Port. Math. 81 (2024), 1–19
DOI 10.4171/PM/2116

Existence of solutions for critical Klein–Gordon equations coupled with Born–Infeld theory in higher dimensions

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Abstract. In this paper, we investigate the existence of nontrivial solutions for nonlinear Klein–Gordon equations coupled with Born–Infeld theory with critical Sobolev exponents by variational methods.

1. Introduction and main results

We consider the existence of solutions for the following critical Klein–Gordon equation coupled with Born–Infeld theory

$$\begin{cases} -\Delta u + (m^2 - \omega^2)u - (2\omega + \phi)\phi u = \lambda f(x,u) + |u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $m, \omega > 0$ are real constants, $u, \phi : \mathbb{R}^N \to \mathbb{R}$, $\Delta_4\phi = \operatorname{div}(|\nabla\phi|^2\nabla\phi)$ and $\beta > 0$. $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Klein–Gordon equations can be used to develop the theory of electrically charged fields (see [13]), and Born–Infeld theory is proposed by Born to overcome the infinite energy problems associated with a point-charge source in the original Maxwell theory (see [3, 4]). The presence of the nonlinear term $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ simulates the interaction between many particles or external nonlinear perturbations.

In recent years, the Born–Infeld nonlinear electromagnetism has regained its importance due to its relevance in the theory of superstring and membranes. When $f(u) = |u|^{p-2}u$, d'Avenia and Pisani [12] established the following system that has infinitely many radially symmetric solutions under the assumptions $|m| > \omega$ and $4 < p < 6$:

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \phi)^2]u = |u|^{q-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

Mathematics Subject Classification 2020: 35A15 (primary); 35B33 (secondary).
Keywords: Klein–Gordon equations, Born–Infeld theory, critical exponents.

Later, more and more scholars considered the systems of a coupled Klein–Gordon equation with Born–Infeld theory by using variational methods. Particularly, Mugnai [21] studied the existence of infinitely many nontrivial radially symmetric solitary waves for the problem (1.2). Yong [27] studied a Klein–Gordon equation coupled with the original Born–Infeld equation, and obtained infinitely many solitary wave solutions. Recently, Wen et al. [25] proved the existence of infinitely many solutions and least energy solution for the following Klein–Gordon equation coupled with Born–Infeld theory:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

For other references related to (1.3), we refer to [9, 23].

It is worth mentioning that Teng and Zhang [24] obtained the existence of solitary wave solutions for the following Klein–Gordon equation coupled with Born–Infeld theory by using variational methods:

$$\begin{cases} \Delta u = [m^2 - (\omega + \phi)^2]u - |u|^{q-2}u - |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

and other problems involving critical exponents can be seen in [6, 7, 14, 16, 17].

In proving our results, we have to deal with various difficulties. For instance, the fact that the problem gets more complicated by the lack of compactness in the entire $\mathbb{R}^N$ space, and we consider more general nonlinearities, even more in the critical case. Hence, it is significant to study the problem (1.1). When $\phi \neq 0$, a process of plugging ϕ into the main equation is used, which can transfer the system into a single equation. This technique was also employed in [1, 8, 9, 11, 18].

Before we state our main results, we assume that f satisfies the following conditions:

- (f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exists $C_0 > 0$ for $2 < p < 2^*$ such that
$$|f(x,t)| \le C_0(1 + |t|^{p-1}), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$
- (f_2) $f(x,t)/t \to 0$, as $t \to 0$, uniformly for $x \in \mathbb{R}^N$.
- (f_3) There exists $2 < \mu < 2^*$ such that $0 < \mu F(x,t) \le f(x,t)t$, for all $t \in \mathbb{R}$, where $F(x,t) = \int_0^1 f(x,\tau t)t\,d\tau$.

The major results hold.

Theorem 1.1. *Assume that* (f_1)–(f_3) *hold. Either* $|m| > |\omega|$ *and* $4 < \mu < 2^*$ *or* $\sqrt{\mu - 2}|m| > \sqrt{2}|\omega|$ *and* $2 < \mu \leq 4$. *Then problem* (1.1) *possesses at least a nontrivial radially symmetric solution provided that*

(i) $N \geq 5$ *and* $N = 4$ *for* $2 < \mu < 2^*$ *if* $\lambda > 0$ *;*

(ii) $N = 3$ *and either* $4 < \mu < 2^*$ *if* $\lambda > 0$ *or* $2 < \mu \leq 4$ *if* λ *is sufficiently large.*

The plan of the paper is as follows. Section 2 is devoted to the variational setting of the problem, and some preliminary results. In Section 3, we analyse the variational structure and present the results on weak solutions satisfying problem (1.1), as well as prove the solution is nontrivial.

2. Preliminaries

Considering the Hilbert space $H^1(\mathbb{R}^N)$ defined by

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\},$$

with the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

And the space

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

is equipped with the norm

$$\|u\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Denote by $\mathcal{D}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with the norm

$$\|\phi\|_{\mathcal{D}} = \left(\int_{\mathbb{R}^N} |\nabla\phi|^2\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} |\nabla\phi|^4\mathrm{d}x\right)^{\frac{1}{4}}.$$

From [15, Proposition 8], we infer that the Banach space $\mathcal{D}(\mathbb{R}^N)$ is continuously embedded in $L^\infty(\mathbb{R}^N)$. And we know that $\mathcal{D}(\mathbb{R}^N) \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$, $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ are continuous. The best Sobolev constant S is given by

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}}. \tag{2.1}$$

Due to the variational characteristic of problem (1.1), its weak solutions $(u,\phi)\in H^1(\mathbb{R}^N)\times\mathcal{D}(\mathbb{R}^N)$ are critical points of the functional given by

$$\begin{aligned}I(u,\phi)=&\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}(m^2-\omega^2)u^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}(2\omega+\phi)\phi u^2\mathrm{d}x\\&-\frac{1}{8\pi}\int_{\mathbb{R}^N}|\nabla\phi|^2\mathrm{d}x-\frac{\beta}{16\pi}\int_{\mathbb{R}^N}|\nabla\phi|^4\mathrm{d}x-\lambda\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x\\&-\frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x.\end{aligned}\tag{2.2}$$

However, since the functional I is strongly indefinite, we need the following technical results to reduce the study of (2.2) to the study of a functional in the only variable u.

