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Lipschitz continuity for elliptic free boundary problems with Dini mean oscillation coefficients

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Abstract. We establish local interior Lipschitz continuity of the solutions of a class of free boundary elliptic problems assuming the coefficients of the equation of Dini mean oscillation in at least one direction. The novelty in this regularity result lies in the fact that it allows discontinuous coefficients in all but one variable.

1. Introduction

Throughout this paper, we denote by Ω a bounded domain in $\mathbb{R}^n$ and by $\mathbf{A}(x) = (a_{ij}(x))$ an $n \times n$ matrix that satisfies for some positive constant $\lambda \in (0,1)$

$$\sum_{i,j} |a_{ij}(x)| \leq \lambda^{-1}, \quad \text{for a.e. } x \in \Omega, \tag{1.1}$$

$$\mathbf{A}(x) \cdot \xi \cdot \xi \geq \lambda |\xi|^2, \quad \text{for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^n, \tag{1.2}$$

$\mathbf{f} : \Omega \to \mathbb{R}^n$ is a vector function such that $\mathbf{f}(x) = (f_1(x), \ldots, f_n(x))$ and

$$\exists \overline{f} > 0 : \|\mathbf{f}\|_\infty \leq \overline{f}. \tag{1.3}$$

We consider the following problem

$$(\mathrm{P}) \begin{cases} \text{Find } (u,\chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ \text{(i)} \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0 \text{ a.e. in } \Omega, \\ \text{(ii)} \quad \operatorname{div}(\mathbf{A}(x)\nabla u + \chi \mathbf{f}(x)) = 0 \quad \text{in } H_0^{-1}(\Omega). \end{cases}$$

This class covers a set of various problems including the heterogeneous dam problem [1, 4, 9, 15], in which case Ω represents a porous medium with permeability matrix $\mathbf{A}(x)$, and $\mathbf{f}(x) = \mathbf{A}(x)\mathbf{e}$, with $\mathbf{e} = (0, \ldots, 0, 1)$. A second example is the lubrication

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problem [2] which is obtained when $\mathbf{A}(x) = h^3(x)\mathbf{I_2}$ and $\mathbf{f}(x) = h(x)\mathbf{e}$, where $\mathbf{I_2}$ is the 2×2 identity matrix, and $h(x)$ is a scalar function related to the Reynolds equation. A third example is the aluminum electrolysis problem [3] which corresponds to $\mathbf{A}(x) = k(x)\mathbf{I_2}$ and $\mathbf{f}(x) = h(x)\mathbf{e}$, with $k(x)$ and $h(x)$ two given scalar functions.

We observe that if $\mathbf{f} \in L^q_{\text{loc}}(\Omega)$ for some $q > n$, then so is $\chi\mathbf{f}$, and by taking into account the assumptions (1.1)–(1.2) and equation (P)(ii), we infer from [14, Theorem 8.24] that $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0, 1)$. In this paper, we will improve this regularity by showing that under suitable assumptions, we actually have $u \in C^{0,1}_{\text{loc}}(\Omega)$. We observe that this regularity is optimal due to the gradient discontinuity across the free boundary which is the interface that separates the sets $\{u = 0\}$ and $\{u > 0\}$ from each other. Moreover, Lipschitz continuity is not only interesting by itself, but is also of particular importance in the analysis of the free boundary (see for example [5, 6]).

Before stating our main result, we need to introduce a definition.

Definition 1.1. (i) We say that a function $\omega : (0, 1] \to [0, \infty)$ satisfies the Dini condition if

$$\int_0^1 \frac{\omega(r)}{r} dr < \infty.$$

(ii) A function $f \in L^1(\Omega)$ is of partial Dini mean oscillation with respect to $x' = (x_1, \ldots, x_{n-1})$ in an open ball $B \subset\subset \Omega$, if the function $\omega_f : (0, 1] \to [0, \infty)$ defined by

$$\omega_f(r) = \sup_{x\in B} \fint_{B_r(x)} \left| f(y) - \fint_{B'_r(x')} f(z', y_n) dz' \right| dy,$$

$B'_r(x') = \{y' \in \mathbb{R}^{n-1} : |y' - x'| < r\}$ satisfies the Dini condition.

(iii) For each $\mathbf{f} \in L^1(\Omega)$, we define the following functions:

$$\varpi_{\mathbf{f}}(t) = \int_0^t \frac{\omega_{\mathbf{f}}(s)}{s} ds \quad \text{and} \quad \Phi_{\mathbf{f}}(t) = \varpi_{\mathbf{f}}(t) + \omega_{\mathbf{f}}(t).$$

Remark 1.1. We observe that if a function $f \in L^1(\Omega)$ is such that for each a_n, the function $x' \to f(x', a_n)$ is Hölder continuous, i.e.,

$$|f(x', a_n) - f(y', a_n)| \le C|x' - y'|^\alpha$$

for some $\alpha \in (0, 1)$, then it is easy to verify that $\omega_f(r) \le 2Cr^\alpha$ for any $r \in (0, 1]$, which leads to

$$\int_0^1 \frac{\omega_{\mathbf{f}}(r)}{r} dr \le \int_0^1 2Cr^{\alpha-1} dr = \frac{2C}{\alpha} < \infty$$

Hence, f is of partial Dini mean oscillation with respect to x' in any open ball $B \subset\subset \Omega$.

