implies the bound

$$\|(r, v)(t)\|^2_{\mathcal{H}^{2k}} \lesssim e^{\int_0^T C(A)B(s)\,ds}\|(r, v)(t)(0)\|^2_{\mathcal{H}^{2k}}. \tag{1.13}$$

Remark 1.2. These energies are constructed in an explicit fashion only for integer k. Nevertheless, as a consequence, in our analysis in the last section of the paper, it follows that bounds of the form (1.13) hold also for all noninteger $k > 0$. However, we do this using a mechanism which is akin to a paradifferential expansion, without constructing an explicit energy functional as provided by the above theorem in the integer case.

A consequence of the last result is the following continuation criteria for solutions to (1.1), which holds regardless of whether k is an integer:

[5] Recall that we can harmlessly assume A small.

Theorem 4. *Let k be as in (1.12). Then the $\mathbf{H}^{2k}$ solutions to (1.1) given by Theorem 2 can be continued for as long as A remains bounded and $B \in L^1_t$.*

Here we implicitly make a topological assumption and exclude the possibility that two gas bubbles at some point touch each other, or that the free boundary self-intersects. This latter possibility is prohibited at small scales by our result, but certainly not at large scales.

This result is consistent with the standard continuation results for quasilinear hyperbolic systems in the absence of a free boundary. But for the physical vacuum free boundary problem, this work is the *first* where anything close to such a continuation result has been proved.

1.6. Historical comments

The study of the compressible Euler evolutions has a long history, and also considerable interest from the physical side. Allowing for vacuum states introduces many added layers of difficulty to the problem, whose nature greatly depends on the behavior of the sound speed near the vacuum boundary. Within this realm, physical vacuum represents the natural boundary condition for compressible gases. Below we begin with a brief discussion of the broader context, and then we focus on the problem at hand.

1.6.1. Compressible Euler flows. The compressible Euler equations are classically considered as a symmetric hyperbolic system, and as such, local well-posedness has long been known; see e.g. [14] and also the Euler-oriented analysis in [20]. The local solutions can be obtained using the energy method, and relying solely on the energy requires initial data local regularity $(\rho_0, v_0) \in H^s$ with $s > \frac{d}{2} + 1$, with the continuation criteria

$$\int_0^\infty \|\nabla(\rho, v)\|_{L^\infty} < \infty.$$

By now it is known that these results can be improved by taking advantage of Strichartz estimates for wave equations. In the irrotational case, for instance, the result of [25] applies directly and yields the sharp local well-posedness result, for[6] $s > \frac{d+1}{2}$. In the rotational case, it is not yet clear what would be the optimal condition on the vorticity which would allow for a similar improvement; see the results in [9] and [28].

1.6.2. Vacuum states in compressible Euler flows. Vacuum states correspond to allowing for the density to vanish in some regions. Here, one should think of having a particle region Ω_t, and a vacuum region, separated by a moving *free boundary* $\Gamma_t = \partial\Omega_t$. There are two major physical scenarios, distinguished by the boundary behavior of the density ρ, or equivalently of the sound speed c_s:

(1) fluid flows, where the pressure is constant on the free boundary, describing a balance of forces, and the density and implicitly the sound speed are assumed to have a nonzero, positive limit there;

[6] Here, $d = 3, 4, 5$.

(2) gas flows, where the density decays to zero near the free boundary; this is our main focus in this paper.

Both are free boundary problems associated to compressible Euler, but their natures are very different. Fluid flows were considered in [4] and [16], and also the incompressible limit was investigated in [17].

Now we turn our attention to our present interest, namely the gas flows. Heuristically one distinguishes several potential scenarios when comparing the sound speed c_s with the distance d_Γ to the vacuum boundary:

(a) Rapid decay corresponds to

$$c_s \lesssim d_{\Gamma_t}.$$

In this case the vacuum boundary evolves linearly, and internal waves cannot reach the boundary arbitrarily fast. Thus this geometry persists at least for a short time, and the local well-posedness problem can even be studied using the standard tools of symmetric hyperbolic systems; see for instance [2,8,18], as well as the alternative approach in [1,21] and the one-dimensional analysis in [19]. Thus this case cannot be thought of as a true free boundary problem. Furthermore, after a finite time, the internal waves will reach the boundary [19], and this geometry breaks down.

(b) Slow decay corresponds to

$$c_s \gg d_{\Gamma_t}.$$

This is where the problem indeed becomes a genuine free boundary problem, as internal waves can reach the boundary arbitrarily fast, and then the flow of the free boundary becomes strongly coupled with the internal flow. One might think that there is a range of possible decay rates, for instance like

$$c_s \approx d_{\Gamma_t}^{\beta}, \quad 0 < \beta < 1.$$

However, both physical and mathematical considerations seem to indicate that among these there is a single stable decay rate, which corresponds to $\beta = \frac{1}{2}$. This is commonly referred to as *physical vacuum*. The other values of β are expected to be unstable, with the solutions instantly falling into the stable regime; but this is all a conjecture at this point, and likely there will be significant differences between the cases $\beta < \frac{1}{2}$ and $\beta > \frac{1}{2}$.

1.6.3. The physical vacuum scenario. We turn now our attention to the problem at hand, i.e. the physical vacuum scenario. The easier one-dimensional setting was considered first, in [6] followed by [12]. While some energy estimates are formally obtained in [6] and a procedure to construct solutions is provided, the functional structure there does not provide a direct description of the initial data space. This issue is remedied in [12], which first introduces the Lagrangian counterparts of the scale of spaces we are also using here, and provides both existence and uniqueness results in sufficiently regular spaces.

More recently, the three-dimensional case was considered in several papers. Energy estimates for $\kappa = 1$ were formally derived in [5]. This was followed by an existence proof proposed in [7], which is based a parabolic regularization. However, the functional setting is similar to their prior one-dimensional paper, and some steps are merely claimed rather than proved, for instance the difference bound, which also, as stated, requires additional regularity for the solutions compared to the existence result. Independently, [13] offers an alternative existence and uniqueness proof for arbitrary $\kappa > 0$, this time within the correct scale of weighted Sobolev spaces, using an iterative argument for the existence part, and with a different approach to the energy estimates.

All the results described above are in the Lagrangian setting, and aim to give existence and uniqueness results in sufficiently regular function spaces. In addition to the limitations mentioned above, no attempt is made to provide any continuous dependence results, nor to transfer the results to the physical, Eulerian framework.

By contrast, our results in the present paper are fully developed within the Eulerian setting, at low regularity, in all dimensions and for all $\kappa > 0$. In this context we provide completely new arguments for existence, uniqueness, and continuous dependence of the solutions on the initial data, i.e. a full well-posedness theory in the Hadamard sense. In addition, we prove a family of sharp, scale-invariant energy estimates, which in particular yield optimal continuation criteria at the level of $\|\nabla v\|_{L^\infty}$, consistent with the well-known results for hyperbolic systems in the absence of the free boundary. Despite the fact that we only construct energy functionals corresponding to integer Sobolev spaces, we nevertheless are able to use these estimates to obtain energy estimates in fractional Sobolev spaces as well, nicely completing the theory up to the optimal Sobolev thresholds.

1.7. An outline of the paper

The article has a modular structure, where, for the essential part, only the main results of each section are used later.

1.7.1. Function spaces and interpolation. The starting point, in the next section, is to describe the appropriate functional setting for our analysis, represented by the $\mathcal{H}^{2k}$ scale of weighted Sobolev spaces. These are associated to the singular Riemannian metric (1.8) under the sole assumption that the boundary Γ_t is Lipschitz, with r as a nondegenerate defining function. A similar scale of spaces was introduced in [13] in the Lagrangian setting, though under more regularity assumptions. However, since in the Eulerian setting the boundary is moving, the corresponding state space $\mathbf{H}^{2k}$ for (r, v) is seen here akin to an infinite-dimensional manifold.

We remark on the dual role of r, as a defining function of the boundary implicitly as a weight on one hand, and as one of the dynamical variables on the other hand; for our low-regularity analysis we carefully decouple these two roles, in order to avoid cumbersome bootstrap loops.

Interpolation plays a significant role in our study. First, this occurs at the level of the $\mathcal{H}^{2k}$ scale of spaces, and it allows us to work with fractional Sobolev spaces without having to directly prove energy estimates in the fractional setting, using expansions which are akin to paradifferential ones but done at the level of the nonlinear flow. Second, we also interpolate between the $\mathcal{H}^{2k}$ spaces and the pointwise bounds captured by our control parameters A and B. It is this last tool which allows us to work at low regularity and to obtain sharp, scale-invariant energy estimates.

1.7.2. The linearized equation and transition operators. In Section 3 we consider the linearized equation, which is modeled as a linear evolution in the time-dependent weighted L^2 space $\mathcal{H}$. We view this as the main tool in the analysis of the nonlinear evolution, rather than the direct nonlinear energy estimates as in all prior work (except for [13], to some extent). This later helps us not only to prove nonlinear energy estimates for single solutions, but also to compare different solutions, which is critical both for our uniqueness proof and for our construction of rough solutions as strong limits of smooth solutions. We remark that at the level of the linearized variables (s, w), there is no longer any boundary condition on the moving free boundary Γ_t; this is closely related to the prior comment about uncoupling the roles of r.

Next, using the linearized equation, we obtain the *transition operators* L_1 and L_2, which act at the level of the two linearized variables s, respectively w, and should be though of as the degenerate elliptic leading spatial part of the wave evolution for s, respectively $\nabla \cdot w$. We call them transition operators because they tie the successive spaces $\mathcal{H}^{2k}$ and H^{2k+2} on our scale in a coercive, invertible fashion. These operators play a leading role in both the higher-order energy estimates and in the regularization used for our construction of regular solutions.

1.7.3. Difference estimates and the uniqueness result. The aim of Section 4 is to construct a nonlinear difference functional which allows us to track the distance between two solutions roughly at the level of the $\mathcal{H}$ norm. This is akin to the difference bounds in a weaker topology which are common in the study of quasilinear problems.

This is one of the centerpieces of our analysis, and to the best of our knowledge this is the first time such a construction has been successfully carried out in a free boundary setting. The fundamental difficulty is that we are seeking to not only compare functions on different domains, but also to track the evolution in time of this distance. This difficulty is translated into the nonlinear character of our difference functional; some delicate, careful choices are made there, which ultimately allow us to propagate this distance forward in time.

1.7.4. Higher-order energy estimates. The aim of Section 5 is to establish energy estimates in integer-index Sobolev spaces on our $\mathcal{H}^{2k}$ scale. We define the nonlinear energy functionals E^{2k} using suitable vector fields applied to the equation. This energy has two components, a wave component and a transport component, which correspond to the heuristic (partial) decoupling of the evolution into a wave part for r and $\nabla \cdot v$ and a transport part for the vorticity ω. Our proof of the energy estimates is split in a modular fashion into two parts, where we successively (i) prove the coercivity of our energy functional and (ii) track the time evolution of the energy.

The coercivity bound is obtained inductively in k, using the transition operators L_1 and L_2 as key tools. The main part of the proof of the propagation bound happens at the level of the wave component, where we identify Alihnac-style "good variables" (s_{2k}, w_{2k}), which are shown to solve the linearized equation modulo perturbative source terms.

Our energy functionals are to some extent the Eulerian counterparts of energies previously constructed in [7, 13] in the Lagrangian setting and at higher regularity. They are closer in style to [7], though the coercivity part is largely missing there and as a consequence some of the functional setting is incomplete/incorrect. The analysis in [13], on the other hand, corresponds to combining the two steps above. This leads to a more comprehensive energy functional, where the coercivity part is relatively straightforward, but instead moves the difficulty to the propagation part, which becomes considerably more complex.

1.7.5. Existence of regular solutions. The aim of Section 6 is to prove the existence theorem in the context of regular solutions. The scheme we propose here is constructive, using a time discretization via an Euler-type method to produce good approximate solutions. However, a naive implementation of Euler's method loses derivatives; to rectify this we precede the Euler step by (i) a regularization on a suitable scale and (ii) a separate transport part.[7] The challenge is to control the energy growth at each step of the way. This is more delicate for the regularization, which has to be done carefully using the elliptic transition operators L_1 and L_2.

We note that our construction is very different from any other approaches previously used in analyzing this problem; they all relied on parabolic regularizations. Our construction is simpler and more direct, though not without interesting subtleties. It is also better tailored to the physical structure of the equations, which makes this approach more robust and also successful in the relativistic counterpart of our problem.

1.7.6. Rough solutions as limits of regular solutions. The last section of the paper aims to construct rough solutions as strong limits of smooth solutions. This is achieved by considering a family of dyadic regularizations of the initial data, which generates corresponding smooth solutions. For these smooth solutions we control on one hand higher Sobolev norms $\mathcal{H}^{2N}$, using our energy estimates, and on the other hand the L^2-type distance between consecutive ones, which is at the level of the $\mathcal{H}$ norms. Combining the high- and the low-regularity bounds directly yields rapid convergence in all $\mathbf{H}^{2k_1}$ spaces below the desired threshold, i.e. for $k_1 < k$. To gain strong convergence in $\mathbf{H}^{2k}$ we use frequency envelopes to more accurately control both the low and the high Sobolev norms above. This allows us to bound differences in the strong $\mathbf{H}^{2k}$ topology. A similar argument yields continuous dependence of the solutions in terms of the initial data also in the strong topology, as well as our main continuation result in Theorem 4.

[7] This bit is optional but does simplify the analysis.

2. Function spaces

The aim of this section is to introduce the main function spaces where we will consider the free boundary problem for the compressible gas. These are Sobolev-type spaces of functions inside the gas domain Ω_t, with weights depending on r, or equivalently on the distance to the free boundary. We begin with a more general discussion of weighted Sobolev spaces in Lipschitz domains, and then specialize to the function spaces that are needed in our problem.

2.1. Weighted Sobolev spaces

As a starting point, in a domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary Γ and nondegenerate defining function r we introduce a two-parameter family of weighted Sobolev spaces (see [26, 27] for a more general take on this):

Definition 2.1. Let $\sigma > -\frac{1}{2}$ and $j \geq 0$. Then the space $H^{j,\sigma} = H^{j,\sigma}(\Omega)$ is defined as the space of all distributions in Ω for which the following norm is finite:

$$\|f\|_{H^{j,\sigma}}^2 := \sum_{|\alpha| \leq j} \|r^\sigma \partial^\alpha f\|_{L^2}^2.$$

By complex interpolation, one also defines corresponding fractional Sobolev spaces $H^{s,\sigma}$ for $s \geq 0$ and $\sigma > -\frac{1}{2}$. This yields a double family of interpolation spaces.

Some comments are in order here:

- At this point, all we assume about the geometry of the problem is that the boundary Γ is Lipschitz, and that r is a nondegenerate defining function for Γ, i.e. proportional to the distance to Γ. Different choices for r yield the same space with different but equivalent norms. Without any restriction in generality, we can assume that r is Lipschitz continuous.
- The requirement $\sigma > -\frac{1}{2}$ corresponds to the fact that no vanishing assumptions on the boundary Γ are made for any of the elements in our function spaces.
- If $\sigma = 0$ then one recovers the classical Sobolev spaces $H^{k,0} = H^k$.
- If $j = 0$ these are weighted L^2 spaces, $H^{0,\sigma} = L^2(r^{2\sigma})$.

Next, we establish some key properties of these spaces. First, we have the Hardy-type embeddings (see the book [15] for a broader view):

Lemma 2.2. *Assume that $s_1 > s_2 \geq 0$ and $\sigma_1 > \sigma_2 > -\frac{1}{2}$ with $s_1 - s_2 = \sigma_1 - \sigma_2$. Then we have*

$$H^{s_1,\sigma_1} \subset H^{s_2,\sigma_2}.$$

Proof. By interpolation and reiteration it suffices to prove the result when $s_1 - s_2 = 1$, both integers. Thus we will show that

$$H^{j,\sigma} \subset H^{j-1,\sigma-1}, \quad j \geq 1,\ \sigma > \frac{1}{2}.$$

It suffices to prove the result in dimension $n = 1$; then all the higher dimensions will follow by considering foliations of Ω with parallel one-dimensional lines which are transversal to Γ.

Here, r is the distance function to the boundary of Ω. Setting $\Omega = [0, \infty)$, r is pointwise equivalent to x, and in particular gives

$$\int_{\Omega} (r^{\sigma-1})^2 |\partial_x^{j-1} f|^2 \, dx \approx \int_{\Omega_t} (x^{\sigma-1})^2 |\partial_x^{j-1} f|^2 \, dx.$$

The inclusion follows from the following integration by parts:

$$\begin{aligned}\int_{\Omega} (x^{\sigma-1})^2 |\partial_x^{j-1} f|^2 \, dx &= \int_{\Omega_t} \Big(\frac{x^{2\sigma-1}}{2\sigma-1}\Big)' |\partial_x^{j-1} f|^2 \, dx \\ &= \partial_x^{j-1} f|^2 \Big(\frac{x^{2\sigma-1}}{2\sigma-1}\Big)\Big|_{x\in\partial\Omega} \\ &\quad - \frac{2}{2\sigma-1} \int_{\Omega} x^{2\sigma-1} |\partial_x^{j-1} f| \, |\partial_x^{j} f| \, dx.\end{aligned}$$

The boundary term vanishes, and we can now apply the Cauchy–Schwarz inequality to obtain

$$\|f\|_{H^{j-1,\sigma-1}} \le \frac{2}{2\sigma-1} \|f\|_{H^{j,\sigma}}. \qquad \blacksquare$$

As a corollary of the above lemma we have embeddings into standard Sobolev spaces:

Lemma 2.3. *Assume that* $\sigma > 0$ *and* $\sigma \le j$. *Then we have*

$$H^{j,\sigma} \subset H^{j-\sigma}. \tag{2.1}$$

In particular, by standard Sobolev embeddings, we also have Morrey-type embeddings into C^s spaces:

Lemma 2.4. *We have*

$$H_r^{j,\sigma} \subset C^s, \quad 0 \le s \le j - \sigma - \frac{d}{2},$$

where the equality can hold only if s *is not an integer.*

2.2. Weighted Sobolev norms for compressible Euler

Our starting point here is the conserved energy for our problem, namely

$$E(r, v) = \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} \Big(r^2 + \frac{\kappa+1}{2} r v^2\Big) \, dx.$$

Even more importantly, in our study of the linearized equation (see Section 3), for linearized variables (s, w) we use the weighted L^2-type energy functional

$$E_{\text{lin}}(s, w) = \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} (|s|^2 + \kappa r |w|^2) \, dx.$$

Based on this, we define our baseline space $\mathcal{H}$ with norm

$$\|(s, w)\|_{\mathcal{H}}^2 = E_{\text{lin}}(s, w).$$

In terms of the $H^{s,\sigma}$ spaces discussed earlier, or weighted L^2 spaces, we have

$$\mathcal{H} = H^{0,\frac{1-\kappa}{2\kappa}} \times H^{0,\frac{1}{2\kappa}} = L^2(r^{\frac{1-\kappa}{\kappa}}) \times L^2(r^{\frac{1}{\kappa}}).$$

Next we define a suitable scale of higher-order Sobolev spaces for our problem. To understand the balance between weights and derivatives we consider the leading-order operator, if we write the wave part of our system as a second-order equation for r. At leading order this yields the wave operator

$$D_t^2 - \kappa r \Delta,$$

which is naturally associated with the Riemannian metric (1.8) in Ω_t.

So, to the above L^2-type space $\mathcal{H}$ we need to add Sobolev regularity based on powers of $r\Delta$, or equivalently, relative to the metric g defined above. Hence we define the higher-order Sobolev spaces $\mathcal{H}^{2k}$,

$$\mathcal{H}^{2k} := H^{2k,k+\frac{1-\kappa}{2\kappa}} \times H^{2k,k+\frac{1}{2\kappa}}, \quad k \geq 0$$

of pairs functions defined inside Ω_t. These form a one-parameter family of interpolation spaces. The $\mathcal{H}^{2k}$ spaces have the obvious norm if k is a nonnegative integer; for instance one can set

$$\|(s, w)\|_{\mathcal{H}^{2k}}^2 := \sum_{|\beta| \leq 2k}^{|\beta|-\alpha \leq k} \|r^\alpha \partial^\beta (s, w)\|_{\mathcal{H}}^2,$$

where α is also restricted to nonnegative integers.

On the other hand, if k is not an integer then the corresponding norms are Hilbertian norms defined by interpolation. Since in the Hilbertian case all interpolation methods yield the same result, for the $\mathcal{H}^{2k}$ norm we will use a characterization which is akin to a Littlewood–Paley decomposition, or to a discretization of the J method of interpolation. Precisely, we have the following lemma:

Lemma 2.5. *Let* $0 < k < N$. *Then* $\mathcal{H}^{2k}$ *can be defined as the space of distributions* (s, v) *which admit a representation*

$$(s, w) = \sum_{l=0}^{\infty} (s_l, w_l)$$

with the property that the following norm is finite:

$$|||\{(s_l, w_l)\}|||^2 := \sum_{l=0}^{\infty} 2^{2kl} \|(s_l, w_l)\|_{\mathcal{H}}^2 + 2^{2l(k-N)} \|(s_l, w_l)\|_{\mathcal{H}^{2N}}^2, \tag{2.2}$$

and with equivalent norm defined as

$$\|(s, w)\|_{\mathcal{H}^{2k}}^2 := \inf |||\{(s_l, w_l)\}|||^2,$$

where the infimum is taken with respect to all representations as above.

2.3. The state space $\mathbf{H}^{2k}$

As already mentioned in the introduction, the state space $\mathbf{H}^{2k}$ is defined for $k > k_0$ (i.e. above scaling) as the set of pairs of functions (r, v) defined in a domain Ω_t in $\mathbb{R}^n$ with boundary Γ_t with the following properties:

(i) Boundary regularity: Γ_t is a Lipschitz surface.

(ii) Nondegeneracy: r is a Lipschitz function in $\overline{\Omega}_t$, positive inside Ω_t, and vanishing simply on the boundary Γ_t.

(iii) Regularity: The functions (r, v) belong to $\mathcal{H}^{2k}$.

Since the domain Ω_t itself depends on the function r, one cannot think of $\mathbf{H}^{2k}$ as a linear space, but rather as an infinite-dimensional manifold. As time varies in our evolution, so does the domain, so we are interested in allowing the domain to vary in $\mathbf{H}^{2k}$. However, describing a manifold structure for $\mathbf{H}^{2k}$ is beyond the purposes of our present paper, particularly since the trajectories associated with our flow are merely expected to be continuous with values in $\mathbf{H}^{2k}$. For this reason, here we will limit ourselves to defining a topology on $\mathbf{H}^{2k}$.

Definition 2.6. A sequence (r_n, v_n) converges to (r, v) in $\mathbf{H}^{2k}$ if the following conditions are satisfied:

(i) Uniform nondegeneracy: $|\nabla r_n| \geq c > 0$.

(ii) Domain convergence: $\|r_n - r\|_{\mathrm{Lip}} \to 0$.

(iii) Norm convergence: For each $\varepsilon > 0$ there exist smooth functions $(\tilde{r}_n, \tilde{v}_n)$ in Ω_n, respectively $(\tilde{r}, \tilde{v})$ in Ω so that

$$(\tilde{r}_n, \tilde{v}_n) \to (\tilde{r}, \tilde{v}) \quad \text{in } C^\infty,$$

while

$$\|(\tilde{r}_n, \tilde{v}_n) - (r_n, v_n)\|_{\mathcal{H}^{2k}(\Omega_n)} \leq \varepsilon.$$

We remark that the last condition in particular provides both a uniform bound for the sequence (r_n, v_n) in $\mathcal{H}^{2k}(\Omega_n)$, as well as an equicontinuity-type property, which ensures that a nontrivial portion of their $\mathcal{H}^{2k}$ norms cannot concentrate on thinner layers near the boundary. This is akin to the conditions in the Kolmogorov–Riesz theorem for compact sets in L^p spaces.

This definition will enable us to achieve two key properties of our flow:

- Continuity of solutions (r, v) as functions of t with values in $\mathbf{H}^{2k}$.
- Continuous dependence of solutions $(r, v) \in C_t\mathbf{H}^{2k}$ on the initial data $(r_0, v_0) \in \mathbf{H}^{2k}$.

2.3.1. Sobolev spaces and control norms. An important threshold for our energy estimates corresponds to the uniform control parameters A and B given by (1.10) and (1.11), respectively. Of these, A is at scaling, while B is one-half of a derivative above scaling.

Thus, by Lemma 2.4 we will have the bounds

$$A \lesssim \|(r, v)\|_{\mathbf{H}^{2k}}, \quad k > k_0 = \frac{d+1}{2} + \frac{1}{2\kappa}, \tag{2.3}$$

respectively

$$B \lesssim \|(r, v)\|_{\mathbf{H}^{2k}}, \quad k > k_0 + \frac{1}{2} = \frac{d+2}{2} + \frac{1}{2\kappa}. \tag{2.4}$$

2.3.2. The regularity of the free boundary. Another property to consider for our flow, in dimension $n \geq 2$, is the regularity of the free boundary, as well as the regularity of the velocity restricted to the free boundary. This is given by trace theorems and the embedding (2.1):

Lemma 2.7. *Suppose that $(r, v) \in \mathbf{H}^{2k}$ and that $2k - \frac{1}{\kappa}$ is not an even integer. Then Γ_t has regularity*

$$\Gamma_t \in H^{k - \frac{1}{2\kappa}}.$$

If in addition $\frac{1}{\kappa}$ is also not an odd integer then the velocity restricted to Γ_t has class

$$v \in H^{\frac{k-1}{2} - \frac{1}{2\kappa}}(\Gamma_t).$$

These properties are provided here only for comparison purposes, and play no role in the sequel. This is because in this problem one cannot view the evolution of the free boundary as a stand-alone flow, not even at leading order. In particular, a priori this velocity does not suffice to transport the regularity of Γ_t; instead the boundary evolution should be viewed as a part of the interior evolution. Indeed, we will see that there is some interesting cancellation arising from the structure of the equations which facilitates this.

2.4. Regularization and good kernels

An important ingredient in our construction of solutions to our free boundary evolution is to have good regularization operators associated to each dyadic frequency scale 2^h, $h \geq 0$. These operators will need to accomplish two distinct goals:

- Fixed domain regularization: Given $(s, v) \in \mathcal{H}^{2k}(\Omega)$, construct regularizations (s^h, w^h) within the same $\mathcal{H}^{2j}(\Omega)$ scale of spaces.
- State and domain regularization: Given $(r, v) \in \mathbf{H}^{2k}$, where the first component defines a domain Ω, construct regularizations (r^h, v^h) within the $\mathbf{H}^{2j}$ scale of spaces, where the regularized domains Ω_h are defined by r^h, $\Omega_h := \{x \in \mathbb{R}^d \mid r^h(x) > 0\}$.

We begin with some heuristic considerations and notation. Given a dyadic frequency scale h, our regularizations will need to select frequencies ξ with the property that $r\xi^2 \lesssim 2^{2h}$, which would require kernels on the scale

$$\delta x \approx r^{\frac{1}{2}} 2^{-h}.$$

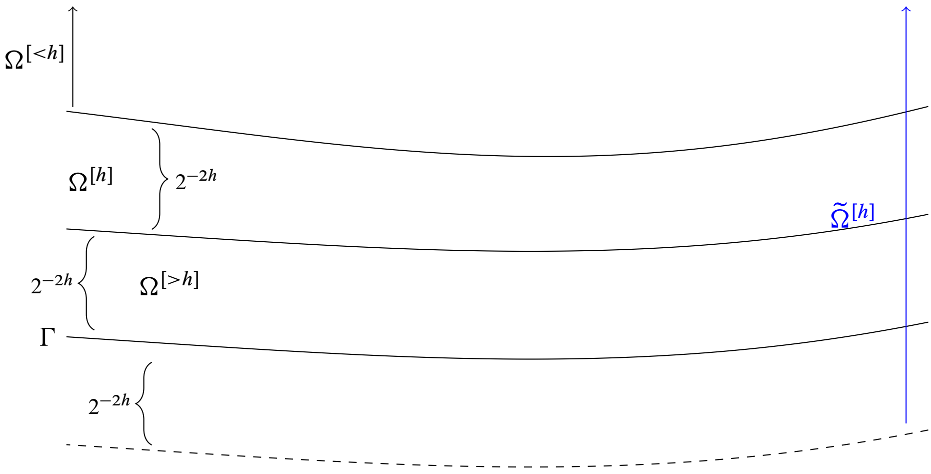

Figure 1. Boundary layers associated to frequency scale 2^h.

However, if we are too close to the boundary, i.e. $r \ll 2^{-2h}$, then we run into trouble with the uncertainty principle, as we would have $\delta x \gg r$. Because of this, we select the spatial scale $r \lesssim 2^{-2h}$ and the associated frequency scale 2^{2h} as cutoffs in this analysis.

