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Curves in the disc, the type B braid group, and a type B zigzag algebra

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Abstract. We construct a finite-dimensional quiver algebra from the non-simply laced type B Dynkin diagram, which we call type B zigzag algebra. This leads to a faithful categorical action of the type B Artin (braid) group $\mathbb{A}(B)$, acting on the homotopy category of its projective modules. This categorical action is also closely related to the topological action of $\mathbb{A}(B)$, viewed as mapping class group of the punctured disc – hence, our exposition can be seen as a type B analogue of Khovanov–Seidel's work.

1. Introduction

In the seminal work [15], Khovanov–Seidel introduce a categorical action of the type A_m Artin group $\mathbb{A}(A_m)$, where the group acts faithfully by exact autoequivalences on the bounded homotopy category $\text{Kom}^b(\mathcal{A}_m\text{-p}_r\text{g}_r\text{mod})$ of projective (graded) modules over the type A_m zigzag algebra $\mathcal{A}_m$. Moreover, they show that this Artin group action on $\text{Kom}^b(\mathcal{A}_m\text{-p}_r\text{g}_r\text{mod})$ is deeply related to the mapping class group action on curves on the punctured disc. More precisely, they construct a map L_A that associates complexes of projective $\mathcal{A}_m$-modules to isotopy classes of curves in the disc and show that L_A intertwines the categorical action of $\mathbb{A}(A_m)$ on complexes with the mapping class action of $\mathbb{A}(A_m)$ on curves. Furthermore, the geometric intersection number between two curves c_1 and c_2 can be computed from the dimension of their corresponding total Hom space $\text{HOM}^*(L_A(c_1), L_A(c_2))$ in $\text{Kom}^b(\mathcal{A}_m\text{-p}_r\text{g}_r\text{mod})$. This may be seen as a bridge connecting two appearances of the same group: the former as the Artin group associated to the type A Coxeter group and the latter as the mapping class group of the punctured disc.

Another family of Artin groups which also appear as mapping class groups are the type B_n Artin groups $\mathbb{A}(B_n)$. To this end, Gadbled–Thiel–Wagner develop a "type B"

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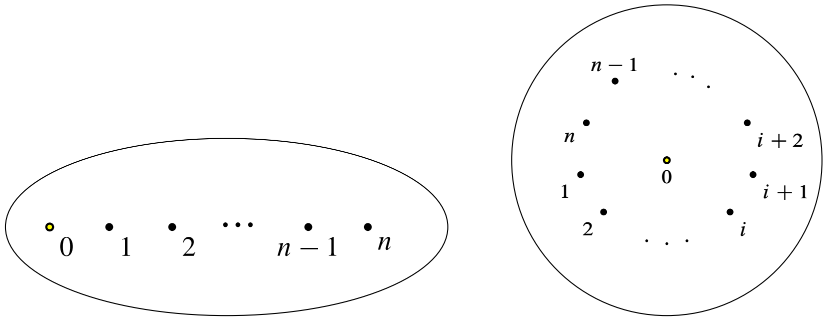


(a) For type B. **(b)** For extended affine type A.

Figure 1.1. Two different affine configurations of the $(n+1)$-punctured disc corresponding to the action of $\mathcal{A}(B_n)$ and $\widehat{\mathcal{A}}(\widehat{A}_{n-1})$, where the puncture labelled "0" is fixed.

analogue of the Khovanov–Seidel story in [9], where they bypass the non-simply laced structure through viewing the type B_n Artin group $\mathcal{A}(B_n)$ as the extended Artin group $\widehat{\mathcal{A}}(\widehat{A}_{n-1})$ of affine type A. Although they are both mapping class groups of an $(n+1)$-punctured disc (fixing one of the punctures), their corresponding natural affine configurations of the disc are different (see Figure 1.1).

The goal of the present paper is to develop a proper (non-simply laced) type B analogue of the stories given by Khovanov–Seidel and Gadbled–Thiel–Wagner. We introduce a (finite-dimensional) quotient of a quiver algebra $\mathcal{B}_n$ over $\mathbb{R}$, which we call type B_n zigzag algebra. Since the type B root system is no longer simply laced, the definition of $\mathcal{B}_n$ will be somewhat subtle – the indecomposable projective $\mathcal{B}_n$-module whose class in the Grothendieck group is a long simple root will only have the structure of a $\mathbb{R}$-vector space, while all the other indecomposable projective $\mathcal{B}_n$-modules whose classes are short simple roots will actually be $\mathbb{C}$-vector spaces. Note that this is somewhat reminiscent of the following non-simply laced extension in quiver theory: by studying representations of K-species [8] instead of quivers (the base field are allowed to be different at each vertex), the finite type K-species are characterised by *all* (including non-simply laced) Dynkin diagrams [6, Theorem B].

The relevance of $\mathcal{B}_n$ to Coxeter theory is provided by the following theorem.

Theorem 1.1 (Theorems 3.13 and 4.6). *The homotopy category* $\mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathit{mod})$ *of projective (bigraded) modules carries a faithful (weak) action of the type B_n Artin group $\mathcal{A}(B_n)$.*

Similar to the works of Khovanov–Seidel [15] and Gadbled–Thiel–Wagner [9], we establish the following result that relates the categorical notions to low-dimensional topology.

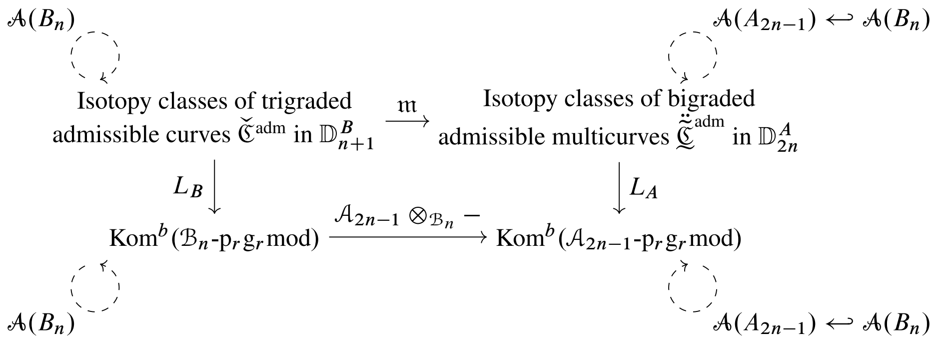


Figure 1.2. The commutative diagram of Theorem 1.3. The map $\mathfrak{m}$ is obtained by lifting curves in $\mathbb{D}^B_{n+1}$ to multicurves in $\mathbb{D}^A_{2n}$ through the double-branched cover $\mathbb{D}^A_{2n} \twoheadrightarrow \mathbb{D}^B_{n+1}$. The map L_B (resp., L_A) is as in Theorem 1.2 (resp., [15, Theorem 4.3]).

Theorem 1.2 (Theorem 5.7 and Proposition 7.3). *There exists a map L_B that associates complexes in $\mathrm{Kom}^b(\mathcal{B}_n\text{-p}_r\mathrm{g}_r mod)$ to curves in the $(n+1)$-punctured disc. This map L_B is $\mathcal{A}(B_n)$-equivariant, intertwining the $\mathcal{A}(B_n)$-action on curves and the $\mathcal{A}(B_n)$-action on complexes in $\mathrm{Kom}^b(\mathcal{B}_n\text{-p}_r\mathrm{g}_r mod)$. Moreover, the (trigraded) intersection number between two curves c_1 and c_2 is given by the Poincaré polynomial of the total Hom space between $L_B(c_1)$ and $L_B(c_2)$.*

One main feature of our work contrasting that of Gadbled–Thiel–Wagner is an explicit realisation of a well-known connection between type B and type A. To explain this, recall that the type B_n Artin group is known to be a (proper) subgroup of the type A_{n-1} Artin group: algebraically the embedding is induced from a folding of Coxeter diagrams [5]; topologically the embedding is obtained by lifting through the double-branched cover of a $(n+1)$-punctured disc $\mathbb{D}^B_{n+1}$ by a $2n$-punctured disc $\mathbb{D}^A_{2n}$ [3]. The topological interpretation induces an $\mathcal{A}(B_n)$-equivariant map $\mathfrak{m}$ that takes curves in the $(n+1)$-punctured disc to *multicurves* in the $2n$-punctured disc, defined by taking the preimage of the covering map. Our work includes a categorical interpretation of this map $\mathfrak{m}$, given by a scalar extension functor.

Theorem 1.3 (Proposition 4.1 and Theorem 5.1). *The type B zigzag algebra $\mathcal{B}_n$ algebra (over $\mathbb{R}$) is isomorphic to Khovanov–Seidel type A zigzag algebra $\mathcal{A}_{2n-1}$ after extending scalars to $\mathbb{C}$, namely,*

$$\mathbb{C} \otimes_{\mathbb{R}} \mathcal{B}_n \cong \mathcal{A}_{2n-1} \quad \textit{as } \mathbb{C}\textit{-algebras}.$$

This induces a scalar extension functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$, which renders the diagram in Figure 1.2 commutative, with all four maps on the square $\mathcal{A}(B_n)$-equivariant.

We would like to mention here that the construction of type B zigzag algebra in this paper can be easily modified to allow for other *Lie-type Dynkin diagrams*, particularly for types C, F_4, and G_2; for the Dynkin diagrams involving edge label 6, one uses a field extension of degree 3 instead[1]. Together with the simply-laced constructions [12, 16], this covers all of the (Lie-type) Dynkin diagrams. To the best of our knowledge, there is no easy generalisation of this construction via finite-dimensional algebras that encapsulates all *Coxeter diagrams*, not even for the finite types H and $I_2(k)$. A different construction via algebra objects in fusion categories that allows for arbitrary Coxeter diagrams can be found in the first author's thesis [11].

Finally, recall that the type A_n zigzag algebra has a geometric origin: it is quasi-isomorphic as a differential graded algebra (dga) with zero differential to the dga associated to an A_n-chain of spherical objects [21]. In particular, it is quasi-isomorphic to the Fukaya A_∞-algebra of a distinguished collection of objects in the Fukaya–Seidel category corresponding to the Milnor fibre of type A singularities [20]. Type B singularities have also been studied, from the symplectic point of view by Arnold [1, 2] as boundary singularities and from the algebraic geometry point of view by Slodowy [23, 24] as simple singularities associated with a $\mathbb{Z}/2\mathbb{Z}$-group action. We expect our type B zigzag algebra to have a similar geometric origin as in type A case. This is an ongoing work of the second author with Shuaige Qiao.

Outline of the paper

Section 2 contains the topological story of this paper – the top row of the commutative diagram in Figure 1.2. We describe the double-branched cover of $\mathbb{D}^B_{n+1}$ by $\mathbb{D}^A_{2n}$, which induces an injection of groups $\Psi : \mathcal{A}(B_n) \hookrightarrow \mathcal{A}(A_{2n-1})$. We state the precise definition of curves and admissible curves in this section and also introduce the notion of trigraded curves – a type B analogue of bigraded curves for type A. The construction of the $\mathcal{A}(B_n)$-equivariant map $\mathfrak{m}$, which lifts trigraded curves to bigraded multicurves, can be found in Section 2.8.

Sections 3 and 4 tell the algebraic story instead – the bottom row of the commutative diagram in Figure 1.2. The definition of type B zigzag algebra $\mathcal{B}_n$ and the proof of the corresponding (weak) categorical action of $\mathcal{A}(B_n)$ on $\text{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\text{mod})$ can be found in Section 3. We then relate our type B zigzag algebra $\mathcal{B}_n$ to the type A zigzag algebra $\mathcal{A}_{2n-1}$ in Section 4, which allows us to obtain the scalar extension functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ and also derive the faithfulness of the $\mathcal{A}(B_n)$ categorical action.

[1] In the early writing of this paper, we have made the (arbitrary) choice of using the field extension $\mathbb{R} \subset \mathbb{C}$ for edge label 4. To be able to deal with both edge labels 4 and 6, it may be more natural to use $\mathbb{Q}$ as the base field to allow for both field extensions of degree 2 and 3.

Section 5 is where we complete the full picture in Figure 1.2 – connecting the top and bottom rows. We recall the $\mathcal{A}(A_{2n-1})$-equivariant map L_A defined in [15] and construct the analogous map L_B for type B. This section also contains the proofs that L_B is $\mathcal{A}(B_n)$-equivariant and that the diagram in Theorem 1.3 commutes, where the latter is the most technical proof of this paper.

Section 6 contains a "decategorified" version of the main theorem (see Theorem 6.3 for the corresponding diagram). Just as the $\mathcal{A}(A_m)$ action on $\text{Kom}^b(\mathcal{A}_m\text{-p}_r\text{g}_r\text{mod})$ categorifies the Burau representation (which can be described as a representation on the first homology of an explicit covering space of $\mathbb{D}^A_{2n}$), we show that the categorical action of $\mathcal{A}(B_n)$ on $\text{Kom}^b(\mathcal{B}_n\text{-p}_r\text{g}_r\text{mod})$ categorifies a representation on (a submodule of) the first homology of an explicit covering space of $\mathbb{D}^B_{n+1}$.

In Section 7, we relate the trigraded intersection numbers of (admissible) curves to the Poincaré polynomial of the total Hom spaces of their corresponding complexes.

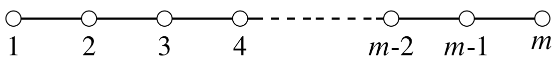
2. Artin groups of type B_n and type A_{2n-1} as mapping class groups

In this section, we will first describe type A and type B Artin groups using generators and relations. After that, we associate these two Artin groups to mapping class groups of surfaces. We then introduce trigraded curves and trigraded intersection numbers as trigraded analogues of bigraded curves and bigraded intersection numbers in [15, Section 3 (b)]. Finally, we construct an $\mathcal{A}(B_n)$-equivariant lift of the isotopy classes of trigraded curves to the isotopy classes of bigraded multicurves.

2.1. Artin groups by generators and relations

An *Artin group* associated to a Coxeter graph Γ is a group defined by generators and relations according to the data of the graph Γ. In this paper, we will only concern ourselves with the Artin groups associated to the type A and type B Coxeter graphs. As such, we will explicitly define them below, and refer the reader to [4, 13] for a more extensive theory on Artin groups.

For $m \geq 2$, the type A_m Artin group $\mathcal{A}(A_m)$ associated to the type A_m Coxeter graph



is the group generated by

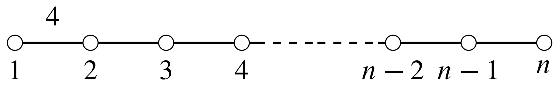
$$\sigma_1^A, \sigma_2^A, \ldots, \sigma_m^A$$

subject to the relations

$$\begin{aligned} \sigma_j^A \sigma_k^A &= \sigma_k^A \sigma_j^A, && \text{for } |j-k| > 1, \\ \sigma_j^A \sigma_{j+1}^A \sigma_j^A &= \sigma_{j+1}^A \sigma_j^A \sigma_{j+1}^A, && \text{for } j = 1, 2, \dots, m-1. \end{aligned}$$

Note that $\mathcal{A}(A_m)$ is the usual $(m+1)$-strand braid group $\mathcal{B}r_{m+1}$.

For $n \geq 2$, the type B_n Artin group $\mathcal{A}(B_n)$ associated to the type B_n Coxeter graph



is the group generated by

$$\sigma_1^B, \sigma_2^B, \dots, \sigma_n^B$$

subject to the relations

$$\begin{aligned} \sigma_1^B \sigma_2^B \sigma_1^B \sigma_2^B &= \sigma_2^B \sigma_1^B \sigma_2^B \sigma_1^B, \\ \sigma_j^B \sigma_k^B &= \sigma_k^B \sigma_j^B, && \text{for } |j-k| > 1, \\ \sigma_j^B \sigma_{j+1}^B \sigma_j^B &= \sigma_{j+1}^B \sigma_j^B \sigma_{j+1}^B, && \text{for } j = 2, 3, \dots, n-1. \end{aligned}$$

2.2. Mapping class groups of discs with marked points

Suppose that $\mathcal{S}$ is a compact, connected, oriented surface, possibly with boundary $\partial\mathcal{S}$, and $\Delta \subset \mathcal{S} \setminus \partial\mathcal{S}$ a finite set of marked points. We denote such a surface as $(\mathcal{S}, \Delta)$, and we will just write $\mathcal{S}$ if the associated Δ is clear from the context. Let $\Delta^{\mathrm{id}} \subset \Delta$ be a subset. Denote by $\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}; \Delta^{\mathrm{id}})$ the group of orientation-preserving diffeomorphisms $f : \mathcal{S} \to \mathcal{S}$ with $f|_{\partial\mathcal{S} \cup \Delta^{\mathrm{id}}} = \mathrm{id}$ and $f(\Delta) = \Delta$. If $\Delta^{\mathrm{id}} = \emptyset$, then we write

$$\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}) := \mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}; \emptyset)$$

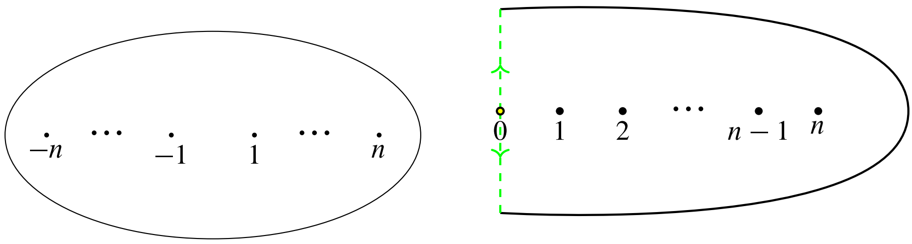
for simplicity. We then define the mapping class group $\mathrm{MCG}(\mathcal{S}, \Delta^{\mathrm{id}})$ of the surface $\mathcal{S}$ with a set Δ of marked points fixing elements in Δ^{id} pointwise by

$$\mathrm{MCG}(\mathcal{S}, \Delta^{\mathrm{id}}) := \pi_0\big(\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}; \Delta^{\mathrm{id}})\big).$$

In a similar fashion, if $\Delta^{\mathrm{id}} = \emptyset$, we denote the mapping class group of $\mathcal{S}$ by

$$\mathrm{MCG}(\mathcal{S}) := \mathrm{MCG}(\mathcal{S}, \emptyset).$$

We will just write $\mathrm{MCG}(\mathcal{S})$ if those conditions are clear from the context. The elements of $\mathrm{MCG}(\mathcal{S})$ are called *mapping classes*. We will see that both Artin groups from Section 2.1 appear as mapping class groups, where we refer the reader to [7] for a more detailed exposition on this.



(**a**) The disc $\mathbb{D}^A_{2n}$ with marked points Δ.

(**b**) The orbit space $\mathbb{D}^B_{n+1} := (\mathbb{D}^A_{2n})/(\mathbb{Z}/2\mathbb{Z})$ with marked points Λ.

Figure 2.1. The affine configurations of the two discs.

2.2.1. Branched covering of $\mathbb{D}^B_{n+1}$ by $\mathbb{D}^A_{2n}$. Consider the following closed disc $\mathbb{D}^A_{2n} := \{z \in \mathbb{C} : \|z\| \leq n+1\}$ embedded in $\mathbb{C}$, equipped with the set

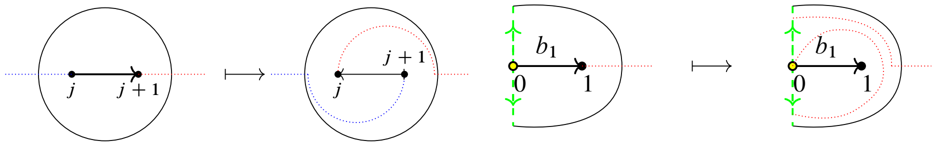
$$\Delta := \{-n, \ldots, -1, 1, \ldots, n\}$$

of $2n$ marked points, as drawn in Figure 2.1a.

Let $r : \mathbb{D}^A_{2n} \to \mathbb{D}^A_{2n}$ be the half-rotation of the disc $\mathbb{D}^A_{2n}$ defined by $r(x) = -x$ for $x \in \mathbb{D}^A_{2n}$. Consider the group $\mathcal{R} \cong \mathbb{Z}/2\mathbb{Z}$ generated by r and its action on $\mathbb{D}^A_{2n}$. It is clear that each $x \in \mathbb{D}^A_{2n} \setminus \{0\}$ has a neighbourhood U_x such that $r(U_x) \cap U_x = \emptyset$. In this way, the quotient map $q_{br} : \mathbb{D}^A_{2n} \to \mathbb{D}^A_{2n}/(\mathbb{Z}/2\mathbb{Z})$ to its orbit space is a normal branched covering with branched point $\{0\}$ [18]. From now on, we will denote $\mathbb{D}^B_{n+1}$ as the orbit space $(\mathbb{D}^A_{2n})/(\mathbb{Z}/2\mathbb{Z})$, and $\Lambda = \{[0], [1], [2], \ldots, [n]\}$ as the set of $n+1$ marked points in $\mathbb{D}^B_{n+1}$. To simplify notation and to help us picture the orbit space $\mathbb{D}^B_{n+1}$, to each equivalence class in $\mathbb{D}^B_{n+1}$ we always pick the element with positive real part as the representative whenever possible (i.e., as long as the equivalence class does not contain points on the imaginary line). This way, we will abuse notation and denote the set of marked points Λ as $\{0, 1, 2, \ldots, n\}$. Figure 2.1b illustrates how we will be picturing $\mathbb{D}^B_{n+1}$, where the two oriented green lines are identified.

2.2.2. Artin groups as mapping class groups. By construction, the marked points on $\mathbb{D}^A_{2n}$ and $\mathbb{D}^B_{n+1}$ are subsets of $\mathbb{Z}$. Therefore, we enumerate the marked points on the disc by increasing sequences of points. Let ϱ_j (resp., b_j) be the horizontal curve connecting the j-th marked point and $(j+1)$-th marked point in $\mathbb{D}^A_{2n}$ (resp., $\mathbb{D}^B_{n+1}$) for $1 \leq j \leq 2n$ (resp., $1 \leq j \leq n+1$).

The group $\mathcal{A}(A_{2n-1})$ is isomorphic to the mapping class group $\mathrm{MCG}(\mathbb{D}^A_{2n})$ of a closed disc $\mathbb{D}^A_{2n}$ with $2n$ marked points. The generator σ^A_j corresponds to the half-twist $[t^A_{\varrho_j}]$ along the arc ϱ_j. Here, $t^A_{\varrho_j}$ is a diffeomorphism in $\mathbb{D}^A_{2n}$ rotating a small open disc enclosing the j-th and $(j+1)$-th marked points anticlockwise by an angle of π,



(a) A half-twist $t^A_{\varrho_j}$ (similarly $t^B_{b_j}$). **(b)** A full twist $(t^B_{b_1})^2$.

Figure 2.2. The twists in $\mathbb{D}^A_{2n}$ and $\mathbb{D}^B_{n+1}$.

permuting the two enclosed marked points, whilst leaving all other marked points fixed; see Figure 2.2a.

On the other hand, the group $\mathcal{A}(B_n)$ is isomorphic to the mapping class group $\mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\})$ of a closed disc $\mathbb{D}^B_{n+1}$ with $n+1$ marked points, fixing the point $\{0\}$ pointwise. The generator σ^B_1 corresponds to the full twist $[(t^B_{b_1})^2]$ along the arc b_1, and for $2 \leq j \leq n$, each generator σ^B_j corresponds to the half-twist $[t^B_{b_j}]$ along the arc b_j. Here, $t^B_{b_j}$ is a diffeomorphism in $\mathbb{D}^B_{n+1}$ rotating a small open disc enclosing the j-th and $(j+1)$-th marked points by an angle of π anticlockwise, as illustrated in Figure 2.2a. As a result, it interchanges the j-th and $(j+1)$-th marked points and leaves the other points fixed pointwise. Consequently, $(t^B_{b_1})^2$ is a diffeomorphism rotating a small open disc enclosing the marked points 0 and 1 anticlockwise by an angle of 2π leaving all the marked points fixed, as shown in Figure 2.2b.

2.2.3. Injection of $\mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\})$ into $\mathrm{MCG}(\mathbb{D}^A_{2n})$. A diffeomorphism f^B in $\mathrm{Diff}(\mathbb{D}^B_{n+1}, \{0\})$ can be lifted to a unique fibre-preserving diffeomorphism f^A in $\mathrm{Diff}(\mathbb{D}^A_{2n})$ via the branched covering map q_{br}. Similarly, an isotopy in $\mathbb{D}^B_{n+1}$ can be lifted to an isotopy in $\mathbb{D}^A_{2n} \setminus \{0\}$. As such, we have a well-defined map Ψ on the mapping class groups from

$$\mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\}) \to \mathrm{MCG}(\mathbb{D}^A_{2n})$$

defined by lifting the mapping class of f^B to the mapping class of f^A. More concretely, using the standard presentation of the groups, Ψ is given by σ^B_1 mapping to σ^A_n and σ^B_j mapping to $\sigma^A_{n+j-1}\sigma^A_{n-(j-1)}$ for $j \geq 2$. In fact, the image of the map Ψ is generated by fibre-preserving mapping classes in $\mathrm{MCG}_p(\mathbb{D}^A_{2n})$. By [3, Theorem 1], we know that any fibre-preserving diffeomorphism f^A which is isotopic to the identity possesses a fibre-preserving isotopy to the identity, which can then be projected to $\mathbb{D}^B_{n+1}$ to get the isotopy

$$f^B \simeq \mathrm{id}\,.$$

Therefore, we have the following well-known result.

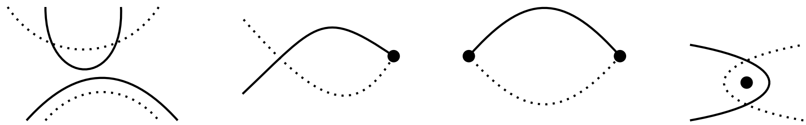


Figure 2.3. The dotted curves and solid curves belong to different multicurves. The multicurves in (a) and (b) do not have minimal intersection, whereas the multicurves in (c) and (d) do.

Proposition 2.1. *The homomorphism* $\Psi : \mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\}) \to \mathrm{MCG}(\mathbb{D}^A_{2n})$ *defined by*

$$\Psi([t^B_{b_i}]) = \begin{cases} [t^A_{\varrho_n}], & \text{for } i = 1, \\ [t^A_{\varrho_{n+i-1}} t^A_{\varrho_{n-(i-1)}}], & \text{for } i \geq 2 \end{cases}$$

is injective.

2.3. Curves and geometric intersection numbers

Here, we collect the definitions of curves and geometric intersection numbers as defined in [15, Section 3a]. Let $(\mathcal{S}, \Delta)$ be a surface with marked points as in Section 2.2. A *curve* c in $(\mathcal{S}, \Delta)$ is a subset of $\mathcal{S}$ that is either a simple closed curve in the interior $\mathcal{S}^o := \mathcal{S} \setminus (\partial\mathcal{S} \cup \Delta)$ of $\mathcal{S}$ and essential (non-nullhomotopic in $\mathcal{S}^o$), or the image of an embedding $\gamma : [0, 1] \to \mathcal{S}$ which is transverse to the boundary $\partial\mathcal{S}$ of $\mathcal{S}$ with its endpoint lying in $\partial\mathcal{S} \cup \Delta$, that is, $\gamma^{-1}(\partial\mathcal{S} \cup \Delta) = \{0, 1\}$. In this way, our defined curves are smooth and unoriented. A *multicurve* in $(\mathcal{S}, \Delta)$ is the union of a finite collection of disjoint curves in $(\mathcal{S}, \Delta)$. We say two curves c_0 and c_1 are *isotopic* if there exists an isotopy in $\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}; \Delta)$ deforming one into the other, denoted by $c_0 \simeq c_1$. Note that the points on $\partial\mathcal{S} \cup \Delta$ cannot move during an isotopy. Therefore, we can partition all curves in $(\mathcal{S}, \Delta)$ into isotopy classes of curves. Two multicurves $\mathfrak{c}_0, \mathfrak{c}_1$ are isotopic if they have the same number of disjoint curves, and each curve in $\mathfrak{c}_0$ is isotopic to one and only one curve in $\mathfrak{c}_1$. Two curves c_0, c_1 are said to have *minimal intersection* if, given two intersection points $z_- \neq z_+$ in $c_0 \cap c_1$, the two arcs $\alpha_0 \subset c_0, \alpha_1 \subset c_1$ with endpoints $z_- \neq z_+$ such that $\alpha_0 \cap \alpha_1 = \{z_-, z_+\}$ do not form an empty bigon (the bigon contains no marked points) unless z_-, z_+ are marked points. Two multicurves $\mathfrak{c}_0, \mathfrak{c}_1$ are said to have *minimal intersection* if any two curves $c_0 \subseteq \mathfrak{c}_0$ and $c_1 \subseteq \mathfrak{c}_1$ have minimal intersection (see Figure 2.3).

Let c_0, c_1 be curves in $(\mathcal{S}, \Delta)$ with $c_0 \cap c_1 \cap \partial\mathcal{S} = \emptyset$. Note that we can always find a curve $c_1' \simeq c_1$ such that c_0 and c_1' have minimal intersection. We define the *geometric intersection number* $I(c_0, c_1) \in \frac{1}{2}\mathbb{Z}$ as follows:

$$I(c_0, c_1) = \begin{cases} 2, & \text{if } c_0, c_1 \text{ are simple closed curves and isotopic;} \\ |(c_0 \cap c_1') \setminus \Delta| + \frac{1}{2}|(c_0 \cap c_1') \cap \Delta|, & \text{if } c_0 \cap c_1' \cap \partial\mathcal{S} = \emptyset. \end{cases}$$

By [15, Lemma 3.2] and [15, Lemma 3.3], the definition is indeed independent of the choice of c_1'. Moreover, note that the definition above does not depend on the orientation of $\mathcal{S}$ and is symmetric. We extend the definition of geometric intersection numbers for multicurves (which do not intersect at $\partial\mathcal{S}$) by just adding up the geometric intersection numbers of each pair of curves $c_0 \subseteq \mathfrak{c}_0$ and $c_1 \subseteq \mathfrak{c}_1$.

2.4. Trigraded curves in $\mathbb{D}^B_{n+1}$

In this subsection, we will extend the notion of bigraded curves and bigraded intersection numbers defined in [15, Section 3d] to *trigraded curves* and *trigraded intersection numbers* (see also Section 2.7).

Let us remind the reader that we equipped the disc $\mathbb{D}^A_{2n}$ with the set of marked points

$$\Delta = \{-n, \dots, -1, 1, \dots, n\}$$

and the disc $\mathbb{D}^B_{n+1}$ with the set of marked points $\Lambda = \{0, 1, \dots, n\}$. Consider another set of marked points $\Delta_0 = \Delta \cup \{0\}$ in the disc $\mathbb{D}^A_{2n}$. Fix the notation as follows:

$$\mathfrak{D}^B_\Lambda := PT(\mathbb{D}^B_{n+1} \setminus \Lambda) \quad \text{and} \quad \mathfrak{D}^A_{\Delta_0} := PT(\mathbb{D}^A_{2n} \setminus \Delta_0),$$

where $PT(\cdot)$ denotes the real projectivisation of the tangent bundle of the respective discs. By taking an oriented trivialisation of its tangent bundle, we can then identify $\mathfrak{D}^A_{\Delta_0} \cong \mathbb{R}\mathrm{P}^1 \times (\mathbb{D}^A_{2n} \setminus \Delta_0)$. In $\mathbb{D}^A_{2n} \setminus \Delta_0$, pick a small loop λ_j winding positively around each puncture $j \in \Delta_0$. In this way, the classes [point $\times \lambda_j$] and [$\mathbb{R}\mathrm{P}^1 \times$ point] form a basis of $H_1(\mathfrak{D}^A_{\Delta_0}; \mathbb{Z})$. Using the universal coefficient theorem for cohomology [10, Theorem 3.2], we consider the covering space $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ of $\mathfrak{D}^A_{\Delta_0}$ classified by the cohomology class $C_0 \in H^1(\mathfrak{D}^A_{\Delta_0}; \mathbb{Z} \times \mathbb{Z})$ defined as follows:

$$C_0([\text{point} \times \lambda_0]) = (0, 0), \tag{2.1}$$
$$C_0([\text{point} \times \lambda_j]) = (-2, 1), \quad \text{for } j = -n, \dots, -1, 1, \dots, n, \tag{2.2}$$
$$C_0([\mathbb{R}\mathrm{P}^1 \times \text{point}]) = (1, 0). \tag{2.3}$$

In fact, $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ is a covering for $\mathfrak{D}^B_\Lambda$ with a group of deck transformations $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, as explained in the following lemma.