Here is the main result of this paper.

Theorem 1.1. *Assume that* **A** *and* **f** *satisfy* (1.1)–(1.3) *and the following conditions:*

$$\forall i, j = 1, \dots, n, \ a_{ij} \text{ is of partial Dini mean oscillation with respect to } x' \text{ in } \Omega, \tag{1.4}$$

$$\forall i = 1, \dots, n, \ f_i \text{ is of partial Dini mean oscillation with respect to } x' \text{ in } \Omega. \tag{1.5}$$

Then for any weak solution of (P)*, we have* $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$.

The novelty in Theorem 1.1 lies in the fact that Lipschitz continuity of weak solutions of problem (P) is obtained even when the entries of the matrix $\mathbf{A}(x)$ and the vector function $\mathbf{f}(x)$ are discontinuous provided they satisfy a Dini mean oscillation condition in at least one direction, i.e., if they are regular in at least one variable. Since problem (P) is invariant by rotation in the sense that it is transformed into a similar problem with different coefficients satisfying the same assumptions as the original ones, it is obvious that we only need to have the Dini mean oscillation condition satisfied in any arbitrary space direction.

We recall that interior Lipschitz continuity for problem (P) was established in [5] and the same method was successfully extended to the quasilinear case in [7] and [8]. Interior and boundary Lipschitz continuity were established in [16] for a wide class of linear elliptic equations under some general assumptions. Recently, in [17], Lipschitz continuity was obtained using a different method based on Harnack's inequality. This approach helped relax some of the assumptions required in [5] and [16] and only required that $\mathbf{A}(x) \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ and $\mathrm{div}(\mathbf{f}) \in L^p_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0, 1)$ and $p > n/(1-\alpha)$.

Lastly, we would like to point out that the assumptions (1.4)–(1.5) were introduced in [13] to obtain C^1- and C^2-regularity of solutions to elliptic equations. In this regard, we also refer the reader to the recent work on gradient estimates for elliptic equations in divergence form with partial Dini mean oscillation coefficients [11].

2. Estimates for the equation $\mathrm{div}(\mathbf{A}(x)\nabla u) = -\mathrm{div}(\mathbf{f})$

Under the assumptions of Theorem 1.1, it is known (see [10, Lemma 2.1]) that any weak solution u of the equation $\mathrm{div}(\mathbf{A}(x)\nabla u) = -\mathrm{div}(\mathbf{f})$ is such that $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$. The main result of this section is a local L^∞-norm estimate of the gradient which will be used in the proof of Theorem 1.1 in Section 3. Needless to say, this estimate is of interest for itself.

Theorem 2.1. *Let $\rho > 0$ be such that $B_{3\rho}(x_0) \subset\subset \Omega$ and $\Phi_{\mathbf{A}}(\rho) \leq C_0^*$ for some positive constant C_0^* depending only on n and λ. Assume that $u \in H^1(B_{3\rho}(x_0))$ is a weak solution of the equation* $\mathrm{div}(\mathbf{A}(x)\nabla u) = -\mathrm{div}(\mathbf{f})$ *in $B_{3\rho}(x_0)$. If moreover, we assume that $f_n \in L^\infty(B_{3\rho}(x_0))$ and $\mathbf{A}$ and $\mathbf{f}$ are of partial Dini mean oscillation with respect to x' in $B_{2\rho}(x_0)$, then $\nabla u \in L^\infty(B_\rho(x_0))$ and we have for some positive constant C_1 depending only on n and λ,*

$$|\nabla u|_{L^\infty(B_\rho(x_0))} \leq 3^{2n} C_1 \big(\rho^{-n}|\nabla u|_{L^1(B_{3\rho}(x_0))} + |f_n|_{L^\infty(B_{3\rho}(x_0))} + 4\Phi_{\mathbf{f}}(\rho)\big).$$

The proof of Theorem 2.1 requires a few lemmas.

Lemma 2.1. *Assume that ω is a Dini function and let $a \in (0,1)$ and $b > 1$ be two given real numbers. Then the function defined by*

$$\widetilde{\omega}(t) = \sum_{i=0}^{\infty} a^i \big(\omega(b^i t)\chi_{\{b^i t \leq 1\}} + \omega(1)\chi_{\{b^i t > 1\}}\big)$$

satisfies

$$\int_0^t \frac{\widetilde{\omega}(s)}{s} ds \leq \frac{1}{1-a}\Big[\varpi(t) + a\varpi(1) + \frac{\omega(1)}{\gamma} t^\gamma\Big] \quad \forall t \in [0,1]$$

where $\gamma = -\frac{\ln(a)}{\ln(b)}$ and $\varpi(t) = \int_0^t \frac{\omega(s)}{s} ds$. In particular, $\widetilde{\omega}$ is also a Dini function.

Proof. First, we recall that the function $\widetilde{\omega}$ was introduced in [12, Lemma 3.1], where an estimate was also given. Nevertheless, our estimate is new and more precise.