Lemma 2.1. *For each $u\in H^1(\mathbb{R}^N)$, there exists a unique $\phi=\phi_u\in\mathcal{D}(\mathbb{R}^N)$, which solves*

$$\Delta\phi+\beta\Delta_4\phi=4\pi(\omega+\phi)u^2.\tag{2.3}$$

Moreover, $-\omega\le\phi_u\le 0$ on the set $\{u\neq 0\}$, where $\omega>0$.

Proof. The existence and uniqueness of $\phi=\phi_u\in\mathcal{D}(\mathbb{R}^N)$ follow from the Lax–Milgram theorem. Arguing as in the proof of [10, Proposition 2.1], fix $u\in H^1(\mathbb{R}^N)$, multiply (2.3) by $(\omega+\phi_u)_-=\min\{\omega+\phi_u,0\}$, to have that

$$\int_{\{\omega+\phi_u<0\}}|\nabla\phi_u|^2\mathrm{d}x+\beta\int_{\{\omega+\phi_u<0\}}|\nabla\phi_u|^4\mathrm{d}x=-4\pi\int_{\{\omega+\phi_u<0\}}(\omega+\phi_u)^2u^2\mathrm{d}x,$$

which implies $-\omega\le\phi_u$, where $u\neq 0$. Then, let $(\phi_u)_+=\max\{\phi_u,0\}$ as a test function in (2.3), we have

$$\begin{aligned}&-\int_{\{\phi_u\ge 0\}}|\nabla\phi_u|^2\mathrm{d}x-\beta\int_{\{\phi_u\ge 0\}}|\nabla\phi_u|^4\mathrm{d}x\\&\quad=4\pi\int_{\{\phi_u\ge 0\}}\omega\phi_u u^2\mathrm{d}x+4\pi\int_{\{\phi_u\ge 0\}}\phi_u^2u^2\mathrm{d}x,\end{aligned}$$

which implies $(\phi_u)_+\equiv 0$. Hence, we obtain $\phi_u\le 0$. ■

By means of Lemma 2.1, Φ can be defined by $\Phi: H^1(\mathbb{R}^N)\to\mathcal{D}(\mathbb{R}^N)$, which is of class C^1 and maps each $u\in H^1(\mathbb{R}^N)$ to the unique solution of $\Delta\phi+\beta\Delta_4\phi=4\pi(\omega+\phi)u^2$. By the definition of Φ, we have

$$I'_\phi(u,\phi_u)=0,\quad\forall u\in H^1(\mathbb{R}^N).$$

Let us consider the functional

$$J: H^1(\mathbb{R}^N)\to\mathbb{R},\quad J(u)=I(u,\phi_u),$$

then, J is $C^1(H^1(\mathbb{R}^N),\mathbb{R})$ and $J'(u) = I'_u(u,\phi_u)$. Multiplying both members of (2.3) by ϕ_u and integrating by parts, we get

$$\frac{1}{4\pi}\int_{\mathbb{R}^N}|\nabla\phi_u|^2\mathrm{d}x + \frac{\beta}{4\pi}\int_{\mathbb{R}^N}|\nabla\phi_u|^4\mathrm{d}x = -\int_{\mathbb{R}^N}(\omega+\phi_u)\phi_u u^2\mathrm{d}x. \tag{2.4}$$

Then, it follows from the definition of I and (2.4) that the functional associated to (1.1) is given by

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}(m^2-\omega^2)u^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}\phi_u^2u^2\mathrm{d}x \\ &\quad + \frac{1}{8\pi}\int_{\mathbb{R}^N}|\nabla\phi_u|^2\mathrm{d}x + \frac{3\beta}{16\pi}\int_{\mathbb{R}^N}|\nabla\phi_u|^4\mathrm{d}x - \lambda\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x \\ &\quad - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x, \end{aligned}$$

and for any $u,v\in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} \langle J'(u),v\rangle &= \int_{\mathbb{R}^N}\langle\nabla u,\nabla v\rangle\mathrm{d}x + \int_{\mathbb{R}^N}(m^2-\omega^2)uv\mathrm{d}x - \int_{\mathbb{R}^N}(2\omega+\phi_u)\phi_u uv\mathrm{d}x \\ &\quad - \lambda\int_{\mathbb{R}^N}f(x,u)v\mathrm{d}x - \int_{\mathbb{R}^N}|u|^{2^*-2}uv\mathrm{d}x. \end{aligned}$$

From [21], we infer that the following results hold.

Lemma 2.2. *The following statements are equivalent:*

(i) $(u,\phi)\in H^1(\mathbb{R}^N)\times\mathcal{D}(\mathbb{R}^N)$ *is a critical point of* I, *i.e.,* (u,ϕ) *is a solution of* (1.1);

(ii) u *is a critical point of* J *and* $\phi=\phi_u$.

Lemma 2.3. *If* $u\in H^1(\mathbb{R}^N)$ *is radially symmetric, then* ϕ_u *obtained by Lemma* 2.1 *is also radially symmetric.*

Proof. The proof is the same as the argument of [12, Lemma 5]. However, we want to state it again for the reader's convenience. Combined with [2], for every field v defined almost everywhere in $\mathbb{R}^N$, and for every $g\in O(N)$, set

$$(T_g v)(x) = v(gx).$$

By the well-known principle of symmetric criticality (see [19]), it is enough to prove that I is T_g-invariant, i.e., for every $u\in H^1(\mathbb{R}^N)$ and $g\in O(N)$,

$$J(T_g u) = J(u).$$

The main point is to prove that, for every $u\in H^1(\mathbb{R}^N)$,

$$\Phi[T_g u] = T_g\Phi[u].$$

It is well known that

$$\Delta T_g \Phi[u] = T_g(\Delta\Phi[u]).$$

Similarly, we can get that

$$\Delta_4 T_g \Phi[u] = T_g(\Delta_4\Phi[u]).$$

It is simple to verify that $\Phi[T_g u]$ and $T_g \Phi[u]$ solve the same equation

$$\Delta\phi + \beta\Delta_4\phi = 4\pi(\omega + \phi)(T_g u)^2.$$

Then, the conclusion follows from the T_g-invariance of the norms in $H^1(\mathbb{R}^N)$, $\mathcal{D}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$. ■