To describe this process, it is convenient to decompose a neighborhood of the boundary Γ into boundary layers. Using Figure 1, we denote the dyadic boundary layer associated to the frequency 2^h by

$$\Omega^{[h]} = \{x \in \Omega,\ r(x) \approx 2^{-2h}\},$$

the corresponding full boundary strip by

$$\Omega^{[>h]} = \{x \in \Omega,\ r(x) \lesssim 2^{-2h}\},$$

and the corresponding interior region by

$$\Omega^{[<h]} = \{x \in \Omega,\ r(x) \gtrsim 2^{-2h}\}.$$

We will also use dyadic enlargements of Ω, denoted by

$$\widetilde{\Omega}^{[h]} = \{x \in \mathbb{R}^d,\ d(x, \Omega) \leq c2^{-2h}\},$$

with a small universal constant c, and

$$\widetilde{\Omega}^{[>h]} = \{x \in \mathbb{R}^d,\ d(x, \Gamma) \leq c2^{-2h}\}.$$

Given a domain Ω with a nondegenerate Lipschitz defining function r, and (s, w) functions in Ω, we will define regularizations (s^h, w^h) associated to the h dyadic scale using smooth kernels K^h,

$$(s^h, w^h)(x) = \Psi^h(s, v) := \int K^h(x, y)(s, w)(y)\, dy.$$

The heuristic discussion above leads to the following notion of *good kernels*:

Definition 2.8. The family of kernels K^h are called good regularization kernels if the following properties are satisfied:

(i) Domain and localization:

$$K^h \colon \tilde{\Omega}^{[h]} \times \Omega \to \mathbb{R}$$

with support properties

$$\operatorname{supp} K^h \subset \{(x,y) \in \tilde{\Omega}^{[h]} \times \Omega^{<h},\ |x-y| \lesssim \delta y^h := 2^{-2h} + 2^{-h} r(y)^{\frac{1}{2}}\}.$$

(ii) Size and regularity:

$$|\partial_x^\alpha \partial_y^\beta K^h(x,y)| \lesssim (2^{-2h} + 2^{-h} r(y)^{\frac{1}{2}})^{-N-|\alpha|-|\beta|}, \quad |\alpha| + |\beta| \le 4N,$$

where N is large enough.

(iii) Approximate identity:

$$\int K^h(x,y)\, dy = 1,$$
$$\int K^h(x,y)(x-y)^\alpha\, dy = 0, \quad 1 \le |\alpha| \le 2N. \tag{2.5}$$

Notably, the first property will allow us to define the regularizations (s^h, w^h) in the extended domain $\tilde{\Omega}^{[h]}$, with dyadic mapping properties as follows:

- For $j < h$, the regularizations (s^h, w^h) in $\Omega^{[j]}$ are determined by (s, w) also in $\Omega^{[j]}$.
- For the h layers, the regularizations (s^h, w^h) in $\tilde{\Omega}^{[>h]}$ are determined by (s, w) only in $\Omega^{[h]}$.

Thus our regularization operators use their inputs only outside the 2^{-2h} boundary layer, but provide outputs in a 2^{-2h} enlargement of the domain Ω. Such a property is critical in order to have good domain regularization properties.

The role of the third property on the other hand is to ensure that polynomials of sufficiently small degree are reproduced by our regularizations. This will later provide good low-frequency bounds for differences of successive regularizations.

Regularization kernels with these properties can be easily constructed:

Lemma 2.9. *Good regularization kernels exist.*

Proof. We outline the steps in the kernel construction, leaving the details for the reader:

(a) We consider a unit vector e which is uniformly transversal to the boundary, outward oriented. Such an e can be chosen locally, and kernels constructed based on a local choice of e can be assembled using a partition of unity in the first variable.

(b) Given such an e, we consider a smooth bump function ϕ with properties as follows:

- the support of ϕ is such that

$$\operatorname{supp}\phi \subset B(e,\delta), \quad \delta \ll 1,$$

- its average is 1:

$$\int \phi(x)\,dx = 1,$$

- and it has zero moments

$$\int x^\alpha \phi(x)\,dx = 0, \quad 1 \le |\alpha| \lesssim N.$$

(c) For each dyadic scale m we consider a shifted regularizing kernel

$$K_0^m(x-y) = 2^{2md}\phi(2^{2m}(x-y))$$

on the 2^{-2m} scale, which is accurate to any order.

Correspondingly, we also consider a partition of unity in Ω,

$$1 = \sum_{m=0}^{\infty} \chi_m,$$

where the functions χ_m select the region $\Omega^{[m]}$ and are smooth on the 2^{-2m} scale. Given a fixed dyadic scale h, we adapt this partition of unity to h,

$$1 = \chi_{>h} + \sum_{m=0}^{h} \chi_m,$$

where the first term $\chi_{>h}$ can be extended by 1 to the exterior of Ω.

(d) We define the regularization kernels

$$K^h(x,y) := \chi_{>h}(x)K_0^h(x-y) + \sum_{m=0}^{h} \chi_m(x)K_0^m(x-y),$$

which are still accurate to any order. It is easily verified that these kernels have the desired properties. ■

Next we prove bounds for our regularizations in $\mathcal{H}^{2k}$ spaces:

Proposition 2.10. *The following estimates hold for good regularization kernels whenever r_1 is a nondegenerate defining function with $|r - r_1| \ll 2^{-2h}$:*

(a) *Regularization bound:*

$$\|\Psi^h(s,w)\|_{\mathcal{H}_{r_1}^{2k+2j}} \lesssim 2^{2jh}\|(s,w)\|_{\mathcal{H}_r^{2k}}, \quad j \ge 0, \tag{2.6}$$

(b) *Difference bound:*

$$\|(\Psi^{h+1} - \Psi^h)(s,w)\|_{\mathcal{H}^{2k+2j}_{r_1}} \lesssim 2^{2jh} \|(s,w)\|_{\mathcal{H}^{2k}_r}, \quad -k \leq j \leq 0, \tag{2.7}$$

(c) *Error bound:*

$$\|(I - \Psi^h)(s,w)\|_{\mathcal{H}^{2k+2j}_r} \lesssim 2^{2jh} \|(s,w)\|_{\mathcal{H}^{2k}_r}, \quad -k \leq j \leq 0. \tag{2.8}$$

Here we recall that the regularized functions $\Psi^h(s,v)$ are defined on the larger domain $\tilde{\Omega}^{[h]}$. This is what allows us to measure them with respect to a perturbed domain $\Omega_1 = \{r_1 > 0\}$ as long as the two boundaries are within $O(2^{-2h})$ of each other.

Proof of Proposition 2.10. By interpolation we can assume that k and j are both integers. Because of the support properties of K_h, we can prove the desired estimate separately in each boundary layer $\Omega^{[l]}$, for $0 \leq l \leq h$, and then separately for $\tilde{\Omega}^{[>h]}$. For instance, in the case of (2.6) we will show that

$$\|\Psi^h(s,w)\|_{\mathcal{H}^{2k+2j}(\Omega^{[l]})} \lesssim 2^{2hj} \|(s,w)\|_{\mathcal{H}^{2k}(\Omega^{[l]})},$$

where the domain-restricted norms are interpreted as the square integral of the appropriate quantities over the restricted domains.[8]

The above localization allows us to fix the r-dependent localization scale $\delta x = 2^{-(h+l)}$ for Ψ^h, which becomes akin to a scaling parameter. Even better, we can localize further to a ball $B_{\delta x} \subset \Omega^{[l]}$ and show that

$$\|\Psi^h(s,w)\|_{\mathcal{H}^{2k+2j}(B_{\delta x})} \lesssim 2^{2jh} \|(s,w)\|_{\mathcal{H}^{2k}(2B_{\delta x})}.$$

Consider one component of the norm on the left, namely the maximal one, and show that

$$\|r_1^{k+j} \partial^{2(k+j)} \Psi^h(s,w)\|_{\mathcal{H}_{r_1}(B_{\delta x})} \lesssim \|r^k \partial^{2k}(s,w)\|_{\mathcal{H}_r(B_{\delta x})}.$$

To avoid distracting technicalities, consider first the case $l < h$, where the weights are constant and can be dropped. Then the above inequality becomes

$$\|\partial^{2(k+j)} \Psi^h u\|_{L^2(B_{\delta x})} \lesssim 2^{2j(h+l)} \|\partial^{2k} u\|_{L^2(2B_{\delta x})}. \tag{2.9}$$

The difficulty here is that we only have control over the derivatives of u (here u can be replaced by either s or w). We can bypass this difficulty using (a higher-order version of) the Poincaré inequality in $B_{\delta r}$, which allows us to find a polynomial P of degree $2k-1$ so that

$$\|\partial^b(u - P)\|_{L^2(B_{\delta x})} \lesssim \delta x^{2k-b} \|\partial^{2k} u\|_{L^2(B_{\delta x})}, \quad 0 \leq b < 2k.$$

[8]In a standard fashion, we also need to allow the domain on the right to be a slight enlargement of the domain on the left.

Property (2.5) shows that $K^h P = P$, therefore in (2.9) we can replace u by $u - P$, for which we have better control of the lower Sobolev norms. Then estimate (2.9) easily follows.

Minor adjustments to this argument are needed in $\Omega^{[h]}$. Then $\delta x \approx 2^{-2h}$, and we can still freeze r in the input region to $r = 2^{-2h}$. On the other hand, in the output region we have $r_1 \lesssim 2^{-2h}$, which still allows us to drop the r_1^k weight. The Poincaré inequality still applies. The only difference is that the weight in the $\mathcal{H}$ norm on the left might be singular. However, this weight is nevertheless square integrable near the boundary, which suffices due to the fact that in effect in $B_{\delta x}$ we can obtain pointwise control for $\partial^{2k+2j}\Psi^h(u - P)$.

Now we consider case (b). There the same localization applies, and the main difference in the proof is that now for a polynomial P of degree at most $2N$ we have

$$(\Psi^{h+1} - \Psi^h)P = 0.$$

This in turn allows us to also substitute u by $u - P$ in (2.9) when j is negative. The rest of the argument is identical.

Finally, for the bound (2.8) we simply add up (2.7) for scales $> h$. ■

Given a rough state $(r, v) \in \mathbf{H}^{2k}$, we can use the above lemma to construct a regularized state (r^h, v^h) as follows:

(a) We define the regularized functions (r^h, v^h) in the larger domain $\tilde{\Omega}^{[h]}$ by

$$(r^h, v^h) = \Psi^h(r, v).$$

(b) We restrict (r^h, v^h) to the set[9] $\Omega_h := \{r^h > 0\}$.

Such a strategy works provided that the domain $\tilde{\Omega}^{[h]}$ is large enough to allow r^h to transition to negative values before reaching the boundary of its domain. We will see that this is indeed true provided that k is above the scaling exponent k_0. Our main result is stated below. For better accuracy, we use the language of frequency envelopes to state it.

Proposition 2.11. *Assume that $k > k_0$. Then given a state $(r, v) \in \mathbf{H}^{2k}$, there exists a family of regularizations $(r^h, v^h) \in \mathbf{H}^{2k}$, so that the following properties hold for a slowly varying frequency envelope $c_h \in \ell^2$ which satisfies*

$$\|c_h\|_{\ell^2} \lesssim_A \|(r, v)\|_{\mathbf{H}^{2k}}. \tag{2.10}$$

(i) *Good approximation:*

$$(r^h, v^h) \to (r, v) \quad \text{in } C^1 \times C^{\frac{1}{2}} \text{ as } h \to \infty,$$

and

$$\|r^h - r\|_{L^\infty(\Omega)} \lesssim 2^{-2(k-k_0+1)h}. \tag{2.11}$$

[9] Here and below we use subscripts for Ω as in $\Omega_* = \{r^* > 0\}$ to indicate the domain associated to a function r^*, and the superscripts $\Omega^{[*]}$ to select various boundary layers.

(ii) *Uniform bound:*

$$\|(r^h, v^h)\|_{\mathbf{H}^{2k}} \lesssim_A \|(r, v)\|_{\mathbf{H}^{2k}}. \tag{2.12}$$

(iii) *Higher regularity:*

$$\|(r^h, v^h)\|_{\mathbf{H}_h^{2k+2j}} \lesssim 2^{2hj} c_h, \quad j > 0. \tag{2.13}$$

(iv) *Low-frequency difference bound:*

$$\|(r^{h+1}, v^{h+1}) - (r^h, v^h)\|_{\mathcal{H}_{\tilde{r}}} \lesssim 2^{-2hk} c_h, \quad |\tilde{r} - r| \ll 2^{-2h}. \tag{2.14}$$

Proof. To start with, we will assume that (r^h, v^h) are defined in the larger set $\tilde{\Omega}^{[h]}$ using good regularization kernels K_h,

$$(r^h, v^h) = \Psi^h(r, v).$$

By Sobolev embeddings we know that

$$(r, v) \in C^{1+k-k_0} \times C^{\frac{1}{2}+k-k_0}(\Omega).$$

This easily implies the uniform bound for (r^h, v^h) in $C^1 \times C^{\frac{1}{2}}(\tilde{\Omega}^{[h]})$, as well as the convergence in the same topology to (r, v) in Ω. It also implies the pointwise bound (2.11). This in turn shows that on the boundary Γ we have $|r^h| \lesssim 2^{-2(k-k_0+1)h}$, therefore the zero set $\Gamma_h = \{r^h = 0\}$ is within distance $2^{-2(k-k_0+1)h}$ from Γ, and thus within $\tilde{\Omega}^{[h]}$. This ensures that (r^h, v^h) restricted to $\Omega_h = \{r^h > 0\}$ is a well-defined state.

Next we consider the bound (2.12). In view of the difference bound (2.11), this is a consequence of (2.6) with $r_1 = r^h$ and $j = 0$.

It remains to prove (2.13) and (2.14). If we were to replace c_h by 1 on the right, this would also follow from Proposition 2.10. To gain the extra decay associated with a frequency envelope, for the functions (r, v) we will use the interpolation space representation given by Lemma 2.5 with N sufficiently large,

$$(r, v) = \sum_{l=0}^{\infty} (s_l, w_l), \tag{2.15}$$

for which the norm in (2.2) is finite. Accordingly, we can choose a slowly varying frequency envelope c_l so that

$$\|(s_l, w_l)\|_{\mathcal{H}} \leq 2^{-2lk} c_l, \quad \|(s_l, w_l)\|_{\mathcal{H}^{2N}} \leq 2^{2l(N-k)} c_l, \tag{2.16}$$

with

$$\sum c_l^2 \lesssim \|(r, v)\|_{\mathbf{H}^{2k}}^2.$$

The frequency envelope c_l above is the one we will use in the proposition. Property (2.10) is then automatically satisfied.

Proof of (2.13). Our starting point is again the decomposition (2.15)–(2.16) for (r, v), but now we separate the contributions of $l \leq k$ and $l > k$.

(a) Low-frequency components $l < k$: Using the Ψ^h bounds in Proposition 2.10, the bounds for (r_l, v_l) carry over to $\Psi^h(r_l, v_l)$, namely

$$\|\Psi^h(s_l, w_l)\|_{\mathcal{H}} \leq 2^{-2lk} c_l, \quad \|\Psi^h(s_l, w_l)\|_{\mathcal{H}^{2N}} \leq 2^{2l(N-k)} c_l.$$

Then by interpolation we have

$$\|\Psi^h(s_l, w_l)\|_{\mathcal{H}^{2k+2j}} \lesssim 2^{2lj} c_l. \tag{2.17}$$

(b) High-frequency components $l \geq k$: Here we discard the $\mathcal{H}^{2N}$ bound, and instead estimate directly

$$\|K_h(s_l, w_l)\|_{\mathcal{H}^{2k+2j}} \lesssim 2^{2h(j+k)} \|(s_l, w_l)\|_{\mathcal{H}} \lesssim 2^{2jh} 2^{2(h-l)j} c_l. \tag{2.18}$$

Combining (2.17) and (2.18), we obtain

$$\|K_h(r, v)\|_{\mathcal{H}^{2k+2j}} \lesssim \sum_{l \leq h} 2^{2lj} c_l + \sum_{l > h} 2^{2jh} 2^{2(h-l)j} c_l \lesssim c_h$$

as needed.

Proof of (2.14). We follow the same strategy as above, where we still can use all the Ψ^h bounds in Proposition 2.10, but with the difference that now we also have access to the difference bound in (2.7).

Starting with the decomposition (2.15)–(2.16) for (r, v), we observe that the $\mathcal{H}$ bound for (r_l, v_l) suffices in the high-frequency case $l \geq h$. It remains to consider the low-frequency case $l < h$, where we will have to rely instead on the $\mathcal{H}^{2N}$ norm. Precisely, by (2.7) we have

$$\|\partial_h K^h(r_l, v_l)\|_{\mathcal{H}} \lesssim 2^{-2Nh} \|(r_l, v_l)\|_{\mathcal{H}^{2N}}, \tag{2.19}$$

which again, combined with (2.16), suffices after dyadic l summation. ■

2.5. Interpolation inequalities

Next we consider L^p interpolation-type inequalities, which are critical to prove our sharp, scale-invariant energy estimates.

For clarity and later use we provide a more general interpolation result. Our main result, which applies in any Lipschitz domain Ω with a nondegenerate defining function r, is as follows:

Proposition 2.12. *Let $\sigma_0, \sigma_m \in \mathbb{R}$ and $1 \leq p_0, p_m \leq \infty$. Define*

$$\theta_j = \frac{j}{m}, \quad \frac{1}{p_j} = \frac{1-\theta_j}{p_0} + \frac{\theta_j}{p_m}, \quad \sigma_j = \sigma_0(1-\theta_j) + \sigma_m \theta_j,$$

and assume that

$$m-\sigma_m-d\Big(\frac{1}{p_m}-\frac{1}{p_0}\Big)>-\sigma_0,\quad \sigma_j>-\frac{1}{p_j}.$$

Then for $0<j<m$ *we have*

$$\|r^{\sigma_j}\partial^j f\|_{L^{p_j}}\lesssim \|r^{\sigma_0}f\|_{L^{p_0}}^{1-\theta_j}\|r^{\sigma_m}\partial^m f\|_{L^{p_m}}^{\theta_j}. \tag{2.20}$$

Remark 2.13. One particular case of the above proposition which will be used later is when $p_0=p_1=p_2=2$, with the corresponding relation between the exponents of the r^{σ_j} weights.

As the objective here is to interpolate between the L^2-type $\mathcal{H}^{m,\sigma}$ norm and L^∞ bounds, we will need the following straightforward consequence of Proposition 2.12:

Proposition 2.14. *Let* $\sigma_m>-\frac{1}{2}$ *and*

$$m-\sigma_m-\frac{d}{2}>0.$$

Define

$$\theta_j=\frac{j}{m},\quad \frac{1}{p_j}=\frac{\theta_j}{2},\quad \sigma_j=\sigma_m\theta_j.$$

Then for $0<j<m$ *we have*

$$\|r^{\sigma_j}\partial^j f\|_{L^{p_j}}\lesssim \|f\|_{L^\infty}^{1-\theta_j}\|r^{\sigma_m}\partial^m f\|_{L^2}^{\theta_j}.$$

We will also need the following two variations of Proposition 2.14:

Proposition 2.15. *Let* $\sigma_m>-\frac{1}{2}$ *and*

$$m-\frac{1}{2}-\sigma_m-\frac{d}{2}>0.$$

Define

$$\sigma_j=\sigma_m\theta_j,\quad \theta_j=\frac{2j-1}{2m-1},\quad \frac{1}{p_j}=\frac{\theta_j}{2}.$$

Then for $0<j<m$ *we have*

$$\|r^{\sigma_j}\partial^j f\|_{L^{p_j}}\lesssim \|f\|_{\dot C^{\frac{1}{2}}}^{1-\theta_j}\|r^{\sigma_m}\partial^m f\|_{L^2}^{\theta_j}.$$

And the second variation:

Proposition 2.16. *Let* $\sigma_m>\frac{m-2}{2}$ *and*

$$m-\frac{1}{2}-\sigma_m-\frac{d}{2}>0.$$

Define

$$\sigma_j = \sigma_m\theta_j - \frac{1}{2}(1-\theta_j), \quad \theta_j = \frac{j}{m}, \quad \frac{1}{p_j} = \frac{\theta_j}{2}.$$

Then for $0 < j < m$ *we have*

$$\|r^{\sigma_j}\partial^j f\|_{L^{p_j}} \lesssim \|f\|_{\tilde{C}^{0,\frac{1}{2}}}^{1-\theta_j} \|r^{\sigma_m}\partial^m f\|_{L^2}^{\theta_j}.$$

Here, the role of the lower bound on σ_m is to ensure that $\sigma_j > -\frac{1}{p_j}$ for all intermediate j, where the $j = 1$ constraint is the strongest.

We will use the last two propositions for (r, v), where the pointwise bound comes from the control norms A and B.

Proof of Proposition 2.12. We begin with several simplifications. First we note that it suffices to prove the case $m = 2$ and $j = 1$. Then the general case follows by reiteration. Indeed, the case $m = 2$ allows us to compare any three consecutive norms

$$\|r^{\sigma_{j+1}}\partial^{j+1} f\|_{L^{p_{j+1}}} \leq \|r^{\sigma_j}\partial^j f\|_{L^{p_j}}^{\frac{1}{2}} \|r^{\sigma_{j+2}}\partial^{j+2} f\|_{L^{p_{j+2}}}^{\frac{1}{2}},$$

and then the main estimates (2.20) follow from combining the above bounds.

A second simplification is to observe that we can also reduce the problem to the one-dimensional case, which we state in the following lemma:

Lemma 2.17. *Let* $p_j \in [1, \infty]$, *and* $\sigma_j \in \mathbb{R}$ *with* $j = \overline{0, 2}$, *so that*

$$\frac{1}{p_2} + \frac{1}{p_0} = \frac{2}{p_1}, \quad \text{and} \quad \sigma_0 + \sigma_2 = 2\sigma_1,$$

and with

$$2 - d\left(\frac{1}{p_2} - \frac{1}{p_0}\right) > \sigma_2 - \sigma_0, \quad \sigma_1 > -\frac{1}{p_1}.$$

Then the following inequality holds:

$$\|x^{\sigma_1}\partial f\|_{L^{p_1}} \lesssim \|x^{\sigma_0} f\|_{L^{p_0}}^{\frac{1}{2}} \|x^{\sigma_2}\partial^2 f\|_{L^{p_2}}^{\frac{1}{2}}. \tag{2.21}$$

To see that the n-dimensional case reduces to the one-dimensional case, we consider a constant vector field X which is transversal to the boundary, apply (2.21), with x replaced by r, on every X line Ω_y in Ω, where y denotes the transversal direction. We raise it to the power p and integrate in y. This yields

$$\begin{aligned}
\|r^{\sigma_1} Xf\|_{L^{p_1}(\Omega)}^{p_1} &\lesssim \int \|r^{\sigma_0} f\|_{L^{p_0}(\Omega_y)}^{\frac{p_1}{2}} \|r^{\sigma_2} X^2 f\|_{L^{p_2}(\Omega_y)}^{\frac{p_1}{2}} \, dy \\
&\lesssim \|r^{\sigma_0} f\|_{L^{p_0}(\Omega)}^{\frac{p_1}{2}} \|r^{\sigma_2} X^2 f\|_{L^{p_2}(\Omega)}^{\frac{p_1}{2}},
\end{aligned}$$

where at the second step we have used the Hölder inequality. The full n-dimensional bound is obtained by applying the above estimate for a finite number of vector fields X which (i) are transversal to the boundary and (ii) span $\mathbb{R}^n$. It remains to prove the last Lemma 2.17:

Proof of Lemma 2.17. This interpolation inequality is a weighted Gagliardo–Nirenberg–Sobolev inequality; see[23]. One main ingredient in the original proof given in [23] for the unweighted case, is the following inequality due to P. Ungar:

Proposition 2.18. *On an interval I, whose length is denoted by λ, one has*

$$\|u_x\|^{p_1}_{L^{p_1}(I)} \lesssim \lambda^{1+p_1-\frac{p_1}{p_2}} \|u_{xx}\|^{p_1}_{L^{p_2}(I)} + \lambda^{-(1+p_1-\frac{p_1}{p_2})} \|u\|^{p_1}_{L^{p_0}(I)},$$

where $p_j \in [1, \infty]$, $j = \overline{0, 2}$

The heuristic interpretation of Proposition (2.18) is that the average of the first derivative of a function is controlled by its pointwise values, and its variation is controlled by its second derivative. This observation yields the balance between the parameters m, σ_0, σ_1, and σ_2 in Lemma 2.21. We will use the same result here to prove (2.21).

The first step is to use a dyadic spatial decomposition of $\mathbb{R}^+$, such that the interval I in Proposition 2.18 is fully contained in a generic interval $[r, 2r]$, where $r = 2^k$, and $k \in \mathbb{Z}$. Using Proposition (2.18), we have

$$\begin{aligned} r^{\sigma_1 p_1} \|\partial f\|^{p_1}_{L^{p_1}(I)} &= \|x^{\sigma_1} \partial f\|^{p_1}_{L^{p_1}(I)} \\ &\lesssim r^{p_1(\sigma_1-\sigma_2)} \lambda^{1+p_1-\frac{p_1}{p_2}} \|x^{\sigma_2} \partial^2 f\|^{p_1}_{L^{p_2}(I)} \\ &\quad + r^{(\sigma_1-\sigma_0)p_1} \lambda^{-(1+p_1-\frac{p_1}{p_0})} \|x^{\sigma_0} f\|^{p_1}_{L^{p_0}(I)}. \end{aligned}$$

To get from this inequality to (2.21) it would be convenient to know that the last two terms in the above inequality are comparable in size. One can try to achieve this by increasing the size of the interval I until this is true. The difficulty is when this cannot be done without going past the dyadic interval size. So the natural strategy is to consider the dyadic decomposition of interval $[0, \infty]$ and compare the L^{p_2} and L^{p_0} norms in each of these dyadic intervals.

If on any such dyadic interval we get

$$\begin{aligned} &r^{p_1(\sigma_1-\sigma_2)+1+p_1-\frac{p_1}{p_2}} \|x^{\sigma_2} \partial^2 f\|^{p_1}_{L^{p_2}([r,2r])} \\ &\qquad \geq r^{(\sigma_1-\sigma_0)p_1-(1+p_1-\frac{p_1}{p_2})} \|x^{\sigma_0} f\|^{p_1}_{L^{p_0}([r,2r])} \end{aligned} \tag{2.22}$$

then we subdivide this interval into pieces where these two terms are comparable, and complete the proof of (2.21) within this interval.

Unfortunately this may not be the case in all dyadic subintervals. To rectify this we introduce slowly varying frequency envelopes $\{c^2_k\}$ for $\|x^{\sigma_2} \partial^2 f\|_{L^{p_2}}$, respectively $\{c^0_k\}$ for $\|x^{\sigma_0} f\|_{L^{p_0}}$, so that the following properties hold:

- Control norm:

$$\|x^{\sigma_2} \partial^2 f\|_{L^{p_2}([I_k])} \leq c^2_k \quad \text{and} \quad \|x^{\sigma_0} f\|_{L^{p_0}([I_k])} \leq c^0_k.$$

- l^{p_2} and l^{p_0} summability:

$$\sum_k (c_k^2)^{p_2} \approx \|x^{\sigma_2}\partial^2 f\|_{L^{p_2}}^{p_2} \quad \text{and} \quad \sum_k (c_k^0)^{p_0} \approx \|x^{\sigma_0} f\|_{L^{p_0}}^{p_0}.$$

- Slowly varying:

$$\frac{c_k^0}{c_j^0} \lesssim 2^{\delta|j-k|} \quad \text{and} \quad \frac{c_k^2}{c_j^2} \lesssim 2^{\delta|j-k|}$$

for δ small and positive.

Now, we compare again as in (2.22):

$$\begin{aligned} 2^{k\{p_1(\sigma_1-\sigma_2)+1+p_1-\frac{p_1}{p_2}\}}(c_k^2)^{p_1} &\geq 2^{k\{(\sigma_1-\sigma_0)p_1-(1+p_1-\frac{p_1}{p_2})\}}(c_k^0)^{p_1}, \\ 2^{k\{1+(\sigma_1-\sigma_2)+\frac{1}{p_1}-\frac{1}{p_2}\}}c_k^2 &\geq 2^{k\{(\sigma_1-\sigma_0)-1-(\frac{1}{p_1}-\frac{1}{p_2})\}}c_k^0, \end{aligned} \tag{2.23}$$

which holds if and only if

$$c_k^2 \geq 2^{k\{(\sigma_2-\sigma_0-2)+\frac{1}{p_2}-\frac{1}{p_0}\}}c_k^0.$$

In the dyadic regions where this holds we finish the proof, as discussed above, by subdividing the dyadic intervals and applying Proposition 2.18. To see where the switch happens we observe that c_k^2 is slowly varying whereas the right-hand side of the inequality above decreases exponentially, as k grows. Then we can find a unique k_0 where the two are comparable,

$$c_{k_0}^2 \approx 2^{k_0\{(\sigma_2-\sigma_0-2)+\frac{1}{p_2}-\frac{1}{p_0}\}}c_{k_0}^0. \tag{2.24}$$

Then (2.23) holds for $k \geq k_0$, which implies that

$$\|x^{\sigma_1}\partial f\|_{L^{p_1}(I_k)}^{p_1} \lesssim (c_k^0 c_k^2)^{\frac{p_1}{2}}. \tag{2.25}$$

It remains to consider the case when $k < k_0$, where we are simply going to obtain a pointwise bound for ∂f. Selecting a favorable point $x_0 \in I_{k_0}$, i.e. where

$$\partial f(x_0) \lesssim 2^{-k_0}\int_{I_{k_0}} |\partial f|\, dx \lesssim 2^{-(\frac{1}{p_1}+\sigma_1)k_0}\|x^{\sigma_1}\partial f\|_{L^{p_1}(I_{k_0})}, \tag{2.26}$$

we estimate for $x \in I_{k_1}$ with $k_1 < k_0$:

$$\begin{aligned} |\partial f(x)| &\lesssim |\partial f(x_0)| + \int_x^{x_0} |\partial^2 f|\, dx \\ &\lesssim |\partial f(x_0)| + \sum_{k=k_1}^{k_0}\int_{I_k} |\partial^2 f|\, dx \\ &\lesssim |\partial f(x_0)| + \sum_{k=k_1}^{k_0} 2^{k(\frac{p_2-1}{p_2}-\sigma_2)}\|x^{\sigma_2}\partial^2 f\|_{L^{p_2}(I_k)} \end{aligned}$$

$$
\begin{aligned}
&\lesssim |\partial f(x_0)| + \sum_{k=k_1}^{k_0} 2^{k(\frac{p_2-1}{p_2}-\sigma_2)}(c_k^2) \\
&\lesssim |\partial f(x_0)| + 2^{(k_0-k_1)(-\frac{p_2-1}{p_2}+\sigma_2+2\delta)_+} \cdot 2^{k_0(\frac{p_2-1}{p_2}-\sigma_2)}(c_k^2) \\
&\lesssim |\partial f(x_0)| + (x_0/x)^{(-\frac{p_2-1}{p_2}+\sigma_2+2\delta)_+} \cdot 2^{k_0(\frac{p_2-1}{p_2}-\sigma_2)}(c_k^2).
\end{aligned}
$$

Now we estimate using the bound above

$$
\begin{aligned}
\|x^{\sigma_1}\partial f\|_{L^{p_1}([0,x_0])}^{p_1} &= \int_0^{x_0} x^{p_1\sigma_1}(\partial f)^{p_1}\,dx \\
&\lesssim x_0^{p_1\sigma_1+1}|\partial f(x_0)|^{p_1} + 2^{k_0 p_1(\frac{p_2-1}{p_2}+\sigma_1-\sigma_2)} x_0(c_k^2),
\end{aligned}
\tag{2.27}
$$

where the integral converges since the exponents obey the restriction dictated by the scaling in (1.9), and δ is sufficiently small. To finish the proof we observe that by (2.24) and (2.26), the right-hand side of (2.27) is comparable to the right-hand side of (2.25) when $k = k_0$.