We start by writing $\widetilde{\omega}(t) = \widetilde{\omega}_1(t) + \widetilde{\omega}_2(t)$ for $t \in (0,1)$, where

$$\widetilde{\omega}_1(t) = \sum_{i=0}^{\infty} a^i \omega(b^i t)\chi_{\{b^i t \leq 1\}} \quad \text{and} \quad \widetilde{\omega}_2(t) = \omega(1)\sum_{i=0}^{\infty} a^i \chi_{\{b^i t > 1\}}.$$

Next, let $i_0 = [-\frac{\ln(t)}{\ln(b)}]$ and observe that we have $t \leq b^{-i}$ iff $i \leq i_0$. Then we have

$$\begin{aligned}
\int_0^t \frac{\widetilde{\omega}_1(s)}{s} ds &= \int_0^t a^0 \frac{\omega(b^0 s)}{s}\chi_{\{b^0 s \leq 1\}} ds + \sum_{i=1}^{\infty} a^i \int_0^t \frac{\omega(b^i s)}{s}\chi_{\{b^i s \leq 1\}} ds \\
&= \int_0^t \frac{\omega(s)}{s} ds + \sum_{i=1}^{\infty} a^i \int_0^{tb^i} \frac{\omega(\tau)}{\tau}\chi_{\{\tau \leq 1\}} d\tau \\
&\leq \varpi(t) + \sum_{i=1}^{\infty} a^i \int_0^1 \frac{\omega(\tau)}{\tau} d\tau = \varpi(t) + \frac{a}{1-a}.\varpi(1). \qquad (2.1)
\end{aligned}$$

Given that $i_0 \leq -\frac{\ln(t)}{\ln(b)} < i_0 + 1$, we can write

$$\widetilde{\omega}_2(s) = \omega(1)\sum_{i=0}^{\infty} a^i \chi_{\{i > -\frac{\ln(t)}{\ln(b)}\}} = \omega(1)\sum_{i=i_0+1}^{\infty} a^i = \frac{a^{i_0+1}}{1-a}\omega(1).$$

Moreover, since $a < 1$, we have $a^{i_0+1} \leq a^{-\frac{\ln(t)}{\ln(b)}} = t^\gamma$, which leads to $\widetilde{\omega}_2(s) \leq \frac{\omega(1)}{1-a} t^\gamma$ and

$$\int_0^t \frac{\widetilde{\omega}_2(s)}{s} ds \leq \frac{\omega(1)}{1-a} \int_0^t \frac{s^\gamma}{s} ds \leq \frac{\omega(1)}{\gamma(1-a)} t^\gamma. \tag{2.2}$$

Now, combining (2.1) and (2.2), we obtain

$$\begin{aligned}\int_0^t \frac{\widetilde{\omega}(s)}{s} ds &\leq \varpi(t) + \frac{a}{1-a}.\varpi(1) + \frac{\omega(1)}{\gamma(1-a)} t^\gamma \\ &\leq \frac{1}{1-a}\Big[(1-a)\varpi(t) + a\varpi(1) + \frac{\omega(1)}{\gamma} t^\gamma\Big] \\ &\leq \frac{1}{1-a}\Big[\varpi(t) + a\varpi(1) + \frac{\omega(1)}{\gamma} t^\gamma\Big].\end{aligned}$$

■

The following lemma is a slight improvement of [11, Theorem 1.2] in the sense that it provides a more precise L^∞-estimate of the gradient.

Lemma 2.2. *Let $u \in H^1(B_3)$ be a weak solution of the equation* $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$ *in B_3, with $\mathbf{A}$ and $\mathbf{f}$ satisfying (1.1)–(1.3) in B_3 and both $\mathbf{A}$ and $\mathbf{f}$ are of partial Dini mean oscillation with respect to x' in B_2. Then we have $\nabla u \in L^\infty(B_1)$ with*

$$|\nabla u|_{L^\infty(B_1)} \leq 3^{nk_0} C_1 \big(|\nabla u|_{L^1(B_3)} + \overline{f} + 4\Phi_{\mathbf{f}}(1)\big)$$

where $C_1 = C_1(n, \lambda)$ is a positive constant depending only on n and λ, and k_0 is an integer greater than 1 satisfying

$$\varpi_{\mathbf{A}}(2^{-k_0}) + 2\omega_{\mathbf{A}}(1)\sqrt{2^{-k_0}} \leq 3^{-n-1} C_0^{-1} = C_0^*.$$

Proof. First, we observe that by scaling, we may replace the ball B_3 by B_6 as in [11]. Next, we denote by $C_0 = C_0(n, \lambda)$ the positive constant depending only on n and λ that was introduced in [11, Proof of Theorem 1.2, p. 1520]. Following this reference, we choose $\gamma = \frac{1}{2}$, $0 < \kappa < \min(2^{-1}, C_0^{-2})$ and we denote by $\widetilde{\omega}_{\mathbf{A}}(t)$ the function defined in Lemma 2.1 with $a = \sqrt{\kappa}$ and $b = \frac{1}{\kappa}$. Let now k_0 be a positive integer greater than 1 that satisfies

$$C_0 \int_0^{\frac{1}{2^{k_0}}} \frac{\widetilde{\omega}_{\mathbf{A}}(t)}{t} dt \leq 3^{-n}$$

which, by taking into account the estimate of Lemma 2.1, is true if

$$\frac{C_0}{1-\sqrt{\kappa}}\big(\varpi_{\mathbf{A}}(2^{-k_0}) + \sqrt{\kappa}\varpi_{\mathbf{A}}(1) + 2\omega_{\mathbf{A}}(1)\sqrt{2^{-k_0}}\big) \leq 3^{-n}.$$