In order to overcome the lack of compactness since critical growth and the invariance by translations of J, we shall restrict the functional J on the subspace of radial functions

$$H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u(x) = u(|x|)\},$$

which can be compactly embedded into $L_r^q(\mathbb{R}^N)$, $2 < q < 2^*$. The conclusion is that any critical point $u \in H_r^1(\mathbb{R}^N)$ of $J|_{H_r^1(\mathbb{R}^N)}$ is also a critical point of $J|_{H^1(\mathbb{R}^N)}$ (see [2]). We also denote $\mathcal{D}_r(\mathbb{R}^N)$, defined by

$$\mathcal{D}_r(\mathbb{R}^N) := \{u \in \mathcal{D}(\mathbb{R}^N) : u(x) = u(|x|)\},$$

which is the radial Sobolev subspace of $\mathcal{D}(\mathbb{R}^N)$.

Lemma 2.4. *Suppose f satisfies (f_1)–(f_3). Then there exist a constant $c > 0$ and a sequence $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ such that*

$$J(u_n) \to c, \quad J'(u_n) \to 0, \quad \text{as } n \to \infty. \tag{2.5}$$

Proof. Since (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon|t| + C_\varepsilon|t|^{p-1},$$

and

$$|F(x,t)| \le \frac{\varepsilon}{2}t^2 + \frac{C_\varepsilon}{p}|t|^p.$$

Consequently,

$$\left|\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x\right| \le \frac{\varepsilon}{2}\int_{\mathbb{R}^N} u^2\mathrm{d}x + \frac{C_\varepsilon}{p}\int_{\mathbb{R}^N} |u|^p\mathrm{d}x. \tag{2.6}$$

In view of (2.6), it follows from Lemma 2.1 and the Sobolev inequality that

$$\begin{aligned} J(u) &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} (m^2-\omega^2)u^2 \mathrm{d}x \\ &\quad - \lambda \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x \\ &\geq C_1\|u\|_{H^1}^2 - C_2\|u\|_{H^1}^p - C_3\|u\|_{H^1}^{2^*}. \end{aligned}$$

Thus, there exist $\alpha, \rho > 0$ such that

$$\inf_{\|u\|=\rho} J(u) > \alpha.$$

Moreover, by (f_1)–(f_3), we can infer that there exist C_4, C_5 such that

$$F(x,t) \geq C_4|t|^{\mu} - C_5 t^2. \tag{2.7}$$

It follows from Lemma 2.1 and (2.7) that for $t > 0$ and $u \in H_r^1(\mathbb{R}^N) \setminus \{0\}$

$$\begin{aligned} J(tu) &\leq \frac{t^2}{2}\int_{\mathbb{R}^N} \left[|\nabla u|^2 + (m^2-\omega^2)u^2\right]\mathrm{d}x + \frac{t^2}{2}\omega^2|u|_2^2 \\ &\quad - \lambda\int_{\mathbb{R}^N} F(x,tu)\mathrm{d}x - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x \\ &\leq \frac{t^2}{2}C_6\|u\|_{H^1}^2 - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x \\ &\to -\infty, \end{aligned}$$

as $t \to +\infty$. Consequently, by setting $e = tu$ with t sufficiently large, we have $\|e\| > \rho$ and $J(e) < 0$. ■

It follows from Lemma 2.4 and a variant of the mountain pass theorem [26] that a $(\mathrm{PS})_c$ sequence of the functional J at the level

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)), \quad c \geq \alpha > 0 \tag{2.8}$$

can be constructed, and the set of paths is defined as

$$\Gamma := \left\{\gamma \in C([0,1], H_r^1(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } \gamma(1) = e\right\}.$$

3. Proof of Theorem 1.1

Lemma 3.1. *Under the assumptions of Theorem 1.1 any* $(\mathrm{PS})_c$ *sequence* $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ *satisfying* (2.5) *is bounded. Moreover,* $\{\phi_{u_n}\}$ *is bounded in* $\mathcal{D}_r(\mathbb{R}^N)$.

Proof. Let $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ be a $(\mathrm{PS})_c$ sequence of J. There are two cases need to be considered: either $4 < \mu < 2^*$ or $2 < \mu \leq 4$.

Case (i) $4 < \mu < 2^*$. In view of Lemma 2.1 and (f_3), we obtain

$$\begin{aligned}
c + 1 + \|u_n\|_{H^1} &\geq J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} [|\nabla u_n|^2 + (m^2 - \omega^2)u_n^2]\mathrm{d}x \\
&\quad + \left(\frac{1}{2} + \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \phi_{u_n}^2 u_n^2 \mathrm{d}x + \frac{2}{\mu}\int_{\mathbb{R}^N} \omega\phi_{u_n} u_n^2 \mathrm{d}x \\
&\quad + \frac{1}{8\pi}\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^2 \mathrm{d}x + \frac{3\beta}{16\pi}\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^4 \mathrm{d}x \\
&\quad + \lambda\int_{\mathbb{R}^N} \left[\frac{1}{\mu} f(x, u_n)u_n - F(x, u_n)\right]\mathrm{d}x \\
&\quad + \left(\frac{1}{\mu} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N} |u_n|^{2^*} \mathrm{d}x \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} [|\nabla u_n|^2 + (m^2 - \omega^2)u_n^2]\mathrm{d}x \\
&\quad + \frac{2}{\mu}\int_{\mathbb{R}^N} (\omega + \phi_{u_n})\phi_{u_n} u_n^2 \mathrm{d}x + \frac{1}{8\pi}\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^2 \mathrm{d}x \\
&\quad + \frac{3\beta}{16\pi}\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^4 \mathrm{d}x \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} [|\nabla u_n|^2 + (m^2 - \omega^2)u_n^2]\mathrm{d}x \\
&\quad + \left(\frac{1}{8\pi} - \frac{1}{2\mu\pi}\right)\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^2 \mathrm{d}x \\
&\quad + \left(\frac{3\beta}{16\pi} - \frac{\beta}{2\mu\pi}\right)\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^4 \mathrm{d}x \\
&\geq C_7\|u_n\|_{H^1}^2,
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$.