This concludes the proof of Lemma 2.17. ■

The proof of the Proposition 2.14 follows as a straightforward consequence. ■

Proof of Proposition 2.15. This is largely similar to the proof of Proposition 2.12, so we omit the details and only describe the key differences. The reduction to the case $m = 2$ is similar, using also the $m = 2$ case of Proposition 2.12, at least if we allow p_2 to be arbitrary rather than 2. The one-dimensional reduction is also similar. Thus we are left with having to prove the following analogue of Lemma 2.17:

Lemma 2.19. *Let $p_j \in [1,\infty]$, and $\sigma_j \in \mathbb{R}$ with $j = \overline{1,2}$, so that*

$$
\frac{1}{p_2} = \frac{3}{p_1} \quad \text{and} \quad \sigma_2 = 3\sigma_1,
$$

and with

$$
\frac{3}{2} - \frac{1}{p_2} > \sigma_2, \quad \sigma_1 > -\frac{1}{p_1}.
$$

Then the following inequality holds:

$$
\|x^{\sigma_1}\partial f\|_{L^{p_1}} \lesssim \|f\|_{\dot{C}^{\frac{1}{2}}}^{\frac{2}{3}} \|x^{\sigma_2}\partial^2 f\|_{L^{p_2}}^{\frac{1}{3}}.
$$

This lemma is proved using the following analogue of Proposition 2.18, which is a straightforward exercise:

Proposition 2.20. *On an interval I, whose length is denoted by λ, one has*

$$
\|u_x\|_{L^{p_1}(I)}^{p_1} \lesssim \lambda^{1+p_1-\frac{p_1}{p_2}} \|u_{xx}\|_{L^{p_2}(I)}^{p_1} + \lambda^{-\frac{1}{2}(1+p_1-\frac{p_1}{p_2})} \|u\|_{\dot{C}^{\frac{1}{2}}(I)}^{p_1},
$$

where $p_j \in [1,\infty]$, $j = \overline{0,2}$.

The proof of Proposition 2.15 is concluded. ■

Proof of Proposition 2.16. This is also similar to the proof of Proposition 2.12, so we omit the details and only describe the key differences. The reduction to the case $m = 2$ uses again the $m = 2$ case of Proposition 2.12, and the one-dimensional reduction is also similar. Thus we are left with having to prove the following analogue of Lemma 2.17:

Lemma 2.21. *Let $p_j \in [1, \infty]$, and $\sigma_j \in \mathbb{R}$ with $j = \overline{1,2}$, so that*

$$\frac{1}{p_2} = \frac{2}{p_1} \quad \textit{and} \quad \sigma_2 - \frac{1}{2} = 2\sigma_1,$$

and with

$$2 - \frac{d}{p_2} > \sigma_2 + \frac{1}{2}, \quad \sigma_1 > -\frac{1}{p_1}.$$

Then the following inequality holds:

$$\|x^{\sigma_1} \partial f\|_{L^{p_1}} \lesssim \|f\|^{\frac{1}{2}}_{\tilde{C}^{0,\frac{1}{2}}} \|x^{\sigma_2} \partial^2 f\|^{\frac{1}{2}}_{L^{p_2}}.$$

This lemma is proved in the same fashion as Lemma 2.17 directly using Proposition 2.18 for $f + c$ with well-chosen constants c. ■

3. The linearized equations

This section is devoted to the study of the linearized equations, which have the form

$$\begin{cases} \partial_t s + v \cdot \nabla s + w \cdot \nabla r + \kappa(s \nabla \cdot v + r \nabla \cdot w) = 0, \\ \partial_t w + (v \cdot \nabla) w + (w \cdot \nabla) v + \nabla s = 0. \end{cases} \tag{3.1}$$

Using the material derivative, these equations are written in the form

$$\begin{cases} D_t s + w \cdot \nabla r + \kappa(s \nabla \cdot v + r \nabla \cdot w) = 0, \\ D_t w + (w \cdot \nabla) v + \nabla s = 0. \end{cases} \tag{3.2}$$

Here, (s, w) are functions defined within the time-dependent gas domain Ω. Notably, no boundary conditions on (s, w) are imposed or required on the free boundary Γ.

3.1. Energy estimates and well-posedness

We first consider the question of proving well-posedness and energy estimates for the linearized equations:

Proposition 3.1. *Let (r, v) be a solution to the compressible Euler equations (1.4) in the moving domain Ω_t. Assume that both r and v are Lipschitz continuous, and that r*

vanishes simply on the free boundary. Then the linearized equation (3.2) *is well posed in* $\mathcal{H}$*, and the following energy estimate holds for all solutions* (s, w)*:*

$$\left| \frac{d}{dt} \|(s, w)\|_{\mathcal{H}}^2 \right| \lesssim \|\nabla v\|_{L^\infty} \|(s, w)\|_{\mathcal{H}}^2. \tag{3.3}$$

Here we estimate the absolute value of the time derivative of the linearized energy, in order to guarantee both forward and backward energy estimates; these are both needed to prove well-posedness.

Proof of Proposition 3.1. We recall the time-dependent weighted $\mathcal{H}$ norm,

$$\|(s, w)\|_{\mathcal{H}}^2 = \int r^{\frac{1-\kappa}{\kappa}} (|s|^2 + \kappa r |w|^2)\, dx.$$

To compute its time derivative, we use the material derivative in a standard fashion. For later reference we state the result in the following lemma:

Lemma 3.2. *Assume that the time-dependent domain* Ω_t *flows with Lipschitz velocity* v*. Then the time derivative of the time-dependent volume integral is given by*

$$\frac{d}{dt} \int_{\Omega(t)} f(t, x)\, dx = \int_{\Omega_t} D_t f + f \nabla \cdot v(t)\, dx.$$

Using the above lemma, we compute

$$\begin{aligned} \frac{d}{dt} \|(s, w)\|_{\mathcal{H}}^2 = -\kappa \int_{\Omega_t} & r^{\frac{1-k}{k}} \nabla v (|s|^2 + 2r|w|^2)\, dx \\ - 2 \int_{\Omega_t} & r^{\frac{1-\kappa}{\kappa}} (s(w \cdot \nabla r + \kappa r \nabla \cdot w) + \kappa r w \nabla s)\, dx. \end{aligned}$$

We observe that the last integral is zero. The computation is straightforward and follows from integration by parts:

$$-2 \int r^{\frac{1-\kappa}{\kappa}} (sw \cdot \nabla r + \kappa r \nabla (sw))\, dx = 0,$$

as the boundary terms vanish on Γ.

The first integral includes the bounded term $\nabla \cdot v$. It follows right away that the energy norm will indeed control it, and the desired energy estimate (3.3) follows.

The well-posedness result will follow in a standard fashion from a similar estimate for the adjoint equation, interpreted as a backward evolution in the dual space $\mathcal{H}^*$. We identify $\mathcal{H}^* = \mathcal{H}$ by Riesz's theorem, with respect to the associated inner product in $\mathcal{H}$:

$$\langle (s, w), (\tilde{s}, \widetilde{w}) \rangle_{\mathcal{H}} = \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} (s\tilde{s} + \kappa r w \widetilde{w})\, dx,$$

Then the adjoint system associated to (3.1), with respect to this duality, is easily computed to be

$$\begin{cases} D_t\tilde{s} + \kappa r\nabla\widetilde{w} + \widetilde{w}\nabla r = 0, \\ D_t\widetilde{w} - \widetilde{w}\nabla v + \nabla\tilde{s} = 0. \end{cases} \tag{3.4}$$

Modulo bounded, perturbative terms, this is identical to the direct system (3.2), therefore the backward energy estimate for the adjoint problem (3.4) follows directly from (3.3). ■

In particular, we note that, due to translations in time and space symmetries, the linearized estimate applies to the functions $(s, w) = (\nabla r, \nabla v)$, as well as $(s, w) = (\partial_t r, \partial_t v)$.

3.2. Second-order transition operators

We remark that discarding the ∇v terms from the equations we obtain a reduced linearized equation,

$$\begin{cases} D_t s + w \cdot \nabla r + \kappa r\nabla \cdot w = 0, \\ D_t w + \nabla s = 0, \end{cases}$$

which is also well posed in $\mathcal{H}$. For many purposes it is useful to also rewrite the linearized equation as a second-order evolution. We will only seek to capture the leading part, up to terms of order 1. Starting from the above reduced linearized equation, we compute second-order equations where we discard the ∇v terms arising from commuting D_t and ∇.

Then for s we obtain the reduced second-order equation (which would be exact if v were constant)

$$D_t^2 s \approx L_1 s, \quad L_1 s = \kappa r\Delta s + \nabla r \cdot \nabla s,$$

which for $\kappa = 1$ yields

$$L_1 = \nabla r\nabla.$$

On the other hand, for w we similarly obtain

$$D_t^2 w \approx L_2 w, \quad L_2 w = \kappa\nabla(r\nabla \cdot w) + \nabla(\nabla r \cdot w).$$

The operators L_1 and L_2 will play an important role in the analysis of the energy functionals in the next section. An important observation is that they are symmetric operators in the L^2 spaces which occur in our energy functional E_{lin} and in the norm $\mathcal{H}$. For a more in-depth discussion we separate them:

Lemma 3.3. *Assume that r is Lipschitz continuous in the domain Ω, and nondegenerate on the boundary Γ. Then the operator L_1, defined as an unbounded operator in the Hilbert space $H^{0,\frac{1-\kappa}{\kappa}} = L^2(r^{\frac{1-\kappa}{\kappa}})$, with*

$$\mathcal{D}(L_1) := \left\{f \in L^2(r^{\frac{1-\kappa}{\kappa}}) \mid L_1 f \in L^2(r^{\frac{1-\kappa}{\kappa}}) \text{ in the distributional sense}\right\},$$

is a nonnegative, self-adjoint operator.

The proof is relatively standard and is left for the reader. Later in the paper, see Lemma 5.2, we prove that L_1 is coercive, and that it satisfies good elliptic bounds, which in particular will allow us to identify the domain of $L_2 + L_3$ as

$$D(L_1) = H^{2,\frac{1+\kappa}{2\kappa}},$$

which is the first component of the $\mathcal{H}^2$ space.

Next we turn our attention to the operator L_2. This is also a symmetric operator, this time in the space $L^2(r^{\frac{1}{\kappa}})$, which is the second component of $\mathcal{H}$. However, it lacks full coercivity as $L_2 w$ only controls the divergence of w. For this reason, we will match it with a second operator which controls the curl of w, namely

$$L_3 = \kappa r^{-\frac{1}{\kappa}} \operatorname{div} r^{1+\frac{1}{\kappa}} \operatorname{curl} = \kappa \operatorname{div} r \operatorname{curl} + \nabla r \operatorname{curl},$$

so that $L_2 L_3 = L_3 L_2 = 0$. Then the operator $L_2 + L_3$ has the right properties:

Lemma 3.4. *Assume that r is Lipschitz continuous in Ω, and nondegenerate on the boundary Γ. Then the operator $L_2 + L_3$, defined as an unbounded operator in the Hilbert space $L^2(r^{\frac{1}{\kappa}})$, with*

$$\mathcal{D}(L_2 + L_3) := \left\{ f \in L^2(r^{\frac{1}{\kappa}}) \mid (L_2 + L_3) f \in L^2(r^{\frac{1}{\kappa}}) \text{ in the distributional sense} \right\},$$

is a nonnegative, self-adjoint operator.

Later in the paper, as a consequence of Lemma 5.2, it follows that $L_2 + L_3$ is coercive, and that it satisfies good elliptic bounds, which in particular will allow us to identify the domain of $L_2 + L_3$ as

$$D(L_2 + L_3) = H^{2,\frac{1+3\kappa}{2\kappa}},$$

which is the second component of the $\mathcal{H}^2$ space.

Remark 3.5. For the most part, we will think of L_1 and L_2 in a paradifferential fashion, i.e. with the r-dependent coefficients localized at a lower frequency than the argument. The exact interpretation of this will be made clear later on.

4. Difference bounds and the uniqueness result

Our aim here is to prove L^2 difference bounds for solutions, which could heuristically be seen as integrated[10] versions of the estimates for the linearized equation in the previous section. As a corollary, this will yield the uniqueness result in Theorem 1.

[10] Along a one-parameter C^1 family of solutions.

For this we consider two solutions (r_1, v_1) and (r_2, v_2) for our main system (1.4), and seek to compare them. Inspired by the linearized energy estimate, we seek to produce a similar weighted L^2 bound for the difference

$$(s, w) = (r_1 - r_2, v_1 - v_2).$$

The first difficulty we encounter is that the two solutions may not have the same domain. The obvious solution is to consider the differences within the common domain,

$$\Omega = \Omega_1 \cap \Omega_2.$$

The domain Ω no longer has a C^1 boundary. However, if we assume that the two boundaries Γ_1 and Γ_2 are close in the Lipschitz topology, then Ω still has a Lipschitz boundary Γ which is close to C^1. To measure the difference between the two solutions on the common domain, we introduce the following distance functional:[11]

$$D_{\mathcal{H}}\big((r_1, v_1), (r_2, v_2)\big) = \int_{\Omega_t} (r_1 + r_2)^{\sigma-1}\big((r_1 - r_2)^2 + \kappa(r_1 + r_2)(v_1 - v_2)^2\big)\, dx,$$

where $\sigma = \frac{1}{\kappa}$ throughout the section. We remark that the weight $r_1 + r_2$ vanishes on Γ only at points where Γ_1 and Γ_2 intersect. Away from such points, both $r_1 + r_2$ and $r_1 - r_2$ are nondegenerate; precisely, we have

$$|r_1(x_0) - r_2(x_0)| = r_1(x_0) + r_2(x_0), \quad x_0 \in \Gamma_t.$$

Since both r_1 and r_2 are assumed to be uniformly Lipschitz and nondegenerate, it follows that this relation extends to a neighborhood of x_0,

$$|r_1(x) - r_2(x)| \approx r_1(x_0) + r_2(x_0), \quad |x - x_0| \ll r_1(x_0) + r_2(x_0).$$

Then the first term in $D_{\mathcal{H}}$ yields a nontrivial contribution in this boundary region:

Lemma 4.1. *Assume that r_1 and r_2 are uniformly Lipschitz and nondegenerate, and close in the Lipschitz topology. Then we have*

$$\int_{\Gamma_t} |r_1 + r_2|^{\sigma+2} d\sigma \lesssim D_{\mathcal{H}}\big((r_1, v_1), r_2, v_2)\big). \tag{4.1}$$

One can view the integral in (4.1) as a measure of the distance between the two boundaries, with the same scaling as $D_{\mathcal{H}}$.

Now we can state our main estimate for differences of solutions:

Theorem 5. *Let (r_1, v_1) and (r_2, v_2) be two solutions for system (1.4) in $[0, T]$, with regularity $\nabla r_j \in \widetilde{C}^{0,\frac{1}{2}}$, $v_j \in C^1$, so that r_j are uniformly nondegenerate near the boundary and close in the Lipschitz topology, $j = 1, 2$. Then we have the uniform difference bound*

$$\sup_{t\in[0,T]} D_{\mathcal{H}}\big((r_1, v_1)(t), (r_2, v_2)(t)\big) \lesssim D_{\mathcal{H}}\big((r_1, v_1)(0), (r_2, v_2)(0)\big).$$

[11] We do not prove or claim that this defines a metric.

We remark that

$$D_{\mathcal{H}}\big((r_1, v_1), (r_2, v_2)\big) = 0 \quad \text{iff} \quad (r_1, v_1) = (r_2, v_2).$$

Thus, our uniqueness result in Theorem 1 can be viewed as a consequence of the above theorem.

The remainder of this section is devoted to the proof of the theorem.

4.1. A degenerate difference functional

The distance functional $D_{\mathcal{H}}$ which was introduced above is effective in measuring the distance between the two boundaries because it is nondegenerate at the boundary. This, however, turns into a disadvantage when we seek to estimate its time derivative. For this reason, in the energy estimates for the difference it is convenient to replace it by a seemingly weaker functional, where the weights do vanish on the boundary. Our solution is to replace the $r_1 + r_2$ weights in $D_{\mathcal{H}}$ with symmetric expressions in r_1 and r_2, which agree to second order with $r_1 + r_2$ where $r_1 = r_2$, and also which vanish on $\Gamma_t = \partial\Omega_t$.

Precisely, we will consider the modified difference functional

$$\begin{aligned} &\widetilde{D}_{\mathcal{H}}\big((r_1, v_1), (r_2, v_2)\big) \\ &\quad := \int_{\Omega_t} (r_1 + r_2)^{\sigma-1}\big(a(r_1, r_2)(r_1 - r_2)^2 + \kappa b(r_1, r_2)(v_1 - v_2)^2\big)\, dx, \end{aligned} \tag{4.2}$$

where for now the weights a and b are chosen as follows as functions of $\mu = r_1 + r_2$ and $\nu = r_1 - r_2$:

(1) They are smooth, homogeneous, nonnegative functions of degree 0, respectively 1, even in ν, in the region $\{0 \leq |\nu| < \mu\}$.

(2) They are connected by the relation $\mu a = 2b$.

(3) They are supported in $\{|\nu| < \frac{1}{2}\mu\}$, with $a = 1$ in $|\nu| < \frac{1}{4}\mu$.

For almost all the analysis these conditions will suffice. Later, almost of the end of the section, we will add one additional condition, see (4.12), and show that such a condition can be enforced.

Our objective now is to compare the two difference functionals. Clearly $\widetilde{D}_{\mathcal{H}} \lesssim D_{\mathcal{H}}$. The next lemma proves the converse.

Lemma 4.2. *Assume that $A = A_1 + A_2$ is small. Then*

$$D_{\mathcal{H}}\big((r_1, v_1), (r_2, v_2)\big) \approx_A \widetilde{D}_{\mathcal{H}}\big((r_1, v_1), (r_2, v_2)\big). \tag{4.3}$$

Proof. We need to prove the "$\lesssim$" inequality. To do that, we observe that by foliating $\Omega(t)$ with lines transversal to Γ, the bound (4.3) reduces to the one-dimensional case. Denoting the distance to the boundary by r and the value of $r_1 + r_2$ on the boundary by r_0, we have the relations

$$r_1 + r_2 \approx r + r_0, \quad a \approx \frac{r}{r + r_0}, \quad b \approx r.$$

Then

$$\widetilde{D}_{\mathcal{H}} \approx \int_0^\infty r(r+r_0)^{\sigma-2}\big((r_1-r_2)^2 + \kappa(r_0+r)(v_1-v_2)^2\big)\,dr.$$

On the other hand,

$$D_{\mathcal{H}} \approx \int_0^\infty (r+r_0)^{\sigma-1}\big((r_1-r_2)^2 + \kappa(r_0+r)(v_1-v_2)^2\big)\,dr + r_0^{\sigma+2}.$$

In the region where $r \ll r_0$ we have $|r_1 - r_2| \approx r_0$. Then we can evaluate the first part in the $\widetilde{D}_{\mathcal{H}}$ integral by

$$\int_0^{cr_0} r(r+r_0)^{\sigma-2}(r_1-r_2)^2\,dr \approx r_0^{\sigma+2},$$

thereby obtaining the integral in (4.1). Conversely, we have

$$\int_0^{cr_0} (r+r_0)^{\sigma-1}(r_1-r_2)^2\,dr \approx r_0^{\sigma+2},$$

which gives the desired bound for the missing part of the first term of $D_{\mathcal{H}}$.

It remains to compare the $v_1 - v_2$ terms, where we also need to focus on the region $r \approx r_0$. Denote

$$\bar{v} := \fint_{r\approx r_0} (v_1 - v_2)\,dr$$

for which we can estimate

$$|\bar{v}|^2 \lesssim r_0^{-\sigma-1}\widetilde{D}_{\mathcal{H}}.$$

Then for smaller r we can use the Hölder $C^{\frac{1}{2}}$ norm to estimate

$$|v_1 - v_2|^2 \lesssim |\bar{v}|^2 + Ar_0.$$

Hence

$$\int_0^{r_0} (r+r_0)^{\sigma}(v_1-v_2)^2\,dr \lesssim r_0^{\sigma+1}(|\bar{v}|^2 + Ar_0) \lesssim \widetilde{D}_{\mathcal{H}},$$

as needed. ■

4.2. The energy estimate

The second step in the proof of Theorem 5 is to track the time evolution of the degenerate energy $\widetilde{D}_{\mathcal{H}}$:

Proposition 4.3. *We have*

$$\frac{d}{dt}\widetilde{D}_{\mathcal{H}}\big((r_1,v_1),(r_2,v_2)\big) \lesssim (B_1+B_2)D_{\mathcal{H}}\big((r_1,v_1),(r_2,v_2)\big).$$

In view of Lemma 4.2, the conclusion of the theorem then follows if we apply the Grönwall inequality.

Proof of Proposition 4.3. To compute the time derivative of $\widetilde{D}_{\mathcal{H}}(t)$ we use material derivatives. But we have two of those, D_t^1 and D_t^2, and it is essential to do the computations in a symmetric fashion, so we will use the averaged material derivative

$$D_t = \frac{1}{2}(D_t^1 + D_t^2).$$

Using equations (1.5), we compute difference equations

$$\begin{aligned} D_t(r_1 - r_2) &= -\frac{\kappa}{2}(r_1 - r_2)\nabla(v_1 + v_2) - \frac{\kappa}{2}(r_1 + r_2)\nabla(v_1 - v_2) \\ &\quad - \frac{1}{2}(v_1 - v_2)\nabla(r_1 + r_2), \end{aligned} \tag{4.4}$$

$$D_t(v_1 - v_2) = -\nabla(r_1 - r_2) - \frac{1}{2}(v_1 - v_2)\nabla(v_1 + v_2). \tag{4.5}$$

We will also need a symmetrized sum equation

$$\begin{aligned} D_t(r_1 + r_2) &= -\frac{\kappa}{2}(r_1 + r_2)\nabla(v_1 + v_2) - \frac{\kappa}{2}(r_1 - r_2)\nabla(v_1 - v_2) \\ &\quad - \frac{1}{2}(v_1 - v_2)\nabla(r_1 - r_2). \end{aligned} \tag{4.6}$$

We use these relations to compute the time derivative of the energy, using Lemma 3.2 with $v := \frac{1}{2}(v_1 + v_2)$. We have

$$|\nabla v_1| + |\nabla v_2| \lesssim B := B_1 + B_2,$$

so the contribution of the $\nabla \cdot v$ term is directly estimated by $BD_{\mathcal{H}}(t)$, and so are the contributions of the first term in (4.4), the first two terms in (4.6), as well as the second term in (4.5). Hence we obtain

$$\frac{d}{dt}\widetilde{D}_{\mathcal{H}}(t) = I_1 + I_2 + I_3 + O(B)D_{\mathcal{H}}(t),$$

where the contributions I_j are as follows:

(i) I_1 represents the contributions of the averaged material derivative applied to the first factor $(r_1 + r_2)^{\sigma-1}$ via the third term (4.6), namely

$$\begin{aligned} I_1 = -\frac{\sigma - 1}{2} \int &(r_1 + r_2)^{\sigma-2}\big(a(r_1, r_2)(r_1 - r_2)^2 + \kappa b(r_1, r_2)(v_1 - v_2)^2\big) \\ &\times (v_1 - v_2)\nabla(r_1 - r_2)\, dx. \end{aligned}$$

We separate the two terms as

$$I_1 = J_1^a + O(J_2),$$

where

$$J_1^a = -\frac{\sigma - 1}{2} \int (r_1 + r_2)^{\sigma-2} a(r_1, r_2)(r_1 - r_2)^2(v_1 - v_2)\nabla(r_1 - r_2)\, dx$$

and

$$J_2 = \int (r_1 + r_2)^{\sigma-1} |v_1 - v_2|^3 \, dx.$$

(ii) I_2 represents the contributions of the averaged material derivative applied to the a and b factors via the third[12] terms in (4.4) and (4.6), namely

$$\begin{aligned} I_2 = &-\frac{1}{2} \int (r_1 + r_2)^{\sigma-1} \big(a_\mu(r_1, r_2)(r_1 - r_2)^2 + \kappa b_\mu(r_1, r_2)(v_1 - v_2)^2\big) \\ &\qquad \times (v_1 - v_2)\nabla(r_1 - r_2) \, dx \\ &-\frac{1}{2} \int (r_1 + r_2)^{\sigma-1} \big(a_\nu(r_1, r_2)(r_1 - r_2)^2 + \kappa b_\nu(r_1, r_2)(v_1 - v_2)^2\big) \\ &\qquad \times (v_1 - v_2)\nabla(r_1 + r_2) \, dx. \end{aligned}$$

We also split this into

$$I_2 = J_1^b + J_1^c + O(J_2),$$

where

$$\begin{aligned} J_1^b &= -\frac{1}{2} \int (r_1 + r_2)^{\sigma-1} a_\mu(r_1, r_2)(r_1 - r_2)^2 (v_1 - v_2)\nabla(r_1 - r_2) \, dx, \\ J_1^c &= -\frac{1}{2} \int (r_1 + r_2)^{\sigma-1} a_\nu(r_1, r_2)(r_1 - r_2)^2 (v_1 - v_2)\nabla(r_1 + r_2) \, dx. \end{aligned}$$

(iii) I_3 represents the contribution of the averaged material derivative applied to the quadratic factors $(r_1 - r_2)^2$ and $(v_1 - v_2)^2$ via the second and third terms in (4.4) and the first term in (4.5):

$$\begin{aligned} I_3 = -\kappa \int (r_1 + r_2)^{\sigma-1} \big(&a(r_1, r_2)(r_1 - r_2)(r_1 + r_2)\nabla(v_1 - v_2) \\ &+ 2b(r_1, r_2)(v_1 - v_2)\nabla(r_1 - r_2)\big) \, dx \\ - \int (r_1 + r_2)^{\sigma-1} &a(r_1, r_2)(r_1 - r_2)(v_1 - v_2)\nabla(r_1 - r_2) \, dx. \end{aligned}$$

This is the main term, where we expect to see the same cancellation as in the case of the linearized equation. At this place we need the matching condition between a and b, namely $2b = (r_1 + r_2)a$. Substituting this in and integrating by parts, we obtain an almost full cancellation unless the derivative falls on a, namely

$$I_3 = \kappa \int (r_1 + r_2)^{\sigma} (r_1 - r_2)(v_1 - v_2)\nabla a(r_1, r_2) \, dx = J_1^d + J_1^e,$$

[12] The contributions of the first and second terms in (4.4) and (4.6) are directly bounded by $O(B)D_{\mathcal{H}}(t)$.

where

$$J_1^d = \kappa \int (r_1 + r_2)^\sigma a_\mu(r_1, r_2)(r_1 - r_2)(v_1 - v_2)\nabla(r_1 + r_2)\, dx,$$

$$J_1^e = \kappa \int (r_1 + r_2)^\sigma a_\nu(r_1, r_2)(r_1 - r_2)(v_1 - v_2)\nabla(r_1 - r_2)\, dx.$$

The above analysis shows that

$$\frac{d}{dt}\tilde{D}_{\mathcal{H}}(t) \leq J_1^a + J_1^b + J_1^c + J_1^d + J_1^e + O(J_2) + O(B_1 + B_2)D_{\mathcal{H}}(t).$$

Hence, to prove (4.2), it remains to estimate the error terms,

$$J_1^a + J_1^b + J_1^c + J_1^d + J_2 \lesssim_A (B_1 + B_2)D_{\mathcal{H}}(t).$$

A. The bound for J_2. We begin with the bound for J_2, which is simpler and will also be needed later. As in Lemma 4.2, we can reduce the problem to the one-dimensional case by foliating Ω with parallel lines nearly perpendicular to its boundary Γ. Again denoting the distance to the boundary by r and the value of $r_1 + r_2$ on the boundary by r_0, we have

$$D_{\mathcal{H}}(t) = \int_0^\infty (r + r_0)^{\sigma-1}\big((r_1 - r_2)^2 + (r + r_0)(v_1 - v_2)^2\big)\, dr + r_0^{\sigma+2}.$$

Then, to estimate J_2, it suffices to prove the L^3 bound in the following interpolation lemma:

Lemma 4.4. *Let $\sigma > 0$ and $r_0 > 0$. Then we have the following interpolation bound in $[r_0, \infty)$:*

$$\|r^{\frac{\sigma-1}{3}} w\|_{L^3}^3 \lesssim \|r^{\frac{\sigma}{2}} w\|_{L^2}^2 \|w'\|_{L^\infty}.$$

The lemma is applied with $w = v_1 - v_2$. Note that by direct integration the same bound holds in all dimensions. Thus we obtain the following corollary:

Corollary 4.5. *In the context of our problem we have*

$$\|r^{\frac{\sigma-1}{3}} w\|_{L^3}^3 \lesssim BD_{\mathcal{H}}(t).$$

The same bound also holds if all norms are restricted to any horizontal cylinder (i.e. transversal to Γ).