At this step, we further assume that $\kappa \leq 2^{-2}$, which leads to $1-\sqrt{\kappa} \geq 2^{-1}$, and makes the above inequality hold if

$$2\big(\varpi_{\mathbf{A}}(2^{-k_0}) + \sqrt{\kappa}\varpi_{\mathbf{A}}(1) + 2\omega_{\mathbf{A}}(1)\sqrt{2^{-k_0}}\big) \leq 3^{-n} C_0^{-1}.$$

This in turn remains true if κ and k_0 are chosen such that

$$2\sqrt{\kappa}\varpi_{\mathbf{A}}(1),\ \varpi_{\mathbf{A}}(2^{-k_0}) + 2\omega_{\mathbf{A}}(1)\sqrt{2^{-k_0}} \leq 3^{-n-1}C_0^{-1} = C_0^{*}.$$

If we replace f_1 by f_n, we get the estimate (see [11, p. 1522]) with a positive constant $C_1(n,\lambda)$ depending only on n and λ,

$$3^{-nk_0}|\nabla u|_{L^\infty(B_1)} \leq C_1(n,\lambda)\left(|\nabla u|_{L^1(B_3)} + |f_n|_{L^\infty(B_3)} + \int_0^1 \frac{\widetilde{\omega}_{\mathbf{f}}(t)}{t}dt\right)$$

which can be written by using Lemma 2.1 again as

$$\begin{aligned}|\nabla u|_{L^\infty(B_1)} &\leq 3^{nk_0}C_1\Big(|\nabla u|_{L^1(B_3)} + \overline{f} + \frac{1}{1-\sqrt{\kappa}}(\varpi(1) + a\varpi(1) + 2\omega(1)1^{\gamma})\Big)\\ &\leq 3^{nk_0}C_1\big(|\nabla u|_{L^1(B_3)} + \overline{f} + 4(\omega_{\mathbf{f}}(1) + \varpi_{\mathbf{f}}(1))\big)\\ &= 3^{nk_0}C_1\big(|\nabla u|_{L^1(B_3)} + \overline{f} + 4\Phi_{\mathbf{f}}(1)\big).\end{aligned}$$

■

The following lemma is a slight improvement of [17, Lemma 2.2].

Lemma 2.3. *Assume that u is a nonnegative weak solution of the equation*

$$\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$$

in Ω and let $x_0 \in \Omega$ and $r > 0$ such that $B_{5r}(x_0) \subset\subset \Omega$ and $\overline{B}_r(x_0) \cap \{u = 0\} \neq \emptyset$. Then we have for some positive constant C_2 depending only on n, λ and $\overline{f}$: $\max_{\overline{B_r}(x_0)} u \leq C_2 r$.

Proof. Let $x_1 \in \overline{B}_r(x_0) \cap \{u = 0\}$ and let $\omega_n = |B_1|$ be the measure of the unit ball in $\mathbb{R}^n$. First since $B_{5r}(x_0) \subset\subset \Omega$, it is easy to verify that $B_{4r}(x_1) \subset\subset \Omega$. Next, we observe that since $\mathbf{f} \in L^\infty(\Omega)$, we can apply Harnack's inequality [14, Theorem 8.17–Theorem 8.18, p. 194] to the equation $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$ with $p = n+1$. Therefore, we get for a positive constant C depending only on n

$$\begin{aligned}\max_{\overline{B}_{2r}(x_1)} u &\leq C\Big(\min_{\overline{B}_{2r}(x_1)} u + \frac{1}{\lambda}r^{1-\frac{n}{n+1}}\|\mathbf{f}\|_{n+1,\overline{B}_{2r}(x_1)}\Big)\\ &\leq C\Big(0 + \frac{\overline{f}}{\lambda}r^{\frac{1}{n+1}}.|B_{2r}(x_1)|^{\frac{1}{n+1}}\Big)\\ &= \frac{C\overline{f}2^{\frac{n}{n+1}}\omega_n^{\frac{1}{n+1}}}{\lambda}.r^{\frac{1}{n+1}}.r^{\frac{n}{n+1}} = \frac{C\overline{f}2^{\frac{n}{n+1}}\omega_n^{\frac{1}{n+1}}}{\lambda}r = C_2 r.\end{aligned}$$

Given that $\overline{B}_r(x_0) \subset \overline{B}_{2r}(x_1)$, the lemma follows. ■

The following lemma is a Cacciopoli type lemma.