Lemma 2.2. (1) *Under the action of the rotation group $\mathcal{R}$ generated by the half-rotation r, the quotient map $q : \mathbb{D}^A_{2n} \setminus \Delta_0 \to \mathbb{D}^B_{n+1} \setminus \Lambda$ is a normal covering space with deck transformation group*

$$\mathcal{R} \cong \mathbb{Z}/2\mathbb{Z}.$$

(2) *The composite $\widetilde{\mathfrak{D}}^A_{\Delta_0} \xrightarrow{\mathfrak{p}} \mathfrak{D}^A_{\Delta_0} \xrightarrow{\mathfrak{q}} \mathfrak{D}^B_{\Lambda}$ is a normal covering, where $\mathfrak{q}$ is the normal covering map induced by the quotient map q on the disc component and the identity map on the $\mathbb{R}\mathrm{P}^1$ component.*

(3) *The group of deck transformations for the covering space $\widetilde{\mathfrak{D}}^A_{\Delta_0} \xrightarrow{\mathfrak{q}\circ\mathfrak{p}} \mathfrak{D}^B_{\Lambda}$ is*

$$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Proof. The proofs of (1) and (2) are straightforward, and we leave them to the reader.

We will now prove (3). Since the covering $\mathfrak{q} \circ \mathfrak{p}$ is normal, its deck transformation group G is given by

$$G \cong \frac{\pi_1(\mathfrak{D}^B_{\Lambda})}{(\mathfrak{q}_* \circ \mathfrak{p}_*)(\pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0}))}.$$

Recall $C_0 : H_1(\mathfrak{D}^A_{\Delta_0}) \to \mathbb{Z} \times \mathbb{Z}$ as defined by (2.1)–(2.3). Let

$$\overline{C_0} : \pi_1(\mathfrak{D}^A_{\Delta_0}) \to \mathbb{Z} \times \mathbb{Z}$$

be the map defined by precomposing C_0 with the natural quotient map

$$\pi_1(\mathfrak{D}^A_{\Delta_0}) \twoheadrightarrow H_1(\mathfrak{D}^A_{\Delta_0}).$$

Observe that we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0}) & \xrightarrow{\mathfrak{p}_*} & \pi_1(\mathfrak{D}^A_{\Delta_0}) & \xrightarrow{\overline{C_0}} & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & 1 \\
 & & \| & & \downarrow \mathfrak{q}_* & & \downarrow \widetilde{\mathfrak{q}_*} & & \\
1 & \longrightarrow & \pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0}) & \xrightarrow{\mathfrak{q}_*\circ\mathfrak{p}_*} & \pi_1(\mathfrak{D}^B_{\Lambda}) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow {\scriptstyle \ell_0 \mapsto 1} & & \downarrow & & \\
1 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & = & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array},$$

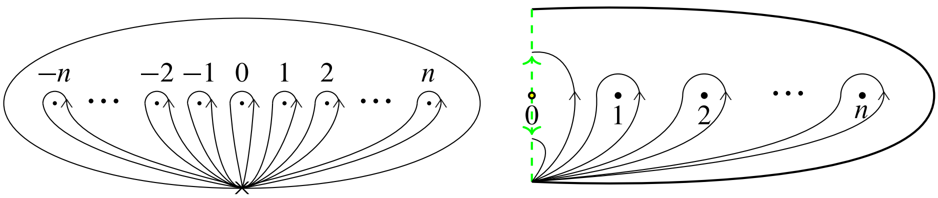


Figure 2.4. The loops chosen for the fundamental groups of $\mathbb{D}^A_{2n} \setminus \Delta_0$ (left) and $\mathbb{D}^B_{n+1} \setminus \Lambda$ (right).

where $\mathbb{Z} \times \mathbb{Z}$ is the deck transformation group of the covering $\mathfrak{p}$, $\mathbb{Z}/2\mathbb{Z}$ is the deck transformation group of the covering $\mathfrak{q}$, and $\widetilde{\mathfrak{q}_*}$ is the map induced by $\mathfrak{q}_*$. We will show that the rightmost column of short exact sequence is left-split; namely, we will construct a map $\widetilde{\varphi} : G \to \mathbb{Z} \times \mathbb{Z}$ such that $\widetilde{\varphi} \circ \widetilde{\mathfrak{q}_*} = \mathrm{id}$:

$$1 \longrightarrow \mathbb{Z} \times \mathbb{Z} \underset{\widetilde{\varphi}}{\overset{\widetilde{\mathfrak{q}_*}}{\rightleftarrows}} G \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

which shows that $G \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as required.

We will first define a map $\varphi : \pi_1(\mathfrak{D}^B_\Lambda) \to \pi_1(\mathfrak{D}^A_{\Delta_0})$ and show that $\overline{C_0} \circ \varphi$ factors uniquely through the quotient G, which we will define to be our map $\widetilde{\varphi} : G \to \mathbb{Z} \times \mathbb{Z}$. We pick loops $\lambda_i \subset \mathbb{D}^A_{2n} \setminus \Delta_0$ and $\ell_i \subset \mathbb{D}^B_{n+1} \setminus \Lambda$ as in Figure 2.4.

The induced map $q_* : \pi_1(\mathbb{D}^A_{2n} \setminus \Delta_0) \to \pi_1(\mathbb{D}^B_{n+1} \setminus \Lambda)$ on the fundamental groups satisfies

$$q_*([\lambda_j]) = \begin{cases} [\ell_0 \circ \ell_0], & \text{for } j = 0, \\ [(\ell_0\ell_1 \cdots \ell_{j-1})\ell_{|j|}(\ell_{j-1}^{-1} \cdots \ell_1^{-1}\ell_0^{-1})], & \text{for } -n \le j \le -1, \\ [\ell_j], & \text{for } 1 \le j \le n. \end{cases}$$

Now, define $\varphi : \pi_1(\mathfrak{D}^B_\Lambda) \to \pi_1(\mathfrak{D}^A_{\Delta_0})$ by sending

$$\begin{cases} [\text{point} \times \ell_0] \mapsto [\text{point} \times \lambda_0], & \\ [\text{point} \times \ell_j] \mapsto [\text{point} \times \lambda_j], & \text{for all } j \in \{1, \ldots, n\}, \\ [\mathbb{R}\mathrm{P}^1 \times \text{point}] \mapsto [\mathbb{R}\mathrm{P}^1 \times \text{point}]. & \end{cases}$$

We claim that $\overline{C_0} \circ \varphi \circ \mathfrak{q}_* = \overline{C_0}$. Firstly, note that

$$\overline{C_0} \circ \varphi \circ \mathfrak{q}_*([\text{point} \times \lambda_0]) = 0 = \overline{C_0}([\text{point} \times \lambda_0]),$$

and

$$\varphi \circ \mathfrak{q}_*([\mathbb{R}\mathrm{P}^1 \times \text{point}]) = [\mathbb{R}\mathrm{P}^1 \times \text{point}]$$

by construction. Moreover, for $j \in \{-n, \dots, -1, 1, \dots, n\}$,

$$(\varphi \circ \mathfrak{q}_*)[\text{point} \times \lambda_j] = \begin{cases} [\text{point}\times\lambda_0\lambda_1\cdots\lambda_{j-1}][\text{point}\times\lambda_{|j|}][\text{point}\times\lambda_0\lambda_1\cdots\lambda_{j-1}]^{-1}, & \text{for } -n\leq j\leq -1, \\ [\text{point}\times\lambda_j], & \text{for } 1\leq j\leq n. \end{cases}$$

Since $\overline{C_0}$ maps to $\mathbb{Z}\times\mathbb{Z}$, which is abelian, we have that, from (2.2),

$$(\overline{C_0}\circ\varphi\circ\mathfrak{q}_*)[\text{point}\times\lambda_j] = \overline{C_0}[\text{point}\times\lambda_{|j|}] = \overline{C_0}[\text{point}\times\lambda_j].$$

This shows that $\overline{C_0}\circ\varphi\circ\mathfrak{q}_*$ and $\overline{C_0}$ agree on all generators of $\pi_1(\mathfrak{D}^A_{\Delta_0})$, and so, they are equal. This implies that

$$(\overline{C_0}\circ\varphi\circ\mathfrak{q}_*\circ\mathfrak{p}_*)(\pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0})) = (\overline{C_0}\circ\mathfrak{p}_*)(\pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0})) = 0.$$

As such, $\overline{C_0}\circ\varphi$ factors uniquely through the quotient G, and we denote this map by

$$\widetilde{\varphi} : G \to \mathbb{Z}\times\mathbb{Z}.$$

By definition, $\widetilde{\mathfrak{q}_*}$ is uniquely determined by the images of [point $\times\lambda_1$], [$\mathbb{R}P^1\times$ point] $\in \pi_1(\mathfrak{D}^A_{\Delta_0})$ under $\mathfrak{q}_*$. It is now easy to see that $\widetilde{\varphi}\circ\widetilde{\mathfrak{q}_*} = \text{id}$ by the construction of $\widetilde{\varphi}$. ■

Remark 2.3. Following the proof of Lemma 2.2 (3), it is easy to see that the covering space $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ of $\mathfrak{D}^B_\Lambda$ is classified by the cohomology class $C^B \in H^1(\mathfrak{D}^B_\Lambda; \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$ defined as follows:

$$\begin{aligned} C^B([\text{point}\times\ell_0]) &= (0,0,1), \\ C^B([\text{point}\times\ell_j]) &= (-2,1,0), \quad \text{for } 1,\dots,n, \\ C^B([\mathbb{R}\text{P}^1\times\text{point}]) &= (1,0,0). \end{aligned}$$

Note that every $f \in \text{Diff}(\mathbb{D}^B_{n+1}, \{0\})$ preserves the class C^B and therefore can be lifted to a unique equivariant diffeomorphism $\check{f}$ of $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ that acts trivially on the fibre of $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ over all points in $T_z\mathbb{D}^B_{n+1}$ for $z\in\partial\mathbb{D}^B_{n+1}$. We will call $\check{f}$ the *preferred lift* of f. Furthermore, every curve c in $\mathbb{D}^B_{n+1}$ admits a canonical section $s_c : c\setminus\Lambda \to \mathfrak{D}^B_\Lambda$ defined by $s_c(z) = T_z c$. We define a *trigrading* of c to be a lift $\check{c}$ of s_c to $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ and a *trigraded curve* to be a pair $(c,\check{c})$ consisting of a curve and its trigrading; we will often just write $\check{c}$ instead of $(c,\check{c})$ when the context is clear. We denote the $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-action on $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ by χ^B. On top of that, we can easily extend the notion of isotopy to the set of trigraded curves, where χ^B and $\text{MCG}(\mathbb{D}^B_{n+1},\{0\})$ have induced actions on the set of isotopy classes of trigraded curves. In particular, for $[f]\in\text{MCG}(\mathbb{D}^B_{n+1},\{0\})$ and a trigraded curve $\check{c}$, $[f](\check{c}) := \check{f}\circ\check{c}\circ f^{-1} : f(c)\setminus\Lambda \to \widetilde{\mathfrak{D}}^A_{\Delta_0}$.

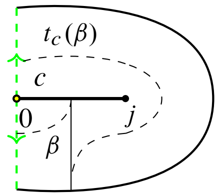


Figure 2.5. The action of full twist around curve joining $\{0\}$ and another point in Λ.

Lemma 2.4. (1) *A curve c admits a trigrading if and only if it is not a simple closed curve.*

(2) *The $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-action on the set of isotopy classes of trigraded curves is free. Equivalently, a trigraded curve $\check{c}$ is never isotopic to $\chi^B(r_1, r_2, r_3)\check{c}$ for any $(r_1, r_2, r_3) \neq 0$.*

Proof. This is essentially the same proof as in [15, Lemmas 3.12 and 3.13]. ∎

Lemma 2.5. (1) *Let c be a curve in $\mathbb{D}^B_{n+1}$ which joins two points of $\Lambda \setminus \{0\}$, $t_c \in \mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\})$ the half-twist along it, and $\check{t}_c$ its preferred lift to $\widetilde{\mathfrak{D}}^A_{\Delta_0}$. Then,*

$$\check{t}_c(\check{c}) = \chi^B(-1, 1, 0)\check{c}$$

for any trigrading $\check{c}$ of c.

(2) *Let c be a curve in $\mathbb{D}^B_{n+1}$ which joins two points of Λ with one of them being $\{0\}$, $t_c \in \mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\})$ the full twist along it, and $\check{t}_c$ its preferred lift to $\widetilde{\mathfrak{D}}^A_{\Delta_0}$. Then, $\check{t}_c(\check{c}) = \chi^B(-1, 1, 1)\check{c}$ for any trigrading $\check{c}$ of c.*

Proof. The proof of (1) is as in [15, Lemma 3.14]. We will now prove (2). Let $\beta : [0, 1] \to \mathbb{D}^B_{n+1} \setminus \Lambda$ be an embedded smooth path from a point $\beta(0) \in \partial\mathbb{D}^B_{n+1}$ to the fixed point $\beta(1) \in c$ of t_c. Note that we have $\check{t}_c(\check{c}) = \chi(r_1, r_2, r_3)\check{c}$ as $t_c(c) = c$. Consider the closed path $\kappa : [0, 2] \to \mathfrak{D}^B_\Lambda$ given by

$$\kappa(t) = \begin{cases} Dt_c(\mathbb{R}\beta'(t)), & \text{if } t \leq 1, \\ \mathbb{R}\beta'(2 - t), & \text{if } t \geq 1, \end{cases}$$

where $\mathbb{R}\beta'(s) \subset T_{\beta(s)}\mathbb{D}^B_{n+1}$. The above situation is illustrated in Figure 2.5. Let $C^B \in H^1(\mathfrak{D}^B_\Lambda; \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ be the cohomology class classifying the covering space $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ of $\mathfrak{D}^B_\Lambda$ as in Remark 2.3. Then, we compute

$$\begin{aligned} (r_1, r_2, r_3) &= -C^B([\kappa]) \\ &= C^B([\mathbb{R}\mathrm{P}^1 \times \text{points}]) + C^B([\text{points} \times \ell_0]) + C^B([\text{points} \times \ell_j]) \\ &= (-1, 1, 1). \end{aligned}$$

Note that $[\kappa]$ only picks up one copy of $[\mathbb{RP}^1 \times \text{points}]$ due to the disc configuration, or more precisely the oriented trivialisation of $\mathbb{D}^B_{n+1}$. ■

2.5. Local index and trigraded intersection numbers

Mimicking the definition of local index for bigraded curves in [15, p. 225], we will define the local index of an intersection between two trigraded curves. Suppose that $(c_0, \check{c}_0)$ and $(c_1, \check{c}_1)$ are two trigraded curves, and $z \in \mathbb{D}^B_{n+1} \setminus \partial\mathbb{D}^B_{n+1}$ is a point where c_0 and c_1 intersect transversally. Take a small circle $\ell \subset \mathbb{D}^B_{n+1} \setminus \Lambda$ around z and an embedded arc $\alpha : [0,1] \to \ell$ which moves clockwise around ℓ such that $\alpha(0) \in c_0$ and $\alpha(1) \in c_1$ and $\alpha(t) \notin c_0 \cup c_1$ for all $t \in (0,1)$. If $z \in \Delta$, then α is unique up to a change of parametrisation; otherwise, there are two choices which can be told apart by their endpoints. Then, take a smooth path $\kappa : [0,1] \to \mathfrak{D}^B_\Lambda$ with $\kappa(t) \in (\mathfrak{D}^B_\Lambda)_{\alpha(t)}$ for all t, going from $\kappa(0) = T_{\alpha(0)}c_0$ to $\kappa(1) = T_{\alpha(1)}c_1$, such that $\kappa(t) \neq T_{\alpha(t)}\ell$ for every t. One can take κ as a family of tangent lines along α which are all transverse to ℓ. After that, lift κ to a path $\check{\kappa} : [0,1] \to \check{\mathfrak{D}}^B_\Lambda$ with $\check{\kappa}(0) = \check{c}_0(\alpha(0))$; subsequently, there exists some $(\mu_1, \mu_2, \mu_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that

$$\check{c}_1(\alpha(1)) = \chi^B(\mu_1, \mu_2, \mu_3)\check{\kappa}(1),$$

as $\check{c}_1(\alpha(1))$ and $\check{\kappa}(1)$ are the lifts of the same point in $\mathfrak{D}^B_\Lambda$. To this end, we define the local index of $\check{c}_0, \check{c}_1$ at z as

$$\mu^{\text{trigr}}(\check{c}_0, \check{c}_1; z) = (\mu_1, \mu_2, \mu_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

It is easy to see that the definition is independent of all the choices made.

The local index has a nice symmetry property similar to [15, p. 227]; see the following lemma.

Lemma 2.6. *If $(c_0, \check{c}_0)$ and $(c_1, \check{c}_1)$ are two trigraded curves such that c_0 and c_1 have minimal intersection, then*

$$\mu^{\text{trigr}}(\check{c}_1, \check{c}_0; z) = \begin{cases} (1,0,0) - \mu^{\text{trigr}}(\check{c}_0, \check{c}_1; z), & \text{if } z \notin \Delta, \\ (0,1,0) - \mu^{\text{trigr}}(\check{c}_0, \check{c}_1; z), & \text{if } z \in \Delta \setminus \{0\}, \\ (1,0,1) - \mu^{\text{trigr}}(\check{c}_0, \check{c}_1; z), & \text{if } z \in \{0\}. \end{cases}$$

Proof. The first two formulae are essentially the same as in [15, p. 227] and can be proven in a similar fashion, which we omit the details. The third formula can be verified using Figure 2.6, as the blue path ℓ in Figure 2.6 contributes $[\mathbb{RP}^1 \times \text{point}] + [\text{point} \times \ell_0]$. ■

Let $\check{c}_0$ and $\check{c}_1$ be two trigraded curves that do not intersect at $\partial\mathbb{D}^B_{n+1}$. Pick a curve $c_1' \simeq c_1$ which intersects minimally with c_0. Then, by Lemma 2.4, c_1' has a unique

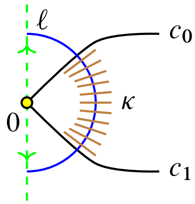


Figure 2.6. Two curves c_0, c_1 intersecting at $\{0\}$.

trigrading $\check{c}_1'$ of c_1' so that $\check{c}_1' \simeq \check{c}_1$. We define the *trigraded intersection number* $I^{\text{trigr}}(\check{c}_0, \check{c}_1) \in \mathbb{Z}[q_1^{\pm 1}, q_2{\pm}1, q_3]/\langle q_3^2 - 1\rangle$ of $\check{c}_0$ and $\check{c}_1$ as follows:

- if $\check{c}_1 \simeq \chi(r_1, r_2, r_3)\check{c}_0$ with $(r_1, r_2, r_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $c_0 \cap c_1 \cap \{0\}$ non-empty, then
$$I^{\text{trigr}}(\check{c}_0, \check{c}_1) = q_1^{r_1} q_2^{r_2} q_3^{r_3}(1 + q_2); \tag{2.4}$$
- otherwise,
$$\begin{aligned} I^{\text{trigr}}(\check{c}_0, \check{c}_1) = (1 + q_3)(1 + q_1^{-1}q_2) &\sum_{z \in (c_0 \cap c_1') \setminus \Delta} q_1^{\mu_1(z)} q_2^{\mu_2(z)} q_3^{\mu_3(z)} \\ + (1 + q_3) &\sum_{z \in (c_0 \cap c_1') \cap \Delta \setminus \{0\}} q_1^{\mu_1(z)} q_2^{\mu_2(z)} q_3^{\mu_3(z)} \\ + (1 + q_1^{-1} q_2 q_3) &\sum_{z \in (c_0 \cap c_1') \cap \{0\}} q_1^{\mu_1(z)} q_2^{\mu_2(z)} q_3^{\mu_3(z)}. \end{aligned}$$

The fact that this definition is independent of the choice of c_1' and is an invariant of the isotopy classes of $(\check{c}_0, \check{c}_1)$ follows similarly as in the case of ordinary geometric intersection numbers.

Remark 2.7. Note that the exceptional case (2.4) in the definition above is motivated by the algebraic correspondence explored in Section 7 (graded HOM space between corresponding irreducible projective modules). We hope to find a more geometric explanation from symplectic geometry in the near future.

Lemma 2.8. *The trigraded intersection number has the following properties:*

(T1) *For any* $f \in \text{Diff}(\mathbb{D}_{n+1}^B, \{0\})$, $I^{\text{trigr}}(\check{f}(\check{c}_0), \check{f}(\check{c}_1)) = I^{\text{trigr}}(\check{c}_0, \check{c}_1)$.

(T2) *For any* $(r_1, r_2, r_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
$$\begin{aligned} I^{\text{trigr}}(\check{c}_0, \chi(r_1, r_2, r_3)\check{c}_1) &= I^{\text{trigr}}(\chi(-r_1, -r_2, r_3)\check{c}_0, \check{c}_1) \\ &= q_1^{r_1} q_2^{r_2} q_3^{r_3} I^{\text{trigr}}(\check{c}_0, \check{c}_1). \end{aligned}$$

(T3) *If c_0, c_1 are not isotopic curves with $\{0\}$ as one of its endpoints, $c_0 \cap c_1 \cap \partial\mathbb{D}^B_{n+1} = \emptyset$, and $I^{\text{trigr}}(\check{c}_0, \check{c}_1) = \sum_{r_1,r_2,r_3} a_{r_1,r_2,r_3} q_1^{r_1} q_2^{r_2} q_3^{r_3}$, then*

$$I^{\text{trigr}}(\check{c}_1, \check{c}_0) = \sum_{r_1,r_2,r_3} a_{r_1,r_2,r_3} q_1^{-r_1} q_2^{1-r_2} q_3^{r_3}.$$

If c_0, c_1 are isotopic curves with $\{0\}$ as one of its endpoints and $c_0 \cap c_1 \cap \partial\mathbb{D}^B_{n+1} = \emptyset$, then $I^{\text{trigr}}(\check{c}_1, \check{c}_0) = I^{\text{trigr}}(\check{c}_0, \check{c}_1)$.

Proof. For (T1) and (T2), these can be proven using a simple topological argument which we omit. For (T3), this is a consequence of Lemma 2.6. We point out that the term $(1 + q_1^{-1} q_2 q_3)$ in the definition of trigraded intersection numbers for two curves that intersect at the point $\{0\}$ is essential for property (T3). ∎

2.6. Admissible curves and normal form in $\mathbb{D}^B_{n+1}$

Following [15, Sections 3b and 3e], we introduce the notion of (trigraded) admissible curves in $\mathbb{D}^B_{n+1}$ and their normal forms. Other than the extra consideration of trigradings, the main difference lies in the normal forms of (trigraded) admissible curves in the region containing the marked point 0; see Figure 2.10.

We fix the set of basic curves $b_1, \dots, b_n$ and choose vertical curves $d_1, \dots, d_n$ as in Figure 2.7, which divide $\mathbb{D}^B_{n+1}$ into regions $D_0, \dots, D_{n+1}$. Note that, unlike in [15], none of our basic curves touches the boundary of the disc $\mathbb{D}^B_{n+1}$.

A curve c is called *admissible* if it is equal to $f(b_j)$ for some $0 \le j \le n$ and some diffeomorphisms $f \in \text{Diff}(\mathbb{D}^B_{n+1}, \{0\})$. Note that the endpoints of c must then lie in $\{0, \dots, n\}$; conversely, all curves which start and end at $\{0, \dots, n\}$ are admissible. Moreover, the two (distinct) orbits $\mathcal{O}([b_1])$ and $\mathcal{O}([b_2])$ under the action of $\mathcal{A}(B_n) \cong \text{MCG}(\mathbb{D}^B_{n+1}, \{0\})$ partition the set of isotopy classes of admissible curves.

If an admissible curve c in its isotopy class has minimal intersection with all the d_j's among its other representatives, then we say that c is in *normal form*. A normal form of c is always achievable by performing an isotopy.

Let c be an admissible curve in normal form. We use the same classification as in [15, Section 3e] to group all connected components of $c \cap D_j$ into finitely many types. For $1 \le j \le n$, the classification is exactly the same as in [15, Section 3e]: there are six types for the case $1 \le j \le n-1$ as depicted in Figure 2.8; whereas for $j = n$, there are two types as shown in Figure 2.9. At $j = 0$, we have two possible types as depicted in Figure 2.10, where they are drawn slightly differently due to the nature of D_0; compare type $2'$ in Figure 2.8 and type $2''$ in Figure 2.10[2]. Note that an admissible

[2]Technically, the difference between type $2'$ and type $2''$ lies in the their trigradings when we consider trigraded curves later on.

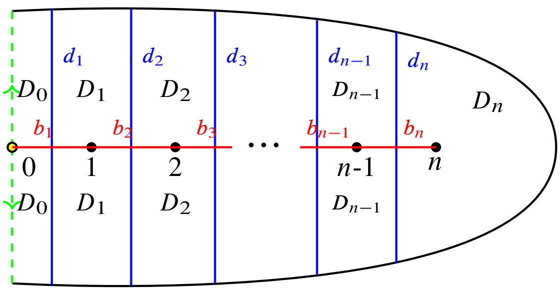


Figure 2.7. The curves b_i and d_i in the aligned configuration with regions D_i.

curve c intersecting all the d_j transversely with each connected component of $c \cap D_j$ belonging to Figures 2.8, 2.9, and 2.10 is already in normal form.

For the rest of this section, c will be an admissible curve in normal form. We call the intersections of c with the curves d_i *crossings* and denote them by

$$cr(c) = c \cap (d_0 \cup d_1 \cup \cdots \cup d_{n-1}).$$

Those intersections $c \cap d_j$ are called *j-crossings of c*. For $0 \leq j \leq n$, the connected components of $c \cap D_j$ are called *segments* of c. If the endpoints of a segment are both crossings, then it is *essential*.

Now, we will study the action of half-twist $t^B_{b_k}$ on normal forms. In general, $t^B_{b_k}(c)$ would not be in normal form even though c is a normal form. Nonetheless, $t^B_{b_k}(c)$ has minimal intersection with all d_j for $j \neq k$. In order to get $t^B_{b_k}(c)$ into a normal form, one just needs to isotope it so that its intersection with d_k is minimal. The same argument used in [15, Proposition 3.17] gives us the following analogous result.

Proposition 2.9. (1) *The normal form of $t^B_{b_k}(c)$ coincides with c outside of $D_{k-1} \cup D_k$. The curve $t^B_{b_k}(c)$ can be brought into normal form by an isotopy inside $D_{k-1} \cup D_k$.*

(2) *Suppose that $t^B_{b_k}(c)$ is in normal form. There is a natural bijection between j-crossings of c and the j-crossings of $t^B_{b_k}(c)$ for $j \neq k$. There is a natural bijection between connected components of intersections of c and $t^B_{b_k}(c)$ inside $D_{k-1} \cup D_k$.*

A connected component of $c \cap (D_{j-1} \cup D_j)$ is called *j-string of c*. Denote by $\mathrm{st}(c, j)$ the set of j-string of c. In addition, we define a *j-string* as a curve in $D_{j-1} \cup D_j$ which is a j-string of c for some admissible curve c in normal form.

Two j-strings are isotopic (equivalently belong to the same isotopy class) if there exists a deformation of one into the other via diffeomorphisms f of $D' = D_{j-1} \cup D_j$ which fix d_{j-1} and d_{j+1} as well as preserve the marked points in D' pointwise;

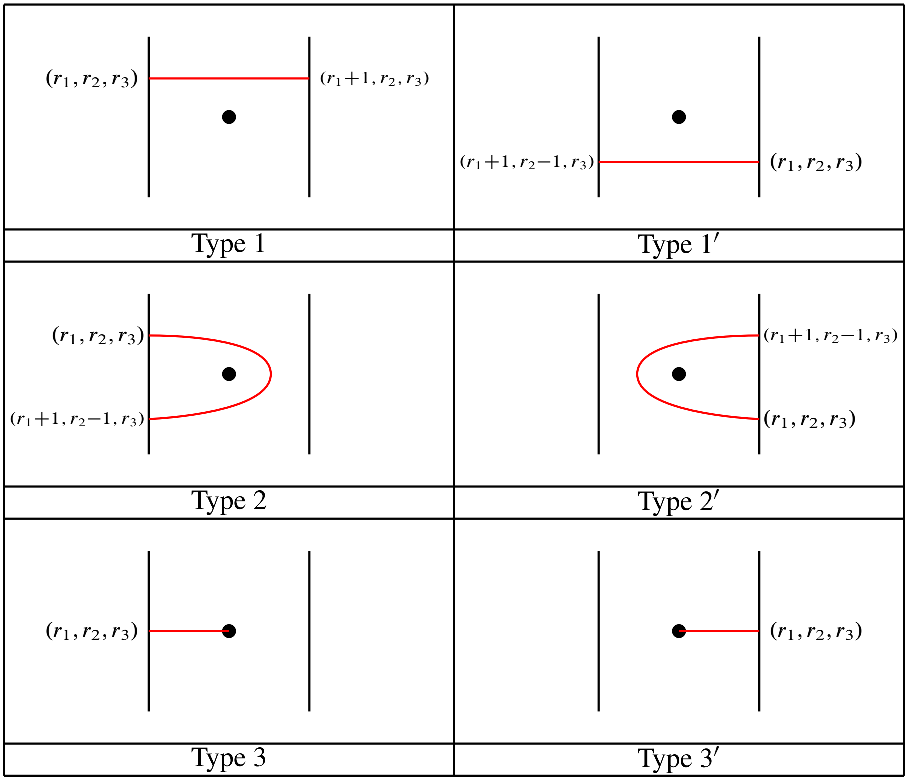


Figure 2.8. The six possible types of connected components $c \cap D_j$, for c in normal form and $1 \leq j < n$.

that is,

$$f(d_{j-1}) = d_{j-1}, \quad f(d_{j+1}) = d_{j+1}, \quad \text{and} \quad f|_{\Delta \cap D'} = \text{id}.$$

For $1 < j < n$, isotopy classes of j-strings can be divided into types as follows: there are five infinite families I_w, II_w, II'_w, III_w, III'_w $(w \in \mathbb{Z})$ and five exceptional types IV, IV′, V, V′, and VI (see Figure 2.11). When $j = n$, there is a similar list, with two infinite families and two exceptional types (see Figure 2.12). The rule for obtaining the $(w+1)$-th from the w-th is by applying $t^B_{b_j}$. For 1-string, there are instead four infinite-family types: II'_w, $\text{II}'_{w+\frac{1}{2}}$, III'_w, $\text{III}'_{w+\frac{1}{2}}$ $(w \in \mathbb{Z})$ and two exceptional types V″ and VI (see Figure 2.13). As for segments of curves, these are drawn slightly different due to the nature of the disc; compare type V′ in Figure 2.11 and type V″ in Figure 2.13. Note that for 1-strings, the rule for obtaining the $(w+1)$-th from the w-th is instead by applying $(t^B_{b_1})^2$.

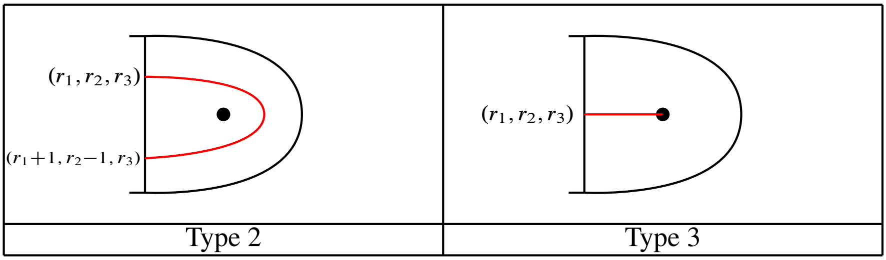


Figure 2.9. The two possible types of connected components $c \cap D_n$.

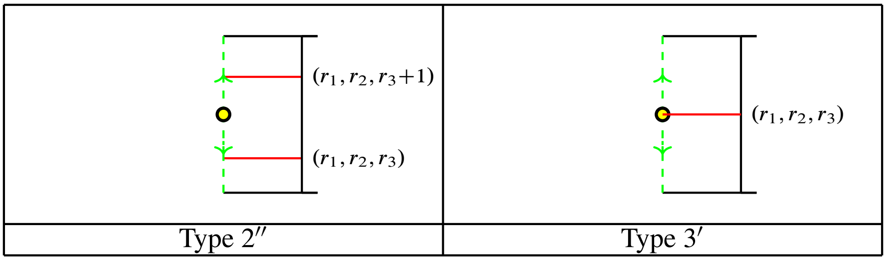


Figure 2.10. The two possible types of connected components $c \cap D_0$.

Based on our definition, j-strings are assumed to be in normal form. As before, we can define *crossings* and *essential segments* of j-string as in the case for admissible curves in normal form and denote the set of crossings of a j-string g by $cr(g)$.

Now, let us adapt the discussion to trigraded curves. Choose trigradings $\check{b}_j$, $\check{d}_j$ of b_j, d_j for $1 \leq j \leq n$ such that

$$I^{\text{trigr}}(\check{d}_j, \check{b}_j) = (1 + q_3)(1 + q_1^{-1} q_2), \quad I^{\text{trigr}}(\check{b}_j, \check{b}_{j+1}) = 1 + q_3.$$

These conditions determine the trigradings uniquely up to an overall shift given by $\chi^B(r_1, r_2, r_3)$.

Suppose that $\check{c}$ is a trigrading of an admissible curve c in normal form. If $a \subset c$ is a connected component of $c \cap D_j$ for some j and $\check{a}$ is $\check{c}|_{a \setminus \Lambda}$, then $\check{a}$ is evidently determined by a together with the local index $\mu^{\text{trigr}}(\check{d}_{j-1}, \check{a}; z)$ or $\mu^{\text{trigr}}(\check{d}_j, \check{a}; z)$ at any point $z \in (d_{j-1} \cup d_j) \cap a$. Moreover, if there is more than one such point, the local indices determine each other.