Case (ii) $2 < \mu \leq 4$. It follows from (2.2) and (2.4) that

$$\begin{aligned}
J(u) = &\frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u|^2 + (m^2 - \omega^2)u^2 - \omega\phi_u u^2]\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (\omega + \phi_u)\phi_u u^2 \mathrm{d}x \\
&- \frac{1}{8\pi}\int_{\mathbb{R}^N} |\nabla\phi_u|^2 \mathrm{d}x - \frac{\beta}{16\pi}\int_{\mathbb{R}^N} |\nabla\phi_u|^4 \mathrm{d}x - \lambda\int_{\mathbb{R}^N} F(x, u)\mathrm{d}x \\
&- \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x
\end{aligned}$$

$$= \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u|^2 + (m^2-\omega^2)u^2 - \omega\phi_u u^2\right]\mathrm{d}x + \frac{\beta}{16\pi}\int_{\mathbb{R}^N} |\nabla \phi_u|^4 \mathrm{d}x$$
$$- \lambda \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x.$$

Thereupon, by applying Lemma 2.1, we have

$$\begin{aligned}
& c + 1 + \|u_n\|_{H^1} \\
&\geq J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \\
&= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + (m^2-\omega^2)u_n^2\right]\mathrm{d}x + \frac{1}{\mu}\int_{\mathbb{R}^N} \phi_{u_n}^2 u_n^2 \mathrm{d}x \\
&\quad - \left(\frac{1}{2}-\frac{2}{\mu}\right)\int_{\mathbb{R}^N} \omega\phi_{u_n} u_n^2 \mathrm{d}x + \frac{\beta}{16\pi}\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^4\mathrm{d}x \\
&\quad + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)\right]\mathrm{d}x + \left(\frac{1}{\mu}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N} |u_n|^{2^*}\mathrm{d}x \\
&\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}\left[\left(\frac{1}{2}-\frac{1}{\mu}\right)(m^2-\omega^2) + \left(\frac{1}{2}-\frac{2}{\mu}\right)\omega^2\right]u_n^2\mathrm{d}x.
\end{aligned}$$

Then, by virtue of $2 < \mu \leq 4$ and $\sqrt{\mu-2}|m| > \sqrt{2}|\omega|$, we can also deduce that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$.

Furthermore, it follows from Lemma 2.1 and the Hölder inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^2\mathrm{d}x + \beta\int_{\mathbb{R}^N} |\nabla\phi_{u_n}|^4\mathrm{d}x \\
&\quad = -4\pi\int_{\mathbb{R}^N} (\omega+\phi_{u_n})\phi_{u_n}u_n^2\mathrm{d}x \\
&\quad \leq -4\pi\int_{\mathbb{R}^N} \omega\phi_{u_n}u_n^2\mathrm{d}x \\
&\quad \leq -4\pi\omega\left(\int_{\mathbb{R}^N} |\phi_{u_n}|^{2^*}\mathrm{d}x\right)^{\frac{1}{2^*}}\left(\int_{\mathbb{R}^N} |u_n|^{\frac{22^*}{2^*-1}}\mathrm{d}x\right)^{\frac{2^*-1}{2^*}} \\
&\quad \leq C\|\phi_{u_n}\|_{\mathcal{D}}\|u_n\|_{H^1}^2.
\end{aligned}$$

Consequently, by Lemma 2.3, $\{\phi_{u_n}\}$ is bounded in $\mathcal{D}_r(\mathbb{R}^N)$. ∎

Lemma 3.2. *If $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^N)$, then, up to a subsequence, $\phi_{u_n} \to \phi_u$ in $\mathcal{D}_r(\mathbb{R}^N)$.*

Proof. Let $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^N)$. Note that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$, then we have

$$u_n \to u \quad \text{in } L_r^q(\mathbb{R}^N) \quad \text{for } 2 < q < 2^*. \tag{3.1}$$

In addition, Lemma 3.1 implies that $\{\phi_{u_n}\}$ is bounded in $\mathcal{D}_r(\mathbb{R}^N)$. Hence, there exists $\phi_{u_0} \in \mathcal{D}_r(\mathbb{R}^N)$ such that $\phi_{u_n} \rightharpoonup \phi_{u_0}$ in $\mathcal{D}_r(\mathbb{R}^N)$, and

$$\phi_{u_n} \rightharpoonup \phi_{u_0} \quad \text{in } L^q_r(\mathbb{R}^N) \quad \text{for } 2 < q < 2^*. \tag{3.2}$$

Next, we can show that $\phi_{u_0} = \phi_u$. Since Lemma 2.1, we know that ϕ_{u_0} satisfies (2.3). Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a test function. By virtue of $\Delta\phi_{u_n} + \beta\Delta_4\phi_{u_n} = 4\pi(\omega + \phi_{u_n})u_n^2$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \langle \nabla\phi_{u_n}, \nabla\varphi\rangle \mathrm{d}x + \beta \int_{\mathbb{R}^N} \langle |\nabla\phi_{u_n}|^2\nabla\phi_{u_n}, \nabla\varphi\rangle \mathrm{d}x \\ = -4\pi \int_{\mathbb{R}^N} \omega u_n^2\varphi \mathrm{d}x - 4\pi \int_{\mathbb{R}^N} \phi_{u_n}u_n^2\varphi \mathrm{d}x.\end{aligned}$$