Lemma 2.4. *Assume that u is a weak solution of the equation* $\mathrm{div}(\mathbf{A}(x)\nabla u) = -\mathrm{div}(\mathbf{f})$ *in* Ω. *For each open ball* $B_r(x_0)$ *such that* $B_{2r}(x_0) \subset \Omega$, *we have*

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{32}{\lambda^4 r^2} \int_{B_{2r}(x_0)} u^2 dx + \frac{8\overline{f}}{\lambda r} \int_{B_{2r}(x_0)} |u| dx + \frac{2^{n+1}\overline{f}^2 \omega_n}{\lambda^2} r^n.$$

Proof. Let $B_r(x_0)$ be an open ball such that $B_{2r}(x_0) \subset \Omega$, and let $\eta \in C_0^\infty(B_{2r}(x_0))$ be a cut-off function such that

$$\eta = 1 \quad \text{in } B_r(x_0), \qquad 0 \leq \eta \leq 1 \qquad \text{and} \qquad |\nabla \eta| \leq \frac{2}{r} \quad \text{in } B_{2r}(x_0).$$

Using $\eta^2 u$ as a test function for equation $\mathrm{div}(\mathbf{A}(x)\nabla u) = -\mathrm{div}(\mathbf{f})$, we get

$$\int_{B_{2r}(x_0)} a(x)\nabla u.\nabla(\eta^2 u) dx = -\int_{B_{2r}(x_0)} \mathbf{f}(x).\nabla(\eta^2 u) dx$$

which can be written as

$$\int_{B_{2r}(x_0)} \eta^2 a(x)\nabla u.\nabla u dx = -\int_{B_{2r}(x_0)} 2\eta.u.a(x)\nabla u.\nabla\eta dx - \int_{B_{2r}(x_0)} 2\eta u.\mathbf{f}(x).\nabla\eta dx - \int_{B_{2r}(x_0)} \eta^2.\mathbf{f}(x).\nabla u dx$$

or by using (1.1)–(1.2)

$$\lambda \int_{B_{2r}(x_0)} \eta^2 |\nabla u|^2 dx \leq \frac{2}{\lambda} \int_{B_{2r}(x_0)} \eta.|\nabla u|.|u|.|\nabla\eta| dx + \int_{B_{2r}(x_0)} 2\eta|u|.|\mathbf{f}|.|\nabla\eta| dx + \int_{B_{2r}(x_0)} \eta.|\nabla u|.\eta.|\mathbf{f}|_\infty dx.$$

By taking into account (1.3) and the fact that $|\nabla\eta| \leq 2/r$, we obtain

$$\int_{B_{2r}(x_0)} \eta^2 |\nabla u|^2 dx \leq \int_{B_{2r}(x_0)} (\eta|\nabla u|).\Big(\frac{4|u|}{\lambda^2 r}\Big) dx + \frac{4\overline{f}}{\lambda r} \int_{B_{2r}(x_0)} |u| dx + \int_{B_{2r}(x_0)} (\eta|\nabla u|).\Big(\frac{\overline{f}}{\lambda}\Big) dx$$

Now, we apply the following Young's type inequality $ab \le \frac{1}{4}a^2 + b^2$ to the first and third integrals of the right-hand side of the previous inequality

$$\begin{aligned}\int_{B_{2r}(x_0)} \eta^2|\nabla u|^2dx \le \frac{1}{4}\int_{B_{2r}(x_0)} \eta^2|\nabla u|^2dx + \frac{16}{\lambda^4r^2}\int_{B_{2r}(x_0)} u^2dx \\ + \frac{4\overline{f}}{\lambda r}\int_{B_{2r}(x_0)} |u|dx + \frac{1}{4}\int_{B_{2r}(x_0)} \eta^2|\nabla u|^2dx \\ + \int_{B_{2r}(x_0)} \frac{\overline{f}^2}{\lambda^2}dx\end{aligned}$$

which leads, since $\eta = 1$ in $B_r(x_0)$, to

$$\int_{B_r(x_0)} |\nabla u|^2dx \le \frac{32}{\lambda^4r^2}\int_{B_{2r}(x_0)} u^2dx + \frac{8\overline{f}}{\lambda r}\int_{B_{2r}(x_0)} |u|dx + \frac{2^{n+1}\omega_n\overline{f}^2}{\lambda^2}r^n.$$

■

Combining Lemmas 2.3 and 2.4, we obtain the following lemma.

Lemma 2.5. *Assume that u is a nonnegative weak solution of the equation*

$$\mathrm{div}(\mathbf{A}(x)\nabla u) = -\mathrm{div}(\mathbf{f})$$

in Ω and let $x_0 \in \Omega$ and $r > 0$ such that $B_{10r}(x_0) \subset\subset \Omega$ and $\overline{B}_{2r}(x_0) \cap \{u = 0\} \neq \emptyset$. Then we have

$$\int_{B_r(x_0)} |\nabla u|dx \le C_3r^n$$

where

$$C_3 = \frac{\omega_n 2^{\frac{n+1}{2}}}{\lambda^2}\sqrt{16C_2^2 + 4C_2\overline{f}\lambda^3 + \overline{f}^2\lambda^2}$$

and C_2 is the constant in Lemma 2.3.