In Figures 2.8, 2.9, and 2.10, we classify the types of pair $(a, \check{a})$ with the local indices. For instance, consider the type $1(r_1, r_2, r_3)$ with $(k-1)$-crossing $z_0 \in d_{k-1} \cap a$ and k-crossing $z_1 \in d_k \cap a$. We have that the local indices at z_0 and at z_1 are

$$\mu^{\text{trigr}}(\check{d}_{k-1}, \check{a}; z_0) = (r_1, r_2, r_3) \quad \text{and} \quad \mu^{\text{trigr}}(\check{d}_k, \check{a}; z_1) = (r_1 + 1, r_2, r_3),$$

respectively.

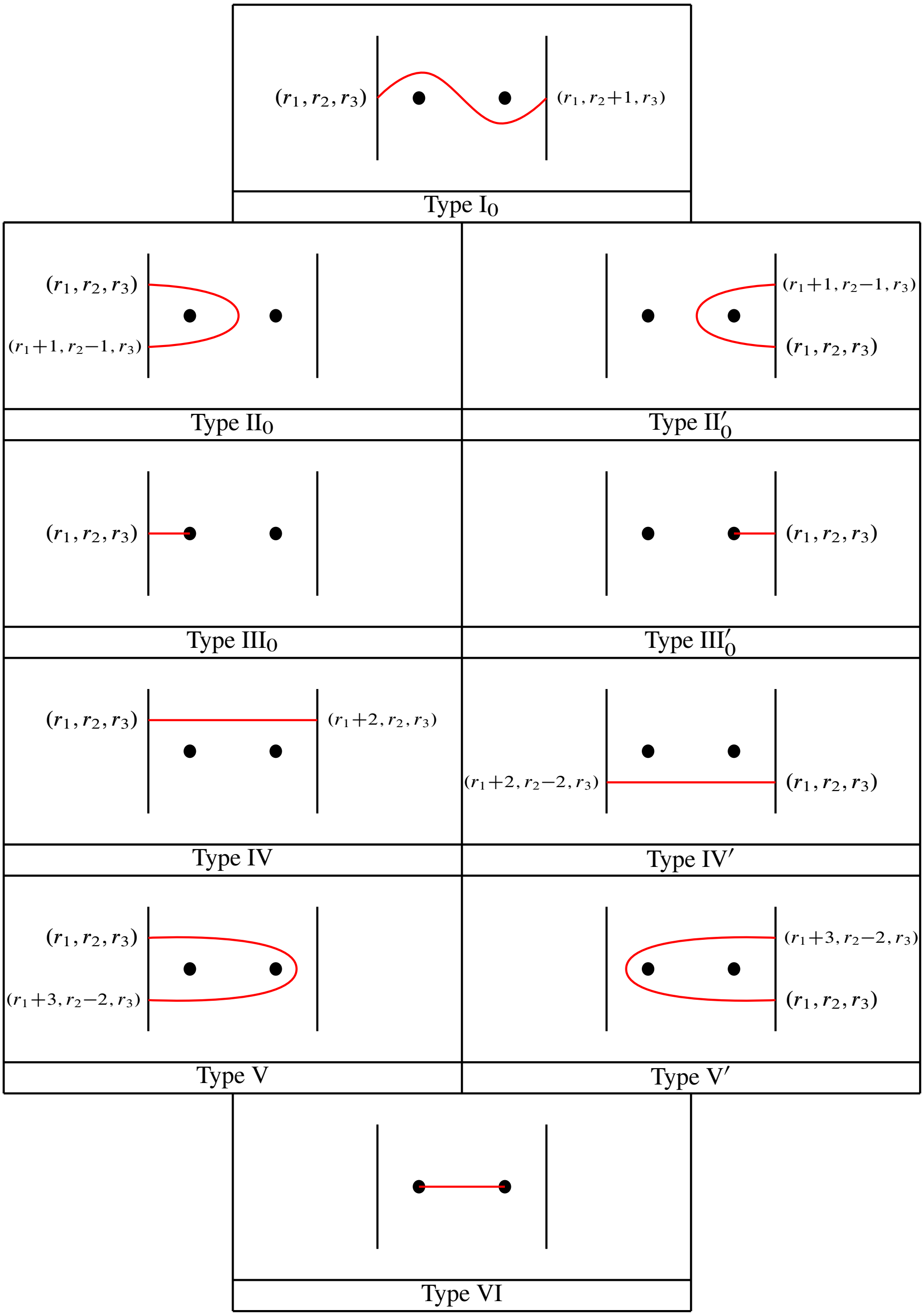


Figure 2.11. The isotopy classes of j-strings for $1 < j < n$.

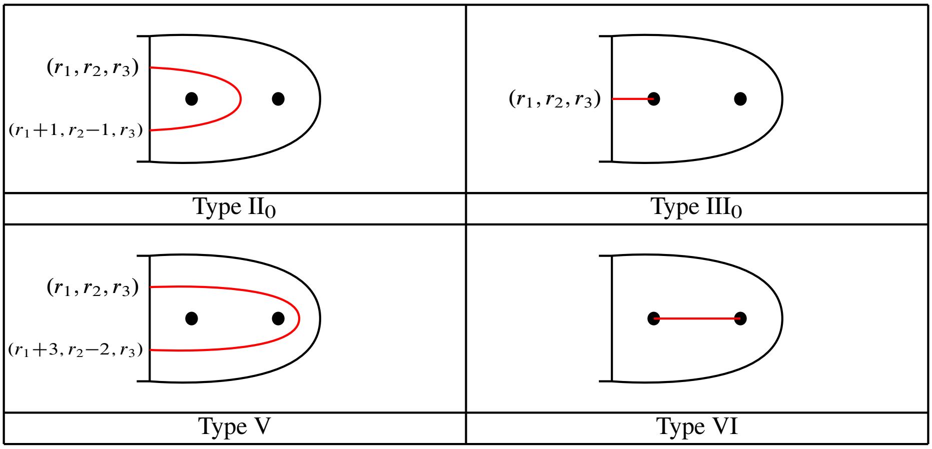


Figure 2.12. The isotopy classes of n-strings.

We recall from Section 2.4 that there is a preferred lift $\check{f} \in \mathrm{Diff}(\widetilde{\mathfrak{D}}^{A}_{\Delta_0})$ of $f \in \mathrm{Diff}(\mathbb{D}^{B}_{n+1}, \{0\})$ which acts as the identity on the boundary. Denote by $\check{t}_{b_j}$ the preferred lift of the twist t_{b_j} along the curve b_j in $\mathbb{D}^{B}_{n+1}$ (see Figure 2.7).

Proposition 2.10. *The diffeomorphisms $\check{t}_{b_j}$ induce a type B_n braid group action on the set of isotopy classes of admissible trigraded curves. Namely, if $\check{c}$ is an admissible trigraded curve, we have the following isotopy relations:*

$$\begin{aligned} \check{t}_{b_1}\check{t}_{b_2}\check{t}_{b_1}\check{t}_{b_2}(\check{c}) &\simeq \check{t}_{b_2}\check{t}_{b_1}\check{t}_{b_2}\check{t}_{b_1}(\check{c}), \\ \check{t}_{b_j}\check{t}_{b_k}(\check{c}) &\simeq \check{t}_{b_k}\check{t}_{b_j}(\check{c}), \qquad \textit{for } |j-k|>1, \\ \check{t}_{b_j}\check{t}_{b_{j+1}}\check{t}_{b_j}(\check{c}) &\simeq \check{t}_{b_{j+1}}\check{t}_{b_j}\check{t}_{b_{j+1}}(\check{c}), \quad \textit{for } j = 2,3,\dots,n. \end{aligned}$$

A crossing of c will be also a *crossing of* $\check{c}$, and we denote the set of crossings of $\check{c}$ by $cr(\check{c})$. Note that as set, $cr(\check{c}) = cr(c)$. However, a crossing of $\check{c}$ comes with a local index in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Moreover, to each crossing y of $\check{c}$ we assign a 4-tuple (y_0, y_1, y_2, y_3), where y_0 denotes the index of the vertical curve which contains the crossing $y \in d_{y_0} \cap c$, and (y_1, y_2, y_3) is the local index (μ_1, μ_2, μ_3) of the crossing y.

We define the *essential segments* of $\check{c}$ as the essential segments of c together with the trigradings which can be obtained from local indices assigned to the ends of the segments.

We also define a *j-string of* $\check{c}$ as a connected component of $\check{c} \cap (D_{j-1} \cup D_j)$ together with the trigrading induced from $\check{c}$. Denote the set of j-string of $\check{c}$ by $\mathrm{st}(\check{c}, j)$.

On top of that, we define a trigraded j-string as a trigraded curve in $D_{j-1} \cup D_j$ that is a connected component of $\check{c} \cap (D_{j-1} \cup D_j)$ for some trigraded curve $\check{c}$.

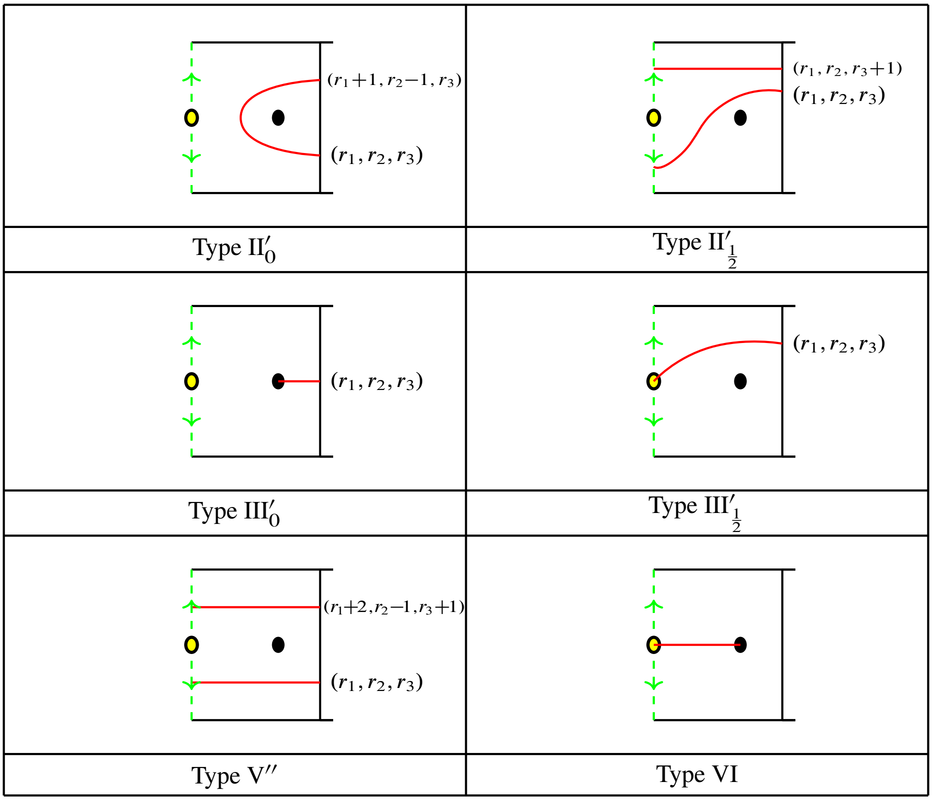


Figure 2.13. The isotopy classes of 1-strings.

In Figures 2.11, 2.12, and 2.13, we depict the isotopy classes of trigraded j-strings. Since j-strings of type VI do not intersect with $d_{j-1} \cup d_{j+1}$, we say that a trigraded j-string $\check{g}$ with the underlying j-string g of type VI has type $\mathrm{VI}(r_1, r_2, r_3)$ if $\check{g} = \chi^B(r_1, r_2, r_3)\check{b}_j$.

The next crucial lemma is the type B analogue of [15, Lemma 3.20], allowing the computation of trigraded intersection numbers between $\check{b}_j$ and any given trigraded curve.

Lemma 2.11. *Let $(c, \check{c})$ be a trigraded curve. Then, $I^{\mathrm{trigr}}(\check{b}_j, \check{c})$ can be computed by adding up contributions from each trigraded j-string of $\check{c}$. For $j > 1$, the contributions are listed in the following table:*

$\mathrm{I}_0(0,0,0)$	$\mathrm{II}_0(0,0,0)$	$\mathrm{II}'_0(0,0,0)$	$\mathrm{III}_0(0,0,0)$
$q_1 + q_2 + q_2q_3 + q_1q_3$	$q_1 + q_2 + q_2q_3 + q_1q_3$	$1 + q_1q_2^{-1} + q_3 + q_1q_2^{-1}q_1q_3$	$q_2 + q_2q_3$

$\mathrm{III}'_0(0,0,0)$	IV	IV′	V	V′	$\mathrm{VI}(0,0,0)$
$1 + q_3$	0	0	0	0	$1 + q_2 + q_3 + q_2q_3$

and the remaining ones can be computed as follows: to determine the contribution of a component of type, say, $\mathrm{I}_u(r_1, r_2, r_3)$*, one takes the contribution of* $\mathrm{I}_0(0,0,0)$ *and multiplies it by* $q_1^{r_1} q_2^{r_2} q_3^{r_3} (q_1^{-1} q_2)^u$*. For* $j = 1$*, the relevant contributions are*

$\mathrm{II}'_0(0,0,0)$	$\mathrm{II}'_{\frac{1}{2}}(0,0,0)$	$\mathrm{III}'_0(0,0,0)$	$\mathrm{III}'_{\frac{1}{2}}(0,0,0)$	V''	$\mathrm{VI}(0,0,0)$
$1 + q_3 + q_1 q_2^{-1} + q_1 q_2^{-1} q_3$	$1 + q_3 + q_1^{-1} q_2 + q_1^{-1} q_2 q_3$	$1 + q_3$	$q_1^{-1} q_2 + q_3$	*0*	$1 + q_2$

and the remaining ones can be computed as follows: to determine the contribution of a component of type, say, $\mathrm{II}'_u(r_1, r_2, r_3)$*, one takes the contribution of* $\mathrm{II}'_0(0,0,0)$ *and multiplies it by* $q_1^{r_1} q_2^{r_2} q_2^{r_2} (q_1^{-1} q_2 q_3)^u$*.*

Proof. Apply Lemma 2.5 as well as $(T2)$ and $(T3)$ of Lemma 2.8. ■

2.7. Bigraded curves and bigraded multicurves in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$

We briefly remind the reader of the definition of a bigraded curve in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$; refer to [15, Section 3d] for a more detailed construction. Consider the projectivisation $\mathfrak{D}^A_\Delta := PT(\mathbb{D}^A_{2n} \setminus \Delta)$ of the tangent bundle of $\mathbb{D}^A_{2n} \setminus \Delta$. The covering $\widetilde{\mathfrak{D}}^A_\Delta$ of $\mathfrak{D}^A_\Delta$ is classified by the cohomology class $C^A \in H^1(\mathfrak{D}^A_\Delta; \mathbb{Z} \times \mathbb{Z})$ defined as follows:

$$C^A([\text{point} \times \lambda_i]) = (-2, 1), \quad \text{for } i = -n, \dots, -1, 1, \dots, n,$$
$$C^A([\mathbb{R}\mathrm{P}^1 \times \text{point}]) = (1, 0).$$

A bigrading of a curve $c \in \mathbb{D}^A_{2n}$ is a lift $\ddot{c}$ of s^A_c to $\widetilde{\mathfrak{D}}^A_\Delta$, where $s^A_c : c \setminus \Delta \to \mathfrak{D}^A_\Delta$ is the canonical section given by $s^A_c(z) = T_z c$. A *bigraded curve* is a pair $(c, \ddot{c})$, where sometimes we abbreviate as $\ddot{c}$.

A *bigraded multicurve* $\ddot{\mathfrak{c}}$ is a union of a finite collection of disjoint bigraded curves. There is an obvious notion of isotopy for bigraded multicurves.

2.8. Lifting of trigraded curves to bigraded multicurves

Our goal is to define a map $\mathfrak{m} : \check{\mathfrak{C}} \to \ddot{\underset{\sim}{\mathfrak{C}}}$ from the set $\check{\mathfrak{C}}$ of isotopy classes of trigraded curves to the set $\ddot{\underset{\sim}{\mathfrak{C}}}$ of isotopy classes of bigraded multicurves. Let c be a curve in $\mathbb{D}^B_{n+1}$ with trigrading $\check{c}$. First, consider the case when $c \cap \{0\} = \emptyset$. Recall the map $q_{br} : \mathbb{D}^A_{2n} \to \mathbb{D}^B_{n+1}$ as defined in Section 2.2.1. Then, $q_{br}^{-1}(c)$ has two connected components in $\mathbb{D}^A_{2n}$; denote them as $\tilde{c}$, $\underset{\sim}{c}$ such that $\tilde{c} \setminus \Delta$ agrees with the curve component of $\mathfrak{p} \circ \check{c}(c \setminus \Lambda)$ and $\underset{\sim}{c} \setminus \Delta$ agrees with the curve component of $\mathfrak{p} \circ \chi^B(0,0,1)\check{c}(c \setminus \Lambda)$. Define $\ddot{\tilde{c}} : \tilde{c} \setminus \Delta \to \widetilde{\mathfrak{D}}^A_\Delta$ as $\ddot{\tilde{c}} := \widetilde{\mathfrak{F}} \circ \check{c} \circ q_{br}|_{\tilde{c} \setminus \Delta}$; similarly, $\ddot{\underset{\sim}{c}} : \underset{\sim}{c} \setminus \Delta \to \widetilde{\mathfrak{D}}^A_\Delta$ is defined as

$$\ddot{\underset{\sim}{c}} := \widetilde{\mathfrak{F}} \circ \chi^B(0,0,1)\check{c} \circ q_{br}|_{\underset{\sim}{c} \setminus \Delta},$$

where $\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{D}}^A_{\Delta_0} \to \widetilde{\mathfrak{D}}^A_{\Delta}$ is the unique map induced by the inclusion $\mathfrak{F} : \mathfrak{D}^A_{\Delta_0} \to \mathfrak{D}^A_{\Delta}$. It is easy to check that these are indeed bigradings of the respective curves. On the other hand, if c contains 0 as one of its endpoints, we define

$$\utilde{c} := \widetilde{c \setminus \{0\}} \amalg \{0\} \amalg \utilde{c \setminus \{0\}},$$

which is just a single connected component. Furthermore, $\ddot{\utilde{c}}$ is defined to be the unique continuous extension of $\ddot{\widetilde{c \setminus \{0\}}} \amalg \ddot{\utilde{c \setminus \{0\}}}$, which is again an easy verification that it is a bigrading of $\utilde{c}$.

In total, we define the map $\mathfrak{m} : \check{\mathfrak{C}} \to \ddot{\utilde{\mathfrak{C}}}$ as follows: for a trigraded curve $(c, \check{c})$ in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$,

$$\mathfrak{m}((c, \check{c})) := \begin{cases} (\utilde{c}, \ddot{\utilde{c}}), & \text{if } c \text{ has } \{0\} \text{ as one of its endpoints,} \\ (\tilde{c}, \ddot{\tilde{c}}) \amalg (\utilde{c}, \ddot{\utilde{c}}), & \text{otherwise.} \end{cases}$$

Due to the isotopy lifting property of the space, $\mathfrak{m}$ is well defined on the isotopy classes of trigraded curves.

Recall the natural induced action of $\mathcal{A}(B_n) \cong \mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\})$ on $\check{\mathfrak{C}}$ given in the paragraph before Lemma 2.4. Since

$$\mathcal{A}(A_{2n-1}) \cong \mathrm{MCG}(\mathbb{D}^A_{2n})$$

acts on $\ddot{\utilde{\mathfrak{C}}}$ [15, Proposition 3.19], there exists an induced action of $\mathcal{A}(B_n)$ on $\ddot{\utilde{\mathfrak{C}}}$ through the injection Ψ as given in Proposition 2.1.

Proposition 2.12. *The map $\mathfrak{m} : \check{\mathfrak{C}} \to \ddot{\utilde{\mathfrak{C}}}$ from isotopy classes of trigraded curves in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ to isotopy classes of bigraded multicurves in $\widetilde{\mathfrak{D}}^A_{\Delta}$ is $\mathcal{A}(B_n)$-equivariant*

$$\begin{array}{ccc} \mathcal{A}(B_n) \circlearrowright & & \circlearrowleft \mathcal{A}(A_{2n-1}) \overset{\Psi}{\hookleftarrow} \mathcal{A}(B_n) \\ \textit{Isotopy classes } \check{\mathfrak{C}} \textit{ of} & \xrightarrow{\mathfrak{m}} & \textit{Isotopy classes } \ddot{\utilde{\mathfrak{C}}} \textit{ of} \\ \textit{trigraded curves in } \mathbb{D}^B_{n+1} & & \textit{bigraded multicurves in } \mathbb{D}^A_{2n}. \end{array}$$

Proof. This follows directly from the definition of $\mathfrak{m}$ and the actions. ∎

2.9. Bigraded intersection number and bigraded admissible multicurves in $\widetilde{\mathfrak{D}}^A_{\Delta}$

The local index for bigraded curves in $\widetilde{\mathfrak{D}}^A_{\Delta}$ is defined in the same spirit as the local index for trigraded curves in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$. For a more detailed explanation, we refer the reader to [15, Section 3d].

We recall from [15, Section 3d] that the bigraded intersection number of two bigraded curves $\ddot{c}_0, \ddot{c}_1$ that do not intersect at $\partial\mathbb{D}^A_{2n}$ is defined by

$$I^{\text{bigr}}(\ddot{c}_0, \ddot{c}_1) = (1 + q_1^{-1}q_2)\Big(\sum_{z\in(c_0\cap c_1')\setminus\Delta} q_1^{\mu_1(z)}q_2^{\mu_2(z)}\Big) + \Big(\sum_{z\in(c_0\cap c_1')\cap\Delta} q_1^{\mu_1(z)}q_2^{\mu_2(z)}\Big).$$

We extend the definition of bigraded intersection number of bigraded curves to bigraded multicurves by adding up the bigraded intersection numbers of each pair of bigraded curves.

To talk about bigraded admissible curves in $\mathbb{D}^A_{2n}$ and their normal forms, we need to fix a set of basic bigraded curves. To do so, first recall the set of trigraded basic curves $(b_j, \check{b}_j)$ and the set of trigraded vertical curves $(d_j, \check{d}_j)$ as defined in Section 2.6. Denote this set of basic trigraded curves as $\check{\mathfrak{B}}$. Consider, for each $(c, \check{c}) \in \check{\mathfrak{B}}$, its lift to bigraded multicurves in $\mathbb{D}^A_{2n}$:

$$\mathfrak{m}(c,\check{c}) = \begin{cases} (\underset{\sim}{\tilde{c}}, \underset{\sim}{\ddot{\tilde{c}}}), & \text{if } (c,\check{c}) = (b_1, \check{b}_1), \\ (\tilde{c}, \ddot{\tilde{c}}) \amalg (\underset{\sim}{c}, \underset{\sim}{\ddot{c}}), & \text{otherwise,} \end{cases}$$

where $\tilde{c}$ denotes the curve whose points have positive real parts, so points in $\underset{\sim}{c}$ have negative real parts. We will fix the set of bigraded basic curves $(\varrho, \ddot{\varrho}_j)$ and bigraded vertical curves $(\theta_j, \ddot{\theta}_j)$ as follows:

- choose $(\theta_n, \ddot{\theta}_n) := (\tilde{d}_1, \ddot{\tilde{d}}_1)$;
- choose $(\theta_{n+j-1}, \ddot{\theta}_{n+j-1}) := (\tilde{d}_j, \ddot{\tilde{d}}_j)$ and
$$(\theta_{n-j+1}, \ddot{\theta}_{n-j+1}) := (\underset{\sim}{d_j}, \underset{\sim}{\ddot{d}_j}), \quad \text{for } 2 \le j \le n;$$
- choose $(\varrho_{n+j-1}, \ddot{\varrho}_{n+j-1}) := (\tilde{b}_j, \ddot{\tilde{b}}_j)$ and
$$(\varrho_{n-j+1}, \ddot{\varrho}_{n-j+1}) := (\underset{\sim}{b_j}, \underset{\sim}{\ddot{b}_j}), \quad \text{for } 2 \le j \le n;$$
- choose $(\varrho_n, \ddot{\varrho}_n) := (\underset{\sim}{\tilde{b}_1}, \underset{\sim}{\ddot{\tilde{b}}_1})$.

Figure 2.14 illustrates the (underlying) basic curves ϱ_j and vertical curves θ_j chosen.

Remark 2.13. We notify the reader of two slight differences here in comparison to [15]: our underlying set of basic curves $\{\varrho_j\}$ chosen does not include the one curve connected to the boundary of the disc; moreover, the bigradings of the basic curves and vertical curves are slightly different. Compare our equations (2.6) and (2.7) to the defining equations in [15, p. 232].

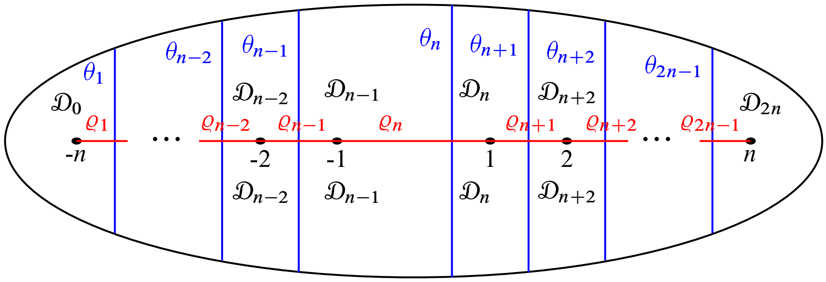


Figure 2.14. The basic curves θ_i and ϱ_i in the aligned configuration with regions $\mathcal{D}_i$ for $\mathbb{D}^A_{2n}$.

Lemma 2.14. *The bigradings we choose for the set of basic curves and vertical curves in $\mathbb{D}^A_{2n}$ satisfy the following properties:*

$$I^{\text{bigr}}(\ddot{\theta}_j, \ddot{\varrho}_j) = 1 + q_1^{-1}q_2, \quad \textit{for } 1 \le j \le 2n-1, \tag{2.5}$$

$$I^{\text{bigr}}(\ddot{\varrho}_j, \ddot{\varrho}_{j+1}) = 1, \quad \textit{for } n \le j \le 2n-2, \tag{2.6}$$

$$I^{\text{bigr}}(\ddot{\varrho}_j, \ddot{\varrho}_{j-1}) = 1, \quad \textit{for } 2 \le j \le n. \tag{2.7}$$

Proof. This follows immediately from the construction. ■

Similar to the curves in $\mathbb{D}^B_{n+1}$, a curve c in $\mathbb{D}^A_{2n}$ is called *admissible* if $c = f(\varrho_j)$ for some $-n \le j \le n$ and some $f \in \text{Diff}(\mathbb{D}^A_{2n}, \partial\mathbb{D}^A_{2n})$. Note that, unlike in [15], admissible curves in $\mathbb{D}^A_{2n}$ will not touch the boundary of the disc (none of our basic curves ϱ_j does). An admissible curve c in its isotopy class that has minimal intersection with all the θ_j's among its other representatives is said to be in *normal form*. We define crossings, essential segments, j-strings, and bigraded j-strings in a similar fashion to the trigraded case (see Section 2.6). In particular, given a j-crossing x of a bigraded curve $\ddot{c}$, we fix $x_0 := j$ and (x_1, x_2) is the local index (μ_1, μ_2) of the crossing x. All these notions can be extended to those for multicurves and bigraded multicurves naturally.

Suppose that c_0 and c_1 are two admissible curves in $\mathbb{D}^B_{n+1}$ intersecting at $z \in \mathbb{D}^B_{n+1}$. If $z = 0$, we require that $c_0 \not\simeq c_1$. Their preimages $q_{br}^{-1}(c_0)$ and $q_{br}^{-1}(c_1)$ in $\mathbb{D}^A_{2n}$ under the map $q_{br} : \mathbb{D}^A_{2n} \to \mathbb{D}^B_{n+1}$ would then intersect minimally. However, if $c_0 \cap c_1 \cap \{0\} \ne \emptyset$ and $c_0 \simeq c_1$, they will not intersect minimally, as illustrated in Figure 2.15.

As such, we obtain the following proposition.

Proposition 2.15. *Let $\check{c}_0$ and $\check{c}_1$ be two trigraded curves intersecting at $z \in \mathbb{D}^B_{n+1}$, with local index $\mu^{\text{trigr}}(\check{c}_0, \check{c}_1, z) = (r_1, r_2, r_3)$. If $c_0 \cap c_1 \cap \{0\} \ne \emptyset$, we require that $c_0 \not\simeq c_1$. If $z \ne 0$, further suppose that*

$$\mathfrak{m}(c_0, \check{c}_0) = (\tilde{c}_0, \check{\tilde{c}}_0) \amalg (\underset{\sim}{c}_0, \check{\underset{\sim}{c}}_0) \quad \textit{and} \quad \mathfrak{m}(c_1, \check{c}_1) = (\tilde{c}_1, \check{\tilde{c}}_1) \amalg (\underset{\sim}{c}_1, \check{\underset{\sim}{c}}_1)$$

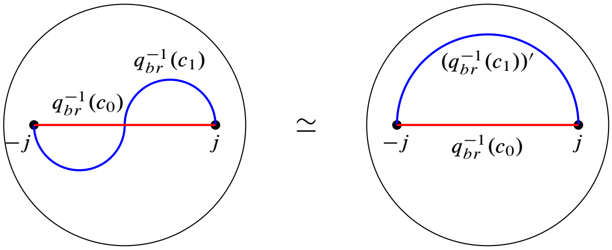


Figure 2.15. The preimages $q_{br}^{-1}(c_0), q_{br}^{-1}(c_1)$ in $\mathbb{D}_{2n}^A$.

such that $\tilde{c}_0 \cap \tilde{c}_1 = \tilde{z}$ *and* $\underset{\sim}{c_0} \cap \underset{\sim}{c_1} = \underset{\sim}{z}$. *Then,*

$$\begin{cases} \mu^{\mathrm{bigr}}(\ddot{\tilde{c}}_0, \ddot{\tilde{c}}_1, \tilde{z}) = (r_1, r_2) = \mu^{\mathrm{bigr}}(\underset{\sim}{\ddot{c}_0}, \underset{\sim}{\ddot{c}_1}, \underset{\sim}{z}), & \textit{for } z \neq 0, \\ \mu^{\mathrm{bigr}}(\underset{\sim}{\ddot{\tilde{c}}_0}, \underset{\sim}{\ddot{\tilde{c}}_1}, 0) = (r_1, r_2), & \textit{for } z = 0. \end{cases}$$

Furthermore, this proposition allows us to relate trigraded intersection numbers and bigraded intersection number in the following way.

Corollary 2.16. *For any trigraded admissible curves* $(c_0, \check{c}_0)$ *and* $(c_1, \check{c}_1)$*,*

$$I^{\mathrm{trigr}}(\check{c}_0, \check{c}_1)|_{q_3=1} = I^{\mathrm{bigr}}(\mathfrak{m}(\check{c}_0), \mathfrak{m}(\check{c}_1)).$$

In particular,

$$\frac{1}{2} I^{\mathrm{trigr}}(\check{c}_0, \check{c}_1)|_{q_1=q_2=q_3=1} = I(\mathfrak{m}(c_0), \mathfrak{m}(c_1));$$

i.e., $\frac{1}{2} I^{\mathrm{trigr}}(\check{c}_0, \check{c}_1)|_{q_1=q_2=q_3=1}$ *counts the geometric intersection number of the lifts of* c_0 *and* c_1 *in* $\mathbb{D}_{2n}^A$ *under the map* $\mathfrak{m}$.