Proof. We think of this as some version of a Hardy-type inequality. The proof is based on a similar argument, seen before in Section 2. We interpret r as being pointwise equivalent with x and get

$$\|r^{\frac{\sigma-1}{3}} w\|_{L^3(0,\infty)} \sim \|x^{\frac{\sigma-1}{3}} w\|_{L^3(0,\infty)}.$$

To get the result we integrate by parts and use the Hölder inequality as follows:

$$\int_0^\infty x^{\sigma-1} w^3\, dx = -\frac{3}{\sigma}\int_0^\infty x^\sigma w^2 w'\, dx.$$

Since we assumed that $w' \in L^\infty(0, \infty)$, we indeed get

$$\|r^{\frac{\sigma-1}{3}} w\|_{L^3(0,\infty)} \leq \frac{3}{\sigma}\|w'\|_{L^\infty}\|r^{\frac{\sigma}{2}} w\|_{L^2}^2. \qquad \blacksquare$$

B. The bound for J_1^a, J_1^b, J_1^c, J_1^d, and J_1^e. We group like terms and set

$$J_1^a + J_1^b + J_1^e := J_1^-, \quad J_1^c + J_1^d := J_1^+,$$

where we can express J_1^- and J_1^+ in the form

$$J_1^- = \int (r_1 + r_2)^{\sigma-2} a^{\pm}(r_1, r_2)(r_1 - r_2)^2 (v_1 - v_2)\nabla(r_1 - r_2)\, dx,$$

with a^- smooth and 0-homogeneous,

$$a^-(r_1,r_2) = -\frac{\sigma-1}{2} a(r_1,r_2) - \frac{1}{2}(r_1+r_2) a_\mu(r_1,r_2) + \kappa(r_1+r_2)^2(r_1-r_2)^{-1} a_\nu(r_1,r_2),$$

respectively

$$J_1^+ = \int (r_1 + r_2)^{\sigma} a^+(r_1, r_2)(r_1 - r_2)(v_1 - v_2)\nabla(r_1 + r_2)\, dx,$$

with a^+ smooth and -1-homogeneous,

$$a^+(r_1, r_2) = \left(\kappa - \frac{1}{2}\right) a_\mu(r_1, r_2).$$

Here we have used the fact that a is 0-homogeneous, which yields $\mu a_\mu + \nu a_\nu = 0$. Also we remark that a_μ vanishes in a conical neighborhood of $\nu = 0$, therefore we can also think of the J_1^+ integrand as being at least cubic in $r_1 - r_2$.

Heuristically, one might think that after another round of integration by parts one might place the derivative in J_1^- either on $v_1 - v_2$, in which case we get good Grönwall terms, or on $r_1 + r_2$, where we just discard it and reduce the problem to estimating an integral of the form

$$J_1 = \int_\Omega (r_0 + r)^{\sigma-3} |v_1 - v_2|\, |r_1 - r_2|^3\, dx.$$

Unfortunately such a strategy works only if $\kappa \in (0, 1]$; for larger κ a problem arises, having to do with potentially large contributions within a thin boundary layer.

Instead, to address the full range of κ, we will develop the idea of separating a carefully selected boundary layer, where we provide a direct argument, whereas outside this boundary layer we can use the simpler integration by parts idea above.

To understand our choice of the boundary layer, we consider first the much simpler case when $r_1 - r_2 = 0$ and $\nabla(r_1 - r_2) = 0$ on the boundary, where $r_0 = 0$ and

$$|\nabla(r_1 - r_2)| \lesssim B r^{\frac{1}{2}}, \quad |r_1 - r_2| \lesssim B r^{\frac{3}{2}}. \tag{4.7}$$

Then the estimate for J_1 above reduces to the one-dimensional case, where we can simply argue by the Hölder inequality:

$$\begin{aligned} J_1 &\lesssim \|r^{\frac{\sigma-1}{3}}(v_1 - v_2)\|_{L^3} \|r^{\frac{2}{9}\sigma - \frac{8}{9}}(r_1 - r_2)\|^3_{L^{\frac{9}{2}}} \\ &\lesssim \|r^{\frac{\sigma-1}{3}}(v_1 - v_2)\|_{L^3} \|r^{\frac{\sigma-1}{2}}(r_1 - r_2)\|^{\frac{4}{3}}_{L^2} \|r^{-\frac{3}{2}}(r_1 - r_2)\|^{\frac{2}{3}}_{L^\infty} \|(r_1 - r_2)/r\|_{L^\infty} \\ &\lesssim B D_{\mathcal{H}}. \end{aligned} \tag{4.8}$$

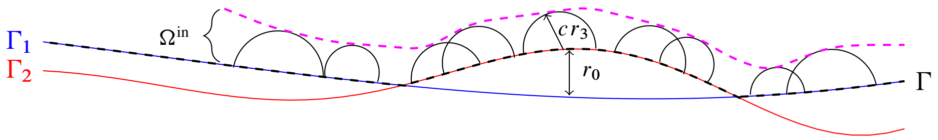

Figure 2. The boundary layer of variable thickness cr_3.

Unfortunately, in general the bound (4.7) will not hold, and we will separate the region where it holds and the region where it does not hold.

Our boundary layer will depend on B, and will roughly be defined as the complement of the region where (4.7) holds, with the additional proviso that it must have thickness at least r_0. For a rigorous definition, we start with the function r_3 defined on the boundary Γ of Ω as follows:

$$r_3 = Cr_0 + (B^{-1}r_0)^{\frac{2}{3}} + (B^{-1}|\nabla(r_1 - r_2)|)^2, \tag{4.9}$$

where C is a fixed large universal constant. Then we define our boundary layer, based on Figure 2, as

$$\Omega^{\text{in}} = \Omega \cap \bigcup_{x\in\Gamma} B(x, cr_3(x)),$$

as well as its enlargement

$$\tilde{\Omega}^{\text{in}} = \Omega \cap \bigcup_{x\in\Gamma} B(x, 4cr_3(x)).$$

Here, c is a small universal constant.

We want this boundary layer to have a locally uniform geometry. This is ensured by a slowly-varying-type property of the function r_3.

Lemma 4.6. *We have*

$$|r_3(x) - r_3(y)| \lesssim r_3^{\frac{1}{2}}(x)|x-y|^{\frac{1}{2}} + |x-y| + r_0^{\frac{1}{2}} r_3^{\frac{1}{2}}.$$

Proof. We consider each of the three components of r_3 in (4.9). For the first one we simply use the Lipschitz bound for r_0. For the second one, we use the $\tilde{C}^{0,\frac{1}{2}}$ bound on ∇r_1 and ∇r_2 to estimate

$$\begin{aligned} |r_0(x) - r_0(y)| &\lesssim |x-y|\,|\nabla r_0| + B(|x-y|^{\frac{3}{2}} + r_0(x)^{\frac{1}{2}}|x-y|) \\ &\lesssim B(|x-y| r_3^{\frac{1}{2}} + |x-y|^{\frac{3}{2}}), \end{aligned}$$

which suffices. Finally, for the last term we have

$$|\nabla(r_1 - r_2)(x) - \nabla(r_1 - r_2)(y)| \lesssim B(r_0^{\frac{1}{2}}(x) + |x-y|^{\frac{1}{2}}),$$

which is again enough. ∎

This property ensures that Ω^{in} and $\widetilde{\Omega}^{\text{in}}$ are separate:

Lemma 4.7. *There exists a smooth cut-off function $0 \le \chi \le 1$ in Ω with the following properties:*

(a) *Support: $\chi = 1$ in Ω^{in} and $\chi = 0$ in $\Omega \setminus \widetilde{\Omega}^{\text{in}}$.*

(b) *Regularity: $|\partial^\alpha \chi(x)| \lesssim (r_1 + r_2)^{-|\alpha|}$.*

Proof. For $y \in \Omega$ we define the function

$$G(y) = \min_{x \in \Gamma} |x - y| r_3(x)^{-1}$$

so that Ω^{in}, $\widetilde{\Omega}^{\text{in}}$ are described by

$$\Omega^{\text{in}} = \{G(y) \le c\}, \quad \widetilde{\Omega}^{\text{in}} = \{G(y) \le 4c\}.$$

Then we can use the function G to describe the separation between Ω^{in} and $\Omega \setminus \widetilde{\Omega}^{\text{in}}$. Precisely, it suffices to show that we can control the Lipschitz constant for G in the transition region,

$$c \le G(y) \le 4c \quad \Rightarrow \quad |\nabla G(y)| \lesssim r^{-1}.$$

Since G is an infimum, it suffices to show the same for each of its defining functions. Equivalently, it suffices to show that if y is in the transition region then

$$c \le |x - y| r_3(x)^{-1} \le 4c \quad \Rightarrow \quad r(y) \lesssim r_3(x).$$

Let z be the closest point to y on the boundary, so that $r(y) \approx |y - z|$. Then the first relation implies that

$$|x - z| \le 8c r_3(x).$$

Since c is small, Lemma 4.6 shows that $r_3(z) \approx r_3(x)$. Since we are in the transition region, we must also have

$$|x - z| \ge c r_3(z),$$

as needed. ∎

Finally, we verify that we have good control over $r_1 - r_2$ on the outer region:

Lemma 4.8. *The good bound (4.7) holds outside Ω^{in}.*

Proof. Let $y \notin \Omega^{\text{in}}$, and x be the closest point to y on the boundary. Then

$$r(y) \approx |x - y| \ge c r_3(x).$$

Using the $\widetilde{C}^{0,\frac{1}{2}}$ bound for $\nabla(r_1 - r_2)$ along the $[x, y]$ line, we have

$$|\nabla(r_1 - r_2)(z) - \nabla(r_1 - r_2)(x)| \lesssim B(r_0(x)^{\frac{1}{2}} + |z - x|^{\frac{1}{2}}).$$

If we use this directly we obtain

$$\begin{aligned}|\nabla(r_1-r_2)(y)| &\lesssim |\nabla(r_1-r_2)(x)| + B(r_0(x)^{\frac12} + |z-x|^{\frac12})\\ &\lesssim B(r_3(x)^{\frac12} + |y-x|) \lesssim Br(y)^{\frac12}.\end{aligned}$$

If instead we integrate it between x and y then we obtain

$$\begin{aligned}|(r_1-r_2)(y)| &\lesssim r_0(x) + |x-y|\,|\nabla(r_1-r_2)(x)| + B(r_0(x)^{\frac12}|x-y| + |x-y|^{\frac32})\\ &\lesssim Br_3(x)^{\frac32} + Br_3(x)^{\frac12}|x-y| + B(r_0(x)^{\frac12}|x-y| + |x-y|^{\frac32})\\ &\lesssim Br_3^{\frac32}.\end{aligned}$$

■

Now we use the cutoff χ to split each of the above integrals in two, and estimate each of them in turn.

B.1. The estimate in the outer region. Here we insert the cutoff $(1-\chi)$ in each of the two integrals $J_1^{\pm}$, and integrate by parts in J_1^-. Precisely, the outer part of J_1^- is

$$J_1^{-,\text{out}} = \int (1-\chi)(r_1+r_2)^{\sigma-2}a^-(r_1,r_2)(r_1-r_2)^2(v_1-v_2)\nabla(r_2-r_1)\,dx.$$

The ν-dependent part of the integrand is

$$\nu^2 a^-(\mu,\nu)\nabla\nu.$$

In order to be able to integrate by parts, we define a function $c(\mu,\nu)$ in the region of interest $|\nu|<\mu$ by

$$\partial_\nu c(\mu,\nu) = \nu^2 a^-(\mu,\nu), \quad c(\mu,0)=0.$$

By definition, c is smooth, homogeneous of order three, and satisfies

$$|c(\mu,\nu)| \lesssim \nu^3, \quad |\partial_\mu c(\mu,\nu)| \lesssim \mu^{-1}\nu^3.$$

Then we can write

$$\nu^2 a(\mu,\nu)\nabla\nu = \nabla c(\mu,\nu) - \partial_\mu c(\mu,\nu)\nabla\mu.$$

We substitute this in $J_1^{-,\text{out}}$ to obtain

$$\begin{aligned}J_1^{-,\text{out}} = \frac{\sigma-1}{2}&\int (1-\chi)(r_1+r_2)^{\sigma-2}(v_1-v_2)\nabla c\,dx\\ &+\int (1-\chi)(r_1+r_2)^{\sigma-2}c_\mu(v_1-v_2)\nabla\mu\,dx.\end{aligned}$$

In the first integral we integrate by parts. If the derivative falls on v_1-v_2 we get a Grönwall term. Else, it falls on μ, which we discard, or on χ, where we use Lemma 4.7. Hence we obtain

$$J_1^{-,\text{out}} \lesssim \int_{\Omega\setminus\Omega^{\text{in}}} (r_0+r)^{\sigma-3}|v_1-v_2|\,|r_1-r_2|^3\,dx + O(B_1+B_2)D_{\mathcal{H}}(t).$$

In view of Lemma 4.8, we can estimate the integral as in (4.8) and conclude.

The argument for $J_1^{b,\text{out}}$ is similar but simpler, as no integration by parts is needed.

B.2. The estimate in the boundary layer region. To fix scales, we use the slowly varying property of r_3 in Lemma 4.6 to partition $\widetilde{\Omega}^{\text{in}}$ into cylinders C_{x_0} centered at some point $x_0 \in \Gamma$, with radius $4cr_3(x_0)$ and similar height, and correspondingly, we partition our integrals using an appropriate locally finite partition of unity,

$$\chi = \sum \chi_{x_0},$$

where each χ_{x_0} is smooth on the $r_3(x_0)$ scale. Within this cylinder we will think of r_3 as a constant, $r_3 = r_3(x_0)$.

Denoting

$$J_1^{-,x_0} = \int_{C_{x_0}} \chi_{x_0}(r_1 + r_2)^{\sigma-2} a^-(r_1, r_2)(r_1 - r_2)^2 (v_1 - v_2)\nabla(r_1 - r_2)\, dx,$$

and similarly for J_1^+, our objective will be to show that in each such component we have

$$J_1^{\pm,x_0} \lesssim BD_{\mathcal{H}}^{x_0}, \tag{4.10}$$

where $D_{\mathcal{H}}^{x_0}$ denotes the integral in $D_{\mathcal{H}}$ but with the added cutoff χ_{x_0}. After summation over x_0 this will give the desired estimate. We will consider separately the cases when B is small or large.

As a prerequisite to the proof of (4.10), we consider pointwise difference bounds within C_{x_0}. We begin with $r_1 - r_2$. By construction, within C_{x_0} we have

$$|\nabla(r_1 - r_2)| \lesssim Br_3^{\frac{1}{2}}, \quad |r_1 - r_2| \lesssim Br_3^{\frac{3}{2}}.$$

In particular, this yields

$$r_0 \lesssim Br_3^{\frac{3}{2}},$$

and the improved pointwise bound

$$|r_1 - r_2| \lesssim r_0 + Brr_3^{\frac{1}{2}}, \tag{4.11}$$

where we observe that r_0 need not be constant on the boundary within C_{x_0}.

Depending on the relative size of B and r_3 we will distinguish two scenarios:

Lemma 4.9. *One of the following two scenarios applies in C_{x_0}:*

(a) *Either $r_0(x_0) \ll r_3$, in which case we must have $B\sqrt{r_3} \lesssim 1$,*

(b) *or $r_0 \approx r_3$, in which case we must have $B\sqrt{r_0} \gtrsim 1$.*

We will refer to the first case as the *small B case* and the second as the *large B case*.

Proof of Lemma 4.9. We start by comparing $r_0(x_0)$ with r_3. If $r_3 \approx r_0$, then we must have

$$r_0 \gtrsim (B^{-1}r_0)^{\frac{2}{3}},$$

and further $B \gtrsim (r_0)^{-\frac{1}{2}}$, which places us in case (b).

If $r_3 \gg r_0(x_0)$, then we have two nonexclusive possibilities. Either we have

$$r_3 \approx (B^{-1} r_0)^{\frac{2}{3}} \gg r_0,$$

which yields $B^2 \approx r_0^2 r_3^{-3} \ll r_3^{-1}$, placing us in case (a), or we have

$$r_3 \approx B^{-2} |\nabla(r_1 - r_2)|^2 \lesssim B^{-2},$$

which places us again in case (b). ∎

In addition to bounds for $r_1 - r_2$, we also need bounds for $v_1 - v_2$. We will show that within the same cylinder we have a good uniform bound for $v_1 - v_2$:

Lemma 4.10. *Within C_{x_0} we have*

$$|v_1 - v_2| \lesssim B r_3 + (D_{\mathcal{H}}^{x_0})^{\frac{1}{2}} r_3^{-\frac{\sigma+1}{2}} r_3^{-\frac{d-1}{2}}.$$

Proof. Denote by $(v_1 - v_2)_{\text{avg}}$ the average of $v_1 - v_2$ in the region

$$\tilde{C}_{x_0} = C_{x_0} \cap \{r \gtrsim \tfrac{1}{2} r_3(x)\},$$

which represents an interior portion of C_{x_0} away from the boundary. We estimate this using the distance $D_{\mathcal{H}}^{x_0}$, where we observe that within $\tilde{C}_{x_0}$ we have $b \approx r_3$. Then we obtain

$$r_3^d r_3^\sigma (v_1 - v_2)_{\text{avg}} \lesssim D_{\mathcal{H}}^{x_0}.$$

To obtain the full bound for $v_1 - v_2$ we combine this with the B Lipschitz bound, which yields

$$|v_1 - v_2| \lesssim B r_3 + |(v_1 - v_2)_{\text{avg}}|$$

within the full cylinder C_{x_0}. ∎

B.2.a. The case of large B. We recall that in this case we have the relations $r_3 = r_0$ and $B\sqrt{r_0} \gtrsim 1$. Consider J_1^{-,x_0} first. We discard the gradient terms, bound $r_1 - r_2$ by r_0, and use Lemma 4.10 for $v_1 - v_2$. This yields

$$J_1^{-,x_0} \lesssim r_0^d r_0^\sigma (B r_0 + (D_{\mathcal{H}}^{x_0})^{\frac{1}{2}} r_0^{-\frac{\sigma+1}{2}} r_0^{-\frac{d-1}{2}}).$$

On the other hand, a localized version of (4.1) yields

$$r_0^{\sigma+2} \lesssim r_0^{-(d-1)} D_{\mathcal{H}}^{x_0}.$$

Combining the last two bounds gives

$$J_1^{-,x_0} \lesssim D_{\mathcal{H}}^{x_0} (B + r_0^{-\frac{1}{2}}) \lesssim B D_{\mathcal{H}}^{x_0},$$

as needed. The argument for J_1^{+,x_0} is identical.

B.2.b. The case of small B. We recall that this corresponds to $r_0 \ll r_3$ and $B\sqrt{r_3} \lesssim 1$. This is the more difficult case.

The first observation concerning the cylinder C_{x_0} is that $r_1 - r_2$ is large there on average, of size $Br_3^{\frac{3}{2}}$. This is reflected in a bound from below for $D_{\mathcal{H}}^{x_0}$:

Lemma 4.11. *Assume we are in the small B case. Then we have*

$$B^2 r_3^{\sigma+3} r_3^{d-1} \lesssim D_{\mathcal{H}}^{x_0}.$$

Proof. We approximate $r_1 - r_2$ near x_0 with its linear expansion,

$$(r_1 - r_2)(y) = r_0 + \nabla(r_1 - r_2)(x_0)(y - x_0) + O(B(r_0^{\frac{1}{2}} + |x_0 - y|^{\frac{1}{2}})|x_0 - y|).$$

Within C_{x_0} this can be simplified to

$$(r_1 - r_2)(y) = r_0 + \nabla(r_1 - r_2)(x_0)(y - x_0) + O(Br_3^{\frac{1}{2}}|x_0 - y|).$$

Now we consider a small interior ball

$$B = B(x_0 + 2rN, r), \quad r_0 < r < cr_3,$$

where we have $a \approx 1$ and $r_1 + r_2 \approx r$, and use $D_{\mathcal{H}}^{x_0}$ to estimate

$$r^{\sigma-1} \int_B |r_0 + \nabla(r_1 - r_2)(x_0)(y - x_0)|^2 \, dy \lesssim r^{\sigma-1} r^d (Br^{\frac{3}{2}})^2 + D_{\mathcal{H}}^{x_0}.$$

The integral on the left is easily evaluated, to get

$$r^{\sigma-1} r^d (r_0^2 + r^2 |\nabla(r_1 - r_2)(x_0)|^2) \lesssim r^{\sigma-1} r^d (Br^{\frac{3}{2}})^2 + D_{\mathcal{H}}^{x_0}.$$

We can compare the constants on the left and the first term on the right. We know that

$$r_3 \approx \max\{(B^{-1} r_0)^{\frac{2}{3}}, (B^{-1}|\nabla(r_1 - r_2)(x_0)|)^2\}.$$

If the first quantity on the right is larger, then

$$r_0 = Br_3^{\frac{3}{2}}$$

and we obtain

$$r^{\sigma-1} r^d (Br_3^{\frac{3}{2}})^2 \lesssim r^{\sigma-1} r^d (Br^{\frac{3}{2}})^2 + D_{\mathcal{H}}^{x_0}.$$

Choosing $r = cr_3$ with a small constant c, the first term on the right is absorbed on the left and we arrive at the desired conclusion.

If the second quantity on the right is larger, then

$$|\nabla(r_1 - r_2)(x_0)| = Br_3^{\frac{1}{2}},$$

and we obtain

$$r^{\sigma-1} r^d r^2 (Br_3^{\frac{1}{2}})^2 \lesssim r^{\sigma-1} r^d (Br^{\frac{3}{2}})^2 + D_{\mathcal{H}}^{x_0}.$$

Hence we can conclude exactly as before. ∎

The above lemma allows us to slightly improve Lemma 4.10 to the following lemma:

Lemma 4.12. *Assume that B is small. Then within C_{x_0} we have*

$$|v_1 - v_2| \lesssim (D_{\mathcal{H}}^{x_0})^{\frac{1}{2}} r_3^{-\frac{\sigma+1}{2}} r_3^{-\frac{d-1}{2}}.$$

We are now ready to estimate the first integral:

$$\begin{aligned}
J_1^{-,x_0} &\lesssim \int_{C_{x_0}} (r_0 + r)^{\sigma-2}(r_1 - r_2)^2 |v_1 - v_2|\, |\nabla(r_2 - r_1)|\, dx \\
&\lesssim B r_3^{\frac{1}{2}} r_3^{-\frac{\sigma+1}{2}} r_3^{-\frac{d-1}{2}} (D_{\mathcal{H}}^{x_0})^{\frac{1}{2}} \int_{C_{x_0}} (r_0 + r)^{\sigma-2}(r_0 + Brr_3^{\frac{1}{2}})^2\, dr\, dx_0 \\
&\lesssim B r_3^{\frac{1}{2}} r_3^{-\frac{\sigma+1}{2}} r_3^{-\frac{d-1}{2}} (D_{\mathcal{H}}^{x_0})^{\frac{1}{2}} \left(\int r_0^{\sigma+1}\, dx_0 + r_3^{d-1} B^2 r_3^{\sigma+2} \right) \\
&\lesssim B r_3^{\frac{1}{2}} r_3^{-\frac{\sigma+1}{2}} r_3^{-\frac{d-1}{2}} (D_{\mathcal{H}}^{x_0})^{\frac{1}{2}} r_3^{d-1} ((B r_3^{\frac{3}{2}})^{\sigma+1} + B^2 r_3^{\sigma+2}) \\
&\lesssim (D_{\mathcal{H}}^{x_0})^{\frac{1}{2}} B^2 r_3^{\frac{d-1}{2}} r_3^{\frac{\sigma+3}{2}} ((B\sqrt{r_3})^{\sigma} + B\sqrt{r_3}) \\
&\lesssim B D_{\mathcal{H}}^{x_0}.
\end{aligned}$$

It remains to estimate J_1^{+,x_0}, which we recall here:

$$J_1^{+,x_0} = C \int \chi_{x_0} \nu \mu^{\sigma} a_{\mu} (v_1 - v_2) \nabla \mu\, dx, \quad C = \kappa - \frac{1}{2}.$$

Aside from the obvious cancellation when $\kappa = \frac{1}{2}$, we would like to integrate by parts in order to move the derivative away from μ. To implement this integration by parts, we need an auxiliary function $c(\mu, \nu)$ so that

$$\partial_{\mu} c(\mu, \nu) = \mu^{\sigma} a_{\mu}.$$

Suppose we have such a function c which is smooth, homogeneous of order σ, and supported in $|\mu| \lesssim |\nu| < \mu$. Then integration by parts yields

$$\begin{aligned}
J_1^{+,x_0} &= C \int \chi_{x_0} \nu c_{\mu}(\mu, \nu)(v_1 - v_2) \nabla \mu\, dx \\
&= -C \int \chi_{x_0} \nu c(\mu, \nu) \nabla \cdot (v_1 - v_2)\, dx \\
&\quad - C \int \chi_{x_0} (c(\mu, \nu) + \nu c_{\nu}(\mu, \nu))(v_1 - v_2) \nabla \nu\, dx \\
&\quad - C \int \nu c(\mu, \nu)(v_1 - v_2) \nabla \chi_{x_0}\, dx.
\end{aligned}$$

In the first integral we bound $\nabla \cdot (v_1 - v_2)$ by B, and then bound the rest by $D_{\mathcal{H}}^{x_0}$ since $\mu \approx \nu$ in the support of the integrand. The second integral is similar to J_1^{a,x_0}. Finally, in the third integral the gradient of χ_{x_0} yields an r_3^{-1} factor, and we can estimate it using Lemma 4.12 and the bound (4.11) for $r_1 - r_2$ by

$$
\begin{aligned}
&\lesssim r_3^{-1} \int_{C_{x_0}} (r_1 - r_2)^{\sigma+2} |v_1 - v_2| \, dx \\
&\lesssim r_3^{d-1} (r_3 r_0^{\sigma+2} + (B\sqrt{r_3})^{\sigma+2} r_3^{\sigma+3}) (D_{\mathcal{H}}^{x_0})^{\frac{1}{2}} r_3^{-\frac{\sigma+1}{2}} r_3^{-\frac{d-1}{2}} \\
&\lesssim B D_H^{x_0},
\end{aligned}
$$

where at the last step we bound $r_0 \lesssim B r_3^{\frac{3}{2}}$ twice, $r_0 \le r_3$ for the rest of r_0, and then use Lemma 4.11; the powers of r_3 will all cancel, as predicted by scaling considerations.

It remains to show that we can find such a function c. This is where a convenient choice of a helps. Precisely, we want a to be nonnegative, even in ν, supported in $|\nu| < \mu$ and equal to 1 when $|\nu| \ll \mu$. To avoid boundary terms in the integration by parts, we will choose c with similar support. But we also want c to be smooth and homogeneous, and then we will have an issue at $\mu = 0$, unless we can arrange for c to also be supported away from $\mu = 0$. But this will happen only if

$$
\int \mu^\sigma a_\mu \, d\mu = 0. \tag{4.12}
$$

Lemma 4.13. *There exists a good choice for a which satisfies* (4.12).

Proof. We will take advantage of the fact that the function μ^σ is increasing, as follows. We start with a choice a_0 for a which is nonincreasing. That would make the integral in (4.12) positive. To correct this we use a nonnegative, compactly supported bump function a_1. Its contribution will be negative, as it can be seen integrating by parts:

$$
\int \mu^\sigma a_{1,\mu} \, d\mu = -\frac{1}{\sigma+1} \int \mu^{\sigma+1} a_1 \, d\mu.
$$

Then we choose $a = a_0 + C a_1$, with $C > 0$ chosen so that the two contributions to the integral in (4.12) cancel. ∎

The proof of Proposition 4.3 is concluded. ∎

5. Energy estimates for solutions

Our objective here is to prove Theorem 3. More precisely, we aim to establish uniform control over the $\mathbf{H}^{2k}$ norm of the solutions (v, r) in terms of the similar norm of the initial data, with growth estimated in terms of the control parameters A, B. The key to this is to characterize these norms using energy functionals constructed with suitable vector fields naturally associated to the evolution.

5.1. The div-curl decomposition

A first step in our analysis is to understand the structure of our system of equations. In the nondegenerate case, it is known that at leading order the compressible Euler equations decouple into a wave equation for $(r, \nabla \cdot v)$ and a transport equation for $\omega = \operatorname{curl} v$. We will show that the same happens here. Of course, algebraically the computations are identical. However, interpreting the coupling terms as perturbative is far more delicate in the present context.