Proof. From Lemmas 2.3 and 2.4, we have

$$\max_{\overline{B_{2r}(x_0)}} u \le C_2(2r) \tag{2.3}$$

and

$$\int_{B_r(x_0)} |\nabla u|^2dx \le \frac{32}{\lambda^4r^2}\int_{B_{2r}(x_0)} u^2dx + \frac{8\overline{f}}{\lambda r}\int_{B_{2r}(x_0)} udx + \frac{2^{n+1}\overline{f}^2\omega_n}{\lambda^2}r^n. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla u|^2 dx &\le \frac{32}{\lambda^4 r^2}\int_{B_{2r}(x_0)} C_2^2 (2r)^2 dx \\
&\quad + \frac{8\overline{f}}{\lambda r}\int_{B_{2r}(x_0)} C_2(2r)dx + \frac{2^{n+1}\omega_n \overline{f}^2}{\lambda^2} r^n \\
&\le \frac{128}{\lambda^4}\omega_n C_2^2 (2r)^n + \frac{16\overline{f}}{\lambda}\omega_n C_2 (2r)^n + \frac{2^{n+1}\overline{f}^2\omega_n}{\lambda^2} r^n \\
&= \frac{2^{n+1}\omega_n}{\lambda^4}\left[64C_2^2 + 8C_2\overline{f}\lambda^3 + \overline{f}^2\lambda^2\right] r^n. \qquad (2.5)
\end{aligned}$$

We conclude by using the Cauchy–Schwarz inequality and taking into account (2.5),

$$\int_{B_r(x_0)} |\nabla u| dx \le \left(\int_{B_r(x_0)} |\nabla u|^2 dx\right)^{\frac{1}{2}} .|B_r(x_0)|^{\frac{1}{2}} \le C_3 r^n. \qquad \blacksquare$$

Proof of Theorem 2.1. We observe that the function $v(y) = \frac{u(x_0+\rho y)}{\rho}$ satisfies the equation $\mathrm{div}(\mathbf{A}_\rho(x)\nabla v) = -\mathrm{div}(\mathbf{f}_\rho)$ in B_3, where $\mathbf{F}_\rho(y) = \mathbf{F}(x_0 + \rho y)$. Moreover, it is obvious that $\mathbf{A}_\rho$ and $\mathbf{f}_\rho$ satisfy the assumption of Lemma 2.2 in B_3. Therefore, we get the estimate

$$|\nabla v|_{L^\infty(B_1)} \le 3^{nk_0} C_1 \big(|\nabla v|_{L^1(B_3)} + |f_{\rho n}|_{L^\infty(B_3)} + 4\Phi_{\mathbf{f}_\rho}(1)\big) \qquad (2.6)$$

where C_1 is a positive constant depending only on n and λ, k_0 is an integer satisfying

$$\varpi_{\mathbf{A}_\rho}(2^{-k_0}) + 2\omega_{\mathbf{A}_\rho}(1)\sqrt{2^{-k_0}} \le C_0^*$$

and C_0 is the positive constant depending only on n and λ from [11, p. 1520].

If we observe that $\omega_{\mathbf{f}_\rho}(s) = \omega_{\mathbf{f}}(\rho s)$, $\varpi_{\mathbf{f}_\rho}(s) = \varpi_{\mathbf{f}}(\rho s)$, and we choose $k_0 = 2$, this condition reduces to $\varpi_{\mathbf{A}}(2^{-2}\rho) + \omega_{\mathbf{A}}(\rho) \le C_0^*$, which is in particular true if $\Phi_{\mathbf{A}}(\rho) \le C_0^*$.

Finally, since $\Phi_{\mathbf{f}_\rho}(s) = \Phi_{\mathbf{f}}(\rho s)$, the estimate of Theorem 2.1 follows from (2.6).

$\blacksquare$

3. Proof of Theorem 1.1

Let $\varepsilon > 0$, $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$, and let C_0^* be the positive constant depending only on n and λ that was introduced in Theorem 2.1. Since $\lim_{t\to 0}\Phi_{\mathbf{F}}(t) = 0$, there exists $t_0 \in (0, 1)$ such that

$$\Phi_{\mathbf{A}}(t) \le C_0^* \quad \forall t \in (0, t_0). \qquad (3.1)$$

We fix $\varepsilon_0 = \frac{3}{4}t_0$ and assume that $\varepsilon < \varepsilon_0$. We shall prove that ∇u is bounded in $\Omega_{41\varepsilon}$ by a constant depending only on n, λ, $\overline{f}$, $\Phi_{\mathbf{f}}(1)$, $|\nabla u|_{L^1(\Omega)}$ and ε, provided $\varepsilon < \varepsilon_0$.

Let $x_0 \in \Omega_{41\varepsilon}$. We distinguish two cases:

(i) $\underline{B_{3\varepsilon}(x_0) \subset \{u > 0\}}$:

Since $\chi = 1$ a.e. in $B_{3\varepsilon}(x_0)$, u satisfies the equation $\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f})$ in $B_{3\varepsilon}(x_0)$. We also have by (1.3)–(1.5) that $\|f_n\|_\infty \le \overline{f}$ and $\mathbf{A}$ and $\mathbf{f}$ are of partial Dini mean oscillation with respect to x' in $B_{2\varepsilon}(x_0)$. Moreover, we have from (3.1) $\Phi_{\mathbf{A}}(\varepsilon) \le C_0^*$. Therefore, by Theorem 2.1 applied with $\rho = \varepsilon$, we get for a positive constant $C_1 = C_1(n, \lambda)$

$$|\nabla u|_{L^\infty(B_\varepsilon(x_0))} \le 3^{2n} C_1\big(\varepsilon^{-n}|\nabla u|_{L^1(\Omega)} + |f_n|_{L^\infty(B_{3\varepsilon}(x_0))} + 4\Phi_{\mathbf{f}}(\varepsilon)\big).$$

