

Deformations of modified r -matrices and cohomologies of related algebraic structures

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Abstract. Modified r -matrices are solutions of the modified classical Yang–Baxter equation, introduced by Semenov-Tian-Shansky, and play important roles in mathematical physics. In this paper, first we introduce a cohomology theory for modified r -matrices. Then we study three kinds of deformations of modified r -matrices using the established cohomology theory, including algebraic deformations, geometric deformations and linear deformations. We give the differential graded Lie algebra that governs algebraic deformations of modified r -matrices. For geometric deformations, we prove the rigidity theorem and study when is a neighborhood of a modified r -matrix smooth in the space of all modified r -matrix structures. In the study of trivial linear deformations, we introduce the notion of a Nijenhuis element for a modified r -matrix. Finally, applications are given to study deformations of the complement of the diagonal Lie algebra and compatible Poisson structures.

1. Introduction

In the seminal work [26], Semenov-Tian-Shansky showed that solutions of the modified classical Yang–Baxter equation, which we call modified r -matrices in this paper, play an important role in studying solutions of Lax equations [24–26]. Furthermore, modified r -matrices are intimately related to particular factorization problems in the corresponding Lie algebras and Lie groups. This factorization problem was considered by Reshetikhin and Semenov-Tian-Shansky in the framework of the enveloping algebra of a Lie algebra with a modified r -matrix to study quantum integrable systems [23]. Any modified r -matrix induces a post-Lie algebra [1], and a factorization theorem for group-like elements of the completion of the Lie enveloping algebra of a post-Lie algebra was established by Ebrahimi-Fard, Mencattini and Munthe-Kaas in [9, 10]. Recently, the global factorization theorem for a Rota–Baxter Lie group was given in [15]. Moreover, modified r -matrices are also useful for the construction of flat metrics and Frobenius manifolds [27], and compatible Poisson structures [18]. Note that in the associative algebra context, such objects are called modified Rota–Baxter algebras by Zhang, Gao and Guo [30, 31].

A classical approach to study a mathematical structure is to associate to it invariants. Among these, cohomology theories occupy a central position as they enable for example to control deformations or extension problems. Note that the cohomology theory for a

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skew-symmetric classical r -matrix was studied in [29] under the general framework of relative Rota–Baxter operators (also called \mathcal{O} -operators [17]). The first purpose of this paper is to study the cohomology theory for a modified r -matrix. In [26], Semenov-Tian-Shansky showed that a modified r -matrix $R : \mathfrak{g} \rightarrow \mathfrak{g}$ on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ induces a new Lie algebra \mathfrak{g}_R in which the Lie bracket $[\cdot, \cdot]_R$ is given by

$$[x, y]_R = [R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}.$$

In [2], Bordemann showed that the induced Lie algebra \mathfrak{g}_R represents on \mathfrak{g} . We use the corresponding Chevalley–Eilenberg cohomology [4] of the Lie algebra \mathfrak{g}_R with coefficients in \mathfrak{g} to define the cohomology of the modified r -matrix R . It is well known that there is a one-to-one correspondence between the modified r -matrix R and the Rota–Baxter operator B of weight 1 via the relation $R = \text{Id} + 2B$. The cohomology theory of the latter was given in [16] and the Van Est type theorem was established. We also show that the cohomology of the modified r -matrix $R = \text{Id} + 2B$ and the cohomology of the Rota–Baxter operator B are isomorphic.

The concept of a formal deformation of an algebraic structure began with the seminal work of Gerstenhaber [13, 14] for associative algebras. Nijenhuis and Richardson extended this study to Lie algebras [20, 21]. There is a well-known slogan, often attributed to Deligne, Drinfeld and Kontsevich: every reasonable deformation theory is controlled by a differential graded Lie algebra, determined up to quasi-isomorphism. This slogan has been made into a rigorous theorem by Lurie and Pridham [19, 22]. It is also meaningful to deform *maps* compatible with given algebraic structures. Recently, the deformation theory of morphisms was developed in [3, 11, 12], the deformation theories of \mathcal{O} -operators on Lie algebras and associative algebras were developed in [6, 29]. The second purpose of the paper is to study deformation theories of modified r -matrices. We study three kinds of deformations of a modified r -matrix R :

- (algebraic deformations) first we consider an algebraic deformation $R + R'$ for a certain linear map R' , and show that this kind of deformations are governed by a differential graded Lie algebra. This fulfill the general slogan for the deformation theory proposed by Deligne, Drinfeld and Kontsevich;
- (geometric deformations) then we consider a smooth geometric deformation R_t such that $R_0 = R$ using the approach developed by Crainic, Schätz and Struchiner in [5]. We show that the tangent space $T_R \text{Orb}_R$ of the orbit Orb_R is the space of 2-coboundaries $B^2(R)$. Consequently, the condition $H^2(R) = 0$ will imply a certain rigidity theorem, and the condition $H^3(R) = 0$ will imply the space of modified r -matrices on the Lie algebra \mathfrak{g} is a manifold in a neighborhood of R . We also give the necessary and sufficient condition on a 2-cocycle giving a geometric deformation using the Kuranishi map;
- (linear deformation) next we study a linear deformation $R + t\hat{R}$. In particular, trivial linear deformations lead to the concept of Nijenhuis elements for a modified r -matrix. If $x \in \mathfrak{g}$ is a Nijenhuis element, then ad_x is a Nijenhuis operator on the Lie algebra \mathfrak{g}_R .

Note that certain particular deformations of classical r -matrices are considered in [28] in the study of integrable infinite-dimensional systems.

The paper is organized as follows. In Section 2, we define the cohomology of a modified r -matrix R using the Chevalley–Eilenberg cohomology of the Lie algebra \mathfrak{g}_R with coefficients in \mathfrak{g} . In Section 3, we construct a differential graded Lie algebra that governs algebraic deformations of a modified r -matrix. In Section 4, we study geometric deformations of a modified r -matrix. In Section 5, we study linear deformations of a modified r -matrix. In Section 6, we study deformations of the complement of the diagonal Lie algebra and compatible Poisson structures as applications.

2. Cohomologies of modified r -matrices

In this section, we establish the cohomology theory of a modified r -matrix R using the Chevalley–Eilenberg cohomology of the Lie algebra \mathfrak{g}_R with coefficients in \mathfrak{g} .

Definition 2.1 ([26]). Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear map $R : \mathfrak{g} \rightarrow \mathfrak{g}$ is called a *modified r -matrix* if it is a solution of the following *modified classical Yang–Baxter equation*:

$$[R(x), R(y)]_{\mathfrak{g}} = R([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}) - [x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}. \quad (1)$$

Definition 2.2. Let R and R' be modified r -matrices on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. A *homomorphism* from R to R' is a Lie algebra homomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$\varphi \circ R = R' \circ \varphi.$$

Remark 2.3. The notion of a modified Rota–Baxter operator of weight -1 on an associative algebra was introduced in [8]. More precisely, it is a linear map $P : A \rightarrow A$ on an associative algebra (A, \cdot_A) satisfying

$$P(u) \cdot_A P(v) = P(P(u) \cdot_A v + u \cdot_A P(v)) - u \cdot_A v, \quad \forall u, v \in A.$$

It is straightforward to see that if a linear map $P : A \rightarrow A$ is a modified Rota–Baxter operator of weight -1 on an associative algebra (A, \cdot_A) , then P is a modified r -matrix on the Lie algebra $(A, [\cdot, \cdot]_A)$, where $[\cdot, \cdot]_A$ is the commutator Lie bracket.

Remark 2.4. Let $R : \mathfrak{g} \rightarrow \mathfrak{g}$ be a linear map on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. Under the condition $R^2 = \text{Id}$, the following structures are equivalent:

- R is a modified r -matrix;
- R is a Nijenhuis operator;
- R is a product structure;
- there is a vector space direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of \mathfrak{g} into subalgebras \mathfrak{g}_1 and \mathfrak{g}_2 such that R is given by

$$R(x, u) = (x, -u), \quad \forall x \in \mathfrak{g}_1, u \in \mathfrak{g}_2.$$

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra and R be a modified r -matrix. Semenov-Tian-Shansky showed that $(\mathfrak{g}, [\cdot, \cdot]_R)$ is a Lie algebra which plays important roles in the study of integrable systems [26], where

$$[x, y]_R = [R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}. \quad (2)$$

Recall that a matched pair of Lie algebras consists of Lie algebras $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$, $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$, a representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{h})$ of \mathfrak{g} on \mathfrak{h} and a representation $\varrho : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g})$ of \mathfrak{h} on \mathfrak{g} , such that some compatibility conditions are satisfied. Bordemann further showed that the induced Lie algebra \mathfrak{g}_R represents on \mathfrak{g} which leads to a matched pair of Lie algebras $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}), (\mathfrak{g}, [\cdot, \cdot]_R))$ [2]. Here we give a direct proof to be self-contained.