We begin with a direct computation, which yields the following second-order wave equation for r:

$$D_t^2 r = \kappa r \Delta r + \kappa^2 r |\nabla \cdot v|^2 + \kappa \nabla v (\nabla v)^{\mathsf{T}},$$

with speed of propagation (sound speed)

$$c_s = \kappa r,$$

where $\nabla \cdot v$ corresponds to the (material) velocity

$$-\kappa \nabla \cdot v = r^{-1} D_t r.$$

On the other hand, for the vorticity we obtain the transport equation

$$D_t \omega = -\omega \nabla v - (\nabla v)^{\mathsf{T}} \omega.$$

These two equations are coupled, so it is natural to consider them at matched regularity levels, but we will use different energy functionals to capture their contributions to the energy.

5.2. Vector fields

Our energy estimates will be obtained by applying a number of well-chosen vector fields to the equation in a suitable fashion, so that the differentiated fields obtained as the outcome solve the linearized equation with perturbative source terms. We do this separately for the wave component and for the transport part:

(a) *Vector fields for the wave equation:* Here we use all the vector fields which commute with the wave equation at the leading order. There are two such vector fields, which generate an associated algebra:

(a1) the material derivative D_t: this has order $\frac{1}{2}$,

(a2) the tangential derivatives $\Omega_{ij} = r_i \partial_j - r_j \partial_i$: these have order 1.

We will only use D_t in this article, but note that a similar analysis works for the tangential derivatives.

(b) *Vector fields for the transport equation:* Here we have more flexibility in our choices, again generating an algebra:

(b1) the material derivative D_t: this has order $\frac{1}{2}$, (b2) all regular derivatives ∂, of order 1,

(b3) multiplication by r, which has order -1.

To avoid negative orders here, one may replace r by $r\partial^2$, which has order 1.

5.3. The energy functional

Here we define energy functionals $E^{2k}(r, v)$ of order k, i.e. which involve combinations of vector fields of orders up to k. We will set this up as the sum of a wave and a transport component,

$$E^{2k}(r, v) = E^{2k}_w(r, v) + E^{2k}_t(r, v).$$

(a) The wave energy: Here we want to use operators of the form

$$D_t^j, \quad j \leq 2k$$

applied to the solution (r, v). However, we would like to have these defined in terms of the data at each fixed time, rather than dynamically. Algebraically this is easily achieved by reiterating the equation. We define

$$(r_j, v_j) = (D_t^j r, D_t^j v),$$

which should be viewed as discussed above, as nonlinear[13] functions of (r, v) at fixed time.

One might hope that these functions should be good approximate solutions for the linearized equations. Unfortunately, this is not exactly the case even for (r_1, v_1). This is because, unlike ∂, D_t does not exactly generate an exact symmetry of the equation. The solution to this difficulty is to work with associated *good variables*, obtained by adding suitable corrections to them. We denote these good variables by (s_j, w_j), and define them as follows:

(i) $j = 0$: $(s_0, w_0) = (r, v)$.

(ii) $j = 1$: $(s_1, w_1) = \partial_t(r, v)$.

(iii) $j = 2$: $(s_2, w_2) = (r_2 + \frac{1}{2}|\nabla r|^2, v_2)$.

(iv) $j \geq 3$: $(s_j, w_j) = (r_j - \nabla r \cdot w_{j-1}, v_j)$.

We now define the wave component of the energy as

$$E^{2k}_w(r, v) = \sum_{j \leq k} \|(s_{2j}, w_{2j})\|^2_{\mathcal{H}},$$

[13] Strictly speaking, at leading order these are linear expressions, so the better terminology would be quasilinear.

where we recall that $\mathcal{H}$ defined in (1.7) represents the natural energy functional for the linearized equation. In the sequel we will use these good variables only for even j, but for the sake of completeness we have listed them for all j.

(b) The transport equation: Here we consider a simpler energy, namely

$$E_t^{2k}(r,v) = \|\omega\|^2_{H^{2k-1,k+\frac{1}{\kappa}}},$$

which at leading order scales in the same way as the wave energy above. One can think of this energy as the outcome of applying vector fields up to and including order k to the vorticity ω.

5.4. Energy coercivity

Our goal here is to prove the equivalence of the energy E^{2k} with the $\mathbf{H}^{2k}$ size of (r,v).

Theorem 6. *Let (r,v) be smooth functions in $\overline{\Omega}$ so that r is positive in Ω and uniformly nondegenerate on $\Gamma = \partial\Omega$. Then we have*

$$E^{2k}(r,v) \approx_A \|(r,v)\|^2_{\mathcal{H}^{2k}}.$$

Proof. (a) We begin with the easier part "$\lesssim$". This is obvious for the vorticity component so it remains to discuss the wave component.

We consider the expressions for (s_{2k}, w_{2k}). These are both linear combinations of multilinear expressions in r and ∇v with the following properties:

- They have order $k-1$, respectively $k-\frac{1}{2}$.
- They have exactly $2k$ derivatives.
- They contain at most $k+1$, respectively k factors of r or its derivatives.

These properties suffice in order to be able to distribute the powers of r and use the interpolation inequalities in Proposition 2.14. We will demonstrate this in the case of s_{2k}; the case of w_{2k} is similar. A multilinear expression in s_{2k} has the form

$$M = r^a \prod_{j=1}^{J} \partial^{n_j} r \prod_{l=1}^{L} \partial^{m_l} v,$$

where $n_j \geq 1, m_l \geq 1$,

$$\sum n_j + \sum m_l = 2k,$$

and[14]

$$a + J + L/2 = k+1.$$

We seek to split

$$a = \sum b_j + \sum c_l,$$

[14]Here we allow for $J = 0$ or $K = 0$, in which case the corresponding products are omitted.

and correspondingly

$$M = \prod_{j=1}^{J} r^{b_j} \partial^{n_j} r \prod_{l=1}^{L} r^{c_l} \partial^{m_l} v,$$

so that we can apply our interpolation inequalities from Proposition 2.14, respectively Proposition 2.15. These will give bounds of the form

$$\| r^{b_j} \partial^{n_j} r \|_{L^{p_j}(r^{\frac{1-\kappa}{\kappa}})} \lesssim A^{1-\frac{2}{p_j}} \|(r,v)\|_{\mathcal{H}^{2k}}^{\frac{2}{p_j}}, \qquad \frac{1}{p_j} = \frac{n_j - 1 - b_j}{2(k-1)},$$

respectively

$$\| r^{c_l} \partial^{m_l} r \|_{L^{q_l}(r^{\frac{1-\kappa}{\kappa}})} \lesssim A^{1-\frac{2}{q_l}} \|(r,v)\|_{\mathcal{H}^{2k}}^{\frac{2}{q_l}}, \qquad \frac{1}{q_l} = \frac{m_l - \frac{1}{2} - c_l}{2(k-1)},$$

where the denominators represent the orders of the expressions being measured, so they add up to $k-1$ as needed.

It only remains to verify that the b_j and the c_l can be chosen in the range where our interpolation estimates apply, which is

$$0 \le b_j \le (n_j - 1)\frac{k}{2k-1},$$

respectively

$$0 \le c_l \le (m_l - \tfrac{1}{2})\frac{k+\frac{1}{2}}{2k-\frac{1}{2}}.$$

To verify that we can satisfy these conditions we need

$$\sum (n_j - 1)\frac{k}{2k-1} + \sum (m_l - \tfrac{1}{2})\frac{k+\frac{1}{2}}{2k-\frac{1}{2}} \le a.$$

But the sum on the left is evaluated by

$$\le \left(\sum n_j + \sum m_l - J - L \right) \frac{k}{2k-1} = (2k - J - L)\frac{k}{2k-1} \le (a+k-1)\frac{k}{2k-1} \le a$$

using $a \le k$. Here, equality holds only if $a = k$, $J = 1$, and $L = 0$, i.e. for the leading linear case.

(b) We continue with the "$\gtrsim$" part. To do this we will argue inductively, relating (s_{2j}, w_{2j}) with (s_{2j-2}, w_{2j-2}). This is done using the transition operators L_1 and L_2 introduced earlier.

Lemma 5.1. *For $j \ge 2$ we have a pair of homogeneous recurrence-type relations*

$$s_{2j} = L_1 s_{2j-2} + f_{2j}, \quad w_{2j} = L_2 w_{2j-2} + g_{2j}, \tag{5.1}$$

where f_{2j} and g_{2j} are also multilinear expressions as above, of order $j-1$, respectively $j - \frac{1}{2}$, but with the additional property that they are non-endpoint, i.e. they contain at least two factors of the form $\partial^{2+} r$ or $\partial^{1+} v$.

Proof. We begin with the first relation, for which we first discuss the generic case $j \geq 3$. We begin expanding the expression of s_{2j}, and then continue calculating the left-hand side of (5.1). We have

$$s_{2j} = (\kappa r \Delta + \nabla r \cdot \nabla)(r_{2j-2} - \nabla r \cdot w_{2j-3}) + f_{2j}. \tag{5.2}$$

The left-hand side expands as

$$s_{2j} = r_{2j} - \nabla r \cdot w_{2j-1} = D_t^{2j} r - \nabla r \cdot D_t^{2j-1} v. \tag{5.3}$$

Each of the two terms appearing in the expression above can be further analyzed. For the first term on the right-hand side in (5.3) we have

$$D_t^{2j} r = D_t^{2j-2}(D_t^2 r) = D_t^{2j-2}(\kappa r \Delta r + \kappa^2 r |\nabla \cdot v|^2 + \kappa \nabla v (\nabla v)^{\mathsf{T}}). \tag{5.4}$$

The last two terms already satisfy the non-endpoint property, so we are left to process the first term on the right-hand side of (5.4) further:

$$D_t^{2j-2}(\kappa r \Delta r) = \kappa \sum_{m=0}^{2j-2-m} \binom{2j-2}{m} D_t^{2j-2-m} r D_t^m \Delta r.$$

We note that $D_t^m \Delta r$ gives at least $\partial^{2+} r$ derivatives, and for any $m \neq 2j-2$ the claim is obvious, as we have that one material derivative on r will produce $\partial^{1+} v$ derivatives. Hence, the more difficult case is when $m = 2j-2$; we discuss it further:

$$\kappa r D_t^{2j-2} \Delta r = \kappa r D_t^{2j-3}(D_t \Delta r) = \kappa r D_t^{2j-3}(D_t \Delta r). \tag{5.5}$$

We commute the material derivative with the Laplacian using the formula

$$[D_t, \Delta] = -\Delta v \cdot \nabla - \nabla v \nabla^2, \tag{5.6}$$

and (5.5) gives

$$\begin{aligned} \kappa r D_t^{2j-2} \Delta r &= \kappa r D_t^{2j-3}(D_t \Delta r) \\ &= \kappa r D_t^{2j-3}(\Delta D_t r - \Delta v \cdot \nabla r - \nabla v \nabla^2 r) \\ &= \kappa r D_t^{2j-3}(\Delta D_t r) - \kappa r D_t^{2j-3}(\Delta v \cdot \nabla r) - \kappa r D_t^{2j-3}(\nabla v \nabla^2 r). \end{aligned} \tag{5.7}$$

The last term in the expression above gets absorbed into f_{2j}. For the next-to-last term we have

$$-\kappa r D_t^{2j-3}(\Delta v \nabla r) = -\kappa r \sum_{k=0}^{2j-3} \binom{2j-3}{\kappa} D_t^{2j-3-k}(\Delta v) D_t^k(\nabla r).$$

We distribute and commute all the material derivatives to observe that all but one term is readily in f_{2j} (commuting D_t with ∇, or even better with Δ gives rise to $\nabla v \cdot \nabla$, respectively (5.6) terms, which ensures the non-endpoint property), namely

$$\kappa D_t^{2j-3}(\Delta v) \nabla r.$$

For this we need to commute the material derivatives with Δ:

$$
\begin{aligned}
D_t^{2j-3}(\Delta v)\nabla r &= \Delta(D_t^{2j-3})v\nabla r \\
&= [D_t^{2j-3}, \Delta]v\nabla r + \Delta(D_t^{2j-3}v)\nabla r \\
&= [D_t^{2j-3}, \Delta]v\nabla r + \Delta v_{2j-3}\nabla r.
\end{aligned} \tag{5.8}
$$

The first term above is in f_{2j} and the last term is part of the expression in (5.2).

For the first term in (5.7), we commute D_t^{2j-3} with the Laplacian:

$$
\kappa r D_t^{2j-3}(\Delta D_t r) = \kappa r\{\Delta D_t^{2j-3}(D_t r) + [D_t^{2j-3}, \Delta]D_t r\}.
$$

We observe that the first term on the right-hand side above is $\kappa r\Delta D_t^{2j-3}(D_t r) = \kappa r\Delta r_{2j-2}$ which is one of the terms on the right-hand side of the expansion in (5.2). The last term is included in f_{2j}, as the commutator $[D_t^{2j-3}, \Delta]$, for $j \geq 2$, will produce at least one of each term in $\{\nabla v, \nabla r\}$.

We now deal with the last term in (5.3):

$$
\begin{aligned}
-\nabla r\cdot D_t^{2j-1}v &= -\nabla r\cdot D_t^{2j-2}(-\nabla r) \\
&= \nabla r\cdot D_t^{2j-3}(D_t\nabla r) \\
&= \nabla r\cdot D_t^{2j-3}([D_t, \nabla]r + \nabla D_t r) \\
&= \nabla r\cdot D_t^{2j-3}(-\nabla v\cdot\nabla r + \nabla D_t r).
\end{aligned} \tag{5.9}
$$

For the first term on the right-hand side of (5.9) we get

$$
\begin{aligned}
-\nabla r\cdot D_t^{2j-3}(\nabla v\cdot\nabla r) &= -\nabla r\cdot\sum_{k=0}^{2j-3}\binom{2j-3}{k} D_t^{2j-3-k}(\nabla v)D_t^k\nabla r \\
&= -\nabla r\cdot\sum_{k=0}^{2j-3}\binom{2j-3}{k} D_t^{2j-3-k}(\nabla v)D_t^{k-1}(D_t\nabla r),
\end{aligned}
$$

where we can, by inspection, see that almost all the terms are in f_{2j}, except for the case $k = 0$, i.e. the term $D_t^{2j-3}(\nabla v)\nabla r$. As before, we have

$$
D_t^{2j-3}(\nabla v)\nabla r = [D_t^{2j-3}, \nabla]v\nabla r + \nabla r\nabla D_t^{2j-3}v = [D_t^{2j-3}, \nabla]v\nabla r + \nabla r\nabla v_{2j-3},
$$

where the first term is in f_{2j} and the last one (together with ∇r from (5.9)) gives another term in (5.2), namely

$$
\nabla r\nabla v_{2j-3}\nabla r. \tag{5.10}
$$

Lastly, we return to the last term in (5.9),

$$
\nabla r\cdot D_t^{2j-3}(\nabla D_t r),
$$

which we rewrite as

$$\begin{aligned}\nabla r\cdot D_t^{2j-3}(\nabla D_t r) &= \nabla r\cdot([D_t^{2j-3},\nabla]D_t r+\nabla(D_t^{2j-2}r))\\ &= \nabla r\cdot[D_t^{2j-3},\nabla](D_t r)+\nabla r\cdot\nabla r_{2j-2}.\end{aligned}$$

This finishes the proof of (5.1) for the s_{2j} formula in the case $j\geq 3$: the first term is part of f_{2j} and the last one appears in (5.2).

The argument for the case $j=2$ is similar. The only difference occurs at the very end, where we collect the contribution of the last term in (5.8) (with the corresponding κr factor) and the expression in (5.10) and rewrite them as follows:

$$\kappa r\nabla r\Delta\nabla r+\nabla r\nabla^2 r\nabla r = L_1(\tfrac{1}{2}|\nabla r|^2)+\kappa r|\nabla^2 r|^2,$$

where the last term goes into f_4.

For the w_{2j} there is no difference in the case $j=2$. The formula we are asked to show is

$$w_{2j} = \kappa\nabla(r\nabla\cdot w_{2j-2})+\nabla(\nabla r\cdot w_{2j-2})+g_{2j}. \tag{5.11}$$

As before, we expand the left-hand side of (5.11) and peel off the terms that belong to g_{2j}, and then inspect that the remaining terms match its right-hand side:

$$w_{2j}=D_t^{2j-1}(D_t v)=-D_t^{2j-1}(\nabla r)=-D_t^{2j-2}(D_t\nabla r)=-D_t^{2j-2}(\nabla D_t r+\nabla v\cdot\nabla r),$$

which gives

$$w_{2j} = \kappa\{\nabla D_t^{2j-2}(r\nabla\cdot v)+[D_t^{2j-2},\nabla](r\nabla\cdot v)\}-D_t^{2j-2}(\nabla v\cdot\nabla r) := \mathrm{I}+\mathrm{II}+\mathrm{III}.$$

The commutator term II gets absorbed into g_{2j}. For I we note that all but one of the terms have the non-endpoint property, namely $\kappa\nabla(r\nabla D_t^{2j-2}v)=\kappa\nabla(r\nabla w_{2j-2})$, which is part of the right-hand side of (5.11). Lastly, for III we have

$$D_t^{2j-2}(\nabla v\cdot\nabla r) = \sum_{m=0}^{2j-2-m}\binom{2j-2}{m} D_t^{2j-2-m}(\nabla v)\cdot D_t^m(\nabla r).$$

The case $m=0$ gives

$$D_t^{2j-2}(\nabla v)\cdot\nabla r = (\nabla D_t^{2j-2}v+[D_t^{2j-2},\nabla]v)\cdot\nabla r.$$

The commutator term belongs to g_{2j}, and hence we are left with

$$\nabla v_{2j-2}\cdot\nabla r,$$

which is again part of the right-hand side of (5.11). ∎

To take advantage of the above recurrence lemma, we will need a pair of elliptic estimates for the operators L_1, L_2. There is one small matter to address, which is that we would like these bounds to depend only on our control parameter A, whereas L_2 contains second derivatives of r in the coefficients. This can be readily rectified by replacing L_2 by

$$\tilde{L}_2 = \kappa \nabla r \nabla + \nabla r \nabla,$$

or in coordinates, to avoid ambiguity in notation,

$$(\tilde{L}_2)_{ij} = \kappa \partial_i r \partial_j + \partial_j r \partial_i.$$

We note that the difference between $L_2 w$ and $\tilde{L}_2 w$ is the expression $\nabla^2 r w$, whose contribution can be harmlessly placed in g_{2j} in (5.1).

Set

$$\sigma := \frac{1}{2\kappa}.$$

Then we have the following lemma:

Lemma 5.2. *Assume that A is small. Then the following elliptic estimates hold:*

$$\|s\|_{H^{2,\sigma+\frac{1}{2}}} \lesssim \|L_1 s\|_{H^{0,\sigma-\frac{1}{2}}} + \|s\|_{H^{0,\sigma+\frac{1}{2}}}, \tag{5.12}$$

respectively

$$\|w\|_{H^{2,\sigma+1}} \lesssim \|\tilde{L}_2 w\|_{H^{0,\sigma}} + \|\operatorname{curl} w\|_{H^{1,\sigma+1}} + \|w\|_{H^{0,\sigma+1}} \tag{5.13}$$

and

$$\|w\|_{H^{2,\sigma+1}} \lesssim \|(\tilde{L}_2 + L_3) w\|_{H^{0,\sigma}} + \|w\|_{H^{0,\sigma+1}}. \tag{5.14}$$

Remark 5.3. We note that in essence this estimate has a scale-invariant nature. The lower-order term added on the right plays no role in the proof, and can be dropped if either (s, w) are assumed to have small support (by the Poincaré inequality), or if we use the corresponding homogeneous norms on the left.

We will in fact need a more general result, where the L_1 and $\tilde{L}_2$ operators are replaced by L_1^b and $\tilde{L}_2^b$, respectively, where $b > 0$:

Corollary 5.4. *The results in Lemma 5.2 also hold when L_1 and $\tilde{L}_2$ are replaced by L_1^b and $\tilde{L}_2^b$, for $b > 0$, where*

$$L_1^b = (\kappa r \nabla + (1 + b\kappa)\nabla r) \cdot \nabla, \quad \tilde{L}_2^b := \kappa \nabla r \nabla + (1 + \kappa b) \nabla r \nabla.$$

This is a direct consequence of the proof of Lemma 5.2, rather than of the lemma.

Proof of Lemma 5.2. We first observe that the bound (5.13) is a direct consequence of (5.14) since $L_3 w$ is a function of $\operatorname{curl} w$. Hence it suffices to prove (5.12) and (5.14).

Before we delve fully into the proof, we note that we have the relatively standard weaker elliptic bounds

$$\|s\|_{H^{2,\sigma+\frac{1}{2}}} \lesssim_A \|L_1 s\|_{H^{0,\sigma-\frac{1}{2}}} + \|s\|_{H^{1,\sigma-\frac{1}{2}}},$$

respectively

$$\|w\|_{H^{2,\sigma+1}} \lesssim_A \|(\tilde{L}_2 + L_3)w\|_{H^{0,\sigma}} + \|w\|_{H^{1,\sigma}}.$$

For these bounds we only need integration by parts, treating the first-order terms in both L_1 and $\tilde{L}_2 + L_3$ perturbatively, and using only the pointwise bound for ∇r. We leave this straightforward computation to the reader.

Taking the above bounds into account, our bounds (5.12) and (5.14) reduce to the scale-invariant estimates

$$\|\nabla s\|_{H^{0,\sigma+\frac{1}{2}}} \lesssim \|L_1 s\|_{H^{0,\sigma-\frac{1}{2}}},$$

respectively

$$\|\nabla w\|_{H^{0,\sigma+1}} \lesssim \|(\tilde{L}_2 + L_3)w\|_{H^{0,\sigma}}.$$

We consider (5.12) first, where we proceed using a simple integration by parts. To avoid differentiating r twice, we assume that at some point $\nabla r(x_0) = e_n$. Then in our domain we have

$$|\nabla r - e_n| \lesssim A \ll 1.$$

We compute

$$\begin{aligned}\int r^{\frac{1-\kappa}{\kappa}} (\kappa r \nabla + \nabla r)\nabla s \cdot \partial_n s \, dx &= \int \kappa r^{\frac{1}{\kappa}} \Delta s \partial_n s + r^{\frac{1-\kappa}{\kappa}} (|\partial_n s|^2 + O(A)|\nabla s|^2) \, dx \\ &= \frac{1}{2} \int r^{\frac{1-\kappa}{\kappa}} |\nabla s|^2 + O(A)|\nabla s|^2 \, dx,\end{aligned}$$

which suffices by the Cauchy–Schwarz inequality.

Next we consider the bound (5.13) for the v component, where

$$\begin{aligned} r^{\frac{1}{\kappa}}((\tilde{L}_2 + L_3)w)_i &= \kappa[\partial_i r \partial_j w_j + \partial_j r(\partial_j w_i - \partial_i w_j)] \\ &\quad + \partial_j r \partial_i w_j + \partial_j r(\partial_j w_i - \partial_i w_j) \\ &= \kappa[\partial_j(r^{\frac{1}{\kappa}+1}\partial_j w_i) + r^{\frac{1}{\kappa}}(\partial_i r \partial_j w_j - \partial_j r \partial_i w_j)]. \end{aligned}$$

We use a computation similar to the one before, integrating by parts and using the fact that all the tangential derivatives of r are $O(A)$ and its normal derivative is $1 + O(A)$:

$$\begin{aligned} &\int r^{\frac{1}{\kappa}}(\tilde{L}_2 + L_3)w \cdot \partial_n w \, dx \\ &\quad = \kappa \int -r^{\frac{1}{\kappa}+1}\partial_j w_i \partial_n \partial_j w_i + r^{\frac{1}{\kappa}}\big((\partial_i r \partial_j w_j - \partial_j r \partial_i w_j)\partial_n w_i + O(A)|\nabla w|^2\big) \, dx \\ &\quad = \kappa \int r^{\frac{1}{\kappa}}\Big[\frac{1}{2}\Big(\frac{1}{\kappa} + 1\Big)|\partial_j w_i|^2 + \partial_j w_j \partial_n w_n - \partial_n w_j \partial_j w_n + O(A)|\nabla w|^2\Big] \, dx. \end{aligned}$$

We claim that the above expression can be bounded from below by

$$\geq (1 - O(A)) \int r^{\frac{1}{\kappa}} |\nabla w|^2 \, dx.$$

To see that, we cancel the two $|\partial_n w_n|^2$ terms, and restricting indices below to $k, m \neq n$, we have to show that

$$-\int r^{\frac{1}{\kappa}}(\partial_k w_k \partial_n w_n - \partial_n w_k \partial_k w_n)\, dx \lesssim \frac{1}{2}\int r^{\frac{1}{\kappa}}[|\partial_j w_i|^2 + O(A)|\nabla w|^2]\, dx. \quad (5.15)$$

Indeed, we can bound the expression on the left by Cauchy–Schwarz as

$$\begin{aligned}
&-\int r^{\frac{1}{\kappa}}(\partial_k w_k \partial_n w_n - \partial_n w_k \partial_k w_n)\, dx \\
&\quad \lesssim \frac{1}{2}\int r^{\frac{1}{\kappa}}\left(\left|\sum_{k=1}^{n-1}\partial_k w_k\right|^2 + |\partial_n w_n|^2 + \sum_{k=1}^{n-1}(|\partial_n w_k|^2 + \partial_k w_n|^2)\right) dx.
\end{aligned}$$

If we can establish that the first term on the right admits the equivalent representation

$$\int r^{\frac{1}{\kappa}}\left|\sum_{k=1}^{n-1}\partial_k w_k\right|^2 dx = \int r^{\frac{1}{\kappa}}\left(\sum_{k,m=1}^{n-1}\partial_k w_m \partial_m w_k + O(A)|\nabla w|^2\right) dx,$$

then (5.15) follows by one more application of Cauchy–Schwarz. This last bound, in turn, reduces to the relation

$$I_{km} := \int r^{\frac{1}{\kappa}}(\partial_k w_m \partial_m w_k - \partial_m w_m \partial_k w_k)\, dx = O(A)\int r^{\frac{1}{\kappa}}|\nabla w|^2\, dx. \quad (5.16)$$

In the model case $r = x_n$, the left-hand side is exactly zero, integrating by parts. In the general case, we arrive at almost the same result after a more careful integration by parts:

$$\begin{aligned}
I_{km} &= \int r^{\frac{1}{\kappa}+1}\partial_n(\partial_k w_m \partial_m w_k - \partial_m w_m \partial_k w_k)\, dx + O(A)\int r^{\frac{1}{\kappa}}|\nabla w|^2\, dx \\
&= \int r^{\frac{1}{\kappa}+1}\partial_k(\partial_n w_m \partial_m w_k - \partial_m w_m \partial_n w_k) + \partial_m(\partial_n w_k \partial_k w_m - \partial_k w_k \partial_n w_m)\, dx \\
&\quad + O(A)\int r^{\frac{1}{\kappa}}|\nabla w|^2\, dx \\
&= O(A)\int r^{\frac{1}{\kappa}}|\nabla w|^2\, dx.
\end{aligned}$$

This concludes the proof of (5.16), and thus the proof of the lemma. ■

The above setup suffices in order to prove our coercivity bounds. We will successively establish the estimates

$$\begin{aligned}
&\|(s_{2j-2}, w_{2j-2})\|_{\mathcal{H}^{2k-2j+2}} \\
&\quad \lesssim \|(s_{2j}, w_{2j})\|_{\mathcal{H}^{2k-2j}} + O(A)\|(r,v)\|_{\mathcal{H}^{2k}}, \quad 1 \le j \le k. \quad (5.17)
\end{aligned}$$

Concatenating these bounds we get the desired estimates in the theorem, where the errors are absorbed using the smallness condition $A \ll 1$.

The case $j = k$ follows directly from Lemma 5.2 above, using the interpolation estimates to get smallness for (f_{2k}, g_{2k}), in the sense that

$$\|(f_{2k}, g_{2k})\|_{\mathcal{H}} \lesssim_A A\|(r, v)\|_{\mathcal{H}^{2k}}.$$

The case $2 \leq j < k$ requires an additional argument. Precisely, we will seek to apply Lemma 5.2 to functions (s, w) of the form

$$s = Ls_{2j-2}, \quad w = Lw_{2j-2},$$

where L is any operator in the right class,

$$L = r^a \partial^b, \quad 2a \leq b \leq 2(k - j).$$

To do that, we need to have a good relation between $L(s_{2j}, w_{2j})$ and $L(s_{2j-2}, w_{2j-2})$. To achieve this, we apply L in (5.1). For s_{2j} this yields

$$L_1 L s_{2j-2} = L s_{2j} - [L, L_1] s_{2j-2} - L f_{2j},$$

where we need to examine the commutator term more closely. To keep the analysis simple it suffices to argue by induction on a, beginning with $a = 0$. All terms in the commutator, where at least one r factor gets differentiated twice, are non-endpoint terms, and can be estimated by interpolation. All terms in the commutator where two r factors get differentiated are taken care of by the induction in a. Finally, all terms where only one r term is differentiated are also taken care of by the induction in a, unless $a = 0$. Thus if $a > 0$ then all commutator terms are estimated either as error terms or via the induction hypothesis.

So the only nontrivial case is when $a = 0$. In this case it is convenient to consider a frame (x', x_n) adapted to the free surface, so that

$$|\partial' r| \lesssim A, \quad |\partial_n r - 1| \lesssim A.$$

Then all commutators with tangential derivatives are error terms, and the only nontrivial commutator terms are those with ∂_n. For these, we write modulo good $O(A)$ error terms

$$[\partial_n^b, L_1] \approx b\Delta\partial_n^{b-1} \approx b\nabla r \cdot \nabla\partial_n^b + b\partial_n^{b-1}(\partial')^2.$$

The contribution of the first term on the right can be included in L_1, akin to a conjugation. The contribution of the second term on the right can be viewed as an induction term if we phrase the argument as an induction in the number b of normal derivatives. Then we can write

$$\partial_n^b L_1 \approx L_1^b \partial_n^b,$$

where

$$L_1^b = (\kappa r\nabla + (1 + b\kappa)\nabla r) \cdot \nabla,$$

for which we can still apply the analysis in Lemma 5.2.