Proof. The case when $c_0 \not\simeq c_1$ in $\mathbb{D}_{n+1}^B$ or when at least one of c_0 and c_1 does not have its endpoint at $\{0\}$ follows directly from Proposition 2.15. The other case follows from a direct computation. The last statement relating trigraded intersection number and geometric intersection number follows from the property of bigraded intersection number (see [15, p. 227, property (B1)]). ∎

We will abuse notation and also allow $\mathfrak{m}$ to lift crossings of a trigraded admissible curve $(c, \check{c})$ to crossings of the bigraded admissible multicurves

$$\mathfrak{m}((c, \check{c})) = (\tilde{c}, \ddot{\tilde{c}}) \amalg (\underset{\sim}{c}, \underset{\sim}{\ddot{c}}).$$

Suppose that z is a j-crossing of c for $j > 1$. Then,

$$q_{br}^{-1}(z) = \{\tilde{z}, \underset{\sim}{z}\},$$

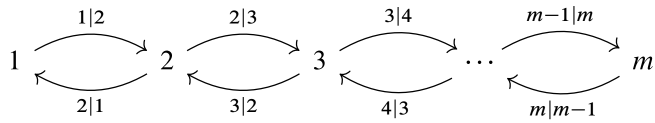


Figure 3.1. The quiver Γ_m.

where $\tilde{z} \in \tilde{c}$ and $\underset{\sim}{z} \in \underset{\sim}{c}$. If z is a 1-crossing of c, then we also have $q_{br}^{-1}(z) = \{\tilde{z}, \underset{\sim}{z}\}$; in this case, we will pick $\tilde{\underset{\sim}{z}}$ to be the unique element in $\{\tilde{z}, \underset{\sim}{z}\} \cap \theta_n$. So, if z is a j-crossing of c with

$$\mu^{\text{trigr}}(\check{d}_k, \check{c}, z) = (r_1, r_2, r_3),$$

we define

$$\mathfrak{m}(z) = \{\tilde{z}, \underset{\sim}{z}\},$$

where both $\tilde{z}$ and $\underset{\sim}{z}$ are with local index (r_1, r_2) for $j > 1$; otherwise, $\mathfrak{m}(z) = \{\tilde{\underset{\sim}{z}}\}$ for $j = 1$, with local index (r_1, r_2) by Proposition 2.15. Let $\check{h}$ be a connected subset of $\check{c}$ within some connected region given by unions of D_j's, equipped with the trigrading given by local indices of crossings of $\check{h}$ induced from crossings of $\check{c}$. We define $\mathfrak{m}(\check{h})$ to consist of $q_{br}^{-1}(\check{h})$, with bigradings given by local indices of crossings of $q_{br}^{-1}(\check{h})$ induced from crossings of $\mathfrak{m}(\check{c})$.

3. Type A_{2n-1} and type B_n zigzag algebras

In this section, we recall the construction of the type A_m zigzag algebra $\mathcal{A}_m$ as given in [15] (with slight change in gradings) and recall the $\mathscr{A}(A_m)$ action on the bounded homotopy category $\text{Kom}^b(\mathcal{A}_m\text{-p}_r\text{g}_r\text{mod})$ of complexes of projective graded modules over $\mathcal{A}_m$. We then construct a type B_n zigzag algebra $\mathcal{B}_n$, following a similar construction, and show that $\mathscr{A}(B_n)$ acts on $\text{Kom}^b(\mathcal{B}_n\text{-p}_r\text{g}_r\text{mod})$. We will assume that the reader is familiar with projective modules over finite-dimensional (graded) algebras and refer the unfamiliar readers to [22, Chapter 1]. Note that, throughout this whole paper, all complexes are bounded.

3.1. Type A_m zigzag algebra $\mathcal{A}_m$

Consider the quiver Γ_m in Figure 3.1.

We can take its path algebra $\mathbb{C}\Gamma_m$ over $\mathbb{C}$; $\mathbb{C}\Gamma_m$ is the $\mathbb{C}$-vector space spanned by the set of all paths in Γ_m, with multiplication given by concatenation of paths (the multiplication is zero if the endpoints do not agree). We denote the constant path at each vertex j by e_j.

In this paper, we will only consider m odd, and the grading we put on $\mathbb{C}\Gamma_m$ will be slightly different to [15]; we set

- the degree of $(j|j-1)$ is 1 for $j > \frac{m+1}{2}$ and 0 for $j \leq \frac{m+1}{2}$,
- the degree of $(j|j+1)$ is 1 for $j < \frac{m+1}{2}$ and 0 for $j \geq \frac{m+1}{2}$,
- the degree of $e_j = 0$ for all j,

where the grading is extended to all paths via multiplication. In this way, this path algebra is $\mathbb{Z}$-graded with the grading shift denoted by $\{-\}$ and unital with a family of pairwise orthogonal primitive central idempotent e_j summing up to the unit element.

Let $\mathcal{A}_m$ be the quotient of the path algebra of the quiver Γ_m by the relations

$$(j|j+1|j) = (j|j-1|j),$$
$$(j-1|j|j+1) = 0 = (j+1|j|j-1)$$

for all $2 \leq j \leq m-1$. It is easy to see that these relations are homogeneous with respect to the above grading so that $\mathcal{A}_m$ is a $\mathbb{Z}$-graded algebra. As a $\mathbb{C}$-vector space, it has dimension $4m-2$ with the following basis:

$$\{e_1, \ldots, e_m, (1|2), \ldots, (m-1|m), (2|1), \ldots, (m|m-1), (1|2|1), \ldots, (m|m-1|m)\}.$$

The indecomposable projective, $\mathbb{Z}$-graded $\mathcal{A}_m$-modules are denoted by $P_j^A := \mathcal{A}_m e_j$, and we denote the (additive) category of projective, graded $\mathcal{A}_m$-modules by $\mathcal{A}_m$-$\mathrm{p_r g_r mod}$. We recall the following results from [15].

Theorem 3.1. *For each j, consider the following complex of graded $(\mathcal{A}_m, \mathcal{A}_m)$-bimodule*

$$\mathcal{R}_j := 0 \to P_j^A \otimes_{\mathbb{C}} {}_j P^A \xrightarrow{\beta_j} \mathcal{A}_m \to 0,$$

with $\mathcal{A}_m$ in cohomological degree 0. Each complex $\mathcal{R}_j$ is invertible in the homotopy category $\mathrm{Kom}^b((\mathcal{A}_m, \mathcal{A}_m)$*-bimod) of graded $(\mathcal{A}_m, \mathcal{A}_m)$-bimodules and satisfies the following relations:*

$$\mathcal{R}_j \otimes \mathcal{R}_k \cong \mathcal{R}_k \otimes \mathcal{R}_j, \quad \text{for } |j-k| > 1,$$
$$\mathcal{R}_j \otimes \mathcal{R}_{j+1} \otimes \mathcal{R}_j \cong \mathcal{R}_{j+1} \otimes \mathcal{R}_j \otimes \mathcal{R}_{j+1}.$$

Proof. See [15, Proposition 2.4 and Theorem 2.5]. ■

Proposition 3.2. *There is a (weak) $\mathcal{A}(A_m)$-action on* $\mathrm{Kom}^b(\mathcal{A}_m\text{-}\mathrm{p_r g_r}mod)$*, where each standard generator σ_j^A of $\mathcal{A}(A_m)$ acts on a complex $M \in$* $\mathrm{Kom}^b(\mathcal{A}_m\text{-}\mathrm{p_r g_r}mod)$ *via $\mathcal{R}_j$:*

$$\sigma_j^A(M) := \mathcal{R}_j \otimes_{\mathcal{A}_m} M.$$

Proof. See [15, Proposition 2.7]. ■

We will abuse notation and use σ_j^A in place of $\mathcal{R}_j$ whenever the context is clear.

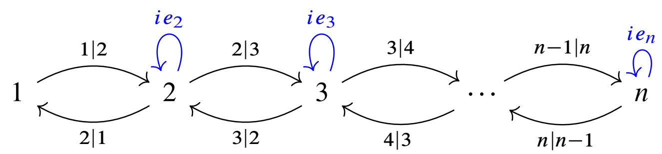


Figure 3.2. The quiver Ω_n.

3.2. Type B_n zigzag algebra $\mathcal{B}_n$

Consider the quiver Ω_n in Figure 3.2.

Take its path algebra $\mathbb{R}\Omega_n$ over $\mathbb{R}$ and consider the two gradings on $\mathbb{R}\Omega_n$ given as follows:

(i) the first grading is defined following the convention in [15], where we set

- the degree of $(j+1|j)$ to be 1 and $(j|j+1)$ to be 0 for all $1 \leq j \leq n-1$, and
- the degree of e_j, ie_j (blue paths in Figure 3.2) to be 0 for all $1 \leq j \leq n$,

extending to all paths.

(ii) The second grading is a $\mathbb{Z}/2\mathbb{Z}$-grading defined by setting

- the degree of ie_j as 1 for all $1 \leq j \leq n$, and
- the degree of all other paths in Figure 3.2 and the constant paths as zero,

extending again to all paths.

We denote a shift in the $\mathbb{Z}$-grading by $\{-\}$ and a shift in the $\mathbb{Z}/2\mathbb{Z}$-grading by $\langle - \rangle$.

We are now ready to define the zigzag algebra of type B_n.

Definition 3.3. The zigzag path algebra of B_n, denoted by $\mathcal{B}_n$, is the quotient algebra of the path algebra $\mathbb{R}\Omega_n$ modulo the usual zigzag relations given by

$$(j|j-1)(j-1|j) = (j|j+1)(j+1|j), \tag{3.1}$$
$$(j-1|j)(j|j+1) = 0 = (j+1|j)(j|j-1) \tag{3.2}$$

for $2 \leq j \leq n-1$, in addition to the relations

$$(ie_j)(ie_j) = -e_j, \quad \text{for } j \geq 2, \tag{3.3}$$
$$(ie_{j-1})(j-1|j) = (j-1|j)(ie_j), \quad \text{for } j \geq 3, \tag{3.4}$$
$$(ie_j)(j|j-1) = (j|j-1)(ie_{j-1}), \quad \text{for } j \geq 3, \tag{3.5}$$
$$(1|2)(ie_2)(2|1) = 0; \tag{3.6}$$
$$(ie_2)(2|1|2) = (2|1|2)(ie_2). \tag{3.7}$$

We will also denote the (non-trivial) loop on vertex j by

$$X_j := (j|j \pm 1)(j \pm 1|j).$$

The relations above are homogeneous with respect to both the $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ gradings, so $\mathcal{B}_n$ is a bigraded algebra.

Proposition 3.4. *As an $\mathbb{R}$-vector space, $\mathcal{B}_n$ has dimension $8n - 6$.*

Proof. Using relations (3.1) to (3.5), the vector subspace (over $\mathbb{R}$) spanned by paths that do not pass through the vertex 1 is isomorphic to $\mathcal{A}_{n-1}$ viewed as an $\mathbb{R}$-vector space, which has $\mathbb{R}$-dimension

$$2(4(n-1) - 2) = 8n - 12.$$

Combined with the remaining relations, one can check that the remaining paths that pass through the vertex 1 (modulo relations) are exactly e_1, $(1|2)$, $(2|1)$, $(1|2)(ie_2)$, $(ie_2)(2|1)$, and X_1. Hence,

$$\dim_{\mathbb{R}}(\mathcal{B}_n) = 8n - 12 + 6 = 8n - 6;$$

in particular, the set

$$\{e_1, \dots, e_n, ie_2, \dots, ie_n, (1|2), \dots, (n-1|n), (2|1), \dots, (n|n-1), (ie_2)(2|1), (1|2)(ie_2),\ (ie_2)(2|3), \dots, (ie_{n-1})(n-1|n), (3|2)(ie_2), \dots, (n|n-1)(ie_{n-1}), X_1, \dots, X_n, (ie_2)X_2, \dots, (ie_n)X_n\}$$

forms an $\mathbb{R}$-basis of $\mathcal{B}_n$. ■

The indecomposable (left) projective, bigraded $\mathcal{B}_n$-modules are given by $P_j^B := \mathcal{B}_n e_j$, and we denote the (additive) category of projective, bigraded $\mathcal{B}_n$-modules by $\mathcal{B}_n$-$\text{p}_r\text{g}_r\text{mod}$. For $j = 1$, P_j^B is naturally a $(\mathcal{B}_n, \mathbb{R})$-bimodule; there is a natural left $\mathcal{B}_n$-action given by multiplication of the algebra and the right $\mathbb{R}$-action induced by the left (commutative) $\mathbb{R}$-action. But for $j \geq 2$, we will endow P_j^B with a right $\mathbb{C}$-action.

To this end, let us view $\mathbb{C}$ as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra over $\mathbb{R}$ by endowing the reals with degree 0 and the complex imaginary i with degree 1 over $\mathbb{Z}/2\mathbb{Z}$ and extend linearly. Note also that (3.3) in Definition 3.3 is analogous to the relation satisfied by the complex imaginary number i. We define a right $\mathbb{C}$-action on P_j^B by $p * (a + ib) = ap + bp(ie_j)$ for $p \in P_j^B, a + ib \in \mathbb{C}$. It follows from the definition that this right action restricted to $\mathbb{R}$ agrees with the natural right (and left) $\mathbb{R}$-action. This makes P_j^B into a bigraded $(\mathcal{B}_n, \mathbb{C})$-bimodule for $j \geq 2$. Dually, we will define ${}_jP^B := e_j\mathcal{B}_n$, where we similarly consider it as a bigraded $(\mathbb{R}, \mathcal{B}_n)$-bimodule for $j = 1$ and as a bigraded $(\mathbb{C}, \mathcal{B}_n)$-bimodule for $j \geq 2$.

Proposition 3.5. *Denote* ${}_jP_k^B := {}_jP^B \otimes_{\mathcal{B}_n} P_k^B$. *We have that*

$$
{}_jP_k^B \cong \begin{cases}
{}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} & \text{as bigraded } (\mathbb{C},\mathbb{C})\text{-bimodules} \\
 & \text{for } j,k \in \{2,\dots,n\} \text{ and } k-j=1, \\
{}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}\{1\} & \text{as bigraded } (\mathbb{C},\mathbb{C})\text{-bimodules} \\
 & \text{for } j,k \in \{2,\dots,n\} \text{ and } j-k=1, \\
{}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}\{1\} & \text{as bigraded } (\mathbb{C},\mathbb{C})\text{-bimodules for } j=k=2,3,\dots,n, \\
{}_{\mathbb{R}}\mathbb{C}_{\mathbb{C}} & \text{as bigraded } (\mathbb{R},\mathbb{C})\text{-bimodules for } j=1 \text{ and } k=2, \\
{}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}}\{1\} & \text{as bigraded } (\mathbb{C},\mathbb{R})\text{-bimodules for } j=2 \text{ and } k=1, \\
{}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}} \oplus {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}\{1\} & \text{as bigraded } (\mathbb{R},\mathbb{R})\text{-bimodules for } j=k=1.
\end{cases}
$$

Proof. The case where $j,k \in \{2,\dots,n\}$ follows as in type A. By identifying ${}_jP_k$ as the $\mathbb{R}$-vector subspace of $\mathcal{B}_n$ spanned by paths starting at vertex j and ending at vertex k, we see that ${}_1P_2$ has basis $\{(1|2),(1|2)(ie_2)\}$, ${}_2P_1$ has basis $\{(2|1),(ie_2)(2|1)\}$, and ${}_1P_1$ has basis $\{e_1, X_1\}$. The fact that the bimodule and bigrading structures agree follows from the definition and is left as a simple exercise to the reader. ∎

Remark 3.6. Note that all the bigraded bimodules in Proposition 3.5 can be restricted to bigraded $(\mathbb{R},\mathbb{R})$-bimodules by identifying ${}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}} \cong \mathbb{R} \oplus \mathbb{R}\langle 1\rangle$. For example, ${}_1P_2^B$ restricted to an $(\mathbb{R},\mathbb{R})$-bimodule is generated by $(1|2)$ and $(1|2)ie_2$, and it is isomorphic to $\mathbb{R} \oplus \mathbb{R}\langle 1\rangle \cong {}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}}$.

Lemma 3.7. *Denote* $\mathbb{K}_1 := \mathbb{R}$ *and* $\mathbb{K}_j := \mathbb{C}$ *when* $j \geq 2$. *The maps*

$$\beta_j : P_j^B \otimes_{\mathbb{K}_j} {}_jP^B \to \mathcal{B}_n \quad \text{and} \quad \gamma_j : \mathcal{B}_n \to P_j^B \otimes_{\mathbb{K}_j} {}_jP^B\{-1\}$$

defined by

$$\beta_j(x \otimes y) := xy,$$

$$
\gamma_j(1) := \begin{cases}
X_j \otimes e_j + e_j \otimes X_j + (j+1|j) \otimes (j|j+1) & \\
\quad +(-ie_{j+1})(j+1|j) \otimes (j|j+1)(ie_{j+1}), & \text{for } j=1, \\
X_j \otimes e_j + e_j \otimes X_j + (j-1|j) \otimes (j|j-1) & \\
\quad +(j+1|j) \otimes (j|j+1), & \text{for } 1<j<n, \\
X_j \otimes e_j + e_j \otimes X_j + (j-1|j) \otimes (j|j-1), & \text{for } j=n
\end{cases}
$$

are $(\mathcal{B}_n,\mathcal{B}_n)$*-bimodule maps.*

Proof. It is obvious from the definition that the β_j are maps of $(\mathcal{B}_n,\mathcal{B}_n)$-bimodules for all j. The fact that γ_j is a $(\mathcal{B}_n,\mathcal{B}_n)$-bimodule map also follows from a tedious check on each basis element, which we will omit and leave it to the reader. ∎

Definition 3.8. Define the following complexes of bigraded $(\mathcal{B}_n, \mathcal{B}_n)$-bimodules:

$$R_j := (0 \to P_j^B \otimes_{\mathbb{K}_j} {}_jP^B \xrightarrow{\beta_j} \mathcal{B}_n \to 0),$$

and

$$R_j' := (0 \to \mathcal{B}_n \xrightarrow{\gamma_j} P_j^B \otimes_{\mathbb{K}_j} {}_jP^B\{-1\} \to 0)$$

for each $1 \leq j \leq n$, with both $\mathcal{B}_n$ in cohomological degree 0, $\mathbb{K}_1 = \mathbb{R}$ and $\mathbb{K}_j = \mathbb{C}$ for $j \geq 2$.

Proposition 3.9. *In the homotopy category* $\mathrm{Kom}^b((\mathcal{B}_n, \mathcal{B}_n)$*-bimod) of complexes of projective graded* $(\mathcal{B}_n, \mathcal{B}_n)$*-bimodules, we have the following isomorphisms:*

$$R_j \otimes R_j' \cong \mathcal{B}_n \cong R_j' \otimes R_j,$$

$$R_j \otimes R_k \cong R_k \otimes R_j, \quad \textit{for } |k - j| > 1,$$

$$R_j \otimes R_{j+1} \otimes R_j \cong R_{j+1} \otimes R_j \otimes R_{j+1}, \quad \textit{for } j \geq 2.$$

Proof. These relations can be verified similarly as in [15, Theorem 2.5]. ∎

3.3. Adjunctions and Dehn twist

To show the last type B_n relation (the 4-braiding relation), we will introduce a larger family of invertible complexes that will aid us in our calculation. This construction mirrors the notion of Dehn twists in topology and uses the theory on adjunctions (we highly recommend [14] for an amazing exposition on expressing adjunctions using planar diagrammatics). Throughout this section, we will denote $\mathbb{K}_1 := \mathbb{R}$ and $\mathbb{K}_j := \mathbb{C}$ for $j \geq 2$.

Definition 3.10. Let $X \in \mathrm{Kom}^b((\mathcal{B}_n, \mathbb{K}_j)\text{-bimod})$ and

$$X^\ell, X^r \in \mathrm{Kom}^b((\mathbb{K}_j, \mathcal{B}_n)\text{-bimod})$$

such that $X^\ell \otimes_{\mathcal{B}_n} -$ and $X^r \otimes_{\mathcal{B}_n} -$ are left and right adjoints of $X \otimes_{\mathbb{K}_j} -$, respectively. We define the twist of X as the complex of $(\mathcal{B}_n, \mathcal{B}_n)$-bimodule

$$\sigma_X := \mathrm{cone}(X \otimes_{\mathbb{K}_j} X^r \xrightarrow{\varepsilon} \mathcal{B}_n),$$

with ε the counit of the adjunction $X \dashv X^r$. Similarly, the dual twist of X is given by

$$\sigma_X' := \mathrm{cone}(\mathcal{B}_n \xrightarrow{\eta} X \otimes_{\mathbb{K}_j} X^\ell)$$

with η the unit of the adjunction $X \vdash X^\ell$. The twist σ_X is said to be spherical if the twist and dual twist are inverses of each other, namely,

$$\sigma_X \otimes_{\mathcal{B}_n} \sigma_X' \cong \mathcal{B}_n \cong \sigma_X' \otimes_{\mathcal{B}_n} \sigma_X.$$

One can verify from the definition of the adjunctions that the twist (resp., dual twist) is uniquely defined up to isomorphism; i.e., $X \cong Y$ implies $\sigma_X \cong \sigma_Y$ (resp., $\sigma'_X \cong \sigma'_Y$). On the other hand, the shift functors [1], {1} and ⟨1⟩ are autoequivalences, so we also have that

$$\sigma_X = \sigma_{X[r]\{s\}\langle t\rangle}.$$

More generally, given a pair of adjunctions (X, X^ℓ, X^r) on X with $Y \cong \Sigma \otimes_{\mathcal{B}_n} X$, where Σ an invertible object in $\mathrm{Kom}^b((\mathcal{B}_n, \mathcal{B}_n)\text{-bimod})$, we also have a pair of adjunctions (Y, Y^ℓ, Y^r) on Y given by

$$Y^\ell := X^\ell \otimes_{\mathcal{B}_n} \Sigma^{-1}, \quad Y^r := X^r \otimes_{\mathcal{B}_n} \Sigma^{-1}.$$

Furthermore, the twists and dual twists are related by

$$\sigma_Y \cong \Sigma \otimes_{\mathcal{B}_n} \sigma_X \otimes_{\mathcal{B}_n} \Sigma^{-1}, \quad \sigma'_Y \cong \Sigma \otimes_{\mathcal{B}_n} \sigma'_X \otimes_{\mathcal{B}_n} \Sigma^{-1}.$$

Lemma 3.11. *The functor $P_j^B \otimes_{\mathbb{K}_j} -$ is a left adjoint of ${}_jP^B \otimes_{\mathcal{B}_n} -$ and a right adjoint of ${}_jP^B\{-1\} \otimes_{\mathcal{B}_n} -$.*

Proof. To show that the $P_j^B \otimes_{\mathbb{K}_j} -$ is a left adjoint to ${}_jP^B \otimes_{\mathcal{B}_n} -$, take the counit to be the functor induced by

$$P_j^B \otimes_{\mathbb{K}_j} {}_jP^B \xrightarrow{\beta_j} \mathcal{B}_n,$$

and the unit is instead induced by $\mathbb{K}_j \xrightarrow{\varphi} {}_jP^B \otimes_{\mathcal{B}_n} P_j^B$, where φ is defined by $\varphi(1) = e_j \otimes_{\mathcal{B}_n} e_j$. To show that ${}_jP^B\{-1\} \otimes_{\mathcal{B}_n} -$ is a left adjoint to $P_j^B \otimes_{\mathbb{K}_j} -$, take the counit to be the functor induced by ${}_jP^B \otimes_{\mathcal{B}_n} P_j^B\{-1\} \xrightarrow{\varphi'} \mathbb{K}_j$, where φ' is defined by

$$\varphi'(e_j \otimes e_j) = 0, \quad \varphi'(X_j \otimes e_j) = 1$$

(note that $X_j \otimes e_j = e_j \otimes X_j$), and the unit is instead induced by $\mathcal{B}_n \xrightarrow{\gamma_j} P_j^B \otimes_{\mathbb{K}_j} {}_jP^B\{-1\}$. We leave the verification of the conditions required to the reader. ■

Using this, we will now prove the last type B_n relation required.

Proposition 3.12. *We have the following isomorphism of* $\mathrm{Kom}^b((\mathcal{B}_n, \mathcal{B}_n)$*-bimod):*

$$R_2 \otimes_{\mathcal{B}_n} R_1 \otimes_{\mathcal{B}_n} R_2 \otimes_{\mathcal{B}_n} R_1 \cong R_1 \otimes_{\mathcal{B}_n} R_2 \otimes_{\mathcal{B}_n} R_1 \otimes_{\mathcal{B}_n} R_2.$$

Proof. We will drop the tensor products for the sake of readability: $R_2R_1R_2R_1 \cong R_1R_2R_1R_2$. Using Proposition 3.9, note that this relation is equivalent to

$$(R_2R_1R_2)R_1(R'_2R'_1R'_2) \cong R_1.$$

By the adjunctions shown in Lemma 3.11, note that R_1 and R'_1 are by definition the same as the twist $\sigma_{P_1^B}$ and dual twist $\sigma'_{P_1^B}$ of P_1^B, respectively. As such, we get that

$$\sigma_{R_2R_1R_2(P_1^B)} \cong (R_2R_1R_2)\sigma_{P_1^B}(R_2'R_1'R_2') \cong \sigma_{P_1^B}.$$

It is now sufficient to show that the complexes $R_2R_1R_2(P_1^B)$ and P_1^B are isomorphic in $\mathrm{Kom}^b((\mathcal{B}_n,\mathbb{R})\text{-bimod})$ up to cohomological or internal gradings shifts. This is shown in the following series computation (note that we have omitted the cohomological grading since it does not matter):

$$\begin{aligned}
R_2(P_1^B) &= 0 \to P_2^B \otimes_{\mathbb{C}} {}_2P_1^B \to P_1^B \to 0 \\
&\cong 0 \to P_2^B\{1\} \xrightarrow{(2|1)} P_1^B \to 0 \quad \text{(by Proposition 3.5)}; \\
R_1R_2(P_1^B) &\cong R_1(0 \to P_2^B\{1\} \xrightarrow{(2|1)} P_1^B \to 0) \\
&= \mathrm{cone}\left\{ \begin{array}{ccc} P_1^B\{1\} \otimes_{\mathbb{R}} {}_1P_2^B & \xrightarrow{\mathrm{id}\otimes(2|1)} & P_1^B \otimes_{\mathbb{R}} {}_1P_1^B \\ \downarrow & & \downarrow \\ P_2^B\{1\} & \xrightarrow{(2|1)} & P_1^B \end{array} \right\} \\
&\cong \mathrm{cone}\left\{ \begin{array}{ccc} P_1^B\{1\} \oplus P_1^B\{1\}\langle 1\rangle & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{smallmatrix}\right]} & P_1^B \oplus P_1^B\{1\} \\ \downarrow{\scriptstyle [(1|2)\ (1|2)i\,]} & & \downarrow{\scriptstyle [\mathrm{id}\ X_1\,]} \\ P_2^B\{1\} & \xrightarrow{(2|1)} & P_1^B \end{array} \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 3.5)} \\
&\cong 0 \to P_1^B\{1\}\langle 1\rangle \xrightarrow{(1|2)i} P_2^B\{1\} \to 0; \\
R_2R_1R_2(P_1^B) &\cong \mathrm{cone}\left\{ \begin{array}{ccc} P_2^B \otimes_{\mathbb{C}} {}_2P_1^B\{1\}\langle 1\rangle & \longrightarrow & P_2^B \otimes_{\mathbb{C}} {}_2P_2^B\{1\} \\ \downarrow & & \downarrow \\ P_1^B\{1\}\langle 1\rangle & \xrightarrow{(1|2)i} & P_2^B\{1\} \end{array} \right\} \\
&\cong \mathrm{cone}\left\{ \begin{array}{ccc} P_2^B\{2\}\langle 1\rangle & \xrightarrow{\left[\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right]} & P_2^B\{1\} \oplus P_2^B\{2\}\langle 1\rangle \\ \downarrow{\scriptstyle (2|1)} & & \downarrow{\scriptstyle [\mathrm{id}\ X_2 i\,]} \\ P_1^B\{1\}\langle 1\rangle & \xrightarrow{(1|2)i} & P_2^B\{1\}. \end{array} \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 3.5)} \\
&\cong P_1^B\{1\}\langle 1\rangle.
\end{aligned}$$

■

Theorem 3.13. *We have a (weak) $\mathcal{A}(B_n)$-action on* $\mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$*, where each standard generator σ_j^B for $j \geq 2$ of $\mathcal{A}(B_n)$ acts on a complex $M \in \mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$ via R_j, and σ_1^B acts via $R_1\langle 1\rangle$:*

$$\sigma_j^B(M) := R_j \otimes_{\mathcal{B}_n} M \quad \text{and} \quad (\sigma_j^B)^{-1}(M) := R'_j \otimes_{\mathcal{B}_n} M,$$
$$\sigma_1^B(M) := R_1\langle 1\rangle \otimes_{\mathcal{B}_n} M \quad \text{and} \quad (\sigma_1^B)^{-1}(M) := R'_1\langle 1\rangle \otimes_{\mathcal{B}_n} M.$$

Proof. This follows directly from Propositions 3.9 and 3.12, where the required relations still hold with the extra third grading shift $\langle 1\rangle$ on R_1 and R'_1. ∎

From now on, we will abuse notation and use σ_j^B and $(\sigma_j^B)^{-1}$ in place of R_j and R'_j (with an extra grading shift $\langle 1\rangle$ for $j = 1$), respectively, whenever it is clear from the context what we mean.

4. Relating categorical type B_n and type A_{2n-1} actions

In Section 2, we have defined $\mathfrak{m}$ that lifts isotopy classes of trigraded curves in $\mathbb{D}^B_{n+1}$ to isotopy classes of bigraded multicurves in $\mathbb{D}^A_{2n}$. Furthermore, we showed that the map $\mathfrak{m}$ is equivariant under the $\mathcal{A}(B_n)$-action. In this section, we will develop the algebraic version of this story. We will first relate our type B_n zigzag algebra $\mathcal{B}_n$ to the type A_{2n-1} zigzag algebra $\mathcal{A}_{2n-1}$ by showing that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathcal{B}_n \cong \mathcal{A}_{2n-1}$$

as graded $\mathbb{C}$-algebras (forgetting the $\mathbb{Z}/2\mathbb{Z}$ grading in $\mathcal{B}_n$). Through this, we have an injection

$$\mathcal{B}_n \hookrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathcal{B}_n \cong \mathcal{A}_{2n-1}$$

as graded $\mathbb{R}$-algebras. Thus, we can relate the two categories $\mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$ and $\mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$ through an extension of scalar $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$. We end this section by showing that the functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ is $\mathcal{A}(B_n)$-equivariant, which also allows us to deduce that the $\mathcal{A}(B_n)$-action on $\mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$ is faithful.

Let Q be a left $\mathbb{C}$-module. Throughout this section, we will denote ${}_{\overline{\mathbb{C}}}Q$ to be the left $\mathbb{C}$-module with a deformed left action, given by multiplication its with complex conjugate

$$a(c) = \bar{a}c. \tag{4.1}$$

Similarly, for Q a right $\mathbb{C}$-module, we use $Q_{\overline{\mathbb{C}}}$ to denote the right $\mathbb{C}$-module with the deformed action.

Proposition 4.1. *Consider the graded $\mathbb{R}$-algebra $\ddot{\mathcal{B}}_n$, where $\ddot{\mathcal{B}}_n$ is just $\mathcal{B}_n$ without the $\mathbb{Z}/2\mathbb{Z}$-grading $\langle -\rangle$. The $\mathbb{Z}$-graded $\mathbb{C}$-algebras $\mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathcal{B}}_n$ and $\mathcal{A}_{2n-1}$ are isomorphic.*

Proof. Note that, for a $\mathbb{C}$-vector space, we have the following decomposition:

$$\begin{aligned}\mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathcal{B}}_n &\cong \bigoplus_{j=2}^{n} \mathbb{C} \otimes_{\mathbb{R}} \left({}_jP_j^B \ \oplus {}_jP_{(j-1)}^B \oplus {}_{(j-1)}P_j^B\right) \oplus \left(\mathbb{C} \otimes_{\mathbb{R}} {}_1P_1^B\right) \\ &\cong \bigoplus_{j=2}^{n} \left(\mathbb{C} \otimes_{\mathbb{R}} {}_jP_j^B\right) \oplus \bigoplus_{j=2}^{n} \left(\mathbb{C} \otimes_{\mathbb{R}} {}_jP_{(j-1)}^B\right) \oplus \bigoplus_{j=2}^{n} \left(\mathbb{C} \otimes_{\mathbb{R}} {}_{(j-1)}P_j^B\right) \\ &\quad \oplus \left(\mathbb{C} \otimes_{\mathbb{R}} {}_1P_1^B\right).\end{aligned}$$

Firstly, for each $j \geq 2$, note that ${}_jP_j^B$ is itself a $\mathbb{C}$-algebra with unit $e_j \otimes 1$. Moreover, after tensoring with $\mathbb{C}$ over $\mathbb{R}$, $\mathbb{C} \otimes_{\mathbb{R}} {}_jP_j^B$ has idempotent

$$\nu_j := \frac{1}{2}(1 \otimes e_j + i \otimes i e_j).$$

We will define a $\mathbb{C}$-linear morphism $\Phi : \mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathcal{B}}_n \to \mathcal{A}_{2n-1}$ by specifying the images of the basis elements of $\mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathcal{B}}_n$. It will be easy to see that Φ is grading preserving, and we leave the routine check that Φ is an algebra morphism to the reader. For $j = 1$,

$$\mathbb{C} \otimes_{\mathbb{R}} {}_1P_1^B \to {}_nP_n^A$$
$$\begin{cases} 1 \otimes \frac{1}{2}X_1 & \mapsto X_n, \\ 1 \otimes e_1 & \mapsto e_n. \end{cases}$$

For $2 \leq j \leq n$,

(1) $\mathbb{C} \otimes_{\mathbb{R}} {}_jP_j^B \to {}_{n-j+1}P_{n-j+1}^A \oplus {}_{n+j-1}P_{n+j-1}^A$

$$\begin{cases} (1 \otimes X_j)\nu_j & \mapsto X_{n-j+1}, \\ (1 \otimes X_j)(1 \otimes e_j - \nu_j) & \mapsto X_{n+j-1}, \\ \nu_j & \mapsto e_{n-j+1}, \\ (1 \otimes e_j - \nu_j) & \mapsto e_{n+j-1}, \end{cases}$$

(2) $\mathbb{C} \otimes_{\mathbb{R}} {}_{j-1}P_j^B \to {}_{n-j+2}P_{n-j+1}^A \oplus {}_{n+j-2}P_{n+j-1}^A$

$$\begin{cases} \big(1 \otimes (j-1|j)\big)\nu_j & \mapsto ((n-j+2) \mid \big(n-j+1)\big), \\ \big(1 \otimes (j-1|j)\big)(1 \otimes e_j - \nu_j) & \mapsto \big((n+j-2) \mid (n+j-1)\big), \end{cases}$$

(3) $\mathbb{C} \otimes_{\mathbb{R}} {}_jP_{j-1}^B \to {}_{n-j+1}P_{n-j+2}^A \oplus {}_{n+j-1}P_{n+j-2}^A$

$$\begin{cases} \nu_j(1 \otimes (j|j-1)) & \mapsto \big((n-j+1) \mid (n-j+2)\big), \\ (1 \otimes e_j - \nu_j)(1 \otimes (j|j-1)) & \mapsto \big((n+j-1) \mid (n+j-2)\big). \end{cases}$$

It is easy to see that this map is surjective and $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathcal{B}}_n) = \dim_{\mathbb{C}}(\mathcal{A}_{2n-1})$. ∎

Let $i : \ddot{\mathcal{B}}_n \hookrightarrow \mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathcal{B}}_n$ be the canonical injection of $\mathbb{Z}$-graded $\mathbb{R}$-algebras. Proposition 4.1 allows us to view $\ddot{\mathcal{B}}_n$ as a $\mathbb{Z}$-graded subalgebra over $\mathbb{R}$ of $\mathcal{A}_{2n-1}$ through $\Phi \circ i$. Thus, every $\mathcal{A}_{2n-1}$-module can be restricted to a $\ddot{\mathcal{B}}_n$-module. In particular, $\mathcal{A}_{2n-1}$ as a $\mathbb{Z}$-graded $(\mathcal{A}_{2n-1}, \mathcal{A}_{2n-1})$-bimodule can be restricted to a $\mathbb{Z}$-graded $(\mathcal{A}_{2n-1}, \ddot{\mathcal{B}}_n)$-bimodule. This gives us an extension of scalar functor $\mathcal{A}_{2n-1} \otimes_{\ddot{\mathcal{B}}_n} -$, sending $\mathbb{Z}$-graded $\ddot{\mathcal{B}}_n$-modules to $\mathbb{Z}$-graded $\mathcal{A}_{2n-1}$-modules.