Proposition 2.5. *Let R be a modified r -matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. Define a linear map $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ by*

$$\rho(x)y = [R(x), y]_{\mathfrak{g}} - R([x, y]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}. \quad (3)$$

Then ρ is a representation of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_R)$ on the vector space \mathfrak{g} .

Proof. For all $x, y, z \in \mathfrak{g}$, by (1) and (3), we have

$$\begin{aligned} & [\rho(x), \rho(y)]z \\ &= \rho(x)\rho(y)z - \rho(y)\rho(x)z \\ &= \rho(x)([R(y), z]_{\mathfrak{g}} - R([y, z]_{\mathfrak{g}})) - \rho(y)([R(x), z]_{\mathfrak{g}} - R([x, z]_{\mathfrak{g}})) \\ &= [R(x), [R(y), z]_{\mathfrak{g}}]_{\mathfrak{g}} - [R(x), R([y, z]_{\mathfrak{g}})]_{\mathfrak{g}} - R([x, [R(y), z]_{\mathfrak{g}}]_{\mathfrak{g}}) \\ &\quad + R([x, R([y, z]_{\mathfrak{g}})]_{\mathfrak{g}}) - [R(y), [R(x), z]_{\mathfrak{g}}]_{\mathfrak{g}} + [R(y), R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}} \\ &\quad + R([y, [R(x), z]_{\mathfrak{g}}]_{\mathfrak{g}}) - R([y, R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}}) \\ &= [[R(x), R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}} - R([R(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}}) - R([x, [R(y), z]_{\mathfrak{g}}]_{\mathfrak{g}}) + [x, [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} \\ &\quad + R([R(y), [x, z]_{\mathfrak{g}}]_{\mathfrak{g}}) - [y, [x, z]_{\mathfrak{g}}]_{\mathfrak{g}} + R([y, [R(x), z]_{\mathfrak{g}}]_{\mathfrak{g}}) \\ &= [[R(x), R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [[x, y]_{\mathfrak{g}}, z]_{\mathfrak{g}} - R([[R(x), y]_{\mathfrak{g}}, z]_{\mathfrak{g}}) - R([[x, R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}}), \end{aligned}$$

and

$$\begin{aligned} & \rho([x, y]_R)z \\ &= \rho([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}})z \\ &= [R([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}), z]_{\mathfrak{g}} - R([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}})z \\ &= [[R(x), R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [[x, y]_{\mathfrak{g}}, z]_{\mathfrak{g}} - R([R(x), y]_{\mathfrak{g}})z - R([x, R(y)]_{\mathfrak{g}})z. \end{aligned}$$

Therefore we have $\rho([x, y]_R) = [\rho(x), \rho(y)]$, which means that ρ is a representation of $(\mathfrak{g}, [\cdot, \cdot]_R)$ on the vector space \mathfrak{g} . ■

Let $d_{\text{CE}}^R : \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^{k+1} \mathfrak{g}, \mathfrak{g})$ be the corresponding Chevalley–Eilenberg coboundary operator of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_R)$ with coefficients in the representation (\mathfrak{g}, ρ) . More precisely, for all $f \in \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$ and $x_1, \dots, x_{k+1} \in \mathfrak{g}$, we have

$$\begin{aligned}
 d_{\text{CE}}^R f(x_1, \dots, x_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \rho(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{k+1}) \\
 &\quad + \sum_{i < j} (-1)^{i+j} f([x_i, x_j]_R, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}) \\
 &= \sum_{i=1}^{k+1} (-1)^{i+1} [R(x_i), f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}} \\
 &\quad - \sum_{i=1}^{k+1} (-1)^{i+1} R([x_i, f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}}) \\
 &\quad + \sum_{i < j} (-1)^{i+j} f([R(x_i), x_j]_{\mathfrak{g}} + [x_i, R(x_j)]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}).
 \end{aligned} \tag{4}$$

Now, we define the cohomology of a modified r -matrix $R : \mathfrak{g} \rightarrow \mathfrak{g}$. Define the space of 0-cochains $C^0(R)$ to be 0 and define the space of 1-cochains $C^1(R)$ to be \mathfrak{g} . For $n \geq 2$, define the space of n -cochains $C^n(R)$ by $C^n(R) = \text{Hom}(\wedge^{n-1} \mathfrak{g}, \mathfrak{g})$.

Definition 2.6. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra and R be a modified r -matrix. The cohomology of the cochain complex $(\bigoplus_{i=0}^{+\infty} C^i(R), d_{\text{CE}}^R)$ is defined to be the *cohomology for the modified r -matrix R* .

Denote the set of n -cocycles by $Z^n(R)$, the set of n -coboundaries by $B^n(R)$ and the n -th cohomology group by

$$H^n(R) = Z^n(R)/B^n(R), \quad n \geq 0.$$

It is obvious that $x \in \mathfrak{g}$ is closed if and only if

$$\text{ad}_x \circ R = R \circ \text{ad}_x,$$

and $f \in \text{Hom}(\mathfrak{g}, \mathfrak{g})$ is closed if and only if

$$\begin{aligned}
 [R(x), f(y)]_{\mathfrak{g}} - R([x, f(y)]_{\mathfrak{g}}) - [R(y), f(x)]_{\mathfrak{g}} + R([y, f(x)]_{\mathfrak{g}}) \\
 = f([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}),
 \end{aligned} \tag{5}$$

for all $x, y \in \mathfrak{g}$.

At the end of this section, we recall the cohomology theory of Rota–Baxter operators given in [16], and establish its relation with the cohomology theory of modified r -matrices.

Definition 2.7. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear map $B : \mathfrak{g} \rightarrow \mathfrak{g}$ is called a *Rota–Baxter operator of weight λ* if

$$[B(x), B(y)]_{\mathfrak{g}} = B([B(x), y]_{\mathfrak{g}} + [x, B(y)]_{\mathfrak{g}} + \lambda[x, y]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}.$$

The following result is well known.

Proposition 2.8. Let \mathfrak{g} be a Lie algebra and $B \in \mathfrak{gl}(\mathfrak{g})$. The linear map $\text{Id} + 2B$ is a modified *r*-matrix on \mathfrak{g} if and only if B is a Rota–Baxter operator of weight 1 on \mathfrak{g} .

Let B be a Rota–Baxter operator of weight 1 on a Lie algebra \mathfrak{g} . Consider the cochain complex $(\bigoplus_{k=1}^{+\infty} C^k(B), d_{\text{CE}}^B)$, where $C^1(B) = \mathfrak{g}$ and $C^k(B) = \text{Hom}(\wedge^{k-1} \mathfrak{g}, \mathfrak{g})$ for $k \geq 2$, and d_{CE}^B is defined by

$$\begin{aligned} d_{\text{CE}}^B f(u_1, \dots, u_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} B([f(u_1, \dots, \hat{u}_i, \dots, u_{k+1}), u_i]_{\mathfrak{g}}) \\ &\quad + \sum_{i=1}^{k+1} (-1)^{i+1} [B(u_i), f(u_1, \dots, \hat{u}_i, \dots, u_{k+1})]_{\mathfrak{g}} \\ &\quad + \sum_{i < j} (-1)^{i+j} f([B(u_i), u_j]_{\mathfrak{g}} - [B(u_j), u_i]_{\mathfrak{g}} \\ &\quad + [u_i, u_j]_{\mathfrak{g}}, u_1, \dots, \hat{u}_i, \dots, \hat{u}_j, \dots, u_{k+1}), \end{aligned}$$

where $f \in C^{k+1}(B)$ and $u_i \in \mathfrak{g}$, $1 \leq i \leq k + 1$.

It was proved in [16] that $(d_{\text{CE}}^B)^2 = 0$. The cohomology of the cochain complex $(\bigoplus_{k=1}^{+\infty} C^k(B), d_{\text{CE}}^B)$ is defined to be the cohomology of the Rota–Baxter operator B .

Theorem 2.9. With the above notations, we have

$$d_{\text{CE}}^R = 2d_{\text{CE}}^B.$$

Consequently, for $k \geq 1$, the k -th cohomology group $H^k(B)$ of a Rota–Baxter operator B is isomorphic with the k -th cohomology group $H^k(R)$ of the modified *r*-matrix $R = \text{Id} + 2B$.