Finally, we consider the case $j = 1$, where the relation in Lemma 5.1 is not exactly true, but it is essentially true once we differentiate at least twice. Precisely, we compute

$$s_2 = \kappa r \Delta r + \frac{1}{2}|\nabla r|^2 + rO(|\nabla v|^2).$$

Instead of comparing s_2 with $L_1 s_0$, we compare Ls_2 with $L_1 L s_0$, where as before $L = r^a \partial^b$. Here we must have $b \geq 2$, so we begin with the case $a = 0$ and $b = 2$. For tangential derivatives we get modulo $O(A)$ error terms

$$\partial^b s_2 \approx L_1 \partial^b s_0,$$

while for normal derivatives

$$\partial_n^b s_2 \approx L_1^b \partial_n^b s_0.$$

From here on the argument is similar to the $j > 2$ case.

The analysis is similar in the case of L_2, which, we recall, has the form

$$L_2 = \nabla(\kappa r \nabla + \nabla r).$$

For this we can write a similar conjugation relation, again modulo $O(A)$ perturbative and induction terms,

$$\partial_n^b L_2 \approx L_2^b \partial_n^b,$$

where

$$L_2^b = \nabla(\kappa r \nabla + (1 + \kappa b)\nabla r).$$

Substituting L_2^b with $\tilde{L}_2^b$, we can then apply the elliptic bounds in Corollary 5.4. ■

5.5. Energy estimates

Here we prove energy estimates in $\mathcal{H}^{2k}$ for solutions (r, v). We recall the equations:

$$\begin{cases} r_t + v\nabla r + \kappa r \nabla v = 0, \\ v_t + (v \cdot \nabla)v + \nabla r = 0, \end{cases}$$

or, with D_t,

$$\begin{cases} D_t r + \kappa r \nabla v = 0, \\ D_t v + \nabla r = 0. \end{cases}$$

We will also use the transport equation for $\omega = \operatorname{curl} v$,

$$D_t \omega = -\omega \cdot \nabla v - (\nabla v)^{\mathsf{T}} \omega.$$

Now we consider the higher Sobolev norms $\mathcal{H}^{2k}$. For these we will prove the following:

Theorem 7. *The energy functional E^{2k} in $\mathcal{H}^{2k}$ has the following two properties:*

(a) *Norm equivalence:*

$$E^{2k}(r,v) \approx_A \|(r,v)\|^2_{\mathcal{H}^{2k}}.$$

(b) *Energy estimate:*

$$\frac{d}{dt}E^{2k}(r,v) \lesssim_A B\|(r,v)\|^2_{\mathcal{H}^{2k}}.$$

The first part of the theorem, i.e. the coercivity, was proved in the previous subsection. To prove the second part of the theorem we will separately estimate the time derivative of each component in E^{2k}. The first step in that is to derive the equations satisfied by the functions used in the definition of the energy.

(I) *The wave component.* Here we will show that (s_{2k}, w_{2k}) is a good approximate solution to the linearized equation:

Lemma 5.5. *Let $k \geq 1$. The functions (s_{2k}, w_{2k}) solve the equations*

$$\begin{cases} D_t s_{2k} + w_{2k}\cdot\nabla r + \kappa r\nabla w_{2k} = f_{2k}, \\ D_t w_{2k} + \nabla s_{2k} = g_{2k}, \end{cases} \tag{5.18}$$

where f_{2k} and g_{2k} are non-endpoint[15] *multilinear expressions in r, ∇v of order $k-\frac{1}{2}$, respectively k, with exactly $2k+1$ derivatives.*

Proof. The assertions about the order and the number of derivatives are obvious. It r-mains to show that no single factor in f_{2k}, respectively g_{2k} has order larger than $k-1$, respectively $k-\frac{1}{2}$. In other words, we want to see that each product in f_{2k}, respectively g_{2k}, has at least two factors of the form $\partial^{2+}r$ or $\partial^{1+}v$.

We begin with f_{2k}:

$$\begin{aligned} f_{2k} &= D_t(D_t^{2k}r - \nabla r\cdot D_t^{2k-1}v) + \nabla r\cdot D_t^{2k}v + \kappa r\nabla D_t^{2k}v \\ &= -\kappa(D_t^{2k}(r\nabla v) - r\nabla D_t^{2k}v) - D_t(\nabla r)D_t^{2k-1}v. \end{aligned}$$

The first term has a commutator structure involving $[D_t^{2k}, r\nabla]$ which yields at least a ∇v coefficient. The same happens with $D_t\nabla r$ in the second term.

We continue with g_{2k}:

$$\begin{aligned} g_{2k} &= D_t^{2k+1}v + \nabla(D_t^{2k}r - \nabla r\cdot D_t^{2k-1}v) \\ &= -D_t^{2k}\nabla r + \nabla D_t^{2k}r - \nabla r\cdot\nabla D_t^{2k-1} + \nabla^2 r\nabla D_t^{2k-1}v. \end{aligned}$$

Here we are commuting D_t^{2k} with ∇, which yields at least a ∇v term. The only case when we do not get the desired structure is if the commutator occurs at the level of the last D_t,

$$[D_t^{2k},\nabla] = [D_t^{2k-1},\nabla]D_t + D_t^{k-1}[D_t,\nabla].$$

[15] We recall that this means that there is no single factor in f_{2k}, respectively g_{2k} which has order larger than $k-1$, respectively $k-\frac{1}{2}$. Equivalently, each of them has at least two $\partial^{2+}r$ or ∂v factors.

The contribution of the first term is always balanced. However, for the second term we have

$$[D_t, \nabla]r = -\nabla v \cdot \nabla r.$$

Thus we get a possibly unbalanced contribution if all of D_t^{2k-1} applies to v. We obtain

$$g_{2k} = \partial_i r \partial_j D_t^{2k-1} v_i - \partial_i r \partial_j D_t^{2k-1} v_i + \text{balanced} = \text{balanced}.$$

The computation for $k = 1$ is similar but simpler, and it is omitted. ∎

(II) *The transport component.* Here, the functions whose weighted L^2 norms we are trying to propagate are denoted by ω_{2k}, and have the form

$$\omega_{2k} = r^a \partial^b \omega, \quad |b| \leq 2k - 1, \quad b - a = k - 1.$$

For these functions we have the following lemma:

Lemma 5.6. *The functions ω_{2k} are approximate solutions for the transport equation*

$$D_t \omega_{2k} = h_{2k}, \tag{5.19}$$

where the h_{2k} are non-endpoint multilinear expressions in r, ∇v of order $2k$ with exactly k derivatives.

Proof. We compute the transport equation

$$D_t \omega_{2k} = h_{2k},$$

where we write schematically

$$h_{2k} = D_t(r^k \partial^{2k-1} \omega) = [D_t, r^k \partial^{2k-1}]\omega - r^k \partial^{2k-1} (\nabla v)^2.$$

This proves that all terms in h_{2k} are balanced, since all commutators include ∇v factors. ∎

To conclude the proof of the energy estimates it remains to bound the time derivative of the linearized energies

$$\|(s_{2k}, w_{2k})\|_{\mathcal{H}}^2, \quad \|\omega_{2k}\|_{L^2_\sigma}^2$$

by $\lesssim_A B\|(r, v)\|_{\mathcal{H}^{2k}}$. In view of our energy estimates for the linearized equation, respectively the transport equation, in order to obtain the desired estimate it suffices to bound the source terms (f_{2k}, g_{2k}), respectively h_{2k}:

Lemma 5.7. *The expressions f and g above satisfy the scale-invariant bounds*

$$\|(f_{2k}, g_{2k})\|_{\mathcal{H}} + \|h_{2k}\|_{H^{0,\sigma}} \lesssim_A B\|(r, v)\|_{\mathcal{H}^{2k}}. \tag{5.20}$$

Proof. This follows using our interpolation inequalities in Propositions 2.14, 2.15, and 2.16, following the same argument as in part (a) of the proof of Theorem 6.

The control parameter A gives L^∞ control at degree 0, i.e. for $\|\nabla r\|_{L^\infty}$ and $\|v\|_{\dot{C}^{\frac{1}{2}}}$, and B gives L^∞ control at degree $\frac{1}{2}$, i.e. for $\|\nabla v\|_{L^\infty}$ and $\|\nabla r\|_{\tilde{C}^{0,\frac{1}{2}}}$.

We consider the factors in each multilinear expression in f_{2k}, g_{2k}, and h_{2k} as follows. The factors of order $-\frac{1}{2}$ (i.e. the r factors) are interpreted as weights, and distributed to the other factors. The factors of order 0 in f_{2k}, g_{2k}, h_{2k} (i.e. ∂r factors) are directly estimated in L^∞ by A and discarded. The factors of maximum order are estimated directly by $\|(r,v)\|_{\mathcal{H}^{2k}}$. The intermediate factors can be estimated in L^p norms in two ways, by interpolating the $\mathcal{H}^{2k}$ norm with A, or by interpolating with B.

Overall the product needs to be estimated in L^2, using exactly one $\|(r,v)\|_{\mathcal{H}^{2k}}$ factor. Then a scaling analysis shows that we will have to use exactly one B norm, i.e. for instance for monomials f_{2k}^m of order m in f_{2k} we have

$$\|f_{2k}^m\|_{H^{0,\sigma-\frac{1}{2}}} \lesssim A^{m-2} B \|(r,v)\|_{\mathcal{H}^{2k}}.$$

This is exactly as in the proof of Theorem 6 (a); the details are left for the reader. ∎

6. Construction of regular solutions

This section contains the first part of the proof of our well-posedness result; precisely, here we give a constructive proof of existence of regular solutions. The rough solutions will be obtained in the following section as unique limits of regular solutions.

Given initial data (r_0, v_0) with regularity

$$(r_0, v_0) \in \mathbf{H}^{2k},$$

where k is assumed to be sufficiently large, we will construct a local-in-time solution with a lifespan depending on the $\mathbf{H}^{2k}$ size of the data. Unlike all prior works on this problem, which use parabolic regularization methods in Lagrangian coordinates, here we propose a new approach, implemented fully within the setting of the Eulerian coordinates.

Our novel method is loosely based on nonlinear semigroup methods, where an approximate solution is constructed by discretizing the problem in time. Then the challenge is to carry out a time step construction which, on one hand, is as simple as possible, but where, on the other hand, the uniform-in-time energy bounds survive. In a classical semigroup approach this would require solving an elliptic free boundary problem, with very precise estimates. At the other extreme, in a pure ODE setting one could simply use an Euler-type method. The Euler method cannot work here, because it would lose derivatives. A better alternative would be to combine an Euler method with a transport part; this would reduce, but not eliminate, the loss of derivatives.

The idea of our approach is to retain the simplicity of the Euler + transport method, while preventing the loss of derivatives by an initial regularization step. Then the regularization step becomes the more delicate part of the argument, because it also needs to have good energy bounds. To achieve that, we carry out the regularization in a paradifferential fashion, but in a setting where we are avoiding the use of complicated classes of pseudo-differential operators. Thus, in a nutshell, our solution is to divide and conquer, splitting the time step into three:

- regularization,
- transport,
- Euler's method,

where the role of the first two steps is to improve the error estimate in the third step.

To summarize, our approach provides a new, simpler method to construct solutions in the context of free boundary problems. Further, we believe it will prove useful in a broader class of problems.

6.1. A few simplifications

To keep our construction as simple as possible, we observe here that we can make a few simplifying assumptions:

(i) By finite speed of propagation and Galilean invariance, we can assume that v vanishes and r is linear outside a small compact set.

(ii) Given the reduction in step (i), the coercivity bound (5.14) proved in Lemma 5.2 carries over to the operator $L_2 + L_3$. This yields a natural div-curl orthogonal decomposition for v in $\mathcal{H}$,

$$v = L_2(L_2 + L_3)^{-1} + L_3(L_2 + L_3)^{-1} v := v_1 + v_2,$$

where the first component is a gradient and the second depends only on curl v. In particular, it follows that we have

$$\begin{aligned} \| \operatorname{curl} v\|^2_{H^{2k-1,\frac{1}{\kappa}}} &= \| \operatorname{curl} v_2\|^2_{H^{2k-1,\frac{1}{\kappa}}} \\ &\approx \sum_{j=0}^{k} \|(L_2 + L_3)^j v_2\|^2_{H^{0,\frac{1}{\kappa}}} \\ &\approx \| \operatorname{curl} v\|^2_{H^{0,\frac{1}{\kappa}}} + \sum_{j=1}^{k} \|L_3^j v\|^2_{H^{0,\frac{1}{\kappa}}}, \end{aligned}$$

where we refer the reader to Lemma 6.5 below for the second step. This allows us to make the simplified choice

$$E_t^{2k}(r, v) = \| \operatorname{curl} v\|^2_{H^{0,\frac{1}{\kappa}}} + \sum_{j=1}^{k} \|L_3^j v\|^2_{H^{0,\frac{1}{\kappa}}} \tag{6.1}$$

for the transport component of the energy.

6.2. Construction of approximate solutions

Given a small time step $\varepsilon > 0$ and initial data $(r_0, v_0) \in \mathbf{H}^{2k}$, we will produce a discrete approximate solution $(r(j\varepsilon), v(j\varepsilon))$, with the following properties:

- Norm bound: We have

$$E^{2k}\big(r((j+1)\varepsilon), v((j+1)\varepsilon)\big) \leq (1 + C\varepsilon)E^{2k}\big(r((j\varepsilon), v(j\varepsilon))\big).$$

- Approximate solution:

$$\begin{cases} r((j+1)\varepsilon) - r(j\varepsilon) + \varepsilon\big[v(j\varepsilon)\nabla r(j\varepsilon) + \kappa r(j\varepsilon)\nabla \cdot v(j\varepsilon)\big] = O(\varepsilon^{1+}), \\ v((j+1)\varepsilon) - v(j\varepsilon) + \varepsilon\big[(v(j\varepsilon)\cdot\nabla)v(j\varepsilon) + \nabla r(j\varepsilon)\big] = O(\varepsilon^{1+}). \end{cases}$$

The first property will ensure a uniform energy bound for our sequence. The second property will guarantee that in the limit we obtain an exact solution. There we can use a weaker topology, where the exact choice of norms is not so important.

Having such a sequence of approximate solutions, it will be a fairly simple matter to produce, as the limit on a subsequence, an exact solution (r, v) on a short time interval which stays bounded in the above topology. The key point is the construction of the above sequence. It suffices to carry out a single step:

Theorem 8. *Let k be a large enough integer such that the following energy bound holds:*

$$E^{2k}(r_0, v_0) \leq M, \tag{6.2}$$

and $\varepsilon \ll 1$. Then there exists a one step iterate (r_1, v_1) with the following properties:

(1) *Norm bound: We have*

$$E^{2k}(r_1, v_1) \leq (1 + C(M)\varepsilon)E^{2k}(r_0, v_0). \tag{6.3}$$

(2) *Approximate solution:*

$$\begin{cases} r_1 - r_0 + \varepsilon[v_0\nabla r_0 + \kappa r_0 \nabla v_0] = O(\varepsilon^2), \\ v_1 - v_0 + \varepsilon[(v_0\cdot\nabla)v_0 + \nabla r_0] = O(\varepsilon^2). \end{cases} \tag{6.4}$$

The remainder of this subsection is devoted to the proof of this theorem.

We begin with a straightforward observation, namely that a direct iteration (Euler's method) loses derivatives. A better strategy would be to separate the transport part; this reduces (halves) the derivative loss, but does not fully eliminate it. However, if we precede this by an initial regularization step, then we can avoid the loss of derivatives altogether. In a nutshell, this will be our strategy. We begin with the outcome of the regularization step.

Proposition 6.1. *Given $(r_0, v_0) \in \mathbf{H}^{2k}$ as in (6.2), there exists a regularization (r, v) with the following properties:*

$$r - r_0 = O(\varepsilon^2), \quad v - v_0 = O(\varepsilon^2), \tag{6.5}$$

respectively

$$E^{2k}(r,v) \le (1+C\varepsilon)E^{2k}(r_0,v_0), \tag{6.6}$$

and

$$\|(r,v)\|_{\mathcal{H}^{2k+2}} \lesssim \varepsilon^{-1}M. \tag{6.7}$$

We postpone for the moment the proof of the proposition, and instead we show how to use it to prove the result in Theorem 8.

Proof of Theorem 8. Here we construct (r_1, v_1) starting from (r, v) given by the last proposition. Naively, the remaining steps are the Euler iteration

$$\begin{cases} r_1 = r - \varepsilon\kappa r\nabla v, \\ v_1 = v - \varepsilon\nabla r, \end{cases}$$

and the flow transport

$$x_1 = x + \varepsilon v(x). \tag{6.8}$$

The important point is that these two steps cannot be carried out separately, as each of them taken alone seems to be unbounded. Instead, taken together there is an extra cancellation to be taken advantage of, which is the direct analogue of a similar cancellation in the energy estimates. Using the transport as above, (r_1, v_1) are defined as

$$\begin{cases} r_1(x_1) = r(x) - \varepsilon\kappa r(x)\nabla v(x), \\ v_1(x_1) = v(x) - \varepsilon\nabla r(x). \end{cases}$$

It remains to show that these have the properties in the proposition. We begin by observing that

$$r_1(x_1) = r(x)(1 + O(\varepsilon)),$$

so these can be used interchangeably as weights. We also have

$$dx_1 = dx(1 + O(\varepsilon)),$$

so the same can be said for the measures of integration.

We successively compute D_t derivatives of (r_1, v_1) in terms of similar derivatives of (r, v). We will work with operators of the form D_t^{2j}. As before, when applied to a data set (r, v), these are interpreted as multilinear partial differential expressions, as if they were applied to a solution and then re-expressed, using the equations, in terms of the initial data. In particular, we recall that the expressions $D_t^{2j} r$ and $D_t^{2j} v$ have orders $(j-2)/2$, respectively $(j-1)/2$.

Switching from derivatives in x to derivatives in x_1 is done by repeated applications of the chain rule, which involves the Jacobian

$$J = (I + \varepsilon Dv)^{-1}.$$

Thus, in this calculation we will not only produce multilinear expressions, but also powers of J. To describe errors, we will enhance our standard notion of order by assigning the order $-\frac{1}{2}$ to ε; this is natural because, as a time step, ε can be thought of as the dual variable to D_t. Such a choice will ensure that the expression $\varepsilon\nabla v$ has order 0, and that all our relations below are homogeneous. Then we have the following lemma:

Lemma 6.2. (a) *The following algebraic relations hold:*

$$\begin{cases} D_t^{2j} r_1(x_1) = D_t^{2j} r(x) + \varepsilon D_t^{2j+1} r(x) + \varepsilon^2 R_{2j}(r, v, \varepsilon\nabla v)(x), \\ D_t^{2j} v_1(x_1) = D_t^{2j} v(x) + \varepsilon D_t^{2j+1} v(x) + \varepsilon^2 V_{2j}(r, v, \varepsilon\nabla v)(x), \end{cases}$$

where R_j and V_j are multilinear expressions in $(r, \nabla v, \varepsilon\nabla v)$ and their derivatives, and also J, with the following properties:

- *v does not appear undifferentiated.*
- *They have order 2, respectively $j + 1/2$.*
- *In addition to powers of J, they contain exactly $2j + 2$ derivatives applied to factors of r, v or $\varepsilon\nabla v$.*
- *They are balanced, i.e. they contain at least two $\partial^{2+} r$ or $\partial^{1+} v$ factors.*

(b) *Similar relations hold for $\omega = \operatorname{curl} v$ and its weighted derivatives ω_{2j},*

$$\omega_{2j,1}(x_1) = \omega_{2j}(x) - \varepsilon h_{2j} - \varepsilon^2 W_{2j}(\omega, v, \varepsilon\nabla v)(x),$$

where h_{2j} is as in (5.19) and W_{2j} has the same properties as R_{2j} and V_{2j} above.

Proof. We prove part (a), as part (b) is similar. As discussed earlier, transcribing the expression $D_t^j r_1(x_1)$ in terms of r and v is based on repeated application of the chain rule, which involves the Jacobian

$$J = (I + \varepsilon D v)^{-1},$$

and yields contributions of order zero. Thus one easily obtains

$$\begin{cases} D_t^j r_1(x_1) = D_t^j r(x) + \varepsilon \tilde{R}_j(r, v, \varepsilon\nabla v)(x), \\ D_t^j v_1(x_1) = D_t^j v(x) + \varepsilon \tilde{V}_j(r, v, \varepsilon\nabla v)(x), \end{cases}$$

where $\tilde{R}_j$ and $\tilde{V}_j$ are multilinear expressions in $(r, \nabla v, \varepsilon\nabla v)$ and with added powers of J and which have order $(j-1)/2$, respectively $j/2$, and exactly $j+1$ derivatives applied to factors of r, v or $\varepsilon\nabla v$.

It remains to identify the coefficients of the ε terms, which are

$$(\tilde{R}_j(r, \nabla v, 0), \tilde{V}_j(r, \nabla v, 0)).$$

Identifying ε with time t, and redenoting $(r_1, v_1) = (r(t), v(t))$, we have

$$(\tilde{R}_j(r, \nabla v, 0), \tilde{V}_j(r, \nabla v, 0)) = \frac{d}{dt}(D_t^j r(x), D_t^j v(x))_{t=0}.$$

But by construction the functions $(r(t), v(t))$ solve the equation at $t = 0$, so the desired identification holds. ■

Returning to the proof of the theorem, we note that the above lemma already gives the bound (6.4) in the uniform topology. It remains to prove the bound (6.3), where we have to compare $E^{2k}(r, v)$ with $E^{2k}(r_1, v_1)$. We recall that these energies have the wave component and the curl component. These are treated in a similar way, so we will focus on the wave component, which is more interesting. For this we need to compare the L^2-type norms of the good variables

$$\|(s_{2k}, w_{2k})\|^2_{\mathcal{H}_r}, \quad \|(s_{1,2k}, w_{1,2k})\|^2_{\mathcal{H}_{r_1}}.$$

The lower-order norms also need to be compared, but that is a straightforward matter. Note that these norms are represented as integrals over different domains. However, we identify these domains via (6.8), and we compare the corresponding densities accordingly.

For exact solutions, the good variables solve the linearized equations with source terms (5.18). For our iteration, the above lemma yields a similar relation with additional source terms,

$$\begin{cases} s_{2k,1} = s_{2k} - \varepsilon(w_{2k} \cdot \nabla r + \kappa r \nabla w_{2k}) - \varepsilon f_{2k} + \varepsilon^2 R_{2k}, \\ w_{2k,1} = w_{2k} - \varepsilon \nabla s_{2k} - \varepsilon g_{2k} + \varepsilon^2 V_{2k}, \end{cases}$$

where f_{2k}, g_{2k} are perturbative source terms as in Lemma 5.5, and (R_{2k}, V_{2k}) are as in the lemma above. The terms (f_{2k}, g_{2k}) satisfy the bound (5.20) in Lemma 5.7, which we recall here:

$$\|(f_{2k}, g_{2k})\|_{\mathcal{H}} \lesssim_A B\|(r, v)\|_{\mathcal{H}^{2k}},$$

which is what allows us to treat them as perturbative.

In a similar fashion, Lemma 5.7 shows that the expressions (R_{2k}, V_{2k}) satisfy

$$\|(R_{2k}, V_{2k})\|_{\mathcal{H}} \lesssim_A B\|(r, v)\|_{\mathcal{H}^{2k+1}}.$$

Since these terms have an ε^2 factor, the bound (6.7) also allows us to treat them as perturbative.

It remains to estimate the main expression, for which we compute

$$\begin{aligned} E_1 &= \|(s_{2k} - \varepsilon(w_{2k} \cdot \nabla r + \kappa r \nabla w_{2k}), w_{2k} - \varepsilon \nabla s_{2k})(x_1)\|^2_{\mathcal{H}_{r_1}} \\ &= \|(s_{2k} - \varepsilon(w_{2k} \cdot \nabla r + \kappa r \nabla w_{2k}), w_{2k} - \varepsilon \nabla s_{2k})\|^2_{\mathcal{H}_r} + C(M)\varepsilon \\ &= \|(s_{2k}, w_{2k})\|^2_{\mathcal{H}} - 2\varepsilon\big\langle(s_{2k}, w_{2k}), (w_{2k} \cdot \nabla r + \kappa r \nabla w_{2k}, \nabla s_{2k})\big\rangle_{\mathcal{H}} \\ &\quad + \varepsilon^2\|(w_{2k} \cdot \nabla r + \kappa r \nabla w_{2k}, \nabla s_{2k})\|^2_{\mathcal{H}} + C(M)\varepsilon. \end{aligned}$$

The second term can be seen to vanish after integrating by parts; this is the same cancellation seen in the proof of the energy estimates for the linearized equation. The third term, on the other hand, can be estimated as an error term via (6.7),

$$\|(w_{2k} \cdot \nabla r + \kappa r \nabla w_{2k}, \nabla s_{2k})\|_{\mathcal{H}} \lesssim \|(s_{2k}, w_{2k})\|^{\frac{1}{2}}_{\mathcal{H}} \|(s_{2k}, w_{2k})\|^{\frac{1}{2}}_{\mathcal{H}^2} \lesssim_M \varepsilon^{-1}.$$

This concludes the proof of the theorem. ■

Now we return to the proof of our regularization result in Proposition 6.1.

Proof of Proposition 6.1. We begin with a heuristic discussion, for which the starting point and the first candidate is the regularization already constructed in Proposition 2.11, with the matched parabolic frequency scale $2^{-2h} = \varepsilon$. This will satisfy properties (6.5) and (6.7), but it is not accurate enough for (6.6).

To improve on this and construct a better regularization we need to understand its effect on the energies, and primarily on the leading energy term which is $\|(s_{2k}, w_{2k})\|^2_{\mathcal{H}}$. For this we need to better understand the expressions for (s_{2k}, w_{2k}). We saw earlier that we have the approximate relations

$$s_{2k} \approx L_1 s_{2k-2}, \quad w_{2k} \approx L_2 w_{2k-2},$$

so one might expect that we have

$$s_{2k} \approx L_1^k r, \quad w_{2k} \approx L_2^k v.$$

However, this is not exactly accurate, as one can see by considering the first relation for $k = 1$. There

$$s_2 = \kappa r \Delta r + \frac{1}{2}|\nabla r|^2,$$

whereas

$$L_1 r = \kappa r \Delta r + |\nabla r|^2.$$

To rectify this discrepancy, we will interpret the operators L_1 and L_2 in a paradifferential fashion, i.e. decouple the r appearing in the coefficients of L_1 and L_2 from the r in the argument of L_1^k. Instead, the r in the coefficients will be harmlessly replaced with a regularized version of itself, call it r_-, and correspondingly L_1 and L_2 will be replaced by L_1^-, L_2^-. Then we will be able to write approximate relations of the form

$$s_2 \approx L_1^-(r - r_-) + s_2^-,$$

and further,

$$s_{2k} \approx (L_1^-)^k(r - r_-) + s_{2k}^-,$$

and similarly for w_{2k}.

Based on these considerations, we will construct our regularization as follows:

- Start with the initial state $(r_0, v_0) \in \mathbf{H}^{2k}$.
- Produce two initial regularizations r_+ and r_- of r_0, on scales $h^+ > h > h^-$, with slightly larger domains, and then restrict them to $\Omega^- = \{r_- > 0\}$.
- Use the self-adjoint operators L_1 and $L_2 + L_3$ associated to r_- to regularize the high-frequency part $(r_+ - r_-, v_+ - v_-)$ within Ω^- below frequency 2^h.
- Obtain the h-scale regularization $(\tilde{r}, \tilde{v})$ of (r_0, v_0) in Ω^-, by adding the low-frequency part (r_-, v_-) to the regularized high-frequency part.
- Decrease $\tilde{r}$ by a small constant $c = O(\varepsilon^4)$ and set $(r, v) = (\tilde{r} - c, \tilde{v})$, in order to ensure that $\Omega := \{r > 0\} \subset \Omega^-$.

1. A formal computation and the good variables. Both to motivate the definition of our regularization and as a tool to prove we have the correct regularization, here we consider the question of comparing the good variables (s^0_{2k}, w^0_{2k}) associated to (r_0, v_0) with $(\tilde{s}_{2k}, \tilde{w}_{2k})$ associated to $(\tilde{r}, \tilde{v})$. The lemma below is purely algebraic, and makes no reference to the relation between (r_0, v_0) and $(\tilde{r}, \tilde{v})$.

Each term in (s_{2k}, w_{2k}) is a multilinear expression of the same order in (r, v), so we will view the difference

$$(s^0_{2k}, w^0_{2k}) - (\tilde{s}_{2k}, \tilde{w}_{2k})$$

as a multilinear expression in $(r_0 - \tilde{r}, v_0 - \tilde{v})$ and $(\tilde{r}, \tilde{v})$. Heuristically, we will think of the first expression as the high-frequency part of (r_0, v_0) and the second expression as the low-frequency part. Since we are working here in high regularity, the intuition is that high–high terms will be better behaved and can be assigned to the error. Explicitly, we write

$$\begin{cases} s^0_{2k} = \tilde{s}_{2k} + Ds_{2k}(\tilde{r}, \tilde{v})(r_0 - \tilde{r}, v_0 - \tilde{v}) + F_{2k}, \\ w^0_{2k} = \tilde{w}_{2k} + Dw_{2k}(\tilde{r}, \tilde{v})(r_0 - \tilde{r}, v_0 - \tilde{v}) + G_{2k}, \end{cases} \tag{6.9}$$

where Ds_{2k} and Dw_{2k} stand for the differentials of s_{2k} and w_{2k} as functions of (r, v). This is akin to a paradifferential expansion of (s^0_{2k}, w^0_{2k}). In this expansion all terms on each line have the same order, which is $k-1$, respectively $k - \frac{1}{2}$, and (F_{2k}, G_{2k}) are at least bilinear in the difference $(r_0 - \tilde{r}, v_0 - \tilde{v})$.