Let $\mathfrak{F}$ denote the functor which forgets the $\mathbb{Z}/2\mathbb{Z}$-grading of the bigraded $\mathcal{B}_n$-modules. We define

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} - := \mathcal{A}_{2n-1} \otimes_{\ddot{\mathcal{B}}_n} (\mathfrak{F}(-)).$$

The proposition below identifies the indecomposable projectives under the functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$.

Proposition 4.2. *Recall the deformed action of $\mathbb{C}$ given in (4.1). We have the following isomorphisms of $\mathbb{Z}$-graded bimodules:*

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} P_1^B \cong (P_n^A)_{\mathbb{R}} \quad \text{as } \mathbb{Z}\text{-graded } (\mathcal{A}_{2n-1}, \mathbb{R})\text{-bimodules},$$

and

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} P_j^B \cong (P_{n-(j-1)}^A)_{\bar{\mathbb{C}}} \oplus P_{n+(j-1)}^A \quad \text{as } \mathbb{Z}\text{-graded } (\mathcal{A}_{2n-1}, \mathbb{C})\text{-bimodules}.$$

Proof. Define $\Phi_1 : \mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} P_1^B \to (P_n^A)_{\mathbb{R}}$ and

$$\Phi_j : \mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} P_j^B \to (P_{n-(j-1)}^A)_{\bar{\mathbb{C}}} \oplus P_{n+(j-1)}^A$$

as the maps given on the basis elements by $a \otimes b \mapsto a\Phi(1 \otimes b)$ and extend linearly. It is easy to check that Φ_1 is a graded $(\mathcal{A}_{2n-1}, \mathbb{R})$-bimodule morphism and Φ_j is a graded $(\mathcal{A}_{2n-1}, \mathbb{C})$-bimodule morphism; the only detail that one should be aware of is that Φ_j maps into $(P_{n-(j-1)}^A)_{\bar{\mathbb{C}}} \oplus P_{n+(j-1)}^A$ instead of $P_{n-(j-1)}^A \oplus P_{n+(j-1)}^A$. The fact that they are isomorphisms follows easily from looking at the dimensions. ∎

It follows from the above proposition that $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ sends projectives to projectives. Therefore, $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ extends to a functor from $\mathrm{Kom}^b(\ddot{\mathcal{B}}_n\text{-p}_r\text{g}_r\text{mod})$ to $\mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$. We will denote the functor

$$\begin{aligned}\mathcal{A}_{2n-1} &\otimes_{\mathcal{B}_n} - \\ &:= \mathcal{A}_{2n-1} \otimes_{\ddot{\mathcal{B}}_n} (\mathfrak{F}(-)) : \mathrm{Kom}^b(\mathcal{B}_n\text{-p}_r\text{g}_r\text{mod}) \to \mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod}).\end{aligned}$$

Recall the injection $\Psi : \mathcal{A}(B_n) \to \mathcal{A}(A_{2n-1})$ as defined in Proposition 2.1; the image of standard generators is explicitly given by

$$\Psi(\sigma_j^B) = \begin{cases} \sigma_n^A, & \text{for } j = 1, \\ \sigma_{n-(j-1)}^A \sigma_{n+(j-1)}^A, & \text{otherwise.} \end{cases}$$

We have previously shown that $\mathcal{A}(B_n)$ acts on $\mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$, and similarly, $\mathcal{A}(A_{2n-1})$ acts on $\mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$. Through Ψ, we have an induced action of $\mathcal{A}(B_n)$ on $\mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$. We will now prove the algebraic version of Proposition 2.12.

Theorem 4.3. *For all $1 \le j \le n$, we have an isomorphism*

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} \sigma_j^B \cong \Psi(\sigma_j^B)_{\mathcal{B}_n}$$

in $\mathrm{Kom}^b((\mathcal{A}_{2n-1}, \mathcal{B}_n)$-bimod). In particular, the functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ is $\mathcal{A}(B_n)$-equivariant

$$\begin{array}{ccc} \mathcal{A}(B_n) \circlearrowright & & \circlearrowleft \mathcal{A}(A_{2n-1}) \overset{\Psi}{\hookleftarrow} \mathcal{A}(B_n) \\ \mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r mod) & \xrightarrow{\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -} & \mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-}\mathrm{p}_r\mathrm{g}_r mod), \end{array}$$

i.e., for any $\sigma \in \mathcal{A}(B_n)$ and any complex $C \in \mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r mod)$,

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} (\sigma \otimes_{\mathcal{B}_n} C) \cong \Psi(\sigma) \otimes_{\mathcal{A}_{2n-1}} (\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} C).$$

Before we start with the proof, we will need the following lemma.

Lemma 4.4. *Recall the deformed $\mathbb{C}$-action given in (4.1). We have that*

$$\mathbb{C} \otimes_{\mathbb{R}} {}_1P^B \cong ({}_nP^A)_{\mathcal{B}_n} \quad \text{as } \mathbb{Z}\text{-graded } (\mathbb{C}, \mathcal{B}_n)\text{-bimodules,} \tag{4.2}$$

$${}_jP^B \cong ({}_{n+(j-1)}P^A)_{\mathcal{B}_n} \quad \text{as } \mathbb{Z}\text{-graded } (\mathbb{C}, \mathcal{B}_n)\text{-bimodules,} \tag{4.3}$$

$${}_jP^B \cong {}_{\bar{\mathbb{C}}}({}_{n-(j-1)}P^A)_{\mathcal{B}_n} \quad \text{as } \mathbb{Z}\text{-graded } (\mathbb{C}, \mathcal{B}_n)\text{-bimodules,} \tag{4.4}$$

$${}_{\mathbb{C}}\mathbb{C} \otimes_{\bar{\mathbb{C}}} \mathbb{C}_{\mathbb{C}} \cong {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \quad \text{as } \mathbb{Z}\text{-graded } (\mathbb{C}, \mathbb{C})\text{-bimodules.} \tag{4.5}$$

Proof. We will only define the maps; the proof that they are isomorphisms with respect to the required structures follows from a simple verification.

For (4.2), take the morphism $\phi_1 : \mathbb{C} \otimes_{\mathbb{R}} {}_1P^B \to ({}_nP^A)_{\mathcal{B}_n}$ as the restriction of Φ constructed in the proof of Proposition 4.1. Note that ϕ_1 does indeed map into $({}_nP^A)_{\mathcal{B}_n}$ since

$$\Phi(c \otimes b) = \Phi(c \otimes e_1 b) = \Phi((1 \otimes e_1)(c \otimes b)) = \Phi(1 \otimes e_1)\Phi(c \otimes b) = e_n \Phi(c \otimes b).$$

For (4.3) and (4.4), consider the morphisms

$$\begin{aligned} \phi_{+j} : {}_jP^B &\to ({}_{n+(j-1)}P^A)_{\mathcal{B}_n} \\ b &\mapsto e_{n+(j-1)}\Phi(1 \otimes b), \end{aligned} \quad \text{and} \quad \begin{aligned} \phi_{-j} : {}_jP^B &\to {}_{\bar{\mathbb{C}}}({}_{n-(j-1)}P^A)_{\mathcal{B}_n} \\ b &\mapsto e_{n-(j-1)}\Phi(1 \otimes b). \end{aligned}$$

Finally, for (4.5), consider the morphism $c : {}_{\mathbb{C}}\mathbb{C} \otimes_{\bar{\mathbb{C}}} \mathbb{C}_{\mathbb{C}} \to \mathbb{C}$ uniquely defined by $1 \otimes 1 \mapsto 1$. ∎

Proof of Theorem 4.3. We will show the required statement by showing it for each generator, namely,

$$\Psi(\sigma_j^B)_{\mathcal{B}} \cong \mathcal{A} \otimes_{\mathcal{B}} \sigma_j^B$$

as complexes of $(\mathcal{A}, \mathcal{B})$-bimodules for each j.

Let $j \geq 2$. Using the relevant isomorphisms in Lemma 4.4, we have the following chain of bimodule isomorphisms:

$$\begin{aligned}
&(P^A_{n-(j-1)} \otimes_{\mathbb{C}} ({}_{n-(j-1)}P^A)_{\mathcal{B}}) \oplus (P^A_{n+(j-1)} \otimes_{\mathbb{C}} ({}_{n+(j-1)}P^A)_{\mathcal{B}}) \\
&\cong (P^A_{n-(j-1)} \otimes_{\overline{\mathbb{C}}} ({}_{n-(j-1)}P^A)_{\mathcal{B}}) \oplus (P^A_{n+(j-1)} \otimes_{\mathbb{C}} ({}_{n+(j-1)}P^A)_{\mathcal{B}}) \quad \text{(by (4.5))} \\
&\cong ((P^A_{n-(j-1)})_{\overline{\mathbb{C}}} \otimes_{\mathbb{C}} {}_jP^B) \oplus (P^A_{n+(j-1)} \otimes_{\mathbb{C}} {}_jP^B) \quad \text{(by (4.3) and (4.4))} \\
&\cong ((P^A_{n-(j-1)})_{\overline{\mathbb{C}}} \oplus P^A_{n+(j-1)}) \otimes_{\mathbb{C}} {}_jP^B.
\end{aligned}$$

Using Proposition 4.2, we have that

$$((P^A_{n-(j-1)})_{\overline{\mathbb{C}}} \oplus P^A_{n+(j-1)}) \otimes_{\mathbb{C}} {}_jP^B \cong \mathcal{A} \otimes_{\mathcal{B}} P^B_j \otimes_{\mathbb{C}} {}_jP^B.$$

Let us denote the composition of this chain of isomorphisms by

$$\begin{aligned}
\Xi : &(P^A_{n-(j-1)} \otimes_{\mathbb{C}} ({}_{n-(j-1)}P^A)_{\mathcal{B}}) \oplus (P^A_{n+(j-1)} \otimes_{\mathbb{C}} ({}_{n+(j-1)}P^A)_{\mathcal{B}}) \\
&\xrightarrow{\cong} \mathcal{A} \otimes_{\mathcal{B}} P^B_j \otimes_{\mathbb{C}} {}_jP^B.
\end{aligned}$$

Since we have that

$$\begin{aligned}
\Psi(\sigma_j^B)_{\mathcal{B}} &= (\sigma^A_{n-(j-1)}\sigma^A_{n+(j-1)})_{\mathcal{B}} \\
&= (P^A_{n-(j-1)} \otimes_{\mathbb{C}} ({}_{n-(j-1)}P^A)_{\mathcal{B}}) \oplus (P^A_{n+(j-1)} \otimes_{\mathbb{C}} ({}_{n+(j-1)}P^A)_{\mathcal{B}}) \\
&\xrightarrow{[\beta^A_{n-(j-1)}\ \beta^A_{n+(j-1)}]} \mathcal{A}_{\mathcal{B}}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A} \otimes_{\mathcal{B}} \sigma_j^B &= \mathcal{A} \otimes_{\mathcal{B}} (P^B_j \otimes_{\mathbb{C}} {}_jP^B \xrightarrow{\beta^B_j} \mathcal{B}) \\
&= \mathcal{A} \otimes_{\mathcal{B}} P^B_j \otimes_{\mathbb{C}} {}_jP^B \xrightarrow{\mathrm{id} \otimes_{\mathcal{B}} \beta^B_j} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{B},
\end{aligned}$$

all that is left is to show that the following diagram commutes:

$$\begin{array}{ccc}
(P^A_{n-(j-1)} \otimes_{\mathbb{C}} ({}_{n-(j-1)}P^A)_{\mathcal{B}}) \oplus (P^A_{n+(j-1)} \otimes_{\mathbb{C}} ({}_{n+(j-1)}P^A)_{\mathcal{B}}) & \xrightarrow{[\beta^A_{n-(j-1)}\ \beta^A_{n+(j-1)}]} & \mathcal{A}_{\mathcal{B}} \\
\downarrow \Xi & & \downarrow \cong \\
\mathcal{A} \otimes_{\mathcal{B}} P^B_j \otimes_{\mathbb{C}} {}_jP^B & \xrightarrow{\mathrm{id} \otimes_{\mathcal{B}} \beta^B_j} & \mathcal{A} \otimes_{\mathcal{B}} \mathcal{B},
\end{array}$$

which we leave for the reader to verify.

The proof of $j = 1$ is simpler and follows from a similar argument. ∎

Remark 4.5. Define $U_j := P_j^B \otimes_j P^B$ and $\mathcal{U}_j := P_j^A \otimes_j P^A$. Proposition 4.2 implies that

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} U_j \cong \begin{cases} (\mathcal{U}_n)_{\mathcal{B}_n}, & \text{for } j = 1, \\ (\mathcal{U}_{n-(j-1)} \oplus \mathcal{U}_{n+(j-1)})_{\mathcal{B}_n}, & \text{otherwise.} \end{cases}$$

When $n = 2$, this also relates our bimodules to the ones given in [17], where

$$\mathcal{U}_2 = \Theta_t \quad \text{and} \quad \mathcal{U}_1 \oplus \mathcal{U}_3 = \Theta_s$$

in [17, Example 2.12] for the A_3 graph (up to a difference in grading).

We may now use this relation to deduce that the categorical action of $\mathcal{A}(B_n)$ is faithful.

Theorem 4.6. *The (weak) action of* $\mathcal{A}(B_n)$ *on the category* $\mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathit{mod})$ *given in Theorem 3.13 is faithful.*

Proof. Assume that we are given $\sigma \in \mathcal{A}(B_n)$ such that

$$\sigma(C) \cong C$$

for all $C \in \mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$. We will show that this implies that σ is the identity. In particular, take

$$C = \bigoplus_{j=1}^{n} P_j^B$$

so that we have

$$\sigma\left(\bigoplus_{j=1}^{n} P_j^B\right) \cong \bigoplus_{j=1}^{n} P_j^B.$$

Applying the functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$, we obtain

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} \sigma\left(\bigoplus_{j=1}^{n} P_j^B\right) \cong \mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} \left(\bigoplus_{j=1}^{n} P_j^B\right) \cong \bigoplus_{j=1}^{2n-1} P_j^A. \tag{4.6}$$

Applying Theorem 4.3 to the LHS of (4.6), we get

$$\begin{aligned} \mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} \sigma\left(\bigoplus_{j=1}^{n} P_j^B\right) &\cong \Psi(\sigma)\left(\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} \left(\bigoplus_{j=1}^{n} P_j^B\right)\right) \\ &\cong \Psi(\sigma)\left(\bigoplus_{j=1}^{2n-1} P_j^A\right). \end{aligned} \tag{4.7}$$

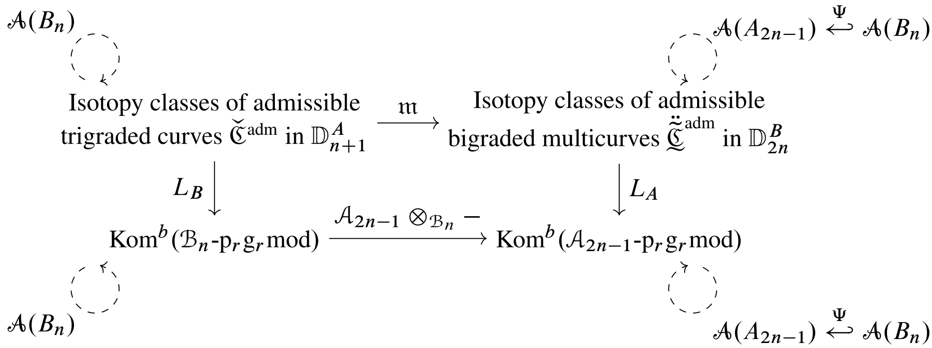


Figure 5.1. The commutative diagram of Theorem 5.1. The first row is from Proposition 2.12 and the second row is from Theorem 4.3.

Combining the two equations (4.6) and (4.7) above, we deduce that

$$\Psi(\sigma)\left(\bigoplus_{j=1}^{2n-1} P_j^A\right) \cong \bigoplus_{j=1}^{2n-1} P_j^A.$$

Since it was shown in [15, Corollary 1.2] that the type A categorical action is faithful, we conclude that

$$\Psi(\sigma) = \text{id}\,.$$

But Ψ is injective, so we must have that $\sigma = \text{id}$ as required. ∎

5. Main theorem

Let us state the main theorem that we aim to prove.

Theorem 5.1. *The diagram in Figure 5.1 is commutative, where the maps $\mathfrak{m}$, L_B, L_A, and $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ are all $\mathcal{A}(B_n)$-equivariant.*

In Section 2, we introduced and showed that $\mathfrak{m}$ is $\mathcal{A}(B_n)$-equivariant. In Section 4, we showed that the functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ is also $\mathcal{A}(B_n)$-equivariant. Thus, the missing pieces are:

(1) the definitions of the maps L_B and L_A,

(2) the fact that the maps L_B and L_A are $\mathcal{A}(B_n)$-equivariant,

(3) the commutativity of the diagram.

We will start by recalling the definition of L_A from [15, Section 4a] and their result that L_A is $\mathcal{A}(A_{2n-1})$-equivariant. We then define L_B in Section 5.2, where the (technical) proof that the diagram commutes and that L_B is $\mathcal{A}(B_n)$-equivariant are given in Section 5.3.

5.1. Complexes associated to admissible multi-curves (type A)

Here, we state the constructions and results shown in [15, Section 4], which can be easily extended to admissible multicurves. Note that we added a subscript L_A instead of L used in [15] to differentiate between type A and type B later on. Let $\ddot{c}$ be a bigraded admissible curve in normal form. We associate to $\ddot{c}$ an object $L_A(\ddot{c})$ in the category $\mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-p}_r\mathrm{g}_r\mathrm{mod})$. Start by defining $L_A(\ddot{c})$ as a bigraded $\mathcal{A}_{2n-1}$-module

$$L_A(\ddot{c}) = \bigoplus_{x \in cr(\ddot{c})} P(x),$$

where $P(x) = P^A_{x_0}[-x_1]\{x_2\}$, with $[-]$ denoting the cohomological degree shift; see the paragraph after Remark 2.13 for the definition of (x_0, x_1, x_2). For every $x, y \in cr(\ddot{c})$, define $\partial_{yx} : P(x) \to P(y)$ by the following rules:

- If x and y are the endpoints of an essential segment and $y_1 = x_1 + 1$, then
  (1) If $x_0 = y_0$ (then it must be that $x_2 = y_2 + 1$), then
  $$\partial_{yx} : P(x) \to P(y) \cong P(x)[-1]\{1\}$$
  is the multiplication on the right by $X_{x_0} \in \mathcal{A}_{2n-1}$.
  (2) If $x_0 = y_0 \pm 1$, then ∂_{yx} is the right multiplication by $(x_0|y_0) \in \mathcal{A}_{2n-1}$.
- Otherwise, $\partial_{yx} = 0$.

We define the differential ∂ as $\partial := \sum_{x,y} \partial_{yx}$. See [15, Lemma 4.1] for a proof that this defines a complex. Moreover, it follows easily that

$$L_A(\chi_A(r_1, r_2)\ddot{c}) \cong L_A(\ddot{c})[-r_1]\{r_2\}. \tag{5.1}$$

For $\ddot{g}$ a bigraded j-string of $\ddot{c}$, we can also assign a complex $L_A(\ddot{g})$ to $\ddot{g}$, whereas a bigraded abelian group, $L_A(\ddot{g}) = \bigoplus_{x \in cr(g)} P^A(x)$, and the differentials are obtained from essential segments of $\ddot{g}$ the same way as for admissible curves. We can easily extend this to define $L_A(\ddot{h})$ for $h \subseteq c$ a connected subset of c such that $h = \bigcup g_{\alpha,j}$ with each $g_{\alpha,j}$ some bigraded j-string of c. The following theorem is proven in [15, Theorem 4.3].

Theorem 5.2. *For a braid $\sigma \in \mathcal{A}(A_m)$ and a bigraded admissible curve $\ddot{c}$ in $\mathbb{D}^A_{m+1}$, we have*

$$\sigma L_A(\ddot{c}) \cong L_A(\sigma(\ddot{c}))$$

in the category $\mathrm{Kom}^b(\mathcal{A}_m\text{-p}_r\mathrm{g}_r\mathrm{mod})$; i.e., L_A is $\mathcal{A}(A_m)$-equivariant.

We extend L_A to admissible multicurves as follows: given bigraded multicurves $\coprod_j \ddot{c}_j$,

$$L_A\Big(\coprod_j \ddot{c}_j\Big) := \bigoplus_j L_A(\ddot{c}_j).$$

It follows easily that this defines a complex, and both (5.1) and Theorem 5.2 still hold for admissible multicurves.

5.2. Complexes associated to admissible curves (type B)

Consider a trigraded admissible curve $\check{c}$. We associate to $\check{c}$ an object $L_B(\check{c})$ in the category $\mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$. Start by defining $L_B(\check{c})$ as a trigraded $\mathcal{B}_n$-module

$$L_B(\check{c}) = \bigoplus_{y \in cr(\check{c})} P(y),$$

where $P(y) = P^B_{y_0}[-y_1]\{y_2\}\langle y_3\rangle$ (see the second paragraph after Proposition 2.10 in Section 2.6 for the definition of y_0, y_1, y_2, and y_3).

We now define maps $\partial_{zy} : P(y) \to P(z)$ for each $y, z \in cr(\check{c})$ using the following rules (note that these are ***not*** the differentials yet):

- If y and z are the endpoints of an essential segment in D_j for $j \geq 1$ and $z_1 = y_1 + 1$, then
  - (1) If $y_0 = z_0$ (then also $y_2 = z_2 + 1$ and $y_3 = z_3$), then

$$\partial_{zy} : P(y) \to P(z) \cong P(y)[-1]\{1\}$$

    is the right multiplication by the element $X_{y_0} \in \mathcal{B}_n$.
  - (2) If $y_0 = z_0 \pm 1$, then ∂_{zy} is the right multiplication by $(y_0|z_0) \in \mathcal{B}_n$.
- Otherwise, $\partial_{yz} = 0$.

We will modify some of these maps before using them as differentials. Define the following equivalence relation on the set of 1-crossings:

$$y \sim y' \iff y \text{ and } y' \text{ are connected by an essential segment in } D_0.$$

Consider the partitioning of the set of 1-crossings using the equivalence relation above. Referring to the possible normal forms in D_0 given by Figure 2.10, every equivalence classes under this relation consists of either one element (type $3'$) or two elements (type $2''$). For each equivalence class $[y]$ of 1-crossings, we modify some of the maps given previously by the following rule:

- If $[y] = \{y\}$, we modify nothing;

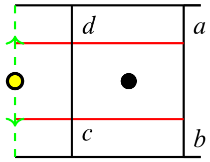
- otherwise, $[y] = \{y, y'\}$ has two distinct 1-crossings. Note that at least one of the 1-crossings must be an endpoint of some essential segment in D_1 (otherwise, c is clearly not be admissible). Up to relabelling, we may assume that y is always a 1-crossing that is an endpoint of an essential segment γ in D_1. Note that γ cannot have both y and y' as endpoints (this will imply that c is a simple closed curve, which is not admissible). As such, let us label the other endpoint of γ connecting y as $z \neq y'$. Consider the two possible cases for z
  (1) z is a 2-crossing:
    (a) if $y_1 = z_1 + 1$, then we have that $\partial_{yz} : P(z) \to P(y)$ is given by the right multiplication by $(2|1)$. We modify
    $$\partial_{y'z} : P(z) \to P(y') \cong P(y)\langle 1\rangle$$
    (which was necessarily 0 previously) so that it is now the right multiplication by $-i(2|1)$;
    (b) otherwise, we have instead $z_1 = y_1 + 1$. In this case,
    $$\partial_{zy} : P(y) \to P(z)$$
    is given by the right multiplication by $(1|2)$. We modify
    $$\partial_{zy'} : P(y') \cong P(y)\langle 1\rangle \to P(z)$$
    (which was necessarily 0 previously) so that it is now the right multiplication by $(1|2)i$.
  (2) z is a 1-crossing:
    (a) if $y_1 = z_1 + 1$, we modify nothing;
    (b) otherwise, we have instead $z_1 = y_1 + 1$. In this case,
    $$\partial_{zy} : P(y) \to P(z)$$
    is given by the right multiplication by X_1. Once again, consider the two possible cases of the equivalence class $[z]$:
      (i) if $[z] = \{z\}$, we modify nothing;
      (ii) otherwise, $[z] = \{z, z'\}$ with $z \neq z'$. We then modify $\partial_{z'y'} : P(y)\langle 1\rangle \cong P(y') \to P(z') \cong P(z)\langle 1\rangle$ (which was necessarily 0 previously) so that it is now the right multiplication by X_1.

Similarly, if the other 1-crossing y' in $[y]$ is also an endpoint of an essential segment in D_1 (distinct from γ), we will repeat the same process above for y'.

Finally, we define the differential as $\partial = \sum_{x,y \in cr(\check{c})} \partial_{xy}$, where ∂_{xy} are the *modified* version above.

Lemma 5.3. *$(L_B(\check{c}),\partial)$ is a complex of projective graded $\mathcal{B}_n$-modules with a grading-preserving differential.*

Proof. For $x, y, z \in cr(\check{c})$ with $x_0, y_0, z_0 \geq 2$, the same argument as in the type A shows that the product of $\partial_{zy}\partial_{yx} : P_x \to P_z$ is always 0. The only occurrence of $\partial_{zy}\partial_{yx} \neq 0$ is when $\partial_{zy} = \partial_{ac}, \partial_{ad}$ and $\partial_{yx} = \partial_{db}, \partial_{cb}$ with a, b, c, d the crossings of the following type of 1-string labelled below:



Note that the two non-zero composition $\partial_{ad}\partial_{db}$ and $\partial_{ac}\partial_{cb}$ always occur as a pair. Moreover, we see that their sum is equal to 0: $\partial_{ad}\partial_{db} + \partial_{ac}\partial_{cb} = X_2 i - X_2 i = 0$, thus, showing that $\partial^2 = 0$ as required. ■

Lemma 5.4. *For any triple (r_1, r_2, r_3) of integers and any trigraded admissible curve $\check{c}$, we have that*

$$L_B(\chi(r_1, r_2, r_3)\check{c}) \cong L_B(\check{c})[-r_1]\{r_2\}\langle r_3\rangle.$$

Proof. This follows directly from the definition. ■

5.3. Some rather technical results

This subsection is where we complete the proof of Theorem 5.1 by proving that the diagram commutes (Proposition 5.5) and that L_B is $\mathcal{A}(B_n)$-equivariant (Theorem 5.7). The proof of these two statements, first of which is technical, will occupy the rest of this section.

Proposition 5.5. *The diagram in Figure 5.1 commutes; i.e., for each trigraded admissible curve $\check{c}$ in $\mathbb{D}^B_{n+1}$, we have that*

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} L_B(\check{c}) \cong L_A(\mathfrak{m}(\check{c}))$$

in $\mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-}\mathrm{p}_r\mathrm{g}_r mod)$.

Remark 5.6. The theorem is stated in the context of homotopy category since the group actions of $\mathcal{A}(A_{2n-1})$ and $\mathcal{A}(B_n)$ live there. In fact, we will show that

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} L_B(\check{c}) \cong L_A(\mathfrak{m}(\check{c}))$$

in the category $\mathrm{Com}^b(\mathcal{B}_n\text{-p}_r\mathrm{g}_r\mathrm{mod})$ of bounded complexes (no homotopy will be required).

Proof of Proposition 5.5. Let x be any j-crossing of $\check{c}$. If $j \geq 2$, we have that $\mathfrak{m}(x)$ consists of an $(n+(j-1))$-crossing $\tilde{x}$ and an $(n-(j-1))$-crossing $\underset{\sim}{x}$ of $\mathfrak{m}(\check{c})$; if $j = 1$, $\mathfrak{m}(x)$ consists of a single n-crossing $\underset{\sim}{\tilde{x}}$ of $\mathfrak{m}(\check{c})$. In either cases, we have isomorphisms $\Phi_j : \mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} P^B(x) \to P^A(\tilde{x}) \oplus P^A(\underset{\sim}{x}) = P^A_{n+(j-1)} \oplus P^A_{n-(j-1)}$ or $\Phi_1 : \mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} P^B(x) \to P^A(\underset{\sim}{\tilde{x}}) = P^A_n$ given in the proof of Proposition 4.2. Putting together these isomorphisms for each crossing x of $\check{c}$, we obtain a cohomological and internal grading-preserving isomorphism of $\mathcal{A}_{2n-1}$-modules between the underlying modules of $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} L_B(\check{c})$ and $L_A(\mathfrak{m}(\check{c}))$; denote this isomorphism by η. Denoting the complexes $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} L_B(\check{c})$ as (Q, δ) and $L_A(\mathfrak{m}(\check{c}))$ as (Q', δ') (so η is an isomorphism from Q to Q'), it follows that η induces an isomorphism of complexes:

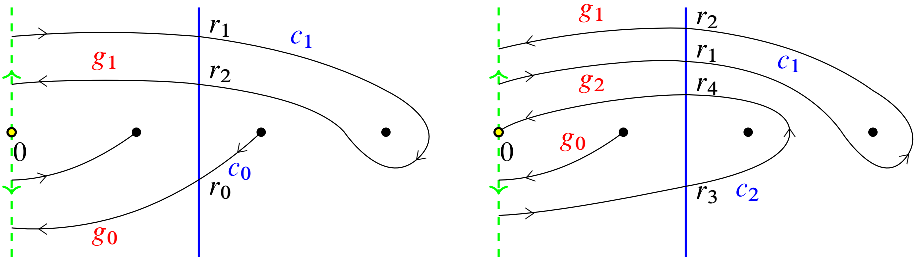
$$(Q, \delta) \cong (Q', \delta_0),$$

with $\delta_0 = \eta\delta\eta^{-1}$. We now aim to show that $(Q', \delta_0) \cong (Q', \delta')$ in $\mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-p}_r\mathrm{g}_r\mathrm{mod})$. In fact, we will show that they are isomorphic in the ordinary category $\mathrm{Com}^b(\mathcal{A}_{2n-1}\text{-p}_r\mathrm{g}_r\mathrm{mod})$ of complexes in $\mathcal{A}_{2n-1}\text{-p}_r\mathrm{g}_r\mathrm{mod}$. Before we proceed with the proof, we will need to introduce some (substantial amount of) notation.