It follows from (3.1), (3.2) and the Hölder inequality that

$$\int_{\mathbb{R}^N} \langle \nabla\phi_{u_n}, \nabla\varphi\rangle \mathrm{d}x \to \int_{\mathbb{R}^N} \langle \nabla\phi_{u_0}, \nabla\varphi\rangle \mathrm{d}x,$$
$$\int_{\mathbb{R}^N} \langle |\nabla\phi_{u_n}|^2\nabla\phi_{u_n}, \nabla\varphi\rangle \mathrm{d}x \to \int_{\mathbb{R}^N} \langle |\nabla\phi_{u_0}|^2\nabla\phi_{u_0}, \nabla\varphi\rangle \mathrm{d}x,$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} \left[(\phi_{u_n}u_n^2 - \phi_{u_0}u^2)\varphi\right]\mathrm{d}x \\ &\quad \le |\phi_{u_n}|_{2^*}|\varphi|_{2^*}|u_n^2 - u^2|_{2^*/(2^*-2)} + \int_{\mathbb{R}^N} (\phi_{u_n} - \phi_{u_0})u^2\varphi \mathrm{d}x.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^N} \phi_{u_n}u_n^2\varphi \mathrm{d}x \to \int_{\mathbb{R}^N} \phi_{u_0}u^2\varphi \mathrm{d}x,$$

and we can easily get

$$\int_{\mathbb{R}^N} \phi_{u_n}^2u_n^2\varphi \mathrm{d}x \to \int_{\mathbb{R}^N} \phi_{u_0}^2u^2\varphi \mathrm{d}x.$$

By the uniqueness result in Lemma 2.1, we have $\phi_{u_0} = \phi_u$. Next, we show $\phi_{u_n} \to \phi_u$ in $\mathcal{D}_r(\mathbb{R}^N)$ actually. Since ϕ_{u_n} and ϕ_u satisfy the equation (2.3), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \left[\nabla(\phi_{u_n} - \phi_u)\nabla v + \beta(|\nabla\phi_{u_n}|^2\nabla\phi_{u_n} - |\nabla\phi_u|^2\nabla\phi_u)\nabla v\right]\mathrm{d}x \\ = -4\pi \int_{\mathbb{R}^N} \left[\omega(u_n^2 - u^2)v + (\phi_{u_n}u_n^2 - \phi_u u^2)v\right]\mathrm{d}x\end{aligned}$$

for each $v \in \mathcal{D}_r(\mathbb{R}^N)$. Letting $v = \phi_{u_n} - \phi_u$ be a test function, and combining with the following inequality:

$$\left[(|x|^{p-2}x - |y|^{p-2}y)(x - y)\right] \ge C_p|x - y|^p, \quad \text{for } x, y \text{ in } \mathbb{R}^N,\ p \ge 2,$$

we deduce that

$$
\begin{aligned}
&C\|\phi_{u_n}-\phi_u\|_{\mathcal{D}}^2\\
&\quad\le \int_{\mathbb{R}^N}|\nabla(\phi_{u_n}-\phi_u)|^2\mathrm{d}x+\beta A_4\int_{\mathbb{R}^N}|\nabla(\phi_{u_n}-\phi_u)|^4\mathrm{d}x\\
&\quad\le \int_{\mathbb{R}^N}\big[|\nabla(\phi_{u_n}-\phi_u)|^2+(|\nabla\phi_{u_n}|^2\nabla\phi_{u_n}-|\nabla\phi_u|^2\nabla\phi_u)(\nabla(\phi_{u_n}-\phi_u))\big]\mathrm{d}x\\
&\quad\le 4\pi\int_{\mathbb{R}^N}\big[|\omega||u_n^2-u^2||\phi_{u_n}-\phi_u|\\
&\qquad+|\phi_{u_n}||\phi_{u_n}-\phi_u|u_n^2+|\phi_u||\phi_{u_n}-\phi_u|u^2\big]\mathrm{d}x\\
&\quad\le 4\pi|\omega||\phi_{u_n}-\phi_u|_{2^*}|u_n^2-u^2|_{2^*/(2^*-1)}+|\phi_{u_n}|_{2^*}|\phi_{u_n}\\
&\qquad-\phi_u|_{2^*}|u_n|^2_{2\cdot 2^*/(2^*-2)}+|\phi_u|_{2^*}|\phi_{u_n}-\phi_u|_{2^*}|u|^2_{2\cdot 2^*/(2^*-2)}.
\end{aligned}
$$

In view of (3.1) and (3.2), we obtain that $\phi_{u_n}\to\phi_u$ in $\mathcal{D}_r(\mathbb{R}^N)$. ■

Lemma 3.3. *Let c be given by (2.8). Then $c<\frac{1}{N}S^{\frac{N}{2}}$, where S is given by (2.1).*

Proof. This proof uses a technique by Bŕezis and Nirenberg [5] and some of its variants. Moreover, similar to the arguments from Miyagaki [20], fixed $R>0$ and $\psi\in C_0^\infty(\mathbb{R}^N)$ is a non-increasing cut-off function such that

$$\psi|B_R=1,\quad 0\le\psi\le 1 \text{ in } B_{2R} \text{ and } \operatorname{supp}\psi\subset B_{2R}.$$

Let $\varepsilon>0$ and define $w_\varepsilon=u_\varepsilon\psi$, where u_ε is the Talenti's function, which can be found in [22], and has the following explicit expression:

$$u_\varepsilon(x)=\frac{[N(N-2)\varepsilon]^{\frac{N-2}{4}}}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}},\quad x\in\mathbb{R}^N,\ \varepsilon>0.$$

Furthermore, let $v_\varepsilon\in C_0^\infty$ be denoted by

$$v_\varepsilon=\frac{w_\varepsilon}{\|w_\varepsilon\|_{L^{2^*}(B_{2R})}}. \tag{3.3}$$