Proof. For $k \geq 1$, define linear maps $\Phi_k : C^k(B) \rightarrow C^k(R)$ by $\Phi_k = 2^{k-2} \text{Id}$. Then the following diagram is commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{d_{\text{CE}}^B} & \text{Hom}(\mathfrak{g}, \mathfrak{g}) & \xrightarrow{d_{\text{CE}}^B} \cdots \xrightarrow{d_{\text{CE}}^B} & \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}) & \xrightarrow{d_{\text{CE}}^B} \cdots \\ & & \frac{1}{2} \text{Id} \downarrow & & \text{Id} \downarrow & & 2^{k-1} \text{Id} \downarrow & \\ 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{d_{\text{CE}}^R} & \text{Hom}(\mathfrak{g}, \mathfrak{g}) & \xrightarrow{d_{\text{CE}}^R} \cdots \xrightarrow{d_{\text{CE}}^R} & \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}) & \xrightarrow{d_{\text{CE}}^R} \cdots \end{array}$$

In fact, for any $f \in \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$, $x_i \in \mathfrak{g}$, $1 \leq i \leq k+1$, we have

$$\begin{aligned}
 & d_{\text{CE}}^R(\Phi_k f)(x_1, \dots, x_{k+1}) \\
 &= 2^{k-1} \left(\sum_{i=1}^{k+1} (-1)^{i+1} ([x_i, f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}} \right. \\
 &\quad \left. + 2[B(x_i), f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}}) \right. \\
 &\quad \left. - \sum_{i=1}^{k+1} (-1)^{i+1} [x_i, f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}} \right. \\
 &\quad \left. - \sum_{i=1}^{k+1} (-1)^{i+1} 2B([x_i, f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}}) \right. \\
 &\quad \left. + \sum_{i < j} (-1)^{i+j} 2f([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}) \right. \\
 &\quad \left. + \sum_{i < j} (-1)^{i+j} 2f([B(x_i), x_j]_{\mathfrak{g}} + [x_i, B(x_j)]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}) \right) \\
 &= 2^k \left(\sum_{i=1}^{k+1} (-1)^{i+1} ([B(x_i), f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}} \right. \\
 &\quad \left. - B([x_i, f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}})) \right. \\
 &\quad \left. + \sum_{i < j} (-1)^{i+j} f([B(x_i), x_j]_{\mathfrak{g}} + [x_i, B(x_j)]_{\mathfrak{g}}) \right. \\
 &\quad \left. + [x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}) \right) \\
 &= \Phi_{k+1}(d_{\text{CE}}^B f)(x_1, \dots, x_{k+1}),
 \end{aligned}$$

which implies that $d_{\text{CE}}^R = 2d_{\text{CE}}^B$ and $H^k(B) \cong H^k(R)$, $k \geq 1$. ■

Example 2.10. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. It is well known that the Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{R})$ is $H = \text{span}\{E_{ii} - E_{i+1i+1} \mid 1 \leq i \leq n-1\}$. Denote the Borel subalgebra of $\mathfrak{sl}(n, \mathbb{R})$ by $B(\mathfrak{sl}(n, \mathbb{R}))$. It is well known that $B(\mathfrak{sl}(n, \mathbb{R})) = H \oplus \text{span}\{E_{ij} \mid i < j\}$. Thus $\mathfrak{sl}(n, \mathbb{R}) = B(\mathfrak{sl}(n, \mathbb{R})) \oplus A$ as vector spaces, where $A = \text{span}\{E_{ij} \mid i > j\}$. Define a linear map $R : \mathfrak{sl}(n, \mathbb{R}) \rightarrow \mathfrak{sl}(n, \mathbb{R})$ by

$$R(x + u) = x - u, \quad \forall x \in B(\mathfrak{sl}(n, \mathbb{R})), u \in A.$$

By Remark 2.4, we obtain that R is a modified r -matrix on the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$. Assume that $a = x + u \in \mathfrak{sl}(n, \mathbb{R})$ where $x \in B(\mathfrak{sl}(n, \mathbb{R}))$ and $u \in A$, such that $d_{\text{CE}}^R a = 0$, that is

$$d_{\text{CE}}^R a(y) = [R(y), a] - R([y, a]) = 0, \quad \forall y \in \mathfrak{sl}(n, \mathbb{R}).$$

- For any $y \in A$, $[R(y), a] - R([y, a]) = 0$ implies that $x \in H$.
- For any $y \in B(\mathfrak{sl}(n, \mathbb{R}))$, $[R(y), a] - R([y, a]) = 0$ implies that $u = 0$.

Thus $d_{\text{CE}}^R a = 0$ if and only if $a \in H$. Therefore, $H^1(R) \cong \mathbb{R}^{n-1}$.

Example 2.11. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, where the Lie bracket is given by $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$ with respect to the basis $\{e, f, h\}$. Then $R : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(2, \mathbb{R})$ defined by

$$R(e, f, h) = (e, f, h) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is a modified r -matrix. Let

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(2, \mathbb{R})$$

satisfy $d_{\text{CE}}^R T = 0$. Then we obtain

$$\begin{aligned} 0 &= [e, T(f)] + [f, T(e)] - R([e, T(f)]) + R([f, T(e)]), \\ 0 &= [e, T(h)] - [h, T(e)] - R([e, T(h)]) + R([h, T(e)]) + 4T(e) \end{aligned}$$

and

$$0 = -[f, T(h)] - [h, T(f)] - R([f, T(h)]) + R([h, T(f)]).$$

Thus we have $t_{11} = t_{21} = t_{31} = 0$ and $t_{22} = t_{13} = 0$. By Example 2.10, we have $B^2(R) = \text{Im } d_{\text{CE}}^R \cong \frac{\mathfrak{g}}{\ker d_{\text{CE}}^R} = \frac{\mathfrak{g}}{H^1(R)} \cong \mathbb{R}^2$. Thus $H^2(R) \simeq \mathbb{R}^2$.

3. Algebraic deformations of modified r -matrices

In this section, we construct a differential graded Lie algebra that governs algebraic deformations of a modified r -matrix.

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra. Further, we consider the graded vector space $C^*(\mathfrak{g}) = \bigoplus_{k=1}^{+\infty} \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$. Define a skew-symmetric bracket operation

$$[[\cdot, \cdot]] : \text{Hom}(\wedge^p \mathfrak{g}, \mathfrak{g}) \times \text{Hom}(\wedge^q \mathfrak{g}, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^{p+q} \mathfrak{g}, \mathfrak{g})$$

by

$$\begin{aligned} & [[f, g]](x_1, x_2, \dots, x_{p+q}) \\ &= \sum_{\sigma \in S(q, 1, p-1)} (-1)^\sigma f([g(x_{\sigma(1)}, \dots, x_{\sigma(q)}), x_{\sigma(q+1)}]_{\mathfrak{g}}, x_{\sigma(q+2)}, \dots, x_{\sigma(p+q)}) \\ & \quad - (-1)^{pq} \sum_{\sigma \in S(p, 1, q-1)} (-1)^\sigma \\ & \quad \cdot g([f(x_{\sigma(1)}, \dots, x_{\sigma(p)}), x_{\sigma(p+1)}]_{\mathfrak{g}}, x_{\sigma(p+2)}, \dots, x_{\sigma(p+q)}) \\ & \quad + (-1)^{pq} \sum_{\sigma \in S(p, q)} (-1)^\sigma [f(x_{\sigma(1)}, \dots, x_{\sigma(p)}), g(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)})]_{\mathfrak{g}}, \quad (6) \end{aligned}$$

for all $f \in \text{Hom}(\wedge^p \mathfrak{g}, \mathfrak{g})$, $g \in \text{Hom}(\wedge^q \mathfrak{g}, \mathfrak{g})$.

Then we have the following theorem characterizing modified r -matrices.

Theorem 3.1. *Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra. Then $(C^*(\mathfrak{g}), \llbracket \cdot, \cdot \rrbracket)$ is a graded Lie algebra and its Maurer–Cartan elements are precisely Rota–Baxter operators of weight 0.*

Moreover, a linear map $R \in \mathfrak{gl}(\mathfrak{g})$ is a modified r -matrix on the Lie algebra \mathfrak{g} if and only if R satisfies the equation

$$\llbracket R, R \rrbracket = 2\pi, \quad (7)$$

where we denote $[\cdot, \cdot]_{\mathfrak{g}}$ by π .

Proof. By [29, Corollary 6.1], $(C^*(\mathfrak{g}), \llbracket \cdot, \cdot \rrbracket)$ is a graded Lie algebra.

For $R \in \mathfrak{gl}(\mathfrak{g})$, we have

$$\llbracket R, R \rrbracket(x, y) = 2(R([R(x), y]_{\mathfrak{g}}) - R([R(y), x]_{\mathfrak{g}}) - [R(x), R(y)]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}.$$

By this equality, we can deduce that on the one hand R is a Rota–Baxter operator of weight 0 if and only if $\llbracket R, R \rrbracket = 0$, i.e., R is a Maurer–Cartan element. On the other hand, R is a modified r -matrix on the Lie algebra \mathfrak{g} if and only if R satisfies (7). ■

Proposition 3.2. *Let R be a modified r -matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. Then $\llbracket R, R \rrbracket$ is in the center of the graded Lie algebra $(C^*(\mathfrak{g}), \llbracket \cdot, \cdot \rrbracket)$.*

Proof. Denote the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$ by π . Since R is a modified r -matrix on the Lie algebra \mathfrak{g} , we have $\llbracket R, R \rrbracket = 2\pi$ via Theorem 3.1. For all $f \in \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$, by (6), we have

$$\begin{aligned} & \llbracket 2\pi, f \rrbracket(x_1, \dots, x_k, x_{k+1}, x_{k+2}) \\ &= 2 \left(\sum_{\sigma \in S(k, 1, 1)} (-1)^{|\sigma|} \pi(\pi(f(x_{\sigma(1)}, \dots, x_{\sigma(k)}), x_{\sigma(k+1)}), x_{\sigma(k+2)}) \right. \\ & \quad - \sum_{\sigma \in S(2, 1, k-1)} (-1)^{|\sigma|} f(\pi(\pi(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}), x_{\sigma(4)}, \dots, x_{\sigma(k+2)}) \\ & \quad \left. + \sum_{\sigma \in S(2, k)} (-1)^{|\sigma|} \pi(\pi(x_{\sigma(1)}, x_{\sigma(2)}), f(x_{\sigma(3)}, \dots, x_{\sigma(k+2)})) \right) \\ &= 0, \end{aligned}$$

which implies that $\llbracket R, R \rrbracket$ is in the center of $C^*(\mathfrak{g})$. ■

We denote $\llbracket R, \cdot \rrbracket$ by d_R . Now we obtain the differential graded Lie algebra that governs algebraic deformations of a modified r -matrix.