Slicing c. Recall that c, being admissible, must have both of its endpoints at two *distinct* marked points, so at least one of its endpoints is not 0, and let m be such an endpoint. Orient the curve c so that it starts from m and ends at its other endpoint. Following this orientation, we can slice c into distinct connected components $c_j \subset c \cap (\bigcup_{j\geq 2} D_j)$ and $g_{j'} \subset c \cap (D_0 \cup D_1)$ (note that $g_{j'}$ are the 1-strings of c) enumerated as follows: following the orientation of c, g_j denotes the $(j-1)$-th 1-string component of c; whereas c_i is the component of c that has a total of i 1-strings components before c_i. In other words, if $m \neq 1$, we start from c_0 to g_0 to c_1, and so on; otherwise, $m = 1$ and we start from g_0 to c_0 to g_1, and so on. Note that if c has no 1-string component, $c = c_0$. Following the same orientation, we will also enumerate the 2-crossings r_t of c (if any), starting the enumeration with $t = 0$ if $m = 1$; otherwise, we start with $t = 1$. Figure 5.2 illustrates two examples with different starting point m.

Now, consider the following subsets of (graded) crossings of $\mathfrak{m}(\check{c})$:

$$\begin{cases} C_j := \mathfrak{m}(\check{c}_j) \cap (\bigcup_i \ddot{\theta}_i), \\ \bar{C}_j := \mathfrak{m}(\check{c}_j) \cap (\bigcup_{i\neq n-1,n+1} \ddot{\theta}_i), \\ G_{j'} := \mathfrak{m}(\check{g}_{j'}) \cap (\ddot{\theta}_{n-1} \cup \ddot{\theta}_n \cup \ddot{\theta}_{n+1}), \\ \bar{G}_{j'} := \mathfrak{m}(\check{g}_{j'}) \cap \ddot{\theta}_n, \\ R_j := \mathfrak{m}(\check{r}_j). \end{cases} \tag{5.2}$$



(a) A slicing with starting point $m = 2 \neq 1$. **(b)** A slicing with starting point $m = 1$.

Figure 5.2. Examples of slicings of curves.

Note that, by definition, the subsets of crossings $\bar{C}_i$, R_j, and $\bar{G}_k$ are pairwise disjoint, and $(\coprod_j \bar{C}_j) \amalg (\coprod_j R_j) \amalg (\coprod_j \bar{G}_j)$ is the set of all crossings of $\mathfrak{m}(\check{c})$. On the other hand, C_j and $G_{j'}$ contain all the crossings of $\mathfrak{m}(c_j)$ and $\mathfrak{m}(g_{j'})$, respectively, which may contain common crossings R_i.

For K a subset of crossings of $\mathfrak{m}(\check{c})$, we define

$$Q'_K := \bigoplus_{x \in K} P^A(x) \subseteq Q'.$$

If K is empty, Q'_K will be the 0 module by convention. Using this, we can decompose Q' as follows: when $m \in \Lambda \setminus \{0, 1\}$, we have

$$Q' = \underbrace{Q'_{\bar{C}_0} \oplus \rlap{$\overbrace{\phantom{Q'_{R_0} \oplus Q'_{\bar{G}_0} \oplus Q'_{R_1}}}^{Q'_{G_0}}$} Q'_{R_0}}_{Q'_{C_0}} \oplus Q'_{\bar{G}_0} \oplus \underbrace{Q'_{R_1} \oplus Q'_{\bar{C}_1} \oplus Q'_{R_2}}_{Q'_{C_1}} \oplus \cdots, \tag{5.3}$$

whereas when $m = 1$, we have instead

$$Q' = \underbrace{Q'_{\bar{G}_0} \oplus \rlap{$\overbrace{\phantom{Q'_{R_1} \oplus Q'_{\bar{C}_1} \oplus Q'_{R_2}}}^{Q'_{C_1}}$} Q'_{R_1}}_{Q'_{G_0}} \oplus Q'_{\bar{C}_1} \oplus \underbrace{Q'_{R_2} \oplus Q'_{\bar{G}_2} \oplus Q'_{R_3}}_{Q'_{G_2}} \oplus \cdots. \tag{5.4}$$

In general, given a decomposition of modules

$$M = \bigoplus_{i \in Y} M_{K_i}$$

and a complex (M, ∂), we can then write ∂ as a block matrix. We will use the notation $\partial_{K_i K_j}$ to denote the block of ∂ that maps from M_{K_j} to M_{K_i}, where we use the shorthand notation ∂_{K_i} for the block of ∂ that maps from M_{K_i} to itself. We will also use the notation $\partial_{\bigoplus_{i\in X\subseteq Y} K_i}$ for the block of ∂ that maps from $\bigoplus_{i\in X} M_{K_i}$ to itself. To illustrate, consider the decomposition of Q' as in (5.3) and let (Q', ∂) be a cochain complex with differential ∂. We will then write the differential ∂ as the matrix

$$\begin{bmatrix} \partial_{\bar{C}_0} & \partial_{R_0\bar{C}_0} & \partial_{\bar{G}_0\bar{C}_0} & \partial_{R_1\bar{C}_0} & \partial_{\bar{C}_1\bar{C}_0} & \cdots \\ \partial_{\bar{C}_0R_0} & \partial_{R_0} & \partial_{\bar{G}_0R_0} & \partial_{R_1R_0} & \partial_{\bar{C}_1R_0} & \cdots \\ \partial_{\bar{C}_0\bar{G}_0} & \partial_{R_0\bar{G}_0} & \partial_{\bar{G}_0} & \partial_{R_1\bar{G}_0} & \partial_{\bar{C}_1\bar{G}_0} & \cdots \\ \partial_{\bar{C}_0R_1} & \partial_{R_0R_1} & \partial_{\bar{G}_0R_1} & \partial_{R_1} & \partial_{\bar{C}_1R_1} & \cdots \\ \partial_{\bar{C}_0\bar{C}_1} & \partial_{R_0\bar{C}_1} & \partial_{\bar{G}_0\bar{C}_1} & \partial_{R_1\bar{C}_1} & \partial_{\bar{C}_1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the blocks ∂_{C_j} corresponding to the summands Q'_{C_j} are the blocks in red and similarly blocks ∂_{G_j} corresponding to the summands Q'_{G_j} are the blocks in blue.

After our lengthy notational digression, let us return to the proof and analyse the difference between the matrices of the two differentials δ_0 and δ' corresponding to the two possible decomposition of Q' as in (5.3) and (5.4). By looking at how the components c_i and g_j are connected, it follows that the non-zero entries of the matrix of δ_0 are all contained in the block matrices $(\delta_0)_{C_k}$ and $(\delta_0)_{G_k}$ for all k; similarly, the connection between the components $\mathfrak{m}(c_i)$ and $\mathfrak{m}(g_j)$ dictates that the non-zero entries of the matrix of δ' are all contained in the block matrices δ'_{C_k} and δ'_{G_k} for all k. One can show from a direct computation that

$$(\delta_0)_{C_k} = \delta'_{C_k}, \quad \text{for all } k. \tag{5.5}$$

So, between the differentials δ_0 and δ', only the block matrices $(\delta_0)_{G_k}$ and δ'_{G_k} may differ. As such, if we were in the case where c has no 1-string, i.e., $c = c_0$, then we are done. In the rest of the proof, we will assume otherwise and treat the rest of the cases.

As we will only need to focus on the module summand Q'_{G_j} later on, for each j, let us simplify both the decompositions of Q' in (5.3) and (5.4) into

$$Q' = Q'_{V_j} \oplus \underbrace{Q'_{R_j} \oplus Q'_{\bar{G}_j} \oplus Q'_{R_{j+1}}}_{Q'_{G_j}} \oplus Q'_{W_j}, \tag{5.6}$$

where Q'_{V_j} (resp., Q'_{W_j}) consists of all the module summands before Q'_{R_j} (resp., after $Q'_{R_{j+1}}$) for both decompositions (5.3) and (5.4). From here on, we will use this simplified decomposition for the matrix of any differential on Q'.

Let $g_0, g_1, \ldots, g_{s-1}$ be the 1-strings in c. To show that $(Q', \delta_0) \cong (Q', \delta')$, by (5.5) and (5.6), it is sufficient to construct a cohomological and internal grading-preserving isomorphism of modules $\mu_j : Q' \to Q'$ for each $0 \leq j \leq s-1$ so that we have an induced chain of isomorphisms in $\mathrm{Com}^b(\mathcal{A}_{2n-1}\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$

$$(Q', \delta_0) \cong (Q', \delta_1) \cong \cdots \cong (Q', \delta_{s-1}) \cong (Q', \delta_s) = (Q', \delta')$$

with $\delta_{j+1} := \mu_j \delta_j \mu_j^{-1}$, where each δ_j for $1 \leq j \leq s$ satisfies the following property:

$$\begin{cases} (\delta_j)_{G_{j-1}} = \delta'_{G_{j-1}}, \\ (\delta_j)_{XY} = (\delta_{j-1})_{XY}, \quad \text{for all } X, Y \in \{V_{j-1}, G_{j-1}, W_{j-1}\} \\ \qquad\qquad\qquad\qquad\quad \text{such that } (X, Y) \neq (G_{j-1}, G_{j-1}). \end{cases} \tag{*}$$

In other words, each μ_j will be constructed in a way that the conjugation of δ_j by μ_j^{-1} *only* alters the differential component $(\delta_j)_{G_j}$ so that

$$(\delta_{j+1})_{G_j} = \mu_j (\delta_j)_{G_j} \mu_j^{-1} = \delta'_{G_j}$$

without affecting the rest of the differential components. In particular, if δ_ℓ satisfies (*) for all $1 \leq \ell \leq j-1$, then we have that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$. Moreover, property (*) will guarantee that $\delta_s = \delta'$.

What remains is to define the required $\mu_j : Q' \to Q'$. We will define μ_j according to the type of 1-string $\check{g}_j$. Within each possible type of 1-string $\check{g}_j$, we will show the following:

(1) For $j = 0$, we show that we can always construct μ_0 to get $(Q'_0, \delta_0) \cong (Q'_1, \delta_1)$ such that $\delta_1 = \mu_0 \delta_0 \mu_0^{-1}$ satisfies (*).

(2) For $j \geq 1$, we show that, given $(Q', \delta_0) \cong \cdots \cong (Q'_j, \delta_j)$ with $\delta_1, \ldots, \delta_j$ satisfying (*) for $j \geq 1$, we can construct $\mu_j : Q' \to Q'$ such that $\delta_{j+1} = \mu_j \delta_j \mu_j^{-1}$ satisfies (*).

By an induction on the total number of 1-strings components (over j), we can always construct a chain of isomorphisms μ_j with property (*) and hence complete the proof of this theorem.

The rest of the proof is an extensive case-by-case analysis. The list below shows that, given each type of $\check{g}_j$, we can construct $\mu_j : Q' \to Q'$ satisfying (*). We will start with the simple cases: types IV, III'_k, and II'_k, followed by types V$''$, $\mathrm{III}'_{k+\frac{1}{2}}$, and $\mathrm{II}'_{k+\frac{1}{2}}$ that require some further analysis; refer to Figure 2.13 for the list of possible types of 1-strings. Within the list, we will omit the gradings when writing out the modules in Q' as it will be obvious from construction that μ_j preserves both the cohomological and internal gradings. We will also use solid arrows for differentials and dashed arrows for the isomorphism $\mu_j : Q' \to Q'$.

(1) $\check{g}_j$ is of type VI: this case is only possible when $j = 0$ and $g_j = c$. But in this case, (Q', δ_0) is given by $0 \to P_n^A \to 0$, which is already the complex (Q', δ') as required ($\delta_0 = 0 = \delta'$).

(2) $\check{g}_j$ is of type III'_k for $k \in \mathbb{Z}$: the case when $k = 0$ is straightforward, where we just pick μ_j to be the identity.

Now, consider the case when $k > 0$. If $j = 0$, we get that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$. For $j \geq 1$, δ_j satisfies (*) by our inductive assumption, so we have that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$. Thus, for all j, we can draw the part of (Q', δ_j) that contains

$$(Q'_{G_i}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$$

as follows:

$$\begin{array}{ccccccccccc} P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \xrightarrow{F} & P_{n-1}^A & \longleftrightarrow & \nabla \\ \oplus & & \oplus & & \oplus & & \oplus & \times & \oplus & \swarrow\!\!\nearrow & \\ P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow & P_{n+1}^A & & \end{array} = (Q', \delta_j),$$

where

$$F = \begin{bmatrix} (n|n-1) & -(n|n-1)i \\ (n|n+1) & (n|n+1)i \end{bmatrix} = \begin{bmatrix} (n|n-1) & 0 \\ 0 & (n|n+1) \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

and where ∇ denotes the rest of the complex (Q', δ_j) containing the modules complement to Q'_{G_j}. In particular, for $j = 0$, ∇ is the part of (Q', δ_0) that contains the module Q'_{W_0}; if $j \geq 1$, then this case is only possible when $j = s - 1$ and ∇ is the part of (Q', δ_0) that contains the module $Q'_{V_{s-1}}$. Nevertheless, the construction of μ_j below depends only on the form above, so the construction will work for all j. Denote

$$M := \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \tag{5.7}$$

and I as the 2×2 identity matrix. We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map, with μ_j acting as the identity map on the rest of the modules in ∇:

$$\begin{array}{ccccccccccc} P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \xrightarrow{F} & P_{n-1}^A & \longleftrightarrow & \nabla \\ \oplus & & \oplus & & \oplus & & \oplus & \times & \oplus & \swarrow\!\!\nearrow & \\ P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow & P_{n+1}^A & & \\ {\scriptstyle 2^{k-1}M}\,\vdots\downarrow & & \vdots\downarrow\,{\scriptstyle 2^{k-2}M} & & {\scriptstyle 2M}\,\vdots\downarrow & & {\scriptstyle M}\,\vdots\downarrow & & {\scriptstyle I}\,\vdots\downarrow & & \\ P_n^A & \xrightarrow{X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{X_n} & P_n^A & \xrightarrow{(n|n-1)} & P_{n-1}^A & \longleftrightarrow & \nabla \\ \oplus & & \oplus & & \oplus & & \oplus & & \oplus & \swarrow\!\!\nearrow & \\ P_n^A & \xrightarrow{X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{X_n} & P_n^A & \xrightarrow{(n|n+1)} & P_{n+1}^A & & \end{array} \begin{array}{l} = (Q', \delta_j) \\ \\ \\ =: (Q', \delta_{j+1}). \end{array}$$

The arrows in the last two rows show the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. Hence, the required condition (*) follows directly.

Now, consider when $k < 0$. As before, we have that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ for all j, so the analysis of the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$ will be the same. Similarly, the construction of μ_i below will work for both cases. We draw the part of (Q', δ_j) that contains

$$(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$$

as follows:

$$\begin{array}{ccccccccccccc}
 & & P^A_{n+1} & \xrightarrow{E} & P^A_n & \xrightarrow{2X_n} & P^A_n & \longrightarrow & \cdots & \longrightarrow & P^A_n & \xrightarrow{2X_n} & P^A_n \\
 & \swarrow\!\!\nearrow & \oplus & \times & \oplus & & \oplus & & & & \oplus & & \oplus & = (Q', \delta_j) \\
\nabla & \longleftrightarrow & P^A_{n-1} & \longrightarrow & P^A_n & \xrightarrow{2X_n} & P^A_n & \longrightarrow & \cdots & \longrightarrow & P^A_n & \xrightarrow{2X_n} & P^A_n
\end{array}$$

with

$$E = \begin{bmatrix} -(n+1|n)i & (n-1|n)i \\ (n+1|n) & (n-1|n) \end{bmatrix}$$

and where ∇ denotes the rest of the complex (Q', δ_j) containing the modules complement to Q'_{G_j}. Denote

$$N := \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \tag{5.8}$$

and I as the 2×2 identity matrix. Note that

$$N \begin{bmatrix} -(n+1|n)i & (n-1|n)i \\ (n+1|n) & (n-1|n) \end{bmatrix} = 2 \begin{bmatrix} (n+1|n) & 0 \\ 0 & (n-1|n) \end{bmatrix}.$$

We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map, with μ_j the identity map on all modules in ∇:

$$\begin{array}{ccccccccccccc}
 & & P^A_{n+1} & \xrightarrow{E} & P^A_n & \xrightarrow{2X_n} & P^A_n & \longrightarrow & \cdots & \longrightarrow & P^A_n & \xrightarrow{2X_n} & P^A_n \\
 & \swarrow\!\!\nearrow & \oplus & \times & \oplus & & \oplus & & & & \oplus & & \oplus & = (Q', \delta_j) \\
\nabla & \longleftrightarrow & P^A_{n-1} & \longrightarrow & P^A_n & \xrightarrow{2X_n} & P^A_n & \longrightarrow & \cdots & \longrightarrow & P^A_n & \xrightarrow{2X_n} & P^A_n \\
 & & \Big\downarrow{\scriptstyle I} & & \Big\downarrow{\scriptstyle 2^{-1}N} & & \Big\downarrow{\scriptstyle 2^{-2}N} & & & & \Big\downarrow{\scriptstyle 2^{k+1}N} & & \Big\downarrow{\scriptstyle 2^{k}N} \\
 & & P^A_{n+1} & \xrightarrow{(n+1|n)} & P^A_n & \xrightarrow{X_n} & P^A_n & \longrightarrow & \cdots & \longrightarrow & P^A_n & \xrightarrow{X_n} & P^A_n \\
 & \swarrow\!\!\nearrow & \oplus & & \oplus & & \oplus & & & & \oplus & & \oplus & =: (Q', \delta_{j+1}). \\
\nabla & \longleftrightarrow & P^A_{n-1} & \xrightarrow{(n-1|n)} & P^A_n & \xrightarrow{X_n} & P^A_n & \longrightarrow & \cdots & \longrightarrow & P^A_n & \xrightarrow{X_n} & P^A_n
\end{array}$$

The arrows in the last two rows show the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. It follows directly that the required condition (*) is satisfied.

(3) $\check{g}_j$ is of type II'_k for $k \in \mathbb{Z}$: as before, by our inductive assumption, we have that

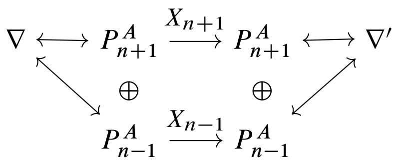
$$(\delta_j)_{G_j} = (\delta_0)_{G_j}$$

for all j, so the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$ will be of the same form. As such, the construction of μ_j below will work for all j.

We will start with $k = 0$. We draw the part of (Q', δ_j) that contains

$$(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$$

as follows:



$$= (Q', \delta_j),$$

where either ∇ or ∇' is the part of (Q', δ_j) that contains the module Q'_{V_j}, whereas the other contains Q'_{W_j}. However, in this case, we already have that

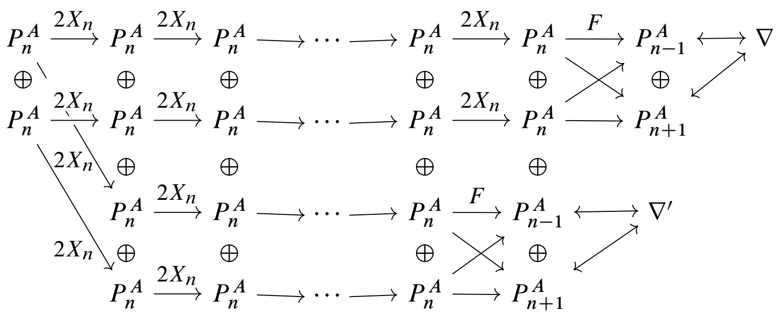
$$(\delta_j)_{G_j} = \delta'_{G_j};$$

thus, we just choose μ_j to be the identity map.

For $k > 0$, the part of (Q', δ_j) containing

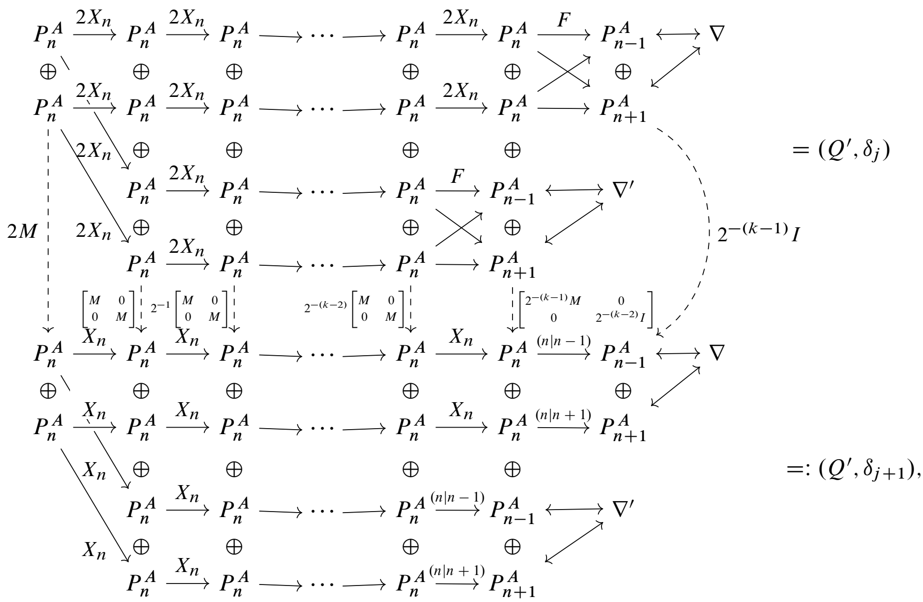
$$(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$$

is as follows:



$$= (Q', \delta_j),$$

where either ∇ or ∇' is the part of (Q', δ_j) that contains the module Q'_{V_j} and the other contains Q'_{W_j}. We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:



where M is as in (5.7). For the rest of the modules in Q', μ_j sends v to $2^{-(k-1)}v$ (resp., v to $2^{-(k-2)}v$) for any v belonging to the modules in ∇ (resp., ∇'). The black arrows in the last four rows show the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. It is easy to see that the required condition (*) is satisfied.

The construction for $k < 0$ is similar, using the map N (from (5.8)) in place of M.

(4) $\check{g}_j$ is of type V$''$: recall the definitions of g_j, c_j, and r_j from the paragraph Slicing c and recall the subsets of crossings of $\mathfrak{m}(\check{c})$ as defined in (5.2). Let h_j be the connected component of $(c \setminus g_j) \cup (g_j \cap d_2)$ that contains the point m so that g_j intersects h_j at the point r_j. To illustrate, $h_0 = c_0$ and $h_1 = c_0 \cup g_0 \cup c_1$ in Figure 5.2a, whereas $h_0 = \emptyset$, $h_1 = g_0 \cup c_1$, and $h_2 = g_0 \cup c_1 \cup g_1 \cup c_2$ in Figure 5.2b. Let $\mathfrak{m}(\check{h}_j) = \tilde{h}_j \amalg \underset{\sim}{h}_j$ and $\mathfrak{m}(r_j) = \tilde{r}_j \amalg \underset{\sim}{r}_j$ so that the curves of $\mathfrak{m}(\check{g}_j)$ and $\tilde{h}_j$ intersect at the point $\tilde{r}_j \in \theta_{n+1}$ and the curves of $\mathfrak{m}(\check{g}_j)$ and $\underset{\sim}{h}_j$ intersect at the point $\underset{\sim}{r}_j \in \theta_{n-1}$.

Now, recall the decomposition of Q' given in (5.6). By definition, $\{\tilde{r}_j, \underset{\sim}{r}_j\}$ is the subset of crossings $R_j \subseteq G_j$. We get that

$$P^A(\tilde{r}_j) \oplus P^A(\underset{\sim}{r}_j) = Q'_{R_j}.$$

First, consider the case when $j = 0$. Then, $g_j = g_0$ is of this type only when $m \in \Lambda \setminus \{0, 1\}$ since $g_0 \cap \{1\} = \emptyset$, so we have

$$Q'_{V_0 \oplus R_0} = Q'_{C_0}.$$

Furthermore, (5.5) implies that $(\delta_0)_{C_0} = \delta'_{C_0}$, giving us

$$(Q'_{V_0} \oplus Q'_{R_0}, (\delta_0)_{V_0 \oplus R_0}) = (Q'_{C_0}, (\delta_0)_{C_0}) = (Q'_{C_0}, \delta'_{C_0}) = L_A(\mathfrak{m}(\check{c}_0))$$

with the last equality following from the definition of $(Q', \delta') = L_A(\mathfrak{m}(\check{c}))$. By definition of h_j, it follows that $c_0 = h_0$, so we can conclude that

$$(Q'_{V_0} \oplus Q'_{R_0}, (\delta_j)_{V_0 \oplus R_0}) = L_A(\mathfrak{m}(\check{h}_0)) = L_A(\tilde{h}_0) \oplus L_A(\underset{\sim}{h_0}),$$

where the last equality follows from the fact that $\mathfrak{m}(\check{h}_0) = \tilde{h}_0 \amalg \underset{\sim}{h_0}$.

Now, consider when $j \geq 1$. In this case, δ_j satisfies property (*) by our induction hypothesis, allowing us to conclude that

$$(\delta_j)_{V_j} = \delta'_{V_j}, \quad (\delta_j)_{V_j R_j} = \delta'_{V_j R_j}, \quad \text{and} \quad (\delta_j)_{R_j V_j} = \delta'_{R_j V_j}.$$

Thus, we have that $(\delta_j)_{V_j \oplus R_j} = \delta'_{V_j \oplus R_j}$, giving us

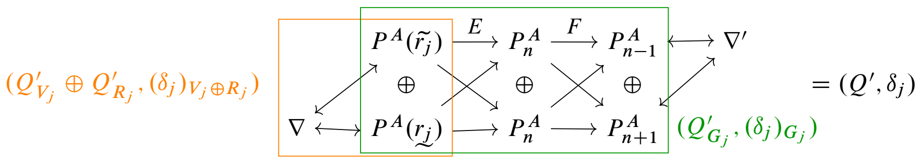
$$\begin{aligned}(Q'_{V_j} \oplus Q'_{R_j}, (\delta_j)_{V_j \oplus R_j}) = (Q'_{V_j} \oplus Q'_{R_j}, \delta'_{V_j \oplus R_j}) &= L_A(\mathfrak{m}(\check{h}_j)) \\ &= L_A(\tilde{h}_j) \oplus L_A(\underset{\sim}{h_j}),\end{aligned}$$

where the second equality follows from the definition of $(Q', \delta') = L_A(\mathfrak{m}(\check{c}))$ and the third equality follows from the fact that $\mathfrak{m}(\check{h}_j) = \tilde{h}_j \amalg \underset{\sim}{h_j}$.

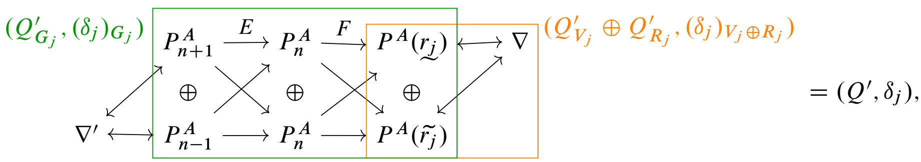
Thus, for all j, we obtain

$$(Q'_{V_j} \oplus Q'_{R_j}, (\delta_j)_{V_j \oplus R_j}) = L_A(\tilde{h}_j) \oplus L_A(\underset{\sim}{h_j}). \tag{5.9}$$

Furthermore, note that $\tilde{h}_j$ and $\underset{\sim}{h_j}$ contain the points $\tilde{r}_j$ and $\underset{\sim}{r_j}$, respectively, so $L_A(\tilde{h}_j)$ contains $P^A(\tilde{r}_j)$ as a submodule and $L_A(\underset{\sim}{h_j})$ contains $P^A(\underset{\sim}{r_j})$ as a submodule. Let us now understand the relation between $(Q'_{G_j}, (\delta_j)_{G_j})$, $(Q'_{V_j} \oplus Q'_{R_j}, (\delta_j)_{V_j \oplus R_j})$, and (Q', δ_j). As before, our inductive hypothesis gives us $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ for all j. So, the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$ will be the same for all j, and it has either of the following two forms:

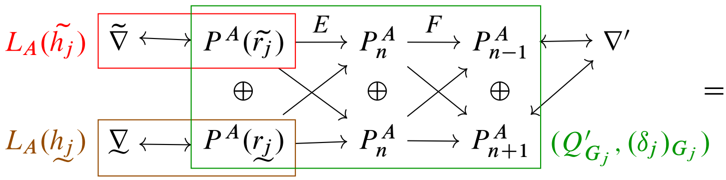


or



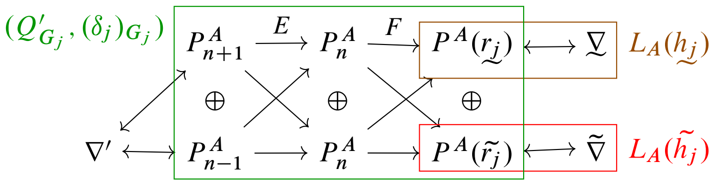
where ∇ denotes the rest of the complex (Q', δ_j) that contains the module Q'_{V_j} and ∇' denotes the rest of the complex (Q', δ_j) that contains the module Q'_{W_j}.

Using (5.9), the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$, $L_A(\widetilde{h}_j)$ and $L_A(\underset{\sim}{h_j})$ has either of the following two forms:



$$= (Q', \delta_j)$$

or



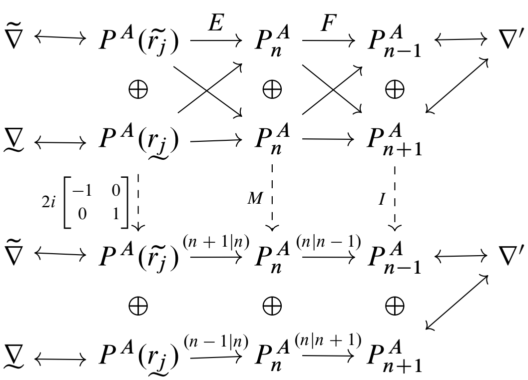
$$= (Q', \delta_j).$$

Thus, for all j, we conclude that (Q', δ_j) must be one of the above two possible forms. It is now sufficient to give a construction of μ_j for each of the two possible forms.

We begin with the construction of μ_j for the first possible form of (Q', δ_j). Firstly, recall M from (5.7), and note that

$$ME = 2i \begin{bmatrix} (n+1|n) & 0 \\ 0 & (n-1|n) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:



$$= (Q', \delta_j) \qquad =: (Q', \delta_{j+1}).$$

For the rest of the modules in Q', we define μ_j as the identity for modules contained in ∇', and μ_j sends v to $-2iv$ (resp., v to $2iv$) for any v belonging to the modules in $\widetilde{\nabla}$ (resp., $\underset{\sim}{\nabla}$). The black arrows in the last two rows show the differential component

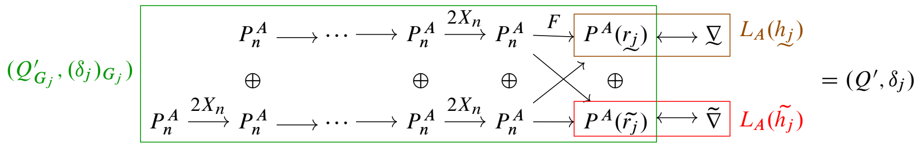
$(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. It is easy to see that δ_{j+1} does indeed satisfy the required property (*).

The construction of μ_j for the second form is similar, changing $\mu_j|_{P_n^A \oplus P_n^A}$ to N from (5.8) instead of M.

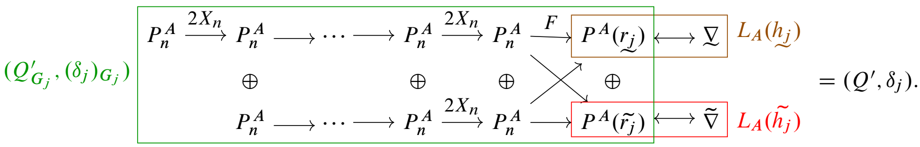
(5) $\check{g}_j$ is of type $\mathrm{III}'_{k+\frac{1}{2}}$ for $k \in Z$: note that the case for $k = 0$ is straightforward: μ_j is just the identity. We will provide the analysis and construction of μ_j for $k > 0$ and $k < 0$.

As in other types, the equation $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ holds for all j by the induction hypothesis. Using the same argument in type V″, one can show that (5.9) holds for all j in this type as well. So, the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$, $L_A(\tilde{h}_j)$ and $L_A(\underset{\sim}{h_j})$ will be of the same form for all j. It follows that the construction of μ_j below will work for all j.