It follows from the estimates given in [5] and as $\varepsilon\to 0$ that

$$\|\nabla v_\varepsilon\|_2^2\le S+O(\varepsilon^{\frac{N-2}{2}}). \tag{3.4}$$

Due to the definition of c, it is sufficient to prove that there exists $\varepsilon>0$ small enough such that

$$\sup_{t\ge 0}J(tv_\varepsilon)<\frac{1}{N}S^{\frac{N}{2}}.$$

Moreover, since Lemma 2.1, we have

$$\frac{1}{16\pi}\int_{\mathbb{R}^N}|\nabla\phi_u|^2\mathrm{d}x+\frac{\beta}{16\pi}\int_{\mathbb{R}^N}|\nabla\phi_u|^4\mathrm{d}x=-\frac{1}{4}\int_{\mathbb{R}^N}(\omega+\phi_u)\phi_u u^2\mathrm{d}x,$$

and substituting this equality into the expression of J, we obtain

$$\begin{aligned}J(u) = &\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}(m^2-\omega^2)u^2\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^N}\phi_u^2u^2\mathrm{d}x\\ &-\frac{1}{16\pi}\int_{\mathbb{R}^N}|\nabla\phi_u|^2\mathrm{d}x - \frac{3}{4}\int_{\mathbb{R}^N}\omega\phi_u u^2\mathrm{d}x - \lambda\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x\\ &-\frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x.\end{aligned}$$

By virtue of Lemma 2.4, we see that $J(tv_\varepsilon) > 0$ for $t > 0$ small, and $J(tv_\varepsilon) \to -\infty$ as $t \to +\infty$. Then there exists $t_\varepsilon > 0$ such that $J(t_\varepsilon v_\varepsilon) = \sup_{t\ge 0} J(tv_\varepsilon) > 0$. Next we will prove that t_ε is upper and lower bounded. For any $\varepsilon > 0$ small enough, we claim that

$$t_\varepsilon \le \left(\int_{B_{2R}}|\nabla v_\varepsilon|^2\mathrm{d}x + \int_{B_{2R}} m^2v_\varepsilon^2\right)^{1/(2^*-2)} = t_0.$$

Indeed, let $\Psi(t) = J(tv_\varepsilon)$. Then, we have

$$\begin{aligned}\Psi'(t) &= \langle J'(tv_\varepsilon), v_\varepsilon\rangle\\ &= \int_{B_{2R}} t\big(|\nabla v_\varepsilon|^2 + [m^2-(\omega+\phi_{tv_\varepsilon})^2]v_\varepsilon^2\big)\mathrm{d}x\\ &\quad -\lambda\int_{B_{2R}} f(x,tv_\varepsilon)v_\varepsilon\mathrm{d}x - t^{2^*-1}\int_{B_{2R}} v_\varepsilon^{2^*}\mathrm{d}x\\ &\le tt_0^{2^*-2} - t^{2^*-1},\end{aligned}$$

which implies that $\Psi'(t) \le 0$ if $t \ge t_0$. Therefore, the claim holds. Moreover, we may assume that there is a positive constant $C_8 > 0$ such that $t_\varepsilon > C_8$ for $\varepsilon > 0$ small. Otherwise, we suppose there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that $t_{\varepsilon_n} \to 0$ as $n \to \infty$. Thus, we have

$$0 < c \le \sup_{t\ge 0} J(tv_\varepsilon) = J(t_{\varepsilon_n}v_\varepsilon) \to 0,$$

which is a contradiction. Consequently, $t_\varepsilon < C_8$. Since (2.7), we have

$$\int_{\mathbb{R}^N} F(x,t_\varepsilon v_\varepsilon)\mathrm{d}x \ge C_4\int_{\mathbb{R}^N}|t_\varepsilon v_\varepsilon|^\mu\mathrm{d}x - C_5\int_{\mathbb{R}^N}|t_\varepsilon v_\varepsilon|^2\mathrm{d}x. \tag{3.5}$$

It follows from (3.4)–(3.5) and Lemma 2.1 that

$$\begin{aligned}J(t_\varepsilon v_\varepsilon) = &\frac{t_\varepsilon^2}{2}\int_{B_{2R}}|\nabla v_\varepsilon|^2\mathrm{d}x + \frac{t_\varepsilon^2}{2}\int_{B_{2R}}(m^2-\omega^2)v_\varepsilon^2\mathrm{d}x\\ &-\frac{t_\varepsilon^2}{4}\int_{B_{2R}}\phi_{t_\varepsilon v_\varepsilon}^2 v_\varepsilon^2\mathrm{d}x - \frac{1}{16\pi}\int_{B_{2R}}|\nabla\phi_{t_\varepsilon v_\varepsilon}|^2\mathrm{d}x\\ &-\frac{3t_\varepsilon^2}{4}\int_{B_{2R}}\omega\phi_{t_\varepsilon v_\varepsilon}v_\varepsilon^2\mathrm{d}x - \lambda\int_{B_{2R}}F(x,t_\varepsilon v_\varepsilon)\mathrm{d}x - \frac{t_\varepsilon^{2^*}}{2^*}\int_{B_{2R}}|v_\varepsilon|^{2^*}\mathrm{d}x\end{aligned}$$

$$\leq \frac{t_\varepsilon^2}{2}\int_{B_{2R}}|\nabla v_\varepsilon|^2\mathrm{d}x+\frac{t_\varepsilon^2}{2}\int_{B_{2R}}m^2v_\varepsilon^2\mathrm{d}x+\frac{3t_\varepsilon^2}{4}\int_{B_{2R}}\omega^2v_\varepsilon^2\mathrm{d}x$$
$$-\lambda\int_{B_{2R}}F(t_\varepsilon v_\varepsilon)\mathrm{d}x$$
$$\leq \frac{1}{N}\left(S+O(\varepsilon^{\frac{N-2}{2}})+\int_{B_{2R}}m^2v_\varepsilon^2\mathrm{d}x\right)^{\frac{N}{2}}+C_9\int_{B_{2R}}v_\varepsilon^2\mathrm{d}x$$
$$-\lambda C_{10}\int_{B_{2R}}|v_\varepsilon|^\mu\mathrm{d}x.$$