Theorem 3.3. *With the above notations, $(C^*(\mathfrak{g}), \llbracket \cdot, \cdot \rrbracket, d_R)$ is a differential graded Lie algebra.*

Moreover, $R + R'$ is still a modified r -matrix on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ if and only if R' is a Maurer–Cartan element of the differential graded Lie algebra $(C^(\mathfrak{g}), \llbracket \cdot, \cdot \rrbracket, d_R)$.*

Proof. It follows from the graded Jacobi identity that d_R is a graded derivation on the graded Lie algebra $(C^*(\mathfrak{g}), [\cdot, \cdot])$. By Proposition 3.2, we have

$$d_R^2 f = \llbracket R, \llbracket R, f \rrbracket \rrbracket = \llbracket \llbracket R, R \rrbracket, f \rrbracket - \llbracket R, \llbracket R, f \rrbracket \rrbracket,$$

which implies that

$$d_R^2 f = \llbracket R, \llbracket R, f \rrbracket \rrbracket = \frac{1}{2} \llbracket \llbracket R, R \rrbracket, f \rrbracket = 0.$$

Therefore, $(C^*(\mathfrak{g}), [\cdot, \cdot], d_R)$ is a differential graded Lie algebra.

Let R' be a linear map from \mathfrak{g} to \mathfrak{g} . Then $R + R'$ is a modified r -matrix if and only if

$$\llbracket R + R', R + R' \rrbracket = 2\pi,$$

that is

$$0 = \llbracket R, R' \rrbracket + \frac{1}{2} \llbracket R', R' \rrbracket = d_R R' + \frac{1}{2} \llbracket R', R' \rrbracket.$$

Thus $R + R'$ is still a modified r -matrix on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ if and only if R' is a Maurer–Cartan element of the differential graded Lie algebra $(C^*(\mathfrak{g}), [\cdot, \cdot], d_R)$. ■

At the end of this section, we establish the relationship between the coboundary operator d_{CE}^R and the differential d_R .

Proposition 3.4. *Let R be a modified r -matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. Then we have*

$$d_{\text{CE}}^R(f) = (-1)^{n-1} \llbracket R, f \rrbracket, \quad \forall f \in \text{Hom}(\wedge^{n-1} \mathfrak{g}, \mathfrak{g}).$$

Proof. For any $f \in \text{Hom}(\wedge^{n-1} \mathfrak{g}, \mathfrak{g})$ and $x_i, 1 \leq i \leq n$, by (6), we have

$$\begin{aligned} & (-1)^{n-1} \llbracket R, f \rrbracket(x_1, \dots, x_n) \\ &= (-1)^{n-1} \left(\sum_{\sigma \in S(n-1, 1)} (-1)^\sigma R([f(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}), x_{\sigma(n)}]_{\mathfrak{g}}) \right. \\ & \quad - (-1)^{n-1} \sum_{\sigma \in S(1, 1, n-2)} (-1)^\sigma f([R(x_{\sigma(1)}), x_{\sigma(2)}]_{\mathfrak{g}}, x_{\sigma(3)}, \dots, x_{\sigma(n)}) \\ & \quad \left. + (-1)^{n-1} \sum_{\sigma \in S(1, n-1)} (-1)^\sigma [R(x_{\sigma(1)}), f(x_{\sigma(2)}, \dots, x_{\sigma(n)})]_{\mathfrak{g}} \right) \\ &= \sum_{i=1}^n (-1)^{i+1} R([f(x_1, \dots, \hat{x}_i, \dots, x_n), x_i]_{\mathfrak{g}}) \\ & \quad + \sum_{i < j} (-1)^{i+j} (f([R(x_i), x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) \\ & \quad - f([R(x_j), x_i]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n)) \\ & \quad + \sum_{i=1}^n (-1)^{i+1} [R(x_i), f(x_1, \dots, \hat{x}_i, \dots, x_n)]_{\mathfrak{g}} \\ &= d_{\text{CE}}^R(f)(x_1, \dots, x_n). \end{aligned}$$

We finish the proof. ■

4. Geometric deformations of modified r -matrices

In this section, we study geometric deformations of modified r -matrices following the approach developed by Crainic, Schatz and Struchiner. We show that the condition $H^2(R) = 0$ will imply a certain rigidity theorem, and the condition $H^3(R) = 0$ will imply the space of modified r -matrices on the Lie algebra \mathfrak{g} is a manifold in a neighborhood of R . We also give the necessary and sufficient condition on a 2-cocycle giving a geometric deformation using the Kuranishi map.

Definition 4.1. Let R be a modified r -matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. A *geometric deformation* of R is a smooth one parameter family of modified r -matrices R_t on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ such that $R_0 = R$.

Definition 4.2. Two geometric deformations R_t and R'_t of R are called *equivalent* if there exists a smooth family of modified r -matrices isomorphisms $\varphi_t : R_t \rightarrow R'_t$ such that $\varphi_0 = \text{Id}$, where φ_t are inner automorphisms of the Lie algebra \mathfrak{g} .

Let R_t be a geometric deformation of R . Denote $\frac{d}{dt}|_{t=0} R_t$ by \dot{R}_0 . Then there is the following proposition.

Proposition 4.3. *With the above notations, \dot{R}_0 is a 2-cocycle in $C^2(R)$. Moreover, if R_t and R'_t are equivalent geometric deformations of R , then $[\dot{R}_0] = [\dot{R}'_0]$ in $H^2(R)$.*

Proof. Since R_t is a geometric deformation of R , for any $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} & [R(x), \dot{R}_0(y)]_{\mathfrak{g}} + [\dot{R}_0(x), R(y)]_{\mathfrak{g}} \\ &= \frac{d}{dt}|_{t=0} [R_t(x), R_t(y)]_{\mathfrak{g}} \\ &= \frac{d}{dt}|_{t=0} (R_t([R_t(x), y]_{\mathfrak{g}} + [x, R_t(y)]_{\mathfrak{g}}) - [x, y]_{\mathfrak{g}}) \\ &= \dot{R}_0([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}) + R([\dot{R}_0(x), y]_{\mathfrak{g}} + [x, \dot{R}_0(y)]_{\mathfrak{g}}). \end{aligned} \quad (8)$$

Thus by (5) and (8), we have $d_{\text{CE}}^R(\dot{R}_0) = 0$.

Assume that φ_t is an isomorphism from R_t to R'_t , that is

$$\varphi_t(R_t(x)) = R'_t(\varphi_t(x)), \quad \forall x \in \mathfrak{g}.$$

Denote $\frac{d}{dt}|_{t=0} \varphi_t$ by $\dot{\varphi}_0$. Then we have $\dot{\varphi}_0(R(x)) + \dot{R}_0(x) = R(\dot{\varphi}_0(x)) + \dot{R}'_0(x)$. Since φ_t are inner automorphisms of the Lie algebra \mathfrak{g} , it follows that $\dot{\varphi}_0$ is an inner derivation of the Lie algebra \mathfrak{g} . Thus there exists $y \in \mathfrak{g}$ such that $\dot{\varphi}_0 = \text{ad}_y$. Therefore, we have

$$[y, R(x)]_{\mathfrak{g}} + \dot{R}_0(x) = R([y, x]_{\mathfrak{g}}) + \dot{R}'_0(x),$$

which implies $\dot{R}_0 - \dot{R}'_0 = d_{\text{CE}}^R(y)$. Thus $[\dot{R}_0] = [\dot{R}'_0]$ in $H^2(R)$. ■

Next, we consider under which conditions does a cocycle $f \in Z^2(R)$ determine a geometric deformation R_t . Define the Kuranishi map $K : Z^2(R) \rightarrow H^3(R)$ by

$$K(f) = \llbracket [f, f] \rrbracket, \quad \forall f \in Z^2(R).$$

Now we give a necessary condition of the above question. The sufficient condition need some preparations and will be given at the end of this section.