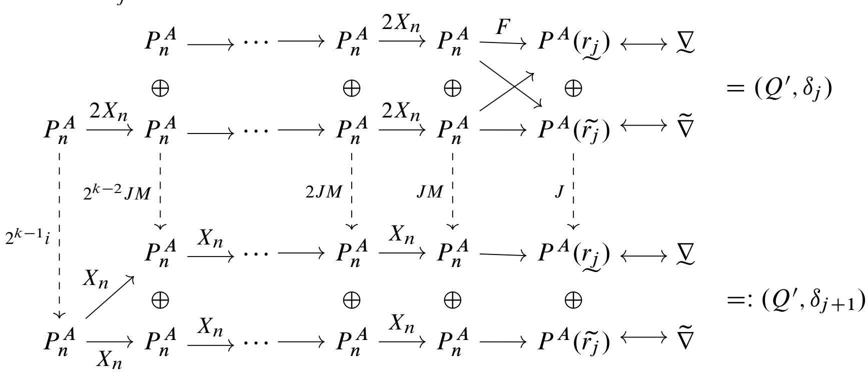
Let us start with $k > 0$. Using the same notation as in the analysis of type V″, we can draw the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$, $L_A(\tilde{h}_j)$ and $L_A(\underset{\sim}{h_j})$ as either of the two forms



or

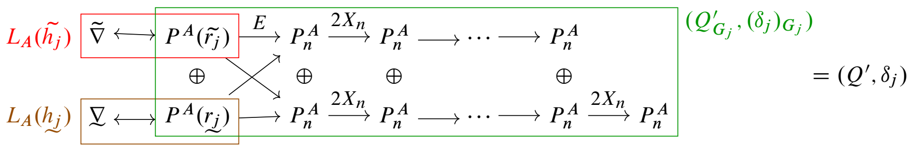


We will construct μ_j for the first form; the second form is a mirrored construction. Define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:

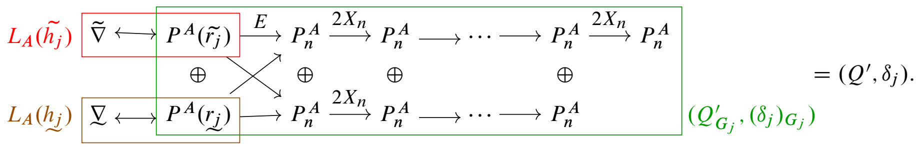


with $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. For the rest of the modules in Q', μ_j sends v to $-v$ (resp., v to v) for any v belonging to the modules in $\underset{\sim}{\nabla}$ (resp., $\tilde{\nabla}$). The black arrows in the last two rows show the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. It is easy to see that μ_j does indeed satisfy the required property (*).

Similarly, for $k<0$, we draw the part of (Q',δ_j) that contains $(Q'_{G_j},(\delta_j)_{G_j}) = (Q'_{G_j},(\delta_0)_{G_j})$, $L_A(\tilde{h}_j)$ and $L_A(\underset{\sim}{h_j})$ as either of the two forms



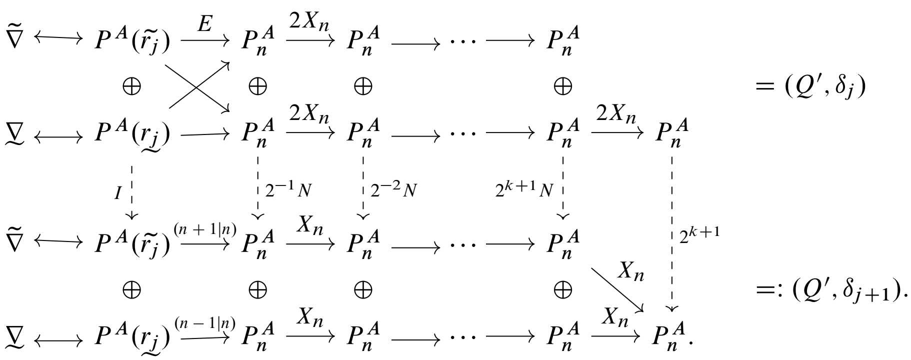
or



Once again, we construct μ_j for the first form; the second form is a mirrored construction. Note that

$$NE = 2\begin{bmatrix} (n+1|n) & 0 \\ 0 & (n-1|n) \end{bmatrix}.$$

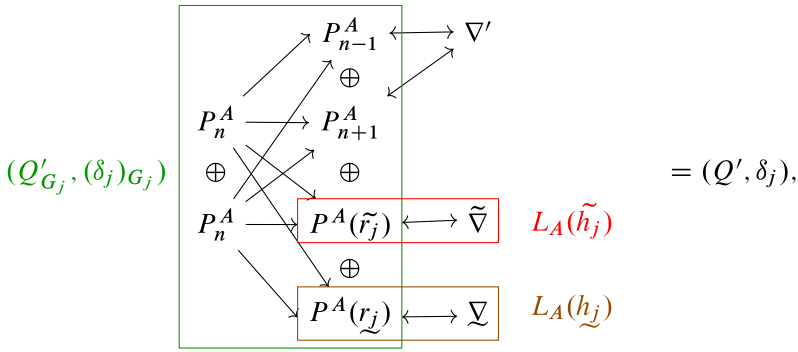
We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map, with μ_j acting as the identity on the rest of the modules in both $\tilde{\nabla}$ and $\underset{\sim}{\nabla}$:



The black arrows in the last two rows show the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. The required condition (*) follows directly.

(6) $\check{g}_j$ is of type $\mathrm{II}'_{k+\frac{1}{2}}$ for $k\in\mathbb{Z}$: as in other types, the equation $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ holds for all j by the induction hypothesis. Using the same argument in type V$''$, one can show that (5.9) holds for all j in this type as well. So, the part of (Q',δ_j) that contains $(Q'_{G_j},(\delta_j)_{G_j})$, $L_A(\tilde{h}_j)$, and $L_A(\underset{\sim}{h_j})$ will be of the same form for all j. It follows that the construction of μ_j below will work for all j.

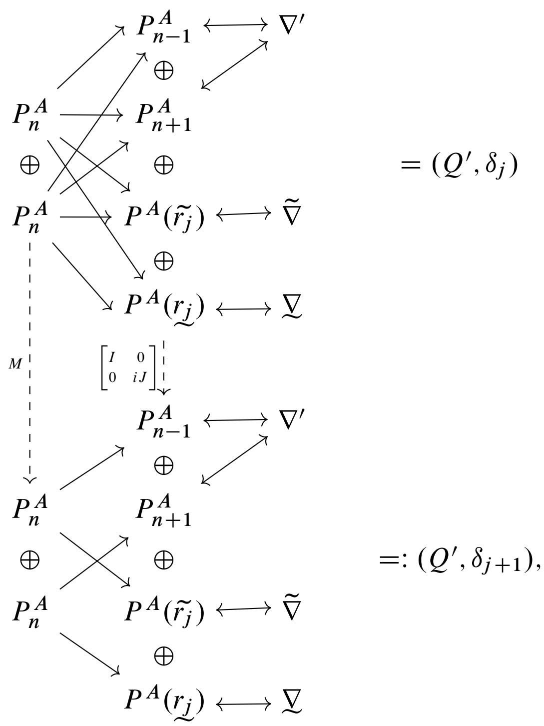
Let us start with $k=0$. With the same notation as in the analysis of type V$''$, we can draw the part of (Q',δ_j) that contains $(Q'_{G_j},(\delta_j)_{G_j})=(Q'_{G_j},(\delta_0)_{G_j})$, $L_A(\tilde{h}_j)$, and $L_A(\underset{\sim}{h_j})$ as follows:



where the first map is given by $\left[\begin{smallmatrix} F \\ F' \end{smallmatrix}\right]$, with F' defined by

$$\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} F' = \begin{bmatrix} (n|n+1) & 0 \\ 0 & (n|n-1) \end{bmatrix} M.$$

We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:

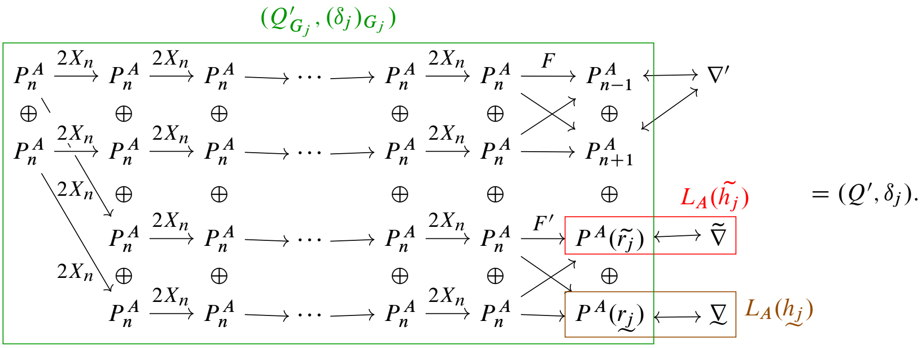


where $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, μ_j is the identity on all modules contained in ∇', and μ_j sends v to $-iv$ (resp., iv) for any v belonging to the modules in $\tilde{\nabla}$ (resp., $\underset{\sim}{\nabla}$). The black arrows in the last four rows show the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. Thus, the required condition (*) is easily satisfied.

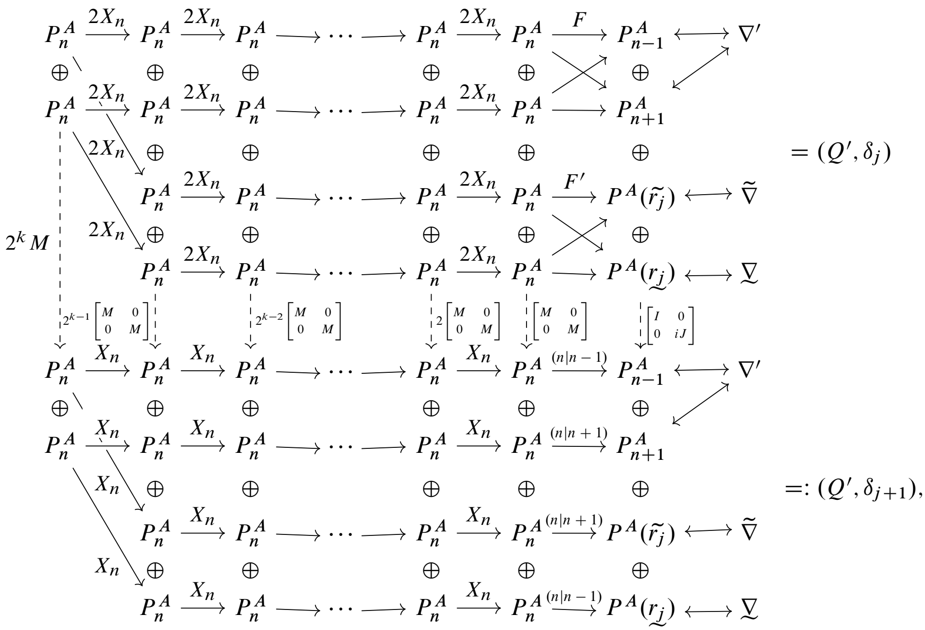
For $k > 0$, we can draw the part of (Q', δ_j) that contains

$$(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j}),$$

$L_A(\tilde{h}_j)$, and $L_A(\underset{\sim}{h_j})$ as follows:



Similarly, we define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:



where μ_j is the identity on all modules contained in ∇' and μ_j sends v to $-iv$ (resp., iv) for any v belonging to the modules in $\widetilde{\nabla}$ (resp., $\underset{\sim}{\nabla}$). The black arrows in the last four rows show the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. Thus, the required condition (*) is easily satisfied. The construction for $k < 0$ is similar, using the map N in place of M.

This completes the list of μ_j for all possible types of 1-string $\check{g}_j$ for all j, thus completing the proof. ∎

Using this result, we can now deduce a type B_n version of Theorem 5.2.

Theorem 5.7. *For $\sigma^B \in \mathcal{A}(B_n)$ and an admissible trigraded curve $\check{c}$ in $\mathbb{D}^B_{n+1}$, we have that*

$$\sigma^B(L_B(\check{c})) \cong L_B(\sigma^B(\check{c})),$$

in the category of $\mathrm{Kom}^b(\mathcal{B}_n\text{-p}_r\text{g}_r\text{mod})$*; i.e., the map L_B is $\mathcal{A}(B_n)$-equivariant.*

Proof. Let $\check{c}$ be a trigraded curve in $\mathbb{D}^B_{n+1}$ and σ^B be an element of $\mathcal{A}(B_n)$. By Proposition 5.5, the diagram in Figure 5.1 commutes. Together with the three other maps being equivariant, we can conclude that

$$\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} (L_B(\sigma^B(\check{c}))) \cong \mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} (\sigma^B(L_B(\check{c}))). \tag{5.10}$$

Recall that the functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ was defined as $\mathcal{A}_{2n-1} \otimes_{\ddot{\mathcal{B}}_n} \mathfrak{F}(-)$ (see the paragraph after Proposition 4.1 for the definitions). Since $\mathcal{A}_{2n-1} \cong \mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathcal{B}}_n$ as graded $\mathbb{C}$-algebras, (5.10) together with the fact that both categories $\mathrm{Kom}^b(\ddot{\mathcal{B}}_n\text{-p}_r\text{g}_r\text{mod})$ and $\mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$ are Krull–Schmidt implies that

$$L_B(\sigma^B(\check{c})) \cong \sigma^B(L_B(\check{c}))\langle s\rangle,$$

with $s = 0$ or 1.

We aim to show that $s = 0$ for all cases. First, consider the case when $c \cap \{0\} = \emptyset$. As $\check{c} \simeq \chi(r_1, r_2, r_3)\beta(\check{b}_2)$ and $P_2^B \cong P_2^B\langle 1\rangle$, it follows easily that $(L_B(\check{c}))\langle 0\rangle \cong (L_B(\check{c}))\langle 1\rangle$, and so, we are done. Now, consider when c has one of its endpoints at 0. Note that it is sufficient to prove the statement for $\sigma^B = \sigma_j^B$ for each j. We assign to each complex C in $\mathrm{Kom}^b(\mathcal{B}_n\text{-p}_r\text{g}_r\text{mod})$ an element of $\mathbb{Z}/2\mathbb{Z}$ denoted by $\mathrm{sgn}(C)$, by taking the sum of the third grading $\langle - \rangle$ over all modules P_1 in C. One can easily show that sgn is invariant under isomorphisms in $\mathrm{Kom}^b(\mathcal{B}_n\text{-p}_r\text{g}_r\text{mod})$, where using Lemma 5.8 below, we get that s must be 0 as required. ∎

Lemma 5.8. *For any trigraded curve $\check{c}$ with one of its endpoints at $\{0\}$ and any generating braid σ_j^B, we have*

$$\mathrm{sgn}(\sigma_j^B(L_B(\check{c}))) = \mathrm{sgn}(L_B(\sigma_j^B(\check{c}))).$$

Proof. For $j \geq 2$, it is clear that

$$\operatorname{sgn}(\sigma_j^B(L_B(\check{c}))) = \operatorname{sgn}(L_B(\check{c})) = \operatorname{sgn}(L_B(\sigma_j^B(\check{c}))).$$

Now, fix $j = 1$. First, consider the case when $\check{c}$ is of type VI, i.e.,

$$\check{c} = \chi^B(r_1, r_2, r_3)\check{b}_1.$$

Then, $\sigma_1^B(\check{c}) = \chi^B(r_1 - 1, r_2 + 1, r_3 + 1)\check{b}_1$ by Lemma 2.5 (2). So,

$$L_B(\sigma_1^B(\check{c})) = P_1^B[-r_1 + 1]\{r_2 + 1\}\langle r_3 + 1\rangle.$$

On the other hand,

$$\begin{aligned}\sigma_1^B(L_B((\check{c}))) &\cong (P_1^B[1]\{1\} \oplus P_1^B[1] \xrightarrow{[X_1\ \mathrm{id}]} P_1^B)[-r_1]\{r_2\}\langle r_3 + 1\rangle \\ &\cong P_1^B[-r_1 + 1]\{r_2 + 1\}\langle r_3 + 1\rangle.\end{aligned}$$

Thus, $\operatorname{sgn}(\sigma_j^B(L_B(\check{c}))) = r_3 + 1 = \operatorname{sgn}(L_B(\sigma_j^B(\check{c})))$.

Otherwise, we analyse sgn based on the number of 2-crossing in $\check{c}$. Note that, for 1-strings $\check{g}$,

$$\operatorname{sgn}(L_B(\sigma_1^B(\check{g}))) = \begin{cases} \operatorname{sgn}(L_B(\check{g})), & \text{when } \check{g} \text{ is of type } \mathrm{II}'_w,\ \mathrm{II}'_{w+\frac{1}{2}},\ \mathrm{III}'_{w+\frac{1}{2}}, \text{ and } \mathrm{V}'', \\ \operatorname{sgn}(L_B(\check{g})) + 1, & \text{when } \check{g} \text{ is of type } \mathrm{III}'_w. \end{cases}$$

Note that σ_1^B would not change the number of 2-crossings of $\check{c}$, and as $\check{c} \cap \{0\} = \{0\}$, $\check{c}$ contains 1-string of type III'_w if and only if $\check{c}$ has an even number of 2-crossings. Since $\operatorname{sgn}(L_B(\sigma_1^B(\check{c})))$ can be computed by summing over all 1-strings of $\check{c}$, we conclude that

$$\operatorname{sgn}(L_B(\sigma_1^B(\check{c}))) = \begin{cases} \operatorname{sgn}(L_B(\check{c})), & \text{if } \check{c} \text{ has an odd number of 2-crossings}, \\ \operatorname{sgn}(L_B(\check{c})) + 1, & \text{if } \check{c} \text{ has an even number of 2-crossings}. \end{cases}$$

On the other hand, note that $\check{c}$ and $\sigma_1^B(\check{c})$ both have an odd number of 1-crossings. Moreover, $\operatorname{sgn}(C\langle 1\rangle) = \operatorname{sgn}(C) + 1$ if and only if C has an odd number of underlying P_1^B's. As such, we have that

$$\operatorname{sgn}(\sigma_1^B(L_B(\check{c}))) = \begin{cases} \operatorname{sgn}(L_B(\check{c})), & \text{if } L_B(\check{c}) \text{ has an odd number of modules } P_2^B, \\ \operatorname{sgn}(L_B(\check{c})) + 1, & \text{if } L_B(\check{c}) \text{ has an even number of modules } P_2^B, \end{cases}$$

since $\sigma_1^B = R_1\langle 1\rangle$, ${}_1P^B \otimes_{\mathcal{B}_n} P_1^B \cong \mathbb{R} \oplus \mathbb{R}\{1\}$, ${}_1P^B \otimes_{\mathcal{B}_n} P_2^B \cong \mathbb{R} \oplus \mathbb{R}\langle 1\rangle$, and ${}_1P^B \otimes_{\mathcal{B}_n} P_j^B = 0$ for all $j \geq 3$ (see Proposition 3.5 and Remark 3.6). Thus, we get that $\operatorname{sgn}(\sigma_1^B(L_B(\check{c}))) = \operatorname{sgn}(L_B(\sigma_1^B(\check{c})))$. ■

6. Categorification of Homological representations

In this section, we will relate the categorical representations of type A_{2n-1} and type B_n Artin groups to their representations on the first homology of surfaces.

Throughout this section, we will use $\mathcal{K}_A := \text{Kom}^b(\mathcal{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$ and $\mathcal{K}_B := \text{Kom}^b(\mathcal{B}_n\text{-p}_r\text{g}_r\text{mod})$ as shorthand notations. We will also use $\mathcal{Z}_A$, $\mathcal{Z}_{B,s}$, and $\mathcal{Z}_{B,r}$ to denote the rings $\mathbb{Z}[q^{\pm 1}]$, $\mathbb{Z}[q^{\pm 1}, s]/\langle s^2 - 1\rangle$, and $\mathbb{Z}[q^{\pm 1}, r]/\langle r^2 - 1\rangle$, respectively. We denote the Grothendieck group of $\mathcal{K}_A$ and $\mathcal{K}_B$ as $K_0(\mathcal{K}_A)$ and $K_0(\mathcal{K}_B)$, respectively; recall that they are the abelian groups freely generated by the isomorphism classes of objects, quotient by the relation

$$\left[\text{cone}\left(A \xrightarrow{f} B\right)\right] = [B] - [A].$$

6.1. Representations on Grothendieck groups

First, consider the Grothendieck group $K_0(\mathcal{K}_A)$. The functor $\{1\}$ makes $K_0(\mathcal{K}_A)$ into a $\mathcal{Z}_A$-module defined by $[X\{1\}] = q[X]$. Note that $K_0(\mathcal{K}_A)$ is isomorphic to the split Grothendieck group $K_0^{\oplus}(\mathcal{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$ (see [19, Theorem 1.1]), so $K_0(\mathcal{K}_A)$ is a free module over $\mathcal{Z}_A$ of rank $2n-1$, generated by $\{[P_j^A] \mid 1 \leq j \leq 2n-1\}$. The action of $\mathcal{A}(A_{2n-1})$ on $\mathcal{K}_A$ preserves cones; namely, for all $\sigma \in \mathcal{A}(A_{2n-1})$,

$$\sigma\Big(\text{cone}\Big(A \xrightarrow{f} B\Big)\Big) \cong \text{cone}\Big(\sigma(A) \xrightarrow{\sigma(f)} \sigma(B)\Big).$$

Moreover, the action commutes with the grading shift functor, so we have an induced $\mathcal{Z}_A$-linear representation of $\mathcal{A}(A_{2n-1})$ on $K_0(\mathcal{K}_A)$, which we denote by

$$\rho_{KA} : \mathcal{A}(A_{2n-1}) \to \text{Aut}_{\mathcal{Z}_A}(K_0(\mathcal{K}_A)).$$

Now, consider the Grothendieck group $K_0(\mathcal{K}_B)$. The functors $\{1\}$ and $\langle 1\rangle$ make $K_0(\mathcal{K}_B)$ into a module over $\mathcal{Z}_{B,s}$. As $P_j^B\langle 1\rangle \cong P_j^B$ for all $j \geq 2$, we have that

$$s[P_j] = [P_j]$$

for all $j \geq 2$. It is easy to see now that

$$K_0(\mathcal{K}_B) \cong \mathcal{Z}_{B,s} \oplus (\mathcal{Z}_{B,s}/\langle s-1\rangle)^{\oplus n-1}$$

as $\mathcal{Z}_B$-modules, generated by $\{[P_j^B] \mid 1 \leq j \leq n\}$. As before, the action of $\mathcal{A}(B_n)$ on $\mathcal{K}_B$ preserves cones, and it commutes with the grading shift functors, so we have an induced $\mathcal{Z}_{B,s}$-linear representation of $\mathcal{A}(B_n)$ on $K_0(\mathcal{K}_B)$, which we denote by

$$\rho_{KB} : \mathcal{A}(B_n) \to \text{Aut}_{\mathcal{Z}_{B,s}}(K_0(\mathcal{K}_B)).$$

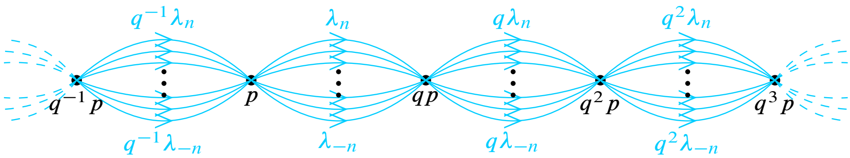


Figure 6.1. The infinite graph of $\Gamma_{\mathbb{Z}}$ homotopy equivalent to $\mathcal{D}_{2n}$.

6.2. Homological representations

It is well known that the reduced Burau representation of $\mathcal{A}(A_{2n-1})$ can be realised as a homological representation (see, for example, [13, Theorems 3.7 and 3.9]). Nonetheless, we will spell out the construction here as it will shed some light on the construction of the homological representation for type B_n and also clarify the relationship between them.

Consider the covering space $\mathcal{D}_{2n}$ classified by the cohomology class

$$C_{\mathcal{D}} \in H^1(\mathbb{D}^A_{2n} \setminus \Delta, \mathbb{Z})$$

defined by $[\lambda_k] \mapsto 1$for all $k \in \Delta = \{-n, \ldots, -1, 1, \ldots, n\}$, where each λ_k is a closed loop around the puncture k. It is easy to see that the space $\mathcal{D}_{2n}$ is homotopy equivalent to the infinite graph $\Gamma_{\mathbb{Z}}$ given in Figure 6.1. The action of $\mathcal{A}(A_{2n-1})$ on $\mathbb{D}^A_{2n} \setminus \Delta$ lifts to an action on $\mathcal{D}_{2n}$ that commutes with the deck transformation group $\mathbb{Z}$, so it induces a $\mathbb{Z}[\mathbb{Z}]$-linear action of $\mathcal{A}(A_{2n-1})$ on $H_1(\mathcal{D}_{2n}, \mathbb{Z})$, which we denote by

$$\rho_{RHA} : \mathcal{A}(A_{2n-1}) \to \mathrm{Aut}_{\mathbb{Z}[\mathbb{Z}]}(H_1(\mathcal{D}_{2n}, \mathbb{Z})).$$

Now, let

$$\Delta_0 := \Delta \cup \{0\},$$

and consider the covering space $\mathfrak{D}_{2n+1}$ of $\mathbb{D}^A_{2n} \setminus \Delta_0$ classified by the cohomology class $C_{\mathfrak{D}} \in H^1(\mathbb{D}^A_{2n} \setminus \Delta_0, \mathbb{Z})$ defined by

$$[\lambda_j] \mapsto \begin{cases} 1, & \text{for } j \neq 0, \\ 0, & \text{for } j = 0, \end{cases}$$

where each λ_j is a closed loop around the puncture j. Note that the composition of coverings

$$\mathfrak{D}_{2n+1} \to \mathbb{D}^A_{2n} \setminus \Delta_0 \to \mathbb{D}^B_{n+1} \setminus \Lambda$$

is a normal covering space of $\mathbb{D}^B_{n+1} \setminus \Lambda$, with its group of deck transformations isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (cf. Lemma 2.2).

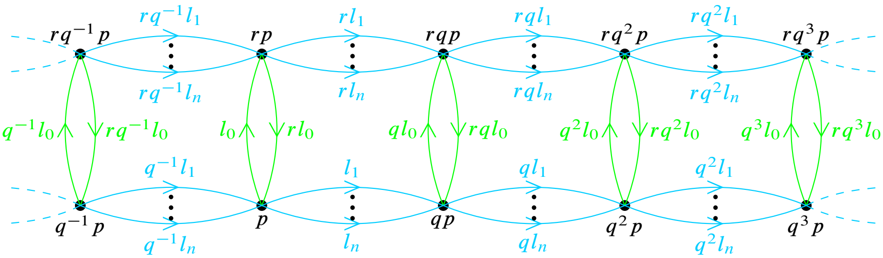


Figure 6.2. The infinite graph $\Gamma_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ that is homotopy equivalent to $\mathcal{D}_{2n+1}$.

Let l_j be a closed loop around each puncture $j \in \Lambda$ of $\mathbb{D}^B_{n+1}$. Note that the following equations hold in $H_1(\mathbb{D}^A_{2n} \setminus \Delta_0, \mathbb{Z})$:

$$\begin{cases} [\lambda_0] = [l_0] + r[l_0], & \\ [\lambda_{-j}] = r[l_j], & \text{for } j > 0, \\ [\lambda_j] = [l_j], & \text{for } j > 0. \end{cases} \tag{6.1}$$

As such, the space $\mathcal{D}_{2n+1}$ is homotopy equivalent to the graph given in Figure 6.2.

The action of $\mathcal{A}(B_n)$ on $\mathbb{D}^B_{n+1} \setminus \Lambda$ lifts to an action on $\mathcal{D}_{2n+1}$ that commutes with deck transformation group

$$\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}] \cong \mathcal{Z}_{B,r},$$

so it induces a $\mathcal{Z}_{B,r}$-linear action on $H_1(\mathcal{D}_{2n+1}, \mathbb{Z})$, which we denote by

$$\widetilde{\rho_{RHB}} : \mathcal{A}(B_n) \to \mathrm{Aut}_{\mathcal{Z}_{B,r}}(H_1(\mathcal{D}_{2n+1}, \mathbb{Z})).$$

6.3. Relating the representations on Grothedieck groups and homology groups

It has been shown in [15, Section 2e.1] that the induced action of $\mathcal{A}(A_{2n-1})$ on the Grothendieck group $K_0(\mathcal{K}_A)$ is isomorphic to the reduced Burau representation, but we will spell it out here before dealing with the type B case. Recall that $\mathcal{H}_n$ and $K_0(\mathcal{K}_B)$ are modules over $\mathcal{Z}_{B,r}$ and $\mathcal{Z}_{B,s}$, respectively, whereas $H_1(\mathcal{D}_{2n}, \mathbb{Z})$ and $K_0(\mathcal{K}_A)$ are modules over $\mathcal{Z}_A$.

Proposition 6.1. *The two $\mathcal{Z}_A$-linear representations*

$$\rho_{KA} : \mathcal{A}(A_{2n-1}) \to \mathrm{Aut}_{\mathcal{Z}_A}(K_0(\mathcal{K}_A))$$

and

$$\rho_{RHA} : \mathcal{A}(A_{2n-1}) \to \mathrm{Aut}_{\mathbb{Z}[\mathbb{Z}]}(H_1(\mathcal{D}_{2n}, \mathbb{Z}))$$

are isomorphic. In particular, the categorical action of $\mathcal{A}(A_{2n-1})$ on $\mathcal{K}_A$ categorifies the reduced Burau representation.

Proof. One can check that $H_1(\mathcal{D}_{2n}, \mathbb{Z})$ is a free module over $\mathbb{Z}[\mathbb{Z}] \cong \mathcal{Z}_A$ of rank $2n-1$, with basis $\{[\gamma_1], \dots, [\gamma_{2n-1}]\}$ defined by

$$[\gamma_j] := \begin{cases} [\lambda_{-1}] - [\lambda_1], & \text{for } j = n, \\ (-1)^{n-j}([\lambda_{-n+j-1}] - [\lambda_{-n+j}]), & \text{for } j \le n-1, \\ (-q)^{n-j}([\lambda_{-n+j}] - [\lambda_{-n+j+1}]), & \text{for } j \ge n+1. \end{cases} \tag{6.2}$$

Similarly, $\{P_1^A, P_2^A, \dots, P_{2n-1}^A\}$ is a $\mathcal{Z}_A$-basis for $K_0(\mathcal{K}_A)$.

Under both of these bases, both ρ_{KA} and ρ_{RHA} are given by the same matrices for each generator of $\mathcal{A}(A_{2n-1})$ as follows:

$$\sigma_1^A \mapsto \begin{bmatrix} -q & -q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}, \quad \sigma_{2n-1}^A \mapsto \begin{bmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & -q \end{bmatrix},$$

$$\sigma_j^A \mapsto \begin{cases} \begin{bmatrix} I_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -q & -q & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}, & \text{for } 2 \le j \le n-1; \\ \begin{bmatrix} I_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -q & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}, & \text{for } j = n; \\ \begin{bmatrix} I_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -q & -q & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}, & \text{for } n+1 \le j \le 2n-2. \end{cases}$$

It follows that ρ_{KA} and ρ_{RHA} are isomorphic representations. ∎

Proposition 6.2. *Under the identification $\mathcal{Z}_{B,s} \cong \mathcal{Z}_{B,r}$ given by $s \mapsto -r$ and $q \mapsto q$, the $\mathcal{Z}_{B,s}$-linear representation $\rho_{KB} : \mathcal{A}(B_n) \to \operatorname{Aut}_{\mathcal{Z}_{B,s}}(K_0(\mathcal{K}_B))$ is isomorphic to a $\mathcal{Z}_{B,s}$-linear subrepresentation of $\widetilde{\rho_{RHB}} : \mathcal{A}(B_n) \to \operatorname{Aut}_{\mathcal{Z}_{B,s}}(H_1(\mathcal{D}_{2n+1}, \mathbb{Z}))$.*

Proof. Consider the sub $\mathcal{Z}_{B,r}$-module

$$\mathcal{H}_n \subseteq H_1(\mathcal{D}_{2n+1}, \mathbb{Z})$$

generated by $\{[\xi_1], \dots, [\xi_n]\}$, where

$$[\xi_j] = \begin{cases} (1-q)[l_0] - (1-r)[l_1], & \text{for } j = 1, \\ (-q)^{1-j}(1-r)([l_{j-1}] - [l_j]), & \text{for } j \geq 2. \end{cases}$$

Note that $[\xi_j] = -r[\xi_j]$ for all $j \geq 2$ so that

$$\mathcal{H}_n \cong \mathcal{Z}_{B,r} \oplus (\mathcal{Z}_{B,r}/\langle r+1\rangle)^{\oplus n-1}$$

as $\mathcal{Z}_{B,r}$-modules.

It is easy to verify on the generators that $\mathcal{H}_n \subseteq H_1(\mathcal{D}_{2n+1}, \mathbb{Z})$ is closed under the action of $\mathcal{A}(B_n)$, so we obtain a $\mathcal{Z}_{B,r}$-linear subrepresentation on $\mathcal{H}_n$, which we denote by $\rho_{RHB} : \mathcal{A}(B_n) \to \mathrm{Aut}_{\mathcal{Z}_{B,r}}(\mathcal{H}_n)$.