By applying the inequality

$$(a+b)^\sigma\leq a^\sigma+\sigma(a+b)^{\sigma-1}b,\quad \text{for } \sigma\geq 1, \text{ and } a,b\geq 0,$$

we have

$$J(t_\varepsilon v_\varepsilon)\leq\frac{1}{N}S^{\frac{N}{2}}+O(\varepsilon^{\frac{N-2}{2}})+C_{11}\int_{B_{2R}}v_\varepsilon^2\mathrm{d}x-\lambda C_{10}\int_{B_{2R}}|v_\varepsilon|^\mu\mathrm{d}x.$$

Next, we will show that

$$\lim_{\varepsilon\to 0}\varepsilon^{-\frac{N-2}{2}}\int_{B_{2R}}(v_\varepsilon^2-\lambda|v_\varepsilon|^\mu)\mathrm{d}x=-\infty.$$

As in [5], one has

$$\int_{B_{2R}}|w_\varepsilon|^{2^*}\mathrm{d}x=[N(N-2)]^{\frac{N}{2}}\int_{\mathbb{R}^N}\frac{1}{(1+|x|^2)^N}\mathrm{d}x+O(\varepsilon^{\frac{N}{2}}). \tag{3.6}$$

On account of (3.3), by using w_ε instead of v_ε, it is sufficient to prove that

$$\lim_{\varepsilon\to 0}\varepsilon^{-\frac{N-2}{2}}\int_{B_R}(w_\varepsilon^2-\lambda|w_\varepsilon|^\mu)\mathrm{d}x=-\infty, \tag{3.7}$$

and

$$\varepsilon^{-\frac{N-2}{2}}\int_{B_{2R}\setminus B_R}(v_\varepsilon^2-\lambda|v_\varepsilon|^\mu)\mathrm{d}x \tag{3.8}$$

is bounded.

Let

$$K_\varepsilon=\varepsilon^{-\frac{N-2}{2}}\int_{B_R}(w_\varepsilon^2-\lambda|w_\varepsilon|^\mu)\mathrm{d}x.$$

Arguing as in the proof of [6, Lemma 7], by substitution of variables, we obtain

$$K_\varepsilon\leq\varepsilon^{\frac{4-N}{2}}\Bigg(C'\int_0^{\frac{R}{\sqrt{\varepsilon}}}\frac{r^{N-1}}{(1+r^2)^{N-2}}\mathrm{d}r$$
$$-\lambda C_{10}\varepsilon^{-\frac{N-2}{4}\mu+\frac{N}{2}-1}\int_0^{\frac{R}{\sqrt{\varepsilon}}}\frac{r^{N-1}}{(1+r^2)^{(N-2)\mu/2}}\mathrm{d}r\Bigg). \tag{3.9}$$

Now we consider the cases $N \geq 5$, $N = 4$ and $N = 3$ respectively as follows:

Case 1: $N \geq 5$.

By means of $\mu > 2$, it is easy to prove that all integrals in (3.9) are convergent as $\varepsilon \to 0$. Moreover, since $2 < \mu < 2^*$, then $-\frac{N-2}{4}\mu + \frac{N}{2} - 1 < 0$. Hence, we have $K_\varepsilon \to -\infty$ as $\varepsilon \to 0$.

Case 2: $N = 4$.

By employing $\mu < 2^* = 4$, and calculating

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^3}{(1+r^2)^2} \mathrm{d}r = \frac{1}{2}\left(\log\left(1 + \frac{R^2}{\varepsilon}\right) + \frac{\varepsilon}{\varepsilon + R^2} - 1\right)$$

and

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^3}{(1+r^2)^4} \mathrm{d}r = \frac{1}{12} - \frac{\varepsilon^2(\varepsilon + 3R^2)}{12(\varepsilon + R^2)^3},$$

we have

$$K_\varepsilon \leq \frac{C'}{2}\left(\log\left(1 + \frac{R^2}{\varepsilon}\right) + \frac{\varepsilon}{\varepsilon + R^2} - 1\right) - \lambda C_{10}\varepsilon^{\frac{2-\mu}{2}}\left(\frac{1}{12} - \frac{\varepsilon^2(\varepsilon + 3R^2)}{12(\varepsilon + R^2)^3}\right).$$

In view of

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{\frac{2-\mu}{2}}}{\log\left(1 + \frac{R^2}{\varepsilon}\right)} = +\infty,$$

we deduce that $K_\varepsilon \to -\infty$ as $\varepsilon \to 0$.

Case 3: $N = 3$.

By simple calculation, one has

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{1+r^2} \mathrm{d}r = \frac{R}{\sqrt{\varepsilon}} - \arctan\left(\frac{R}{\sqrt{\varepsilon}}\right),$$

then, similar arguments to those in the proof of the case $N = 4$, we get

$$\begin{aligned} K_\varepsilon &\leq C'R - C'\varepsilon^{\frac{1}{2}} \arctan\left(\frac{R}{\sqrt{\varepsilon}}\right) - \lambda C_{10}\varepsilon^{\frac{4-\mu}{4}} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^{\frac{\mu}{2}}} \mathrm{d}r \\ &\leq C'R - \lambda C_{10}\varepsilon^{\frac{4-\mu}{4}} \int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^{\frac{\mu}{2}}} \mathrm{d}r. \end{aligned}$$

Next, we will discuss it case by case: either $2 < \mu \leq 4$ or $4 < \mu < 2^*$.

For $4 < \mu < 2^*$, $\frac{4-\mu}{4} < 0$ and the integral

$$\int_0^{\frac{R}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^{\frac{\mu}{2}}} \mathrm{d}r$$

is convergent, as $\varepsilon \to 0$. Thus, we get the conclusion that $K_\varepsilon \to -\infty$ as $\varepsilon \to 0$.

As for $2 < \mu \leq 4$, we know that

$$\int_0^\infty \frac{r^2}{(1+r^2)^{\frac{\mu}{2}}}\mathrm{d}r \geq \frac{\pi}{2} - 1,$$

which implies that

$$K_\varepsilon \leq C_{12} - \left(\frac{\pi}{2} - 1\right)\lambda C_{10}\varepsilon^{\frac{4-\mu}{4}}.$$

Then, we can also obtain $K_\varepsilon \to -\infty$ as $\varepsilon \to 0$ by choosing $\lambda = \varepsilon^{-\frac{1}{2}}$.