Since $\Phi_{\mathbf{f}}(t)$ is nondecreasing and $\varepsilon < \varepsilon_0 < 1$, we obtain

$$|\nabla u|_{L^\infty(B_\varepsilon(x_0))} \le 3^{2n} C_1\big(\varepsilon^{-n}|\nabla u|_{L^1(\Omega)} + \overline{f} + 4\Phi_{\mathbf{f}}(1)\big). \tag{3.2}$$

(ii) $\underline{B_{3\varepsilon}(x_0) \cap \{u = 0\} \ne \emptyset}$:

Let $x \in B_\varepsilon(x_0)$ such that $u(x) > 0$ and let $r(x) = \operatorname{dist}(x, \{u = 0\})$ be the distance function to the set $\{u = 0\}$. Our objective is to estimate $|\nabla u|_{L^\infty(B_{r(x)/3}(x))}$. To do that, we will again use Theorem 2.1.

We claim that $r(x) < 4\varepsilon$ and $\overline{B_{10r(x)}}(x) \subset B_{41\varepsilon}(x_0)$. Indeed, by assumption (ii), there exists $z \in B_{3\varepsilon}(x_0) \cap \{u = 0\}$. So, we get $r(x) \le |x - z| \le |x - x_0| + |x_0 - z| < \varepsilon + 3\varepsilon = 4\varepsilon$.

Then we have for each $y \in \overline{B_{10r(x)}}(x)$

$$|x_0 - y| \le |x_0 - x| + |x - y| < \varepsilon + 10r(x) < \varepsilon + 10(4\varepsilon) = 41\varepsilon.$$

This means that $y \in B_{41\varepsilon}(x_0)$, and therefore $\overline{B_{10r(x)}}(x) \subset B_{41\varepsilon}(x_0)$. In particular, we have $\overline{B_{10r(x)}}(x) \subset \Omega_{41\varepsilon} \subset \Omega$.

Now, we obtain from the previous step that $\overline{B_{r(x)}}(x) \subset \Omega$. Moreover, since $B_{r(x)}(x) \subset \{u > 0\}$, we have by (P)(i) that $\chi = 1$ a.e. in $B_{r(x)}(x)$, which leads by (P)(ii) to

$$\operatorname{div}(\mathbf{A}(x)\nabla u) = -\operatorname{div}(\mathbf{f}) \quad \text{in } B_{r(x)}(x).$$

By (1.3)–(1.5), we also know that $\|f_n\|_\infty \le \overline{f}$ and $\mathbf{A}$ and $\mathbf{f}$ are of partial Dini mean oscillation with respect to x' in $B_{2r(x)/3}(x)$. On the other hand, we have $r(x)/3 \le 4\varepsilon/3 < t_0$, which ensures by (3.1) that $\Phi_{\mathbf{A}}(r(x)/3) \le C_0^*$. Hence, we infer from Theorem 2.1 applied with $\rho = r(x)/3$, that we have for a positive constant $C_1 = C_1(n, \lambda)$

$$|\nabla u|_{L^\infty(B_{r(x)/3}(x))} \le 3^{2n} C_1\bigg(\Big(\frac{r(x)}{3}\Big)^{-n}|\nabla u|_{L^1(B_{r(x)})} + \overline{f} + 4\Phi_{\mathbf{f}}(r(x)/3)\bigg). \tag{3.3}$$

Since $r(x)/3 < t_0 < 1$ and $\Phi_{\mathbf{f}}(t)$ is nondecreasing, we have $\Phi_{\mathbf{f}}(r(x)/3) \le \Phi_{\mathbf{f}}(1)$. Furthermore, we have $\overline{B_{10r(x)}}(x) \subset \Omega$ and $\overline{B}_{2r(x)}(x) \cap \{u = 0\} \neq \emptyset$, whence we can use Lemma 2.5 to improve (3.3) as follows:

$$|\nabla u|_{L^\infty(B_{r(x)/3}(x))} \le 3^{2n} C_1 \big(3^n C_3 + \overline{f} + 4\Phi_{\mathbf{f}}(1)\big).$$

Given that x is arbitrary in $B_\varepsilon(x_0) \cap \{u > 0\}$ and $\nabla u(x) = 0$ a.e. in $B_\varepsilon(x_0) \cap \{u = 0\}$, it turns out that ∇u is uniformly bounded in $B_\varepsilon(x_0)$, with

$$|\nabla u|_{L^\infty(B_\varepsilon(x_0))} \le 3^{2n} C_1 \big(3^n C_3 + \overline{f} + 4\Phi_{\mathbf{f}}(1)\big). \tag{3.4}$$

Finally, by taking into account the fact that $\Omega_{41\varepsilon} \subset \bigsqcup_{0<\varepsilon<\varepsilon_0, x_0 \in \Omega_{41\varepsilon}} B_\varepsilon(x_0)$, we conclude from (3.2) and (3.4) that

$$|\nabla u|_{L^\infty(\Omega_{41\varepsilon})} \le 3^{2n} C_1 \big(\varepsilon^{-n} |\nabla u|_{L^1(\Omega)} + 3^n C_3 + \overline{f} + 4\Phi_{\mathbf{f}}(1)\big)$$

which means that $|\nabla u(x)|$ is uniformly bounded in $\Omega_{41\varepsilon}$ by a constant depending only on n, λ, $\overline{f}$, $\Phi_{\mathbf{f}}(1)$, $|\nabla u|_{L^1(\Omega)}$ and $\varepsilon < d(x, \partial\Omega)/41$, for any small enough $\varepsilon > 0$. ∎