Proposition 4.4. *Assume that there exists a geometric deformation R_t of R on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ such that $\dot{R}_0 = f \in Z^2(R)$, then $K(f) = 0$.*

Proof. Consider the Taylor expansion of R_t around $t = 0$, then we have

$$R_t(x) = R(x) + tf(x) + \frac{t^2}{2}g(x) + o(t^3).$$

Since $[R_t(x), R_t(y)]_{\mathfrak{g}} = R_t([R_t(x), y]_{\mathfrak{g}} + [x, R_t(y)]_{\mathfrak{g}}) - [x, y]_{\mathfrak{g}}$ and $f \in Z^2(R)$, we have

$$\frac{t^2}{2}(\mathfrak{d}_{\text{CE}}^R(g)(x, y) + 2[f(x), f(y)]_{\mathfrak{g}} - 2f([f(x), y]_{\mathfrak{g}} + [x, f(y)]_{\mathfrak{g}})) + o(t^3) = 0. \quad (9)$$

Thus by (6) and (9), we obtain $\llbracket [f, f] \rrbracket = \mathfrak{d}_{\text{CE}}^R g$, which implies that $K(f) = 0$. ■

Let $E \xrightarrow{\pi} M$ be a vector bundle. Assume that there is a smooth action $\cdot : G \times E \rightarrow E$ of a Lie group G on E preserving the zero-section $Z : M \rightarrow E$. It follows that M inherits a G -action. We also denote the action of G on M by $\cdot : G \times M \rightarrow M$. For all $x \in M$, define a smooth map $\mu_x : G \rightarrow M$ by $\mu_x(g) = g \cdot x$. Denote the tangent map from \mathfrak{g} to $T_x M$ by $D(\mu_x)_{e_G}$, where e_G is the unit of G .

Definition 4.5 ([5]). A section $s : M \rightarrow E$ is called *equivariant* if s satisfies

$$s(g \cdot x) = g \cdot s(x), \quad \forall g \in G, x \in M.$$

Denote the zero set of a section $s : M \rightarrow E$ by $z(s) = \{x \in M \mid s(x) = 0\}$. A zero $x \in M$ of s is called *non-degenerate* if the sequence

$$\mathfrak{g} \xrightarrow{D(\mu_x)_{e_G}} T_x M \xrightarrow{D^v(s)_x} E_x$$

is exact, where $D^v(s)_x$ is the vertical derivative of s at x .

Proposition 4.6 ([5]). *Let s be an equivariant section of the vector bundle $E \xrightarrow{\pi} M$ and x be a non-degenerate zero of s . Then there is an open neighborhood U of x and a smooth map $p : U \rightarrow G$ such that for all $m \in U$ with $s(m) = 0$, one has $p(m) \cdot x = m$. In particular, the orbit of x under the action of G and the zero set of s coincide in an open neighborhood of x .*

Proposition 4.7 ([5]). *Let E and F be vector bundles over a smooth manifold M . Let $s \in \Gamma(E)$ be a section and $\phi \in \Gamma(\text{Hom}(E, F))$ be a vector bundle map such that $\phi \circ s = 0$. Suppose that $x \in M$ is $s(x) = 0$ such that*

$$T_x M \xrightarrow{D^v(s)_x} E_x \xrightarrow{\phi_x} F_x$$

is exact. Then $s^{-1}(0)$ is locally a manifold around x of dimension $\dim \ker(D^v(s)_x)$.

Denote the group whose elements are inner automorphisms of a Lie algebra \mathfrak{g} by $\text{InnAut}(\mathfrak{g})$. Then its Lie algebra is the Lie algebra of inner derivations of \mathfrak{g} and denote it by $\text{InnDer}(\mathfrak{g})$. Define an action of $\text{InnAut}(\mathfrak{g})$ on $\text{Hom}(\mathfrak{g}, \mathfrak{g})$ by

$$\cdot : \text{InnAut}(\mathfrak{g}) \times \text{Hom}(\mathfrak{g}, \mathfrak{g}) \rightarrow \text{Hom}(\mathfrak{g}, \mathfrak{g}), \quad A \cdot f = AfA^{-1},$$

for all $A \in \text{InnAut}(\mathfrak{g})$, $f \in \text{Hom}(\mathfrak{g}, \mathfrak{g})$. Assume that R is a modified r -matrix on a Lie algebra \mathfrak{g} , then the orbit $\text{Orb}_R = \{A \cdot R \mid A \in \text{InnAut}(\mathfrak{g})\}$ of R is a manifold. Define a map $\mu_R : \text{InnAut}(\mathfrak{g}) \rightarrow \text{Hom}(\mathfrak{g}, \mathfrak{g})$ by $\mu_R(A) = A \cdot R$. Then $T_R \text{Orb}_R$ is $D(\mu_R)_{e_G}(\text{InnDer}(\mathfrak{g}))$, where $D(\mu_R)_{e_G}$ is the tangent map of μ_R at e_G .

Proposition 4.8. *With the above notations, $T_R \text{Orb}_R$ is $B^2(R)$.*

Proof. Since $T_R \text{Orb}_R = D(\mu_R)_{e_G}(\text{InnDer}(\mathfrak{g}))$, for any $v \in T_R \text{Orb}_R$, there exists $x \in \mathfrak{g}$ such that

$$\begin{aligned} v &= \left. \frac{d}{dt} \right|_{t=0} \exp(t \text{ad}_x) \cdot R \\ &= \left. \frac{d}{dt} \right|_{t=0} (\exp(t \text{ad}_x) R \exp(-t \text{ad}_x)) \\ &= \text{ad}_x R - R \text{ad}_x \\ &= -d_{\text{CE}}^R x. \end{aligned}$$

Thus we have $T_R \text{Orb}_R = B^2(R)$. ■

Theorem 4.9. *Let R be a modified r -matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. If $H^2(R) = 0$, then there exists an open neighborhood $U \subset \text{Hom}(\mathfrak{g}, \mathfrak{g})$ of R and a smooth map $p : U \rightarrow \text{InnAut}(\mathfrak{g})$ such that $p(R') \cdot R = R'$ for every modified r -matrix $R' \in U$.*

Proof. Denote by $M = \text{Hom}(\mathfrak{g}, \mathfrak{g})$ and $E = \text{Hom}(\mathfrak{g}, \mathfrak{g}) \times \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$. Then E is a trivial vector bundle over M with fiber $\text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$. Define an action of $\text{InnAut}(\mathfrak{g})$ on the manifold E by

$$\cdot : \text{InnAut}(\mathfrak{g}) \times E \rightarrow E, \quad A \cdot (f, \alpha) = (AfA^{-1}, A\alpha \circ A^{-1}),$$

for $(f, \alpha) \in E$, $A \in \text{InnAut}(\mathfrak{g})$, where $A\alpha \circ A^{-1}(x, y) = A\alpha(A^{-1}x, A^{-1}y)$ for any $x, y \in \mathfrak{g}$. Define a section $s : M \rightarrow E$ by

$$s(f) = (f, S(f)), \quad \forall f \in M,$$

where $S : \text{Hom}(\mathfrak{g}, \mathfrak{g}) \rightarrow \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ is given by

$$S(f)(x, y) = [f(x), f(y)]_{\mathfrak{g}} - f([f(x), y]_{\mathfrak{g}}) + [x, f(y)]_{\mathfrak{g}} + [x, y]_{\mathfrak{g}},$$

for all $f \in \text{Hom}(\mathfrak{g}, \mathfrak{g})$, $x, y \in \mathfrak{g}$. Then for any $A \in \text{InnAut}(\mathfrak{g})$, $f \in M$ and $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} AS(f) \circ A^{-1}(x, y) &= A([f(A^{-1}x), f(A^{-1}y)]_{\mathfrak{g}}) - f([f(A^{-1}x), A^{-1}y]_{\mathfrak{g}}) \\ &\quad + [A^{-1}x, f(A^{-1}y)]_{\mathfrak{g}} + [A^{-1}x, A^{-1}y]_{\mathfrak{g}}) \\ &= [Af(A^{-1}x), Af(A^{-1}y)]_{\mathfrak{g}} - AfA^{-1}([Af(A^{-1}x), y]_{\mathfrak{g}}) \\ &\quad + [x, Af(A^{-1}y)]_{\mathfrak{g}}) + [x, y]_{\mathfrak{g}} \\ &= S(AfA^{-1})(x, y). \end{aligned}$$

Thus we have

$$A \cdot s(f) = (AfA^{-1}, AS(f) \circ A^{-1}) = s(A \cdot f),$$

which implies that s is an equivariant section.