As a consequence, we have completed the argument of (3.7).

Now, let us estimate for all $N \geq 3$. Fix ε sufficiently small, and by (3.6), we have

$$\begin{aligned}\varepsilon^{-\frac{N-2}{2}} \int_{B_{2R}\setminus B_R} (v_\varepsilon^2 - \lambda |v_\varepsilon|^\mu)\mathrm{d}x &\leq \frac{C'}{\varepsilon^{\frac{N-2}{2}}} \int_{B_{2R}\setminus B_R} \psi^2 w_\varepsilon^2 \mathrm{d}x \\ &\leq C''\varepsilon \|\psi\|_{H^1},\end{aligned}$$

where R large is chosen such that $w_\varepsilon^2 \leq \varepsilon^{\frac{N}{2}}$, for all $|x| \geq R$. Thereupon, the equation (3.8) is bounded. This completes the proof of Lemma 3.3. ∎

Lemma 3.4. *The weak limit* (u, ϕ_u) *solves problem* (1.1).

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence as given in (2.5). From Lemma 3.1, we know that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$, then, up to subsequence, there exists $u \in H_r^1(\mathbb{R}^N)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_r^1(\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L_r^q(\mathbb{R}^N) \text{ for } q \in (2, 2^*). \end{aligned}$$

And we know that

$$\begin{aligned}\langle J'(u_n), v\rangle = &\int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v\rangle \mathrm{d}x + \int_{\mathbb{R}^N} (m^2 - \omega^2) u_n v \mathrm{d}x \\ &- \int_{\mathbb{R}^N} (2\omega + \phi_{u_n})\phi_{u_n} u_n v \mathrm{d}x - \int_{\mathbb{R}^N} f(x, u_n) v \mathrm{d}x \\ &- \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n v \mathrm{d}x.\end{aligned}$$

Moreover,

$$\int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v\rangle \mathrm{d}x \to \int_{\mathbb{R}^N} \langle \nabla u, \nabla v\rangle \mathrm{d}x,$$

which follows from the weak convergence. Furthermore, it follows from the strong convergences in $L^q_r(\mathbb{R}^N)$, $2 < q < 2^*$ and the Hölder inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\phi_{u_n} u_n - \phi_u u) v \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \phi_{u_n} (u_n - u) v \mathrm{d}x + \int_{\mathbb{R}^N} (\phi_{u_n} - \phi_u) u v \mathrm{d}x \\
&\quad \leq |\phi_{u_n}|_{2^*} |u_n - u|_{2 \cdot 2^*/(2^*-2)} |v|_2 + |\phi_{u_n} - \phi_u|_{2^*} |u|_{2 \cdot 2^*/(2^*-2)} |v|_2,
\end{aligned}$$

as $n \to \infty$. Similarly, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\phi_{u_n}^2 u_n - \phi_u^2 u) v \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \phi_{u_n}^2 (u_n - u) v \mathrm{d}x + \int_{\mathbb{R}^N} (\phi_{u_n}^2 - \phi_u^2) u v \mathrm{d}x \\
&\quad \leq |\phi_{u_n}|_{2^*}^2 |u_n - u|_{2 \cdot 2^*/(2^*-2)} |v|_{2 \cdot 2^*/(2^*-2)} \\
&\qquad + |\phi_{u_n} - \phi_u|_{2 \cdot 2^*/(2^*-2)} |\phi_{u_n} + \phi_u|_{2 \cdot 2^*/(2^*-2)} |u|_{2^*} |v|_{2^*},
\end{aligned}$$

as $n \to \infty$. On account of $\{u_n\}$ is bounded in L^{2^*}, then

$$|u_n|^{2^*-2} u_n \rightharpoonup |u|^{2^*-2} u \quad \text{in } (L^{2^*})'.$$

And it follows from (f_1) and (f_2) that

$$\int_{\mathbb{R}^N} f(u_n) v \mathrm{d}x \to \int_{\mathbb{R}^N} f(u) v \mathrm{d}x.$$

This completes the proof. ■

Proof of Theorem 1.1. Now, we show that u is nontrivial. Assume by contradiction that $u \equiv 0$, and hence $\phi_u = 0$. By means of Lemma 2.1, we infer that

$$\begin{aligned}
-\int_{\mathbb{R}^N} (2\omega + \phi_{u_n}) \phi_{u_n} u_n^2 &\leq -2 \int_{\mathbb{R}^N} \omega \phi_{u_n} u_n^2 \mathrm{d}x \\
&\leq C |u_n|_{2 \cdot 2^*/(2^*-1)}^2 \|\phi_{u_n}\|_{\mathcal{D}},
\end{aligned}$$

which converges to zero as $n \to \infty$. And by (f_1) and (f_2), we obtain

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} F(x, u_n) \mathrm{d}x = 0.$$

Let

$$L := \lim_{n \to \infty} \int_{\mathbb{R}^N} [|\nabla u_n|^2 + (m^2 - \omega^2) u_n^2] \mathrm{d}x \quad \text{and} \quad l := \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \mathrm{d}x.$$

Moreover, in view of $\langle J'(u_n), u_n\rangle = o_n(1)$, we deduce that $L = l$. Since (2.1), we have

$$\int_{\mathbb{R}^N} [|\nabla u_n|^2 + (m^2 - \omega^2)u_n^2]\mathrm{d}x \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x \geq S|u_n|_{2^*}^2,$$

which implies $L \geq S^{\frac{N}{2}}$. Consequently,

$$c = \frac{L}{2} - \frac{L}{2^*} \geq \frac{1}{N} S^{\frac{N}{2}},$$

which contradicts with Lemma 3.3. Thus, u is a nontrivial solution of the problem (1.1). ■

Acknowledgements. We thank the anonymous reviewers for carefully reading the manuscript and their valuable suggestions.

Funding. This work was supported by NSFLN(2021-MS-275) and EFLN(LJKQZ-2021093).

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Received 26 July 2022.

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