Remark 3.1. With a slight modification of the proof, it is not difficult to extend Theorem 1.1 to the following problem:

$$\begin{cases} \text{Find } (u, \chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ \text{(i)} \quad u \ge 0, \quad 0 \le \chi \le 1, \quad u(1-\chi) = 0 \;\text{ a.e. in } \Omega, \\ \text{(ii)} \quad \operatorname{div}(\mathbf{A}(x)\nabla u + \chi \mathbf{f}(x)) = -\chi \mathbf{g}(x) \quad \text{in } \Omega, \end{cases}$$

provided that $\mathbf{A}(x)$ satisfies (1.1)–(1.2), $\mathbf{g}(x)$ and $\mathbf{f}(x)$ satisfy (1.3), and the three functions are of partial Dini mean oscillation with respect to x' in Ω.

References

[1] H. W. Alt, Strömungen durch inhomogene poröse Medien mit freiem Rand. *J. Reine Angew. Math.* **305** (1979), 89–115 Zbl 0392.76091 MR 518852

[2] S. J. Alvarez and J. Carrillo, A free boundary problem in theory of lubrication. *Comm. Partial Differential Equations* **19** (1994), no. 11-12, 1743–1761 Zbl 0809.35166 MR 1301172

[3] A. Bermúdez, M. C. Muñiz, and P. Quintela, Existence and uniqueness for a free boundary problem in aluminum electrolysis. *J. Math. Anal. Appl.* **191** (1995), no. 3, 497–527 Zbl 0821.35155 MR 1325815

[4] S. Challal and A. Lyaghfouri, A new formulation of the dam problem. *European J. Appl. Math.* **16** (2005), no. 5, 583–599 Zbl 1119.35120 MR 2207642

[5] S. Challal and A. Lyaghfouri, On the continuity of the free boundary in problems of type $\operatorname{div}(a(x)\nabla u) = -(h(x)\chi(u))_{x_1}$. *Nonlinear Anal.* **62** (2005), no. 2, 283–300 Zbl 1070.35136 MR 2145607

[6] S. Challal and A. Lyaghfouri, On a class of free boundary problems of type $\operatorname{div}(a(X)\nabla u) = -\operatorname{div}(\chi(u)H(X))$. *Differential Integral Equations* **19** (2006), no. 5, 481–516 Zbl 1212.35508 MR 2235138

[7] S. Challal and A. Lyaghfouri, Lipschitz continuity of solutions of a free boundary problem involving the p-Laplacian. *J. Math. Anal. Appl.* **355** (2009), no. 2, 700–707 Zbl 1167.35551 MR 2521745

[8] S. Challal and A. Lyaghfouri, Regularity results for a quasilinear free boundary problem. *Math. Model. Anal.* **25** (2020), no. 3, 338–350 Zbl 1479.35962 MR 4125293

[9] M. Chipot and A. Lyaghfouri, The dam problem for linear Darcy's law and nonlinear leaky boundary conditions. *Adv. Differential Equations* **3** (1998), no. 1, 1–50 Zbl 0947.35177 MR 1608073

[10] J. Choi, Conormal derivative problem for elliptic equations in divergence form with partial Dini mean oscillation coefficients. *Korean J. Math.* **28** (2020), no. 4, 865–875 Zbl 1460.35104 MR 4209257

[11] J. Choi, S. Kim, and K. Lee, Gradient estimates for elliptic equations in divergence form with partial Dini mean oscillation coefficients. *J. Korean Math. Soc.* **57** (2020), no. 6, 1509–1533 Zbl 1458.35079 MR 4169354

[12] H. Dong, Gradient estimates for parabolic and elliptic systems from linear laminates. *Arch. Ration. Mech. Anal.* **205** (2012), no. 1, 119–149 MR 2927619

[13] H. Dong and S. Kim, On C^1, C^2, and weak type-$(1,1)$ estimates for linear elliptic operators. *Comm. Partial Differential Equations* **42** (2017), no. 3, 417–435 Zbl 1373.35117 MR 3620893

[14] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. Second edn., Grundlehren Math. Wiss. 224, Springer, Berlin, 1983 Zbl 0562.35001 MR 737190

[15] A. Lyaghfouri, The inhomogeneous dam problem with linear Darcy's law and Dirichlet boundary conditions. *Math. Models Methods Appl. Sci.* **6** (1996), no. 8, 1051–1077 Zbl 0869.76088 MR 1428145

[16] A. Lyaghfouri, On the Lipschitz continuity of the solutions of a class of elliptic free boundary problems. *J. Appl. Anal.* **14** (2008), no. 2, 165–181 Zbl 1175.35029 MR 2501665

[17] A. Lyaghfouri, A note on Lipschitz continuity of the solutions of a class of elliptic free boundary problems. *Z. Anal. Anwend.* **40** (2021), no. 3, 255–259 Zbl 1471.35109 MR 4284343

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