Since R is a modified r -matrix on the Lie algebra \mathfrak{g} , it follows that $R \in z(s)$. Moreover, since E is a trivial vector bundle, we have $D^v(s)_R = D(S)_R : T_RM \rightarrow E_R$. For any $g \in \text{Hom}(\mathfrak{g}, \mathfrak{g})$, $x, y \in \mathfrak{g}$,

$$\begin{aligned} D(S)_R(g)(x, y) &= \frac{d}{dt} \Big|_{t=0} S(R + tg)(x, y) \\ &= \frac{d}{dt} \Big|_{t=0} \left([R(x) + tg(x), R(y) + tg(y)]_{\mathfrak{g}} \right. \\ &\quad \left. - (R + tg)([R(x) + tg(x), y]_{\mathfrak{g}}) + [x, R(y) + tg(y)]_{\mathfrak{g}} + [x, y]_{\mathfrak{g}} \right) \\ &= [g(x), R(y)]_{\mathfrak{g}} + [R(x), g(y)]_{\mathfrak{g}} - R([g(x), y]_{\mathfrak{g}}) + [x, g(y)]_{\mathfrak{g}} \\ &\quad - g([R(x), y]_{\mathfrak{g}}) + [x, R(y)]_{\mathfrak{g}} \\ &= d_{\text{CE}}^R(g)(x, y). \end{aligned} \tag{10}$$

By Proposition 4.8 and $H^2(R) = 0$, we have that R is a non-degenerate zero of s . By Proposition 4.6, there exists an open neighborhood $U \subset \text{Hom}(\mathfrak{g}, \mathfrak{g})$ of R and a smooth map $p : U \rightarrow \text{InnAut}(\mathfrak{g})$ such that $p(R) \cdot R = R'$ for every modified r -matrix $R' \in U$. ■

Theorem 4.10. *Let R be a modified r -matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. If $H^3(R) = 0$, then the space of modified matrices on the Lie algebra \mathfrak{g} is a manifold in a neighborhood of R whose dimension is $\dim Z^2(R)$.*

Proof. Denote by $M = \text{Hom}(\mathfrak{g}, \mathfrak{g})$, $E = \text{Hom}(\mathfrak{g}, \mathfrak{g}) \times \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ and $F = \text{Hom}(\mathfrak{g}, \mathfrak{g}) \times \text{Hom}(\wedge^3 \mathfrak{g}, \mathfrak{g})$. Then E and F are trivial vector bundles over M with fiber $\text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ and $\text{Hom}(\wedge^3 \mathfrak{g}, \mathfrak{g})$ respectively. Define a smooth map $\phi : E \rightarrow F$ by

$$\phi(f, \alpha) = (f, \llbracket f, \alpha \rrbracket), \quad \forall f \in M, \alpha \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}).$$

Thus ϕ is a vector bundle map.

Moreover, denote the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$ by π , define $s(f) = \pi - \frac{1}{2}[[f, f]]$. By Proposition 3.2, we know that π lies in the center, we have

$$\phi \circ s(f) = \left(f, [[f, \pi]] - \frac{1}{2}[[f, [[f, f]]]] \right) = (f, 0),$$

which implies $\phi \circ s = 0$. Moreover, denote $\phi_R : E_R \rightarrow F_R$ by $\phi(R, \cdot)$, then $\phi_R = d_{\text{CE}}^R$. By (10) and $H^3(R) = 0$, we have that

$$T_R M \xrightarrow{D^v(s)_R} E_R \xrightarrow{\phi_R} F_R$$

is exact. By Proposition 4.7, we obtain that the space of modified r -matrices on the Lie algebra \mathfrak{g} is a manifold in a neighborhood of R , whose dimension is $\dim Z^2(R)$. ■

At the end of this section, we give the sufficient condition on a 2-cocycle to give a geometric deformation. Recall that the necessary condition is given in Proposition 4.4 using the Kuranishi map.

Corollary 4.11. *With the above notations, if $H^3(R) = 0$, then any $f \in Z^2(R)$ gives rise to a geometric deformation of R .*

Proof. Since R is a modified r -matrix and $H^3(R) = 0$, we have that the space W of modified r -matrices on the Lie algebra \mathfrak{g} is a manifold in a neighborhood of R , whose dimension is $\dim Z^2(R)$. Assume $\gamma(t) \in W$, by Proposition 4.3, we have $\dot{\gamma}(0) \in Z^2(R)$. Moreover, $\dim W = \dim Z^2(R)$, then $T_R W = Z^2(R)$. Thus any $f \in Z^2(R)$ gives rise to a geometric deformation of R . ■

5. Linear deformations of modified r -matrices

In this section, we study linear deformations of a modified r -matrix using the established cohomology theory. In particular, a trivial linear deformation leads to a Nijenhuis element for a modified r -matrix R .

Definition 5.1. Let R be a modified r -matrix on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $\hat{R} : \mathfrak{g} \rightarrow \mathfrak{g}$ be a linear map. If there exists a positive number $\varepsilon \in \mathbb{R}$ such that $R_t = R + t\hat{R}$ is still a modified r -matrix on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ for all $t \in (-\varepsilon, \varepsilon)$, we say that \hat{R} generates a *linear deformation* of the modified r -matrix R .

Definition 5.2. Let $R : \mathfrak{g} \rightarrow \mathfrak{g}$ be a modified r -matrix on \mathfrak{g} . Two linear deformations $R_t^1 = R + t\hat{R}_1$ and $R_t^2 = R + t\hat{R}_2$ are said to be *equivalent* if there exists an $x \in \mathfrak{g}$ such that

$$\varphi_t = \text{Id}_{\mathfrak{g}} + t\text{ad}_x,$$

satisfies the following conditions:

- (i) $\varphi_t([y, z]_{\mathfrak{g}}) = [\varphi_t(y), \varphi_t(z)]_{\mathfrak{g}}, \forall y, z \in \mathfrak{g}$,
- (ii) $R_t^2 \circ \varphi_t = \varphi_t \circ R_t^1$.

Theorem 5.3. Let $\hat{R} : \mathfrak{g} \rightarrow \mathfrak{g}$ generate a linear deformation of the modified r -matrix R . Then \hat{R} is a 2-cocycle.

Let R_t^1 and R_t^2 be equivalent linear deformations of R generated by \hat{R}_1 and \hat{R}_2 respectively. Then $[\hat{R}_1] = [\hat{R}_2]$ in $H^2(R)$.

Proof. Since $R_t = R + t\hat{R}$ is a modified r -matrix on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$, we have

$$[R_t(x), R_t(y)]_{\mathfrak{g}} = R_t([R_t(x), y]_{\mathfrak{g}} + [x, R_t(y)]_{\mathfrak{g}}) - [x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}.$$

Consider the coefficients of t and t^2 respectively, we have

$$\begin{aligned} & [\hat{R}(x), R(y)]_{\mathfrak{g}} + [R(x), \hat{R}(y)]_{\mathfrak{g}} \\ &= R([\hat{R}(x), y]_{\mathfrak{g}} + [x, \hat{R}(y)]_{\mathfrak{g}}) + \hat{R}([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}, \end{aligned} \quad (11)$$

and

$$[\hat{R}(x), \hat{R}(y)]_{\mathfrak{g}} = \hat{R}([\hat{R}(x), y]_{\mathfrak{g}} + [x, \hat{R}(y)]_{\mathfrak{g}}). \quad (12)$$

By (11), we deduce that \hat{R} is a 2-cocycle of the modified r -matrix R .

If R_t^1 and R_t^2 are equivalent linear deformations of R , then there exists $x \in \mathfrak{g}$ such that

$$(\text{Id}_{\mathfrak{g}} + t\text{ad}_x)(R + t\hat{R}_1)(u) = (R + t\hat{R}_2)(\text{Id}_{\mathfrak{g}} + t\text{ad}_x)(u), \quad \forall u \in \mathfrak{g},$$

which implies

$$\hat{R}_1(u) - \hat{R}_2(u) = [R(u), x]_{\mathfrak{g}} - R([u, x]_{\mathfrak{g}}), \quad \forall u \in \mathfrak{g}. \quad (13)$$

By (13), we have

$$\hat{R}_1 - \hat{R}_2 = d_{\text{CE}}^R x,$$

where d_{CE}^R is given by (4). Thus $[\hat{R}_1] = [\hat{R}_2]$ in $H^2(R)$. ■

Definition 5.4. A linear deformation of a modified r -matrix R generated by \hat{R} is *trivial* if there exists an $x \in \mathfrak{g}$ such that $\text{Id} + t\text{ad}_x$ is an isomorphism from $R_t = R + t\hat{R}$ to R .

Definition 5.5. Let R be a modified r -matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. An element $x \in \mathfrak{g}$ is called a *Nijenhuis element* associated to R if x satisfies

$$[[x, y]_{\mathfrak{g}}, [x, z]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \quad (14)$$

$$[x, [x, R(y)]_{\mathfrak{g}}]_{\mathfrak{g}} = [x, R([x, y]_{\mathfrak{g}})]_{\mathfrak{g}}, \quad (15)$$

for all $y, z \in \mathfrak{g}$.