Using the set of generators $\{P_1^B, P_2^B, \dots, P_n^B\}$ for $\mathcal{K}_B$, ρ_{KB} is given by the following matrices for each generators of $\mathcal{A}(B_n)$:

$$\rho_{KB}(\sigma_1^B) = \begin{bmatrix} -sq & -(1+s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}, \quad \rho_{KB}(\sigma_n^B) = \begin{bmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & -q \end{bmatrix},$$

$$\rho_{KB}(\sigma_j^B) = \begin{bmatrix} I_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -q & -q & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}, \quad \text{for } j \neq 1, n.$$

One can check that, using the set of generators $\{[\xi_1], \dots, [\xi_n]\}$ for $\mathcal{H}_n$, ρ_{RHB} is defined by associating the above matrices to the generators of $\mathcal{A}(B_n)$ with $s = -r$. Thus, under the identification $\mathcal{Z}_{B,s} \cong \mathcal{Z}_{B,r}$ given by $s \mapsto -r$ and $q \mapsto q$, the two representations are isomorphic. ■

Denote $ev_{\pm 1} : \mathcal{Z}_{B,s} \to \mathcal{Z}_A$ as the $\mathbb{Z}[q^{\pm 1}]$-linear evaluation maps defined by $s \mapsto \pm 1$. Throughout the rest of this section, we will view $H_1(\mathcal{D}_{2n}, \mathbb{Z})$ as a $\mathcal{Z}_{B,s}$-module through ev_{-1} and $K_0(\mathcal{K}_A)$ as a $\mathcal{Z}_{B,s}$-module through ev_1.

The functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ as in Proposition 4.2 preserves cones; hence, it induces a map on the Grothendieck groups $K_0(\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -) : K_0(\mathcal{K}_B) \to K_0(\mathcal{K}_A)$, given by

$$K_0(\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -)([P_j^B]) = \begin{cases} [P_n^A], & \text{for } j = 1, \\ [P_{n-(j-1)}^A] + [P_{n+(j-1)}^A], & \text{otherwise.} \end{cases}$$

On the other hand, the natural inclusion map $\iota : \mathbb{D}^A_{2n} \setminus \Delta_0 \to \mathbb{D}^A_{2n} \setminus \Delta$ induces a map on the homology

$$\iota : H_1(\mathbb{D}^A_{2n} \setminus \Delta_0, \mathbb{Z}) \to H_1(\mathbb{D}^A_{2n} \setminus \Delta, \mathbb{Z})$$

that sends

$$[\lambda_j] \mapsto \begin{cases} 0, & \text{for } j = 0, \\ [\lambda_j], & \text{otherwise.} \end{cases}$$

Thus, ι lifts uniquely to $\widetilde{\iota} : \mathcal{D}_{2n+1} \to \mathscr{D}_{2n}$, which induces a map on the homology $\widetilde{\iota} : H_1(\mathcal{D}_{2n+1}, \mathbb{Z}) \to H_1(\mathscr{D}_{2n}, \mathbb{Z})$. We will now show a "decategorified" analogue of Theorem 5.1.

Theorem 6.3. *Consider the action of $\mathcal{A}(B_n)$ on $\mathcal{H}_n$ and $K_0(\mathcal{K}_B)$ given by $\rho_{RHB} = \widetilde{\rho_{RHB}}|_{\mathcal{H}_n}$ and ρ_{KB}, respectively, as in Proposition 6.2; similarly, consider the action of $\mathcal{A}(B_n) \overset{\Psi}{\hookrightarrow} \mathcal{A}(A_{2n-1})$ on $H_1(\mathscr{D}_{2n}, \mathbb{Z})$ and $K_0(\mathcal{K}_A)$ given by ρ_{RHA} and ρ_{KA}, respectively, as in Proposition 6.1. Then, there exist $\mathcal{Z}_{B,s}$-linear isomorphisms Θ_A and Θ_B such that the following diagram is commutative with all four maps $\mathcal{Z}_{B,s}$-linear and $\mathcal{A}(B_n)$-equivariant:*

$$\begin{array}{ccc}
\mathcal{A}(B_n) \circlearrowright \mathcal{H}_n & \xrightarrow{\quad \widetilde{\iota} \quad} & H_1(\mathscr{D}_{2n}, \mathbb{Z}) \circlearrowleft \mathcal{A}(A_{2n-1}) \overset{\Psi}{\hookleftarrow} \mathcal{A}(B_n) \\
\cong \big\downarrow \Theta_B & & \cong \big\downarrow \Theta_A \\
\mathcal{A}(B_n) \circlearrowright K_0(\mathcal{K}_B) & \xrightarrow{K_0(\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -)} & K_0(\mathcal{K}_A) \circlearrowleft \mathcal{A}(A_{2n-1}) \overset{\Psi}{\hookleftarrow} \mathcal{A}(B_n)
\end{array}.$$

Proof. We use isomorphisms Θ_A and Θ_B which identify the respective generators chosen in the proof of Proposition 6.1 and Proposition 6.2, respectively.

The functor $\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -$ commutes with the grading functor, so $K_0(\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -)$ is $\mathcal{Z}_{B,s}$ linear. The fact that it is $\mathcal{A}(B_n)$-equivariant follows from Theorem 4.3. On the other hand, it follows immediately from the construction of the covering spaces $\mathcal{D}_{2n+1}$ and $\mathscr{D}_{2n}$ that $\widetilde{\iota}$ is a $\mathcal{Z}_{B,s}$-linear map and is $\mathcal{A}(B_n)$-equivariant.

Using (6.1) and (6.2), the restriction of $\widetilde{\iota}$ to $\mathcal{H}_n$ is given by

$$\widetilde{\iota}([\xi_j]) = \begin{cases} [\gamma_n], & \text{for } j = 1, \\ [\gamma_{n-(j-1)}] + [\gamma_{n+(j-1)}], & \text{for } j \geq 2. \end{cases}$$

It now follows immediately that $\Theta_A \circ \widetilde{\iota} = K_0(\mathcal{A}_{2n-1} \otimes_{\mathcal{B}_n} -) \circ \Theta_B$. ∎

7. Trigraded intersection numbers and graded dimensions of homomorphism spaces

In this section, we will relate the trigraded intersection number and the Hom spaces between the corresponding complexes. Throughout this section, we will fix the following shorthand notations:

$$\mathcal{K}_B := \mathrm{Kom}^b(\mathcal{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod}), \quad \mathcal{K}_A := \mathrm{Kom}^b(\mathcal{A}_{2n-1}\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod}),$$
$$\mathcal{B}m := \mathcal{B}_n\text{-mod}, \qquad \mathcal{A}m := \mathcal{A}_{2n-1}\text{-mod}.$$

For $V = \bigoplus_{(r,s)\in\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} V_{(r,s)}\{r\}\langle s\rangle$ a $(\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{R}$-vector space, we denote its *bigraded dimension* as

$$\mathrm{bigrdim}(V) := \sum_{(r,s)\in\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \dim(V_{(r,s)})q_2^r q_3^s.$$

Recall that, for each pair of objects $(C^*, \partial_C), (D^*, \partial_D)$ in $\mathcal{K}_B$, one can consider the internal (bigraded) Hom complex $\mathrm{HOM}^*_{\mathcal{K}_B}(C, D)$ defined as follows: for each cohomological degree $s_1 \in \mathbb{Z}$,

$$\mathrm{HOM}^{s_1}_{\mathcal{K}_B}(C, D) := \bigoplus_{\substack{(s_2,s_3)\in\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\\ m+n=s_1}} \mathrm{Hom}_{\mathcal{B}m}(C^m, D^n\{s_2\}\langle s_3\rangle)\{-s_2\}\langle s_3\rangle$$

is a $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{R}$-vector space and the differentials are given by

$$d(f) = \partial_D \circ f - (-1)^{s_1} f \circ \partial_C$$

for each $f \in \mathrm{HOM}^{s_1}_{\mathcal{K}_B}(C, D)$. It follows that each $H^n(\mathrm{HOM}^*_{\mathcal{K}_B}(C, D))$ is a $(\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{R}$-vector space. We define the *Poincaré polynomial*

$$\mathfrak{P}(C, D) \in \mathbb{Z}[q_1, q_1^{-1}, q_2, q_2^{-1}, q_3]/\langle q_3^2 - 1\rangle$$

of $\mathrm{HOM}^*_{\mathcal{K}_B}(C, D)$ as

$$\mathfrak{P}(C, D) := \sum_{s_1\in\mathbb{Z}} q_1^{s_1}\, \mathrm{bigrdim}_{\mathbb{R}}(H^{s_1}(\mathrm{HOM}^*_{\mathcal{K}_B}(C, D))).$$

Lemma 7.1. *For any trigraded admissible curve $\check{c}$, the following internal Hom complexes are quasi-isomorphic:*

$$(\mathrm{HOM}^*_{\mathcal{K}_B}(P_j^B, L_B(\check{c})), d_C^*) \cong \bigoplus_{\check{g}\in\mathrm{st}(\check{c},j)} (\mathrm{HOM}^*_{\mathcal{K}_B}(P_j^B, L_B(\check{g})), d_G^*),$$

for all $1 \le j \le n$ and $(s_1, s_2, s_3) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

Proof. To simplify notation, denote

$$(C^*, \partial_C^*) := L_B(\check{c}) \quad \text{and} \quad (G^*, \partial_G^*) := \bigoplus_{\check{g} \in \mathrm{st}(\check{c}, j)} L_B(\check{g}).$$

Note that G^* can be obtained from C^* by discarding the modules P_k^B in $L_B(\check{c})$ for $|k - j| > 1$. In particular, for all $m \in \mathbb{Z}$, $C^m = G^m \oplus U^m$, where U^m consists of all indecomposable P_k^B in C^m with $|k - j| > 1$. Using the decomposition above, let us write $\partial_C^m : C^m = G^m \oplus U^m \to G^{m+1} \oplus U^{m+1} = C^{m+1}$ as

$$\partial_C^m = \begin{bmatrix} \tau^m & * \\ * & * \end{bmatrix}$$

so that $\tau^m : G^m \to G^{m+1}$. Also note that the differential

$$\partial_G^m : G^m \to G^{m+1}$$

can be obtained from τ^m by modifying the differentials $P^B(x) \xrightarrow{\partial_{yx}} P^B(y)$ to 0 whenever x and y are crossings of two different j-strings of $\check{c}$. Since

$$\bigoplus_{(s_1,s_2,s_3) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}m}(P_j^B, P_k^B[s_1]\{s_2\}\langle s_3\rangle) = 0$$

for all k such that $|j - k| > 1$, it follows that

$$\bigoplus_{(s_1,s_2,s_3) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}m}(P_j^B, U^{s_1}\{s_2\}\langle s_3\rangle) = 0, \tag{7.1}$$

and thus,

$$\mathrm{HOM}^m_{\mathcal{K}_B}(P_j^B, C^*) = \mathrm{HOM}^m_{\mathcal{K}_B}(P_j^B, G^*)$$

for each $m \in \mathbb{Z}$ as underlying graded vector space. Moreover, we know that $d_G^m = (\partial_G^m \circ -)$ and $d_C^m = (\partial_C^m \circ -)$ by the definition of the HOM complex. But (7.1) allows us to conclude that

$$d_C^m = (\tau^m \circ -).$$

Therefore, to prove the proposition, it is sufficient to show that $d_C^m = (\tau^m \circ -)$ and $d_G^m = (\partial_G^m \circ -)$ have isomorphic kernels and isomorphic images for each $m \in \mathbb{Z}$. For the rest of the proof, let $m \in \mathbb{Z}$ be arbitrary.

Let us first consider the simple case when $j \neq 2$. We claim that

$$d_C^m = d_G^m.$$

Note that when $j \neq 2$, ∂_{yx} in τ^m that are modifed to 0 in ∂_G^m are always right multiplication by loops X_{j-1} or X_{j+1}. But for such maps, the corresponding induced maps on the HOM complex $(\partial_{yx} \circ -)$ are always 0, so $(\tau^m \circ -) = (\partial_G^m \circ -)$ as required.

Now, let us consider the case when $j = 2$. The types of maps

$$\partial_{yx} : P(x) \to P(y)$$

in τ^m that are modifed to 0 in ∂_G^m are of the following types:

(i) $\partial_{yx} = X_1$ or X_3;

(ii) $\partial_{yx} = (1|2)i$ or $\partial_{yx} = -i(2|1)$.

Moreover, ∂_{yx} of type (ii) does not exist in ∂_G^m by the definition of L_B. By the same argument in the case $j \neq 2$, the induced differential in the HOM complex by ∂_{yx} of type (i) is 0. So, we can relate d_C^m and d_G^m as follows:

$$d_C^m = (\tau^m \circ -) = (\partial_G^m \circ -) + (\delta \circ -) = d_G^m + (\delta \circ -), \tag{7.2}$$

where $\delta := \sum \partial_{yx}$, summing over all ∂_{yx}'s in τ^m that are of type (ii).

Before we analyse the kernel and image of both d_C^m and d_G^m, we will consider a decomposition of G^m and G^{m+1} using τ^m. Denote $\mathfrak{G}^m$ and $\mathfrak{G}^{m+1}$ as the subsets of all crossings of $\check{c}$ such that

$$G^m = \bigoplus_{z \in \mathfrak{G}^m} P(z) \quad \text{and} \quad G^{m+1} = \bigoplus_{z \in \mathfrak{G}^{m+1}} P(z).$$

We will reorganise the direct summands of G^m and G^{m+1} in the following way:

(1) Set
- $\alpha = 1$,
- $\epsilon := \delta$,
- $X := \mathfrak{G}^m$,
- $H^m := G^m$,
- $Y = \mathfrak{G}^{m+1}$,
- $H^{m+1} = G^{m+1}$.

(2) If $\epsilon = 0$, then skip to step (3); otherwise, let ∂_{yx} be one of the summands in δ. Consider the smallest subset $X' \subseteq X$ and $Y' \subseteq Y$ such that
- $x \in X'$,
- $y \in Y'$,
- $\partial_{zw} = 0$, X_1, or X_3 whenever $w \in (X')^c$, $z \in Y'$ or $w \in X'$, $z \in (Y')^c$.

We organise the direct summands of H^m in the following way:

$$H^m = Q_\alpha^m \oplus \left(\bigoplus_{x \in (X')^c} P(x) \right) \quad \text{and} \quad H^{m+1} = Q_\alpha^{m+1} \oplus \left(\bigoplus_{y \in (Y')^c} P(y) \right),$$

where $Q^m_\alpha := \bigoplus_{x\in X'} P(x)$ and $Q^{m+1}_\alpha := \bigoplus_{y\in Y'} P(y)$. Let $e = \sum \partial_{yx}$, summing over all $\partial_{yx} = (1|2)i, -i(2|1)$ with $x \in X'$ and $y \in Y'$.
Redefine

- $\alpha := \alpha + 1$,
- $\epsilon := \epsilon - \gamma$,
- $H^m := \bigoplus_{x\in(X')^c} P(x)$,
- $H^{m+1} := \bigoplus_{y\in(Y')^c} P(y)$.

Repeat step (2).

(3) If $H^m \neq 0$, then set $Q^m_\alpha := H^m$; else, if $H^{m+1} \neq 0$, then set $Q^{m+1}_\alpha := H^{m+1}$.

(4) Output $G^m = \bigoplus_{s\in S} Q^m_s$ and $G^{m+1} = \bigoplus_{s'\in S'} Q^m_{s'}$ with the appropriate index sets $S = \{1, \dots, M\}$ and $S' = \{1, \dots, M'\}$.

Now, consider τ^m and ∂^m_G as block matrices corresponding to the decomposition obtained above

$$\tau^m = [(\tau^m)_{s',s}]_{(s',s)\in S'\times S}, \quad \partial^m_G = [(\partial^m_G)_{s',s}]_{(s',s)\in S'\times S}.$$

Note that, by the construction of the decomposition, we have that the block $(\tau^m)_{s',s}$ only has entries X_1, X_3, or 0 for all $s \neq s'$. On the HOM complexes, the decompositions also give us

$$\mathrm{HOM}^m_{\mathcal{K}_B}(P^B_j, C^*) = \mathrm{HOM}^m_{\mathcal{K}_B}(P^B_j, G^*) = \bigoplus_{s\in S} \mathrm{Hom}_{\mathcal{B}^m}(P^B_j, Q^m_s),$$

and

$$\mathrm{HOM}^{m+1}_{\mathcal{K}_B}(P^B_j, C^*) = \mathrm{HOM}^{m+1}_{\mathcal{K}_B}(P^B_j, G^*) = \bigoplus_{s'\in S'} \mathrm{Hom}_{\mathcal{B}^m}(P^B_j, Q^{m+1}_{s'}).$$

Similarly, consider the two differentials d^m_C and d^m_G written as block matrices corresponding to the decompositions

$$d^m_C = [(d^m_C)_{s',s}]_{(s',s)\in S'\times S}, \quad d^m_G = [(d^m_G)_{s',s}]_{(s',s)\in S'\times S}.$$

The construction of the decomposition guarantees the property that

$$(d^m_C)_{s',s} = (d^m_G)_{s',s} = 0$$

whenever $s \neq s'$ (recall that the induced maps $(X_1 \circ -)$ and $(X_3 \circ -)$ are 0). So, to show that $d^m_C = (\tau^m \circ -)$ and $d^m_G = (\partial^m_G \circ -)$ have isomorphic images and isomorphic kernels, it is now sufficient to show them for each block $(d^m_C)_{s',s} = ((\tau^m)_{s',s} \circ -)$ and $(d^m_G)_{s',s} = ((\partial^m_G)_{s',s} \circ -)$, where $s = s'$.

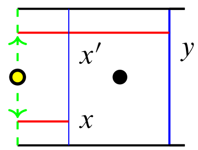


Figure 7.1. The crossings x, x', and y of the partial curve of $\check{c}$.

To simplify notation, for the rest of this proof, we will drop the subscript s and denote

$$Q^m := Q^m_s, \qquad Q^{m+1} := Q^{m+1}_s, \qquad d_C := (d^m_C)_{s,s},$$
$$d_G := (d^m_G)_{s,s}, \qquad \tau := \tau^m_{s,s}, \qquad \text{and} \quad \partial_G := (\partial^m_G)_{s,s}.$$

We will now look at the possible types of maps $\tau : Q^m \to Q^{m+1}$ which give us all possible $d_C = (\tau \circ -)$, where $d_G = (\partial_G \circ -)$ can be obtained by

$$d_G = (\tau \circ -) - (\delta \circ -)$$

(following from (7.2)).

If $d_C = d_G$, i.e., τ has no entry of type (ii) so that $\delta = 0$, then there is nothing left to show. Otherwise, τ contains at least one entry ∂_{yx} of type (ii). The two possibilities of ∂_{yx} of type (ii) are $(1|2)i$ and $-i(2|1)$. We will only explicitly show the classification method used to obtain all possible types of $Q^m \xrightarrow{\tau} Q^{m+1}$ when $\partial_{yx} = (1|2)i$, where the same method can be applied to the case when $\partial_{yx} = -i(2|1)$.

So, let us consider the case when τ has an entry with $\partial_{yx} = (1|2)i$. Recall that, by the definition of L_B, for any ∂_{yx} of type (ii), there must be a corresponding 1-crossing x' of $\check{c}$ such that

- x' and x are connected by an essential segment in D_0,
- x' and y are endpoints of an essential segment of $\check{c}$ in D_1.

So, in the case $\partial_{yx} = (1|2)i$, x' and y are connected through an essential segment in D_1 of type 1 (refer to Figure 2.8) and the map

$$\partial_{yx'} : P(x') \to P(y)$$

is the right multiplication by $(1|2)$. By the construction of Q^m and Q^{m+1}, Q^m must then at least contain the direct summands $P(x)$ and $P(x')$ and Q^{m+1} must at least contain the direct summand $P(y)$, so the differential $Q^m \xrightarrow{\tau} Q^{m+1}$ must contain at least two entries $\partial_{yx} = (1|2)i$ and $\partial_{yx'} = (1|2)$. Thus, the crossings x, x', and y must be contained in the corresponding partial curve of $\check{c}$ in Figure 7.1.

As seen from Figure 7.1 above, x and y are the only crossings that can be in another distinct essential segment of $\check{c}$.

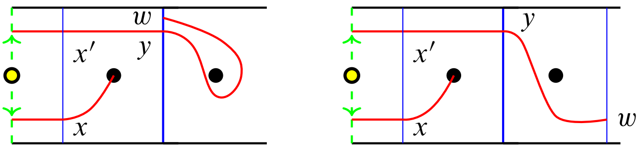


Figure 7.2. The two possible essential segments from y.

Let us now first consider the subcase when x is not in another distinct essential segment. If y is also not in another distinct essential segment, then we have that $Q^m \xrightarrow{\tau} Q^{m+1}$ is of the form

$$\begin{array}{ccc} P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \nearrow_{(1|2)i} & \\ P(x) & & \end{array} .$$

If instead y is part of another essential segment of $\check{c}$ with its other endpoint some crossing w, then the essential segment must be in D_2. Since $P(y)$ is a direct summand of Q^{m+1}, w must have the property

$$w_1 = y_1 - 1$$

so that $P(w)$ is an entry of Q^m and ∂_{yw} is an entry of τ. The only two such possibilities are shown in Figure 7.2.

Now, note that if w is a 2-crossing (left picture in Figure 7.2), then one sees that w cannot be connected to any other crossing z through another distinct essential segment in $\check{c}$ with $P(z)$ a direct summand of Q^{m+1}; if instead w is a 3-crossing (right picture in Figure 7.2), then the only possibility for $P(z)$ to be a direct summand of Q^{m+1} is when w is also a 3-crossing, with w and z endpoints of an essential segment of $\check{c}$ in D_3 of type 2, giving us $\partial_{wz} = X_3$. Recall the chosen decomposition of G^m and G^{m+1}, where $Q^{m+1} \subseteq G^{m+1}$ corresponds to the smallest subset of crossings in $\mathfrak{G}^{m+1}$ which contains y, with the property that maps between the direct summands of the decompositions of G^m and G^{m+1} are either 0 or X_1 or X_3. Thus, $P(z)$ must be excluded from Q^{m+1}. We can therefore conclude that, for the subcase when x is not connected to any other distinct essential segments, we have 3 possible forms for $Q^m \xrightarrow{\tau} Q^{m+1}$:

$$\begin{array}{ccc} P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \nearrow_{(1|2)i} & \\ P(x) & & \end{array} \quad \text{or} \quad \begin{array}{ccc} P(w) & & \\ \oplus & \searrow^{X_2 \text{ or } (3|2)} & \\ P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \nearrow_{(1|2)i} & \\ P(x) & & \end{array} .$$

To analyse the maps d_C and d_G, let us identify the morphism spaces as

$$\begin{aligned}
&\mathrm{HOM}^m_{\mathcal{K}_B}(P_2^B, G^*)\\
&= \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}^m}(P_2^B, (P(x')\oplus P(x)\oplus P(w))\{s\}\langle t\rangle)\{s\}\langle t\rangle\\
&\cong (\mathbb{R}\{(2|1)\}\langle x_3\rangle \oplus \mathbb{R}\{i(2|1)\}\langle x_3+1\rangle)\{x_2+1\}\\
&\quad \oplus (\mathbb{R}\{(2|1)\}\langle x_3'\rangle \oplus \mathbb{R}\{i(2|1)\}\langle x_3'+1\rangle)\{x_2'\}\\
&\quad \oplus Z,
\end{aligned}$$

where $Z = 0$ for the first type of $Q^m \xrightarrow{\tau} Q^{m+1}$, and

$$\begin{aligned}
\mathrm{HOM}^{m+1}_{\mathcal{K}_B}(P_2^B, G^*) &= \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}^m}(P_2^B, P(y)\{s\}\langle t\rangle)\{s\}\langle t\rangle\\
&\cong (\mathbb{R}\{X_2\}\langle y_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y_3+1\rangle)\{y_2+1\}\\
&\quad \oplus (\mathbb{R}\{\mathrm{id}\}\langle y_3\rangle \oplus \mathbb{R}\{i\}\langle y_3+1\rangle)\{y_2\}.
\end{aligned}$$

Using this identification, we can write d_C and d_G as the corresponding matrices

$$d_C = \begin{bmatrix} 1 & 0 & 0 & -1 & e\\ 0 & 1 & 1 & 0 & f\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad d_G = \begin{bmatrix} 1 & 0 & 0 & 0 & e\\ 0 & 1 & 0 & 0 & f\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where d_G is obtained from d_C by removing maps that were induced by $(1|2)i$. It follows that d_C and d_G have the same image and have isomorphic kernels.

Now, consider the other subcase where x is in another essential segment of $\check{c}$ with its other endpoint some crossing y'. Note that since x is already part of an essential segment in D_0, the essential segment connecting x and y' can only be in D_1. As before, we must have

$$y_1' = x_1 - 1$$

so that $P(y')$ is a direct summand of Q^{m+1} and that $\partial_{y'x}$ is an entry of τ. Furthermore, if x and y' are connected by the essential segment of type 2 in Figure 2.8, then $\partial_{y'x} = X_1$. Therefore, such $P(y')$ is excluded from Q^{m+1}. Collecting the results, the only possible essential segment connecting x and y' with $y_1' = x_1 - 1$ and $\partial_{y'x} \neq X_1$ is the essential segment of type 1. Thus, the crossings x, x', y, and y' must be contained in the corresponding partial curve of $\check{c}$ as in Figure 7.3.

The same analysis in the previous subcase on the possible essential segments connected to y can be applied similarly to the crossings y and y' here. Thus, we conclude that, for this subcase,

$$Q^m \xrightarrow{\partial_C^m} Q^{m+1}$$

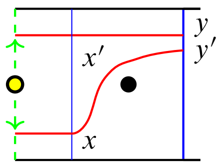


Figure 7.3. The crossings x, x', y, and y' when x is in another essential segment.

is equal to one of the following 6 types (there are 3 possible combinations of X_2 and $(3|2)$ maps in the rightmost diagram):

$$\begin{array}{ccc} P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \searrow\!\!\nearrow (1|2)i & \oplus \\ P(x) & \xrightarrow[(1|2)]{} & P(y') \end{array}, \qquad \begin{array}{ccc} P(z) & & \\ \oplus & \searrow^{X_2 \text{ or } (3|2)} & \\ P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \searrow\!\!\nearrow (1|2)i & \oplus \\ P(x) & \xrightarrow[(1|2)]{} & P(y') \end{array} \quad \text{or} \quad \begin{array}{ccc} P(z) & & \\ \oplus & \searrow^{X_2 \text{ or } (3|2)} & \\ P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \searrow\!\!\nearrow (1|2)i & \oplus \\ P(x) & \xrightarrow[(1|2)]{} & P(y') \\ \oplus & \nearrow_{X_2 \text{ or } (3|2)} & \\ P(z') & & \end{array}$$

swapping x with x' (and correspondingly y with y') if necessary. Let us again identify the morphism spaces as

$$\begin{aligned} \mathrm{HOM}^m_{\mathcal{K}_B}(P_2^B, G^*) &= \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}m}(P_2^B, (P(x')\oplus P(x)\oplus P(z))\{s\}\langle t\rangle)\{s\}\langle t\rangle \\ &\cong (\mathbb{R}\{(2|1)\}\langle x_3\rangle \oplus \mathbb{R}\{i(2|1)\}\langle x_3+1\rangle)\{x_2+1\} \\ &\quad \oplus (\mathbb{R}\{(2|1)\}\langle x'_3\rangle \oplus \mathbb{R}\{i(2|1)\}\langle x'_3+1\rangle)\{x'_2\} \\ &\quad \oplus Z, \end{aligned}$$

where $Z = 0$ for the first type of $Q^m \xrightarrow{\tau} Q^{m+1}$, and

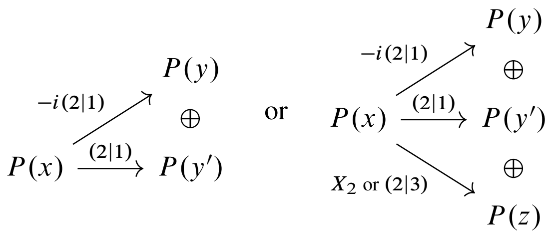
$$\begin{aligned} \mathrm{HOM}^{m+1}_{\mathcal{K}_B}(P_2^B, G^*) &= \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}m}(P_2^B, P(y)\{s\}\langle t\rangle)\{s\}\langle t\rangle \\ &\cong (\mathbb{R}\{X_2\}\langle y_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y_3+1\rangle)\{y_2+1\} \\ &\quad \oplus (\mathbb{R}\{X_2\}\langle y'_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y'_3+1\rangle)\{y'_2+1\} \\ &\quad \oplus (\mathbb{R}\{\mathrm{id}\}\langle y_3\rangle \oplus \mathbb{R}\{i\}\langle y_3+1\rangle)\{y_2\} \\ &\quad \oplus (\mathbb{R}\{\mathrm{id}\}\langle y'_3\rangle \oplus \mathbb{R}\{i\}\langle y'_3+1\rangle)\{y'_2\} \\ &= (\mathbb{R}\{X_2\}\langle y_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y_3+1\rangle)\{y_2+1\} \\ &\quad \oplus (\mathbb{R}\{X_2\}\langle y'_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y'_3+1\rangle)\{y'_2+1\} \oplus V. \end{aligned}$$

Writing d_C and d_G as the corresponding matrix, we get

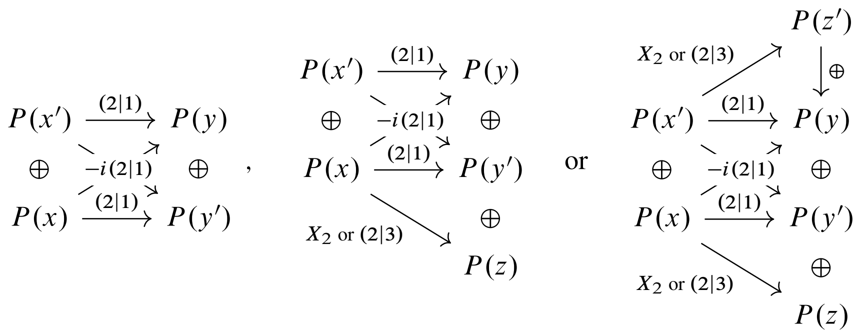
$$d_C = \begin{bmatrix} 1 & 0 & 0 & -1 & e' \\ 0 & 1 & 1 & 0 & f' \\ 0 & -1 & 1 & 0 & g' \\ 1 & 0 & 0 & 1 & h' \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad d_G = \begin{bmatrix} 1 & 0 & 0 & 0 & e' \\ 0 & 1 & 0 & 0 & f' \\ 0 & 0 & 1 & 0 & g' \\ 0 & 0 & 0 & 1 & h' \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which also have the same image and same kernel. Thus, for $\partial_{yx} = (1|2)i$, all possible cases of d_C and d_G have isomorphic images and isomorphic kernels as required.

Applying the same classification method to the case when $\partial_{yx} = -i(2|1)$, the possible types of $Q^m \xrightarrow{\tau} Q^{m+1}$ are given by



when y is not part of another distinct essential segment of $\check{c}$, and



when y is part of another distinct essential segment of $\check{c}$, where as before we swap y with y' (correspondingly x with x') if necessary. By identifying the morphism spaces and comparing the corresponding matrices of d_C and d_G as before, it follows that d_C and d_G have isomorphic images and isomorphic kernels. This covers all cases of δ_C and δ_G, thus completing the proof. ■

Lemma 7.2. *The Poincaré polynomial* $\mathfrak{P}(P_j^B, L_B(\check{g}))$ *of* $\mathrm{HOM}^*_{\mathcal{K}_B}(P_j^B, L_B(\check{g}))$ *is equal to the trigraded intersection number* $I^{\mathrm{trigr}}(\check{b}_j, \check{g})$ *for any trigraded* j*-string* $\check{g}$.

Proof. This follows exactly as in [15, Lemma 4.11]. ■

Proposition 7.3. *For any σ and τ in $\mathcal{A}(B_n)$ and any $1 \leq j, k \leq n$, the Poincaré polynomial of*

$$\mathrm{HOM}^*_{\mathcal{K}_B}(\sigma(P_j^B), \tau(P_k^B))$$

is equal to the trigraded intersection number $I^{\text{trigr}}(\check{\sigma}(\check{b}_j), \check{\tau}(\check{b}_k))$.

Proof. By Lemma 2.11, we get that

$$I^{\text{trigr}}(\check{b}_j, \check{c}) = \sum_{\check{g} \in \mathrm{st}(\check{c}, j)} I^{\text{trigr}}(\check{b}_j, \check{g}).$$

Using Lemma 7.1, we instead get that

$$\mathfrak{P}(P_j^B, L_B(\check{c})) = \sum_{\check{g} \in \mathrm{st}(\check{c}, j)} \mathfrak{P}(P_j^B, L_B(\check{g})).$$

By Lemma 7.2, each $\mathfrak{P}(P_j^B, L_B(\check{g})) = I^{tri}(\check{b}_j, \check{g})$; thus, we can conclude that

$$I^{\text{trigr}}(\check{b}_j, \check{c}) = \mathfrak{P}(P_j^B, L_B(\check{c})).$$

The proposition now follows from the fact that the categorical action of $\mathcal{A}(B_n)$ respects morphism spaces, and similarly, the topological action of $\mathcal{A}(B_n)$ respects trigraded intersection number. ∎

Remark 7.4. Note that we can also use Proposition 7.3 to prove the faithfulness of the $\mathcal{A}(B_n)$ categorical action. The proof is similar to [15, the paragraph before Section 5] modulo the centre of $\mathcal{A}(B_n)$, which is an easy check that elements of the centre act by shifting degrees and therefore are not isomorphic to the identity functor.

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