The high–high terms (F_{2k}, G_{2k}) will play a perturbative role in our analysis. This leaves us with the terms which are linear in the difference, i.e. the low–high terms involving the two differentials Ds_{2k} and Dw_{2k}. We will further simplify this by observing that the low–high terms where the low-frequency factor is differentiated (i.e. has order > 0) are also favorable. This leaves us only with low–high terms with top order in the high-frequency factor in the leading part. These terms are identified in the following lemma:

Lemma 6.3. *We have the algebraic relations*

$$\begin{cases} Ds_{2k}(\tilde{r}, \tilde{v})(r_0 - \tilde{r}, v_0 - \tilde{v}) = (L_1(\tilde{r}))^k (r_0 - \tilde{r}) + \widetilde{F}_{2k}, \\ Dw_{2k}(\tilde{r}, \tilde{v})(r_0 - \tilde{r}, v_0 - \tilde{v}) = (L_2(\tilde{r}))^k (v_0 - \tilde{v}) + \widetilde{G}_{2k}, \end{cases} \tag{6.10}$$

where the error terms $(\widetilde{F}_{2k}, \widetilde{G}_{2k})$ *are linear in* $(r_0 - \tilde{r}, v_0 - \tilde{v})$,

$$\widetilde{F}_{2k} = D^1_{2k}(\tilde{r}, \tilde{v})(r_0 - \tilde{r}, v_0 - \tilde{v}), \quad \widetilde{G}_{2k} = D^2_{2k}(\tilde{r}, \tilde{v})(r_0 - \tilde{r}, v_0 - \tilde{v}),$$

whose coefficients are multilinear differential expressions in $(\tilde{r}, \tilde{v})$ *which contain at least one factor with order* > 0, *i.e.* $\partial^{2+}\tilde{r}$ *or* $\partial^{1+}\tilde{v}$.

We remark that combining (6.9) and (6.10) we obtain the expansion

$$\begin{cases} s^0_{2k} = \tilde{s}_{2k} + (L_1(\tilde{r}))^k (r_0 - \tilde{r}) + F_{2k} + \widetilde{F}_{2k}, \\ w^0_{2k} = \tilde{w}_{2k} + (L_2(\tilde{r}))^k (v_0 - \tilde{v}) + G_{2k} + \widetilde{G}_{2k}, \end{cases} \tag{6.11}$$

where all terms on each line are multilinear expressions in $(r_0 - \tilde{r}, v_0 - \tilde{v})$ and $(\tilde{r}, \tilde{v})$ of order $k-1$, respectively $k - \frac{1}{2}$, and whose multilinear error terms have either

(a) (high–high) two difference factors, i.e. (F_{2k}, G_{2k}), or

(b) (low–high balanced) exactly one difference factor, and at least one nondifference factor with order > 0, i.e. $(\widetilde{F}_{2k}, \widetilde{G}_{2k})$.

One should think of the above expansions as paradifferential linearizations, but implemented without using the paraproduct formalism.

Proof of Lemma 6.3. Our starting point is provided by relations (5.1), differentiated with respect to (r, v). This yields

$$Ds_{2j} = L_1(r) Ds_{2j-2} - DL_1(r) s_{2j-2} + Df_{2j}, \quad j \geq 2.$$

Since the expression f_{2j} is balanced, its differential can be included in D^1_{2k}. Similarly, the second expression on the right also has terms of order > 0 in (r, v). Thus we get

$$Ds_{2j} = L_1(r) Ds_{2j-2} + \widetilde{F}_{2j}, \quad j \geq 2. \tag{6.12}$$

Next we turn our attention to the case $j = 1$, where we have

$$s_2 = \kappa r \Delta r - \frac{1}{2} |\nabla r|^2 + f_2,$$

therefore

$$Ds_2 = \kappa r \Delta + \kappa \Delta r - \nabla r \nabla + Df_2,$$

where the second and fourth terms are admissible errors, so we also get (6.12). Then the conclusion of the lemma follows by reiterated use of (6.12). The argument for w_{2k} is similar. ∎

2. Regularizations for (r_0, v_0). We begin with the dyadic frequency scale h matching the time step ε, in a parabolic fashion, namely $2^{-2h} = \varepsilon$. As mentioned earlier, the direct regularization (r^h, v^h) of (r_0, v_0) given by Proposition 2.11 is not a sufficiently accurate regularization, in that it satisfies properties (6.5) and (6.7), but not necessarily (6.6).

Nevertheless, we will still use Proposition 2.11 to bracket our desired regularization as follows. Starting with the frequency scale h we define a lower- and a higher-frequency scale

$$1 \ll h^- < h < h^+,$$

where h^- and h^+ will be chosen later to satisfy a specific set of constraints. We remark for now that this is a soft choice, in that there is a large range of parameters that will work.

Correspondingly we consider the regularizations given by Proposition 2.11, denoted by

$$(r_+, v_+) = (r^{h^+}, v^{h^+}), \quad (r_-, v_-) = (r^{h^-}, v^{h^-}).$$

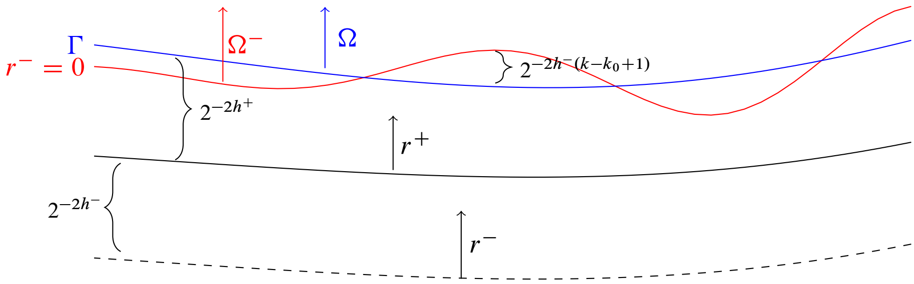

Figure 3. Domains associated with the regularization scheme.

These regularizations are defined on the enlarged domains $\tilde{\Omega}^{[h^+]}$, respectively $\tilde{\Omega}^{[h^-]}$; see Figure 3. We will use them on the domain $\Omega^- = \{r_- > 0\}$. By Proposition 2.11, this domain's boundary is at distance at most $2^{-2h^-(k-k_0+1)}$ from the original boundary Γ_0. To ensure that (r_+, v_+) are defined on this domain, we will impose the constraint

$$h^+ < h^-(k - k_0 + 1). \tag{6.13}$$

We will think of (r_-, v_-) as a "sub"-regularization, which has to be a part of $(\tilde{r}, \tilde{v})$, and of (r_+, v_+) as a "super"-regularization, in that $(\tilde{r}, \tilde{v})$ will be a regularization of it. We arrive at (r, v) in two steps:

(i) We define our first regularization $(\tilde{r}, \tilde{v})$ as smooth functions in Ω^- as follows:

$$\tilde{r} := r_- + \chi_\varepsilon(L_1(r_-))(r_+ - r_-),$$
$$\tilde{v} := v_- + \chi_\varepsilon((L_2 + L_3)(r_-))(v_+ - v_-),$$

where $\chi_\varepsilon(\lambda) := \chi(\lambda\varepsilon)$, with χ a smooth, positive bump function with values in $(0, 1)$ and the following asymptotics:

$$\begin{aligned} \chi(\lambda) &\approx 1 - \lambda \quad \text{near } \lambda = 0, \\ \chi(\lambda) &\approx 1/\lambda \quad \text{near } \lambda = \infty. \end{aligned} \tag{6.14}$$

(ii) The functions $(\tilde{r}, \tilde{v})$ in Ω^- are not yet the desired regularizations as $\tilde{r}$ does not vanish on the boundary Ω^-. If it were negative there, we would simply restrict them to $\Omega = \{r > 0\}$. Unfortunately, all we know is that for some large C we have

$$|\tilde{r}| \ll 2^{-2Ch} \quad \text{on } \Gamma^-.$$

Then we define

$$(r, v) := (\tilde{r} - 2^{-2Ch}, \tilde{v})$$

restricted to $\Omega = \{r > 0\}$ as our final regularization.

3. Bounds for the regularization $(\tilde{r}, \tilde{v})$. To start with, we have the bounds for $(r^{\pm}, v^{\pm})$ from Proposition 2.11. So here we consider the bounds for $(\tilde{r}, \tilde{v})$.

Lemma 6.4. *Assume that $\|(r_0, v_0)\|_{\mathbf{H}^{2k}} \leq M$. Then the following estimates hold for $(\tilde{r}, \tilde{v})$ in Ω^-:*

$$\|(\tilde{r}, \tilde{v})\|_{\mathcal{H}^{2k+2j}_{r_-}} \lesssim_M 2^{2hj}, \quad j = 0, 1, \tag{6.15}$$

respectively

$$\|(r_+ - \tilde{r}, v_+ - \tilde{v})\|_{\mathcal{H}^{2k-2}_{r_-}} \lesssim_M 2^{-2h}. \tag{6.16}$$

Proof. With $L_1 = L_1(r_-)$ and similarly for L_2 and L_3, we have the obvious bounds

$$\begin{aligned}\|(L_1^{k+j}\tilde{r}, (L_2 + L_3)^{k+j}\tilde{v})\|_{\mathcal{H}} &\lesssim 2^{2hj}(\|(L_1^k r_+, (L_2 + L_3)^k v_+)\|_{\mathcal{H}} \\ &\qquad + \|(L_1^k r_-, (L_2 + L_3)^k v_-)\|_{\mathcal{H}}) \\ &\lesssim_M 2^{2hj}.\end{aligned}$$

Then (6.15) follows from elliptic bounds for L_1, respectively $L_2 + L_3$, which for convenience we collect in the next lemma:

Lemma 6.5. *Assume that r satisfies*

$$\|(r, 0)\|_{\mathcal{H}^{2k}} \leq M$$

and

$$\|(r, 0)\|_{\mathcal{H}^{2k+2j}} \leq M 2^{2hj}, \quad 0 < j \leq N.$$

Then we have the estimates

$$\|(s, w)\|_{\mathcal{H}^{2k}} \lesssim_M \sum_{l=0}^{k} \|(L_1^l s, (L_2 + L_3)^l w)\|_{\mathcal{H}}, \tag{6.17}$$

respectively

$$\|(s, w)\|_{\mathcal{H}^{2k+2j}} \lesssim_M \varepsilon^{-2j} \sum_{l=0}^{k+j} \|(L_1^l s, (L_2 + L_3)^l w)\|_{\mathcal{H}}, \quad 0 < j \leq N. \tag{6.18}$$

Proof. The estimates in (6.17), respectively (6.18) will follow from the bounds

$$\|(s, w)\|_{\mathcal{H}^{2m}} \lesssim_M \|(L_1 s, (L_2 + L_3)w)\|_{\mathcal{H}^{2m-2}}, \quad 1 \leq m \leq k - 1, \tag{6.19}$$

respectively

$$\begin{aligned}\|(s, w)\|_{\mathcal{H}^{2k+2j}} \lesssim_M\ & \|(L_1 s, (L_2 + L_3)w)\|_{\mathcal{H}^{2k+2j-2}} \\ &+ \sum_{l=0}^{j-1} 2^{-2h(j-l)} \|(s, w)\|_{\mathcal{H}^{2m+2l}}, \quad j \geq 1.\end{aligned} \tag{6.20}$$

The bounds for s and the bounds for w are independent of each other. As the arguments are similar, we will prove the bounds for s and leave the bounds for w for the reader. We begin with (6.19), where we have to estimate

$$\|s\|_{H^{2m,m+\sigma}}, \quad \sigma = \frac{\kappa - 1}{2\kappa}.$$

To achieve this we will inductively bound the norms

$$\|s\|_{H^{m+a,a+\sigma}}, \quad a = \overline{0, m}.$$

For the induction step, we need to bound

$$\|Ls\|_{H^{2,\sigma}},$$

where $L = r^{a-1}\partial^{m-2+a}$ is an operator of order $m-1$. By Lemma 5.2 we have

$$\|Ls\|_{H^{2,\sigma+1}} \lesssim \|L_1 Ls\|_{H^{0,\sigma}} \lesssim \|L_1 s\|_{H^{2m-2,\sigma+m}} + \|[L, L_1]s\|_{H^{0,\sigma}}.$$

The commutator $[L, L_1]$ has order m, but at most $2m-1$ derivatives. Hence by the Hölder inequality and interpolation we can estimate

$$\|[L, L_1]s\|_{H^{0,\sigma}} \lesssim \|s\|_{H^{m+a-1,a+\sigma-1}}. \tag{6.21}$$

Thus we obtain

$$\|s\|_{H^{m+a,a+\sigma}} \lesssim \|L_1 s\|_{H^{m-2,m-1+\sigma}} + \|s\|_{H^{m+a-1,a+\sigma-1}},$$

which concludes the induction step.

It remains to consider the initial case $a = 0$, where we simply take $L = \partial^{m-1}$. Here we argue as in the proof of Theorem 6, more precisely the bound (5.17); in an adapted frame we split the derivatives into normal and tangential, $L = \partial_n^b \partial_\tau^c$, and conjugate

$$LL_1 = L_1^b L + R,$$

where the remainder R has $O(A)$ contributions only,

$$\|Rs\|_{H^{0,\sigma}} \lesssim_M A\|s\|_{H^{m,\sigma}}. \tag{6.22}$$

Applying Lemma 5.2 for L_1^b we obtain

$$\|s\|_{H^{m,\sigma}} \lesssim \|L_1 s\|_{H^{m-2,m-1+\sigma}} + A\|s\|_{H^{m,\sigma}},$$

where the error term on the right can be absorbed on the left.

Now turning our attention to the s component of (6.20), the argument is entirely similar, with a slight modification in the commutator bounds (6.21) and (6.22). These are in turn replaced by

$$\|[L, L_1]s\|_{H^{0,\sigma}} \lesssim \|s\|_{H^{j+k+a-1,a+\sigma-1}} + \sum_{l=0}^{j-1} 2^{-2h(l-j)} \|s\|_{H^{2k+2l,k+l+\sigma}}, \quad L = r^{a-1}\partial^{k+j-2+a},$$

respectively

$$\|Rs\|_{H^{0,\sigma}} \lesssim_M A\|s\|_{H^{2k+2l,\sigma}} + \sum_{l=0}^{j-1} 2^{-2h(l-j)} \|s\|_{H^{2k+2l,k+l+\sigma}}, \qquad L = \partial^{k+j-1}.$$

The $O(A)$ terms in the last bound arise exactly as before when exactly one L derivative applies to the r factor in L_1. All other contributions have fewer derivatives on s, and are estimated by the Hölder inequality and Sobolev embeddings. The negative 2^{-k} powers only arise when more than $2k$ derivatives apply to the r factors in L_1, which means that fewer derivatives apply to s. The details are somewhat tedious but routine, and are omitted. ∎

We now return to the proof of Lemma 6.4, and turn our attention to the bound (6.16). We have

$$(r_+ - \tilde{r}, v_+ - \tilde{v}) = \big((I - \chi_\varepsilon(L_1(r_-)))(r_+ - r_-), (I - \chi_\varepsilon(L_2 + L_3)(r_-))(v_+ - v_-)\big).$$

Hence, given the properties of χ_ε, and the above lemma, we have the $\mathcal{H}$ bound

$$\begin{aligned} &\|(r_+ - \tilde{r}, v_+ - \tilde{v})\|_{\mathcal{H}^{2k-2}_{r_-}} \\ &\quad\lesssim 2^{-2h} \|(L_1(r_-)^k (r_+ - r_-), (L_2 + L_3)(r_-)^k (v_+ - v_-)\|_{\mathcal{H}_{r_-}} \\ &\quad\lesssim_M 2^{-2h}. \end{aligned}$$ ∎

4. Comparing the energies for (r_0, v_0) and $(\tilde{r}, \tilde{v})$. Here, the first energy is taken in the domain Ω_0, while the second is taken in Ω^-. Our objective is to prove the following result:

Lemma 6.6. *Assume that k is large enough, and that h^+ and h^- are suitably chosen relative to h. Then we have*

$$E^{2k}(\tilde{r}, \tilde{v}) \le (1 + C\varepsilon) E^{2k}(r_0, v_0). \tag{6.23}$$

The proof below consists of several steps, each of which will require various constraints on h^+ and h^-. These are then collected at the end of the proof in (6.35). For orientation, one could simply think of the case $h^- = h/2$ and $h^+ = Ch$ with $C \approx k - k_0$.

Proof of Lemma 6.6. These energies have two components: the wave energy and the transport energy. We will focus on the wave component in the sequel, as the argument for the transport part is similar but considerably simpler. For the wave component we need to compare the good variables (s^0_{2k}, w^0_{2k}), respectively $(\tilde{s}_{2k}, \tilde{w}_{2k})$, associated to (r_0, v_0), respectively $(\tilde{r}, \tilde{v})$, and their $\mathcal{H}$ norms,

$$\|(s^0_{2k}, v^0_{2k})\|^2_{\mathcal{H}_r} \quad \text{vs.} \quad \|(\tilde{s}_{2k}, \tilde{v}_{2k})\|^2_{\mathcal{H}_{r_-}}. \tag{6.24}$$

We note that in the second expression we are using the $\mathcal{H}_{r_-}$ norm, as r_- is the defining function for the domain Ω^- where $(\tilde{s}_{2k}, \tilde{v}_{2k})$ are defined. As we seek to compare functions on different domains, it is natural to restrict them to a common domain. To understand this choice, we recall that the two free boundaries Γ^0 and Γ^- are at distance $\ll 2^{-2h^-(k-k_0+1)}$ of each other, and the two weights are at a similar distance within the common domain,

$$|r - r_-| \ll 2^{-2h^-(k-k_0+1)}.$$

For the difference of the two weights to only yield $O(\varepsilon)$ errors, we will restrict our comparison to the region $\Omega_0^{[<h^-(k-k_0)+1]-h}$, where we have

$$|r - r_-| \ll \varepsilon r \quad \text{in } \Omega_0^{[<h^-(k-k_0+1)-h]}.$$

Outside this region we will simply neglect the contribution to the first norm in (6.24). On the other hand, we will seek to make the second norm small in this region. For this to work, we first need to make sure that the neglected region is within the $(\tilde{r}, \tilde{v})$ boundary layer, which has width 2^{-2h}. Thus we require that

$$2h < h^-(k - k_0 + 1).$$

But in addition to that, we also want the second norm to be ε small in this region. Within a fixed layer $\Omega^{-,[h_1]}$ with $h_1 < h$ this norm is

$$\|(\tilde{s}_{2k}, \tilde{v}_{2k})\|^2_{\mathcal{H}_{r_-}(\Omega^{-,[h_1]})} \lesssim_M 2^{\frac{2}{\kappa}(h-h_1)}, \tag{6.25}$$

which is a consequence of the fact that we are integrating a function which is smooth on the 2^{-2h} scale, over a thinner region. This is $\varepsilon^2 = 2^{-4h}$ small if

$$h_1 > h(1 + 2\kappa). \tag{6.26}$$

Hence we obtain

$$\|(\tilde{s}_{2k}, \tilde{v}_{2k})\|_{\mathcal{H}_{r_-}(\Omega_0^{[>h^-(k-k_0+1)-h]})} \lesssim \varepsilon^2, \tag{6.27}$$

provided that

$$h^-(k - k_0 + 1) > 2h(1 + \kappa). \tag{6.28}$$

Within $\Omega_0^{[<h^-(k-k_0+1)-h]}$ we use Lemma 6.3, more precisely its consequence (6.11), in order to compare (s^0_{2k}, w^0_{2k}), respectively $(\tilde{s}_{2k}, \widetilde{w}_{2k})$. There we seek to estimate the errors perturbatively. We begin with (F_{2k}, G_{2k}):

Lemma 6.7. *Assume that $(r_0, v_0) \in \mathbf{H}^{2k}$, with size M and that $(\tilde{r}, \tilde{v})$ are defined as above. Then we have the error bounds*

$$\|(F_{2k}, G_{2k})\|_{\mathcal{H}_r(\Omega_0^{[<h^-(k-k_0+1)-h]})} \lesssim_M \varepsilon^2.$$

The proof of this lemma is similar to the proof of Lemma 5.7, using interpolation inequalities, and is omitted. Here, the region where we evaluate the norm is less important, and serves only to ensure that r_0 and r_- are both defined and comparable there. The gain comes from the fact that the difference $(r - \tilde r, v - \tilde v)$ is small at low frequency, which comes from (6.16) combined with the bounds for the differences $(r_0 - r_+, v_0 - v_+)$ in Proposition 2.11. The power ε^2 requires $k > k_0 + 2$, but one can gain more if k is assumed to be larger.

Next we consider the expressions $(\tilde F_{2k}, \tilde G_{2k})$:

Lemma 6.8. *Assume that* $(r_0, v_0) \in \mathbf{H}^{2k}$*, with size* M *and that* $(\tilde r, \tilde v)$ *are defined as above. Then we have the error bounds*

$$\|(\tilde F_{2k}, \tilde G_{2k})\|_{\mathcal{H}_r(\Omega_0^{[<h^-(k-k_0+1)-h]})} \lesssim \|(r_+ - \tilde r, v_+ - \tilde v)\|_{\mathcal{H}^{2k-1}} + \varepsilon^2 C(M). \tag{6.29}$$

Proof. We recall that the expressions $(\tilde F_{2k}, \tilde G_{2k})$ are balanced multilinear expressions in $(\tilde r, \tilde v)$, respectively $(r_0 - \tilde r, v_0 - \tilde v)$, linear in the second component, containing exactly $2k$ derivatives, and of order $k - 1$, respectively $k - \frac{1}{2}$. The fact that they are balanced allows us to estimate them using the Hölder inequality and interpolation as in Lemma 5.7, by

$$\|(\tilde F_{2k}, \tilde G_{2k})\|_{\mathcal{H}} \lesssim_{A_0, \tilde A} (B_0 + A_0 \tilde B)\|(\tilde r, \tilde v)\|_{\mathcal{H}^{2k-1}} + \tilde B\|(r_0 - \tilde r, v_0 - \tilde v)\|_{\mathcal{H}^{2k-1}},$$

where A_0, B_0, respectively $\tilde A$, $\tilde B$ are control parameters associated to $(\tilde r, \tilde v)$, respectively $(r_0 - \tilde r, v_0 - \tilde v)$.

Here, the first component $(\tilde r, \tilde v)$ is localized at frequencies below 2^h, while the second is localized at frequencies above 2^h. In particular, it follows that A_0, B_0 are small,

$$A_0 + B_0 \lesssim \varepsilon^2,$$

so their contributions go into the second term on the right in (6.29).

On the other hand, $\tilde A$ and $\tilde B$ are merely bounded $\lesssim_M 1$. We split

$$(r_0 - \tilde r, v_0 - \tilde v) = (r_0 - r_+, v_0 - v_+) + (r_+ - \tilde r, v_+ - \tilde v).$$

The first term is localized at frequencies $\geq 2^{h^+}$ so using the bounds in Proposition 2.10 we have

$$\|(r_0 - r_+, v_0 - v_+)\|_{\mathcal{H}^{2k-1}} \lesssim_M 2^{-h^+},$$

which can be made smaller than ε^2 if $h^+ > 4h$. The proof of the lemma is concluded. ∎

Using the above two lemmas together with (6.27), we obtain our first relation between the two energies,

$$\begin{aligned}\|(\tilde s_{2k}, \tilde v_{2k})\|^2_{\mathcal{H}_{r_-}} \leq \|(s^0_{2k}, v^0_{2k})\|^2_{\mathcal{H}_r} + M\|(r_+ - \tilde r, v_+ - \tilde v)\|_{\mathcal{H}^{2k-1}} + C(M)\varepsilon \\ - 2\big\langle\big(L_1(\tilde r)^k(r_0 - \tilde r_1), L_2(\tilde r)^k(v_0 - \tilde v)\big), (\tilde s_{2k}, \tilde w_{2k})\big\rangle_{\mathcal{H}_{r_-}(\Omega_0^{[<2h^-(k-k_0)]})}.\end{aligned}$$

This is not yet satisfactory, but we can improve it further. We first observe that in the above inner product we can harmlessly replace the operators $L_1(\tilde r)$ and $L_2(\tilde r)$ by $L_1(r_-)$ and $L_2(r_-)$ respectively. Precisely, we have the difference bound

$$\|(L_1(\tilde r)^k - L_1(r_-)^k)(r_0 - \tilde r), (L_2(\tilde r)^k - L_2(r_-)^k)(v_0 - \tilde v))\|_{\mathcal{H}_{r_-}(\Omega_0^{[<2h^-(k-k_0)]})} \lesssim_M \varepsilon.$$

This is a consequence of interpolation inequalities and the Hölder inequality due to the fact that both differences $\tilde{r} - r_-$ and $((r_0 - \tilde{r}), (v_0 - \tilde{v}))$ are concentrated at high frequencies and have small $O(\varepsilon^C)$ pointwise size. The details are left for the reader. We arrive at

$$\begin{aligned} \|(\tilde{s}_{2k}, \tilde{v}_{2k})\|^2_{\mathcal{H}_{r_-}} &\\ \leq \|(s^0_{2k}, v^0_{2k})\|^2_{\mathcal{H}_r} &+ M\|(r_+ - \tilde{r}, v_+ - \tilde{v})\|_{\mathcal{H}^{2k-1}} + C(M)\varepsilon \\ &- 2\big\langle\big(L_1(r_-)^k(r_0 - \tilde{r}), L_2(r_-)^k(v_0 - \tilde{v})\big), (\tilde{s}_{2k}, \tilde{w}_{2k})\big\rangle_{\mathcal{H}_{r_-}(\Omega_0^{[<2h^-(k-k_0)]})}. \end{aligned} \tag{6.30}$$

A second simplification is that we can replace (r_0, v_0) by (r_+, v_+) in the inner product. For this we need to show that

$$\big\langle\big(L_1(r_-)^k(r_+ - r_0), L_2(r_-)^k(v_+ - v_0)\big), (\tilde{s}_{2k}, \tilde{w}_{2k})\big\rangle_{\mathcal{H}_{r_-}(\Omega_0^{[<2h^-(k-k_0)]})} \lesssim_M \varepsilon.$$

We first insert a cut-off $\chi^{[2h^-(k-k_0)]}$ function in the differences on the left, associated to the same boundary layer, which equals 1 further inside and 0 closer to the boundary. This is allowed because the second factor in the inner product is already ε small in the cut-off region, while the first one is still bounded in $\mathcal{H}_{r^-}$ in the same region, provided that the cutoff is lower frequency than the $r_+ - r_0$ frequency,

$$h^-(k - k_0) < h^+. \tag{6.31}$$

One should compare this to (6.13); together these bounds give the allowed range for h^+. With this substitution, we are left with proving that

$$\big\langle\big(L_1(r_-)^k\delta^+ r, L_2(r_-)^k\delta^+ v\big), (\tilde{s}_{2k}, \tilde{w}_{2k})\big\rangle_{\mathcal{H}_{r_-}(\Omega^-)} \lesssim_M \varepsilon,$$

where

$$\delta^+ r = \chi^{[2h^-(k-k_0)]}(r_+ - r_0), \quad \delta^+ v = [\chi^{[2h^-(k-k_0)]}(v_+ - v_0)].$$

Since L_1 and L_2 are self-adjoint, we can move one of them to the right. This becomes

$$\big\langle\big(L_1(r_-)^{k-1}\delta^+ r, L_2(r_-)^{k-1}\delta^+ v\big), \big(L_1(r_-)\tilde{s}^1_{2k}, L_2(r_-)\tilde{w}^1_{2k}\big)\big\rangle_{\mathcal{H}_{r_-}(\Omega^-)} \lesssim_M \varepsilon.$$

Now the left factor has size 2^{-2h^+} and the right factor has size 2^{2h}. This yields an ε^2 gain provided that

$$h^+ > 4h. \tag{6.32}$$

Thus, we can replace (r_0, v_0) by (r_+, v_+) in (6.30), to obtain

$$\begin{aligned} \|(\tilde{s}_{2k}, \tilde{v}_{2k})\|^2_{\mathcal{H}_{r_-}} &\\ \leq \|(s^0_{2k}, v^0_{2k})\|^2_{\mathcal{H}_r} &+ C(M)\|(r_+ - \tilde{r}, v_+ - \tilde{v})\|_{\mathcal{H}^{2k-1}_{r_-}} + C(M)\varepsilon \\ &- 2\big\langle\big(L_1(r_-)^k(r_+ - \tilde{r}), L_2(r_-)^k(v_+ - \tilde{v})\big), (\tilde{s}_{2k}, \tilde{w}_{2k})\big\rangle_{\mathcal{H}_{r_-}(\Omega_0^{[<2h^-(k-k_0)]})}. \end{aligned}$$

Once this is done, the expression on the left in the inner product is defined on the entire domain Ω^-, and we can harmlessly extend the inner product to the full region as the expression on the right in the inner product is already ε small there. We get

$$\begin{aligned}
&\|(\tilde{s}_{2k}, \tilde{w}_{2k})\|^2_{\mathcal{H}_{r_-}} \\
&\quad \le \|(s^0_{2k}, v^0_{2k})\|^2_{\mathcal{H}_r} + C(M)\|(r_+ - \tilde{r}, v_+ - \tilde{v})\|_{\mathcal{H}^{2k-1}} + C(M)\varepsilon \\
&\quad\quad - 2\big\langle\big(L_1(r_-)^k(r_+ - \tilde{r}), L_2(r_-)^k(v_+ - \tilde{v})\big), (\tilde{s}_{2k}, \tilde{w}_{2k})\big\rangle_{\mathcal{H}_{r_-}(\Omega^-)}. \qquad (6.33)
\end{aligned}$$

The next step is to apply the expansion (6.11) for the expression on the right in the inner product to write

$$\begin{cases}
\tilde{s}_{2k} = s^-_{2k} + (L_1(r_-))^k(\tilde{r} - r_-) + F^-_{2k} + \tilde{F}^-_{2k}, \\
\tilde{w}_{2k} = w^-_{2k} + (L_2(r_-))^k(\tilde{v} - v_-) + G^-_{2k} + \tilde{G}^-_{2k}.
\end{cases}$$

By the counterpart of Lemma 6.7 the error terms (F^-_{2k}, G^-_{2k}) will be ε small, so their contribution to (6.33) can be included in the expression $C(M)\varepsilon$.