Let \hat{R} generate a trivial linear deformation of a modified r -matrix R on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. Then there exists $x \in \mathfrak{g}$ such that

$$\begin{aligned} (\text{Id} + t\text{ad}_x)[y, z]_{\mathfrak{g}} &= [y + t[x, y]_{\mathfrak{g}}, z + t[x, z]_{\mathfrak{g}}]_{\mathfrak{g}}, \\ R(y + t[x, y]_{\mathfrak{g}}) &= (\text{Id} + t\text{ad}_x)(R(y) + t\hat{R}(y)), \end{aligned}$$

for all $y, z \in \mathfrak{g}$. Therefore, we have

$$[[x, y]_{\mathfrak{g}}, [x, z]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \quad [x, \widehat{R}(y)]_{\mathfrak{g}} = 0, \quad R([x, y]_{\mathfrak{g}}) = [x, R(y)]_{\mathfrak{g}} + \widehat{R}(y).$$

Thus a trivial linear deformation gives rise to a Nijenhuis element.

Theorem 5.6. *Let R be a modified r -matrix on a Lie algebra \mathfrak{g} . Then for any Nijenhuis element $x \in \mathfrak{g}$, $R_t = R + t d_{\text{CE}}^R x$ is a trivial linear deformation of the modified r -matrix R .*

Proof. Denote by $\widehat{R} = d_{\text{CE}}^R x$. To show that R_t is a linear deformation of R , it suffices to show that (11) and (12) hold. Note that (11) means that \widehat{R} is closed, which holds naturally since now $\widehat{R} = d_{\text{CE}}^R x$ is exact. Thus, we need to verify that equation (12) holds. For any $y, z \in \mathfrak{g}$, by (4), we obtain $\widehat{R}(y) = [R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}})$. Moreover, by (1), (14) and (15), it follows that

$$\begin{aligned} & [R([y, x]_{\mathfrak{g}}), R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}} \\ & \stackrel{(1),(14)}{=} R([R([y, x]_{\mathfrak{g}}), [z, x]_{\mathfrak{g}}]_{\mathfrak{g}} + [[y, x]_{\mathfrak{g}}, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}) \\ & = R([R([y, x]_{\mathfrak{g}}), [z, x]_{\mathfrak{g}}]_{\mathfrak{g}}) + R([[y, x]_{\mathfrak{g}}, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}) \\ & = R([[R([y, x]_{\mathfrak{g}}), z]_{\mathfrak{g}}, x]_{\mathfrak{g}}) + R([z, [R([y, x]_{\mathfrak{g}}), x]_{\mathfrak{g}}]) \\ & \quad + R([[y, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}}) + R([y, [x, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}}) \\ & \stackrel{(14),(15)}{=} R([[R([y, x]_{\mathfrak{g}}), z]_{\mathfrak{g}}, x]_{\mathfrak{g}}) + R([x, [z, [x, R(y)]_{\mathfrak{g}}]_{\mathfrak{g}}]) \\ & \quad + R([[y, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}}) - R([x, [y, [x, R(z)]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}}), \\ & - [[R(y), x]_{\mathfrak{g}}, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}} \\ & = -[[R(y), R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}} - [R(y), [x, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} \\ & \stackrel{(1)}{=} -[R([R(y), [z, x]_{\mathfrak{g}}]_{\mathfrak{g}}), x]_{\mathfrak{g}} - [R([y, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}} \\ & \quad + [[y, [z, x]_{\mathfrak{g}}]_{\mathfrak{g}}, x]_{\mathfrak{g}} - [R(y), [x, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} \\ & \stackrel{(14),(15)}{=} -[x, [x, R([R(y), z]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} - [R([z, [R(y), x]_{\mathfrak{g}}]_{\mathfrak{g}}), x]_{\mathfrak{g}} \\ & \quad - [R([y, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}} + [[y, [z, x]_{\mathfrak{g}}]_{\mathfrak{g}}, x]_{\mathfrak{g}} + [x, [x, [R(y), R(z)]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}} \\ & \quad + [[x, [R(y), x]_{\mathfrak{g}}]_{\mathfrak{g}}, R(z)]_{\mathfrak{g}} \\ & \stackrel{(1),(15)}{=} [x, [x, R([y, R(z)]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} - [x, [x, [y, z]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}} + [[y, [z, x]_{\mathfrak{g}}]_{\mathfrak{g}}, x]_{\mathfrak{g}} \\ & \quad - [R([z, [R(y), x]_{\mathfrak{g}}]_{\mathfrak{g}}), x]_{\mathfrak{g}} - [R([y, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}} \\ & \quad - [[x, R([x, y]_{\mathfrak{g}})]_{\mathfrak{g}}, R(z)]_{\mathfrak{g}}, \end{aligned}$$

and

$$\begin{aligned} & - [R([y, x]_{\mathfrak{g}}), [R(z), x]_{\mathfrak{g}}]_{\mathfrak{g}} \\ & \stackrel{(1)}{=} [R([[R(z), y]_{\mathfrak{g}}, x]_{\mathfrak{g}}), x]_{\mathfrak{g}} + [R([y, [R(z), x]_{\mathfrak{g}}]_{\mathfrak{g}}), x]_{\mathfrak{g}} + [R([z, R([y, x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}} \\ & \quad - [[z, [y, x]_{\mathfrak{g}}]_{\mathfrak{g}}, x]_{\mathfrak{g}} + [R(z), [x, R([y, x]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}}. \end{aligned}$$

By (14) and the above equations, we have

$$\begin{aligned}
 & [\widehat{R}(y), \widehat{R}(z)]_{\mathfrak{g}} - \widehat{R}([\widehat{R}(y), z]_{\mathfrak{g}} + [y, \widehat{R}(z)]_{\mathfrak{g}}) \\
 &= [[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}} \\
 &\quad - [R([[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), z])_{\mathfrak{g}}, x]_{\mathfrak{g}} + R([[[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), z]_{\mathfrak{g}}, x]_{\mathfrak{g}}) \\
 &\quad - [R([y, [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}} + R([[[y, [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}}) \\
 &= -[[R(y), x]_{\mathfrak{g}}, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}} + [R([y, x]_{\mathfrak{g}}), R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}} \\
 &\quad - [R([y, x]_{\mathfrak{g}}), [R(z), x]_{\mathfrak{g}}]_{\mathfrak{g}} - [R([[[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), z])_{\mathfrak{g}}, x]_{\mathfrak{g}} \\
 &\quad + R([[[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), z]_{\mathfrak{g}}, x]_{\mathfrak{g}}) \\
 &\quad - [R([y, [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}} + R([[[y, [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}}) \\
 &= 0.
 \end{aligned}$$

Thus $R_t = R + t d_{\text{CE}}^R x$ is a linear deformation of the modified r -matrix R . Since $x \in \mathfrak{g}$ is a Nijenhuis element, we have $(\text{Id} + t \text{ad}_x)[y, z]_{\mathfrak{g}} = [y + t[x, y]_{\mathfrak{g}}, z + t[x, z]_{\mathfrak{g}}]_{\mathfrak{g}}$ and $R \circ (\text{Id} + t \text{ad}_x) = (\text{Id} + t \text{ad}_x) \circ (R + t d_{\text{CE}}^R x)$. Thus for any Nijenhuis element $x \in \mathfrak{g}$, $R_t = R + t d_{\text{CE}}^R x$ is a trivial linear deformation of the modified r -matrix R . ■

At the end of this section, we consider the relation between linear deformations of modified r -matrices and linear deformations of the induced Lie algebras. Recall that a skew-symmetric bilinear map $\omega : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ generates a linear deformation of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ if $[\cdot, \cdot]_t = [\cdot, \cdot]_{\mathfrak{g}} + t\omega$ defines a Lie algebra structure on \mathfrak{g} for all $t \in (-\varepsilon, \varepsilon)$.

Proposition 5.7. *Let \widehat{R} generate a linear deformation of a modified r -matrix R on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. Then ω defined by*

$$\omega(x, y) = [\widehat{R}(x), y]_{\mathfrak{g}} + [x, \widehat{R}(y)]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g},$$

generates a linear deformation of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_R)$ given by the modified r -matrix R , which is exactly the one associated to the linear deformation of the modified r -matrix R .

Proof. It is obvious that

$$[x, y]_{R_t} = [R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}} + t([\widehat{R}(x), y]_{\mathfrak{g}} + [x, \widehat{R}(y)]_{\mathfrak{g}}) = [x, y]_R + t\omega(x, y).$$

Since $[\cdot, \cdot]_{R_t}$ are Lie algebra structures, we have that ω generates a linear deformation of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_R)$ given by the modified r -matrix R . ■

The notion of a Nijenhuis operator on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ was given in [7], which gives rise to a trivial linear deformation of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$.

Definition 5.8 ([7]). Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear map $N : \mathfrak{g} \rightarrow \mathfrak{g}$ is called *Nijenhuis operator* if

$$[N(x), N(y)]_{\mathfrak{g}} = N([N(x), y]_{\mathfrak{g}} + [x, N(y)]_{\mathfrak{g}}) - N^2([x, y]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}.$$

Theorem 5.9. *Let $x \in \mathfrak{g}$ be a Nijenhuis element associated to a modified matrix R . Then ad_x is a Nijenhuis operator on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_R)$.*

Proof. For any $x, y, z \in \mathfrak{g}$, by (14) and (15), we have

$$\begin{aligned}
 & [\text{ad}_x y, \text{ad}_x z]_R \\
 &= [R([x, y]_{\mathfrak{g}}), [x, z]_{\mathfrak{g}}]_{\mathfrak{g}} + [[x, y]_{\mathfrak{g}}, R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}} \\
 &= [[R([x, y]_{\mathfrak{g}}), x]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [x, [R([x, y]_{\mathfrak{g}}), z]_{\mathfrak{g}}]_{\mathfrak{g}} + [[x, R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}}, y]_{\mathfrak{g}} \\
 &\quad + [x, [y, R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} \\
 &= -[[x, [x, R(y)]_{\mathfrak{g}}]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [x, [R([x, y]_{\mathfrak{g}}), z]_{\mathfrak{g}}]_{\mathfrak{g}} + [[x, [x, R(z)]_{\mathfrak{g}}]_{\mathfrak{g}}, y]_{\mathfrak{g}} \\
 &\quad + [x, [y, R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} \\
 &= -[x, [[x, R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}}]_{\mathfrak{g}} + [x, [R([x, y]_{\mathfrak{g}}), z]_{\mathfrak{g}}]_{\mathfrak{g}} + [x, [[x, R(z)]_{\mathfrak{g}}, y]_{\mathfrak{g}}]_{\mathfrak{g}} \\
 &\quad + [x, [y, R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}}
 \end{aligned}$$

and

$$\begin{aligned}
 & \text{ad}_x([\text{ad}_x y, z]_R + [y, \text{ad}_x z]_R) - \text{ad}_x^2([y, z]_R) \\
 &= [x, [R([x, y]_{\mathfrak{g}}), z]_{\mathfrak{g}}]_{\mathfrak{g}} + [[x, y]_{\mathfrak{g}}, R(z)]_{\mathfrak{g}} + [x, [R(y), [x, z]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}} \\
 &\quad + [x, [y, R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} - [x, [x, [R(y), z]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}} - [x, [x, [y, R(z)]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}}.
 \end{aligned}$$

Thus $[\text{ad}_x y, \text{ad}_x z]_R = \text{ad}_x([\text{ad}_x y, z]_R + [y, \text{ad}_x z]_R) - \text{ad}_x^2([y, z]_R)$, which implies that ad_x is a Nijenhuis operator on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_R)$. \blacksquare

6. Applications

In this section, we give some applications of the above deformation theories, including deformations of complements of the diagonal Lie algebra \mathfrak{g}_{Δ} and compatible Poisson structures.

6.1. Deformations of complements

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra, then we have a direct-product Lie algebra structure $[\cdot, \cdot]_{\oplus}$ on $\mathfrak{g} \oplus \mathfrak{g}$, where

$$[(x_1, y_1), (x_2, y_2)]_{\oplus} = ([x_1, x_2]_{\mathfrak{g}}, [y_1, y_2]_{\mathfrak{g}}), \quad \forall x_i, y_i \in \mathfrak{g}, i = 1, 2.$$

Define the subspace \mathfrak{g}_{Δ} by $\mathfrak{g}_{\Delta} = \{(x, x) \mid \forall x \in \mathfrak{g}\}$ and the subspace $\mathfrak{g}_{-\Delta} = \{(x, -x) \mid \forall x \in \mathfrak{g}\}$. It is obvious that \mathfrak{g}_{Δ} is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$, while $\mathfrak{g}_{-\Delta}$ is not a Lie subalgebra. To find a complement of \mathfrak{g}_{Δ} which is also a Lie subalgebra, it is natural to consider the graph of certain linear map from $\mathfrak{g}_{-\Delta}$ to \mathfrak{g}_{Δ} . It is known that a complement of \mathfrak{g}_{Δ} is isomorphic to a graph of a linear map from $\mathfrak{g}_{-\Delta}$ to \mathfrak{g}_{Δ} . Let $R \in \mathfrak{gl}(\mathfrak{g})$ be a linear map. Define a linear map $\hat{R} : \mathfrak{g}_{-\Delta} \rightarrow \mathfrak{g}_{\Delta}$ by

$$\hat{R}(x, -x) = (-R(x), -R(x)), \quad \forall x \in \mathfrak{g}.$$

Proposition 6.1. *With the above notations, the graph $\mathcal{G}(\widehat{R}) := \{\widehat{R}u + u \mid u \in \mathfrak{g}_{-\Delta}\}$ is a Lie subalgebra of $(\mathfrak{g} \oplus \mathfrak{g}, [\cdot, \cdot]_{\oplus})$ if and only if R is a modified r -matrix.*

Proof. For all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} & [(-R(x), -R(x)) + (x, -x), (-R(y), -R(y)) + (y, -y)]_{\oplus} \\ &= ([x, y]_{\mathfrak{g}}, [x, y]_{\mathfrak{g}}) + ([R(x), R(y)]_{\mathfrak{g}}, [R(x), R(y)]_{\mathfrak{g}}) \\ &\quad + (-[R(x), y]_{\mathfrak{g}}, [R(x), y]_{\mathfrak{g}}) + (-[x, R(y)]_{\mathfrak{g}}, [x, R(y)]_{\mathfrak{g}}) \\ &= ([x, y]_{\mathfrak{g}} + [R(x), R(y)]_{\mathfrak{g}}, [x, y]_{\mathfrak{g}} + [R(x), R(y)]_{\mathfrak{g}}) \\ &\quad + (-[R(x), y]_{\mathfrak{g}} - [x, R(y)]_{\mathfrak{g}}, [R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}). \end{aligned}$$

Thus $\mathcal{G}(\widehat{R})$ is a Lie subalgebra if and only if

$$R([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}) = [x, y]_{\mathfrak{g}} + [R(x), R(y)]_{\mathfrak{g}},$$

i.e., R is a modified r -matrix. ■

Proposition 6.2. *Let R be a modified r -matrix. Then $(\mathfrak{g}_{\Delta}, \mathcal{G}(\widehat{R}))$ is a matched pair of Lie algebras.*

Proof. It is obvious that $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\Delta} \oplus \mathcal{G}(\widehat{R})$ since $\mathfrak{g}_{\Delta} \cap \mathcal{G}(\widehat{R}) = 0$. Then the conclusion follows from the fact that both \mathfrak{g}_{Δ} and $\mathcal{G}(\widehat{R})$ are Lie subalgebras. ■

Summarizing the above studies, we have the following conclusion.

Theorem 6.3. *Let R_t be a geometric deformation of a modified r -matrix R . Then $\mathcal{G}(\widehat{R}_t)$ is a deformation of the complement $\mathcal{G}(\widehat{R})$. Moreover, $(\mathfrak{g}_{\Delta}, \mathcal{G}(\widehat{R}_t))$ are matched pairs of Lie algebras.*

6.2. Compatible Poisson structures

A *compatible Poisson structure* consists of two Poisson structures π, π' on a manifold M such that $\pi + \pi'$ is also a Poisson structure on the manifold M .

Let R be a modified r -matrix on a Lie algebra \mathfrak{g} . Then $(\mathfrak{g}, [\cdot, \cdot]_R)$ is a Lie algebra and we denote by $(\mathfrak{g}^*, \{\cdot, \cdot\}_R)$ the corresponding linear Poisson manifold.

Proposition 6.4. *Let R be a modified r -matrix on a Lie algebra \mathfrak{g} and $R_t = R + t\widehat{R}$ be a linear deformation of R . For any $t_1, t_2 \in \mathbb{R}$, $\{\cdot, \cdot\}_{R_{t_1}}$ and $\{\cdot, \cdot\}_{R_{t_2}}$ are compatible Poisson structures on \mathfrak{g}^* .*

Proof. By the fact that $R + \frac{t_1+t_2}{2}\widehat{R}$ is also a modified r -matrix on the Lie algebra \mathfrak{g} , we have

$$[x, y]_{R_{t_1}} + [x, y]_{R_{t_2}} = 2\left(\left[R(x) + \frac{t_1+t_2}{2}\widehat{R}(x), y\right]_{\mathfrak{g}} + \left[x, R(y) + \frac{t_1+t_2}{2}\widehat{R}(y)\right]_{\mathfrak{g}}\right),$$

which implies that $[\cdot, \cdot]_{R_{t_1}} + [\cdot, \cdot]_{R_{t_2}}$ is also a Lie bracket on the Lie algebra \mathfrak{g} by (2). Thus, for any $t_1, t_2 \in \mathbb{R}$, $\{\cdot, \cdot\}_{R_{t_1}}$ and $\{\cdot, \cdot\}_{R_{t_2}}$ are compatible Poisson structures on \mathfrak{g}^* . ■

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