For the contribution of $(\tilde{F}^-_{2k}, \tilde{G}^-_{2k})$ we integrate by parts one instance of L_1, respectively L_2, to bound it by

$$\begin{aligned}
&\|(L_1(r_-)^{k-1}(r_+ - \tilde{r}), L_2(r_-)^{k-1}(v_+ - \tilde{v}))\|_{\mathcal{H}_{r_-}} \|(L_1(r_-)\tilde{F}^-_{2k}, L_2(r_-)\tilde{G}^-_{2k})\|_{\mathcal{H}_{r_-}} \\
&\quad \lesssim_M \|(r_+ - \tilde{r}, v_+ - \tilde{v})\|_{\mathcal{H}^{2k-2}_{r_-}} \|(\tilde{r} - r_-, \tilde{v} - v_-)\|_{\mathcal{H}^{2k+1}_{r_-}} \\
&\quad \lesssim_M 2^{-2h}\|(\tilde{r} - r_-, \tilde{v} - v_-)\|_{\mathcal{H}^{2k+1}_{r_-}}.
\end{aligned}$$

Finally, for the contribution of (s^-_{2k}, w^-_{2k}) we can integrate again by parts to obtain

$$\big\langle(r_+ - \tilde{r}, v_+ - \tilde{v}), \big(L_1(r_-)^k s^-_{2k}, L_2(r_-)^k w^-_{2k}\big)\big\rangle_{\mathcal{H}_{r_-}(\Omega^-)} \lesssim_M \varepsilon,$$

provided that

$$(k-1)h > kh^-.$$

Thus (6.33) becomes

$$\begin{aligned}
&\|(\tilde{s}_{2k}, \tilde{v}_{2k})\|^2_{\mathcal{H}_{r_-}} \\
&\quad \le \|(s^0_{2k}, v^0_{2k})\|^2_{\mathcal{H}_r} + C(M)\varepsilon \\
&\quad\quad + C(M)(\|(r_+ - \tilde{r}, v_+ - \tilde{v})\|_{\mathcal{H}^{2k-1}_{r_-}} + 2^{-2h}\|(\tilde{r} - r_-, \tilde{v} - v_-)\|_{\mathcal{H}^{2k+1}_{r_-}}) \\
&\quad\quad - 2\big\langle\big(L_1(r_-)^k(r_+ - \tilde{r}_1), L_2(r_-)^k(v_+ - \tilde{v})\big), \\
&\quad\quad\quad \big(L_1(r_-)^k(\tilde{r} - r_-), L_2(r_-)^k(\tilde{v} - v_-)\big)\big\rangle_{\mathcal{H}_{r_-}}.
\end{aligned}$$

Now our choice of $(\tilde{r}, \tilde{w})$ guarantees that the inner product is positive. Combining the above bound with its counterpart for the transport energy (this is where our choice (6.1) simplifies matters), we further obtain

$$\begin{aligned}
&E^{2k}(\tilde{r}, \tilde{v}) \\
&\quad \le E^{2k}(r^0, v^0) + C(M)\varepsilon \\
&\quad\quad + C(M)(\|(r_+ - \tilde{r}, v_+ - \tilde{v})\|_{\mathcal{H}^{2k-1}_{r_-}} + 2^{-2h}\|(\tilde{r} - r_-, \tilde{v} - v_-)\|_{\mathcal{H}^{2k+1}_{r_-}}) - 2I, \qquad (6.34)
\end{aligned}$$

where

$$I = \big\langle \big(L_1(r_-)^k(r_+ - \tilde r_1), (L_2 + L_3)(r_-)^k(v_+ - \tilde v)\big), \\ (L_1(r_-))^k(\tilde r - r_-), (L_2 + L_3)(r_-)^k(\tilde v - v_-)\big\rangle_{\mathcal{H}_{r_-}}$$

is still positive. Finally, we use the positivity of I to estimate the two remaining terms on the right. Precisely, using properties (6.14) of the multiplier χ in the definition of $(\tilde r, \tilde v)$, as well as the ellipticity of L_1, respectively $L_2 + L_3$ in the two components of $\mathcal{H}$, we have

$$I \gtrsim 2^{2h}\|(r_+ - \tilde r, v_+ - \tilde v)\|^2_{\mathcal{H}^{2k-1}_{r_-}} + 2^{-2h}\|(r_- - \tilde r, v_- - \tilde v)\|^2_{\mathcal{H}^{2k+1}_{r_-}}.$$

Hence, applying the Cauchy–Schwarz inequality in (6.34) we finally obtain

$$E^{2k}(\tilde r, \tilde v) \le E^{2k}(r^0, v^0) + C(M)\varepsilon,$$

as desired.

This concludes the proof of (6.23), provided that the scales h^+ and h^- were chosen so that the constraints (6.13), (6.28), (6.31), (6.32) are all satisfied. We recall them all here:

$$\begin{gathered} h^- < \frac{k-1}{k}h, \quad h^+ > 4h, \\ h^-(k-k_0) > h\Big(1 + \frac{1}{\kappa}\Big), \\ h^-(k-k_0) < h^+ < h^-(k-k_0+1). \end{gathered} \tag{6.35}$$

Then the parameters h^+ and h^- can be chosen e.g. as follows:

(a) set $h^- = h/2$,

(b) take k large enough so that the second constraint holds,

(c) choose h^+ in the range given by the third constraint. ■

5. Comparing the energies of $(\tilde r, \tilde v)$ and (r, v). To recall our setting here, the functions $(\tilde r, \tilde v)$ are defined in the domain Ω^- and are localized at frequency $\le 2^h$ scale, but cannot be thought of as a state because $\tilde r$ does not vanish on the boundary Γ^-. Instead, we have

$$|\tilde r| \lesssim 2^{-2(k-k_0)h^-} \quad \text{on } \Gamma^-.$$

To rectify this, we decrease $\tilde r$ by a small constant and set

$$(r, v) = (\tilde r - c, \tilde v), \quad c = 2^{-2(k-k_0)h^-},$$

so that the level set $\Gamma = \{r = 0\}$ is fully contained within Ω^-. Then we aim to prove that the energies do not change much:

Lemma 6.9. *We have the energy bound*

$$E^{2k}(r, v) \lesssim E^{2k}(\tilde r, \tilde v) + O_M(\varepsilon).$$

Proof. We separate a boundary layer $\Omega^{[>h_1]}$, with $h_1 > h$ to be chosen later, where we verify directly that the norm on the left is $O(\varepsilon)$. Outside this layer, we directly compare the associated good variables.

For the first step we use (6.25), which suffices if we impose constraint (6.26), which we recall here:

$$h_1 > h\Big(1 + \frac{1}{\kappa}\Big).$$

For the second step, we simply note that the good variables are identical except for the r factors, where we replace r_1 by $r_1 - c$. Hence it suffices to ensure that

$$c \lesssim \varepsilon r_1 \quad \text{in } \Omega^{[<h_1]},$$

which yields

$$(k - k_0)h^- > h + h_1.$$

These two constraints for h_1 are again compatible if k is large enough. The proof of the lemma is concluded. ∎

Now combining the outcomes of Lemmas 6.6 and 6.9, it follows that our final regularization (r, v) satisfies the bound (6.6). It also satisfies (6.5) and (6.7) due to Lemma 6.4; there one can harmlessly substitute the weight r_- by r since (r, v) are smooth on the ε^2 scale, which is larger than c. Thus the proof of Proposition 6.1 is concluded. ∎

6.3. Construction of regular exact solutions

Here we use the approximate solutions above. Given initial data (r_0, v_0) so that

$$\|(r_0, v_0)\|_{\mathbf{H}^{2k}} \leq M,$$

applying the successive iterations above we obtain approximate solutions $(r^\varepsilon, v^\varepsilon)$ defined at ε steps, so that

$$E^{2k}(r^\varepsilon, v^\varepsilon)((j+1)\varepsilon) \lesssim (1 + C(M)\varepsilon)E^{2k}(r^\varepsilon, v^\varepsilon)(j\varepsilon).$$

By the discrete Grönwall inequality, it follows that these approximate solutions are defined uniformly up to a time $T = T(M)$, with uniform bounds

$$\|(r^\varepsilon, v^\varepsilon)\|_{\mathbf{H}^{2k}} \lesssim_M 1, \quad t \in [0, T]. \tag{6.36}$$

On the other hand, in a weaker topology we have

$$(r^\varepsilon, v^\varepsilon)((j+1)\varepsilon) - (r^\varepsilon, v^\varepsilon)(j\varepsilon) = O(\varepsilon).$$

Hence by Arzelà–Ascoli we get uniform convergence on a subsequence to a function (r, v) in a C^j norm, uniformly in t. Passing to the limit in the relation (6.4), it follows that (r, v) solves our equation. Finally, taking weak limits in the norms in (6.36) we also obtain an energy bound on (r, v),

$$\|(r, v)(t)\|_{\mathbf{H}^{2k}} \lesssim_M 1, \quad t \in [0, T].$$

7. Rough solutions

Our goal in this section is to construct rough solutions as limits of smooth solutions, and conclude the proof of Theorem 2. In terms of a general outline, the argument here is relatively standard, and involves the following steps:

(1) We regularize the initial data.

(2) We prove uniform bounds for the regularized solutions.

(3) We prove convergence of the regularized solutions in a weaker topology.

(4) We prove convergence in the strong topology by combining the weak difference bounds with the uniform bounds in a frequency envelope fashion.

The main difficulty we face is that our phase space is not linear, and at each stage we have to compare functions on different domains.

7.1. Regularizing the initial data

Given rough initial data $(r_0, v_0) \in \mathbf{H}^{2k}$, our first task is to construct an appropriate family of regularized data, depending smoothly on the regularization parameter. Here, it suffices to directly use the family of regularizations provided by Proposition 2.11.

7.2. Uniform bounds and the lifespan of regular solutions

Once we have the regularized data sets (r_0^h, v_0^h), we consider the corresponding smooth solutions (r^h, v^h) generated by the smooth data (r_0^h, v_0^h). A priori these solutions exist on a time interval that depends on h. Instead, we would like to have a lifespan bound which is independent of h. To obtain this, we use a bootstrap argument for our control parameter B for (r^h, v^h), which depends on h and t.

For a large parameter B_0, to be chosen later, we will make the bootstrap assumption

$$B(t, h) \leq 2B_0, \quad t \in [0, T],\ 0 \leq h \leq h_0.$$

The solutions (r^h, v^h) can be continued for as long as this is satisfied. We will prove that we can improve this bootstrap assumption provided that T is small enough, $T \leq T_0$, but with T_0 independent of $h \leq h_0$. Here, h_0 is finite but arbitrarily large; its role is simply to ensure that we run the bootstrap argument on finitely many quantities at once.

Our choice of T_0 will be quite straightforward:

$$T_0 \leq \frac{1}{B_0}.$$

In view of our energy estimates in Theorem 3 and the Grönwall inequality, this guarantees uniform energy bounds for the solutions (r^h, v^h) in all integer Sobolev spaces $\mathcal{H}^{2l}$ in $[0, T]$.

We remark that the bound (2.12) does not directly propagate unless k is an integer. Indeed, in that case one could immediately close the bootstrap at the level of the $\mathcal{H}^{2k}$

norm using the embeddings (2.3) and (2.4). The goal of the argument that follows is to establish the $\mathcal{H}^{2k}$ bound for noninteger k, by working only with energy estimates for integer indices.

Combining Theorem 3 with (2.13) we obtain the higher energy bound in $[0, T]$,

$$\|(r^h, v^h)\|_{\mathbf{H}_h^{2k+2j}} \lesssim 2^{2hj} c_h, \quad j > 0,\ j + k \in \mathbb{N}. \tag{7.1}$$

Next we consider the bound (2.14), which we reinterpret in a discrete fashion as a difference bound

$$D((r_0^h, v_0^h), (r_0^{h+1}, v_0^{h+1})) \lesssim 2^{-4hk} c_h^2.$$

We can also propagate this bound by Theorem 5, to obtain, also in $[0, T]$, the estimate

$$D((r^h, v^h), (r^{h+1}, v^{h+1})) \lesssim 2^{-4hk} c_h^2. \tag{7.2}$$

Our objective now is to combine the bounds (7.1) and (7.2) to obtain a uniform $\mathcal{H}^{2k}$ bound

$$\|(r^h, v^h)\|_{\mathbf{H}^{2k}} \lesssim M := \|(r_0, v_0)\|_{\mathbf{H}^{2k}}. \tag{7.3}$$

To prove this, we would naively like to consider a representation of the form

$$(r^h, v^h) = (r^1, v^1) + \sum_{l=1}^{h-1} (r^{l+1} - r^l, v^{l+1} - v^l),$$

where we can estimate the successive terms in both $\mathcal{H}$ and $\mathcal{H}^{2N}$. The difficulty we face is that these functions have different domains. Hence the first step is to use the bounds (7.1) and (7.2) to compare these domains.

Lemma 7.1. *Assume that r^h and r^{h+1} are nondegenerate, and that (7.2) holds. Then we have*

$$d(\Gamma_h, \Gamma_{h+1}) \lesssim 2^{-h(2+\delta)}, \quad \delta > 0.$$

Proof. We use the uniform nondegeneracy property for the functions r_h to compare these domains. If $r = d(\Gamma_h, \Gamma_{h+1})$, then we can find a ball B_{cr} in the common domain so that

$$r^h, r^{h+1}, |r^h - r^{h+1}| \approx r \quad \text{in } B_{cr}.$$

Then we obtain

$$r^{d+1+\frac{1}{\kappa}} \lesssim D((r^h, v^h), (r^{h+1}, v^{h+1})) \lesssim 2^{-4hk} c_h^2,$$

or equivalently

$$r^{2\kappa_0} \lesssim 2^{-4hk} c_h^2.$$

Since $k > k_0$, we obtain

$$r \lesssim 2^{-h(2+\delta)}, \quad \delta > 0. \qquad \blacksquare$$

Now we return to our expansion for (r^h, v^h). To compare functions which are defined on a common domain, we replace the functions (r^l, v^l) with their regularizations $\Psi^l(r^l, v^l)$. Their domain includes an additional 2^{-2l} boundary layer, which by the previous Lemma 6.9 suffices in order to cover the domain Ω_h for all $h > l$. Then we write

$$(r^h, v^h) = \Psi^0(r^0, v^0) + \sum \Psi^{l+1}(r^{l+1}, v^{l+1}) - \Psi^l(r^l, v^l) + (I - \Psi^h)(r^h, v^h),$$

and claim that this decomposition is as in Lemma 2.5.

The first term is trivial. For the last one we use the boundedness of Ψ^h in $\mathcal{H}^{2k}$ and the bound (2.19) integrated in h to write

$$\|(I - \Psi^h)(r^h, v^h)\|_{\mathcal{H}^{2N}} \lesssim \|(r^h, v^h)\|_{\mathcal{H}^{2N}},$$

respectively

$$\|(I - \Psi^h)(r^h, v^h)\|_{\mathcal{H}} \lesssim 2^{-2Nh}\|(r^h, v^h)\|_{\mathcal{H}^{2N}},$$

for a fixed large enough integer N, which together suffice in order to place this term into (s^h, w^h), with norm c_h.

For later use, we state the remaining bound for intermediate l as a separate result:

Lemma 7.2. *For any nondegenerate r with $|r - r^l| \ll 2^{-2l}$ we have the difference bounds*

$$\|\Psi^{l+1}(r^{l+1}, v^{l+1}) - \Psi^l(r^l, v^l)\|_{\mathcal{H}_r} \lesssim 2^{-2lk} c_l,$$
$$\|\Psi^{l+1}(r^{l+1}, v^{l+1}) - \Psi^l(r^l, v^l)\|_{\mathcal{H}_r^{2N}} \lesssim 2^{2l(N-k)} c_l.$$

As a corollary of this lemma, we remark that via Sobolev embeddings we also get uniform difference bounds:

Corollary 7.3. *In the region $\tilde{\Omega}^{[l]}$ have*

$$\|\Psi^{l+1}(r^{l+1}, v^{l+1}) - \Psi_l(r^l, v^l)\|_{C^{\frac{3}{2}} \times C^1} \lesssim 2^{-2\delta l}, \quad \delta > 0.$$

This will serve later in the study of convergence of the regularized solutions.

Proof of Lemma 7.2. We split

$$\Psi^{l+1}(r^{l+1}, v^{l+1}) - \Psi^l(r^l, v^l) = (\Psi^{l+1} - \Psi^l)(r^{l+1}, v^{l+1}) - \Psi^l(r^{l+1} - r^l, v^{l+1} - v^l).$$

For the first term we use again the boundedness of Ψ^l and then (2.19) to conclude that

$$\|(\Psi^{l+1} - \Psi^l)(r^{l+1}, v^{l+1})\|_{\mathcal{H}^{2N}} \lesssim \|(r^{l+1}, v^{l+1})\|_{\mathcal{H}^{2N}} \lesssim 2^{2l(N-k)} c_l$$

and

$$\|(\Psi^{l+1} - \Psi^l)(r^{l+1}, v^{l+1})\|_{\mathcal{H}} \lesssim 2^{-2lN}\|(r^{l+1}, v^{l+1})\|_{\mathcal{H}^{2N}} \lesssim 2^{-2lk} c_l,$$

as needed.

For the second term we use again the $\mathcal{H}^{2N}$ boundedness, but for the $\mathcal{H}$ bound we use instead the difference bound (7.2) together with the $\mathcal{H}$ bound

$$\|\Psi^l(r^{l+1} - r^l, v^{l+1} - v^l)\|_{\mathcal{H}}^2 \lesssim D((r^{l+1}, v^{l+1}), (r^l, v^l)),$$

and conclude using (7.2). ∎

By the above lemma we can place the telescopic term into (s^l, w^l), with norm c_l, and thus, by Lemma 2.5, we obtain the desired bound (7.3) and conclude our bootstrap argument.

7.3. The limiting solution

Here we show that the limit

$$(r, v) = \lim_{h\to\infty} (r^h, v^h)$$

exists, first in a weaker topology and then in the strong $\mathbf{H}^{2k}$ topology.

As before, the smooth solutions (r^h, v^h) do not have common domains. However, by Lemma 7.1 the limit

$$\Omega = \lim_{h\to\infty} \Omega_h$$

exists, has a Lipschitz boundary Γ, and further we have

$$d(\Gamma, \Gamma_h) \lesssim 2^{-h(2+\delta)}.$$

For this reason, it is convenient to consider instead the limit

$$(r, v) = \lim_{h\to\infty} \Psi^h(r^h, v^h),$$

where the functions on the right are all defined in Ω. Indeed, by Lemma 7.2 we see that we have convergence in $\mathcal{H}$, and, by interpolation, in $\mathcal{H}^{2k_1}$ for all $k_1 < k$.

To obtain convergence in $\mathcal{H}_r^{2k}$, we write

$$(r, v) = \Psi^0(r^0, v^0) + \sum_{j=0}^{\infty} \Psi^{l+1}(r^{l+1}, v^{l+1}) - \Psi^l(r^l, v^l),$$

and view the telescopic sum as a generalized Littlewood–Paley decomposition of (r, v). Then Lemma 7.2 shows that (r, v) is in $\mathcal{H}^{2k}$, with norm

$$\|(r, v)\|_{\mathcal{H}^{2k}} \lesssim \|c_h\|_{l^2}.$$

We also see that we have convergence in $\mathcal{H}^{2k}$, namely

$$\|\Psi^l(r^l, v^l) - (r, v)\|_{\mathcal{H}^{2k}} \lesssim \|c_{\geq l}\|_{l^2} \to 0.$$

We also show that we have strong convergence of (r^h, v^h) in $\mathcal{H}^{2k}$ in the sense of Definition 2.6. Indeed, it suffices to compare it with the constant sequence $\Psi^l(r^m, v^m)$. Then for $l \geq m$ we have

$$\|(r^l, v^l) - \Psi^m(r^m, v^m)\|_{\mathcal{H}^{2k}} \lesssim \|c_{\geq m}\|_{l^2} \to 0.$$

The same relations also show the continuity of (r, v) in $\mathbf{H}^{2k}$ as functions of time.

7.4. Continuous dependence

We consider a sequence of initial data $(r_0^{(n)}, v_0^{(n)})$ which converges to (r_0, v_0) in $\mathbf{H}^{2k}$ in the sense of Definition 2.6, and will show that the corresponding solutions $(r^{(n)}, v^{(n)})$ converge to (r, v).

The first observation is that the $\mathbf{H}^{2k}$ convergence implies $\mathbf{H}^{2k}$ uniform boundedness for $(r_0^{(n)}, v_0^{(n)})$, which in turn implies a uniform lifespan bound for the solutions, as well as a uniform bound in $\mathbf{H}^{2k}$.

Our strategy to prove convergence is to compare this family of solutions with the limit (r, v) via the regularizations used in the construction of rough solutions. Precisely, denote by $(r_0^{(n),h}, v_0^{(n),h})$, respectively (r_0^h, v_0^h) the regularized data sets, for which we have the obvious convergence

$$(r_0^{(n),h}, v_0^{(n),h}) \to (r_0^h, v_0^h) \quad \text{in } C^\infty.$$

These are also uniformly bounded in $\mathbf{H}^{2k}$ and thus have a uniform lifespan.

Denoting by $c_h^{(n)}$ corresponding frequency envelopes for $(r_0^{(n)}, v_0^{(n)})$, we have the difference bounds

$$\|\Psi^h(r^{(n),h}, v^{(n),h}) - (r^{(n)}, v^{(n)})\|_{\mathcal{H}^{2k}} \lesssim c_{\geq h}^{(n)}.$$

To finish the proof we need to establish two facts:

- For each $\varepsilon > 0$, the frequency envelopes $c_h^{(n)}$ can be chosen so that[16]

$$\limsup_{h\to\infty} \sup_n c_{\geq h}^{(n)} \leq \varepsilon.$$

- We have the C^∞ convergence

$$\Psi^h(r^{(n),h}, v^{(n),h}) \to \Psi^h(r^h, v^h).$$

(i) *Equicontinuity of frequency envelopes:* This is easily achieved via the decomposition

$$(r_0^{(n)}, v_0^{(n)}) = (r_0^{\text{smooth}}, v_0^{\text{smooth}}) + O_{\mathcal{H}^{2k}}(\varepsilon),$$

which holds for each ε. The smooth part yields envelopes which are uniformly decreasing, and the error term yields ε-sized envelopes.

(ii) C^∞ *convergence:* Here we have uniform $\mathcal{H}^{2N}$ bounds for the sequence $(r^{(n),h}, v^{(n),h})$, as well as weak convergence, in the sense that

$$D((r^{(n),h}, v^{(n),h}), (r^h, v^h)) \to 0.$$

The last property implies domain convergence. Then we have L^2 convergence away from a 2^{-2h} boundary layer, which in turn shows convergence of the regularizations in C^∞.

[16]One can do better than that and ensure that the limit is zero, but that is not needed for our argument.

7.5. The lifespan of rough solutions

Here we complete the proof of our last result in Theorem 4. Thus, we consider rough initial data $(r_0, v_0) \in \mathbf{H}^{2k}$ and a corresponding solution (r, v) in a time interval $[0, T)$ with the property that

$$\int_0^T B(t)\, dt = C < \infty.$$

By the local well-posedness result, in order to prove the theorem it suffices to show that we have a uniform bound

$$\sup_{t\in[0,T]} \|(r, v)\|_{\mathbf{H}^{2k}} < \infty.$$

We consider the regularized data (r_0^h, v_0^h) and the corresponding solutions (r^h, v^h). By the continuous dependence theorem we know that these solutions converge to (r, v) in $[0, T)$, and in particular their lifespans T^h satisfy

$$\liminf_{h\to\infty} T^h \geq T.$$

What we do not have is a uniform bound for their corresponding control parameters B^h. To rectify this, we consider a large parameter h_0, to be chosen later, and we will show that, for $h > h_0$, the solutions (r^h, v^h) persist up to time T with uniform bounds

$$\int_0^T B^h(t)\, dt \leq 2C, \quad h \geq h_0. \tag{7.4}$$

If that were the case, then by the local well-posedness proof it would follow that the solutions (r^h, v^h) remain uniformly bounded in $\mathbf{H}^{2k}$ and converge to (r, h), thereby concluding the proof.

To establish the bound (7.4) we will run a bootstrap argument. Precisely, we assume that on a time interval $[0, T_0]$ with $T_0 < T$ we have a uniform bound

$$\int_0^{T_0} B^h(t)dt \leq 4C, \quad h \geq h_0.$$

Then we will show that in effect we must have the better bound

$$\int_0^{T_0} B^h(t)dt \leq 2C, \quad h \geq h_0. \tag{7.5}$$

That would suffice, for then the local well-posedness argument would yield a uniform bound for (r^h, v^h) in $\mathbf{H}^{2k}$ and thus allow us to expand the interval $[0, T_0]$ on which the bootstrap assumption holds, uniformly with respect to $h \geq h_0$.

Our goal now is to compare B^h and B. Precisely, we aim to show that

$$B^h \lesssim C_1 B + C_2 2^{-\delta h}, \quad \delta > 0, \tag{7.6}$$

with a universal constant C_1, but C_2 depending both on the initial data size and on C above. This suffices to establish (7.5), because we are allowed to choose the threshold h_0 sufficiently large, depending on parameters which are fixed in the problem.

The tools we have at our disposal are

(i) a high-frequency bound, provided by our energy estimates in Theorem 3, namely (7.1),

(ii) the difference bound (7.2).

The constants in both bounds depend exactly on the $\mathbf{H}^{2k}$ norm initial data and on C above.

The difficulty we have in comparing B^h and B is that the two solutions are supported in different domains Ω, respectively Ω_h. However, the difference bound (7.2) allows us to apply Lemma 7.1 to conclude that the two domains are at distance $\lesssim 2^{-h(2+\delta)}$. Thus, rather than comparing (r, v) and (r^h, v^h), it is better to compare their regularizations $\Psi^h(r, v)$ and $\Psi^h(r^h, v^h)$, which are defined on 2^{-2h} enlargements of the domains, which in particular cover the union of Ω and Ω_h. By a slight abuse of notation, we will identify their domains.

We begin with $\Psi^h(r, v)$, for which we have the straightforward bound

$$\|\nabla\Psi^h(r, v)\|_{\tilde{C}^{0,\frac{1}{2}}\times L^\infty} \leq C_1 B. \tag{7.7}$$

This is where the universal constant C_1 appears.

Next we compare $\Psi^h(r, v)$ and $\Psi^h(r^h, v^h)$. Here we take a telescopic sum,

$$\Psi^h(r, v) - \Psi^h(r^h, v^h) = \sum_{l=h}^{\infty} \Psi^h(r^{l+1}, v^{l+1}) - \Psi^h(r^l, v^l).$$

Using the difference bound (7.2), we can estimate the successive terms in all $\mathcal{H}^{2m}_{r^h}$ norms,

$$\|\Psi^h(r^{l+1}, v^{l+1}) - \Psi^h(r^l, v^l)\|_{\mathcal{H}^{2m}_{r^h}} \lesssim c_l 2^{-2kl} 2^{2mh}, \quad m \geq 0,$$

which after summation yields

$$\|\Psi^h(r, v) - \Psi^h(r^l, v^l)\|_{\mathcal{H}^{2m}_{r^h}} \lesssim c_l 2^{-2kh} 2^{2mh}, \quad m \geq 0.$$

Now we can use Sobolev embeddings to conclude that[17]

$$\|\nabla(\Psi^h(r, v) - \Psi^h(r^h, v^h))\|_{\tilde{C}^{0,\frac{1}{2}}\times L^\infty} \lesssim 2^{-\delta h}. \tag{7.8}$$

Finally, using (7.1), we compare $\Psi^h(r^h, v^h)$ with (r^h, v^h), also estimating the low frequencies,

$$\|(r^h, v^h) - \Psi^h(r^h, v^h)\|_{\mathcal{H}^{2m}_{r^h}} \lesssim c_l 2^{-2kh} 2^{2mh}, \quad m \geq 0.$$

[17] From Sobolev embeddings we get in effect a $\dot{C}^{\frac{1}{2}}$ bound for the first component.

Using Sobolev embeddings again, we conclude that

$$\|\nabla((r^h, v^h) - \Psi^h(r^h, v^h))\|_{\tilde{C}^{0,\frac{1}{2}} \times L^\infty} \lesssim 2^{-\delta h}. \tag{7.9}$$

Now (7.6) is obtained by combining (7.7), (7.8), and (7.9), and the proof of the theorem is concluded.

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